J. Inv. Ill-Posed Problems **18** (2011), 997–1011
DOI 10.1515/JIIP.2011.017

Some regularization methods for a thermoacoustic inverse problem

Barbara Kaltenbacher and Wolfgang Polifke

Abstract. In this paper we consider the thermoacoustic inverse problem of identifying the oscillatory heat release from pressure measurements. We consider the spatially one-dimensional and time harmonic case. Three different regularization methods for the stable solution of this ill-posed inverse problem are proposed: Lavrent'ev's method, regularization by discretization, and a method based on an explicit formula combined with regularized numerical differentiation. For these methods, the results of numerical experiments are documented.

Keywords. Combustion, thermoacoustic inverse problem, regularization, Volterra integral equation.

2010 Mathematics Subject Classification. 80A25, 45D05, 65F22, 80A23.

Dedicated to Prof. Dr. Michael V. Klibanov on the occasion of his 60th birthday

1 Introduction

In combustion technology, unsteady fluctuations of the heat release rate can adversely affect pollutant emissions, noise production, and combustion stability. The physical background for the latter phenomena is the fact that oscillatory heat release acts as a monopole source of sound in compressible flows. Unfortunately, it is quite difficult in experiment to determine the spatio-temporal distribution of heat release rate in a combustor even with advanced measurement techniques. For perfectly premixed combustors with good optical access, imaging of OH^* chemiluminescence over the entire extent of the flame does provide a qualitative measure of the spatial distribution of heat release rate and its fluctuations [1, 18]. However, this measurement technique is in general not applicable to technically relevant combustor configurations. On the other hand, it is fairly easy to measure the oscillatory pressure p at the combustor walls with wall-mounted microphones or pressure transducers. Therefore, the possibility of recovering oscillatory heat release distribution solely from acoustic pressure measurements has been considered [2, 13, 19, 22]. However, this represents an ill-posed problem and therefore has to be regularized, see e.g. [4, 5, 10, 15, 17, 23, 24].

Indeed, Bala Subrahmanyam et al. [2] show that the thermo-acoustic inverse problem can be formulated as a Fredholm integral equation of the first kind, which involves the Green's function for acoustic wave propagation. Furthermore, they provide a re-formulation of the thermo-acoustic inverse problem in terms of a Volterra integral equation of the second kind, with the *integral* over the fluctuating heat release rate as the unknown, thus leading to a problem of numerical differentiation. More recently, Pfeifer et al. [19] proposed an algorithm for the determination of location and strength of monopole sources in a closed chamber by evaluation of pressure signals measured by wall-mounted microphones. Based on the theory of near-field acoustic holography, the acoustic field in the chamber was represented by Green's Functions, taking into account also higher-order, possibly evanescent acoustic modes. In this manner, a linear mapping from N sound sources to M sensor positions was formulated. This mapping was inverted with Tikhonov regularization and truncated singular value decomposition (SVD), thus providing a solution to the thermo-acoustic inverse problem for the configuration considered.

The aim of this paper is to develop and apply more problem adapted regularization strategies than Tikhonov regularization or truncated SVD. Indeed, the formulations we will consider allow to make use of the problem structure as a Volterra integral equation or an inverse problem for an ordinary differential equation (ODE).

In Section 2 we specify the underlying model for our thermoacoustic inverse problem. Section 3 provides a discussion of some regularization strategies taking into account the Volterra structure of the problem. It also contains a new approach, that is based on an explicit formula for the inverse problem by solution of some ODE, as well as numerical differentiation. Finally, in Section 4, we compare the proposed methods in numerical tests.

2 Preliminaries

The oscillatory heat release q appears as a source term in the acoustic wave equation for the pressure fluctuations p, which for example in the limit of vanishing Mach number $M \equiv \overline{u}/\overline{c} \to 0$ (where $\overline{u}$ is a characteristic flow velocity and $\overline{c}$ the average speed of sound) reads as

$$\frac{1}{\overline{c}^2}\frac{\partial^2 p}{\partial t^2} - \bar{\rho}\,\nabla\cdot\left(\frac{1}{\bar{\rho}}\nabla p\right) = \frac{\kappa-1}{\overline{c}^2}\frac{\partial q}{\partial t},$$

where $\bar{\rho}$ is the average density, and κ the ratio of specific heats. We therefore refer to, e.g., [9, 11, 21, 25] for important results on uniqueness and stability of inverse source problems for second order hyperbolic equations.

