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Department of Mechanical Engineering
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Krylov Subspace Methods for Model Reduction of MIMO Quadratic-Bilinear Systems

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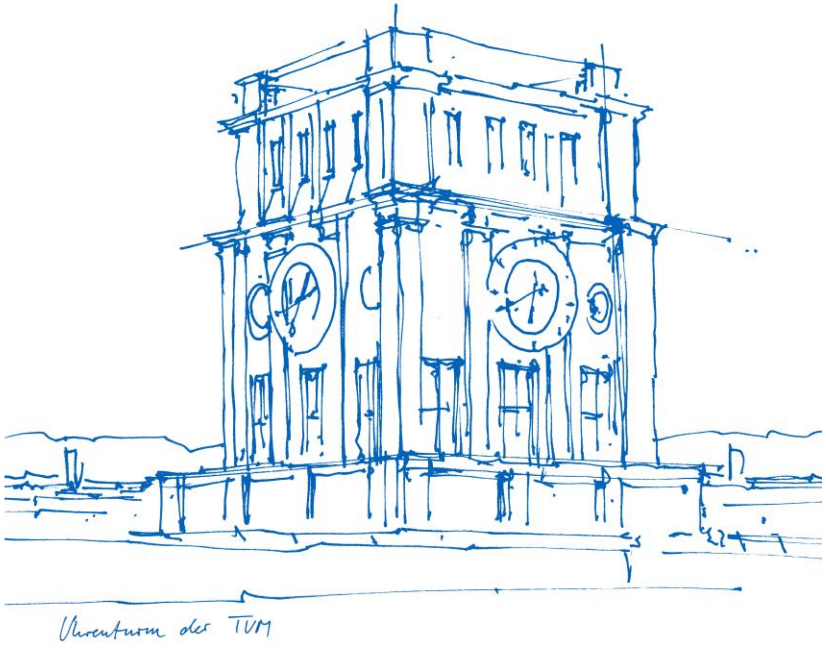

Motivation

Given a large-scale nonlinear control system of the form

$\det(\mathbf{E}) \neq 0$

$$\mathbf{\Sigma} : \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

$\mathbf{x}(t) \in \mathbb{R}^n$

with $\mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{f}(\mathbf{x}(t)) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{q \times n}$

Simulation, design, control and optimization cannot be done efficiently!

MOR

Reduced order model (ROM)

$$\mathbf{\Sigma}_r : \begin{cases} \mathbf{E}_r\dot{\mathbf{x}}_r(t) = \mathbf{f}_r(\mathbf{x}_r(t)) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_r(t) = \mathbf{C}_r\mathbf{x}_r(t), \quad \mathbf{x}_r(0) = \mathbf{x}_{r,0} \end{cases}$$

$\mathbf{x}_r(t) \in \mathbb{R}^r, \; r \ll n$

with $\mathbf{E}_r \in \mathbb{R}^{r \times r}$, $\mathbf{f}_r(\mathbf{x_r}(t)) : \mathbb{R}^r \rightarrow \mathbb{R}^r$ and $\mathbf{B}_r \in \mathbb{R}^{r \times m}$, $\mathbf{C}_r \in \mathbb{R}^{q \times r}$

Goal:
$\mathbf{y}_r(t) \approx \mathbf{y}(t)$

Challenges of Nonlinear Model Order Reduction

Nonlinear systems can exhibit complex behaviours

- Strong nonlinearities
- Multiple equilibrium points
- Limit cycles
- Chaotic behaviours

Input-output behaviour of nonlinear systems **cannot** be described with transfer functions, the state-transition matrix or the convolution integral (only possible for special cases)

Choice of the reduced order basis

- Projection basis should comprise most dominant directions of the state-space
- Different existing approaches:
  - Simulation-based methods
  - System-theoretic techniques

Expensive evaluation of $\mathbf{f}(\mathbf{V}\mathbf{x}_r)$

- Vector of nonlinearities **f** still has to be evaluated in full dimension
- Approximation of **f** by so-called hyper-reduction techniques:
  → EIM, DEIM, GNAT, ECSW…

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State-of-the-Art: Overview

Reduction of nonlinear (parametric) systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ Simulation-based:
  - POD, TPWL
  - Reduced Basis, Empirical Gramians

Reduction of bilinear systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{N}\mathbf{x}u + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ Carleman bilinearization (approx.)
- ! Large increase of dimension: $n + n^2$
- ☑ Generalization of well-known methods:
  - Balanced truncation
  - **Krylov subspace methods**
  - **$\mathcal{H}_2$ (pseudo)-optimal approaches**

Reduction of quadratic-bilinear systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{H}(\mathbf{x} \otimes \mathbf{x}) + \mathbf{N}\mathbf{x}u + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

- ☑ **Quadratic-bilinearization** (no approx.!)
- ☑ Minor increase of dimension: $2n,\ 3n$
- ☑ Generalization of well-known methods:
  - **Krylov subspace methods**
  - **$\mathcal{H}_2$-optimal approaches**
- ☐ Reduction methods for **MIMO** models

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Quadratic-Bilinearization Process

SISO Quadratic-bilinear system:

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,\mathbf{x}\otimes\mathbf{x} + \mathbf{N}\,\mathbf{x}\,u + \mathbf{b}\,u$$

$$y = \mathbf{c}^T\,\mathbf{x}$$

Quadratic terms of **x**

States-input coupling

$\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n\times n}$

$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{b}, \mathbf{c} \in \mathbb{R}^{n}$

Objective: Bring general nonlinear systems to the quadratic-bilinear (QB) form

1. **Polynomialization:** Convert nonlinear system into an equivalent polynomial system
2. **Quadratic-bilinearization:** Convert the polynomial system into a QBDAE

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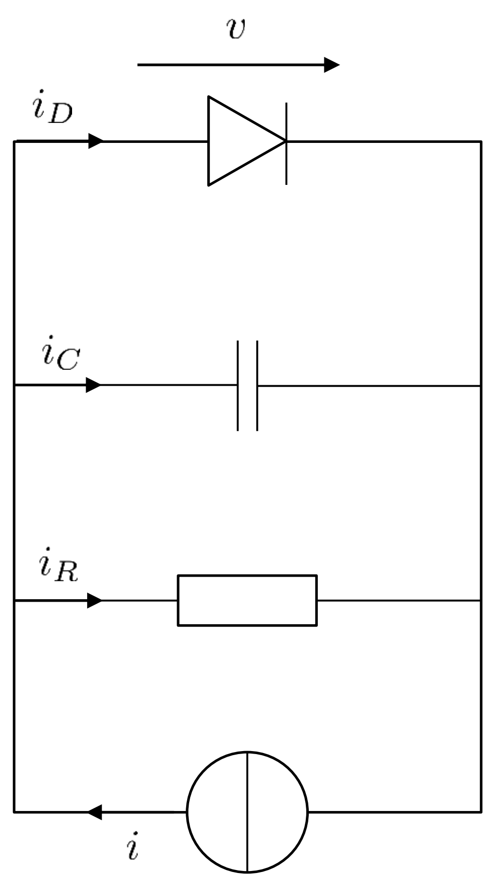
Quadratic-Bilinearization Process – Example



$$i_C + i_R + i_D = i \qquad \text{with} \quad \begin{cases} i_C = C\dot{v} \\ i_R = \dfrac{v}{R} \\ i_D = e^{\alpha v} - 1 \end{cases}$$

Nonlinear ODE: $\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - e^{\alpha v} + 1 + i\right)$

1 **Polynomialization step:** Introduce new variable and its Lie derivative

$$w = e^{\alpha v} - 1$$

$$\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - w + i\right)$$

$$\begin{aligned} \dot{w} &= (\alpha e^{\alpha v})\dot{v} \\ &= \frac{\alpha}{C}\left(-\frac{vw}{R} - w^2 + wi - \frac{v}{R} - w + i\right) \end{aligned}$$

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Quadratic-Bilinearization Process – Example

2 **Quadratic-bilinearization step:** Convert polynomial system into a QBDAE

$$\dot{v} = \frac{1}{C}\left(-\frac{v}{R} - w + i\right)$$

$$\dot{w} = \frac{\alpha}{C}\left(-\frac{vw}{R} - w^2 + wi - \frac{v}{R} - w + i\right)$$

$$\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\mathbf{E}}\underbrace{\begin{bmatrix}\dot{v}\\ \dot{w}\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix}-\frac{1}{RC} & -\frac{1}{C}\\ -\frac{\alpha}{RC} & -\frac{\alpha}{C}\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}v\\ w\end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & -\frac{\alpha}{RC} & 0 & -\frac{\alpha}{C}\end{bmatrix}}_{\mathbf{H}}\underbrace{\begin{bmatrix}v^2\\ vw\\ vw\\ w^2\end{bmatrix}}_{\mathbf{x}\otimes\mathbf{x}} + \underbrace{\begin{bmatrix}0 & 0\\ 0 & \frac{\alpha}{C}\end{bmatrix}}_{\mathbf{N}}\underbrace{\begin{bmatrix}v\\ w\end{bmatrix}}_{\mathbf{x}}\underbrace{i}_{u} + \underbrace{\begin{bmatrix}\frac{1}{C}\\ \frac{\alpha}{C}\end{bmatrix}}_{\mathbf{b}}\underbrace{i}_{u}$$

