Trace and flux *a priori* error estimates in finite-element approximations of Signorni-type problems

OLAF STEINBACH

Institut für Numerische Mathematik, TU Graz, Steyrergasse 30, 8010 Graz, Austria
o.steinbach@tugraz.at

BARBARA WOHLMUTH

AND

LINUS WUNDERLICH*

*M2 – Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3,
85748 Garching, Germany*
*Corresponding author: linus.wunderlich@ma.tum.de   wohlmuth@ma.tum.de

Variational inequalities play an important role in many applications and are an active research area. Optimal *a priori* error estimates in the natural energy norm do exist, but only very few results are known for different norms. Here, we consider as prototype a simple Signorini problem, and provide new optimal order *a priori* error estimates for the trace and the flux on the Signorini boundary. The *a priori* analysis is based on a continuous and a discrete Steklov–Poincaré operator, as well as on Aubin–Nitsche-type duality arguments. Numerical results illustrate the convergence rates of the finite-element approach.

Keywords: anisotropic norms; Lagrange multiplier; Schur complement; Signorini boundary conditions; Steklov–Poincaré operator.

1. Introduction

Signorini-type problems are nonlinear boundary value problems that can be regarded as a simplified scalar model of elastic contact problems which are of interest in many engineering applications; see Laursen (2002) and Wriggers (2002). Signorini and contact problems share a similar formulation, and their approximation remains a challenging task due to the nonlinear boundary condition. *A priori* error estimates in the $H^1(\Omega)$-norm for such problems were investigated over many years; see Scarpini & Vivaldi (1977), BenBelgacem & Renard (2003) for Signorini and BenBelgacem *et al.* (1999), Lhalouani & Sassi (1999), Hild (2000) for contact problems. Optimal *a priori* error estimates for two body contact problems in the $H^1(\Omega)$-norm were established in Hüeber & Wohlmuth (2005), and more recently reconsidered in Hild & Renard (2012) and Drouet & Hild (2015). However, the optimal order *a priori* analysis for different norms of interest is still missing.

In this work, we restrict ourselves to the Poisson equation with unilateral Signorini boundary conditions, and provide optimal-order convergence rates in norms associated with the Signorini boundary Γ_S. More precisely, we consider *a priori* error estimates for the trace in the $H_{00}^{1/2}(\Gamma_S)$-norm and for the Lagrange multiplier, i.e., the flux, in the $H^{-1/2}(\Gamma_S)$-norm. As a corollary, we show improved *a priori* estimates in the L^2-norm for the primal variable on Ω and for the dual variable on Γ_S. While convergence rates for traces can often be established using estimates in the domain, these rates are typically

not optimal. To the best of our knowledge, no optimal-order error estimates in different norms than $H^1(\Omega)$ have been proved so far. The order of the finite-element approximation in the $L^2(\Omega)$-norm is firstly addressed in the early paper by Natterer (1976). However, the theoretical results are limited to very special situations. A generalization can be found in Coorevits *et al.* (2001) and Suttmeier (2008), but for a straightforward application to Signorini problems, the required dual regularity is lacking, so we do not follow these ideas. Recently introduced techniques allow optimal estimates on interfaces and boundaries for linear problems under moderately stronger regularity assumptions; see Apel *et al.* (2012), Melenk & Wohlmuth (2012), Melenk *et al.* (2013), Waluga & Wohlmuth (2013), Apel *et al.* (2015) and Larson & Massing (2014). These techniques can also be used to compensate a lack of regularity in the dual problem; see Horger *et al.* (2013). A reformulation of the primal variational inequality on the boundary, as applied in Spann (1993), Eck *et al.* (1999), Steinbach (2014), and a Strang lemma for variational inequalities allow us to use these techniques for the nonlinear Signorini problem.

This article is structured as follows: in the next section we state the Signorini-type problem and its discretization as a primal formulation. In Section 3, two reformulations which play an important part in the analysis are briefly recalled; namely, a saddle point problem and a variational formulation of the Schur complement. Since the Galerkin formulation of the continuous Schur complement differs from the discrete Schur complement, a Strang lemma is applied in Section 4, and the error is related to the difference of two Steklov–Poincaré operators. In Section 5, a rate for the primal error in the $H_{00}^{1/2}(\Gamma_S)$-norm is proved, based on anisotropic norms and dual problems with local data. As a corollary, improved rates for the $L^2(\Omega)$-norm are shown. The results are extended in Section 6, where optimal rates for the Lagrange multiplier in the natural $H^{-1/2}(\Gamma_S)$ and also in the stronger $L^2(\Gamma_S)$-norm are derived. Finally in Section 7, numerical results are presented which confirm the new theoretical *a priori* bounds and illustrate some additional aspects.

2. Problem setting and main result

We consider the Poisson equation with Signorini-type boundary conditions. The partial differential equation is defined in a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We assume Ω to be polyhedral, convex and bounded. The boundary $\Gamma := \partial\Omega$ is divided into two disjoint open parts $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_S$, such that Γ_D has a positive Lebesgue measure. For simplicity of notation, we assume Γ_S to be one facet of the boundary Γ. For $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_S)$, we consider Equations (2.1a) to (2.1c):

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{2.1b}$$

$$\partial_n u \leqslant 0, \quad u \leqslant g, \quad (u - g)\partial_n u = 0 \quad \text{on } \Gamma_S. \tag{2.1c}$$

The problem can be regarded as a simplified contact problem where the constraints on Γ_S play the role of a nonpenetration condition.

We assume that g is positive in a neighbourhood of $\partial\Gamma_S$ and that the actual contact set $\Gamma^{\text{act}} := \{x \in \Gamma_S : u(x) = g(x)\}$ is a compact subset of Γ_S.

REMARK 2.1 In general, weak solutions of Dirichlet–Neumann problems with smooth data can be represented as a series of singular components and a smooth part. The first singular component has typically a regularity of $H^{3/2-\varepsilon}(\Omega)$. However, due to the sign-condition of the Signorini boundary, the regularity is improved. As long as no jump of the outer unit normal is present at the boundary of $\Gamma^{\text{act}} \subset \Gamma_S$, the stress intensity factor associated with the first singular component has to be zero. We are interested in

the effects of the approximation caused by the Signorini boundary condition, so let us assume that these singular parts do not appear at any other part of the boundary. Hence, we assume f to be sufficiently smooth and the solution to be $H^{5/2-\varepsilon}(\Omega)$-regular; see Moussaoui & Khodja (1992).

2.1 *Weak formulations*

The nonlinear Signorini boundary condition yields a constrained minimization problem, e.g., Glowinski (1984) and Kinderlehrer & Stampacchia (2000). Let $V := \{v \in H^1(\Omega): v|_{\Gamma_D} = 0\}$ and denote the trace space of V restricted to Γ_S as $W := H_{00}^{1/2}(\Gamma_S)$. For simplicity of notation, we omit the trace operator whenever there is no ambiguity. We define the convex set of admissible functions by $K := \{v \in V: v|_{\Gamma_S} \leqslant g\}$, the bilinear form $a(u, v) := \int_\Omega \nabla u^{\mathrm{T}} \nabla v \, \mathrm{d}x$ and the linear form $f(v) := \int_\Omega fv \, \mathrm{d}x$.

The weak solution $u \in K$ then satisfies the variational inequality

$$a(u, v - u) \geqslant f(v - u), \quad v \in K. \tag{2.2}$$

For the discretization, we assume a family of shape-regular simplicial triangulations $\mathcal{T}_h$. We denote by N_{V_h} the number of vertices of the triangulation, except the ones on $\bar{\Gamma}_D$ and by N_{M_h} the number of vertices on Γ_S. Note that, since the Signorini boundary is a facet of the polyhedral domain, both boundary parts are exactly represented by the triangulation. We define the discrete primal space using first-order conforming finite elements $V_h := \{v_h \in C(\bar{\Omega}): v_h|_T \in \mathbb{P}_1(T), T \in \mathcal{T}_h, v_h|_{\Gamma_D} = 0\}$, spanned by the nodal Lagrange basis φ_i, $i = 1, \ldots, N_{V_h}$, and denote the discrete trace space restricted to Γ_S by W_h. For simplicity, let us assume that g is affine and strictly positive. The discretization of (2.2) then reads: find $u_h \in K_h := \{v_h \in V_h: v_h|_{\Gamma_S} \leqslant g\}$, such that

$$a(u_h, v_h - u_h) \geqslant f(v_h - u_h), \quad v_h \in K_h. \tag{2.3}$$

In the more general case, a suitable approximation of g can be used in the definition of K_h. We note that since g is affine one has $K_h \subset K$.

2.2 *Main results*

$H^1(\Omega)$-error estimates of order h for contact problems are given in Hüeber & Wohlmuth (2005) under some regularity assumption on the active set, as well as more recently in Hild & Renard (2012) and Drouet & Hild (2015) under weaker assumptions on the solution.

The previously used assumption is a certain regularity for the active set Γ^{act}, in order to exclude a fractal active set, see e.g., Wohlmuth (2011, Assumption 4.4). Given $\Sigma_h := \{x \in \Gamma_S : \mathrm{dist}\,(x, \partial \Gamma^{\mathrm{act}}) \leqslant 2h\}$, under the assumption

$$\|u - g\|_{L^2(\Sigma_h)} \leqslant ch^{2-\varepsilon}|u|_{H^{2-\varepsilon}(\Gamma_S)}, \quad u \in H^{2-\varepsilon}(\Gamma_S), \tag{2.4}$$

optimal $H^1(\Omega)$-error estimates can be derived more easily. For $d = 2$, this is fulfilled if $\partial \Gamma^{\mathrm{act}}$ contains a finite number of points. For $d = 3$, the abstract condition (2.4) is implied by the following criterion based on Γ^{act}; see Li *et al.* (2009, Lemma 2.1). The active set fulfils a cone property, has a piecewise C^1 boundary and there exists a $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$ and $x \in \partial \Gamma^{\mathrm{act}}$, one has $x + \delta n \notin \Gamma^{\mathrm{act}}$, where n is the outer unit normal of $\partial \Gamma^{\mathrm{act}}$ in Γ_S. See Fig. 1 for an illustration of the regularity condition. Note that due to recently introduced techniques in Drouet & Hild (2015), the regularity assumption is no longer necessary.
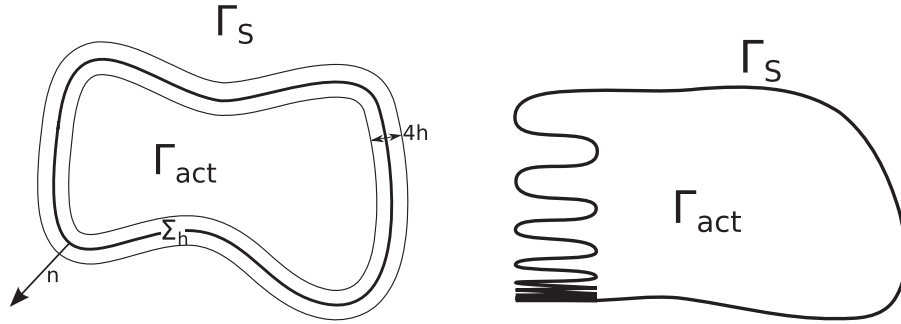
FIG. 1. Illustration of the regularity assumption. Left: fulfilled condition and an illustration of the set Σ_h. Right: violated criterion on Γ^{act}.