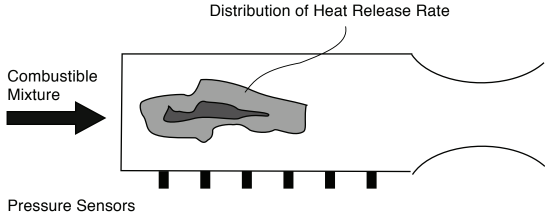

Figure 1. Experimental setup.

Following [2], we work in frequency domain, assuming time harmonic behaviour at frequency ω, and formulate the problem as a one-dimensional differential equation

$$p_{xx} + Z_1 p_x + Z_2 p = Z_3(ikq + Mq_x), \quad x \in (0, L), \tag{2.1}$$

which is justified in an appropriate experimental setup as shown in Figure 1 at frequencies below the cut-on frequency of non-plane acoustic modes, cf. [2]. Here p denotes the acoustic pressure, q the heat release, and the constants Z_1, Z_2, Z_3 are given by

$$Z_1 = -\frac{2kM}{1 - M^2}\, i$$
$$Z_2 = \frac{k^2}{1 - M^2}$$
$$Z_3 = -\frac{\gamma - 1}{\overline{c}(1 - M^2)}$$

where γ is the ratio of specific heats, $k = \omega/\overline{c}$ the wave number, $M = \overline{u}/\overline{c}$ the mean Mach number, $\overline{u}$ the mean axial velocity, $\overline{c}$ the mean speed of sound, and L the length of the combustor.

In here, to keep notation similar to existing literature on this application, we write a subscript x for the derivative with respect to space, although (2.1) is obviously an ordinary differential equation. Considering (2.1) as an ODE for q, we see that q is only uniquely determined if we specify in addition to (2.1) an initial value for q. We will simply set

$$q(0) = 0, \tag{2.2}$$

which is physically justified by the fact that $q(0)$ can be regarded as a selectable offset value.

Unique identifiability of q from measurements of p follows from the Picard–Lindelöf theorem provided p is sufficiently smooth. One can even derive the following explicit formula

$$\begin{aligned} q(x) \\ &= \exp(-i(k/M)x)(q(0) + 1MZ_3(-p_x(0) + (i(k/M) - Z_1)p(0))) \\ &\quad + 1MZ_3(p_x(x) - (i(k/M) - Z_1)p(x)) \\ &\quad + 1MZ_3(-(k/M)^2 - Z_1 i(k/M) + Z_2)\int_0^x \exp(i(k/M)(\xi - x))p(\xi)\,d\xi\,. \end{aligned} \tag{2.3}$$

This expression contains derivatives of the data p and is therefore not directly applicable for stable inversion, but it may be used for defining a reconstruction method if regularization is applied, see Subsection 3.3 below.

3 Regularization methods for the identification of the heat release

The problem of identifying the heat release from pressure measurements according to (2.1) is ill-posed in the sense that small perturbations in the data can lead to large deviations in the solution. As a matter of fact, in place of the exact pressures p only measured values p^δ are available, which are contaminated with noise (indicated here and below by a superscript δ). Consequently, regularization has to be applied. In this section we will propose three different methods for this purpose.

For appropriately choosing regularization parameters, we will assume that the noise level δ_p in

$$|p(x) - p^\delta(x)| \le \delta_p \tag{3.1}$$

is known.

3.1 Lavrent'ev's method

Integrating twice with respect to space, we can reformulate (2.1) as a Volterra integral equation of the first kind

$$\int_0^x \mathbf{k}(x,\xi)q(\xi)\,d\xi = f(x)\,, \quad x \in (0,L)\,, \tag{3.2}$$

where

$$\begin{aligned} f(x) = -p(x) - \int_0^x (Z_1 + Z_2(x-\xi))p(\xi)\,d\xi + (1 + Z_1 x)p(0) \\ + x p_x(0) + Z_3 M x q(0) \end{aligned} \tag{3.3}$$

and the kernel takes the simple (convolution type) form

$$\mathbf{k}(x, \xi) = -Z_3(M + ik(x - \xi)) \tag{3.4}$$

cf. [2].

The degree of ill-posedness of (3.2) is given by the smoothing properties of the integral operator

$$T : v \mapsto \int_0^{\cdot} \mathbf{k}(\cdot, \xi) v(\xi)\, d\xi\,, \tag{3.5}$$

cf. [12]. Due to the fact that by $\mathbf{k}(x, x) = -Z_3 M > 0$, (3.2) can be written as a (well-posed) second kind Volterra integral equation

$$\mathbf{k}(x, x) q(x) + \int_0^x \frac{\partial \mathbf{k}}{\partial x}(x, \xi) q(\xi)\, d\xi = f_x(x) \tag{3.6}$$

containing first derivatives of the data in f, we can regard (3.5) as a *one-smoothing* operator which implies that (3.2) is as ill-posed as one numerical differentiation.

