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Optimal Wire Ordering and Spacing in Low Power Semiconductor Design

Dedicated to Prof. Martin Grötschel on the occasion of his 60th birthday.

Abstract A key issue for high integration circuit design in the semiconductor industry are power constraints that stem from the need for heat removal and reliability or battery lifetime limitations. As the power consumption depends heavily on the capacitances between adjacent wires, determining the optimal ordering and spacing of parallel wires is an important issue in the design of low power chips.

As it turns out, optimal wire spacing is a convex optimization problem whereas the optimal wire ordering is combinatorial in nature, containing (a special class of) the Minimum Hamilton Path problem. While the latter is $\mathcal{NP}$-hard in general, the present paper provides an $\mathcal{O}(N \log N)$ algorithm that solves the coupled ordering and spacing problem for N parallel wires to optimality.

Keywords Optimal Wire Placement · Convex Programming · Combinatorial Optimization · Hamilton Path

Mathematics Subject Classification (2000) 90C27 · 90C25 · 90C90

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1 Introduction

One of the key issues of miniaturization in integrated circuit design are power constraints that result from the need of heat reduction and reliability or battery lifetime limitations. As the power consumption depends heavily on the switching capacitances between adjacent wires, determining the optimal ordering and spacing of parallel wires according to their switching frequencies is a key issue in the design of low power chips.

Wire ordering and spacing according to various objectives have a long history in electronic design automation; see [9,10]. Wire ordering, although with different design objectives, appears in [17,7] and also (in a more limited context) in [6]. Wire spacing with respect to switching activities was first considered in [8]. Due to the apparent intrinsic complexity of the problem, most known algorithms for this and related problems are based on heuristics or exhaustive search techniques, an overview can be found in [18]. The present paper derives an efficient rigorous algorithm for power optimal wire ordering and spacing under distance requirements as side constraints.

We begin by presenting and analyzing a mathematical model that captures all major aspects of the wire ordering and spacing problem in integrated circuit design. As the first step towards the algorithm we then consider the wire spacing problem separately. As it turns out, the underlying convex optimization problem can essentially be solved analytically. This can be utilized to relate the combined wire placement problem to a specific kind of Minimum Hamilton Path (MHP) (or Traveling Salesman, TSP) Problem. While the general MHP is notoriously $\mathcal{NP}$-hard, our algorithm for the combined ordering and spacing problem for N parallel wires relies on strong new structural results and will be shown to run in total worst-case $\mathcal{O}(N \log N)$ time.

The paper is organized as follows. Section 2 contains some relevant background information on low power semiconductor design to motivate and justify the mathematical model for the wire ordering and spacing problem. The precise mathematical problem and the main result are stated in Section 3. Section 4 deals with the wire spacing problem, solves the underlying convex optimization problem and derives an efficient algorithm for that problem. Based on these results, Section 5 will then derive structural combinatorial results that can be used to relate the combined wire ordering and spacing problem to the solution of a certain class of Minimum Hamilton Path problems, resulting in the asserted $\mathcal{O}(N \log N)$ algorithm. Section 6 contains some final remarks.

2 Low Power Semiconductor Design

We will now briefly give some background information on wire ordering and spacing in low power semiconductor design and introduce a first mathematical description of the optimal placement problem that is still formulated from a 'natural application specific perspective'. The further abstracted and finalized mathematical model will be given in Section 3.

Naturally, we do not aim at a comprehensive treatment of all relevant aspects from electrical engineering here but will concentrate on those that motivate and justify the concise mathematical model for the wire ordering and spacing problem given in the subsequent Section 3. For further information see [11,12,14,18] and the references quoted there. (Readers less interested in this background information may skip this Section and continue directly with the 'hard core' mathematical optimization problem in Section 3.)

2.1 Power Loss in Semiconductor Circuits

A semiconductor circuit is composed of a bottom layer containing the transistors and possibly other electronic components, and several stacked metal layers containing the wires which interconnect the elements of the bottom layer. Within a single metal layer all wires are of the same thickness and the wires are all parallel, while the directions of the wires in two metal layers positioned on top of each other are orthogonal, cf. Figure 1 for an illustration of the wires on three metal layers.