In the following, $\varepsilon \in (0, \frac{1}{2}]$ is fixed. Generic constants $0 < c, C < \infty$ are independent of the mesh size, but possibly dependent on the mesh regularity. Moreover, some of our generic constants are not robust with respect to ε.

The main result of this paper is summarized in the following theorem and proved in the following sections.

THEOREM 2.2 Let u be the solution of (2.2) and u_h be the solution of (2.3). Assuming $u \in H^{5/2-\varepsilon}(\Omega)$, $\varepsilon \in (0, \frac{1}{2}]$ fixed, we get

$$\|u - u_h\|_{H^{1/2}_{00}(\Gamma_S)} \leqslant c h^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

Note that the constant c depends on ε and tends to infinity if ε tends to zero.

Based on this trace estimate, we can easily improve the $L^2(\Omega)$-estimate, up to the order $h^{3/2-\varepsilon}$. Additionally, we show optimal approximation results for the boundary flux $\partial_n u|_{\Gamma_S}$ in the natural $H^{-1/2}(\Gamma_S)$-norm.

3. Equivalent reformulations

A crucial role in our analysis play three different but equivalent variational formulations. Since g is affine, the pointwise condition $u_h \leqslant g$ can be reformulated in a variationally consistent way, using a biorthogonal dual basis. This choice yields the second variational formulation, a saddle point formulation, where the primal solution as well as the flux on the Signorini boundary are unknowns. The third formulation, a variational formulation for the Schur complement posed on Γ_S, is adequate to bound the primal trace error. However, the Schur complement of the discrete formulation differs from the Galerkin discretization of the continuous Schur complement.

3.1 *Saddle point formulation*

The second formulation, a saddle point problem, is widely used for Signorini type as well as contact problems. It can be obtained from the theory of constrained optimization; see for example, Ekeland & Temam (1999) and Ito & Kunisch (2008). Associated to the dual space of W, $M := H^{-1/2}(\Gamma_S)$, is the convex cone $M^+ := \{\mu \in M : \langle v, \mu \rangle_{\Gamma_S} \geqslant 0, v \in W, v \geqslant 0\}$, where $\langle \cdot, \cdot \rangle_{\Gamma_S}$ denotes the duality pairing between $H^{1/2}_{00}(\Gamma_S)$ and $H^{-1/2}(\Gamma_S)$.

The saddle point problem reads: find $(u, \lambda) \in V \times M^+$, such that

$$a(u, v) + \langle v, \lambda \rangle_{\Gamma_S} = f(v), \quad v \in V, \tag{3.1a}$$

$$\langle u, \mu - \lambda \rangle_{\Gamma_S} \leqslant \langle g, \mu - \lambda \rangle_{\Gamma_S}, \quad \mu \in M^+. \tag{3.1b}$$

Let the vertices be enumerated such that the first N_{M_h} vertices lie on Γ_S. Associated to the primal Lagrange basis functions φ_i, $i = 1, \dots, N_{M_h}$, which do not vanish on Γ_S, are biorthogonal basis functions $\psi_i \in L^2(\Gamma_S)$, $i = 1, \dots, N_{M_h}$, satisfying $\langle \varphi_j, \psi_i \rangle_{\Gamma_S} = \delta_{ij} \langle \varphi_j, 1 \rangle_{\Gamma_S}$. The standard choice are piecewise linear basis functions with the same support on Γ_S as the corresponding primal basis function. This uniquely determines the basis by an inversion of the local mass matrix; see, for example, Wohlmuth (2001). The discrete dual space M_h is spanned by the biorthogonal basis functions $\psi_i \in L^2(\Gamma_S)$, and a uniform inf–sup stability for the discrete spaces V_h and M_h holds, see Wohlmuth (2000). The convex cone M^+ is discretized as the positive span of the biorthogonal basis functions, i.e., $M_h^+ := \{\sum_{i=1}^{N_{M_h}} \alpha_i \psi_i, \alpha_i \geqslant 0\}$. We note that a crosspoint modification is in practice not required due to our assumption that Γ^{act} is a compact subset of Γ_S.

The discretized saddle point formulation of (3.1) then reads: find $(u_h, \lambda_h) \in V_h \times M_h^+$, such that

$$a(u_h, v_h) + \langle v_h, \lambda_h \rangle_{\Gamma_S} = f(v_h), \quad v_h \in V_h, \tag{3.2a}$$

$$\langle u_h, \mu_h - \lambda_h \rangle_{\Gamma_S} \leqslant \langle g, \mu_h - \lambda_h \rangle_{\Gamma_S}, \quad \mu_h \in M_h^+. \tag{3.2b}$$

We point out that $M_h \subset M$, but the discrete cone M_h^+ is not included in M^+. The equivalence of the primal formulation (2.3) with a pointwise constraint and the weak constraint of the saddle point problem holds since g is affine. In the more general case, one can define K_h as $\{v_h \in V_h : \langle v_h, \lambda_h \rangle_{\Gamma_S} \leqslant \langle g, \lambda_h \rangle_{\Gamma_S}, \lambda_h \in M_h^+\}$. Then (2.3) and (3.2) are still equivalent.

3.2 *Reformulation as a Schur complement system*

Due to the fact that the inequality constraint is solely located on the boundary, we can easily rewrite (3.1) and (3.2) as Schur complement systems. On the continuous level, we define the Steklov–Poincaré operator $S : W \to M$ by solving the Dirichlet problem

$$-\Delta w_z = 0 \ \text{in} \ \Omega, \quad w_z = 0 \ \text{on} \ \Gamma_D, \quad w_z = z \ \text{on} \ \Gamma_S,$$

for any $z \in W$ and defining $Sz := \partial_n w_z|_{\Gamma_S}$. The continuous Newton potential $Nf = -\partial_n \hat{w}_f|_{\Gamma_S} \in M$ is defined based on the solution of the homogeneous Dirichlet problem $-\Delta \hat{w}_f = f$ in Ω and $\hat{w}_f = 0$ on $\partial \Omega$. Based on these operators, we can formulate the Schur complement system, which is a variational inequality on the Signorini boundary. The primal trace $u_S := u|_{\Gamma_S} \in K_S := \{v \in H_{00}^{1/2}(\Gamma_S) : v \leqslant g\}$ solves

$$\langle v - u_S, Su_S \rangle_{\Gamma_S} \geqslant \langle v - u_S, Nf \rangle_{\Gamma_S}, \quad v \in K_S. \tag{3.3}$$

An equivalent characterization of the Steklov–Poincaré operator is possible, as the Lagrange multiplier $\lambda_z = -Sz$ of a saddle point problem, where $(w_z, \lambda_z) \in V \times M$ solves

$$a(w_z, v) + \langle v, \lambda_z \rangle_{\Gamma_S} = 0, \quad v \in V, \tag{3.4a}$$

$$\langle w_z, \mu \rangle_{\Gamma_S} = \langle z, \mu \rangle_{\Gamma_S}, \quad \mu \in M, \tag{3.4b}$$

which corresponds to weakly imposed Dirichlet conditions; see Babuška (1973). The continuous Newton potential can also be defined as the Lagrange multiplier of an analogue saddle point formulation with a nontrivial right-hand side in (3.4a), but a trivial one in (3.4b). The Steklov–Poincaré operator and the Newton potential map Dirichlet data and volume data to Neumann data, respectively. They have several applications, for example, in domain decomposition and boundary element methods; see Quarteroni & Valli (1999), Toselli & Widlund (2005) and Steinbach (2008).

By using a mixed finite-element approximation to the above Dirichlet problem (3.4), we can define a mesh-dependent Steklov–Poincaré operator $S_h : W \to M_h$ by $S_h z := -\lambda_{z,h}$, where $(w_{z,h}, \lambda_{z,h}) \in V_h \times M_h$ solves

$$a(w_{z,h}, v_h) + \langle v_h, \lambda_{z,h} \rangle_{\Gamma_S} = 0, \quad v_h \in V_h, \tag{3.5a}$$

$$\langle w_{z,h}, \mu_h \rangle_{\Gamma_S} = \langle z, \mu_h \rangle_{\Gamma_S}, \quad \mu_h \in M_h. \tag{3.5b}$$

An analogue discretization yields a mesh-dependent Newton potential $N_h f$. Denote by W_h the trace space of V_h. Up to scaling factors, the matrix formulation for $S_h|_{W_h}$ and $N_h f$ coincide with the discrete Schur complement system of the matrix formulation of (2.3) by construction. The uniform continuity of S_h directly follows from the saddle point theory using the inf–sup stability of the discrete spaces, while the uniform W_h-ellipticity follows using basic properties of discrete harmonic functions, e.g., Toselli & Widlund (2005, Lemma 4.10). Precisely, one has $\langle v_h, S_h v_h \rangle_{\Gamma_S} = a(w_{v,h}, w_{v,h})$, where $w_{v,h} \in V_h$ is the discrete harmonic extension of $v_h \in W_h$; hence $\langle v_h, S_h v_h \rangle_{\Gamma_S} = |w_{v,h}|^2_{H^1(\Omega)} \geqslant c\|v_h\|^2_{H^{1/2}_{00}(\Gamma_S)}$.

The Schur complement system of (2.3) can be represented as an approximative discretization of (3.3). For $K_{S,h} := \{v_h \in W_h : v_h \leqslant g\}$ find $u_{S,h} \in K_{S,h}$ such that

$$\langle v_h - u_{S,h}, S_h u_{S,h} \rangle_{\Gamma_S} \geqslant \langle v_h - u_{S,h}, N_h f \rangle_{\Gamma_S}, \quad v_h \in K_{S,h}. \tag{3.6}$$

The three weak formulations (2.2), (3.1) and (3.3) are equivalent as well as the three discrete variational problems (2.3), (3.2) and (3.6).

