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Approximations of the GOCE error variance-covariance matrix for least-squares estimation of height datum offsets

Research Article

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Abstract:
One main geodetic objective of the European Space Agency's satellite mission GOCE (gravity field and steady-state ocean circulation explorer) is the contribution to global height system unification. This can be achieved by applying the Geodetic Boundary Value Problem (GBVP) approach. Thereby one estimates the unknown datum offsets between different height networks (datum zones) by comparing the physical (e.g. orthometric) height values H of benchmarks in different datum zones to the corresponding values derived from the difference between ellipsoidal heights h (e.g. determined by means of global navigation satellite systems) and geoid heights N. In the ideal case, i.e. neglecting data errors, the misfit between H and $(h-N)$ is constant inside one datum zone and represents the datum offset. In practise, the accuracy of the offset estimation depends on the accuracy of the three quantities H, h and N, where the latter can be computed from the combination of a GOCE-derived Global Potential Model (GPM) for the long to medium wavelength and terrestrial data for the short wavelength content. Providing an optimum estimation of the datum offsets along with realistic error estimates, theoretically, requires propagation of the full error variance and covariance information of the GOCE spherical harmonic coefficients to geoid heights, respectively geoid height differences. From a numerical point of view, this is a very demanding task which cannot simply be run on a single PC. Therefore it is worthwhile to investigate on different levels of approximation of the full variance-covariance matrix (VCM) with the aim of minimizing the numerical effort. In this paper, we compare the estimation error based on three levels of approximation, namely (1) using the full VCM, (2) using only elements of the dominant m-block structure of the VCM and (3) using only the main diagonal of the VCM, i.e. neglecting all error covariances between the spherical harmonic coefficients. We show that the m-block approximation gives almost the same result as provided by the full VCM. The diagonal approximation however over- or underestimates the geoid height error, depending on the geographic location and therefore is not regarded to be a suitable approximation.

Keywords:
GOCE • height system unification • variance-covariance matrix • vertical datum

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1. Introduction

Traditionally, vertical reference frames are based on levelling networks. Thereby only height differences between benchmarks of a network are observed. Determination of absolute height values requires fixing the height of one or several datum points in the network. A suitable selection is to tie the height values to mean sea level as observed at a single or a set of tide gauge stations. All benchmarks referring to one and the same set of tide gauge stations belong to the same datum zone. Due to local and regional variations in dynamic ocean topography, i.e. due to varying heights of the mean sea surface above a common equipotential surface, there exist datum offsets between different vertical reference frames referring to different tide gauges. For example the height values in Germany historically are related to the tide gauge in Amsterdam, while the French height system refers

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to a tide gauge in Marseille. Comparison of the height values in both national systems with the common adjustment of the European Vertical Reference Frame shows a datum offset of about 50 cm between the two systems (see Sacher et al. 1999). This is caused by about a 50 cm difference of the mean dynamic topography between the North Sea (in the vicinity of Amsterdam) and the Mediterranean Sea (in the vicinity of Marseille). Such offsets can cause severe problems in engineering or scientific applications where height values from both systems need to be integrated. Therefore it is desirable to refer all vertical frames to one and the same datum, globally.

Basically there are three methods for height system unification (see, e.g., Rummel 2001). The first one is connection of different datums by spirit levelling. This corresponds to a common adjustment of all networks as provided for Europe by the European Vertical Reference Frame. Practically, however, this method is restricted to individual continents and is not suitable for global height system unification. The second method is ocean levelling, where one determines the mean dynamic ocean topography, i.e. the height of the mean sea surface above a conventional equipotential surface. This surface can be realised by a global geoid model, e.g. based on the measurements of the GOCE mission. The third and last method applies the same principle to continental areas. This is the Geodetic Boundary Value Problem (GBVP) approach, where orthometric (or normal) heights of different datum zones are compared to the difference between ellipsoidal heights and geoid (or quasi-geoid) heights. The ellipsoidal heights are determined by means of geometric methods like global navigation satellite system (GNSS) or satellite laser ranging (SLR), while the geoid (or quasi-geoid) heights are determined from the solution of a GBVP. In practise the geoid (or quasi-geoid) heights are derived from a combination of a Global Geopotential Model (GPM) and terrestrial data in the neighbourhood of the computation point.

The GBVP-approach as described, e.g., by Rummel and Teunissen (1988) or Xu and Rummel (1991) estimates the datum offsets in a least-squares adjustment. In order to derive both, an optimum result as well as a realistic quality description of the offsets, proper description of all error contributions is required. This comprises all observations, i.e. orthometric (or normal) heights, ellipsoidal heights and geoid (or quasi-geoid) heights.

The aim of the European Space Agency's (ESA) satellite gravity mission GOCE (gravity field and steady-state ocean circulation explorer) is to provide a precise, homogenous and high resolution global geoid (ESA, 1999). This gives the main contribution to the geoid (or quasi-geoid) height. Proper quality description therefore requires propagating the errors of the spherical harmonic coefficients of a GOCE-derived GPM to geoid (or quasi-geoid) height differences. The full resolution of GOCE based GPMs can be expected to be in the range of around degree and order 200-250. The number of coefficients of a model up to degree and order 250 is 63001 and therefore the size of the full variance-covariance matrix (VCM) of such a GPM is 63001 x 63001 which requires about 32 GB of storage space. Obviously propagation of such a large matrix is a numerically demanding and time consuming task. Therefore it is worthwhile to study approximations of the full VCM with the aim of reducing considerably the numerical effort while at the same time keeping the approximation error small. In this study we compare the error propagation of height system unification based on three levels of approximation: (1) using the full VCM, (2) using only the dominant block diagonal structure and (3) using the error variances only. These approximations are well known in geodesy. They are described for example in Haagmans and van Gelderen (1991) where comparisons of different approximations are carried out employing the global model GEM-T1 (March et al., 1988).