Equivalent **representation**

Dimension slightly **increased**

Transformation **not unique**

The matrix **H** can be seen as a **tensor**

Variational Analysis of Nonlinear Systems

[Rugh '81]

Assumption: Nonlinear system can be broken down into a series of homogeneous subsystems that depend nonlinearly from each other (Volterra theory)

For an input of the form $\alpha u(t)$, we assume that the response should be of the form

$$\mathbf{x}(t) = \alpha \mathbf{x}_1(t) + \alpha^2 \mathbf{x}_2(t) + \alpha^3 \mathbf{x}_3(t) + \ldots$$

Inserting the assumed input and response in the QB system and comparing coefficients of α^k, we obtain the variational equations:

$$\begin{aligned}
\mathbf{E}\dot{\mathbf{x}}_1 &= \mathbf{A}\mathbf{x}_1 + \mathbf{b}u \\
\mathbf{E}\dot{\mathbf{x}}_2 &= \mathbf{A}\mathbf{x}_2 + \mathbf{H}\mathbf{x}_1 \otimes \mathbf{x}_1 + \mathbf{N}\mathbf{x}_1 u \\
\mathbf{E}\dot{\mathbf{x}}_3 &= \mathbf{A}\mathbf{x}_3 + \mathbf{H}\left(\mathbf{x}_1 \otimes \mathbf{x}_2 + \mathbf{x}_2 \otimes \mathbf{x}_1\right) + \mathbf{N}\mathbf{x}_2 u \\
&\vdots \\
\mathbf{E}\dot{\mathbf{x}}_k &= \mathbf{A}\mathbf{x}_k + \sum_{i=1}^{k-1} \mathbf{H}\left(\mathbf{x}_i \otimes \mathbf{x}_{k-i}\right) + \mathbf{N}\mathbf{x}_{k-1} u, \quad k = 4, 5, 6, \ldots
\end{aligned}$$

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Generalized Transfer Functions (SISO)

[Rugh '81]

Series of generalized transfer functions can be obtained via the growing exponential approach:

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

1st subsystem:

$$G_1(s_1) = -\mathbf{c}^T(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

2nd subsystem:

$$G_2(s_1,s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$\mathbf{H}$ is symmetric

$$\mathbf{H}(\mathbf{u}\otimes\mathbf{v}) = \mathbf{H}(\mathbf{v}\otimes\mathbf{u})$$

$$G_2(s_1,s_2) = -\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b}\otimes\mathbf{A}_{s_2}^{-1}\mathbf{b}) - \frac{1}{2}\mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$$

$s_1 = s_2 = \sigma$

$$G_2(\sigma,\sigma) = -\mathbf{c}^T\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma}^{-1}\mathbf{b}) - \frac{1}{2}\mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b}\right]$$

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Moments of QB-Transfer Functions

Taylor coefficients of the transfer function: $G(s) = \underbrace{G(s_0)}_{m_0} + \underbrace{\frac{dG(s_0)}{ds}}_{m_1}(s - s_0) + \underbrace{\frac{1}{2!}\frac{d^2G(s_0)}{ds^2}}_{m_2}(s - s_0)^2 + \ldots$

1st subsystem: $G_1(s_1) = -\mathbf{c}^T(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{b} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{b}$

$$\mathbf{A}_s = \mathbf{A} - s\mathbf{E}$$

$$\frac{\partial}{\partial s}\mathbf{A}_s^{-1}(s) = -\mathbf{A}_s^{-1}\frac{\partial \mathbf{A}_s}{\partial s}\mathbf{A}_s^{-1} = \mathbf{A}_s^{-1}\mathbf{E}\mathbf{A}_s^{-1}$$

$$\frac{\partial G_1}{\partial s_1} = -\mathbf{c}^T\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}$$

2nd subsystem: $G_2(s_1, s_2) = -\frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{b}) - \mathbf{N}(\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b})\right]$

$$\begin{aligned}\frac{\partial G_2}{\partial s_1} = &- \mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\ &- \mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{H}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\ &+ \frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{E}\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{b} + \mathbf{A}_{s_2}^{-1}\mathbf{b}] \\ &+ \frac{1}{2}\mathbf{c}^T\mathbf{A}_{s_1+s_2}^{-1}\mathbf{N}[\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{b}]\end{aligned}$$

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Krylov subspaces for SISO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Multimoments approach [Gu '11, Breiten '12]:

$$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_{\mathrm{lin}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{b}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{q}})$$

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma}^{-1}\mathbf{b})\right\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{2\sigma}^{-T}\mathbf{N}^{T}\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{2\sigma}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma}^{-T}\mathbf{c})\right\}$$

$$G_1(\sigma_i) = G_{1,r}(\sigma_i) \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i) \qquad \frac{\partial}{\partial s_j}G_2(\sigma_i,\sigma_i) = \frac{\partial}{\partial s_j}G_{2,r}(\sigma_i,\sigma_i)$$

- Quadratic and bilinear dynamics are treated **separately**
- **Higher-order moments** can be matched
- **3** Krylov directions per shift

Hermite approach [Breiten '15]:

$$\mathrm{span}(\mathbf{V}) = \mathrm{span}(\mathbf{V}_{\mathrm{lin}}) \cup \mathrm{span}(\mathbf{V}_{\mathrm{qb}})$$

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{\sigma}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma}^{-1}\mathbf{b}\right]\right\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\ldots,k}\left\{\mathbf{A}_{2\sigma}^{-T}\mathbf{c},\ \mathbf{A}_{2\sigma}^{-T}\left[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma}^{-T}\mathbf{c}) - \frac{1}{2}\mathbf{N}^{T}\mathbf{A}_{2\sigma}^{-T}\mathbf{c}\right]\right\}$$

$$G_1(\sigma_i) = G_{1,r}(\sigma_i) \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i) \qquad \frac{\partial}{\partial s_j}G_2(\sigma_i,\sigma_i) = \frac{\partial}{\partial s_j}G_{2,r}(\sigma_i,\sigma_i)$$

- Quadratic and bilinear dynamics are treated **as one**
- **Only 0th and 1st moments** can be matched
- **2** Krylov directions per shift

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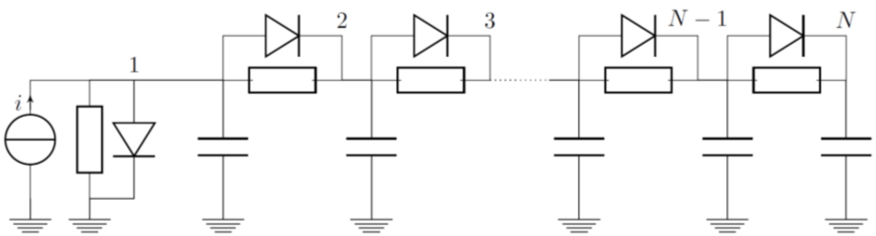
Numerical Examples: SISO RC-Ladder

SISO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Input/Output: $u(t) = e^{-t}; \quad y(t) = v_1(t)$

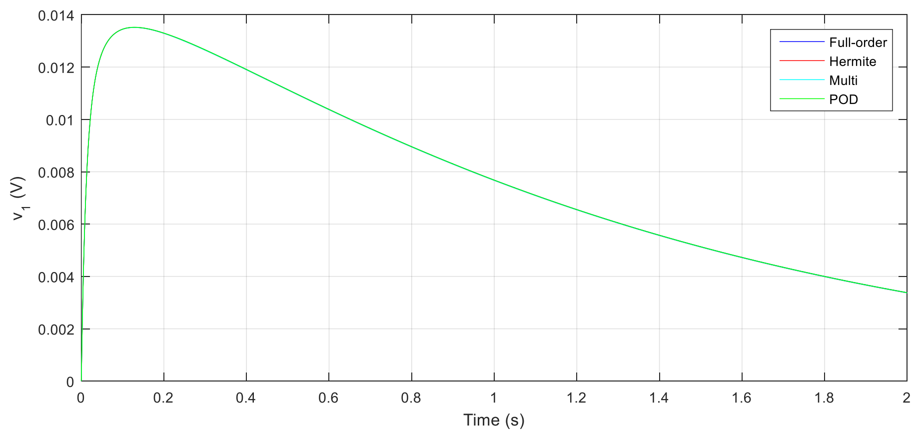
Reduction information:

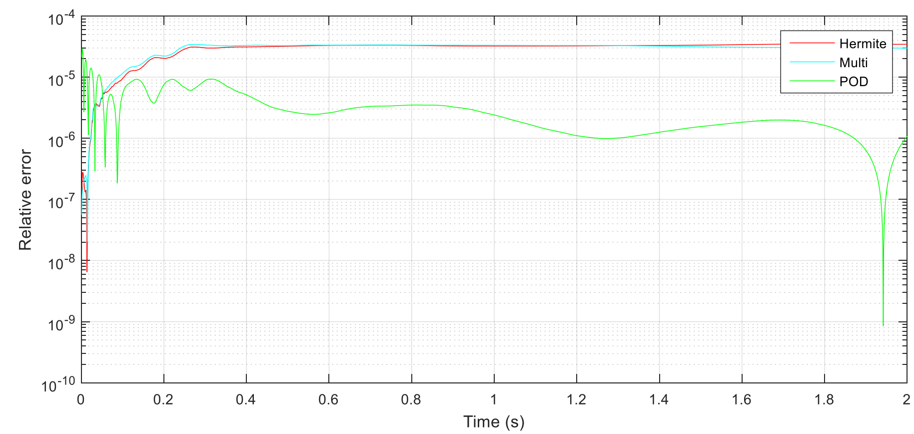
$n = 1000;$ Shifts s_0 gotten from IRKA

$t_{\mathrm{sim,orig}} = 17.6$ s

$r_{\mathrm{her}} = 12$ $\qquad r_{\mathrm{multi}} = 18$

$t_{\mathrm{sim,her}} = 0.116$ s $\qquad t_{\mathrm{sim,multi}} = 0.122$ s





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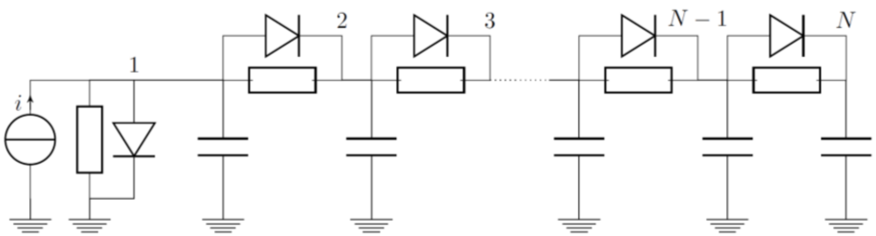
Numerical Examples: SISO RC-Ladder

SISO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Input/Output: $u(t) = {}^1\!/_2 \left[\cos\left(2\pi\,{}^t\!/_{10}\right) + 1\right]$

$y(t) = v_1(t)$

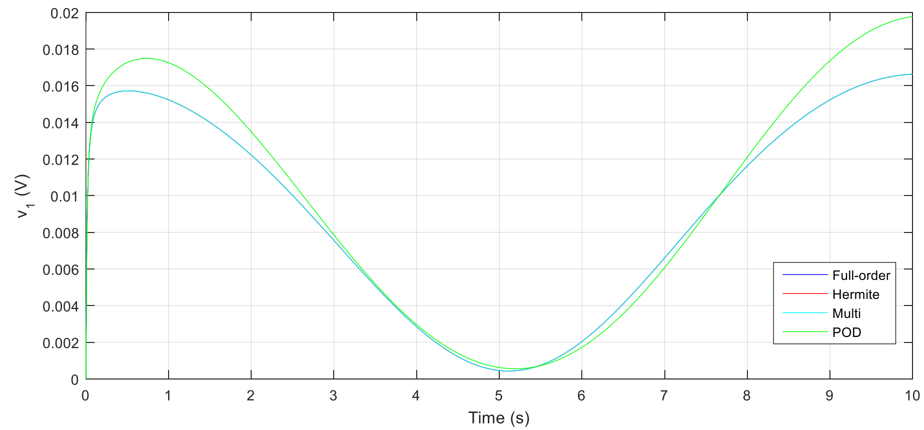
Reduction information:

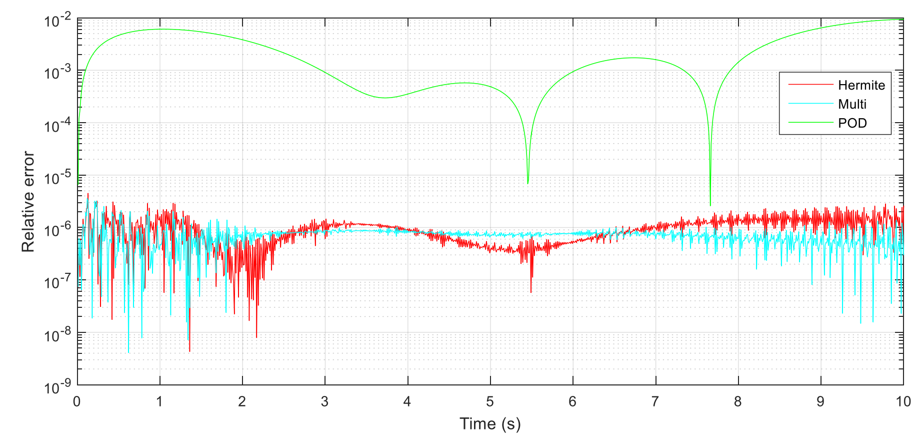
$n = 1000$; Shifts s_0 gotten from IRKA

$t_{\text{sim,orig}} = 25.5$ s

$r_{\text{her}} = 12$ $r_{\text{multi}} = 18$

$t_{\text{sim,her}} = 0.468$ s $t_{\text{sim,multi}} = 0.788$ s





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MIMO quadratic-bilinear systems

MIMO Quadratic-bilinear system:

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,\mathbf{x}\otimes\mathbf{x} + \sum_{j=1}^{m}\mathbf{N}_j\,\mathbf{x}\,u_j + \mathbf{B}\,\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\,\mathbf{x}$$

One bilinear matrix for each input

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n\times n}$

$\mathbf{H} \in \mathbb{R}^{n\times n^2}$: Hessian tensor

$\mathbf{B} \in \mathbb{R}^{n\times m}, \ \mathbf{C} \in \mathbb{R}^{p\times n}$

$$\bar{\mathbf{N}} = [\mathbf{N}_1 \ \ \mathbf{N}_2 \ \ \ldots \ \ \mathbf{N}_m] \in \mathbb{R}^{n\times n\cdot m}$$

$$\mathbf{E}\,\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{H}\,(\mathbf{x}\otimes\mathbf{x}) + \bar{\mathbf{N}}\,(\mathbf{u}\otimes\mathbf{x}) + \mathbf{B}\,\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\,\mathbf{x}$$

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Transfer matrices of a MIMO QB system

Generalized transfer matrices can be obtained similarly via the growing exponential approach:

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

1st subsystem:

$$\mathbf{G}_1(s_1) = -\mathbf{C}(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{B} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{B}$$

2nd subsystem:

$$\mathbf{G}_2(s_1, s_2) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m \otimes (\mathbf{A}_{s_1}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B}))\right]$$

$s_1 = s_2 = \sigma$

$$\mathbf{G}_2(\sigma, \sigma) = -\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) - \bar{\mathbf{N}}\left(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}\right)\right]$$

Transfer matrices with

$$\dim(\mathbf{G}_1(s)) = (p, m)$$
$$\dim(\mathbf{G}_2(s_1, s_2)) = (p, m^2)$$

The quadratic term **cannot be simplified**

$$\mathbf{H}(\mathbf{U} \otimes \mathbf{V}) \neq \mathbf{H}(\mathbf{V} \otimes \mathbf{U})$$

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Moments of QB-Transfer Matrices

1st subsystem: $\mathbf{G}_1(s_1) = -\mathbf{C}(\mathbf{A} - s_1\mathbf{E})^{-1}\mathbf{B} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{B}$

$$\mathbf{A}_s = \mathbf{A} - s\mathbf{E}$$

$$\frac{\partial}{\partial s}\mathbf{A}_s^{-1}(s) = -\mathbf{A}_s^{-1}\frac{\partial \mathbf{A}_s}{\partial s}\mathbf{A}_s^{-1} = \mathbf{A}_s^{-1}\mathbf{E}\mathbf{A}_s^{-1}$$

$$\frac{\partial \mathbf{G}_1}{\partial s_1} = -\mathbf{C}\mathbf{A}_{s_1}^{-1}\mathbf{E}\mathbf{A}_{s_1}^{-1}\mathbf{B}$$

2nd subsystem: $\mathbf{G}_2(s_1, s_2) = -\frac{1}{2}\mathbf{C}\mathbf{A}_{s_1+s_2}^{-1}\left[\mathbf{H}(\mathbf{A}_{s_1}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_2}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B} \otimes \mathbf{A}_{s_1}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m \otimes (\mathbf{A}_{s_1}^{-1}\mathbf{B} + \mathbf{A}_{s_2}^{-1}\mathbf{B}))\right]$

$$\begin{aligned}\frac{\partial \mathbf{G}_2}{\partial s_1}(\sigma,\sigma) = &-\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &-\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B} + \mathbf{A}_{\sigma}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &+\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\mathbf{E}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{B}) \\ &+\frac{1}{2}\mathbf{C}\mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_{\sigma}^{-1}\mathbf{E}\mathbf{A}_{\sigma}^{-1}\mathbf{B})\end{aligned}$$

