Chair of Automatic Control

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Interpolation-based model reduction of nonlinear systems

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Motivation for Nonlinear Model Order Reduction

Given a large-scale nonlinear control system of the form

$$\det(\mathbf{E}) \neq 0$$

$$\boldsymbol{\Sigma}: \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

$$\mathbf{x}(t) \in \mathbb{R}^n$$

with $\mathbf{E} \in \mathbb{R}^{n\times n}$, $\mathbf{f}(\mathbf{x}(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{q\times n}$

Simulation, design, control and optimization cannot be done efficiently!

MOR

Reduced order model

$$\boldsymbol{\Sigma}_r: \begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{f}_r(\mathbf{x}_r(t)) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t), \quad \mathbf{x}_r(0) = \mathbf{x}_{r,0} \end{cases}$$

$$\mathbf{x}_r(t) \in \mathbb{R}^r, \; r \ll n$$

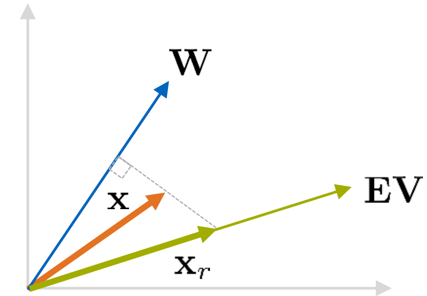
with $\mathbf{E}_r \in \mathbb{R}^{r\times r}$, $\mathbf{f}_r(\mathbf{x_r}(t)) : \mathbb{R}^r \rightarrow \mathbb{R}^r$ and $\mathbf{B}_r \in \mathbb{R}^{r\times m}$, $\mathbf{C}_r \in \mathbb{R}^{q\times r}$

Goal:
$\mathbf{y}_r(t) \approx \mathbf{y}(t)$

Projective nonlinear MOR

Approximation in the subspace $\mathcal{V} = \mathrm{span}(\mathbf{EV})$

$$\mathbf{x} = \mathbf{V}\mathbf{x}_r + \mathbf{e}, \qquad \mathbf{V} \in \mathbb{R}^{n \times r}$$



Procedure:

1. Replace $\mathbf{x}$ by its approximation
2. Reduce the number of equations (via projection with $\mathbf{\Pi} = \mathbf{EV}(\mathbf{W}^T\mathbf{EV})^{-1}\mathbf{W}^T$
3. Petrov-Galerkin condition

$$\underbrace{\mathbf{W}^T \mathbf{E} \mathbf{V}}_{\mathbf{E}_r} \dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{x}_r)}_{\mathbf{f}_r(\mathbf{x}_r)} + \underbrace{\mathbf{W}^T \mathbf{B}}_{\mathbf{B}_r} \mathbf{u}$$

$$\mathbf{y} \approx \mathbf{y}_r = \underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r} \mathbf{x}_r$$

Model Order Reduction (MOR)

Large-scale nonlinear model

$$\boldsymbol{\Sigma} : \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

$\mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{f}(\mathbf{x}(t)) : \mathbb{R}^n \to \mathbb{R}^n$

$\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{q \times n}$

$r \ll n$

MOR

Projection

$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$

$\mathbf{E}_r = \mathbf{W}^T \mathbf{E}\, \mathbf{V}$, $\mathbf{f}_r = \mathbf{W}^T \mathbf{f}(\mathbf{V}\mathbf{x}_r)$, $\mathbf{B}_r = \mathbf{W}^T \mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\, \mathbf{V}$

Reduced order model (ROM)

$$\boldsymbol{\Sigma}_r : \begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{f}_r(\mathbf{x}_r(t)) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t), \quad \mathbf{x}_r(0) = \mathbf{x}_{r,0} \end{cases}$$

$\mathbf{E}_r \in \mathbb{R}^{r \times r}$, $\mathbf{f}_r(\mathbf{x}_r(t)) : \mathbb{R}^r \to \mathbb{R}^r$

$\mathbf{B}_\mathbf{r} \in \mathbb{R}^{r \times m}$, $\mathbf{C}_r \in \mathbb{R}^{q \times r}$

Challenges of Nonlinear Model Reduction

- Nonlinear systems can exhibit complex behaviours
  - Multiple equilibria
  - Stable, unstable or semi-stable limit cycles
  - Chaotic behaviours

- Input-output behaviour of nonlinear systems **cannot** be described with the help of transfer functions, the state-transition matrix or the convolution (only possible for special cases)

- Choice of the reduced order basis
  - Projection bases should comprise the most dominant directions of the state-space
  - Existing approaches:
    - → Simulation-based methods
    - → Volterra-based approaches
    - → Quadratic-bilinear-based techniques

- Expensive evaluation of the full-order vector of nonlinearities $\mathbf{f}(\mathbf{V}\mathbf{x}_r(t))$
  - Approximation by so-called hyper-reduction techniques: EIM, DEIM, Gappy-POD, GNAT, ECSW, …

Overview of existing nonlinear model reduction methods

- Classification in
  1. Simulation- or trajectory-based methods
  2. Volterra-based approaches (bilinear)
  3. Polynomialization- and variational analysis-based techniques (quadratic-bilinear)

  or

  a) Time domain approaches (Simulation- or trajectory-based approaches)
  b) Frequency domain approaches (Interpolation-based methods: bilinear & QBMOR)

  or

  i. Strong nonlinear approaches (POD, NL-BT, Empirical Gramians, TPWL, QBMOR)
  ii. Weakly nonlinear approaches (Bilinear models)

- Methods:
  1. POD, Nonlinear Balanced Truncation (NL-BT), Empirical Gramians, TPWL
  2. Bilinear systems (BT, bilinear RK, BIRKA, Loewner Framework,…)
  3. Quadratic-bilinear (BT, RK)

Overview

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Simulation-based methods

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

- Proper Orthogonal Decomposition (POD)
- Nonlinear Balanced Truncation
- Empirical Gramians
- Trajectory piecewise linear approximation (TPWL)

Volterra-based methods

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

- Bilinear Balanced Truncation
- Bilinear Rational Krylov
- Bilinear IRKA
- Bilinear Loewner Framework

