Vertical datum unification for the International Height Reference System (IHRS)

Laura Sánchez, Deutsches Geodätisches Forschungsinstitut, Technische Universität München, Arcisstr. 21, 80333 München, Germany, lm.sanchez@tum.de
Michael G. Sideris, Department of Geomatics Engineering, University of Calgary, 2500 University Drive NW, Calgary, Alberta, T2N 1N4, Canada, sideris@ucalgary.ca

Vertical datum parameters

The International Association of Geodesy (IAG) released in July 2015 a resolution for the definition and realisation of an International Height Reference System (IHRS). According to this resolution, the IHRS coordinates are potential differences referring to the level surface of the Earth's gravity field realised by the conventional value W_0 = 62 636 853.4 m^{2}s^{-2}. A main component of the IHRS is the integration of the existing height systems into the global one; i.e., existing vertical coordinates should be referred to one and the same reference level realised by the conventional W_0. This procedure is known as vertical datum unification and its main result are the vertical datum parameters, i.e., the potential differences between the local and the global reference levels (Fig. 1):

$$\delta W_{0i} = W_0 - W_{0i} \qquad \text{Eq. [1]}$$

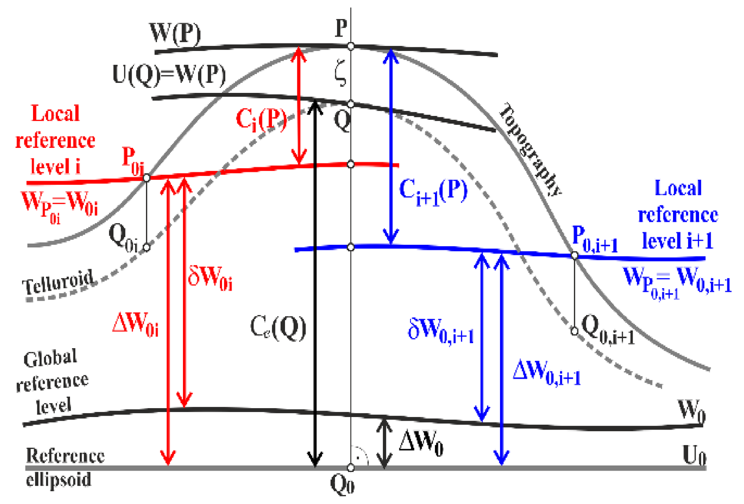


Fig. 1: Vertical datum parameters for the local height systems i and i+1.

Observation equations for the vertical datum unification

The estimation of the vertical datum parameters is based on the comparison of the height anomaly ζ (or geoid undulation N) obtained from the solution of the geodetic boundary value problem (GBVP) with the height anomaly ζ (or geoid undulation N) derived from combining satellite-based ellipsoidal heights (h) with levelling-based geopotential numbers (C) or physical heights (H^* or H). For a general formulation, independent of normal or orthometric heights, the observation equations are given here in terms of potential quantities:

$$l_P = h(P)\gamma - C_i(P) - T_i(P) + \Delta W_0 = \left(1 + f_{0i}(P)\right)\delta W_{0i}(P) + \sum_{j=1, j\neq i}^{J} f_{0j}(P)\delta W_{0j}(P) + v_P \qquad \text{Eq. [2]}$$

– $\Delta W_0 = W_0 - U_0$ is the potential difference between the IHRS W_0 value and the normal potential U_0 of the GRS80 ellipsoid.

– δW_{0i} denotes the vertical datum parameters (cf. Eq. [1]).

– $T_i(P)$ is the usual result obtained by solving the GBVP using (biased) gravity anomalies (Δg_i) referring to the vertical datum i:

$$T_i(P) = \frac{G\delta M}{R} + \sum_{i=1}^{J}\frac{R}{4\pi}\iint_{\sigma_i}\left(\Delta g_i(P_k) + g_{1i}(P_k)\right) S\left(\psi(P,P_k)\right)d\sigma \qquad \text{Eq. [3]}$$

– $\sum_{j=1, j\neq i}^{J} f_{0j}(P)\delta W_{0j}(P) + v_P$ with $f_{0i}(P) := \frac{1}{2\pi\gamma}\iint_{\sigma_i} S\left(\psi_{P,P_k}\right)d\sigma$ denotes an *indirect bias term* caused by the effects of the level differences δW_{0i} on the boundary values (i.e., Δg_i).