In Section 2, an overview is given on the GBVP-approach to height system unification which comprises the basic idea as well as the formalism that needs to be applied. This leads to a least-squares model for estimating datum offsets and to the corresponding error propagation. Section 3 focuses on error propagation of a GOCE derived spherical harmonic GPM and gives a detailed description of the three levels of approximation. In addition, geoid height errors based on the three different approximations are presented for a GOCE and, for comparison, for a GRACE-based GPM. These errors are used in the synthetic example of height system unification given in Section 4. Thereby the error contribution of a spherical harmonic GPM is determined for the case of 5 datum points in Europe which shall be connected to the datum point in Amsterdam. Section 5 summarises the results and presents the conclusions. Finally it should be mentioned, that, in the sequel, we do not differentiate between geoid and quasi-geoid heights respectively between orthometric and normal heights. The basic principle is the same for both types of heights and geoids. For the long-wavelength contribution which is derived from a GPM, the difference between geoid and quasi-geoid is related to the selection of the proper location of the computation point (on the surface of the ellipsoid or on the telluroid) and to the treatment of the masses outside the geoid. Both aspects will not be considered here in detail and we will loosely talk about geoid heights, while we are aware of the fact that also quasi-geoid heights exist and modifications need to be applied in the practical evaluation of both quantities.

2. Height System Unification Using the GBVP-Approach

Least-squares estimation of datum offsets based on the solution of the GBVP was developed by Rummel and Teunissen (1988) and various test cases were simulated by Xu and Rummel (1991) and Xu (1992). In the following we present the basic relations of this approach. Accordingly, height system unification is closely related to GNSS-Levelling (also see Gruber et al., 2012). Thereby orthometric heights H_k , derived from a combination of spirit levelling and gravimetry and related to some datum k, are compared to orthometric heights $H=h-N$, determined from the combination of ellipsoidal heights h (derived from geodetic space techniques like GPS) and geoid heights N. If a global geoid model is used to provide a common height reference surface, the comparison of

H and H_k directly reveals the datum offset of datum zone $\Delta\Omega_k$ with respect to the global datum. Figure 1 shows the basic geometric relations. Thereby N_k is derived from GNSS-Levelling, i.e. $N_k = h - H_k$, and represents some kind of biased geoid, which acts as height reference surface in datum zone k; it is offset from a global geoid by the unknown datum offset. The biased geoid is only defined inside the respective datum zone. The datum offset of a certain datum zone with respect to a global geoid can be derived according to

$$
\begin{aligned}
N_k - N &= N_0 + N_{k0} \\
(h - H_k) - N &= N_0 + N_{k0}
\end{aligned}
\tag{1}
$$

where the biased geoid height N_k is derived from GNSS-Levelling and the datum offset is split into the offset N_0 of an arbitrarily chosen conventional global reference datum ($k{=}0$) and the relative offset N_{k0} of datum k with respect to the global reference datum.

Equation (1) is the linear model for estimation of datum offsets, with the unknown offsets on the right hand side and the observations (ellipsoidal heights, leveled orthometric heights and geoid heights) on the left hand side. The model can also be formulated in relative sense to determine only datum differences between the different datum zones. Then the global offset N_0 drops out.

In the ideal case, the global geoid N is determined by means of satellite techniques; thus it is independent on terrestrial data and therefore also free from any datum definition. Given a set of spherical harmonic coefficients $\left\{\Delta\bar{C}_{lm}, \Delta\bar{S}_{lm}\right\}$ of the disturbing gravity field, e.g. from the GOCE mission, the geoid height can be computed according to (see e.g. Heiskanen and Moritz, 1967)

$$
N^{\mathrm{GOCE}}(P) = R \sum_{l=2}^{L} \sum_{m=0}^{l} \left(\Delta\bar{C}_{lm} \cos m\lambda_p + \Delta\bar{S}_{lm} \sin m\lambda_P\right) \bar{P}_{lm}\left(\cos\theta_P\right) \tag{2}
$$

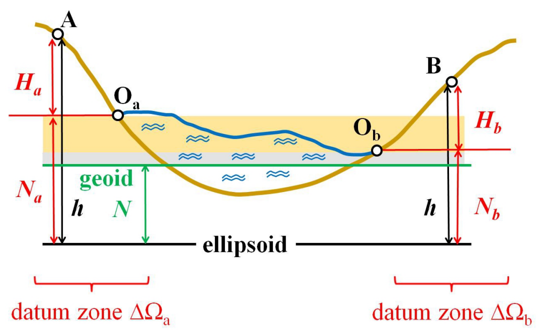

Figure 1. Geometric relations between a global geoid, ellipsoidal heights (h), orthometric heights (H_a and H_b) and biased geoid heights (N_a and N_b). The latter two quantities refer to two different datum zones $\Delta\Omega_a$ and $\Delta\Omega_b$ which are related to two different tide gauge stations (datum points) O_a and O_b. The gray area marks the datum offset of a arbitrarily chosen reference datum zone (here datum zone $\Delta\Omega_b$) and the global geoid; the yellow area marks the relative offsets of any other datum zone (here datum zone $\Delta\Omega_a$) to this reference datum zone.