Quadratic-bilinear methods

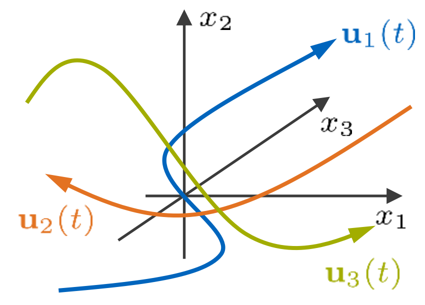
$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{H}(\mathbf{x} \otimes \mathbf{x}) + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

- Balanced Truncation for QBDAEs
- Two-sided Rational Krylov for QBDAEs

Proper Orthogonal Decomposition (POD)

Starting point:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$



1. Choose suitable training input signals $\mathbf{u}_1(t), \mathbf{u}_2(t), \ldots, \mathbf{u}_t(t)$

2. Take snapshots from simulated full-order state trajectories

$$\underset{(n,t\cdot N)}{\mathbf{X}} = \left[\mathbf{x}^{\mathbf{u}_1}(t_1),\, \mathbf{x}^{\mathbf{u}_1}(t_2),\, \cdots,\, \mathbf{x}^{\mathbf{u}_1}(t_N) \quad \mathbf{x}^{\mathbf{u}_2}(t_1),\, \mathbf{x}^{\mathbf{u}_2}(t_2),\, \cdots\right]$$

3. Perform singular value decomposition (SVD) of the snapshot matrix $\mathbf{X}$

$$\mathbf{X} = \underset{(n,n)}{\mathbf{M}} \; \underset{(n,n)}{\mathbf{\Sigma}} \; \underset{(n,t\cdot N)}{\mathbf{N}^T} \approx \underset{(n,r)}{\mathbf{M}_r} \; \underset{(r,r)}{\mathbf{\Sigma}_r} \; \underset{(r,t\cdot N)}{\mathbf{N}_r^T}$$

4. Reduced order basis: $\mathbf{V} = \mathbf{M}_r \in \mathbb{R}^{n\times r}$

Advantages

- Straightforward data-driven method
- Error bound for approximation error
- Optimal in least squares sense:

$$\min_{\mathrm{rank}(\mathbf{X}_r)=r} ||\mathbf{X} - \mathbf{X}_r||_2$$

Drawbacks

- Simulation of full-order model for different input signals required
- SVD of large snapshot matrix $\mathbf{X}$
- Training input dependency

Trajectory Piecewise-Linear Approximation (TPWL)

Starting point:
$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

1. Linearize original nonlinear model along simulated state trajectory

Weighted sum of s linearized models

$$\mathbf{E}\dot{\mathbf{x}}(t) = \sum_{i=1}^{s} \omega_i(\mathbf{x}) \, \left(\mathbf{f}(\mathbf{x}_i) + \mathbf{A}_i(\mathbf{x} - \mathbf{x}_i)\right) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Jacobi matrix
$$\mathbf{A}_i = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_i}$$

2. Reduce linearized models with well-known linear model reduction techniques (e.g. POD, Balanced Truncation, Rational Krylov, …)

3. Construct reduced order model as weighted sum of linearized reduced models:

$$\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \sum_{i=1}^{s} \omega_i(\mathbf{x}_r)\mathbf{W}^T\left(\mathbf{f}(\mathbf{x}_i) - \mathbf{A}_i\mathbf{x}_i\right) + \sum_{i=1}^{s} \omega_i(\mathbf{x}_r)\,\left(\mathbf{W}^T\mathbf{A}_i\mathbf{V}\mathbf{x}_r\right) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}_r(t) = \mathbf{C}\mathbf{V}\mathbf{x}_r(t)$$

Weighting functions
$$\sum_{i=1}^{s} \omega_i(\mathbf{x}_r(t)) = 1, \quad \omega_i(\mathbf{x}_r(t)) \geq 0$$

Trajectory Piecewise-Linear Approximation (TPWL)

Offline stage

1. Simulation of full-order model for several appropriate training input signals
2. Selection of linearization points (number s and distance δ) and linearization at selected points
3. Reduction of all linearized models
4. Choice of weighting function (e.g. Gaussian, sinc squared, trapezoidal, triangular, …)

Online stage

1. Calculation of the weights according to the current state
2. Computation of reduced model as convex combination of linearized reduced models

Advantages	Drawbacks
• Strong nonlinear approach • Linear model reduction techniques can be used • No hyper-reduction step necessary	• Simulation, linearization and reduction of full-order models • Many degrees of freedom (s, δ, ω_i) • Training input dependency

Trajectory Piecewise-Linear Approximation (TPWL)

Variations and extensions of the TPWL approach

- Fast approximate simulation
  - select the linearization points using the linearized or the reduced trajectory

- Reduction of the linearized models
  - Using global projection matrices: $\mathbf{V} = \left[\mathbf{V}_1^{(1)}\ \mathbf{V}_1^{(2)}\ \mathbf{V}_2^{(1)}\ \mathbf{V}_2^{(2)}\ \ldots\ \mathbf{V}_s^{(1)}\ \mathbf{V}_s^{(2)}\right]$

$$\mathrm{span}\{\mathbf{V}_i^{(1)}\} = \mathcal{K}_r\left((\mathbf{A}_i - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}_i - s_0\mathbf{E})^{-1}\mathbf{B}\right)$$

$$\mathrm{span}\{\mathbf{V}_i^{(2)}\} = \mathcal{K}_r\left((\mathbf{A}_i - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}_i - s_0\mathbf{E})^{-1}(\mathbf{f}(\mathbf{x}_i) - \mathbf{A}_i\mathbf{x}_i)\right)$$

  - Using local projection matrices: $\mathbf{V}_1 = \left[\mathbf{V}_1^{(1)}\ \mathbf{V}_1^{(2)}\right], \ldots, \mathbf{V}_s = \left[\mathbf{V}_s^{(1)}\ \mathbf{V}_s^{(2)}\right]$

    → Computation of state transformations to common subspace are necessary

- Generation of stable TPWL reduced models

- Reduction of nonlinear, parametric models using TPWL + pMOR by Matrix Interpolation