While most "classical" regularization methods like Tikhonov regularization would destroy the non-anticipatory structure of problems of the form (3.2), several regularization methods have been proposed and analyzed in the literature, that respect the causal behaviour of Volterra type integral equations, see e.g., the survey paper [12]. Perhaps the most well-known among them is *Lavrent'ev's method*, which, given a small *regularization parameter* $\alpha > 0$ defines the solution q_α to the second kind (hence well-posed) Volterra integral equation

$$\alpha q(x) + \int_0^x \mathbf{k}(x, \xi) q(\xi)\, d\xi = f^\delta(x) \tag{3.7}$$

as a regularized approximation to the solution of (3.2). For a convergence analysis of this method we refer to [3] and further references in [12]. The choice of the regularization parameter α is crucial for an appropriate trade off between stability and approximation and should be chosen in dependence of the noise level δ on the right hand side f, which derives from the noise level δ_p, δ_0 in the data $p(x)$, $x \in [0, L]$, $p_x(0)$. If the derivative $p_x(0)$ has to be computed numerically by a one sided difference quotient $D_h^0 p^\delta$

$$\text{(a)} \qquad p_x(0) = \underbrace{\frac{p(h) - p(0)}{h}}_{=: D_h^0 p} + O(h) \text{ if } p \in C^2 \quad \text{or}$$

$$\text{(b)} \qquad p_x(0) = \underbrace{\frac{-p(2h) + 4p(h) - 3p(0)}{2h}}_{=: D_h^0 p} + O(h^2) \text{ if } p \in C^3\,, \tag{3.8}$$

then an optimal choice of the stepsize

$$h \sim \delta_p^{1/2} \text{ in case (a)} \quad \text{and} \quad h \sim \delta_p^{1/3} \text{ in case (b)} \tag{3.9}$$

yields

$$\delta_0 \sim \delta_p^{1/2} \text{ in case (a)} \quad \delta_0 \sim \delta_p^{1/3} \text{ in case (b)}$$

for

$$|p_x(0) - p_x^\delta(0)| \le \delta_0\,, \tag{3.10}$$

where $p_x^\delta(0) =: D_h^0 p^\delta$.

By (3.3), (2.2), we can use

$$\delta = (2 + 2Z_1 L + Z_2 L^2)\delta_p + L\delta_0$$

as noise bound in

$$|f(x) - f^\delta(x)| \le \delta$$

provided that (3.1), (3.10) holds.

According to Theorem 1 in [12] (quoted from [3]) $\alpha = \alpha(\delta)$ should be chosen such that

$$\alpha \to 0 \quad \text{and} \quad \delta/\alpha \to 0 \quad \text{as } \delta \to 0.$$

Then the solution $q^\delta_{\alpha(\delta)}$ of (3.7) with noisy data in the right hand side is guaranteed to converge to the exact q as $\delta \to 0$. The discrepancy principle is an a posteriori parameter choice strategy that is well known to yield convergence with optimal rates for many different regularization methods. Note, however, that the discrepancy principle in general does not yield convergence of Lavrent'ev's method (cf. Remark 7 in [20]). For a convergence analysis of modified versions of Lavrent'ev's method with the discrepancy principle see [7, 20] and the references therein.

Since $f(0) = 0$ according to (3.3), method (3.7) yields a solution q_α^δ with $q_\alpha^\delta(0) = 0$ for any $\alpha > 0$, which fits to (2.2). Therewith the problem of a boundary layer at $x = 0$ (as it often arises in the context of Lavrent'ev's method) does not occur here due to (2.2).