This term **cannot be simplified**

$$\mathbf{H}(\mathbf{U} \otimes \mathbf{V}) \neq \mathbf{H}(\mathbf{V} \otimes \mathbf{U})$$

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Block-Multimoments approach (MIMO)

Algorithm 1 QB Multimoment Matching (MIMO)

Input: $\mathbf{E}$, $\mathbf{A}$, $\mathbf{H}$, $\bar{\mathbf{N}}$, $\mathbf{B}$, $\mathbf{C}$, shift σ, reduced order of first transfer function q_1 and of the second transfer function q_2

Output: Projection matrices $\mathbf{V}$, $\mathbf{W}$

$\mathbf{V}_1 = \mathcal{K}_{q_1}(\mathbf{A}_\sigma^{-1}\mathbf{E}, \mathbf{A}_\sigma^{-1}\mathbf{B})$

$\mathbf{W}_1 = \mathcal{K}_{q_1}(\mathbf{A}_{2\sigma}^{-T}\mathbf{E}^T, \mathbf{A}_{2\sigma}^{-T}\mathbf{C}^T)$

linear

for $i = 1 : q_2$ **do**

$\quad \mathbf{V}_2^i = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\bar{\mathbf{N}}(\mathbf{I}_m \otimes (\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B})\right)$

$\quad \mathbf{W}_2^i = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{I}_m \otimes (\mathbf{A}_{2\sigma}^{-1}\mathbf{E})^{i-1}\mathbf{A}_{2\sigma}^{-1}\mathbf{B})\right)$

bilinear

$\quad$ **for** $j = 1 : \min(q_2 - i + 1, i)$ **do**

$\quad\quad \mathbf{V}_3^{i,j} = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{H}((\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes (\mathbf{A}_\sigma^{-1}\mathbf{E})^{j-1}\mathbf{A}_\sigma^{-1}\mathbf{B})\right)$

$\quad\quad \mathbf{W}_3^{i,j} = \mathcal{K}_{q_2-i+1}\left(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{H}^{(2)}((\mathbf{A}_\sigma^{-1}\mathbf{E})^{i-1}\mathbf{A}_\sigma^{-1}\mathbf{B} \otimes (\mathbf{A}_{2\sigma}^{-1}\mathbf{E})^{i-1}\mathbf{A}_{2\sigma}^{-1}\mathbf{B})\right)$

quadratic

$\quad$ **end for**

end for

$\text{span}(\mathbf{V}) = \text{span}(\mathbf{V}_1) \cup \bigcup_i \text{span}(\mathbf{V}_2^i) \cup \bigcup_{i,j} \text{span}(\mathbf{V}_3^{i,j})$

$\text{span}(\mathbf{W}) = \text{span}(\mathbf{W}_1) \cup \bigcup_i \text{span}(\mathbf{W}_2^i) \cup \bigcup_{i,j} \text{span}(\mathbf{W}_3^{i,j})$

$$\frac{\partial^i \mathbf{G}_1}{\partial s_1^i}(\sigma) = \frac{\partial^i \mathbf{G}_{1,r}}{\partial s_1^i}(\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^i \mathbf{G}_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i \mathbf{G}_{1,r}}{\partial s_1^i}(2\sigma), \qquad i = 0, \ldots, q_1 - 1$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j}\mathbf{G}_2(\sigma,\sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j}\mathbf{G}_{2,r}(\sigma,\sigma), \qquad i + j \leq 2q_2 - 1$$

$$\text{span}(\mathbf{V}) = \text{span}(\mathbf{V}_{\text{lin}}) \cup \text{span}(\mathbf{V}_{\text{b}}) \cup \text{span}(\mathbf{V}_{\text{q}})$$

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Krylov subspaces for MIMO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Block tensor-based approach:

$$\begin{aligned}\operatorname{span}(\mathbf{V}) \supset \operatorname{span}_{i=1,\dots,k}\{&\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \mathbf{A}_{\sigma_i}^{-1}\mathbf{E}\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}, \dots, (\mathbf{A}_{\sigma_i}^{-1}\mathbf{E})^m\mathbf{A}_{\sigma_i}^{-1}\mathbf{B},\\ &\mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B} \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B}) - \bar{\mathbf{N}}(\mathbf{I}_m \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B})\right]\}\end{aligned}$$

$$\begin{aligned}\operatorname{span}(\mathbf{W}) \supset \operatorname{span}_{i=1,\dots,k}\Big\{&\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T, \mathbf{A}_{\sigma_i}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B} \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T),\\ &\mathbf{A}_{\sigma_i}^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{I}_m \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T)\Big\}\end{aligned}$$

$$\frac{\partial^l \mathbf{G}_1}{\partial s^l}(\sigma_i) = \frac{\partial^l \mathbf{G}_{1,r}}{\partial s^l}(\sigma_i) \qquad l = 0, \dots, m$$

$$\mathbf{G}_1(2\sigma_i) = \mathbf{G}_{1,r}(2\sigma_i)$$

$$\mathbf{G}_2(\sigma_i, \sigma_i) = \mathbf{G}_{2,r}(\sigma_i, \sigma_i)$$

$$\frac{\partial \mathbf{G}_2}{\partial s_j}(\sigma_i, \sigma_i) = \frac{\partial \mathbf{G}_{2,r}}{\partial s_j}(\sigma_i, \sigma_i) \qquad j = 1, 2$$

- **Subsystem interpolation**
- **(m+1) + 4** moments matched
- **(m+1)·m + m² = m + 2m²** columns per shift

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Krylov subspaces for MIMO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Tangential tensor-based approach:

$$\text{span}(\mathbf{V}) \supset \text{span}_{i=1,\dots,k}\left\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i, \mathbf{A}_{\sigma_i}^{-1}\mathbf{E}\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i, \dots, (\mathbf{A}_{\sigma_i}^{-1}\mathbf{E})^m\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i,\right.$$
$$\left.\mathbf{A}_{2\sigma_i}^{-1}\left[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i) - \bar{\mathbf{N}}(\mathbf{r}_i \otimes \mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i)\right]\right\}$$

$$\text{span}(\mathbf{W}) \supset \text{span}_{i=1,\dots,k}\left\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i, \mathbf{A}_{\sigma_i}^{-T}\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{B}\mathbf{r}_i \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i),\right.$$
$$\left.\mathbf{A}_{\sigma_i}^{-T}\bar{\mathbf{N}}^{(2)}(\mathbf{r}_i \otimes \mathbf{A}_{2\sigma_i}^{-T}\mathbf{C}^T\mathbf{l}_i)\right\}$$

$$\left[\frac{\partial^l \mathbf{G}_1}{\partial s^l}(\sigma_i)\right]\mathbf{r}_i = \left[\frac{\partial^l \mathbf{G}_{1,r}}{\partial s^l}(\sigma_i)\right]\mathbf{r}_i \qquad l = 0, \dots, m$$

$$\mathbf{l}_i^T\left[\mathbf{G}_1(2\sigma_i)\right] = \mathbf{l}_i^T\left[\mathbf{G}_{1,r}(2\sigma_i)\right]$$

$$\left[\mathbf{G}_2(\sigma_i,\sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) = \left[\mathbf{G}_{2,r}(\sigma_i,\sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i)$$

$$\mathbf{l}_i^T\left[\frac{\partial \mathbf{G}_2}{\partial s_j}(\sigma_i,\sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) = \mathbf{l}_i^T\left[\frac{\partial \mathbf{G}_{2,r}}{\partial s_j}(\sigma_i,\sigma_i)\right](\mathbf{r}_i \otimes \mathbf{r}_i) \qquad j = 1, 2$$

- **Tangential sub-system interpolation**
- **(m+1) + 4** moments matched
- **3** columns per shift

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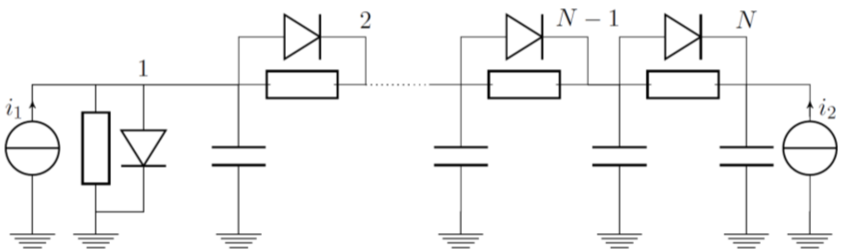
Numerical Examples: MIMO RC-Ladder