4. Application of a Strang lemma

While u solves the variational inequality (3.3) with the operators S and N, the discrete solution u_h solves the variational inequality (3.6) with the mesh-dependent operators S_h and N_h. In this subsection, we show that the $H^{1/2}_{00}(\Gamma_S)$ error can be bounded by two terms. The first term is the $H^{-1/2}(\Gamma_S)$ norm of the difference between $Nf - S(u|_{\Gamma_S}) = \lambda$ and $N_h f - S_h(u|_{\Gamma_S}) =: \tilde{\lambda}_h \in M_h$. Note that $\tilde{\lambda}_h$ is the discrete dual solution of the linear saddle point problems defining the Dirichlet–Neumann map; see (3.5). Associated with $\tilde{\lambda}_h$ is $\tilde{u}_h = \hat{w}_{f,h} + w_{u|_{\Gamma_S},h}$, and $(\tilde{u}_h, \tilde{\lambda}_h) \in V_h \times M_h$ solves

$$a(\tilde{u}_h, v_h) + \langle v_h, \tilde{\lambda}_h \rangle_{\Gamma_S} = f(v_h), \quad v_h \in V_h,$$
$$\langle \tilde{u}_h, \mu_h \rangle_{\Gamma_S} = \langle u, \mu_h \rangle_{\Gamma_S}, \quad \mu_h \in M_h.$$

The second term is the discretization error of the variational inequality on the boundary (3.3). Let $\bar{u}_h \in K_{S,h}$ be such that

$$\langle v_h - \bar{u}_h, S\bar{u}_h \rangle_{\Gamma_S} \geqslant \langle v_h - \bar{u}_h, Nf \rangle_{\Gamma_S}, \quad v \in K_{S,h}. \tag{4.1}$$

LEMMA 4.1 The trace error of the Signorini problem (2.2) can be bounded by

$$\|u - u_h\|_{H_{00}^{1/2}(\Gamma_S)} \leqslant c\|\lambda - \tilde{\lambda}_h\|_{H^{-1/2}(\Gamma_S)} + c\|u - \bar{u}_h\|_{H_{00}^{1/2}(\Gamma_S)}.$$

Proof. The proof of this lemma follows the lines of Of *et al.* (2015, Theorem 3.2). Since the proof is fundamental, we work it out. We start with the trivial triangle inequality

$$\|u - u_h\|_{H_{00}^{1/2}(\Gamma_S)} \leqslant \|u - \bar{u}_h\|_{H_{00}^{1/2}(\Gamma_S)} + \|\bar{u}_h - u_h\|_{H_{00}^{1/2}(\Gamma_S)}.$$

For the second term $\bar{u}_h - u_h$, we use the W_h-ellipticity of the mesh-dependent Steklov–Poincaré operator and apply the variational inequalities (3.6) and (4.1):

$$\begin{aligned}
c\|\bar{u}_h - u_h\|_{H_{00}^{1/2}(\Gamma_S)}^2 &\leqslant \langle \bar{u}_h - u_h, S_h(\bar{u}_h - u_h)\rangle_{\Gamma_S} \\
&\leqslant \langle \bar{u}_h - u_h, S_h\bar{u}_h\rangle_{\Gamma_S} + \langle \bar{u}_h - u_h, Nf - N_hf\rangle_{\Gamma_S} - \langle \bar{u}_h - u_h, S\bar{u}_h\rangle_{\Gamma_S} \\
&= \langle \bar{u}_h - u_h, Nf - S\bar{u}_h - (N_hf - S_h\bar{u}_h)\rangle_{\Gamma_S} \\
&\leqslant \|\bar{u}_h - u_h\|_{H_{00}^{1/2}(\Gamma_S)}\|Nf - S\bar{u}_h - (N_hf - S_h\bar{u}_h)\|_{H^{-1/2}(\Gamma_S)}.
\end{aligned}$$

Using the boundedness of the operators and once again the triangle inequality, we get

$$\begin{aligned}
\|\bar{u}_h - u_h\|_{H_{00}^{1/2}(\Gamma_S)} &\leqslant c\|Nf - Su - (N_hf - S_hu)\|_{H^{-1/2}(\Gamma_S)} + c\|S(u - \bar{u}_h)\|_{H^{-1/2}(\Gamma_S)} + c\|S_h(u - \bar{u}_h)\|_{H^{-1/2}(\Gamma_S)} \\
&\leqslant c\|\lambda - \tilde{\lambda}_h\|_{H^{-1/2}(\Gamma_S)} + c\|u - \bar{u}_h\|_{H_{00}^{1/2}(\Gamma_S)}.
\end{aligned}$$

A bound of $u - \bar{u}_h$ can be shown using Falk's lemma; see Falk (1974, Theorem 1), which is an analogue result to Céa's lemma for variational inequalities. Since g is affine, the discretization of the variational inequality is conforming in the sense that $K_{S,h} \subset K_S$, and Falk's lemma reads

$$\|u - \bar{u}_h\|_{H_{00}^{1/2}(\Gamma_S)} \leqslant c \inf_{v_h \in K_{S,h}} (\|u - v_h\|_{H_{00}^{1/2}(\Gamma_S)} + \langle \lambda, u - v_h\rangle_{\Gamma_S}^{1/2}). \qquad (4.2)$$

$\square$

LEMMA 4.2 Let $u \in K_S$ be the solution to the variational inequality on the boundary (3.3) and $\bar{u}_h \in K_{S,h}$ be the Galerkin approximation; see Equation (4.1). Assuming $u \in H^{5/2-\varepsilon}(\Omega)$, we get

$$\|u - \bar{u}_h\|_{H_{00}^{1/2}(\Gamma_S)} \leqslant ch^{3/2-\varepsilon}|u|_{H^{5/2-\varepsilon}(\Omega)}.$$

Proof. This type of estimate was already considered in the context of boundary element methods, in Spann (1993, Theorem 3.1) and Steinbach (2014, Section 3), where additional assumptions on the boundary of the active set were made. To keep this article self-contained, we present a proof based on techniques for $H^1(\Omega)$-estimates.

In this proof, only the Signorini boundary Γ_S is considered, so any notation refers to $\mathbb{R}^{d-1}$. We introduce the triangulation $\mathcal{T}_h^S$ on the Signorini boundary which is induced by the triangulation of Ω. We note that the induced triangulation on the Signorini boundary is also shape-regular and denote by h_T the diameter of an element $T \in \mathcal{T}_h^S$.

We use recently introduced techniques from Drouet & Hild (2015) based on nonstandard L^1- and L^2-estimates. We use Falk's lemma (4.2) with $v_h = J_hu \in K_{S,h}$, where for $d = 2$, J_h is the piecewise linear
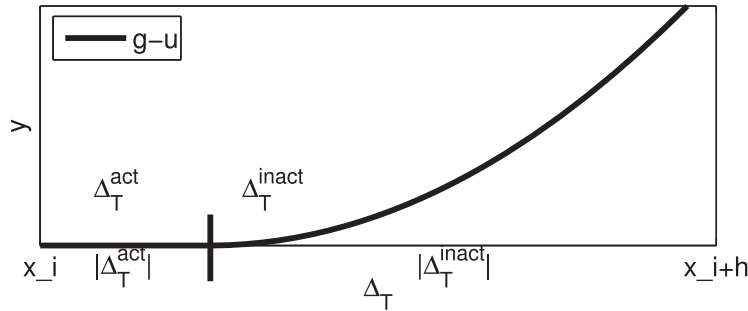
FIG. 2. Active and inactive area within one element T for $d = 2$, i.e., $\Delta_T = T$.

nodal interpolation operator and for $d = 3$, J_h is the Chen–Nochetto operator, see Chen & Nochetto (2000). It remains to bound

$$\langle \lambda, u - J_h u \rangle_{\Gamma_S} = \sum_{T \in \mathcal{T}_h^S} \int_T \lambda(u - J_h u) \, \mathrm{d}\Gamma.$$

One of the main ideas of this proof is to derive two estimates for each element, where dependent on the measure of the active area in a surrounding patch one of the two estimates is applied. Given any $T \in \mathcal{T}_h^S$, we set $\Delta_T = T$ for $d = 2$ and $\Delta_T \subset \Gamma_S$ as the patch surrounding T for $d = 3$.

On the patch, we define the local active area $\Delta_T^{\mathrm{act}} = \Delta_T \cap \Gamma^{\mathrm{act}}$ and the local inactive area $\Delta_T^{\mathrm{inact}} = \Delta_T \setminus \Delta_T^{\mathrm{act}}$; see Fig. 2.

Note that by construction only the elements with $|\Delta_T^{\mathrm{act}}| > 0$ and $|\Delta_T^{\mathrm{inact}}| > 0$ are of interest, otherwise either λ or $u - J_h u$ is equal to zero. Recently developed nonstandard estimates for u and λ (see Drouet & Hild, 2015) yield

$$\int_T \lambda(u - J_h u) \, \mathrm{d}\Gamma \leqslant c \min \left(\frac{|\Delta_T|^{1/2}}{|\Delta_T^{\mathrm{inact}}|^{1/2}}, \frac{|\Delta_T|^{1/2}}{|\Delta_T^{\mathrm{act}}|^{1/2}} \right) h_T^{3-2\varepsilon} \left(|\lambda|_{H^{1-\varepsilon}(\Delta_T)}^2 + |u|_{H^{2-\varepsilon}(\Delta_T)}^2 \right).$$

Since $|\Delta_T^{\mathrm{act}}| + |\Delta_T^{\mathrm{inact}}| = |\Delta_T|$, one of the measures is greater than or equal to $|\Delta_T|/2$. Summing over the elements and applying the trace inequality yields the desired estimate. □

5. *A priori* estimate of the primal trace

In this section, an upper bound for $\|\lambda - \tilde{\lambda}_h\|_{H^{-1/2}(\Gamma_S)}$ is shown, which concludes the primal trace estimate in Lemma 2.2. The Lagrange multiplier arises from a linear Dirichlet problem with a weak enforcement of the boundary values, which is covered by the problem formulation in Melenk & Wohlmuth (2012). However, the required regularity of $B_{2,1}^{5/2}(\Omega)$ is not given in our case. Thus, we have to generalize these results. We follow the lines of Melenk & Wohlmuth (2012), but will not work with the Besov space $B_{2,1}^{5/2}(\Omega)$. Reducing the regularity from $B_{2,1}^{5/2}(\Omega)$ to $H^{5/2-\varepsilon}(\Omega)$ automatically results in a reduced convergence order, but we do not lose a log-term.

The first two subsections collect some technical tools for the proof which is carried out in Sub-section 5.3. Firstly, for a Scott–Zhang operator, we show optimal approximation results in anisotropic
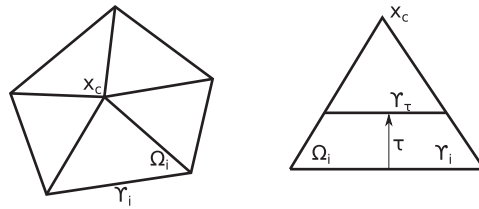
FIG. 3. Left: decomposition of a 2D domain into the patches. Right: one patch after a suitable rotation.

norms. Secondly, for two dual problems, estimates in these norms are shown. As a corollary of the main result, we show improved rates in the $L^2(\Omega)$-norm.