3.2 Regularization by discretization

On one hand, for numerical computations the infinite dimensional problem (3.2) or its regularized version (3.7) has to be discretized. On the other hand, discretization leads to a finite-dimensional and therewith stable problem. Therefore *regularization by discretization* is another option here, especially in view of the fact that we deal with a mildly ill-posed problem (recall that the Volterra integral operator is one-smoothing). To obtain convergence as $\delta \to 0$, it is essential to choose an appropriate kind of discretization. It has been shown (cf. [14]) that a combination of collocation with the trapezoidal rule

$$h\left(\frac{1}{2}\mathbf{k}(x_k, x_k)q_k + \sum_{j=1}^{k-1}\mathbf{k}(x_k, x_j)q_j + \frac{1}{2}\mathbf{k}(x_k, x_0)q_0\right) = f^\delta(x_k),$$
$$k = 1, \dots, N,\ h := \frac{L}{N},\ x_k = kh$$

for finding approximations $q_j \approx q(jh),\ j = 1, \dots, N$ is numerically unstable, whereas combining collocation with the midpoint rule

$$h\sum_{j=1}^{k}\mathbf{k}(x_k, x_{j-1/2})q_{j-1/2} = f^\delta(x_k), \tag{3.11}$$
$$k = 1, \dots, N,\ h := \frac{L}{N},\ x_k = kh,\ x_{j-1/2} = \left(j - \frac{1}{2}\right)h$$

for $q_{j-1/2} \approx q((j - \frac{1}{2})h),\ j = 1, \dots, N$ leads to a convergent method at least for exact data. The step size h acts as a regularization parameter in place of α appearing in Lavrent'ev's method and should be chosen such that

$$h \to 0 \quad \text{and} \quad \delta/h \to 0 \quad \text{as } \delta \to 0, \tag{3.12}$$

to yield convergence, see [8]. Convergence and convergence rates for regular solutions can be obtained if in place of the a priori choice (3.12), the discrepancy principle is used

$$\begin{aligned} h_* = \max\{h = \beta^{-l}L \mid \\ &\max_{k\in\{1,\dots,N_*\}} |[(Tq_h^\delta)(x_k) - (Tq_h^\delta)(x_{k-1})] - [f^\delta(x_k) - f^\delta(x_{k-1})]| \\ &\le 2\tau\delta\,,\ l \in \mathbb{N}_0\}\,, \end{aligned} \tag{3.13}$$

where $\beta > 1$ and $\tau > 1$ are a priori fixed constants, (in our computations we used $\beta = 2$ and $\tau = 1.1$) and $N_* = L/h_*$. see [8], where also well-definedness of h_* according to (3.13) is shown.

3.3 An explicit formula using numerical differentiation of the data

Another possibility for transforming (3.2) to a well posed problem is to use its differentiated version (3.6) applying regularized numerical differentiation (see, e.g., [5, 6, 16]) to obtain a stable approximation of f_x from the given data f^δ. In our case, this is equivalent to applying only one integration with respect to space in (2.1), which, in its turn, is equivalent to regarding (2.1) as differential equation for q

$$Z_3(ikq + Mq_x) = -f_{xx}, \tag{3.14}$$

with f as in (3.3). This ODE (3.14) can be solved explicitly for q, which gives the solution formula (2.3). To be able to apply this formula to noisy data p^δ, we have to approximate the derivative p_x by a difference quotient with appropriately chosen stepsize. Therewith we end up with the method

$$\begin{aligned} q(x_k) &\approx q_k^\delta \\ &:= \exp(-i(k/M)x_k)(q(0) + 1MZ_3(-D_h^0 p^\delta + (i(k/M) - Z_1)p^\delta(0))) \\ &\quad + 1MZ_3((D_h p^\delta)_k - (i(k/M) - Z_1)p^\delta(x_k)) \qquad (3.15) \\ &\quad + 1MZ_3(-(k/M)^2 - Z_1 i(k/M) + Z_2)Q(\exp(i(k/M)(\cdot - x))p^\delta(\cdot); x_k) \end{aligned}$$

where

$$(D_h p^\delta)_j = \frac{p(x_{j+1/2}) - p(x_{j-1/2})}{h} \approx p_x(x_j),$$

D_h^0 is as in (3.8), and $Q(g; x)$ denotes some quadrature rule with error of order $O(h^2)$ for approximating $\int_0^x g(\xi)\,d\xi$ (we here used the midpoint rule again). The stepsize h is chosen according to (3.9), depending on whether

$$\text{(a) } p \in C^2 \quad \text{or} \quad \text{(b) } p \in C^3,$$

which yields

$$|q(x_k) - q_k^\delta| \sim \delta_p^{1/2} \text{ in case (a)} \quad |q(x_k) - q_k^\delta| \sim \delta_p^{1/3} \text{ in case (b).}$$

4 Numerical experiments

In our computations, we used the values $M = 0.1$, $k = 0.5$, $\gamma = 1.2$ taken from [2].