MIMO RC-Ladder model:



Nonlinearity: $g(x) = e^{40x} + x - 1$

Inputs/Outputs: $\mathbf{u}(t) = \sin(2t) \cdot [1 \;\; 1]^T$

$\mathbf{y}(t) = [v_1(t) \;\; v_{N-1,N}]^T$

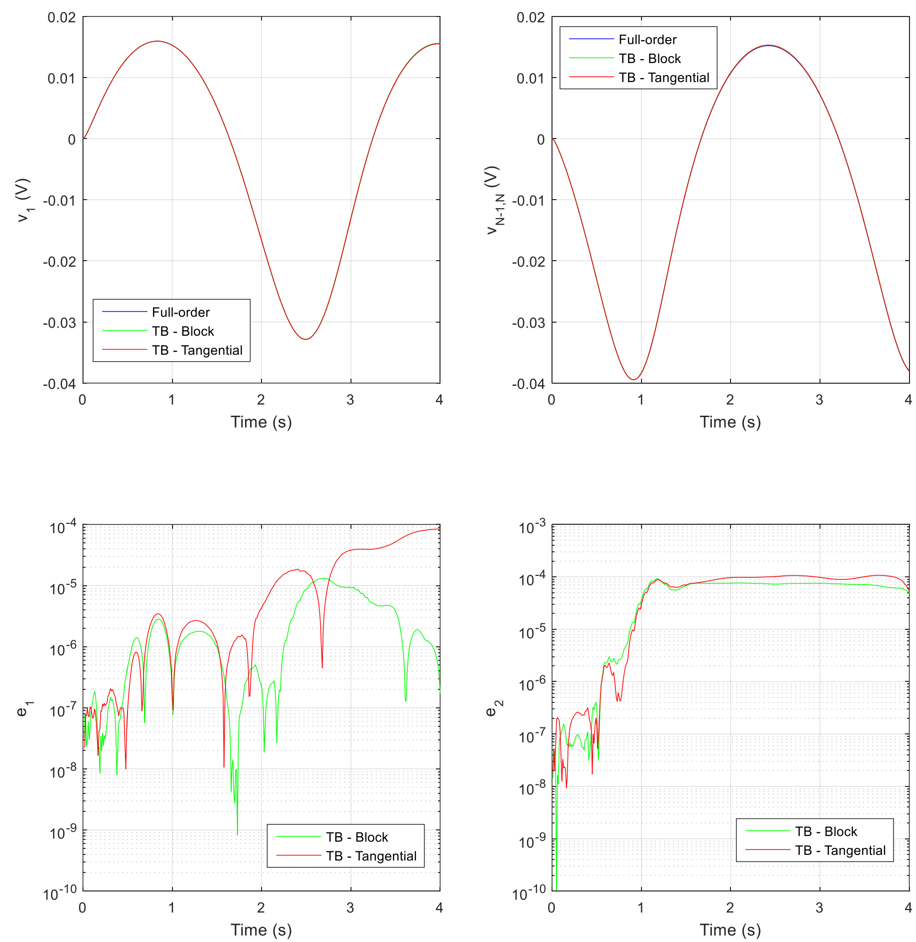
Reduction information:

$n = 800;$ Shifts s_0 gotten from IRKA

$t_{\text{sim,orig}} = 17.4$ s

$r_{\text{block}} = 30$ $\quad$ $r_{\text{tang}} = 21$

$t_{\text{sim,block}} = 0.232$ s $\quad$ $t_{\text{sim,tang}} = 0.109$ s



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Numerical Examples: FitzHugh-Nagumo

$$\epsilon \frac{\partial v}{\partial t}(x,t) = \epsilon^2 \frac{\partial^2 v}{\partial x^2}(x,t) + f(v(x,t)) - w(x,t) + g$$
$$\frac{\partial w}{\partial t}(x,t) = hv(x,t) - \gamma w(x,t) + g$$

Nonlinearity: $f(v) = v(v-0.1)(1-v)$

Inputs: $\mathbf{u}(t) = \begin{bmatrix} 5 \cdot 10^4 t^3 e^{-15t} \\ 1 \end{bmatrix}$

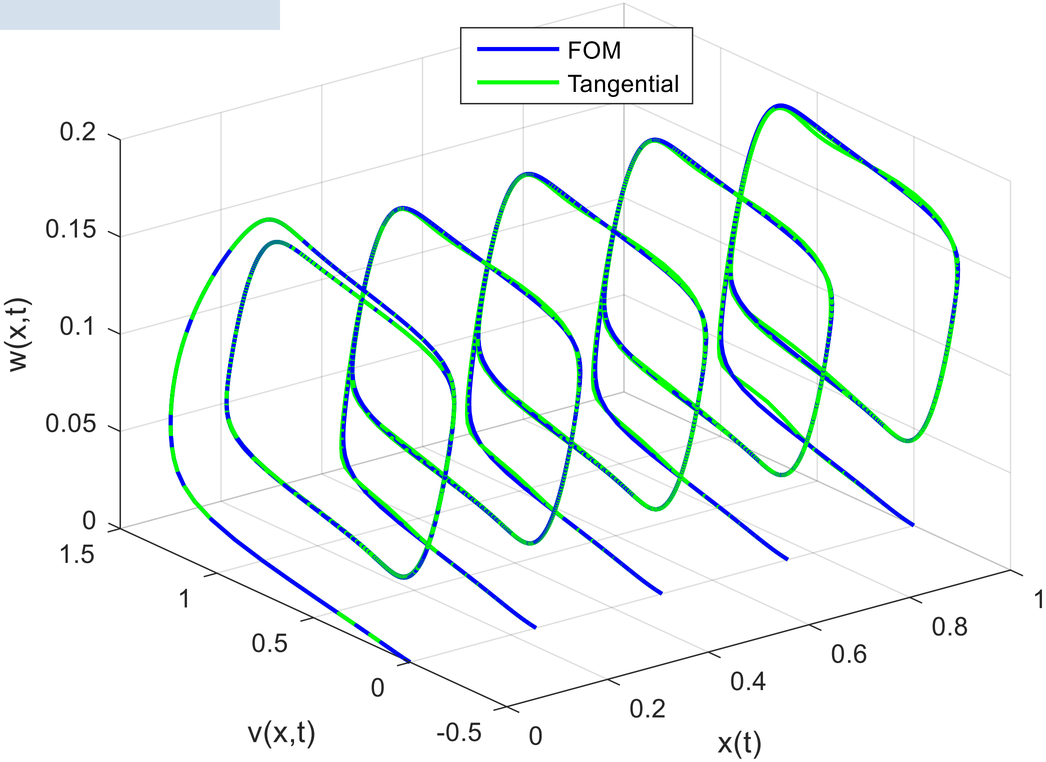
Reduction information:

$n = 1500;\quad$ Shifts s_0 gotten from IRKA

$t_{sim,orig} = 518$ s

$r_{\text{tang}} = 15$

$t_{\text{sim,tang}} = 0.631$ s



Conclusions

Summary:

- Many smooth nonlinear systems can be equivalently transformed into QB systems
- QB systems can be described by generalized **transfer functions**
- Systems theory and Krylov subspaces for SISO QB systems
- Systems theory for MIMO QB systems
- Krylov subspaces were extended to **MIMO case**

Conclusions:

- Transfer matrices make Krylov subspace methods more complicated in MIMO case
- **Tangential directions**: good option
- **Choice of shifts and tangential directions** plays an important role

Outlook

Next steps:

- **Optimal** choice of **shifts**
  - Comparison with T-QB-IRKA
  - Shifts gotten from T-QB-IRKA

- **Stability preserving** methods

- Other **benchmark** models
  - Nonlinear heat transfer
  - Electrostatic beam
  - Navier-Stokes equation

Thank you for your attention!