wherein R is Earth's mean radius and $\bar{P}_{lm}\left(\cos\theta_P\right)$ are fully-normalised associated Legendre functions of spherical harmonic degree l and order m. Thereby we neglect the fact, that the geoid is not directly connected to the outer gravity field and in practice corrections for the topographic masses would have to be considered. As indicated in Eq. (2), the resolution of the satellite based geoid model is limited by the maximum spherical harmonic degree L and does, even in case of GOCE, not exceed 200-250, which corresponds to a spatial resolution between 80 and 100 km. The finer details of the gravity field are not represented in the GPM, but can still account to residual geoid heights in the order of several decimetres on global average (based, e.g., on the degree variance model of Tscherning and Rapp, 1974) or even one metre and more for individual points (see e.g.·Gruber et al., 2012). Therefore, the residual geoid height N^{res}, which corresponds to the omission error of the satellite based geoid model, must be determined from high resolution terrestrial data. Based on the well-known integral formula of Stokes (see Heiskanen and Moritz, 1967), the residual geoid height can be computed according to

$$
N^{\mathrm{res}}(P) = \frac{R}{4\pi\gamma} \int_{\Omega} S^{\mathrm{res}}(\psi_{PQ}) \left(\Delta g_k + \frac{2}{R} C_{k0}\right) \mathrm{d}\Omega_Q. \tag{3}
$$

In practice the integration is limited to some distance from the computation point, thus avoiding the necessity to provide terrestrial gravity anomalies Δg_k globally. Two important modifications of the original Stokes equation are indicated in Eq. (3):

1. Stokes' function $S\left(\psi_{PQ}\right)$ is replaced by the modified version $S^{\mathrm{res}}\left(\psi_{PQ}\right)$ and
2. the (unbiased) gravity anomalies Δg are expressed as the sum of biased anomalies Δg_k (referring to vertical datum zone k) and the gravity anomaly effect of the corresponding vertical datum offset (expressed in terms of the geopotential difference C_{k0}).

Modification (a) is introduced to ensure proper filtering of the long wavelength of the anomaly signal, thus preventing long wavelength terrestrial information to enter the residual geoid height. One very simple such modification is the one introduced by Wong and Gore (1969) which limits the Stokes kernel to spherical harmonic degrees above the maximum degree L of the employed global satellite model. Further modifications are possible to reduce the artificial effect caused by the sharp spectral cut-off of the

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Wong-and-Gore-modified kernel function at degree L, or to minimise the truncation error when limiting the integration radius to some relatively small distance around the computation point. We will however not discuss those, but only refer to the Wong and Gore modification which can be interpreted as a kind of worst-case modification causing significant oscillations in the residual geoid height, cf. Gerlach and Rummel (2012).

Modification (b) expresses the fact, that terrestrial gravity anomalies are indirectly influenced by height datum offsets. This is because orthometric heights of the data points are required for the computation of free-air gravity anomalies to reduce the surface gravity values down to the geoid. Since the orthometric heights refer to some local or regional datum and therefore are, in general, biased with respect to some global conventional datum, also the gravity anomalies are biased. Therefore the terrestrial anomalies are denoted Δg_k, which indicates relation to datum k. Formally, the anomaly offset is corrected by adding, in Eq. (3), the second term in parenthesis which is the datum offset C_{k0} expressed in terms of geopotential values and transferred to units of gravity by multiplication with $2/R$. Because the datum offset is not known beforehand, in practice, only biased anomalies can be used in the evaluation of Stokes's integral. Separating the integration into two terms we can write

$$\begin{aligned} &\frac{R}{4\pi}\int_{\Phi} S^{\mathrm{res}}\left(\psi_{PQ}\right)\left(\Delta g_k + \frac{2}{R}C_{k0}\right)\mathrm{d}\Omega_Q = \\ &\frac{R}{4\pi}\int_{\Omega} S^{\mathrm{res}}\left(\psi_{PQ}\right)\Delta g_k \mathrm{d}\Omega_Q + \frac{R}{4\pi}\int_{\Phi} S^{\mathrm{res}}\left(\psi_{PQ}\right)\frac{2}{R}C_{k0}\mathrm{d}\Omega_Q = \\ &N^{\mathrm{Stokes}} + N^{\mathrm{ind}} \end{aligned} \tag{4}$$

The first term takes care of the integration of biased terrestrial gravity anomalies, while the second is the indirect bias term which contains the unknown datum offsets and therefore needs to be moved to the right hand side of the least-squares model given in Eq. (1). This computationally complicates the adjustment process, because Stokes's function needs to be integrated individually over all datum zones, which requires knowledge of the segmentation of the different height datum zones. However, when employing a satellite model from GOCE, which implies all spherical harmonic degrees below L being removed from the residual Stokes's function $S^{\mathrm{res}}\left(\psi_{PQ}\right)$, the amplitude of the indirect bias term can be reduced significantly. As shown by Gerlach and Rummel (2012) the term can even be neglected for height system unification with GOCE as it can be expected not to exceed the level of 1 cm. Therefore Eq. (1) reduces to

$$(h - H_k) - \left(N^{\mathrm{GOCE}} + N^{\mathrm{Stokes}}\right) = N_0 + N_{k0} \tag{5}$$