As a first test example, we considered

$$p(x) = \exp(i(\Omega/L)x)$$

with the exact solution according to (2.3) given by

$$\begin{aligned} q(x) &= \exp(-i(k/M)x)q(0) \\ &\quad + \frac{Z_2 - (\Omega/L)^2 + Z_1 i\Omega/L}{i((k/M) + (\Omega/L))MZ_3}(\exp(i(\Omega/L)x) - \exp(-i(k/M)x)) \end{aligned} \tag{4.1}$$

Note that availability of an analytic formula for the solution helps to avoid an inverse crime (i.e. produce non-representative numerical results by restriction of the whole problem to a finite dimensional subspace). Synthetic noise of relative level $\frac{\|p^\delta - p\|}{\|p\|} = \delta_p * 0.01$ with $\delta_p = 14, 12, 1, 2,$ and 4 in the data is generated by adding rescaled standard normally distributed random numbers to the exact values of p. Figure 2 shows the respective results for the methods described in Subsections 3.1, 3.2, 3.3 for test example (4.1) with $L = 1$, $\Omega = 2\pi$.

To compare the methods directly, we plot their performance at different relative noise levels according to the formula

$$\frac{\|p^\delta - p\|}{\|p\|} = 10^{-\mathrm{SNR}/20}$$

(cf. [19]) see, Figure 3.

As a second test example, we consider a heat release distribution with two peaks

$$q(x) = A_1 \exp\left(\frac{(x - x_1)^2}{\sigma_1^2}\right) + A_2 \exp\left(\frac{(x - x_2)^2}{\sigma_2^2}\right) \tag{4.2}$$

with corresponding pressure distribution according to (2.1) computed by finite differences on a fine grid in order to avoid an inverse crime.

Figure 4 shows the respective results for the methods described in Subsections 3.1, 3.2, 3.3 for test example (4.2) with $x_1 = L/3$, $x_2 = 3L/4$, $\sigma_1 = 1.\mathrm{e}{-3}$, $\sigma_2 = 3.\mathrm{e}{-3}$, $A_1 = 100$, $A_2 = 50$.

In Figure 5, we show a comparison of the methods at different noise levels.

Methods (3.11) and (3.15) appear to be more robust against noise as compared to (3.7) for the smoother test example (4.1). For the less regular test example (4.2), performance was worse for all methods, as expected from the well-known fact that