The power consumption of a semiconductor circuit is usually decomposed into two parts, a static component attributed to leakage at transistor level and a dynamic component caused by switching capacitances between adjacent wires and short circuit currents.[1] As of today, capacitances between adjacent wires account for the major part of total power consumption, and although leakage has increased over the past years, simultaneously a relatively increasing fraction of the capacitances has moved from transistors to wires. This aspect is of special importance at low ambient temperature or in low leakage circuits frequently found in mobile devices, where in non-idle phases the dynamic component exceeds the static component by several orders of magnitude.

Energy loss caused by interconnects in semiconductor circuits is mainly due to capacitances emerging between neighboring wires whenever their relative voltage changes. Over the last years, the demand for ever higher integration densities has substantially increased, and for technical reasons this requires the typical on-chip wire's thickness to become larger than both its width and the wire distances. This technological change results in a change of relevance of the different kinds of occurring capacitances. While in the past the highest fraction of the sum of capacitances was caused by the coupling between different layers and the bottom areas of the substrate, now the edge-to-edge capacitances within one layer dominate; see [15].

Figure 1 shows the capacitances in a layout simulated with the software package QuickCap (see [4], now distributed by Magma Design Automation). The test setup comprises a wire (gray) on the third metal layer of a typical 0.13 µm process embedded into a fully crowded proximity. The capacitances between the wire and its immediate neighbors clearly dominate all other capacitance components. The fraction may reach up to 40 % on each side in case of less population in the layers above and below.

[1] Short circuits are due to transistor behavior: A transistor 'opens' when a certain voltage is available on one of its gates. On the other hand, the transistor 'closes' when that voltage drops below a certain threshold, which is usually lower than the 'opening threshold'. As a consequence, whenever a switch occurs, some transistors may be open at the same time, as power rises or falls, which may lead to short-time short circuits.

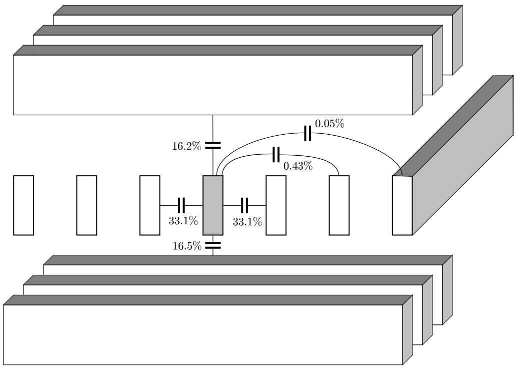

Fig. 1 Capacitances in a 0.13 µm process.

The power loss caused by capacitances between adjacent wires depends on two different factors. As the electric field between two wires remains constant as long as the voltages of the wires do not change, electric power is lost only when a signal transition occurs on one of the wires. The frequency of such a toggle is called the *switching frequency* of a wire w and is modeled as a positive number $\alpha(w)$. If the switches lead to changes between zero voltage and the operating voltage U that is constant throughout the whole part of the chip, the (suitably normalized) number $\alpha(w)$ can be interpreted as the probability of a signal transition on the wire w at any of the given periods of time. For an existing layout, this value can be derived by a simulation; it then represents the actual number of toggles of a wire.

To build up the field, energy is required that is directly proportional to the capacitance value. Half of the energy is dissipated as heat, and the other half is stored in the field. If the field diminishes, the energy stored in the field is dissipated as heat as well. The capacitance between two adjacent wires and thus the energy required for a signal transition is directly proportional to the quotient of the surface area and the distance of the two wires.

We will assume that the physical dimensions of the wires are fixed i.e., only the distances matter. More specifically, the power loss for a wire w depends only on the distances x_{left} and x_{right} to the two neighboring wires on each side, and, up to a constant, can be expressed as

$$\alpha(w) \cdot \left(\frac{1}{x_{\text{left}}} + \frac{1}{x_{\text{right}}} \right),$$

where two neighbors at distances x_{left} and x_{right} are present. (Here we tacitly assume that neighboring wires do not switch simultaneously.) Hence we can specify a wire ordering and spacing by the relative positions of the wires i. e. , by associating with each wire a real number in a given interval $[0, r]$. We assume here that r represents the actual available space i. e. , the given space reduced by the wire widths, so that we can regard a wire as having zero width. Naturally, wires must not be placed too closely together, so a minimum distance d must be enforced.