Symbolically replacing the observations by vector $\boldsymbol{y}$ and the unknowns by vector $\boldsymbol{x}$, we get the linear model

$$\boldsymbol{y} = \mathrm{A}\boldsymbol{x} \tag{6}$$

with design matrix A and least-squares solution

$$\hat{x} = \left(A^T \boldsymbol{Q}_{yy}{}^{-1}\mathrm{A}\right)^{-1}\mathrm{A}^T\boldsymbol{Q}_{yy}{}^{-1}\boldsymbol{y} \tag{7}$$

where $\boldsymbol{Q}_{yy}$ is the error VCM of the observations. The estimated error VCM of the unknowns is given by the inverse of the normal equation matrix, i.e. by

$$\boldsymbol{Q}_{\hat{x}\hat{x}} = \left(\mathrm{A}^T\boldsymbol{Q}_{yy}{}^{-1}\mathrm{A}\right)^{-1} \tag{8}$$

Because ellipsoidal heights, orthometric heights and geoid heights from GOCE are all determined independent of each other employing different types of data it can be assumed that the corresponding errors are uncorrelated. Neglecting as well possible correlations between orthometric heights and geoid heights computed from Stokes integration (where possibly the same terrestrial gravity data enters), the VCM of the observation errors can be written as

$$\boldsymbol{Q}_{yy} = \boldsymbol{Q}_{hh} + \boldsymbol{Q}_{HH} + \boldsymbol{Q}_{NN} + \boldsymbol{Q}_{gg} \tag{9}$$

Thereby $\boldsymbol{Q}_{hh}$ is the error-VCM of the ellipsoidal heights, $\boldsymbol{Q}_{HH}$ the error-VCM of the orthometric heights, $\boldsymbol{Q}_{NN}$ is the VCM of geoid height errors derived from the global satellite model and $\boldsymbol{Q}_{gg}$ is the VCM of the residual geoid height errors resulting from integration of the errors of the terrestrial gravity anomalies. Proper description of all error contributions is required for proper relative weighting of the observations and to achieve the optimum estimate for the unknowns as well as a realistic description of the error of the estimated unknowns. In this paper we only focus on the error contribution of the satellite based global geoid model, i.e. on $\boldsymbol{Q}_{NN}$. All other error contributions are not treated here.

3. Error Propagation from Global Potential Models

As mentioned in Section 2, quality assessment of height unification is based on propagation of error variances and covariances of all required quantities. In the context of height unification with GOCE, error variances and covariances of GOCE-derived geoid heights at or between datum points (or an adequate set of connection points) in different datum zones must be known. This information can be derived from the full error VCM of GOCE spherical harmonic coefficients, e.g., provided by ESA through the GOCE Virtual Archive at http://eo-virtual-archive1.esa.int. Based on the spherical harmonic synthesis given in Eq. (2), error propagation yields for the error covariance of the geoid height between two stations P and Q (see, e.g., Haagmans and van Gelderen, 1991)

$$\mathrm{Cov}(N_P,N_Q) = R^2 \sum_{m=0}^{L}\sum_{k=0}^{L}\Bigg[\left(\sum_{l=m}^{L}\sum_{n=k}^{L}\mathrm{Cov}(\bar{C}_{lm},\bar{C}_{nk})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nk}(\cos\vartheta_Q)\right)\cos m\lambda_P\cos k\lambda_Q +$$
$$+\left(\sum_{l=m}^{L}\sum_{n=k}^{L}\mathrm{Cov}(\bar{S}_{lm},\bar{C}_{nk})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nk}(\cos\vartheta_Q)\right)\sin m\lambda_P\cos k\lambda_Q +$$
$$+\left(\sum_{l=m}^{L}\sum_{n=k}^{L}\mathrm{Cov}(\bar{C}_{lm},\bar{S}_{nk})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nk}(\cos\vartheta_Q)\right)\cos m\lambda_P\sin k\lambda_Q +$$
$$+\left(\sum_{l=m}^{L}\sum_{n=k}^{L}\mathrm{Cov}(\bar{S}_{lm},\bar{S}_{nk})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nk}(\cos\vartheta_Q)\right)\sin m\lambda_P\sin k\lambda_Q\Bigg] \tag{10}$$

where error covariances between two different spherical harmonic coefficients, e.g. a sine-coefficient and a cosine-coefficient) are denoted $\mathrm{Cov}(\bar{S}_{lm},\bar{C}_{nk})$; the spherical harmonic degree number is indicated by letters l and n, the order number by letters m and k and $\{\vartheta,\lambda\}$ are the spherical coordinates latitude and longitude of the computation points P and Q.

Since the full error VCM of GOCE has an expected size of $8\cdot(L+1)^4$ which corresponds to roughly 13 GB (for L=200) respectively 32 GB (for L=250), error propagation is a computationally demanding task, which cannot simply be run on an ordinary personal computer in an efficient way. Therefore one has to either make use of super computers or computer clusters or one has to take into account certain approximations. In the following, such approximations will be applied to the computation of geoid height error covariances and it will be investigated if different levels of approximation lead to significantly different estimates of height datum offsets.