– l_P and v_P represent the observables and the stochastic residuals, respectively.

At each point P referring to the vertical datum i, an observation equation like Eq. [2] can be formulated. At those points referring to two neighbouring vertical datums i, $i+1$ (see Fig. 1), the observation equation takes the form:

$$l_P = C_{i+1}(P) - C_i(P) = \delta W_{0i} - \delta W_{0,i+1} + v_P \qquad \text{Eq. [4]}$$

Least-squares estimation of the vertical datum parameters

Equations [2] and [4] can be solved by a least-squares adjustment. The functional and stochastic models are given by:

$$\mathbf{A\,x} - \mathbf{v} = \mathbf{l} \qquad \text{Eq. [5]}$$

$$E\{\mathbf{v}\} = 0; \quad E\{\mathbf{v}\mathbf{v}^T\} = \mathbf{C}_l = \mathbf{C}_{h(P)} + \mathbf{C}_{C_i(P)} + \mathbf{C}_{T_i(P)} \qquad \text{Eq. [6]}$$

$\mathbf{A}$ is the design matrix containing the coefficients of the unknowns in the observation equations; $\mathbf{x}$ is the vector of the unknowns; $\mathbf{v}$ contains the residuals; $\mathbf{l}$ contains the left-hand side elements in Eqs. [2] and [4]; $E\{\bullet\}$ is the expectation operator; $\mathbf{C}$ represents the variance-covariance matrices of the input data. The least-squares solution provides estimates for the vertical datum parameters δW_{0i} and the corresponding variance-covariance matrix:

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A}\right)^{-1}\left(\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{l}\right); \quad \mathbf{C_x} = \left(\mathbf{A}^T\mathbf{C}_l^{-1}\mathbf{A}\right)^{-1} \qquad \text{Eq. [7]}$$

Effect of omission error and indirect bias term on datum unification in North America

The disturbing potential $T_i(P)$ (Eq. [3]) is typically computed by the remove-compute-restore procedure:

$$T_i(P) = T_{GGM}(P) + T_{i,res}(P) \qquad \text{Eq. [8]}$$

$T_{GGM}(P)$ is inferred form a global gravity model (GGM) of maximum degree and order N_{max}. $T_{i,res}(P)$ is estimated by evaluating Eq. [3] using residual gravity anomalies $\Delta g_{i,res}=\Delta g_i - \Delta g_{GGM}$ and the residual kernel function $S_{res}(\psi) = S(\psi) - S_{Nmax}(\psi)$. If the datum unification is performed using a GGM with an N_{max} in the 180 to 220 range, and local gravity and topography data, the following questions arise:

Q1. will $T_{i,res}(P)$ in Eq. [8], or omission error, be small enough to ignore? and

Q2. will the last term of the indirect bias in Eq. [2] be small enough to omit?

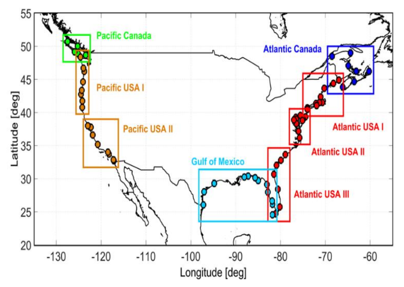
To answer Q1, Table 1 summarises the omission error of the DIR5 model (Bruinsma et al., 2013) at different tide gauges in North America. Although this error decreases when averaged over many points, it can reach several decimetres and cannot be omitted. In other words, the local disturbing potential $T_i(P)$ at individual datum regions should always be computed by combining a GGM with the available local gravity and topography data.

To answer Q2, Fig. 2 shows the indirect bias term computed with the full-unmodified kernel function $S(\psi)$ and residual kernel functions $S_{res}(\psi)$ of various truncation degrees N_{max}. Results indicate that although this term can reach over 4 m^{2}s^{-2} (i.e., > 40 cm in ζ), it drops below 0.1 m^{2}s^{-2} (~1 cm) for N_{max} ≥ 180. It can therefore be concluded that the indirect bias is indeed negligible if a GGM of N_{max} ≥ 180 is used for the determination of the disturbing potential.