- Reduction of nonlinear DAE models (e.g electrostatic beam, IMTEK) using TPWL

Overview

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Simulation-based methods	Volterra-based methods	Quadratic-bilinear methods
$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$	$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$	$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{H}(\mathbf{x} \otimes \mathbf{x}) + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$
Proper Orthogonal Decomposition (POD)	Bilinear Balanced Truncation	Balanced Truncation for QBDAEs
Nonlinear Balanced Truncation	Bilinear Rational Krylov	Two-sided Rational Krylov for QBDAEs
Empirical Gramians	Bilinear IRKA	
Trajectory piecewise linear approximation (TPWL)	Bilinear Loewner Framework	

Carleman linearization

Starting point:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Goal: Approximation of (weakly) nonlinear systems by Carleman linearization

- Taylor series representation:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \underbrace{\frac{\mathbf{f}^{(1)}(\mathbf{x}_0)}{1!}}_{\mathbf{A}_1}(\mathbf{x}-\mathbf{x}_0) + \underbrace{\frac{\mathbf{f}^{(2)}(\mathbf{x}_0)}{2!}}_{\mathbf{A}_2}(\mathbf{x}-\mathbf{x}_0)^2 + \underbrace{\frac{\mathbf{f}^{(3)}(\mathbf{x}_0)}{3!}}_{\mathbf{A}_3}(\mathbf{x}-\mathbf{x}_0)^3 + \cdots$$

Assumptions:
- $\mathbf{x}_0 = \mathbf{0}$
- $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$

$$\mathbf{A}_1 \in \mathbb{R}^{n\times n}, \quad \mathbf{A}_2 \in \mathbb{R}^{n\times n^2}, \quad \mathbf{A}_3 \in \mathbb{R}^{n\times n^3}, \quad \vdots$$

$$\mathbf{x}^{(1)} = \mathbf{x} \in \mathbb{R}^{n}, \quad \mathbf{x}^{(2)} = \mathbf{x}\otimes\mathbf{x} \in \mathbb{R}^{n^2}, \quad \mathbf{x}^{(3)} = \mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x} \in \mathbb{R}^{n^3}, \quad \vdots$$

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x}\otimes\mathbf{x}) + \mathbf{A}_3(\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}) + \cdots = \sum_{k=1}^{\infty}\mathbf{A}_k\mathbf{x}^{(k)} \approx \sum_{k=1}^{N}\mathbf{A}_k\mathbf{x}^{(k)}$$

- State-space model:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \sum_{k=1}^{N}\mathbf{A}_k\mathbf{x}^{(k)} + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Carleman bilinearization

Starting point:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Goal: Bilinear model

- Consider differential equations for $\mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \ldots, \mathbf{x}^{(N)}$

$$\begin{aligned}\mathbf{E}^{(2)} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{(2)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}\mathbf{x} \otimes \mathbf{E}\mathbf{x}) &= \mathbf{E}\dot{\mathbf{x}} \otimes \mathbf{E}\mathbf{x} + \mathbf{E}\mathbf{x} \otimes \mathbf{E}\dot{\mathbf{x}} \\ &= \left(\sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}\mathbf{u}\right) \otimes \mathbf{E}\mathbf{x} + \mathbf{E}\mathbf{x} \otimes \left(\sum_{k=1}^{N} \mathbf{A}_k \mathbf{x}^{(k)} + \mathbf{B}\mathbf{u}\right) \\ &= \sum_{k=1}^{N-1} \left[\mathbf{A}_k \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{A}_k\right] \mathbf{x}^{(k+1)} + \left[\mathbf{B} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{B}\right] \mathbf{x} \otimes \mathbf{u}\end{aligned}$$

$$\mathbf{E}^{(3)} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{(3)} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}\mathbf{x} \otimes \mathbf{E}\mathbf{x} \otimes \mathbf{E}\mathbf{x})$$

$$\vdots$$

- Bilinear model:

$$\begin{aligned}\mathbf{E}^{\otimes}\dot{\mathbf{x}}^{\otimes} &= \mathbf{A}^{\otimes}\mathbf{x}^{\otimes} + \mathbf{N}^{\otimes}\mathbf{x}^{\otimes}\mathbf{u} + \mathbf{B}^{\otimes}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}^{\otimes}\mathbf{x}^{\otimes}\end{aligned}$$

with

$$\mathbf{x}^{\otimes} = \begin{bmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \mathbf{x}^{(N)} \end{bmatrix}$$

Carleman bilinearization: example

Starting point:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Carleman linearization $N = 2$

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x}\otimes\mathbf{x}) + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Carleman bilinearization

$$\mathbf{x}^{\otimes} = \begin{bmatrix}\mathbf{x}^{(1)}\\ \mathbf{x}^{(2)}\end{bmatrix}$$

$$\begin{aligned}\mathbf{E}^{(2)}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^{(2)} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}\mathbf{x}\otimes\mathbf{E}\mathbf{x}) = \mathbf{E}\dot{\mathbf{x}}\otimes\mathbf{E}\mathbf{x} + \mathbf{E}\mathbf{x}\otimes\mathbf{E}\dot{\mathbf{x}})\\ &= (\mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x}\otimes\mathbf{x}) + \mathbf{B}\mathbf{u})\otimes\mathbf{E}\mathbf{x}\\ &\quad + \mathbf{E}\mathbf{x}\otimes(\mathbf{A}_1\mathbf{x} + \mathbf{A}_2(\mathbf{x}\otimes\mathbf{x}) + \mathbf{B}\mathbf{u})\\ &= [\mathbf{A}_1\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{A}_1]\,\mathbf{x}\otimes\mathbf{x} + [\mathbf{B}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{B}]\,\mathbf{x}\otimes\mathbf{u}\end{aligned}$$

Bilinear model:

$$\begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\otimes\mathbf{E}\end{bmatrix}\dot{\mathbf{x}}^{\otimes} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{0} & \mathbf{A}_1\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{A}_1\end{bmatrix}\mathbf{x}^{\otimes} + \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{B}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{B} & \mathbf{0}\end{bmatrix}\mathbf{x}^{\otimes}\mathbf{u} + \begin{bmatrix}\mathbf{B}\\ \mathbf{0}\end{bmatrix}\mathbf{u}$$
$$\mathbf{y} = \begin{bmatrix}\mathbf{C} & \mathbf{0}\end{bmatrix}\mathbf{x}^{\otimes}$$

State-Space Representation of Bilinear Systems

Consider bilinear SISO systems of the form

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{N}\,\mathbf{x}\,u + \mathbf{b}\,u$$

$$y = \mathbf{c}^T\,\mathbf{x}$$

with $\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}$ and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$.