Backup

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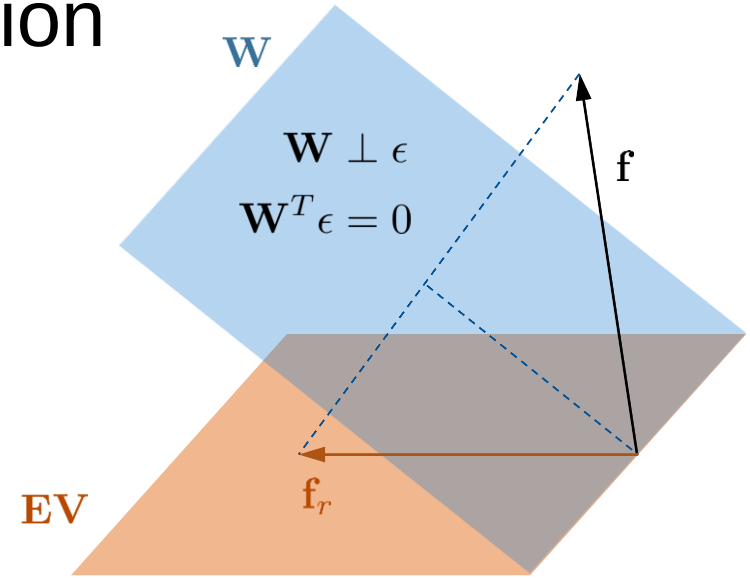
Projective Model Order Reduction



Assumption: State trajectory $\mathbf{x}(t)$ does not transit all regions of the state-space equally often, but mainly stays in a subspace of lower dimension

Approximation in the subspace $\mathcal{V} = \mathrm{span}(\mathbf{EV})$

$$\mathbf{x} = \mathbf{V} \; \mathbf{x}_r + \mathbf{e} \;, \quad \mathbf{V} \in \mathbb{R}^{n \times r}$$

Procedure:

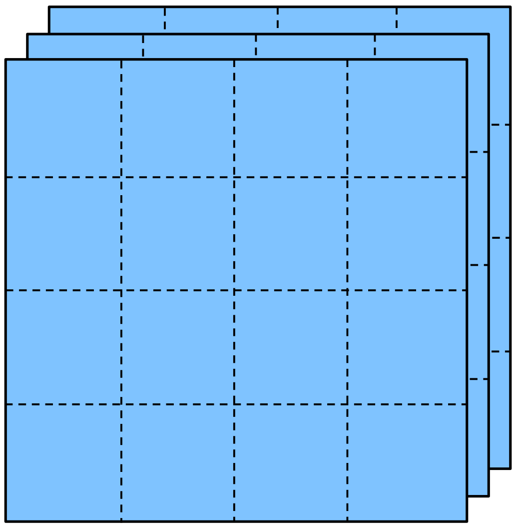
1. Replace $\mathbf{x}$ by its approximation
2. Reduce the number of equations (via projection with $\mathbf{\Pi} = \mathbf{EV}(\mathbf{W}^T\mathbf{EV})^{-1}\mathbf{W}^T$)
3. Petrov-Galerkin condition

$$\underbrace{\mathbf{W}^T \; \mathbf{E} \; \mathbf{V}}_{\mathbf{E}_r} \; \dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T \; \mathbf{f}(\mathbf{V}\,\mathbf{x}_r)}_{\mathbf{f}_r(\mathbf{x}_r)} + \underbrace{\mathbf{W}^T \; \mathbf{B}}_{\mathbf{B}_r} \; \mathbf{u}$$

$$\mathbf{y}_r = \underbrace{\mathbf{C} \; \mathbf{V}}_{\mathbf{C}_r} \; \mathbf{x}_r$$

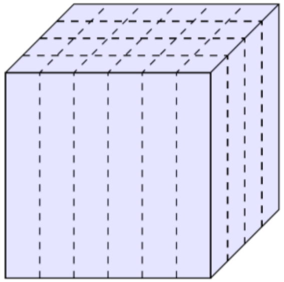
Tensors

Definition:

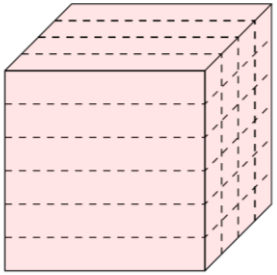


Three-dimensional figure

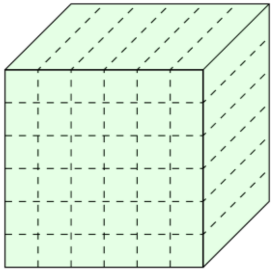
Matricizations:



1-mode: layers are put **side by side**



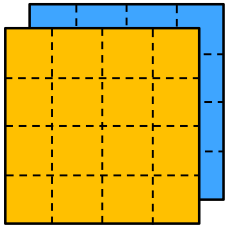
2-mode: **transposed** layers are put side by side



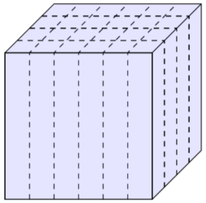
3-mode: fibers on the **depth** are put side by side

Matricization example

$$\mathcal{H}_{(:,:,1)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathcal{H}_{(:,:,2)} = \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$$

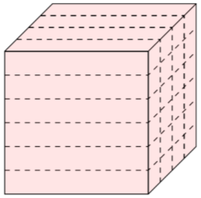


$$\mathbf{H}^{(1)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$



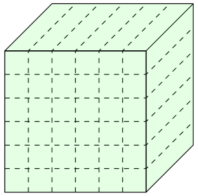
1-mode: layers are put **side by side**

$$\mathbf{H}^{(2)} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$



2-mode: **transposed** side by side

$$\mathbf{H}^{(3)} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$



3-mode: fibers on the **depth** side by side

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Kronecker product

Definition:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}$$

Most used:

$$\mathbf{x} \otimes \mathbf{x} = \begin{bmatrix} x_1^2 \\ x_2 x_1 \\ x_3 x_1 \\ \vdots \\ x_n^2 \end{bmatrix} \qquad \mathbf{u} \otimes \mathbf{x} = \begin{bmatrix} u_1 x_1 \\ u_2 x_1 \\ u_3 x_1 \\ \vdots \\ u_m x_n \end{bmatrix}$$

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Polynomialization Process

Algorithm 2.1 : Polynomialization procedure [20]

Data : $\mathbf{X} = [\dot{x}_1, \dot{x}_2, \dots \dot{x}_n]$, the list of symbolic expressions of the ODEs

Result : $\mathbf{Y}_{var}$, the set of new variables; $\mathbf{Y}_{expr}$, the set of expressions of the new variables; $\mathbf{X}$, the list of symbolic expressions of the polynomial ODEs.

begin
- Initialize $\mathbf{Y}_{var} \leftarrow \{\}$, $\mathbf{Y}_{expr} \leftarrow \{\}$;
- **while** *there is in* $\mathbf{X}$ *at least one non-polynomial function of* $\mathbf{x}$ *or any of the variables in* $\mathbf{Y}_{var}$ **do**
  - Pick from $\mathbf{X}$ a nonlinear function $g(\mathbf{x})$ that is not a polynomial function of $\mathbf{x}$;
  - Define a new state variable $v = g(\mathbf{x})$;
  - Add v into $\mathbf{Y}_{var}$ and $g(\mathbf{x})$ into $\mathbf{Y}_{expr}$;
  - Compute the symbolic expression of $\dot{v} = \frac{dg}{d\mathbf{x}}\dot{\mathbf{x}}$;
  - Add the symbolic expression of $\dot{v}$ to $\mathbf{X}$;
  - In $\mathbf{X}$, replace the occurrences of expressions in $\mathbf{Y}_{expr}$ by corresponding variables in $\mathbf{Y}_{var}$;

- $e^{\alpha x}$ (Typical diode I-V characteristic curve) [20]:

$$v = e^{\alpha x} \Rightarrow v' = \alpha e^{\alpha x} = \alpha v$$

- $\frac{1}{x+k}$ [20]:

$$v = \frac{1}{x+k} \Rightarrow v' = -\frac{1}{(x+k)^2} = -v^2$$

- x^α (Going from a monomial to quadratic expressions) [20]:

$$v_1 = x^\alpha \Rightarrow v_1' = \alpha x^{\alpha-1} = \alpha v_1 \underbrace{x^{-1}}_{v_2} = \alpha v_1 v_2$$
$$v_2 = x^{-1} \Rightarrow v_2' = -x^{-2} = -v_2^2$$

- $\ln(x)$ [20]:

$$v_1 = \ln(x) \Rightarrow v_1' = x^{-1} = v_2$$
$$v_2 = x^{-1} \Rightarrow v_2' = -x^{-2} = -v_2^2$$

- $\tan^{-1}(kx)$ (Can represent a saturation curve):

$$v_1 = \tan^{-1}(kx) \Rightarrow v_1' = \underbrace{\frac{k}{k^2x^2+1}}_{v_2}$$
$$v_2 = \frac{k}{k^2x^2+1} \Rightarrow v_2' = -\frac{2k^3x}{((kx)^2+1)^2} = -2kxv_2^2$$

Polynomialization Process

$$v_1 = \frac{1}{1+v_2} \longrightarrow \dot{v}_1 = \left(\frac{1}{1+v_2}\right)' \dot{v}_2 = -\frac{1}{(1+v_2)^2} v_2 v_1 = -v_1^3 v_2$$

$$\dot{x} = \frac{1}{1+e^x}$$

$$v_2 = e^x \longrightarrow \dot{v}_2 = v_2 \dot{x} = v_2 v_1$$

$$\dot{x} = v_1$$

$$\dot{\mathbf{x}}_{pol} = \begin{bmatrix} \dot{x} \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1^3 v_2 \\ v_1 v_2 \end{bmatrix}$$

Equivalent representation

Quadratic-Bilinearization Process

Algorithm 2.2 : Quadratic-bilinearization procedure [20]