Consequently, the f_{0i} coefficients in Eq. [2] can be set equal to zero, resulting in a much simpler system of observation equations. This yields a solution where the datum parameters are the weighted mean of all individual station $\delta W_{0i}(P)$ values. Fig. 3 shows the North American vertical datum parameters with respect to the IHRS W_0 reference level that were used to obtain the results shown in Table 1 and Fig. 2.

Table 1: Statistics of the DIR5 model omission error computed at tide gauges using local data. Map shows the geographic location of the tide gauges.

Region	Mean [m^{2}s^{-2}]	σ [m^{2}s^{-2}]	Min [m^{2}s^{-2}]	Max [m^{2}s^{-2}]
Atlantic Canada	0.75	3.00	-3.53	4.52
Pacific Canada	-2.21	2.56	-5.82	0.24
Atlantic USA I	1.05	1.29	-1.38	2.81
Atlantic USA II	-1.60	2.54	-5.59	1.56
Atlantic USA III	0.25	2.89	-3.91	4.62
Pacific USA I	-1.66	2.96	-7.12	3.11
Pacific USA II	-1.12	1.80	3.49	2.32
Gulf of Mexico	-0.23	1.59	-3.70	2.86



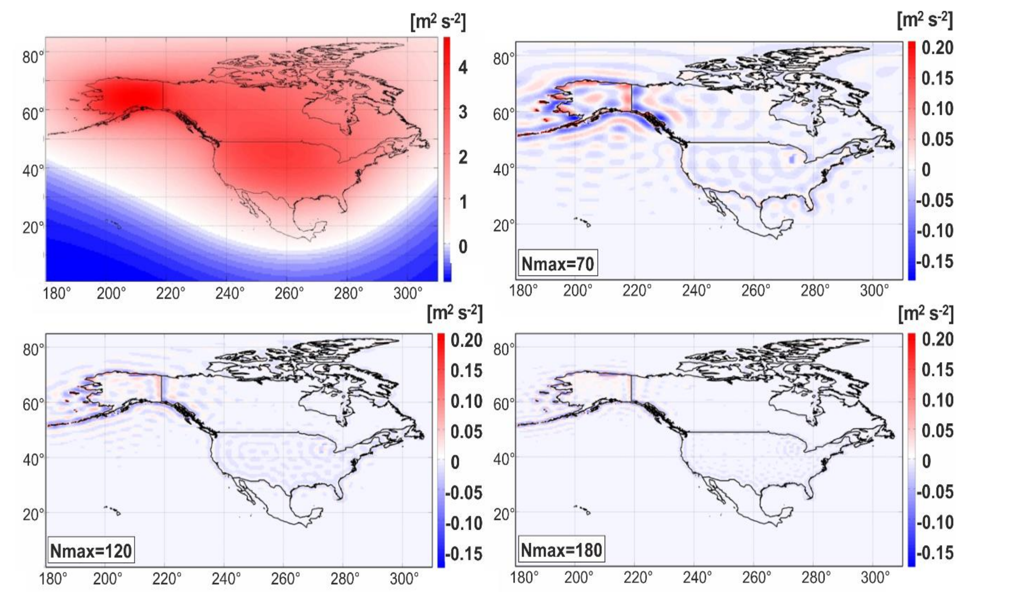


Fig. 2: Indirect bias term computed with the full-unmodified kernel function (left, above) and residual kernel functions of various truncation degrees N_{max}.

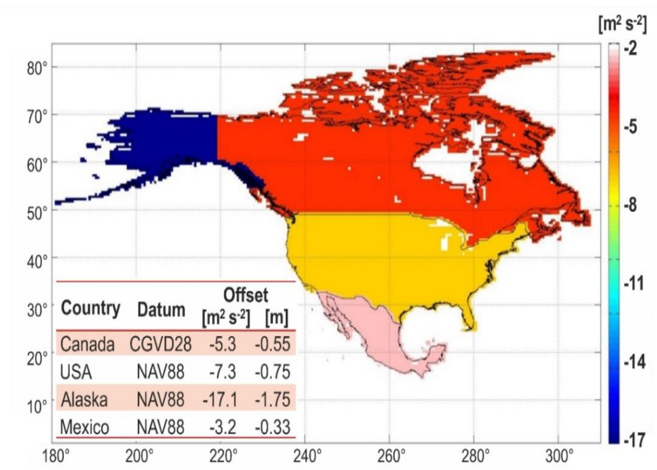


Fig. 3: North American vertical datum parameters with respect to the IHRS reference level W_0= 62 636 853.4 m^{2}s^{-2}.