5.1 *Anisotropic norms and quasi-interpolation results*

Estimating the dual solution on the boundary can be related to bounds of the primal solution in a neighbourhood of Γ. We define strips around the boundary of width δ by $\mathfrak{S}(\delta) := \{x \in \Omega : \mathrm{dist}(x, \Gamma) \leqslant \delta\}$. Using a dual Neumann problem with local volume data, we can relate the dual error to the primal error on a strip $\mathfrak{S}(ch)$. As a technical tool to derive local error estimates for the dual problems on these strips, we use anisotropic norms as in Melenk & Wohlmuth (2012), Melenk *et al.* (2013) and Waluga & Wohlmuth (2013). We simplify the original definition, which was based on a technical decomposition of the domain into 'cylinders'. Instead, we use an intuitive decomposition into triangles and pyramids, based on the faces of the polygonal domain.

For a formal definition, we first decompose the domain Ω into a set of patches which are triangles if $d = 2$ and pyramids if $d = 3$. Each patch is supposed to connect one facet with the barycentre of Ω. Since Ω is convex, the barycentre x_c lies in the interior of Ω. Let an enumeration of the facets be given by γ_i, $i = 1, \ldots, N_\gamma$, and consider one facet γ_i. The patch Ω_i is the triangle and pyramid in 2D and 3D, respectively, with γ_i as the base side and x_c as the top. Obviously, $\bar{\Omega} = \cup_{i=1}^{N_\gamma} \bar{\Omega}_i$; see Fig. 3. For each patch Ω_i, we define the anisotropic norm $L(p, 2; \Omega_i)$ based on a decomposition of the patch into a $(d - 1)$-dimensional part parallel to the facet γ_i and the one-dimensional distance to the facet. Given $i \in \{1, \ldots, N_\gamma\}$, without any loss of generality, we assume that γ_i lies in the $x_1, \ldots, x_{d-1}$-plane and Ω lies in the positive half-space $\{(x', \tau), x' \in \mathbb{R}^{d-1}, \tau \geqslant 0\}$. We denote by $\gamma_\tau := \{(x', \tau) \in \Omega_i, x' \in \mathbb{R}^{d-1}\}$ for $\tau \geqslant 0$, the part in Ω_i parallel to γ_i, having distance τ to γ_i. We have $\gamma_\tau = \emptyset$ for $\tau < 0$ and $\tau \geqslant D$, where D is the diameter of Ω. By the Fubini–Tonelli formula, the integral over Ω_i can be decomposed as

$$\int_{\Omega_i} v \, \mathrm{d}x = \int_{\tau=0}^{D} \int_{\gamma_\tau} v \, \mathrm{d}\mu \, \mathrm{d}\tau,$$

where $\mathrm{d}\mu$ denotes the $(d - 1)$-dimensional Lebesgue measure. We define anisotropic norms $L(p, 2; \Omega_i)$, $1 \leqslant p \leqslant \infty$, by

$$\|v\|_{L(p,2;\Omega_i)}^p := \int_{\tau=0}^{D} \left(\int_{\gamma_\tau} v^2 \, \mathrm{d}\mu \right)^{p/2} \mathrm{d}\tau, \quad 1 \leqslant p < \infty,$$

$$\|v\|_{L(\infty,2;\Omega_i)} := \sup_{\tau \in (0,D)} \left(\int_{\gamma_\tau} v^2 \, \mathrm{d}\mu \right)^{1/2}.$$

Adding the components of each patch, we define anisotropic norms $L(p, 2)$ on Ω:

$$\|v\|_{L(p,2)}^p := \sum_{i=1}^{N_\gamma} \|v\|_{L(p,2;\Omega_i)}^p, \quad 1 \leqslant p < \infty,$$

$$\|v\|_{L(\infty,2)} := \max_{i=1,\dots,N_\gamma} \|v\|_{L(\infty,2;\Omega_i)}.$$

Note that the patches cover Ω without any overlap and the $L(2, 2)$-norm coincides with the $L^2(\Omega)$-norm.

The Hölder inequality $\int_\Omega fg \, \mathrm{d}x \leqslant \|f\|_{L(p,2)} \|g\|_{L(q,2)}$ for $1/p + 1/q = 1$ follows from the one-dimensional Hölder inequality. Furthermore, an interpolation result analogue to L^p spaces is valid; see e.g., Tartar (2007).

LEMMA 5.1 For $1 < p < \infty$ and $1/p + 1/p' = 1$, one has

$$L(p, 2) = (L(1, 2), L(\infty, 2))_{1/p', p}.$$

Proof. For the convenience of the reader, we sketch the main steps. Consider any patch Ω_i, $i \in \{1, \dots, N_\gamma\}$. For any $1 \leqslant q \leqslant \infty$ and $v \in L(q, 2; \Omega_i)$, $I = (0, D)$, consider $f_v \in L^q(I)$ which is defined for almost every $\tau \in I$ by $f_v(\tau) := \|v\|_{L^2(\gamma_\tau)}$. One has $\|v\|_{L(q,2;\Omega_i)} = \|f_v\|_{L^q(I)}$, and we can show the equality of the two K-functionals

$$K(t, v; L(1, 2; \Omega_i), L(\infty, 2; \Omega_i)) = \inf_{v=v_0+v_1} (\|v_0\|_{L(1,2;\Omega_i)} + t\|v_1\|_{L(\infty,2;\Omega_i)}),$$

$$K(t, f_v; L^1(I), L^\infty(I)) = \inf_{f_v=f_0+f_1} (\|f_0\|_{L^1(I)} + t\|f_1\|_{L^\infty(I)}),$$

and use the standard L^p-interpolation $L^p(I) = (L^1(I), L^\infty(I))_{1/p', p}$; see Tartar (2007, Lemma 22.6).

On the one hand, any decomposition $f_v = f_0 + f_1$ directly implies a decomposition by $v_i(x', \tau) := v(x', \tau) f_i(\tau) / f_v(\tau)$ for $x' \in \mathbb{R}^{d-1}$. The case $f_v(\tau) = 0$ is trivial and can be excluded. One has $v = v_0 + v_1$ and $f_{v_i} = f_i$. As a consequence

$$K(t, v; L(1, 2; \Omega_i), L(\infty, 2; \Omega_i)) \leqslant K(t, f_v; L^1(I), L^\infty(I)).$$

On the other hand, for any decomposition $v = v_0 + v_1$ one has

$$f_{v_0}(\tau) + f_{v_1}(\tau) = \|v_0\|_{L^2(\gamma_\tau)} + \|v_1\|_{L^2(\gamma_\tau)} \geqslant \|v_0 + v_1\|_{L^2(\gamma_\tau)} = f_v(\tau).$$

Hence, the decomposition of f_v by $f_v = f_0 + f_1$,

$$f_i(\tau) := f_{v_i}(\tau) \frac{f_v(\tau)}{f_{v_0}(\tau) + f_{v_1}(\tau)} \leqslant f_{v_i}(\tau), \quad i = 0, 1,$$

yields $\|f_0\|_{L^1(I)} \leqslant \|v_0\|_{L(1,2)}$ as well as $\|f_1\|_{L^\infty(\Omega)} \leqslant \|v_1\|_{L(\infty,2)}$. This implies

$$K(t, v; L(1, 2; \Omega_i), L(\infty, 2; \Omega_i)) \geqslant K(t, f_v; L^1(I), L^\infty(I))$$

and concludes the equality of both K-functionals.

Since the patches cover Ω without any overlap, the interpolation property for $L(p, 2)$ follows. $\square$

As a preliminary to our analysis, we state approximation results of a Scott–Zhang-type quasi-interpolation operator in the anisotropic norms. We consider $P_h \colon V \to V_h$ as in Scott & Zhang (1990), based on the biorthogonal basis on Γ_S, preserving the homogeneous Dirichlet data on Γ_D. The boundary values are preserved such that $P_h v|_{\Gamma_D} = 0$ and $\langle P_h v, \mu_h \rangle_{\Gamma_S} = \langle v, \mu_h \rangle_{\Gamma_S}$ for $\mu_h \in M_h$. On Γ_S, optimal order L^2-approximation properties

$$\|v - P_h v\|_{L^2(\Gamma_S)} \leqslant c h^{2-\varepsilon} |v|_{H^{2-\varepsilon}(\Gamma_S)} \tag{5.1}$$

for $v \in V \cap H^{5/2-\varepsilon}(\Omega)$ are given. An approximation result in the $L(q, 2)$-norm is given by the following lemma.

LEMMA 5.2 For $v \in V \cap H^{5/2-\varepsilon}(\Omega)$, and $1/2 \geqslant \varepsilon > 0$, one has

$$\|\nabla(v - P_h v)\|_{L(q,2)} \leqslant c h \|v\|_{H^{5/2-\varepsilon}(\Omega)}, \quad \text{with } q = 1/\varepsilon.$$

Proof. Since the $L(2, 2)$-norm coincides with the $L^2(\Omega)$-norm, we have the standard approximation result

$$\|\nabla(v - P_h v)\|_{L(2,2)} \leqslant c h |v|_{H^2(\Omega)}.$$

For $q > 2$, we show the estimate by an interpolation argument, using the $L(2, 2)$ and the $L(\infty, 2)$ estimate. For the $L(\infty, 2)$ norm, we can easily adapt the proof in Melenk & Wohlmuth (2012, Lemma 4.1) using local approximation results of the Scott–Zhang operator (Scott & Zhang, 1990, Equation 4.3). For any patch Ω_i, $i \in \{1, \ldots, N_\gamma\}$ and $\tau > 0$, we first define two strips around γ_τ. A strip of width 2δ is defined by $\mathfrak{S}_i(\delta, \tau) := \{x \in \Omega : \text{dist}(x, \gamma_\tau) \leqslant \delta\}$ and a discrete neighbourhood can be constructed by the elements intersecting $\gamma_\tau \colon \mathcal{I}_\tau := \{T \in \mathcal{T}_h : \gamma_\tau \cap \bar{T} \neq \emptyset\}$. Note that we cannot expect $\mathfrak{S}_i(\delta, \tau) \subset \Omega_i$, but this inclusion is not necessary for our analysis. Using these strips, local estimates of the Scott–Zhang operator yield

$$\|\nabla(v - P_h v)\|_{L^2(\gamma_\tau)}^2 \leqslant c \sum_{T \in \mathcal{I}_\tau} \left(\frac{1}{h} \|\nabla(v - P_h v)\|_{L^2(T)}^2 + h \|\nabla^2(v - P_h v)\|_{L^2(T)}^2 \right)$$

$$\leqslant c h |v|_{H^2(\mathfrak{S}_i(\bar{c}h, \tau))}^2 \leqslant c h^2 \|v\|_{B_{2,1}^{5/2}(\Omega)}^2,$$

where in the last step (Li *et al.*, 2009, Lemma 2.1) was used. Consequently, we have

$$\|\nabla(v - P_h v)\|_{L(\infty,2)} \leqslant c h \|v\|_{B_{2,1}^{5/2}(\Omega)}.$$

To show this estimate also for interpolation spaces, we can apply the interpolation property (Tartar, 2007, Lemma 22.3). By the reiteration theorem and Lemma 5.1, we have the interpolation representations $L(q, 2) = (L(2, 2), L(\infty, 2))_{1-2\varepsilon, q}$ as well as the similar term $H^{5/2-\varepsilon}(\Omega) = (H^2(\Omega), B_{2,1}^{5/2}(\Omega))_{1-2\varepsilon, 2} \subset (H^2(\Omega), B_{2,1}^{5/2}(\Omega))_{1-2\varepsilon, q}$. As a consequence, the stated estimate is also valid in the interpolated spaces. $\qquad\square$

5.2 *Dual problems*

In this subsection, we follow the lines of Melenk & Wohlmuth (2012, Section 5) and define a dual Dirichlet problem with locally supported data. For $v \in L^2(\Omega)$, supp$v \subset \overline{\mathfrak{S}(h)}$, we denote by $T^D v$ the

solution operator of

$$-\Delta w = v \ \text{in} \ \Omega, \quad w = 0 \ \text{on} \ \Gamma, \tag{5.2}$$

i.e., $T^D v = w$.