- Many (weakly) nonlinear systems can be approximated by bilinear systems through Carleman bilinearization

  Drawback: Dimension of the bilinear model is significantly higher than the original state dimension → only applicable for medium-sized (weakly) nonlinear systems

- Linear in input and linear in state, but not jointly linear in state and input

- Advantage: Close relation to linear systems, a lot of well-known concepts can be extended, e.g. transfer functions, Gramians, Sylvester and Lyapunov equations.

Output response and Transfer Functions of Bilinear Systems

Some background on Volterra theory

- Output response expressed by Volterra series: $y(t) = \sum_{k=1}^{\infty} y_k(t)$

$$y_k(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_k} \underbrace{\mathbf{c}^T e^{\mathbf{E}^{-1}\mathbf{A}t_k} \mathbf{E}^{-1}\mathbf{N} \cdots \mathbf{E}^{-1}\mathbf{N}\, e^{\mathbf{E}^{-1}\mathbf{A}t_1} \mathbf{E}^{-1}\mathbf{b}}_{g_k(t_1,\ldots,t_k)}\, u(t - t_1 - \ldots - t_k) \ldots u(t - t_k)\, \mathrm{d}t_k \ldots \mathrm{d}t_1$$

Impulse response / kernel of kth degree

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_k} g_k(t_1, \ldots, t_k)\, u(t - t_1 - \ldots - t_k) \ldots u(t - t_k)\, \mathrm{d}t_k \ldots \mathrm{d}t_1$$

- Multivariable Laplace-transform:

$$G_1(s_1) = \mathbf{c}^T (s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$G_2(s_1, s_2) = \mathbf{c}^T (s_2\mathbf{E} - \mathbf{A})^{-1}\, \mathbf{N}\, (s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$G_3(s_1, s_2, s_3) = \mathbf{c}^T (s_3\mathbf{E} - \mathbf{A})^{-1}\, \mathbf{N}\, (s_2\mathbf{E} - \mathbf{A})^{-1}\, \mathbf{N}\, (s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

$$\vdots$$

$$G_k(s_1, \ldots, s_k) = \mathbf{c}^T (s_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{N} \cdots \mathbf{N}\, (s_2\mathbf{E} - \mathbf{A})^{-1}\, \mathbf{N}\, (s_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$$

Model Reduction of Bilinear Systems

Volterra-based methods

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}\mathbf{u} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Bilinear Balanced Truncation
- [Al-Baiyat '93], [Benner/Damm '11]
- Solution of two bilinear Lyapunov equations

Bilinear Rational Krylov
- [Phillips '00], [Bai/Skoogh '06], [Breiten/Damm '10]
- Multimoment-Matching for bilinear systems

Bilinear IRKA
- [Zhang/Lam '02], [Benner/Breiten '12], [Flagg '12]
- H2-optimal model reduction for bilinear systems

Bilinear Loewner Framework
- [Flagg '12], [Antoulas '14]
- Data-driven interpolation-based approach

MOR for Bilinear Systems: Multimoment-Matching

Multimoments for bilinear systems: [Bai/Skoogh '06], [Breiten/Damm '10]

- Transfer function:

$$G_k(s_1,\ldots,s_k) = \mathbf{c}^T(s_k\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}\cdots\mathbf{N}(s_2\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}(s_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

  - Make use of Neumann expansion
  - Expansion in a multivariable Maclaurin series

$$G_k(s_1,\ldots,s_k) = \sum_{l_k=1}^{\infty}\cdots\sum_{l_1=1}^{\infty} m(l_1,\ldots,l_k)\cdot(s_1-\sigma_1)^{l_1-1}\cdots(s_k-\sigma_k)^{l_k-1}$$

- Multimoments:

$$m(l_1,\ldots,l_k) = (-1)^k\mathbf{c}^T(\mathbf{A}-\sigma_k\mathbf{E})^{-l_k}\mathbf{N}\cdots\mathbf{N}(\mathbf{A}-\sigma_2\mathbf{E})^{-l_2}\mathbf{N}(\mathbf{A}-\sigma_1\mathbf{E})^{-l_1}\mathbf{b}$$

- Markov parameters:

$$m^\infty(l_1,\ldots,l_k) = \mathbf{c}^T\mathbf{A}^{l_k-1}\mathbf{N}\cdots\mathbf{N}\mathbf{A}^{l_2-1}\mathbf{N}\mathbf{A}^{l_1-1}\mathbf{b}$$

with $G_k(s_1,\ldots,s_k) = \sum_{l_k=1}^{\infty}\cdots\sum_{l_1=1}^{\infty} m^\infty(l_1,\ldots,l_k)\cdot s_1^{-l_1}\cdots s_k^{-l_k}$

MOR for Bilinear Systems: Multimoment-Matching

Multimoment-Matching: [Bai/Skoogh '06], [Feng/Benner '07], [Breiten/Damm '10]

1. Calculation of the Krylov subspaces:

$$\mathrm{span}\{\mathbf{V}^{(1)}\} = \mathcal{K}_{r_1}\left((\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{b}\right)$$

$$\mathrm{span}\{\mathbf{V}^{(2)}\} = \mathcal{K}_{r_2}\left((\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(1)}\mathbf{U}^T\right)$$

$$\vdots$$

$$\mathrm{span}\{\mathbf{V}^{(j)}\} = \mathcal{K}_{r_j}\left((\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(j-1)}\mathbf{U}^T\right), \quad j = 2, \ldots, J$$

$$\mathrm{span}\{\mathbf{V}\} = \bigcup_{j=1}^{J} \mathrm{colspan}\{\mathbf{V}^{(j)}\}$$

2. Computation of the reduced order model:

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V}, \quad \mathbf{b}_r = \mathbf{W}^T\mathbf{b}, \quad \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$$

Example:

- 1st subsystem: $r_1 = 4, \ \sigma_1$

$$\mathbf{V}^{(1)} = \left[(\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{b}, \ (\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{E}(\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{b}, \ \ldots\right]$$

$$\mathbf{W}^{(1)} = \left[(\mathbf{A}-\sigma_1\mathbf{E})^{-T}\mathbf{c}, \ (\mathbf{A}-\sigma_1\mathbf{E})^{-T}\mathbf{E}^T(\mathbf{A}-\sigma_1\mathbf{E})^{-T}\mathbf{c}, \ \ldots\right]$$

$m(l_1) = m_r(l_1)$
for $l_1 = 1, \ldots, r_1$

- 2nd subsystem: $r_2 = 2, \ \sigma_2$

$$\mathbf{V}^{(2)} = \left[(\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(1)}\mathbf{U}^T, \ (\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{E}(\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(1)}\mathbf{U}^T\right]$$

$$\mathbf{W}^{(2)} = \left[(\mathbf{A}-\sigma_2\mathbf{E})^{-T}\mathbf{N}^T\mathbf{W}^{(1)}\mathbf{U}^T, \ (\mathbf{A}-\sigma_2\mathbf{E})^{-T}\mathbf{E}^T(\mathbf{A}-\sigma_2\mathbf{E})^{-T}\mathbf{N}^T\mathbf{W}^{(1)}\mathbf{U}^T\right]$$

$m(l_1, l_2) = m_r(l_1, l_2)$
for $l_1 = 1, \ldots, r_1$
$l_2 = 1, \ldots, r_2$

MOR for Bilinear Systems: Multimoment-Matching

Multimoment-Matching: [Bai/Skoogh '06], [Feng/Benner '07], [Breiten/Damm '10]

1. Calculation of the Krylov subspaces:

$$\mathrm{span}\{\mathbf{V}^{(1)}\} = \mathcal{K}_{r_1}\left((\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{b}\right)$$

$$\mathrm{span}\{\mathbf{V}^{(2)}\} = \mathcal{K}_{r_2}\left((\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_2\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(1)}\mathbf{U}^T\right)$$

$$\vdots$$

$$\mathrm{span}\{\mathbf{V}^{(j)}\} = \mathcal{K}_{r_j}\left((\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(j-1)}\mathbf{U}^T\right), \; j = 2, \ldots, J$$

$$\mathrm{span}\{\mathbf{V}\} = \bigcup_{j=1}^{J} \mathrm{colspan}\{\mathbf{V}^{(j)}\}$$

2. Computation of the reduced order model:

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V}, \quad \mathbf{b}_r = \mathbf{W}^T\mathbf{b}, \quad \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$$

Open questions/problems:

- How to choose the expansion points?
  → Optimal expansion points via $\mathcal{H}_2$-optimal model reduction (bilinear IRKA)
- How many moments should be matched per subsystem?
- How many subsystems are necessary for a good approximation?
- Error bounds?

MOR for Bilinear Systems: $\mathcal{H}_2$-optimal model reduction

- $\mathcal{H}_2$-norm of a MIMO bilinear system:

$$||\mathbf{\Sigma}||^2_{\mathcal{H}_2} := \mathrm{tr}\left(\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{(2\pi)^k}\mathbf{G}_k(j\omega_1,\ldots,j\omega_k)\mathbf{G}_k^*(j\omega_1,\ldots,j\omega_k)\mathrm{d}\omega_k\cdots\mathrm{d}\omega_1\right)$$

Alternative calculation via

$$||\mathbf{\Sigma}||^2_{\mathcal{H}_2} = \mathrm{tr}\left(\mathbf{CPC}^T\right) = \mathrm{tr}\left(\mathbf{B}^T\mathbf{QB}\right)$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the solutions of the following bilinear Lyapunov equations:

$$\mathbf{APE} + \mathbf{EPA}^T + \sum_{k=1}^{m}\mathbf{N}_k\mathbf{PN}_k^T + \mathbf{BB}^T = \mathbf{0}$$

$$\mathbf{A}^T\mathbf{QE} + \mathbf{E}^T\mathbf{QA} + \sum_{k=1}^{m}\mathbf{N}_k^T\mathbf{QN}_k + \mathbf{C}^T\mathbf{C} = \mathbf{0}$$

$$\Rightarrow ||\mathbf{\Sigma}||^2_{\mathcal{H}_2} = (\mathrm{vec}(\mathbf{I}_q))^T(\mathbf{C}\otimes\mathbf{C})\left(-\mathbf{A}\otimes\mathbf{E} - \mathbf{E}\otimes\mathbf{A} - \sum_{k=1}^{m}\mathbf{N}_k\otimes\mathbf{N}_k\right)^{-1}(\mathbf{B}\otimes\mathbf{B})\mathrm{vec}(\mathbf{I}_m)$$

- Error system: $\mathbf{\Sigma}_e = \mathbf{\Sigma} - \mathbf{\Sigma}_r$

$$\mathbf{E}_e = \begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}_r\end{bmatrix},\ \mathbf{A}_e = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_r\end{bmatrix},\ \mathbf{N}_{k,e} = \begin{bmatrix}\mathbf{N}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{N}_{k,r}\end{bmatrix},\ \mathbf{B}_e = \begin{bmatrix}\mathbf{B}\\ \mathbf{B}_r\end{bmatrix},\ \mathbf{C}_e = \begin{bmatrix}\mathbf{C} & -\mathbf{C}_r\end{bmatrix}$$

MOR for Bilinear Systems: $\mathcal{H}_2$-optimal model reduction

- $\mathcal{H}_2$-norm of error system:

$$E^2 = ||\boldsymbol{\Sigma}_e||^2_{\mathcal{H}_2} = ||\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r||^2_{\mathcal{H}_2} = \mathrm{tr}\left(\mathbf{C}_e\mathbf{P}_e\mathbf{C}_e^T\right) = \mathrm{tr}\left(\mathbf{B}_e^T\mathbf{Q}_e\mathbf{B}_e\right)$$

where $\mathbf{P}_e$ and $\mathbf{Q}_e$ are the solutions of the following bilinear Lyapunov equations:

$$\mathbf{A}_e\mathbf{P}_e\mathbf{E}_e + \mathbf{E}_e\mathbf{P}_e\mathbf{A}_e^T + \sum_{k=1}^{m}\mathbf{N}_{k,e}\mathbf{P}_e\mathbf{N}_{k,e}^T + \mathbf{B}_e\mathbf{B}_e^T = \mathbf{0}$$

$$\mathbf{A}_e^T\mathbf{Q}_e\mathbf{E}_e + \mathbf{E}_e^T\mathbf{Q}_e\mathbf{A}_e + \sum_{k=1}^{m}\mathbf{N}_{k,e}^T\mathbf{Q}_e\mathbf{N}_{k,e} + \mathbf{C}_e^T\mathbf{C}_e = \mathbf{0}$$