Data : $\mathbf{X} = [\dot{x}_1, \dot{x}_2, \dots \dot{x}_n]$, the list of symbolic expressions of the ODEs

Result : $\mathbf{Y}_{var}$, the set of new variables; $\mathbf{Y}_{expr}$, the set of expressions of the new variables; $\mathbf{X}$, the list of symbolic expressions of the polynomial ODEs.

begin
  Initialize $\mathbf{Y}_{var} \leftarrow \{\}$, $\mathbf{Y}_{expr} \leftarrow \{\}$;
  while *there is in* $\mathbf{X}$ *at least one nonlinear or non-quadratic term of* $\mathbf{x}$ *or any of the variables in* $\mathbf{Y}_{var}$ **do**
    Pick a monomial $m(\mathbf{x})$ from $\mathbf{X}$ that has degree greater than 2;
    Find a decomposition of $m(\mathbf{x})$, i.e., find $g(\mathbf{x})$ and $h(\mathbf{x})$ that satisfy $m(\mathbf{x}) = g(\mathbf{x}) \times h(\mathbf{x})$;
    Define a new state variable $v = g(\mathbf{x})$;
    Add v into $\mathbf{Y}_{var}$ and $g(\mathbf{x})$ into $\mathbf{Y}_{expr}$;
    Compute the symbolic expression of $\dot{v} = \frac{dg}{d\mathbf{x}}\dot{\mathbf{x}}$;
    Add the symbolic expression of $\dot{v}$ to $\mathbf{X}$;
    for *all monomials* $m(\mathbf{x})$ **do**
      if $m(\mathbf{x})$ *is linear or quadratic in terms of* $\mathbf{x}$ *or any of the variables in* $\mathbf{Y}_{var}$ **then**
        Replace $m(\mathbf{x})$ as a linear or quadratic term;

Kernels

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$$\mathbf{E}\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) + \mathbf{b}u(t) \tag{2.68}$$

$$\mathbf{E}\dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) + \mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_1(t)) + \mathbf{N}\mathbf{x}_1(t)u(t) \tag{2.69}$$

As the first subsystem is linear, its time response can be easily calculated by means of the transition matrix $\mathbf{\Phi}(t) = e^{(\mathbf{E}^{-1}\mathbf{A}t)}$, as represented in the following:

$$\begin{aligned}\mathbf{x}_1(t) &= \int_{-\infty}^{\infty} \underbrace{e^{(\mathbf{E}^{-1}\mathbf{A}\sigma)}}_{\mathbf{\Phi}(\sigma)} \mathbf{E}^{-1}\mathbf{b} \cdot u(t-\sigma)\, d\sigma = \int_{-\infty}^{\infty} \underbrace{\mathbf{\Phi}(\sigma)\mathbf{E}^{-1}\mathbf{b}}_{\mathbf{f}_1(\sigma)} \cdot u(t-\sigma)\, d\sigma \\ &= \int_{-\infty}^{\infty} \mathbf{f}_1(\sigma) \cdot u(t-\sigma)\, d\sigma\end{aligned} \tag{2.70}$$

This result can be worked on in order to be inserted on the equation for the second subsystem. Equation (2.71) shows the Kronecker product that is necessary.

$$\begin{aligned}\mathbf{x}_1(t) \otimes \mathbf{x}_1(t) &= \int_{-\infty}^{\infty} \mathbf{f}_1(\sigma) \cdot u(t-\sigma)\, d\sigma \otimes \int_{-\infty}^{\infty} \mathbf{f}_1(\sigma) \cdot u(t-\sigma)\, d\sigma \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}_1(\sigma_1) \otimes \mathbf{f}_1(\sigma_2) \cdot u(t-\sigma_1)u(t-\sigma_2)\, d\sigma_1 d\sigma_2\end{aligned} \tag{2.71}$$

Now, rearranging (2.69) and knowing that $\mathbf{x}_2(t)$ does not depend on $\mathbf{x}_1(t)$, one gets a system which can be interpreted as linear on the input $\mathbf{u}^*(t) = [u(t) \quad 1]^T$, as rewritten in the following:

$$\begin{aligned}&\mathbf{E}\dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) + \mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_1(t)) + \mathbf{N}\mathbf{x}_1(t)u(t) \Rightarrow \\ &\mathbf{E}\dot{\mathbf{x}}_2(t) = \mathbf{A}\mathbf{x}_2(t) + \underbrace{\begin{bmatrix}\mathbf{N}\mathbf{x}_1(t) & \mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_1(t))\end{bmatrix}}_{\mathbf{B}^*(t)} \underbrace{\begin{bmatrix} u(t) \\ 1 \end{bmatrix}}_{\mathbf{u}^*(t)}\end{aligned} \tag{2.72}$$

Hence, it is possible to proceed with it the same way as done with the first subsystem, that is, calculating its time response by means of transition matrix and the results of Equation (2.71).

$$\begin{aligned}
\mathbf{x}_2(t) &= \int_{-\infty}^{\infty} \underbrace{e^{(\mathbf{E}^{-1}\mathbf{A}\sigma)}}_{\boldsymbol{\Phi}(\sigma)} \mathbf{E}^{-1}\mathbf{B}^*(\sigma) \cdot \mathbf{u}^*(t-\sigma)\, d\sigma \\
&= \int_{-\infty}^{\infty} \boldsymbol{\Phi}(\sigma)\mathbf{E}^{-1}[\mathbf{H}(\mathbf{x}_1(t) \otimes \mathbf{x}_1(t)) + \mathbf{N}\mathbf{x}_1(t)u(t-\sigma)]\, d\sigma \\
&= \int_{-\infty}^{\infty} \boldsymbol{\Phi}(\sigma)\mathbf{E}^{-1} \Bigg[\mathbf{H}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}_1(\sigma_1) \otimes \mathbf{f}_1(\sigma_2) \cdot u(t-\sigma_1)u(t-\sigma_2)\, d\sigma_1 d\sigma_2\right) \\
&\qquad +\mathbf{N}\left(\int_{-\infty}^{\infty} \mathbf{f}_1(\sigma_1) \cdot u(t-\sigma_1)\, d\sigma_1\right) u(t-\sigma)\Bigg]\, d\sigma \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{\Phi}(\sigma)\mathbf{E}^{-1}\mathbf{H}(\mathbf{f}_1(\sigma_1) \otimes \mathbf{f}_1(\sigma_2)) \cdot u(t-\sigma_1)u(t-\sigma_2)\, d\sigma_1 d\sigma_2 d\sigma \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{\Phi}(\sigma)\mathbf{E}^{-1}\mathbf{N}\mathbf{f}_1(\sigma_1) \cdot u(t-\sigma_1)u(t-\sigma)\, d\sigma_1 d\sigma \qquad (2.73)
\end{aligned}$$

Finally, one can define the second order kernel [32, §§3.4] as

$$\mathbf{f}_2(\sigma_1, \sigma_2) = \mathbf{\Phi}(\sigma_2)\mathbf{E}^{-1}\mathbf{N}\mathbf{f}_1(\sigma_1) + \int_{-\infty}^{\infty} \mathbf{\Phi}(\sigma)\mathbf{E}^{-1}\mathbf{H}(\mathbf{f}_1(\sigma_1) \otimes \mathbf{f}_1(\sigma_2))\, d\sigma \tag{2.74}$$

Such as the time response of the second order subsystem can be written as represented in Equation (2.75) [32, §3.4].

$$\mathbf{x}_2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{f}_2(\sigma_1, \sigma_2) u(t - \sigma_1) u(t - \sigma_2)\, d\sigma_1 d\sigma_2 \tag{2.75}$$

This procedure can be done with virtually all remaining subsystems of higher order, but in this thesis, only the first two subsystems are of great importance.