Approximations

According to Sneeuw (2000), selection of a nominal circular orbit with constant inclination leads to a linear system of observation equation, where each individual spherical harmonic order m is decoupled from coefficients of all other orders. Therefore the normal equation matrix and the error VCM of the estimated potential coefficients shows a block diagonal structure, provided that the matrices are ordered with respect to spherical harmonic order m (hereafter called m-order). Because the orbit of GOCE is almost circular and the inclination almost constant, one can expect, that the error VCM is block-diagonal dominant.

Therefore, as a first level of approximation, all covariances between coefficients of different order m ($m \neq k$) will be neglected. This results in an enormous reduction of computational load. In this case, the VCM is split into individual blocks, each with a maximum number of coefficients of $(L+1)^2$. In addition, covariances between sine and cosine coefficients are neglected. This leads to the following formulation

$$\mathrm{Cov}(N_P,N_Q) \approx R^2 \sum_{m=0}^{L}\Bigg[\left(\sum_{l=m}^{L}\sum_{n=m}^{L}\mathrm{Cov}(\bar{C}_{lm},\bar{C}_{nm})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nm}(\cos\vartheta_Q)\right)\cos m\lambda_P\cos m\lambda_Q +$$
$$+\left(\sum_{l=m}^{L}\sum_{n=m}^{L}\mathrm{Cov}(\bar{S}_{lm},\bar{S}_{nm})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{nm}(\cos\vartheta_Q)\right)\sin m\lambda_P\sin m\lambda_Q\Bigg] \tag{11}$$

This level of approximation is called the *m-block approach*.

In a second level of approximation all of the error covariances are neglected. This is called the *diagonal approach* because only the elements along the main diagonal of the VCM, i.e. only the error variances $\mathrm{Var}(\bar{C}_{lm})$ and $\mathrm{Var}(\bar{S}_{lm})$, are used. Accordingly, Eq. (11) further reduces to

$$\mathrm{Cov}(N_P,N_Q) \approx R^2 \sum_{m=0}^{L}\Bigg[\left(\sum_{l=m}^{L}\mathrm{Var}(\bar{C}_{lm})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{lm}(\cos\vartheta_Q)\right)\cos m\lambda_P\cos m\lambda_Q +$$
$$+\left(\sum_{l=m}^{L}\mathrm{Var}(\bar{S}_{lm})\bar{P}_{lm}(\cos\vartheta_P)\bar{P}_{lm}(\cos\vartheta_Q)\right)\sin m\lambda_P\sin m\lambda_Q\Bigg] \tag{12}$$

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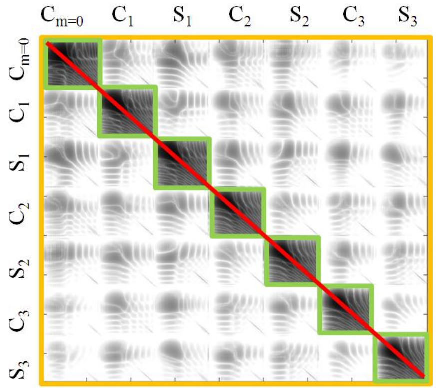

Figure 2. Error variance-covariance matrix of GOCE spherical harmonic coefficients (TIM3 model up to maximum degree L = 250) in m-order for the lowest orders m = {0,1,2,3}

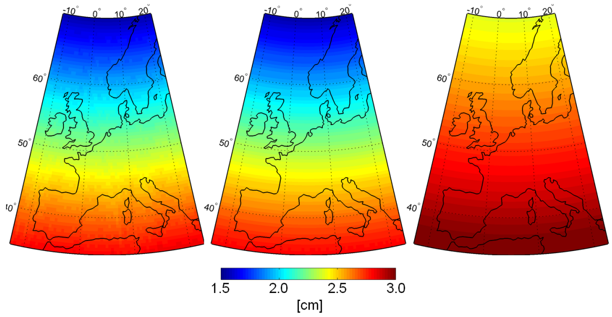

Figure 3. Geoid height error standard deviation on a 30' x 30' geographic grid in Europe from the GOCO02s spherical harmonic model (up to maximum spherical harmonic degree 180) based on three levels of approximation: full variance-covariance matrix (left), m-blocks approximation (middle) and diagonal approximation (right).

Figure 2 illustrates the three levels of approximation for a small section of the full error VCM of the GOCE model TIM3 (Pail et al., 2010a). The magnitude of the covariances is indicated by the gray scale value of the image with black representing large values and white representing no correlation. Visual inspection proofs that the GOCE error VCM indeed is block diagonal dominant. Using the full matrix for error propagation (corresponding to Eq. (10)) is represented by the orange box which encloses all matrix elements. The *m-block* approach described in Eq. (11) is represented by the elements along the dark blocks inside the green boxes. The diagonal approach described in Eq. (12) is represented by the red line, which actually only contains the main diagonal of the full VCM.