Assume the reduced model $\boldsymbol{\Sigma}_r$ is given by ist eigenvalue decomposition:

$$\mathbf{E}_r^{-1}\mathbf{A}_r = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}, \quad \tilde{\mathbf{N}}_k = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{N}_{k,r}\mathbf{R}, \quad \tilde{\mathbf{B}} = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{B}_r, \quad \tilde{\mathbf{C}} = \mathbf{C}_r\mathbf{R}$$

$$\Rightarrow \quad E^2 = \mathrm{f}(\mathbf{A}, \boldsymbol{\Lambda}, \mathbf{N}_k, \tilde{\mathbf{N}}_k, \mathbf{B}, \tilde{\mathbf{B}}, \mathbf{C}, \tilde{\mathbf{C}}) \rightarrow \min$$

Optimization parameters
$\boldsymbol{\Lambda}, \tilde{\mathbf{N}}_k, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}$

- Necessary conditions for $\mathcal{H}_2$-optimality:

1. $\dfrac{\partial E^2}{\partial \tilde{C}_{ij}} \overset{!}{=} 0 \Leftrightarrow \mathbf{G}(-\overline{\lambda}_{r,i})\tilde{\mathbf{B}}_i^T = \mathbf{G}_r(-\overline{\lambda}_{r,i})\tilde{\mathbf{B}}_i^T$

2. $\dfrac{\partial E^2}{\partial \tilde{B}_{ij}} \overset{!}{=} 0 \Leftrightarrow \tilde{\mathbf{C}}_i^T\mathbf{G}(-\overline{\lambda}_{r,i}) = \tilde{\mathbf{C}}_i^T\mathbf{G}_r(-\overline{\lambda}_{r,i})$

3. $\dfrac{\partial E^2}{\partial \lambda_{r\,i}} \overset{!}{=} 0 \Leftrightarrow \tilde{\mathbf{C}}_i^T\mathbf{G}'(-\overline{\lambda}_{r,i})\tilde{\mathbf{B}}_i^T = \tilde{\mathbf{C}}_i^T\mathbf{G}_r'(-\overline{\lambda}_{r,i})\tilde{\mathbf{B}}_i^T$

4. $\dfrac{\partial E^2}{\partial \tilde{N}_{k,ij}} \overset{!}{=} 0$

MOR for Bilinear Systems: $\mathcal{H}_2$-optimal model reduction

Bilinear IRKA approach

Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (BIRKA)

Input: $\mathbf{E}, \mathbf{A}, \mathbf{N}_k, \mathbf{B}, \mathbf{C}, \mathbf{E}_r, \mathbf{A}_r, \mathbf{N}_{k,r}, \mathbf{B}_r, \mathbf{C}_r$

Output: $\mathbf{E}_r^{\text{opt}}, \mathbf{A}_r^{\text{opt}}, \mathbf{N}_{k,r}^{\text{opt}}, \mathbf{B}_r^{\text{opt}}, \mathbf{C}_r^{\text{opt}}$

1: **while** (change in $\mathbf{\Lambda} > \epsilon$) **do**

2: $\quad \mathbf{E}_r^{-1}\mathbf{A}_r = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$, $\tilde{\mathbf{N}}_k = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{N}_{k,r}\mathbf{R}$, $\tilde{\mathbf{B}} = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{B}_r$, $\tilde{\mathbf{C}} = \mathbf{C}_r\mathbf{R}$

3: $\quad \text{vec}(\mathbf{V}) = \left(-\mathbf{\Lambda} \otimes \mathbf{E} - \mathbf{E} \otimes \mathbf{A} - \sum_{k=1}^{m} \tilde{\mathbf{N}}_k \otimes \mathbf{N}_k\right)^{-1} (\tilde{\mathbf{B}} \otimes \mathbf{B})\text{vec}(\mathbf{I}_m)$

4: $\quad \text{vec}(\mathbf{W}) = \left(-\mathbf{\Lambda} \otimes \mathbf{E} - \mathbf{E} \otimes \mathbf{A}^T - \sum_{k=1}^{m} \tilde{\mathbf{N}}_k^T \otimes \mathbf{N}_k^T\right)^{-1} (\tilde{\mathbf{C}}^T \otimes \mathbf{C})^T\text{vec}(\mathbf{I}_q)$

5: $\quad \mathbf{V} = \text{orth}(\mathbf{V})$, $\mathbf{W} = \text{orth}(\mathbf{W})$

6: $\quad \mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V}$, $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}$

7: **end while**

8: $\mathbf{E}_r^{\text{opt}} = \mathbf{E}_r$, $\mathbf{A}_r^{\text{opt}} = \mathbf{A}_r$, $\mathbf{N}_{k,r}^{\text{opt}} = \mathbf{N}_{k,r}$, $\mathbf{B}_r^{\text{opt}} = \mathbf{B}_r$, $\mathbf{C}_r^{\text{opt}} = \mathbf{C}_r$

MOR for Linear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

- Duality: Krylov subspaces with Sylvester equations

$$\mathrm{span}\{\mathbf{V}\} = \mathcal{K}_r\left((\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{B}\right)$$

$$\mathrm{span}\{\mathbf{W}\} = \mathcal{K}_r\left((\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{C}^T\right)$$

$\Leftrightarrow$

$$\mathbf{AV} - \mathbf{EVS} = \mathbf{BL}$$

$$\mathbf{A}^T\mathbf{W} - \mathbf{E}^T\mathbf{W}\mathbf{S}^T = \mathbf{C}^T\mathbf{L}$$

$\lambda_i(\mathbf{S}) = s_0$: shifts

$\mathbf{L}$: tangential directions

- $\mathcal{H}_2$-optimality vs. $\mathcal{H}_2$ pseudo-optimality

$\mathcal{H}_2$-optimality

- Problem:

$$\|\mathbf{G}-\mathbf{G}_r\|_{\mathcal{H}_2} = \min_{\dim(\tilde{\mathbf{G}}_r)=r} \left\|\mathbf{G}-\tilde{\mathbf{G}}_r\right\|_{\mathcal{H}_2}$$