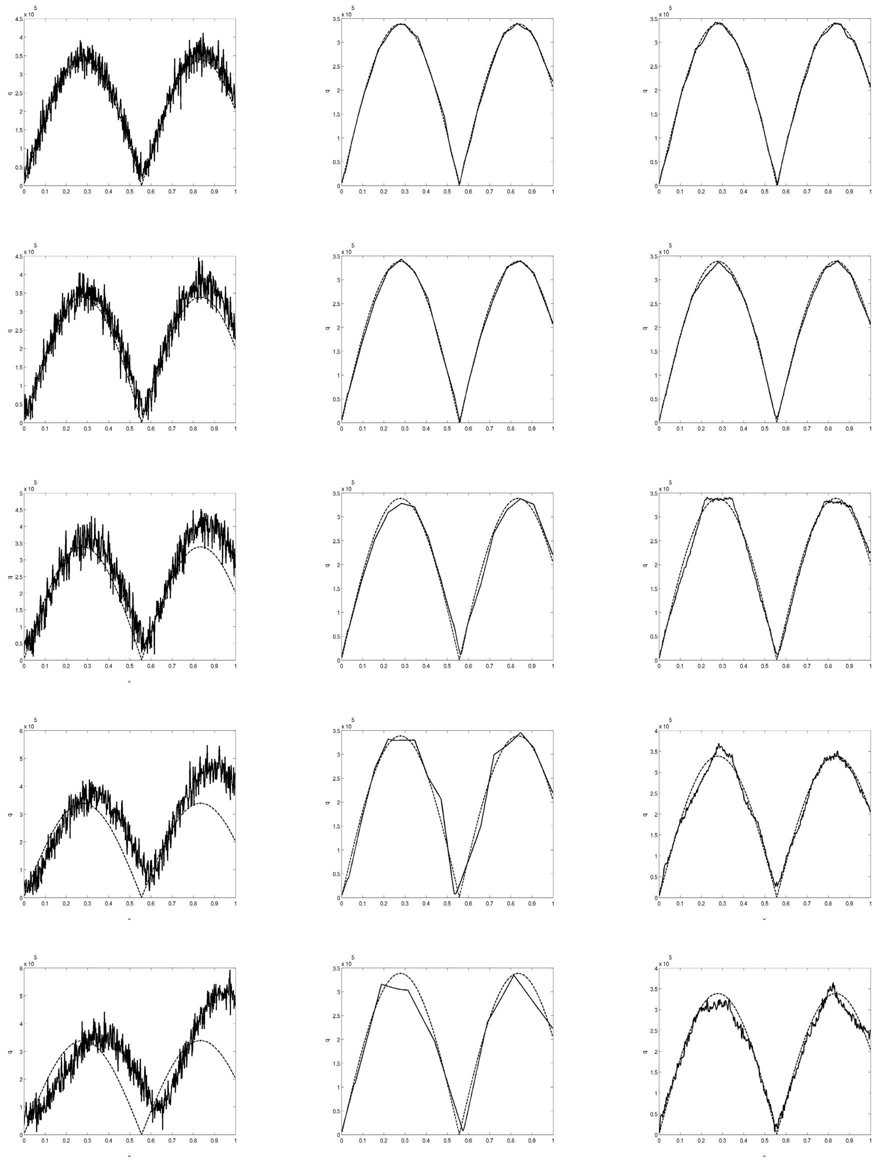

Figure 2. Method (3.7) (left) (3.11) (middle), (3.15) (right) for test example (4.1) applied to data with 14, 12, 1, 2, and 4 per cent noise (solid line) versus exact solution (dashed line).

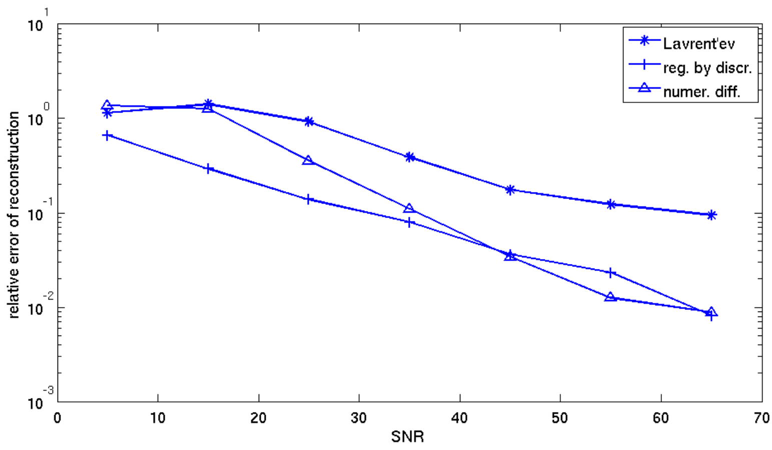

Figure 3. Relative error $\frac{\|q^{rec}-q\|}{\|q\|}$ for test example (4.1) in the reconstruction q^{rec} for SNR between 5 and 65

convergence of regularization methods depends on smoothness of the solution. Method (3.11) very well locates the peaks and to some extent even their heights, but yields somewhat oscillatory solutions. Methods (3.7) and (3.15) behave similarly for (4.2) at lower noise levels, where they succeed in avoiding oscillations but give poor approximations to the peak heights. For the highest noise level, method (3.15) misses the peaks completely. In summary, method (3.11) performs best for the two test cases considered.

5 Conclusions and outlook

In this paper we considered the inverse thermoacoustic problem of identifying the oscillatory heat release distribution from acoustic pressure measurements in the spatially one-dimensional time harmonic case. We applied three different regularization methods and compared their numerical behaviour on synthetic data.

Future research will be devoted to the spatially higher dimensional case in general combustor geometries, as well as with applications to realistic data representative of turbulent combustion.

In the 2- and 3-dimensional situation, (2.1) becomes a damped Helmholtz equation for p with a term of the form $Z_3(ikq + \vec{b}\nabla q)$ on the right hand side. Note that also in this case, techniques of numerical differentiation applied to p, in combination with numerical solution methods for a transport equation for q, as well as regularization by discretization, are expected to yield efficient and robust re-