3.1. Geoid height error variances from GOCE

Based on the three levels of approximation discussed above, geoid height error variances on a 30' x 30' geographic grid in Europe were computed from the error VCM of a GOCE model (here we used the combined field GOCO02s (Pail et al., 2010b), which would be the model of choice in practical applications) up to spherical harmonic degree and order 180. The results (in terms of error standard deviations) are shown in Fig. 3. The left picture is based on the propagation of the full VCM, the central picture is based on the m-block approximation and the right picture is based on the diagonal approach. The full matrix and the m-block approach provide almost identical results with only some minor small scale differences. In contrast, the diagonal approach yields significantly larger values. The error is overestimated by a factor of about 1.2 to 1.5. Figure 4 shows the relative error of the two approximation schemas in terms of standard deviations. The left panel in Fig. 4 reveals small error correlations along the orbit tracks which are neglected in the m-block result. They amount to maximum values of about 1% of the total error standard deviation. The right panel in Fig. 4 shows the corresponding relative errors of the diagonal approximation which reveals a strong miscalling of up to 50% in the northern latitudes. This fits to the scaling factor of 1.5 described above. The computations were repeated for a global grid of computation points to give an overview on the differences between the full and the diagonal approach. Figure 5 shows the corresponding results, i.e. the geoid height error standard deviation derived from the full VCM in the top panel, the result based on the diagonal approach in the middle and in the lower panel the scaling factor between the two solutions. As expected, the diagonal approach provides a solution with no longitude dependence. The values are more homogenous as compared to the full solution, i.e. the data distribution with denser orbit tracks in higher latitudes is not clearly represented in the diagonal solution. Evaluation of a location dependent scaling coefficient (lower panel) shows that the diagonal approach tends to overestimate the error variance in the northern hemisphere, while it underestimates the error in the southern hemisphere. Due to a slight eccentricity of the GOCE orbit, the picture is not symmetric (or anti-symmetric) with respect to the equator, but a small southward shift is visible in the top and lower panels. Comparing the numerical values of the European example in Fig. 3 with those of the global example in Fig. 5, one must be aware, that the maximum degree and order used for error propagation is limited to 180 for the European example, while for the global case the full resolution was used, i.e. degree and order 250. The global case is only shown for illustration of the error behaviour and employs the full set of coefficients, while the European example will be used in a synthetic example of height system unification in Section 4; therefore, as discussed in Gruber et al. (2012), the maximum degree is limited to the significant resolution of the model at around degree 180. Coefficients above this degree do not represent the full gravity field signal anymore.

3.2. Geoid height error covariances from GOCE and GRACE

In addition to error variances, also error covariances have been computed for the European example. The two-dimensional covari-

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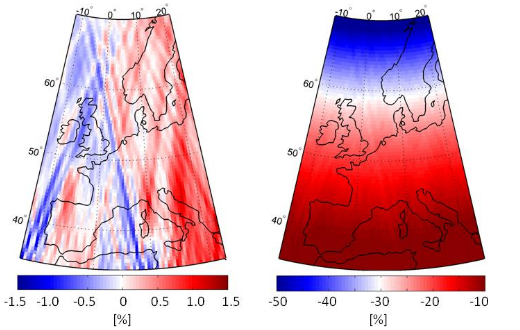

Figure 4. Relative error of the standard deviations shown in Fig. 3. Left: relative error of the m-block approximation. Right: relative error of the diagonal approximation. Both are given as percentage numbers. Note the different colorbars in both panels.

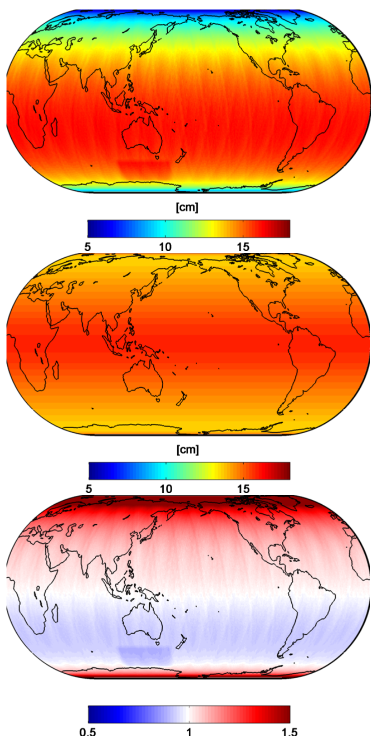

Figure 5. Global 30' x 30' grid of geoid height error standard deviations from the GOCO02s model (up to spherical harmonic degree 250) using the full variance-covariance matrix (upper panel), using the diagonal approximation (middle panel) and the scaling factor between the two (lower panel).

ance functions are shown in Fig. 6; the origin of the functions is located in the center of the area close to Amsterdam. The three panels represent the full (left), the m-block (middle) and the diagonal (right) approach. All of the three functions are quite similar, with the diagonal approach being almost exactly isotropic while the two other solutions show slightly un-isotropic features (compare the function values along a North-South axis with those along an East-West axis). Also the magnitude of the function values seems to decay a bit faster in case of the full and the m-block solutions as compared to the diagonal approach. Figure 7 shows the relative errors of the two approximations computed by dividing the difference between full and approximate solution (left: m-block approximation; right: diagonal approximation) by the value of the variance at the origin of the full-approach solution. The figure confirms that there are only slight differences of below 1% between the full and the m-block solutions. The diagonal approach however yields relative differences of up to 40% concentrated around the origin of the function.