- Necessary conditions for local $\mathcal{H}_2$-optimality (SISO): (Meier-Luenberger)

$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$$

$$G'(-\overline{\lambda}_{r,i}) = G_r'(-\overline{\lambda}_{r,i})$$

- $\mathbf{G}_r$ minimizes the $\mathcal{H}_2$ error locally within the set of all ROMs of order r

$\mathcal{H}_2$ pseudo-optimality

- Problem: $\Lambda = \{\lambda_1, \dots, \lambda_r\},\ \lambda_i \in \mathbb{C}^-$

$$\|\mathbf{G}-\mathbf{G}_r\|_{\mathcal{H}_2} = \min_{\tilde{\mathbf{G}}_r\in\mathcal{G}(\Lambda)} \left\|\mathbf{G}-\tilde{\mathbf{G}}_r\right\|_{\mathcal{H}_2}$$

- Necessary **and** sufficient condition for global $\mathcal{H}_2$ pseudo-optimality:

$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$$

- Pseudo-optimal means optimal in a certain subset
- $\mathbf{G}_r$ minimizes the $\mathcal{H}_2$ error globally within the subset of all ROMs of order r with poles Λ

MOR for Linear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

Notation:

Gramian	$\mathbf{A}_r\mathbf{P}_r\mathbf{E}_r^T + \mathbf{E}_r\mathbf{P}_r\mathbf{A}_r^T + \mathbf{B}_r\mathbf{B}_r^T = \mathbf{0}$	(known)
Scalar product	$\mathbf{A}\mathbf{X}\mathbf{E}_r^T + \mathbf{E}\mathbf{X}\mathbf{A}_r^T + \mathbf{B}\mathbf{B}_r^T = \mathbf{0}$	(unknown)
Krylov	$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{S} = \mathbf{B}\mathbf{L}$	(known)
Projection	$\mathbf{B}_\perp = \mathbf{B} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{B}_r$	(known)

New conditions for pseudo-optimality [Wolf '14]:

Let $\mathbf{V}$ be a basis of a Krylov subspace. Let $\mathbf{G}_r(s)$ be the reduced model obtained by projection with $\mathbf{W}$. Then, the following conditions are equivalent:

i) $\mathbf{S} = -\mathbf{P}_r\mathbf{A}_r^T\mathbf{E}_r^{-T}\mathbf{P}_r^{-1}$

ii) $\mathbf{E}_r^{-1}\mathbf{B}_r + \mathbf{P}_r\mathbf{L}^T = \mathbf{0}$

iii) $\mathbf{S}\mathbf{P}_r + \mathbf{P}_r\mathbf{S}^T - \mathbf{P}_r\mathbf{L}^T\mathbf{L}\mathbf{P}_r = \mathbf{0}$

iv) $\mathbf{X} = \mathbf{V}\mathbf{P}_r$

v) $\mathbf{A}\widehat{\mathbf{P}}\mathbf{E}^T + \mathbf{E}\widehat{\mathbf{P}}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{B}_\perp\mathbf{B}_\perp^T$

vi) $\mathbf{P}_r^{-1} = \mathbf{E}_r^*\widehat{\mathbf{Q}}_r\mathbf{E}_r$

MOR for Linear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

PORK: Pseudo-optimal rational Krylov

Algorithm 1 Pseudo-optimal rational Krylov (PORK)

Input: $\mathbf{V}$, $\mathbf{S}$, $\mathbf{L}$, $\mathbf{C}$, such that $\mathbf{AV} - \mathbf{EVS} = \mathbf{BL}$ is satisfied

Output: $\mathcal{H}_2$ pseudo-optimal reduced model $\mathbf{G}_r(s) = \mathbf{C}_r\left(s\mathbf{E}_r - \mathbf{A}_r\right)^{-1}\mathbf{B}_r$

1: $\mathbf{P}_r^{-1} = \text{lyap}\left(-\mathbf{S}^T, \mathbf{L}^T\mathbf{L}\right)$

2: $\mathbf{B}_r = -\left(\mathbf{P}_r^{-1}\right)^{-1}\mathbf{L}^T$

3: $\mathbf{A}_r = \mathbf{S} + \mathbf{B}_r\mathbf{L}$, $\mathbf{E}_r = \mathbf{I}$, $\mathbf{C}_r = \mathbf{CV}$

Advantages and properties of PORK:

- ROM is globally optimal within a subset: $\left\|\mathbf{G} - \mathbf{G}_r\right\|_{\mathcal{H}_2} = \min_{\widetilde{\mathbf{G}}_r \in \mathcal{G}(\Lambda)} \left\|\mathbf{G} - \widetilde{\mathbf{G}}_r\right\|_{\mathcal{H}_2}$
- Eigenvalues of ROM: $\Lambda(\mathbf{S}) = \Lambda(-\mathbf{E}_r^{-1}\mathbf{A}_r)$

  → choice of the shifts is twice as important
- Stability preservation in the ROM can be ensured
- Low numerical effort required: solution of a Lyapunov equation and a linear system of equations, both of reduced order.

MOR for Bilinear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

- Duality: Bilinear Krylov subspaces with bilinear Sylvester equations [Flagg '12]

$$\text{span}\{\mathbf{V}^{(1)}\} = \mathcal{K}_{r_1}\left((\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_1\mathbf{E})^{-1}\mathbf{B}\right)$$

$$\text{span}\{\mathbf{V}^{(j)}\} = \mathcal{K}_{r_j}\left((\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\sigma_j\mathbf{E})^{-1}\mathbf{N}\mathbf{V}^{(j-1)}\mathbf{U}^T\right), \; j=2,\dots,J$$

$$\text{span}\{\mathbf{W}^{(1)}\} = \mathcal{K}_{r_1}\left((\mathbf{A}-\sigma_1\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-\sigma_1\mathbf{E})^{-T}\mathbf{C}^T\right)$$

$$\text{span}\{\mathbf{W}^{(j)}\} = \mathcal{K}_{r_j}\left((\mathbf{A}-\sigma_j\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-\sigma_j\mathbf{E})^{-T}\mathbf{N}^T\mathbf{W}^{(j-1)}\mathbf{U}^T\right), \; j=2,\dots,J$$

$$\Updownarrow$$

$$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{S} - \mathbf{N}\mathbf{V}\mathbf{U}^T = \mathbf{B}\mathbf{L}$$
$$\mathbf{A}^T\mathbf{W} - \mathbf{E}^T\mathbf{W}\mathbf{S}^T - \mathbf{N}^T\mathbf{W}\mathbf{U}^T = \mathbf{C}^T\mathbf{L}$$

$\lambda_i(\mathbf{S}) = s_0$: shifts

$\mathbf{L}$: tangential directions

$\mathbf{U}^T$: weights

Can we derive new conditions for pseudo-optimality for bilinear systems?