Let us close this section with some comments on our assumptions. One of our assumptions is that we only consider parallel wires of the same length. While wires within one layer are generally parallel, the assumption of equal length is certainly not satisfied for all such wires. However, local patches where our assumption holds may easily be identified and real-world circuits contain a considerable amount of such areas. Application of our method to these patches may require some extra 'detour wires' (cf. [18] for details on this approach) or the use of a permutation network (see Section 6), but according to recent studies (cf. [7]) both do not add considerable overhead in practice.

In reality there may be special wires that are wider and require a larger minimum distance to be manufacturable than others, for instance for power supply. Often these power supply wires build a ring around the entire core of the chip and are not subject to optimization. In some (larger) circuits (and only on a fraction of the existing layers), power meshes may even be drawn through the chip. They can then be regarded as defining 'natural boundaries' of the relevant problems here. Hence, our assumptions still allow for a wide range of applicability of the results of this paper in practice; results of a practical study will be given in Section 6.

2.2 A Model for Wire Placement

We consider a scenario involving N parallel wires which are regarded as being enclosed between two static wires with switching frequencies 0. On a chip these boundary wires could be power or shield wires.

In the following let $N \in \mathbb{N}$, and let $w_1, \dots, w_N$ denote different *(proper) parallel wires*. Further, let w_0 and w_{N+1} be two additional *'dummy wires'*, and set $W = \{w_1, \dots, w_N\}$ and $\widehat{W} = W \cup \{w_0, w_{N+1}\}$.

Let $r \in]0, \infty[$ be the given *spacing range*, and let $d \in]0, r]$ be the *minimum accepted inter wire distance*. Then, a *wire placement* is a map $\varphi : \widehat{W} \to [0, r]$ such that $\varphi(w_0) = 0$, $\varphi(w_{N+1}) \geq \varphi(w)$ for $w \in W$ and

$$|\varphi(w) - \varphi(w')| \geq d$$

for $w, w' \in \widehat{W}$ with $w \neq w'$. As it turns out, the underlying optimization problem can be described best in terms of the two separate tasks of wire ordering and wire spacing.

A *wire ordering* is a bijection $\pi : \widehat{W} \to \{0, 1, \dots, N, N+1\}$ such that $\pi(w_0) = 0$ and $\pi(w_{N+1}) = N + 1$. Let $\mathcal{P}_N$ denote the set of all wire orderings for a given number N. An admissible *wire spacing* is a function $\delta : \{0, 1, \dots, N, N+1\} \to [0, r]$ with $\delta(0) = 0$ and

$$\delta(j) + d \leq \delta(k)$$

for any $j, k \in \{0, 1, \dots, N+1\}$ with $j < k$. Let $\mathcal{D}_N(r, d)$ denote the set of all admissible wire spacings. Note that the above constraints are already implied by the conditions on all pairs of adjacent positions.

Of course, any pair (π, δ) of a wire ordering and a wire spacing constitutes a wire placement φ via $\varphi = \delta \circ \pi$ and vice versa. Hence we will not distinguish between them and, in particular, also speak of (π, δ) as a wire placement.

Finally, let $\alpha : W \to [0, \infty[$ encode the *switching frequencies* of the proper wires. The set of all such functions will be denoted by $\mathcal{A}_N$.

Then, the *power loss* $L(\pi, \delta)$ of a wire placement (π, δ) is given by

$$L(\pi,\delta) = \sum_{w \in W} \alpha(w) \left(\frac{1}{\delta(\pi(w)) - \delta(\pi(w)-1)} + \frac{1}{\delta(\pi(w)+1) - \delta(\pi(w))} \right),$$

and the *optimal wire placement* problem (OWP) is the following task: Given $N \in \mathbb{N}$, $r, d \in]0, \infty[$, and $\alpha \in \mathcal{A}_N$, find $\pi^* \in \mathcal{P}_N$ and $\delta^* \in \mathcal{D}_N(r,d)$ such that

$$L(\pi^*, \delta^*) = \min \{ L(\pi, \delta) : \pi \in \mathcal{P}_N \ \wedge \ \delta \in \mathcal{D}_N(r,d) \},$$

or decide that no such minimum exists.