In contrast to Melenk & Wohlmuth (2012), we cannot assume $B_{2,1}^{5/2}(\Omega)$-regularity for the solution of (2.2), but only $H^{5/2-\varepsilon}(\Omega)$-regularity. Naive interpolation of the final estimate does not yield optimal results, but an additional log-term. For optimal results, we need the stronger estimate given in the following lemmas. In the next lemma, we state a regularity estimate in a weighted Sobolev space using the local support of the data of the dual problem. Based on this estimate, we then state an approximation result for the Galerkin approximation of the dual solution in an anisotropic norm.

LEMMA 5.3 For $v \in L^2(\Omega)$, $\operatorname{supp} v \subset \overline{\mathfrak{S}(h)}$ and $w := T^D v$ there exists $0 < \tilde{c} < \infty$ independent of v and h, such that

$$\|\delta_\Gamma^{1/2-\varepsilon/2} \nabla^2 w\|_{L^2(\Omega \setminus \mathfrak{S}(\tilde{c}h))} \leqslant c h^{1/2-\varepsilon/2} \|v\|_{L^2(\Omega)},$$

where δ_Γ is the distance function to Γ.

Proof. We follow the idea of Melenk & Wohlmuth (2012, Lemma 5.4), but instead of several local translations of w, we consider a global scaling of the coordinate system. To exploit the local data of the dual problem, we choose a sufficiently large scaling factor such that the transformation of w is harmonic in a neighbourhood of Ω. This allows us to apply interior regularity results for the transformation of w (see Gilbarg & Trudinger, 2001, Theorem 8.8):

$$\|\nabla^2 z\|_{L^2(B_1)} \leqslant c \|z\|_{H^1(B_{1+\rho})}, \tag{5.3}$$

for $-\Delta z = 0$ on $B_{1+\rho}$, a ball of radius $1 + \rho$ for a fixed $\rho > 0$.

Without loss of generality, assume that the barycentre of Ω is the origin of the coordinate system. For sufficiently small h, we define a neighbourhood of Ω by a scaling $\tilde{\Omega} := \{(1 + 4C_1 h)x : x \in \Omega\}$. Since we estimate w only on $\Omega \setminus \mathfrak{S}(\tilde{c}h)$, where $\tilde{c}$ is selected later, we can choose the scaling factor appropriately. The constant C_1 is sufficiently large, but fixed and independent of h, such that, for $x \in \mathfrak{S}(h)$, one has $(1 + 2C_1 h)x \notin \Omega$. We scale w to a function on this neighbourhood by $\tilde{w} : \tilde{\Omega} \to \mathbb{R}$, $\tilde{w}(x) := w(x/(1 + 4C_1 h))$.

Note that the introduced scaling preserves harmonic functions, more precisely for $x \in \Omega$ and $h < 1/(2C_1)$, we have $(1 + C_1 h)/(1 + 4C_1 h)x \in \Omega \setminus \mathfrak{S}(h)$, and thus

$$\Delta \tilde{w} = 0 \ \text{at} \ (1 + C_1 h)x, \quad x \in \Omega.$$

Since the scaling factor is uniformly bounded, it also preserves Sobolev norms, i.e.,

$$c\|\tilde{w}\|_{H^\sigma(\tilde{\Omega})} \leqslant \|w\|_{H^\sigma(\Omega)} \leqslant C\|\tilde{w}\|_{H^\sigma(\tilde{\Omega})}, \quad \sigma \in \{0, 3/2\}.$$

To apply the transformation $\tilde{w}$, we choose $\tilde{c}$ sufficiently large such that the transformation of $\Omega \setminus \mathfrak{S}(\tilde{c}h)$ is a subset of $\Omega \setminus \mathfrak{S}(h)$, and thus

$$\|\delta_\Gamma^{1/2-\varepsilon/2} \nabla^2 w\|_{L^2(\Omega \setminus \mathfrak{S}(\tilde{c}h))} \leqslant c \|(\delta_\Gamma + h)^{1/2-\varepsilon/2} \nabla^2 \tilde{w}\|_{L^2(\Omega \setminus \mathfrak{S}(h))}, \quad \text{for} \ x \in \Omega.$$

Standard interior regularity (5.3) yields $\|\nabla^2 \tilde{w}\|_{L^2(B_r)} \leqslant c r^{-1/2+\varepsilon/2} \|\tilde{w}\|_{H^{3/2+\varepsilon/2}(B_{r(1+\rho)})}$ for a fixed $\rho > 0$ and any concentric balls of radius r and $r(1 + \rho)$, such that $B_{r(1+\rho)} \subset \Omega$. A covering of $\Omega \setminus \mathfrak{S}(h)$ using balls

of centre x_i and radii $r_i \sim h + \delta_\Gamma(x_i)$ shows

$$\|(\delta_\Gamma + h)^{1/2-\varepsilon/2}\nabla^2\tilde{w}\|_{L^2(\Omega\setminus\mathfrak{S}(h))} \leqslant c\|\tilde{w}\|_{H^{3/2+\varepsilon/2}(\Omega)}.$$

Details on the Besicovitch covering theorem can be found in Evans & Gariepy (1992, Section 1.5.2) and Melenk (2002, Chapter 5).

An analogue computation as in Melenk & Wohlmuth (2012, Lemma 5.4), where the case $\varepsilon = 0$ was considered, concludes the proof . We bound the K-functional of the fractional Sobolev space $(H^1(\Omega), H^2(\Omega))_{1/2+\varepsilon/2,2} = H^{3/2+\varepsilon/2}(\Omega)$ by

$$\|\tilde{w}\|^2_{H^{3/2+\varepsilon/2}(\Omega)} = \int_{t=0}^h (t^{-1/2-\varepsilon/2}K(t,\tilde{w}))^2\frac{\mathrm{d}t}{t} + \int_{t=h}^1 (t^{-1/2-\varepsilon/2}K(t,\tilde{w}))^2\frac{\mathrm{d}t}{t}$$

$$\leqslant \int_{t=0}^h (t^{-1/2-\varepsilon/2}K(t,\tilde{w}))^2\frac{\mathrm{d}t}{t} + \int_{t=h}^1 t^{-1-\varepsilon}\,\mathrm{d}t \sup_{t>0}(t^{-1/2}K(t,\tilde{w}))^2. \tag{5.4}$$

Again applying the interior regularity (5.3), we get $\|\tilde{w}\|_{H^2(\Omega)} \leqslant ch^{-1/2}\|w\|_{H^{3/2}(\Omega)}$, which yields $K(t,\tilde{w}) \leqslant ct\|\tilde{w}\|_{H^2(\Omega)} \leqslant cth^{-1/2}\|w\|_{H^{3/2}(\Omega)}$. Substituting this upper bound in the first integral of (5.4) and observing $\sup_{t>0}(t^{-1/2}K(t,\tilde{w})) \leqslant \|w\|_{B^{3/2}_{2,\infty}(\Omega)}$, yields

$$\|\tilde{w}\|_{H^{3/2+\varepsilon/2}(\Omega)} \leqslant ch^{-\varepsilon/2}\|w\|_{B^{3/2}_{2,\infty}(\Omega)}.$$

Finally, Melenk & Wohlmuth (2012, Lemma 5.2) states $\|w\|_{B^{3/2}_{2,\infty}(\Omega)} \leqslant ch^{1/2}\|v\|_{L^2(\Omega)}$, which concludes the proof. □

REMARK 5.4 A closer look reveals that the upper bound depends on ε. More precisely if ε tends to zero it tends to infinity. The first term on the right-hand side of (5.4) is robust in ε, but the constant in the second term can only be bounded by $\varepsilon^{-1/2}$, and in the limit case $\varepsilon = 0$ a log-term in h appears.

Using local error estimates and the weighted regularity result proved above, we show an approximation result for the Galerkin approximation of the dual problem in anisotropic norms.

LEMMA 5.5 Given $v \in L^2(\Omega)$ with supp$v \subset \overline{\mathfrak{S}(h)}$, consider $w = T^D v$ and the Galerkin approximation $w_h \in V_h \cap H_0^1(\Omega)$. For $1 < p = (1-\varepsilon)^{-1} \leqslant 2$, one has the following approximation property:

$$\|\nabla(w - w_h)\|_{L(p,2)} \leqslant ch^{3/2-\varepsilon}\|v\|_{L^2(\Omega)}.$$

Proof. We show the estimate on each patch $\Omega_i, i \in \{1, \ldots, N_\gamma\}$. In the definition of the norm, we decompose the integral in τ from 0 to D into two parts and find

$$\|\nabla(w - w_h)\|^p_{L(p,2;\Omega_i)} = \int_{\tau=0}^{\tilde{c}_1 h} \|\nabla(w - w_h)\|^p_{L^2(\gamma_\tau)}\,\mathrm{d}\tau + \int_{\tau=\tilde{c}_1 h}^D \|\nabla(w - w_h)\|^p_{L^2(\gamma_\tau)}\,\mathrm{d}\tau,$$

where $\tilde{c}_1$ has to be adapted to the constant $\tilde{c}$ resulting from the previous lemma.