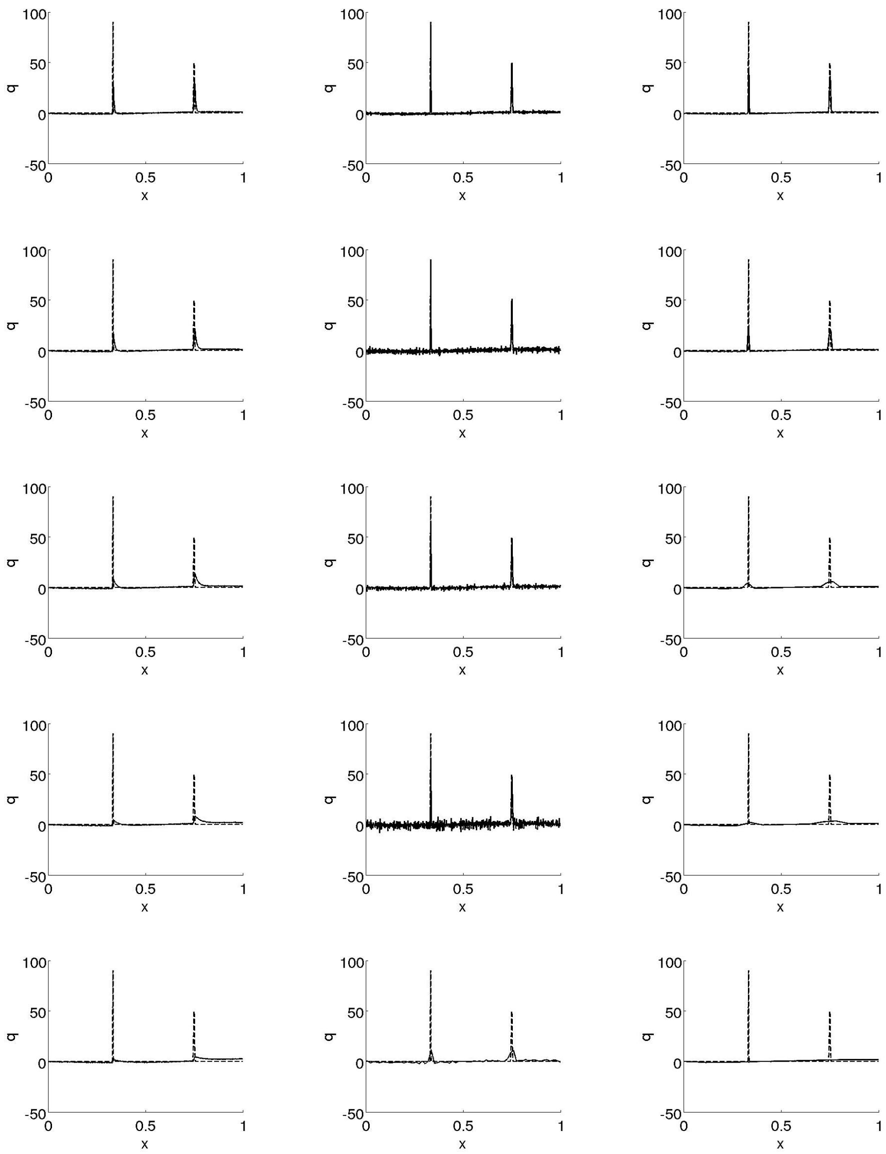

Figure 4. Method (3.7) (left) (3.11) (middle), (3.15) (right) for test example (4.2) applied to data with 14, 12, 1, 2, and 4 per cent noise (solid line) versus exact solution (dashed line).

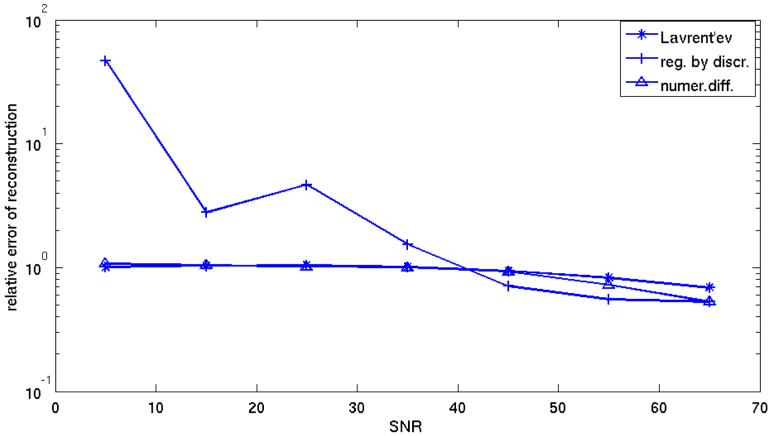

Figure 5. Relative error $\frac{\|q^{\text{rec}}-q\|}{\|q\|}$ for test example (4.2) in the reconstruction q^{rec} for SNR between 5 and 65.

construction methods. Due to their relation to (3.11), (3.15), the present paper might serve as a first study for the higher dimensional case. On the other hand, already this 1-d setting corresponds to a practically relevant experimental setup and therefore provides valuable information.