Note that the specific set W does not play any role. All that matters are the switching frequencies associated with the wires. Also the function π can be identified with a permutation on $\{1, \ldots, N\}$. The concise mathematical formulation of OWP given in the next section uses this abstraction and describes the task in terms of the variables x_i for $i = 1, \ldots, N+1$, that are related to the functions π and δ through

$$x_{\pi(w)} = \delta(\pi(w)) - \delta(\pi(w)-1).$$

The switching frequencies will be encoded by a vector $(s_1, \ldots, s_N)$ where $s_i = \alpha(w_i)$ for $i = 1, \ldots, N$.

3 The Main Result

Let $\mathcal{S}_N$ denote the *symmetric group* on N elements. We are dealing with the following mathematical optimization problem.

Problem: Optimal Wire Placement (OWP)
Instance: $N \in \mathbb{N}$; $s_1, \ldots, s_N \in [0, \infty[$; $d, r \in]0, \infty[$.
Question: Decide whether there exists a solution (π, x) of

$$\begin{aligned} \min \ & \sum_{i=1}^{N} s_{\pi(i)} \left(\frac{1}{x_i} + \frac{1}{x_{i+1}} \right) \\ \text{s. th.} \ & \sum_{i=1}^{N+1} x_i \le r \\ & x_i \ge d \quad (i = 1, \ldots, N+1) \\ & \pi \in \mathcal{S}_N, \end{aligned}$$

and, if so, give one.

When the permutation π is fixed, we are confronted with an instance of Optimal Wire Spacing (OWS); the input is the same but the objective is just to find optimal wire distances x_i; see Section 4. While Optimal Wire Spacing is a nonlinear programming problem under linear side constraints, the minimization over all permutations makes Optimal Wire Placement a combinatorial optimization problem with nonlinear objective function. The main contribution of this paper is to show that both problems can be solved efficiently. We state our results for the real RAM model.

Theorem 1 *Both problems,* Optimal Wire Spacing *and* Optimal Wire Placement*, can be solved in* $\mathcal{O}(N \log N)$ *time.*

Similar results can be obtained for the binary Turing machine model when the input is restricted to the rationals and the output is computed up to a precision given as part of the input; see the end of Section 4 for some additional remarks.

4 The Wire Spacing Problem

We will now consider the wire spacing problem separately. Hence a permutation π is given and is fixed throughout this section. Without loss of generality we may assume that π is the identity. Hence, an instance consists of $N \in \mathbb{N}$, $s_1, \ldots, s_N \in [0, \infty[$ and $d, r \in]0, \infty[$, and the task is to solve

$$\min \sum_{i=1}^{N} s_i \left(\frac{1}{x_i} + \frac{1}{x_{i+1}} \right) \qquad \text{s. th.} \quad \sum_{i=1}^{N+1} x_i \leq r \quad \text{and} \quad x_1, \ldots, x_{N+1} \geq d.$$

In the following we work with a reformulation which uses $s_0 = s_{N+1} = 0$ and $q_i = s_{i-1} + s_i$ for $i = 1, \ldots, N+1$.

Problem: Optimal Wire Spacing (OWS)
Instance: $N \in \mathbb{N}$; $q_1, \ldots, q_{N+1} \in [0, \infty[$; $d, r \in]0, \infty[$.
Question: Decide whether there exists a solution of

$$\begin{aligned} \min \quad & \sum_{i=1}^{N+1} \frac{q_i}{x_i} \\ \text{s. th.} \quad & \sum_{i=1}^{N+1} x_i \leq r \\ & x_i \geq d \quad (i = 1, \ldots, N+1), \end{aligned}$$

and, if so, give one.