The first term is an integral over a strip of width $\mathcal{O}(h)$. The Hölder inequality with the exponents $2/p$, $2/(2-p)$ and the Fubini–Tonelli formula obviously yield, for $p = (1-\varepsilon)^{-1}$,

$$\int_{\tau=0}^{\tilde{c}_1 h} \|\nabla(w-w_h)\|_{L^2(\gamma_\tau)}^p \, \mathrm{d}\tau \leqslant ch^{(2-p)/2} \left(\int_{\tau=0}^{\tilde{c}_1 h} \|\nabla(w-w_h)\|_{L^2(\gamma_\tau)}^2 \, \mathrm{d}\tau \right)^{p/2}$$

$$\leqslant ch^{p(1/2-\varepsilon)} \|\nabla(w-w_h)\|_{L^2(S(\tilde{c}_1 h))}^p.$$

Since Ω is convex, we have $\|\nabla(w-w_h)\|_{L^2(\mathfrak{S}(\tilde{c}_1 h))} \leqslant \|\nabla(w-w_h)\|_{L^2(\Omega)} \leqslant ch\|v\|_{L^2(\Omega)}$, which gives

$$\int_{\tau=0}^{\tilde{c}_1 h} \|\nabla(w-w_h)\|_{L^2(\gamma_\tau)}^p \, \mathrm{d}\tau \leqslant ch^{p(3/2-\varepsilon)} \|v\|_{L^2(\Omega)}^p.$$

The second integral is estimated using a local approximation property and the regularity result given in Lemma 5.3. First, we insert $\tau^{1/2}\tau^{-1/2}$ and use the Hölder inequality with the same exponents as before:

$$\int_{\tau=\tilde{c}_1 h}^{D} \tau^{-1/2}\tau^{1/2} \|\nabla(w-w_h)\|_{L^2(\gamma_\tau)}^p \, \mathrm{d}\tau$$

$$\leqslant \left(\int_{\tau=\tilde{c}_1 h}^{D} \tau^{-1/(2-p)} \, \mathrm{d}\tau \right)^{(2-p)/2} \left(\int_{\tau=\tilde{c}_1 h}^{D} \tau^{1/p} \|\nabla(w-w_h)\|_{L^2(\gamma_\tau)}^2 \, \mathrm{d}\tau \right)^{p/2}$$

$$\leqslant ch^{-p\varepsilon/2} \|\tau^{1/2-\varepsilon/2}\nabla(w-w_h)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_1 h))}^p.$$

Again, we note that the value of the constant does not stay bounded for $\varepsilon \to 0$.

Based on the discussion in Melenk & Wohlmuth (2012, Section 5.1.2), we derive the bound

$$\|\tau^{1/2-\varepsilon/2}\nabla(w-w_h)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_1 h))}$$
$$\leqslant c\|\tau^{1/2-\varepsilon/2}\nabla(w-I_h w)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_2 h))} + c\|\tau^{-1/2-\varepsilon/2}(w-w_h)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_2 h))} \tag{5.5}$$

for an arbitrary but fixed $\tilde{c}_2$ if $\tilde{c}_1$ is chosen sufficiently large. Here, I_h denotes the standard nodal interpolation operator. This estimate is based on local approximation properties found in Wahlbin (1991, 1995) and a Besicovitch covering argument.

To estimate the first term, we exploit the regularity result derived in Lemma 5.3. Based on $\tilde{c}$, which is given from the previous lemma, we can choose $\tilde{c}_2$ and $\tilde{c}_1$ sufficiently large such that

$$\|\tau^{1/2-\varepsilon/2}\nabla(w-I_h w)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_2 h))} \leqslant ch\|\tau^{1/2-\varepsilon/2}\nabla^2 w\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}h))}$$

$$\leqslant ch^{3/2-\varepsilon/2}\|v\|_{L^2(\Omega)}.$$

Using the convexity of Ω the second term of (5.5) can be bounded easily by

$$\|\tau^{-1/2-\varepsilon/2}(w-w_h)\|_{L^2(\Omega\setminus\mathfrak{S}(\tilde{c}_2 h))} \leqslant ch^{-1/2-\varepsilon/2}\|w-w_h\|_{L^2(\Omega)} \leqslant ch^{3/2-\varepsilon/2}\|v\|_{L^2(\Omega)}. \qquad \square$$

The previously shown bounds in anisotropic norms are sufficient to show primal estimates in a neighbourhood of the boundary. For a final bound of the Lagrange multiplier, we also need to consider a dual problem with Neumann data, as defined in Melenk & Wohlmuth (2012, Section 5.2). Given

$v \in L^2(\Omega)$, supp$v \subset \overline{\mathfrak{S}(h)}$, define w_v^N such that

$$-\Delta w_v^N = v - \frac{1}{|\Omega|} \int_\Omega v \, \mathrm{d}x \text{ in } \Omega, \quad \partial_n w_v^N = 0, \text{ on } \Gamma, \quad \int_\Omega w_v^N \, \mathrm{d}x = 0. \qquad (5.6)$$

Denote by V_h^{-1} the space of discrete functions without any restriction of the boundary values. Using the same arguments as before, we can adapt the proof of Melenk & Wohlmuth (2012, Lemma 5.7), and show the following statement based on the dual Neumann problem.

COROLLARY 5.6 Let $u \in V \cap H^{5/2-\varepsilon}(\Omega)$ and $u_h^N \in V_h^{-1}$ satisfy the orthogonality condition

$$a(u - u_h^N, v_h) = 0 \text{ for } v_h \in V_h^{-1} \text{ and } \int_{\mathfrak{S}(h)} u - u_h^N \, \mathrm{d}x = 0;$$

then,

$$\|u - u_h^N\|_{L^2(\mathfrak{S}(h))} \leqslant ch^{5/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)},$$
$$|u - u_h^N|_{H^{1/2}(\Gamma)} \leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

5.3 *Error bound for the Dirichlet–Neumann map*

With the results of the previous subsection, we can estimate the $H^{-1/2}(\Gamma_S)$-error of the Dirichlet–Neumann map $Nf - S(u|_{\Gamma_S})$ and the mesh-dependent Dirichlet–Neumann map $N_h f - S_h(u|_{\Gamma_S})$ (see Section 3.2), in two steps. This bound is the last step to show the primal estimate in Theorem 2.2. Firstly, we relate the error of the dual variable to the error of the primal variable in a small strip around Γ using the dual Neumann problem (5.6). Secondly, the error in the strip is estimated using the dual Dirichlet problem (5.2) and the approximation results derived in the anisotropic norms.

THEOREM 5.7 Assuming the solution u of the Signorini problem (2.2) to be in $H^{5/2-\varepsilon}(\Omega)$, $\varepsilon \in (0, \frac{1}{2}]$ fixed, then one has

$$\|\lambda - \tilde\lambda_h\|_{H^{-1/2}(\Gamma_S)} \leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

Proof. The proof is divided into two steps. Firstly, we bound the dual error by the primal error on a small neighbourhood of the boundary. Secondly, we bound the primal error on a small strip using the anisotropic estimates stated in Lemmas 5.2 and 5.5.

To be more precise, the first step is to show the upper bound

$$\|\lambda - \tilde\lambda_h\|_{H^{-1/2}(\Gamma_S)} \leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)} + c\frac{1}{h} \|u - \tilde u_h\|_{L^2(\mathfrak{S}(h))}. \qquad (5.7)$$

We use the saddle point formulation to represent the dual error by discrete harmonic functions on the domain. Using the stability of the harmonic extension and an inverse trace inequality, we can relate the dual error to the primal error on the strip $S(h)$.

We start using the uniform inf–sup stability in the $H^{-1/2}(\Gamma_S)$-norm to get

$$\|\lambda - \tilde\lambda_h\|_{H^{-1/2}(\Gamma_S)} \leqslant c \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_{H^{-1/2}(\Gamma_S)} + c \sup_{z_h \in W_h} \frac{\langle z_h, \lambda - \tilde\lambda_h\rangle_{\Gamma_S}}{\|z_h\|_{H_{00}^{1/2}(\Gamma_S)}}$$

$$\leqslant ch^{3/2-\varepsilon} \|\lambda\|_{H^{1-\varepsilon}(\Gamma_S)} + c \sup_{z_h \in W_h} \frac{a(\tilde u_h - u, \mathcal{E}_h z_h)}{\|z_h\|_{H_{00}^{1/2}(\Gamma_S)}},$$

where $\mathcal{E}_h z_h \in V_h$ is the discrete harmonic extension of $z_h^\Gamma \in H^{1/2}(\Gamma)$, which is the trivial extension to Γ of $z_h \in W_h \subset H_{00}^{1/2}(\Gamma_S)$.

We replace u by a discrete function $u_h^N \in V_h^{-1}$ satisfying the requirements of Corollary 5.6. We also use the fact that $\mathcal{E}_h z_h$ and $\tilde{u}_h - u_h^N$ are discrete harmonic to see

$$\sup_{z_h \in W_h} \frac{a(\tilde{u}_h - u, \mathcal{E}_h z_h)}{\|z_h\|_{H_{00}^{1/2}(\Gamma_S)}} = \sup_{z_h \in W_h} \frac{a(\tilde{u}_h - u_h^N, \mathcal{E}_h z_h)}{\|z_h\|_{H_{00}^{1/2}(\Gamma_S)}} \leqslant c|\tilde{u}_h - u_h^N|_{H^{1/2}(\Gamma)}.$$

Using an inverse inequality, we get

$$|\tilde{u}_h - u_h^N|_{H^{1/2}(\Gamma)} \leqslant c\frac{1}{h}\|\tilde{u}_h - u_h^N\|_{L^2(\mathfrak{S}(h))}$$
$$\leqslant c\frac{1}{h}\|u - u_h^N\|_{L^2(\mathfrak{S}(h))} + c\frac{1}{h}\|u - \tilde{u}_h\|_{L^2(\mathfrak{S}(h))}.$$

Now, Corollary 5.6 results in (5.7).

To bound $\|u - \tilde{u}_h\|_{L^2(\mathfrak{S}(h))}$, we employ different Galerkin orthogonalities to get a suitable representation of the error in the whole domain based on the solution of the dual problem. Applying Green's formula, we obtain the representation of the local error $e_h := u - \tilde{u}_h$:

$$\|e_h\|_{L^2(\mathfrak{S}(h))} = \sup_{\|v\|_{L^2(\mathfrak{S}(h))}=1} (e_h, v)_{L^2(\Omega)} = \sup_{\|v\|_{L^2(\mathfrak{S}(h))}=1} (e_h, -\Delta(T^D v))_{L^2(\Omega)}$$
$$= \sup_{\|v\|_{L^2(\mathfrak{S}(h))}=1} a(T^D v, e_h) - \langle e_h, \partial_n(T^D v)\rangle_{\Gamma_S},$$

where $T^D v \in H_0^1(\Omega)$ is the solution to the dual problem (5.2).