Vertical datum unification in South America

The formulation of the observation equations [2] and [4] implies the availability of terrestrial gravity anomalies, levelling-based geopotential numbers, ellipsoidal heights from GNSS on land and from satellite altimetry in oceans, and border levelling points with geopotential numbers referring to neighbouring vertical datums. According to the geodetic data available in South America, this study is based on

– 14 observation equations of the type Eq. [2] in the marine areas nearby the reference tide gauges (Fig. 4a);

– 663 observation equations of the type Eq. [2] at the reference stations of the continental reference frame SIRGAS (Fig. 4b);

– 7 observation equations of the type Eq. [4]: connections between Ecuador and Colombia, Colombia and Venezuela, Venezuela and Brazil, and Brazil and Argentina (Fig. 4c).

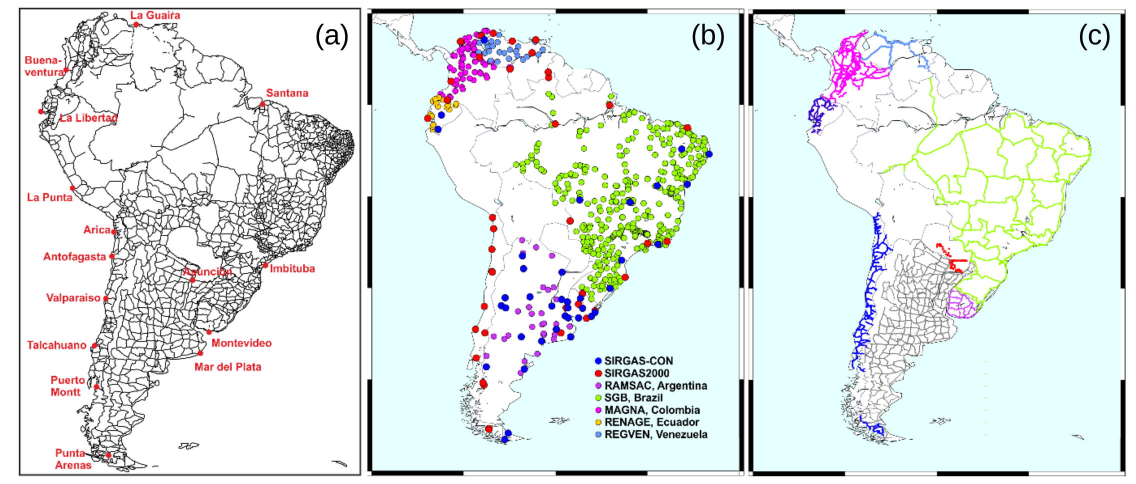


Fig. 4: Geodetic data available for the vertical datum unification in South America: (a) Levelling networks and reference tide gauges; (b) Geometric reference stations; (c) Levelling surveys provided towards a new adjustment of the vertical networks.

In a first estimation, Eq. [7] was solved by employing the data currently in use in the different South American countries; i.e., referred to different ITRF solutions and reference epochs, given in different tide systems, etc. A second estimation was made based on standardised data. The purpose was to compare the vertical datum parameters obtained using the available raw geodetic data (without further processing) and those parameters obtained with harmonised (or standardised) geodetic data. The accuracy of the different observables was taken into account to build the covariance matrices in Eq. [6]. Figures 5 and 6 show adjustment residuals and the estimated vertical datum parameters for the existing South American height systems. The accuracy was assessed to be ±0.5 m^{2}s^{-2} in those regions with a high number of observations; i.e., Argentina, Brazil (Imbituba), Colombia, Ecuador, Uruguay and Venezuela. In regions with a small number of observations, like the northern part of Brazil (Santana), Bolivia, Peru and the southern part of Chile (Punta Arenas), the accuracy was not better than ±2 m^{2}s^{-2} to ±4 m^{2}s^{-2}.