Bibliography

[1] B. O. Ayoola, R. Balachandran, J. H. Frank, E. Mastorakos and C. F. Kaminski, Spatially resolved heat release rate measurements in turbulent premixed flames, *Combust. and Flame* **144** (2006), 1–16.

[2] P. Bala Subrahmanyam, R. I. Sujith and M. Ramakrishna, Determination of unsteady heat release distribution from acoustic pressure measurements: A reformulation of the inverse problem, *J. Acoust. Soc. Am.* **114** (2003), 686–696.

[3] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, 1991.

[4] H. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

[5] C. W. Groetsch, *Inverse Problems in the Mathematical Sciences*, Vieweg, Braunschweig, 1993.

[6] M. Hanke and O. Scherzer, Inverse problems light: Numerical Differentiation, *Amer. Math. Monthly* **108** (2001), 512–521.

[7] J. Janno and U. Tautenhahn, Scale-type-estimates for a generalized method of Lavrent'ev regularization, *J. Inv. Ill-Posed Problems* **11** (2003), 161–190.

[8] B. Kaltenbacher, A convergence analysis of the midpoint rule for first kind Volterra integral equations with noisy data, *Journal of Integral Equations and Applications (special issue dedicated to Ch. W. Groetsch)* **22** (2010), 313–340.

[9] M. A. Kazemi and M. V. Klibanov, Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities, *Applicable Analysis* **50** (1993), 93–102.

[10] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, New York, 1996.

[11] M. V. Klibanov, Inverse Problems and Carleman Estimates, *Inverse Problems* **8** (1992), 575–596.

[12] K. Lamm, P., A survey of regularization methods for first-kind Volterra equations, in: *Surveys on Solution Methods for Inverse Problems*, pp. 53–82, Springer, Vienna, New York, 2000.

[13] T. C. Lieuwen, Y. Neumeier and B. T. Zinn, Determination of the Unsteady Combustion Process Driving in an Unstable Combustor from Acoustic Pressure Distribution Measurements, *J. Propul. Power* **15** (1999), 613–616.

[14] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[15] A. K. Louis, *Inverse und schlecht gestellte Probleme*, Teubner, Stuttgart, 1989.

[16] S. Lu and S. V. Pereverzev, Numerical Differentiation from a view point of Regularization Theory, *Math. Comp.* **75** (2006), 1853–1870.

[17] V. A. Morozov, *Regularization Methods for Ill-Posed Problems*, CRC Press, Boca Raton, 1993.

[18] C. O. Paschereit, B. B. H. Schuermans, W. Polifke and O. Mattson, Measurement of transfer matrices and source terms of premixed flames, *J. Eng. Gas Turbines and Power* **124** (2002), 239–247, Originally published as ASME 99-GT-133.

[19] C. Pfeifer, J. P. Moenck, L. Enghardt and C. O. Paschereit, Localization of combustion noise sources in enclosed flames, in: *Int. Conf. on Jets, Wakes, and Separated Flows*, Technical University of Berlin, Berlin, Germany, 2008.

[20] R. Plato, On the discrepancy principle for iterative and parametric methods to solve linear ill-posed equations, *Numer. Math.* **75** (1996), 99–120.

[21] J.-P. Puel and M. Yamamoto, On a global estimate in a linear inverse hyperbolic problem, *Inverse Problems* **12** (1996), 995–1002.

[22] M. K. Ramachandra and W. C. Strahle, Acoustic signature from flames as a combustion diagnostic tool, in: *20th Aerospace Sciences Meeting*, AIAA 82-0039, Orlando, FL, USA, 11–14 Jan. 1982.

[23] A. N. Tikhonov and V. A. Arsenin, *Methods for Solving Ill-Posed Problems*, Nauka, Moscow, 1979.

[24] G. Vainikko and A. Y. Veterennikov, *Ill-Posed Problems with A Priori Information*, VSP, Zeist, 1995.

[25] M. Yamamoto, Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method, *Inverse Problems* **11** (1995), 481–496.

Received September 28, 2010.

Author information

Barbara Kaltenbacher, Institut für Mathematik und Wissenschaftliches Rechnen, Universität Graz, Heinrichstr. 36, A-8010 Graz, Austria.
E-mail: barbara.kaltenbacher@uni-graz.at

Wolfgang Polifke, Fachgebiet für Thermodynamik, Technische Universität München, Boltzmannstr. 15, D-85748 Garching, Germany.
E-mail: polifke@td.mw.tum.de