Let us introduce the conforming finite-element approximation of $w := T^D v$ as $w_h \in V_h \cap H_0^1(\Omega)$, and denote $\lambda_w := -\partial_n w|_{\Gamma_S}$. We recall the following orthogonality results: using the Galerkin orthogonality in the domain for the variational inequality (2.3), one has $a(w_h, e_h) = 0$, since $\text{Tr} w_h = 0$. We recall, that the definition of the Scott–Zhang operator P_h (see Section 5.1), guarantees $\langle u - P_h u, \mu_h\rangle_{\Gamma_S} = 0$ as well as $\langle P_h u - \tilde{u}_h, \mu_h\rangle_{\Gamma_S} = 0$ for $\mu_h \in M_h$. We can then conclude with

$$a(w - w_h, P_h u - \tilde{u}_h) + \langle P_h u - \tilde{u}_h, \lambda_w\rangle_{\Gamma_S} = 0.$$

For $1/p + 1/q = 1$, we find using the terms discussed above

$$a(w, e_h) + \langle e_h, \lambda_w\rangle_{\Gamma_S} = a(w - w_h, u - P_h u) + \inf_{\mu_h \in M_h}\langle u - P_h u, \lambda_w - \mu_h\rangle_{\Gamma_S}$$
$$\leqslant \|\nabla(w - w_h)\|_{L(p,2)}\|\nabla(u - P_h u)\|_{L(q,2)} + \|u - P_h u\|_{L^2(\Gamma_S)}\inf_{\mu_h \in M_h}\|\lambda_w - \mu_h\|_{L^2(\Gamma_S)}.$$

The convexity of Ω guarantees $\lambda_w \in H^{1/2}(\Gamma_S)$ with $\|\lambda_w\|_{H^{1/2}(\Gamma_S)} \leqslant c\|v\|_{L^2(\Omega)}$. Setting $q = \varepsilon^{-1}$, $p = (1 - \varepsilon)^{-1}$, the best approximation of the dual space, Equation (5.1) and Lemmas 5.2 and 5.5 yield the result. □

Summarizing the results of Lemmas 4.1, 4.2 and Theorem 5.7 shows the *a priori* result for the primal variable of Theorem 2.2.

5.4 *An improved result on the $L^2(\Omega)$-error*

Based on Natterer (1976), a convergence order $h^{3/2}$ in the $L^2(\Omega)$-norm was stated in Coorevits *et al.* (2001). However, the required $H^2(\Omega)$-regularity of the dual problem is very strong, since the dual problem is a variational inequality. Based on the improved trace estimate, we can show almost the same order without involving a dual inequality problem.

COROLLARY 5.8 Let u be the solution of (2.2) and u_h be the solution of (2.3). Assuming $u \in H^{5/2-\varepsilon}(\Omega)$, $\varepsilon \in (0, 1/2]$ fixed, we get

$$\|u - u_h\|_{L^2(\Omega)} \leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

Proof. The proof is based on an Aubin–Nitsche-type argument using a linear dual problem with homogeneous Dirichlet conditions. Due to the nonlinear Signorini condition, an additional error term on Γ_S needs to be bounded.

Let $w \in H_0^1(\Omega)$ solve $-\Delta w = u - u_h$ in Ω. Since Ω is convex, one has $\|w\|_{H^2(\Omega)} \leqslant c\|u - u_h\|_{L^2(\Omega)}$ and $\|\partial_n w_v\|_{L^2(\Gamma_S)} \leqslant \|w\|_{H^2(\Omega)}$. Applying Green's formula yields

$$\|u - u_h\|_{L^2(\Omega)}^2 = \int_\Omega \nabla w^{\mathrm{T}} \nabla (u - u_h) \, \mathrm{d}x - \langle u - u_h, \partial_n w_v \rangle_{\Gamma_S}.$$

The first term can be bounded as it is standard in Aubin–Nitsche arguments, due to the homogeneous Dirichlet values of w. For the second term, we use the trace estimate provided in Theorem 2.2:

$$
\begin{aligned}
\langle u - u_h, \partial_n w \rangle_{\Gamma_S} &\leqslant \|u - u_h\|_{L^2(\Gamma_S)} \|\partial_n w\|_{L^2(\Gamma_S)} \\
&\leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)} \|w\|_{H^2(\Omega)}. \qquad \square
\end{aligned}
$$

REMARK 5.9 We note that, in the proof of the $L^2(\Omega)$-norm, we use the trivial bound $\|u - u_h\|_{L^2(\Gamma_S)} \leqslant \|u - u_h\|_{H^{1/2}(\Gamma_S)}$. Thus, an extra $h^{1/2}$ would be gained, if a higher-order $L^2(\Gamma_S)$ bound was available.

6. Lagrange multiplier estimates

The $H^{-1/2}(\Gamma_S)$-norm for the Lagrange multiplier of the Signorini problem arising in the saddle point formulation (3.1) can be estimated using similar arguments as those used in Theorem 5.7. Due to the given primal estimate of Theorem 2.2, no estimate on a strip is needed here. By standard techniques, the $L^2(\Gamma_S)$-norm can also be estimated.

THEOREM 6.1 Let (u, λ) be the solution of the saddle point formulation (3.1). If the regularity requirement $u \in H^{5/2-\varepsilon}(\Omega)$, $\varepsilon \in (0, \frac{1}{2}]$ fixed, holds, then

$$\|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma_S)} \leqslant ch^{3/2-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)},$$

$$\|\lambda - \lambda_h\|_{L^2(\Gamma_S)} \leqslant ch^{1-\varepsilon} \|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

Proof. The first line of the saddle point problem (3.1a) and its Galerkin discretization (3.2a) yield $a(u - u_h, v_h) + \langle v_h, \lambda - \lambda_h \rangle_{\Gamma_S} = 0$ for $v_h \in V_h$. Similar arguments as in the proof of Theorem 5.7 give

$$\|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma_S)} \leqslant ch^{3/2-\varepsilon}\|\lambda\|_{H^{1-\varepsilon}(\Gamma_S)} + c \sup_{z_h \in W_h} \frac{a(u_h - u, \mathcal{E}_h z_h)}{\|z_h\|_{H_{00}^{1/2}(\Gamma_S)}}$$

$$\leqslant ch^{3/2-\varepsilon}\|\lambda\|_{H^{1-\varepsilon}(\Gamma_S)} + c|u_h - u_h^N|_{H^{1/2}(\Gamma)},$$

where we exploit the fact that $a(u, \mathcal{E}_h z_h) = a(u_h^N, \mathcal{E}_h z_h)$ and a stability estimate for discrete harmonic functions; see Toselli & Widlund (2005, Lemma 4.10). It is important to note that $u_h^N \in V_h^{-1}$, which is defined as in the proof of Theorem 5.7, only depends on u, not on u_h or $\tilde{u}_h$. Nevertheless, $u_h - u_h^N$ is discrete harmonic due to the Galerkin approximation of the saddle point problem. Using Corollary 5.6 and the primal estimate of Theorem 2.2, we conclude

$$|u_h - u_h^N|_{H^{1/2}(\Gamma)} \leqslant |u - u_h|_{H_{00}^{1/2}(\Gamma_S)} + |u - u_h^N|_{H^{1/2}(\Gamma)}$$

$$\leqslant ch^{3/2-\varepsilon}\|u\|_{H^{5/2-\varepsilon}(\Omega)}.$$

The remaining error estimate in the $L^2(\Gamma_S)$-norm follows by an inverse inequality and the best approximation properties:

$$\|\lambda - \lambda_h\|_{L^2(\Gamma_S)} \leqslant \inf_{\mu_h \in M_h}(\|\lambda - \mu_h\|_{L^2(\Gamma_S)} + \|\mu_h - \lambda_h\|_{L^2(\Gamma_S)})$$

$$\leqslant c \inf_{\mu_h \in M_h}\left(\|\lambda - \mu_h\|_{L^2(\Gamma_S)} + \frac{1}{\sqrt{h}}\|\mu_h - \lambda\|_{H^{-1/2}(\Gamma_S)}\right) + \frac{c}{\sqrt{h}}\|\lambda - \lambda_h\|_{H^{-1/2}(\Gamma_S)}. \qquad \square$$

7. Numerical results

We chose an example with an analytically known solution on $\Omega = (0, 1.4 + e/2.7) \times (0, 0.5)$, where $\Gamma_S = (0, 1.4 + e/2.7) \times \{0\}$. The choice of the domain was done in order to have an easy representation of the solution with an asymmetry over the Signorini boundary. We chose the volumetric and Dirichlet boundary data as well as the initial gap $g(x) = 1$ according to the exact solution, which is constructed as follows.

In polar coordinates, the singular component (see also Remark 2.1) is given by $u_{\mathrm{sing}}(r, \theta) = r^{3/2}\sin(\frac{3}{2}\theta)$, which we will also denote $u_{\mathrm{sing}}(x, y)$ in Cartesian coordinates. As this singular component has a one-sided active area, we need to modify the function to ensure the condition that the active set Γ^{act} is a compact subset of the Signorini boundary Γ_S. The singular function is translated such that the transmission point between the active and inactive part is at $x_l := 0.2 + 0.3/\pi \approx 0.295$. A spline of polynomial order four is used as a cut-off function u_{cut}. Adding a weighted reflection of this function, we get a function with a compact contact area. The second transmission point is set to $x_r := 1.2 - 0.3/\pi \approx 1.105$. For some scalar weight $a > 0$ (in the examples $a = 0.7$), the solution is given by

$$u(x, y) := (u_{\mathrm{sing}}(x - x_l, y)u_{\mathrm{cut}}(x) + au_{\mathrm{sing}}(x_r - x, y)u_{\mathrm{cut}}(1.4 - x))(1 - y^2) + 1.$$

For the right-hand side $f := -\Delta u$, the Dirichlet data $u_D := u|_{\Gamma_D}$ and $g(x) = 1$, the solution satisfies the Signorini-type problem (2.2). The actual contact area is given by $\Gamma^{\mathrm{act}} = [0.2 + 0.3/\pi, 1.2 - 0.3/\pi]$.
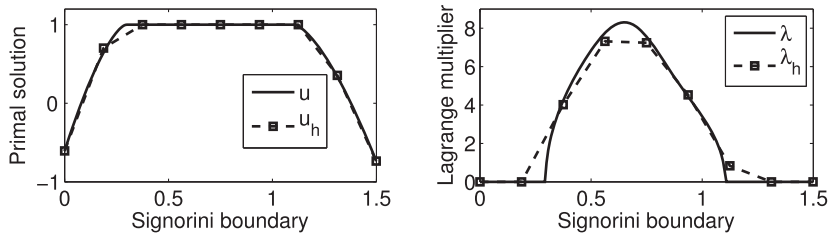
FIG. 4. Exact solution and finite-element approximation on level 2. Values for the primal solution (left) and the dual solution (right).
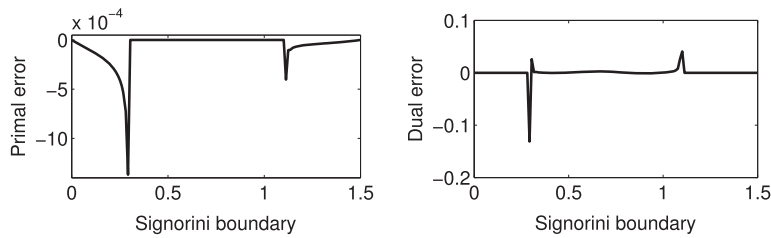

FIG. 5. Discretization error displayed at the Signorini boundary. Error of the primal variable (left) and of the dual variable (right) at level $k = 6$.