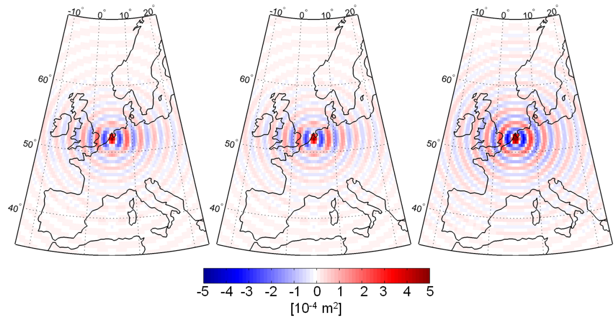

Figure 6. Geoid height error covariance function on a 30' x 30' geographic grid in Europe from the GOCO02s spherical harmonic model (up to maximum spherical harmonic degree 180) based on three levels of approximation: full variance-covariance matrix (left), m-blocks approximation (middle) and diagonal approximation (right).

Due to the fact, that all of the solutions provide an almost isotropic error behaviour, for reasons of comparison the same computations have been repeated using the error VCM of the GRACE spherical harmonic model ITG-GRACE2010s (see Mayer-Gürr, 2006 and http://www.igg.uni-bonn.de/apmg/index.php?id=itg-grace2010) up to degree and order 120. The results are shown in Fig. 8 in the same order as in the GOCE case of Fig. 6. Again the full and the m-block solutions show similar error behaviour while the diagonal solution is significantly different. The latter is not perfectly, but at least close to being isotropic, while the full and the m-block approaches result in the well-known striping pattern of GRACE errors (see, e.g., Swenson and Wahr, 2006). The figures show that depending on the satellite technique used to derive the global gravity field model, the error behavior can be quite different and different levels of approximation of the full error VCM might lead to either only slightly or significantly different covariance functions. The question of how significant the differences are in case of least-squares estimation of height datum offsets shall be answered in the following section.

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Table 1. Geoid height difference (with respect to the datum point in Amsterdam) error standard deviation from the GOCO02s model (up to spherical harmonic degree 180) based on three levels of approximation: full variance-covariance matrix (top row), m-blocks approximation (mid row) and diagonal approximation (bottom row). Units are centimetre. In addition the relative error is provided in percentage.

	Tredge	Newlyn	Trieste	Marseille	Cascais
GOCE full	3.1 (0.0%)	3.2 (0.0%)	3.3 (0.0%)	3.3 (0.0%)	3.5 (0.0%)
GOCE m-blocks	3.1 (0.1%)	3.2 (0.1%)	3.3 (0.1%)	3.4 (0.3%)	3.5 (0.2%)
GOCE diagonal	3.9 (26.1%)	3.8 (20.7%)	3.9 (18.4%)	3.9 (17.3%)	4.0 (15.4%)

Table 2. Geoid height difference (with respect to the datum point in Amsterdam) error standard deviation from ITG-GRACE2010s (up to spherical harmonic degree 120) based on three levels of approximation: full variance-covariance matrix (top row), m-blocks approximation (mid row) and diagonal approximation (bottom row). Units are centimetre. In addition the relative error is provided in percentage.

	Tredge	Newlyn	Trieste	Marseille	Cascais
GRACE full	2.8 (0.0%)	2.3 (0.0%)	3.1 (0.0%)	3.1 (0.0%)	3.1 (0.0%)
GRACE m-blocks	3.0 (7.7%)	2.4 (6.5%)	3.3 (5.8%)	3.4 (7.9%)	3.4 (8.2%)
GRACE diagonal	2.2 (21.8%)	2.2 (5.9%)	2.5 (18.5%)	2.6 (16.1%)	2.8 (9.4%)

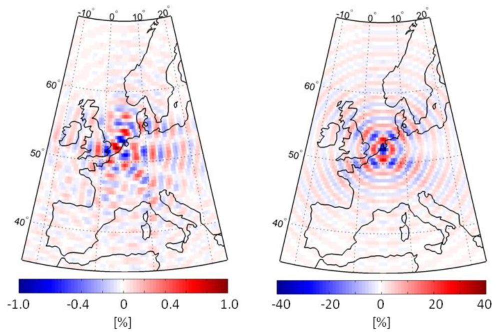

Figure 7. Relative error of the covariances shown in **Fig. 6**. Left: relative error of the m-block approximation. Right: relative error of the diagonal approximation. Both are given as percentage numbers. Note the different colorbars in both panels.

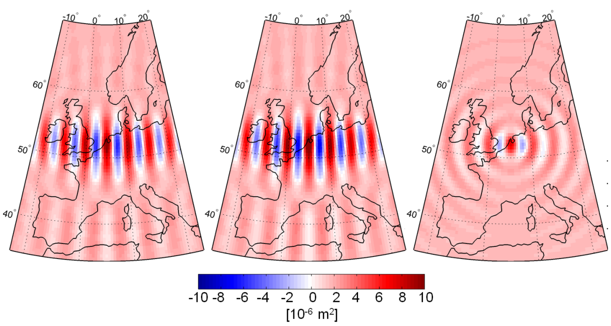

Figure 8. Geoid height error covariance function on a 30' x 30' geographic grid in Europe from the spherical harmonic model ITG-GRACE2010s (up to maximum spherical harmonic degree 120) based on three levels of approximation: full variance-covariance matrix (left), m-blocks approximation (middle) and diagonal approximation (right).