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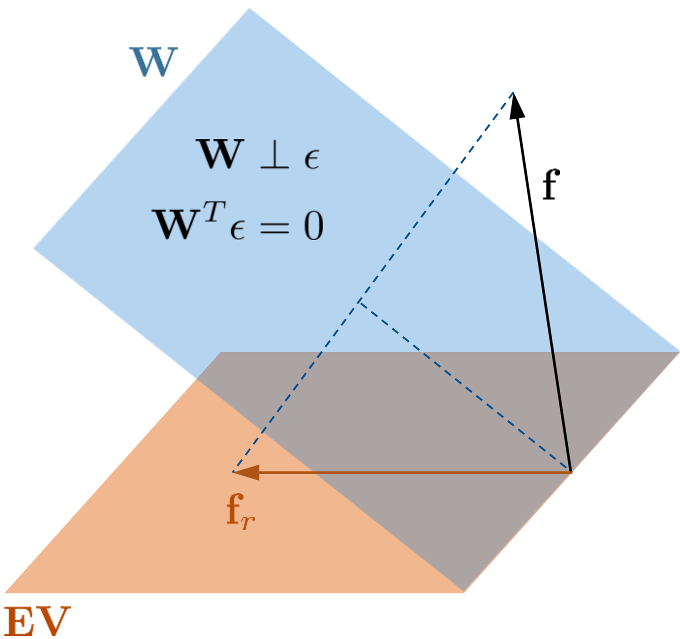
Projective Model Order Reduction



$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{H}(\mathbf{x}\otimes\mathbf{x}) + \mathbf{N}\mathbf{x}u + \mathbf{b}u$$
$$y = \mathbf{c}^T\mathbf{x}$$

$\mathbf{x} \approx \mathbf{V}\mathbf{x}_r$

$\mathbf{W}^T\cdot$

$$\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r = \mathbf{A}\mathbf{V}\mathbf{x}_r + \mathbf{H}(\mathbf{V}\mathbf{x}_r\otimes\mathbf{V}\mathbf{x}_r) + \mathbf{N}\mathbf{V}\mathbf{x}_r u + \mathbf{b}u + \epsilon$$
$$y_r = \mathbf{c}^T\mathbf{V}\mathbf{x}_r$$

$$\mathbf{E}_r\dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{H}_r(\mathbf{x}_r\otimes\mathbf{x}_r) + \mathbf{N}_r\mathbf{x}_r u + \mathbf{b}_r u$$
$$y_r = \mathbf{c}_r^T\mathbf{x}_r$$

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$$
$$\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$$
$$\mathbf{H}_r = \mathbf{W}^T\mathbf{H}(\mathbf{V}\otimes\mathbf{V})$$
$$\mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V}$$
$$\mathbf{b}_r = \mathbf{W}^T\mathbf{b}$$
$$\mathbf{c}_r = \mathbf{V}^T\mathbf{c}$$

Multimoments approach (SISO)

[Breiten ’12]

Algorithm 1 QB Multimoment Matching (SISO)

Input: $\mathbf{E}$, $\mathbf{A}$, $\mathbf{H}$, $\mathbf{N}$, $\mathbf{b}$, $\mathbf{c}$, shift σ, reduced order of first transfer function q_1 and of the second transfer function q_2

Output: Projection matrices $\mathbf{V}$, $\mathbf{W}$

$\mathbf{V}_1 = \mathcal{K}_{q_1}(\mathbf{A}_\sigma^{-1}\mathbf{E}, \mathbf{A}_\sigma^{-1}\mathbf{b})$

$\mathbf{W}_1 = \mathcal{K}_{q_1}(\mathbf{A}_{2\sigma}^{-T}\mathbf{E}^T, \mathbf{A}_{2\sigma}^{-T}\mathbf{c})$

for $i = 1 : q_2$ **do**

  $\mathbf{V}_2^i = \mathcal{K}_{q_2-i+1}(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{N}\mathbf{V}_1(:,i))$

  $\mathbf{W}_2^i = \mathcal{K}_{q_2-i+1}(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{N}^T\mathbf{W}_1(:,i))$

  for $j = 1 : \min(q_2 - i + 1, i)$ **do**

    $\mathbf{V}_3^{i,j} = \mathcal{K}_{q_2-i+1}(\mathbf{A}_{2\sigma}^{-1}\mathbf{E}, \mathbf{A}_{2\sigma}^{-1}\mathbf{H}(\mathbf{V}_1(:,i) \otimes \mathbf{V}_1(:,j)))$

    $\mathbf{W}_3^{i,j} = \mathcal{K}_{q_2-i+1}(\mathbf{A}_\sigma^{-T}\mathbf{E}^T, \mathbf{A}_\sigma^{-T}\mathbf{H}^{(2)}(\mathbf{V}_1(:,i) \otimes \mathbf{W}_1(:,j)))$

  end for

end for

$\text{span}(\mathbf{V}) = \text{span}(\mathbf{V}_1) \cup \bigcup_i \text{span}(\mathbf{V}_2^i) \cup \bigcup_{i,j} \text{span}(\mathbf{V}_3^{i,j})$

$\text{span}(\mathbf{W}) = \text{span}(\mathbf{W}_1) \cup \bigcup_i \text{span}(\mathbf{W}_2^i) \cup \bigcup_{i,j} \text{span}(\mathbf{W}_3^{i,j})$

linear

bilinear

quadratic

$$\frac{\partial^i G_1}{\partial s_1^i}(\sigma) = \frac{\partial^i G_{1,r}}{\partial s_1^i}(\sigma), \qquad i = 0, \dots, q_1 - 1$$

$$\frac{\partial^i G_1}{\partial s_1^i}(2\sigma) = \frac{\partial^i G_{1,r}}{\partial s_1^i}(2\sigma), \qquad i = 0, \dots, q_1 - 1$$

$$\frac{\partial^{i+j}}{\partial s_1^i s_2^j}G_2(\sigma,\sigma) = \frac{\partial^{i+j}}{\partial s_1^i s_2^j}G_{2,r}(\sigma,\sigma), \qquad i + j \le 2q_2 - 1$$

$$\text{span}(\mathbf{V}) = \text{span}(\mathbf{V}_{\text{lin}}) \cup \text{span}(\mathbf{V}_{\text{b}}) \cup \text{span}(\mathbf{V}_{\text{q}})$$

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Hermite approach (SISO)

Theorem: Two-sided rational interpolation [Breiten '15]

Let $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$ be nonsingular, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V},\ \mathbf{H}_r = \mathbf{W}^T\mathbf{H}(\mathbf{V}\otimes\mathbf{V}),\ \mathbf{N}_r = \mathbf{W}^T\mathbf{N}\mathbf{V},$ $\mathbf{b}_r = \mathbf{W}^T\mathbf{b},\ \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$ with $\mathbf{V},\ \mathbf{W}\in\mathbb{R}^{n\times r}$ having full rank such that

$$\mathrm{span}(\mathbf{V}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{\sigma_i}^{-1}\mathbf{b},\ \mathbf{A}_{2\sigma_i}^{-1}[\mathbf{H}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}) - \mathbf{N}\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}]\}$$

$$\mathrm{span}(\mathbf{W}) \supset \mathrm{span}_{i=1,\dots,k}\{\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c},\ \mathbf{A}_{\sigma_i}^{-T}[\mathbf{H}^{(2)}(\mathbf{A}_{\sigma_i}^{-1}\mathbf{b}\otimes\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c}) - \frac{1}{2}\mathbf{N}^T\mathbf{A}_{2\sigma_i}^{-T}\mathbf{c})]\}$$

with $\sigma_i \notin \{\Lambda(\mathbf{A},\mathbf{E}),\Lambda(\mathbf{A}_r,\mathbf{E}_r\}$.

Then:

$$G_1(\sigma_i) = G_{1,r}(\sigma_i) \qquad G_1(2\sigma_i) = G_{1,r}(2\sigma_i)$$

$$G_2(\sigma_i,\sigma_i) = G_{2,r}(\sigma_i,\sigma_i) \qquad \frac{\partial G_2}{\partial s_j}(\sigma_i,\sigma_i) = \frac{\partial G_{2,r}}{\partial s_j}(\sigma_i,\sigma_i)$$

Krylov subspaces for MIMO systems

$$\mathbf{A}_{s_0} = \mathbf{A} - s_0\mathbf{E}$$

Pseudolinear approach:

$$\text{span}(\mathbf{V}) \supset \text{span}\left\{\mathbf{A}_\sigma^{-1}\mathbf{B}, \mathbf{A}_\sigma^{-1}\mathbf{E}\mathbf{A}_\sigma^{-1}\mathbf{B}\right\}$$

$$\text{span}(\mathbf{W}) \supset \text{span}\left\{\mathbf{A}_{2\sigma}^{-T}\mathbf{C}^T, \mathbf{A}_{2\sigma}^{-T}\mathbf{E}^T\mathbf{A}_{2\sigma}^{-T}\mathbf{C}^T\right\}$$

$$\mathbf{G}_1(\sigma) = \mathbf{G}_{1,r}(\sigma) \qquad \mathbf{G}_1(2\sigma) = \mathbf{G}_{1,r}(2\sigma) \qquad \mathbf{G}_2(\sigma,\sigma) = \mathbf{G}_{2,r}(\sigma,\sigma)$$

$$\frac{\partial \mathbf{G}_1}{\partial s}(\sigma) = \frac{\partial \mathbf{G}_{1,r}}{\partial s}(\sigma) \qquad \frac{\partial \mathbf{G}_1}{\partial s}(2\sigma) = \frac{\partial \mathbf{G}_{1,r}}{\partial s}(2\sigma) \qquad \frac{\partial \mathbf{G}_2}{\partial s_j}(\sigma,\sigma) = \frac{\partial \mathbf{G}_{2,r}}{\partial s_j}(\sigma,\sigma)$$

- **2m** columns per shift
- **7** moments matched

V and **W** do not have any nonlinear information!

Stability issues