This choice of the contact area was made to ensure that no vertex of the mesh coincides with its boundary. The domain yields an asymmetry of the contact area. The desired regularity $u \in H^{5/2-\varepsilon}(\Omega)$ is given by construction. We start from a coarse, quadrilateral, initial mesh of 4×2 elements and refine uniformly.

The exact solution on the Signorini boundary as well as a coarse finite-element approximation are displayed in Fig. 4. In Fig. 5, the error distribution restricted to Γ_S on a fine finite-element grid is shown. Since the discrete Lagrange multiplier is based on a biorthogonal basis and hence is discontinuous, a post-processing is applied for the visualization and the error computation. Instead of $\lambda_h = \sum_{i=1}^{N_{M_h}} \lambda_i \psi_i \in M_h$, we represent the Lagrange multiplier as

$$\hat{\lambda}_h = \sum_{i=1}^{N_{M_h}} \lambda_i \varphi_i \in W_h.$$

As was shown in Hüeber (2008, Section 3.3), the order of convergence of $\hat{\lambda}_h$ is the same as for λ_h. Although the proof was shown for rates up to the order h, it can be easily extended to our situation.

The error distribution reflects the singularities of the solution at $\partial \Gamma^{\text{act}}$. We observe two peaks of large errors at the boundary of the active set caused by the reduced regularity at these points. The error in the interior of the domain is of a similar order, hence the overall error is not dominated by the error on the boundary.

In Table 1, the computed L^2-norms of the error as well as the estimated rate of convergence are depicted for each level k. Errors in fractional Sobolev norms are given in Fig. 6. The $L^2(\Gamma_S)$-,

TABLE 1 *Relative errors of the primal and dual solution at different mesh levels k and an averaged numerical convergence order*

k	$\|\lambda - \lambda_h\|_{L^2(\Gamma_S)}$		$\|u - u_h\|_{L^2(\Gamma_S)}$		$\|u - u_h\|_{L^2(\Omega)}$	
1	3.2629e−01	—	2.1724e−01		1.0050e−01	
2	1.2955e−01	1.33	4.2717e−02	2.35	2.5761e−02	1.96
3	4.4331e−02	1.44	7.0192e−03	2.48	6.2041e−03	2.01
4	1.8560e−02	1.38	2.3468e−03	2.18	1.5550e−03	2.00
5	1.5159e−02	1.11	9.3812e−04	1.96	4.0559e−04	1.99
6	5.8243e−03	1.16	1.8083e−04	2.05	9.8738e−05	2.00
7	2.7746e−03	1.15	4.0096e−05	2.07	2.4336e−05	2.00
8	1.9410e−03	1.06	1.7967e−05	1.94	6.4165e−06	1.99
9	9.8497e−04	1.05	4.6480e−06	1.94	1.5986e−06	1.99
10	4.0873e−04	1.07	9.7558e−07	1.97	3.8972e−07	2.00
11	1.7042e−04	1.09	1.9775e−07	2.01	9.5737e−08	2.00
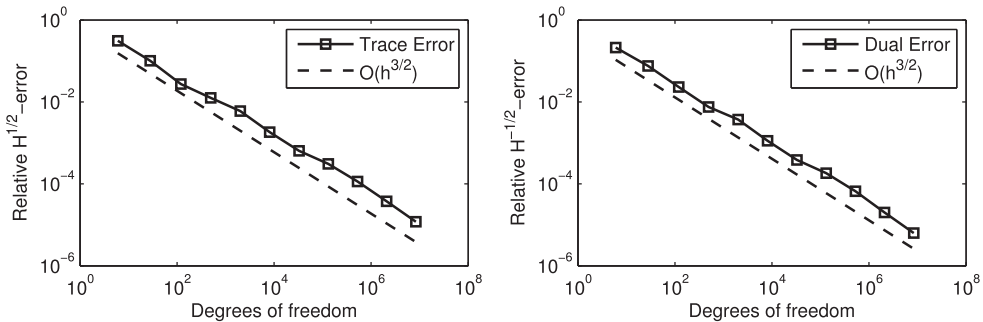
FIG. 6. Estimated convergence rates in fractional Sobolev spaces. Left: $H_{00}^{1/2}(\Gamma_S)$ norm for the primal solution. Right: $H^{-1/2}(\Gamma_S)$ norm for the dual solution.

$L^2(\Omega)$- and $H^1(\Omega)$-norms were computed by an adaptive integration to guarantee reliable results for the nonsmooth solution. The dual norm $H^{-1}(\Gamma_S)$ was estimated as the norm of the dual space to a fine finite-element space. To be more precise, the post-processed Lagrange multiplier on each level $k = 1, \ldots, 11$ was prolongated to level 15. On this level, we replace λ by the piecewise linear interpolation and compute $\hat{\lambda}_{h_k} - \mathcal{I}_{15}\lambda \in W_{h_{15}}$. Note that we have $\hat{\lambda}_{h_k} \in W_{h_k} \subset W_{h_{15}}$ due to the post-processing as described above. The $H^{-1}(\Gamma_S)$-norm is approximated by the dual norm of $W_{h_{15}}$, i.e.,

$$\|\lambda - \lambda_{h_k}\|_{H^{-1}(\Gamma_S)} \approx \sup_{w_{h_{15}} \in W_{h_{15}}} \frac{\int_{\Gamma_S}(\hat{\lambda}_{h_k} - \mathcal{I}_{15}\lambda)w_{h_{15}}\,\mathrm{d}x}{\|w_{h_{15}}\|_{H^1(\Gamma_S)}}.$$

The fractional-order Sobolev spaces $H_{00}^{1/2}(\Gamma_S)$ and $H^{-1/2}(\Gamma_S)$ were bounded using their interpolation property, i.e.,

$$\|v\|_{H_{00}^{1/2}(\Gamma_S)} \leqslant \|v\|_{H^1(\Gamma_S)}^{1/2}\|v\|_{L^2(\Gamma_S)}^{1/2}, \quad \|v\|_{H^{-1/2}(\Gamma_S)} \leqslant \|v\|_{H^{-1}(\Gamma_S)}^{1/2}\|v\|_{L^2(\Gamma_S)}^{1/2}.$$

TABLE 2 *Distance of the transmission points x_l and x_r to the discrete transmission points $x_{l,h}$ and $x_{r,h}$ on level k, compared to the mesh size h*

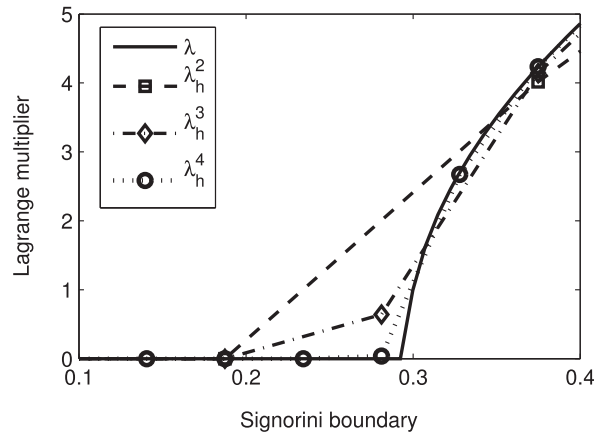| k	$|x_l - x_{l,h}|$	$|x_l - x_{l,h}|/h$	$|x_r - x_{r,h}|$	$|x_r - x_{r,h}|/h$
1	7.9339e−02	0.21	1.9989e−02	0.05
2	7.9339e−02	0.42	1.9989e−02	0.11
3	1.4369e−02	0.15	1.9989e−02	0.21
4	1.4369e−02	0.31	2.6865e−02	0.57
5	9.0579e−03	0.39	3.4384e−03	0.15
6	2.6556e−03	0.23	3.4384e−03	0.29
7	3.2012e−03	0.55	3.4384e−03	0.59
8	2.7280e−04	0.09	5.1006e−04	0.17
9	2.7280e−04	0.19	5.1006e−04	0.35
10	2.7280e−04	0.37	2.2203e−04	0.30
11	9.3246e−05	0.25	1.4402e−04	0.39

FIG. 7. Zoom of dual solution and approximations at levels 2–4 around the left transmission point.

The averaged convergence rates α_k, as given in Table 1 were computed in comparison with the first solution, by the formula

$$\left(\frac{\text{err}_1}{\text{err}_k}\right) = \left(\frac{1}{2}\right)^{\alpha_k(k-1)}.$$

We observe optimal-order convergence rates in the L^2-norms, which is as expected from our theory for the Lagrange multiplier, whereas for the $L^2(\Omega)$ and the $L^2(\Gamma_S)$-norm, we obtain better rates, than given by the theory. A closer look reveals that the convergence rates from level to level for the values on Γ_S vary more strongly. This is related to the fact that the discrete resolution of the active set is restricted to the vertices of the finite-element mesh. Depending on the quality of the approximation of the active set, the rates for values on Γ_S can be larger or smaller than expected. In Fig. 6, we see that the averaging described above is a reasonable estimate for the convergence rate.

We are also interested in a good resolution of the actual contact set, so we take a closer look at the solution near the boundary of Γ^{act}. The discrete active set is taken as the coincidence set of the primal solution $\Gamma_h^{\mathrm{act}} := \{x \in \Gamma_S : u_h(x) = 1\}$. In Table 2, the distance between the transmission points and the discrete transmission points is shown and compared with the mesh size. We note that the distance is always smaller than the mesh size. Since no vertex matches with a transmission point, this is the best we can expect. Figure 7 shows the dual solution and some finite-element approximations on the Signorini boundary.

8. Conclusion

In this work, we proved optimal-order convergence in the $H_{00}^{1/2}(\Gamma_S)$-norm for a standard finite-element approximation of Signorini problems. Based on this estimate, an optimal-order error bound for the Lagrange multiplier, i.e., the flux, was derived in the $H^{-1/2}(\Gamma_S)$-norm, and an improved bound for the primal error in the $L^2(\Omega)$-norm was shown as a side result.

Our analysis is based on a variational formulation of the continuous and the discrete Schur complement system which are variational inequalities posed on $H_{00}^{1/2}(\Gamma_S)$. The difficulties arising from the nonlinearity could be handled by a Strang lemma, resulting in two terms. One term is a Galerkin discretization error on Γ_S, which could be bounded by standard techniques. To bound the second term, a trace error of a linear problem posed on the whole domain, modern duality techniques with local estimates are adapted to the given situation.

A numerical example confirmed the optimal bounds and showed an accurate resolution of the active set. It also revealed that a gap remains between the theoretical and numerical results in the $L^2(\Omega)$-norm. As it was noted in Section 5.4, improved bounds in the $L^2(\Gamma_S)$-norm would directly imply improved bounds in the $L^2(\Omega)$-norm.

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