4. GOCE Error Contribution to Datum Unification: a Synthetic Example

The error covariance functions derived in the previous section shall be used in a synthetic example of height system unification in Europe. The geographic distribution of the 6 tide-gauge stations acting as datum points of the related national vertical reference frames is given in Fig. 9. In our example, the datum points in Tredge (Norway), Newlyn (United Kingdom), Cascais (Portugal), Marseille (France) and Trieste (the tide gauge is located in Italy, but only the former Austrian reference frame and some frames along the Adriatic sea are linked to Trieste, while the Italian datum refers to a tide gauge in Geneva) shall be linked to the datum point in Amsterdam (the Netherlands). Therefore only relative datum offsets are to be determined and the observation equation reads (given here, e.g., for the datum offset between Amsterdam (A) and Trieste (T))

$$\left(h_A - H_A - N_A^{\mathrm{GOCE}} - N_A^{\mathrm{Stokes}}\right) - \left(h_T - H_T - N_T^{\mathrm{GOCE}} - N_T^{\mathrm{Stokes}}\right) = N_{TA} \quad (13)$$

Therefore the error of the geoid height difference between Amsterdam and Trieste is required. Given the error variances $\mathrm{Var}(N)$ at both locations (taken from Fig. 3) as well as the error covariances $\mathrm{Cov}(N_A, N_T)$ between Amsterdam and Trieste (taken from Fig. 6) the error of the geoid height difference can be computed from

$$\mathrm{Var}(N_A - N_T) = \mathrm{Var}(N_A) + \mathrm{Var}(N_T) - 2\mathrm{Cov}(N_A, N_T) \quad (14)$$

The error standard deviation of the geoid height difference, i.e. the error contribution of the global satellite model to height system unification at all of the 5 datum points with respect to Amsterdam are given in Table 1 (for the GOCE case) and for comparison also in

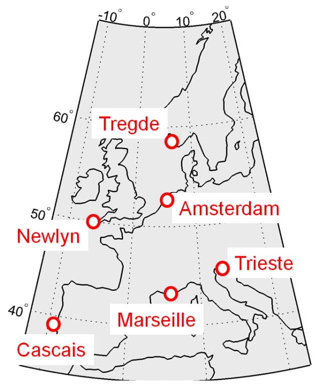

Figure 9. Geographic distribution of 6 tide gauge stations which are used as datum points for different vertical reference frames in Europe. The datum points are used in the synthetic example presented in Section 4.

Table 2 (for the GRACE case). For GOCE all errors are in the range between 3 and 4 centimetre. There are only marginal differences between the full and the m-block solution, while the diagonal approach overestimates the error by around 20% (with a maximum of 26%). As shown in Section 3.2, the amount of over- or underestimation of the geoid height error depends on the geographic location. Thus the numbers given here are only valid for the European test scenario. In the Southern hemisphere, the diagonal approach will underestimate the error variance.

It is also interesting to compare GOCE and GRACE. In the latter case (see Table 2) the deviations between the full and the m-block solution are larger as in case of GOCE. This means that the correlations outside the m-blocks are larger in case of GRACE. However, in least-squares adjustment a relative error of the weights in the order of some few percentages (6% to 8% in this example) might still be acceptable. The result of the diagonal solution does not seem to be too bad either, with relative errors in the order of 10-20% (the maximum value of 22% is even smaller than in the GOCE case), but referring to the strong striping pattern, the results are depending on the relative location of the datum points involved. Therefore it cannot be concluded in general that the diagonal approximation works better in case of GRACE then in case of GOCE.

5. Summary and Conclusions

Global height system unification is one of the main geodetic applications of ESA's GOCE mission. GOCE provides a precise and global vertical reference surface of medium resolution to which all vertical datum zones could be linked. The corresponding datum offsets can be estimated in a least-squares adjustment. The observations comprise levelled heights referring to the different datum zones, ellipsoidal heights from space-geodetic techniques and GOCE derived geoid (or quasi-geoid) heights (extended with terrestrial data for the high frequency portion of the signal). The best possible estimate for the datum offsets (along with a realistic error description) requires having available realistic measures of the error variances and covariances at and between the observation stations. The full error variance-covariance information of GOCE is available from ESA. The corresponding matrix, however, is too big to be efficiently exploited on an ordinary personal computer. Therefore it is worthwhile to investigate suitable approximations of the full variance-covariance matrix of GOCE. Besides the full matrix, we have used two approaches: the first takes only the elements of the dominant block-diagonal structure (*m-block approach*) into account (neglecting correlations between coefficients of different spherical harmonic order m), while the second neglects all error correlations and only takes the variances along the main diagonal into account (*diagonal approach*). We find that the m-block approach gives only minor deviations from the full matrix solution, while the diagonal approach over- or underestimates the error variance by up to 50%, depending on the geographic location. In a synthetic example, the GOCE contribution to the quality of datum offsets, estimated between some selected regional datum points in Europe, was investigated. We find that the error estimates of datum offsets changes for less than 0.3% in the m-block approximation, while the diagonal approach leads to an approximation error of up to 26%. In conclusion, the m-block approach seems to be a suitable approximation of the full GOCE error variance-covariance matrix, while the diagonal approach seems to be too coarse. Future investigations comprising the propagation of all relevant error sources will show, if the diagonal approach still can prove to provide reasonable datum offset estimates. Indeed, it would be the numerically simplest approach to apply.

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