MOR for Bilinear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

Notation:

Gramian	$\mathbf{A}_r\mathbf{P}_r\mathbf{E}_r^T + \mathbf{E}_r\mathbf{P}_r\mathbf{A}_r^T + \mathbf{N}_r\mathbf{P}_r\mathbf{N}_r^T + \mathbf{B}_r\mathbf{B}_r^T = \mathbf{0}$	(known)
Scalar product	$\mathbf{A}\mathbf{X}\mathbf{E}_r^T + \mathbf{E}\mathbf{X}\mathbf{A}_r^T + \mathbf{N}\mathbf{X}\mathbf{N}_r^T + \mathbf{B}\mathbf{B}_r^T = \mathbf{0}$	(unknown)
Krylov	$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{S} - \mathbf{N}\mathbf{V}\mathbf{U}^T = \mathbf{B}\mathbf{L}$	(known)
Projection	$\mathbf{B}_\perp = \mathbf{B} - \mathbf{E}\mathbf{V}\mathbf{E}_r^{-1}\mathbf{B}_r$	(known)

New conditions for pseudo-optimality for bilinear systems:

Let $\mathbf{V}$ be a basis of a Krylov subspace. Let Σ_r be the reduced model obtained by projection with $\mathbf{W}$. Then, the following conditions are equivalent:

i) $\mathbf{S} = -\mathbf{P}_r\mathbf{A}_r^T\mathbf{E}_r^{-T}\mathbf{P}_r^{-1}$

ii-1) $\mathbf{E}_r^{-1}\mathbf{B}_r + \mathbf{P}_r\mathbf{L}^T = \mathbf{0}$

ii-2) $\mathbf{E}_r^{-1}\mathbf{N}_r\mathbf{P}_r + \mathbf{P}_r\mathbf{U} = \mathbf{0}$

iii) $\mathbf{S}\mathbf{P}_r + \mathbf{P}_r\mathbf{S}^T - \mathbf{P}_r\mathbf{L}^T\mathbf{L}\mathbf{P}_r + \mathbf{P}_r\mathbf{U}\mathbf{N}_r^T\mathbf{E}_r^{-T} = \mathbf{0}$

iv) $\mathbf{X} = \mathbf{V}\mathbf{P}_r$

v)-vi) Work In Progress (WIP)

MOR for Bilinear Systems: $\mathcal{H}_2$ pseudo-optimal reduction

BIPORK: Bilinear pseudo-optimal rational Krylov

Algorithm 1 Bilinear pseudo-optimal rational Krylov (BIPORK)

Input: $\mathbf{V}$, $\mathbf{S}$, $\mathbf{U}$, $\mathbf{L}$, $\mathbf{C}$, such that $\mathbf{AV} - \mathbf{EVS} - \mathbf{NVU}^T = \mathbf{BL}$ is satisfied

Output: $\mathcal{H}_2$ pseudo-optimal reduced model $\mathbf{\Sigma}_r$

1: $\mathbf{P}_r^{-1}$: solution of bil. Lyap. equation: $\mathbf{S}^T\mathbf{P}_r^{-1} + \mathbf{P}_r^{-1}\mathbf{S} - \mathbf{U}\mathbf{P}_r^{-1}\mathbf{U}^T - \mathbf{L}^T\mathbf{L} = \mathbf{0}$

2: $\mathbf{N}_r = -(\mathbf{P}_r^{-1})^{-1}\mathbf{U}\mathbf{P}_r^{-1}$

3: $\mathbf{B}_r = -\left(\mathbf{P}_r^{-1}\right)^{-1}\mathbf{L}^T$

4: $\mathbf{A}_r = \mathbf{S} + \mathbf{B}_r\mathbf{L} + \mathbf{N}_r\mathbf{U}^T$, $\mathbf{E}_r = \mathbf{I}$, $\mathbf{C}_r = \mathbf{CV}$

Summary and Outlook

Summary:

- **Goal:** Reduction of high dimensional nonlinear systems
- Simulation-based, Volterra-based and quadratic-bilinear-based approaches
- Model reduction for bilinear systems (BT, Krylov, BIRKA, Loewner)
- $\mathcal{H}_2$ pseudo-optimal model reduction for bilinear systems
  - Derivation of new conditions for $\mathcal{H}_2$ pseudo-optimality for bilinear systems
  - Bilinear pseudo-optimal Rational Krylov (BIPORK)

Outlook:

- Solution of bilinear Lyapunov equations with BIPORK:

  BI-LR-ADI = RKSM + BIPORK
- Cumulative reduction of bilinear systems
- Quadratic-bilinear MOR
  - Stability-preserving two-sided rational Krylov for QBDAEs?
  - IRKA for QBDAEs? Algorithm for choosing optimal expansion points?

Discussion and open problems

Feedback and hints relating the following topics are welcome:

- Numerical solvers (direct and/or indirect) for nonlinear matrix equations
  a) Direct solvers
     - Direct solvers for bilinear Sylvester and Lyapunov equations

  b) Indirect solvers
     - Bilinear low-rank ADI method
     - Bilinear Extended Krylov Subspace Method (EKSM)
     - Other Krylov-based iterative solvers, e.g. CG, PCG, BiCG, BiCGstab

- Error bounds for bilinear systems
  – Existing approaches or literature?

- Nonlinear, *parametric* benchmarks
  – Parametric Nonlinear RC-Ladder?
  – Parametric Nonlinear Heat Transfer (IMTEK)?
  – …

Thank you for your attention!