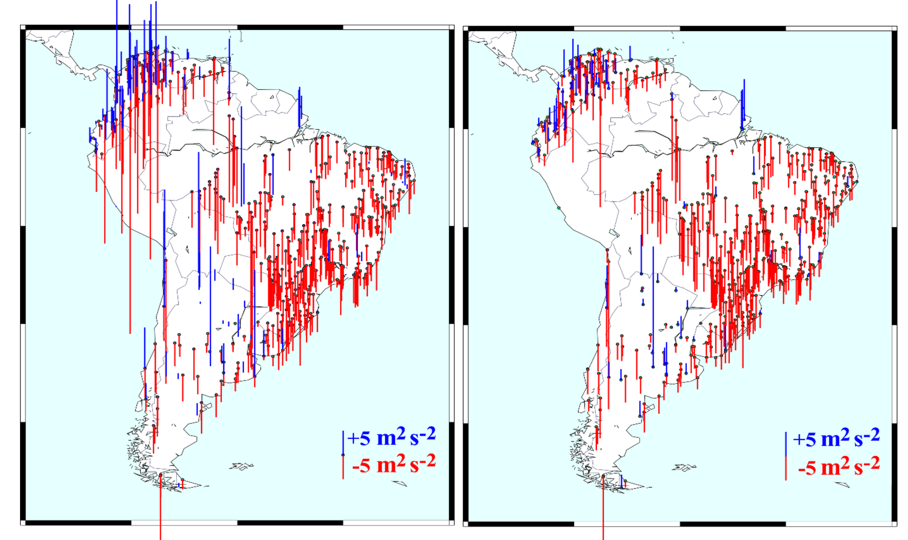


Fig. 5: Residuals obtained after the estimation of the vertical datum parameters by employing the geodetic data currently in use (left) and the geodetic data homogenised in this study (right).

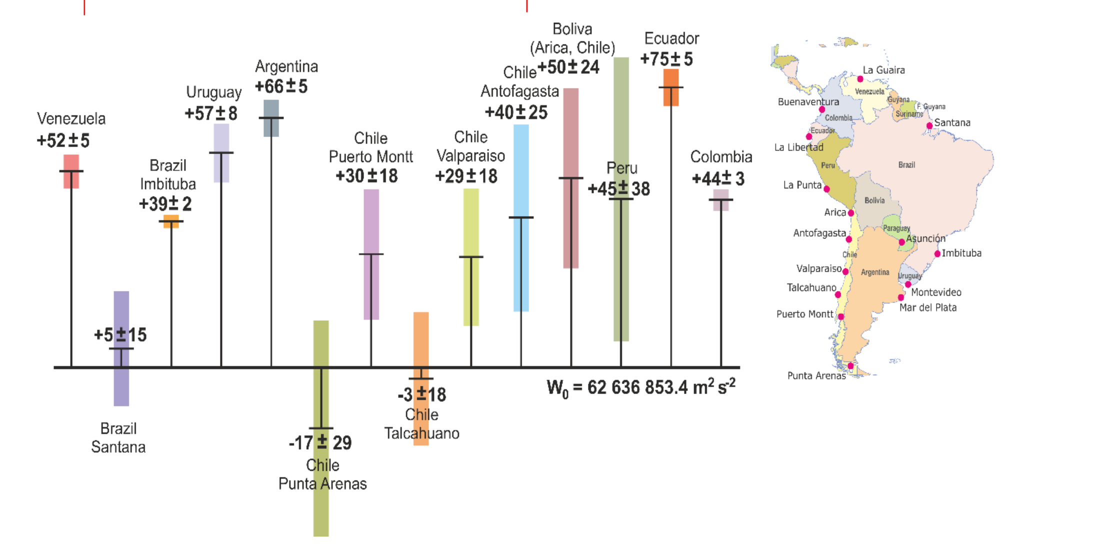


Fig. 6: South American vertical datum parameters [in cm] with respect to the IHRS reference level W_0 = 62 636 853.4 m^{2}s^{-2}. Coloured bars show the standard deviations.

Further reading: *Sánchez L., Sideris M.G.*: **Vertical datum unification for the International Height Reference System (IHRS)** . Geophysical Journal International 209(2), 570-586, 10.1093/gji/ggx025 , 2017