In a given instance of OWS the parameters $q_1, \ldots, q_{N+1}$ do not explicitly rely on $s_1, \ldots, s_{N+1}$ and can hence be ordered without loss of generality. Note, however, that a different order of the switching frequencies leads to a different set of $q_1, \ldots, q_{N+1}$.

Since the feasible region of an instance of OWS is compact, in fact a simplex P, on which the objective function F is continuous, the minimum is indeed attained unless P is empty. But $P = \emptyset$ if and only if $r < (N+1)d$. For $r = (N+1)d$, there exists only the trivial solution with all distances equal to

d. Hence we may always assume that $r > (N+1)d$. Also, we may require (and will do so for convenience) that $q_1, \ldots, q_{N+1} > 0$ rather than $q_1, \ldots, q_{N+1} \geq 0$ since for $q_1 = \cdots = q_{N+1} = 0$, any feasible spacing is optimal, and $q_{i_0} = 0$ for some but not all $i_0 \in \{1, \ldots, N+1\}$ implies $x^*_{i_0} = d$ for each optimal solution x^* of the given instance.

The following lemma characterizes optimal wire spacings.

Lemma 1 *Let $(N, q_1, \ldots, q_{N+1}, r, d)$ be an instance of OWS with $r > (N+1)d$, and $q_1, \ldots, q_{N+1} > 0$. Then the objective function F is strictly convex on the feasible region P, and the minimum of F over P is uniquely determined.*

For $x = (x_1, \ldots, x_{N+1})^T$ define

$$D(x) = \big\{ i \in \{1, \ldots, N+1\} : x_i = d \big\} \quad \text{and} \quad R(x) = \{1, \ldots, N+1\} \setminus D.$$

Then a vector $x^* = (x^*_1, \ldots, x^*_{N+1})^T$ is the optimal solution, if and only if

$$d < x^*_k = \frac{\sqrt{q_k}\big(r - |D(x^*)|d\big)}{\sum_{i \in R(x^*)} \sqrt{q_i}} \leq \frac{\sqrt{q_k}}{\sqrt{q_j}} d \tag{1}$$

for all $j \in D(x^*)$ and $k \in R(x^*)$.

Proof Let u_i denote the ith standard unit vector in $\mathbb{R}^{N+1}$, set $\mathbb{1} = \mathbb{1}_{N+1} = u_1 + \ldots + u_{N+1}$ and denote the objective function of OWS by $F = F(x)$. Of course, F is differentiable on the set P of feasible points, and we have for $i, j = 1, \ldots, N+1$

$$\frac{\partial F}{\partial x_i} = -\frac{q_i}{x_i^2} \quad \wedge \quad \frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} \dfrac{2q_i}{x_i^3} & \text{for } i = j; \\ 0 & \text{else.} \end{cases}$$

Hence, F is strictly convex on P, and its minimum over P is unique. By the Karush-Kuhn-Tucker Theorem, a feasible vector $x^* = (x^*_1, \ldots, x^*_{N+1})^T$ is optimal if and only if there exist non negative Lagrange multipliers $\lambda_0, \lambda_1, \ldots, \lambda_{N+1}$ such that

$$\begin{aligned} \frac{q_i}{(x^*_i)^2} = -\nabla F(x^*)^T u_i = \lambda_0 - \lambda_i & \qquad (i = 1, \ldots, N+1) \\ \lambda_0 \left(r - \mathbb{1}^T x^*\right) = \quad 0 & \\ \lambda_i \left(x^*_i - d\right) = \quad 0 & \qquad (i = 1, \ldots, N+1). \end{aligned}$$

Let x^* be the optimal solution. Since all q_i's are positive, we have $\lambda_0 > 0$, and hence

$$\sum_{i=1}^{N+1} x^*_i = r.$$

As a consequence, $r > (N+1)d$ implies that $R(x^*) \neq \emptyset$.

Now, let $i, k \in R(x^*)$. Then $x^*_i, x^*_k > d$, hence $\lambda_i = \lambda_k = 0$ and therefore

$$0 = \lambda_i = \lambda_0 - \frac{q_i}{(x^*_i)^2} = \lambda_0 - \frac{q_k}{(x^*_k)^2} = \lambda_k, \quad \text{thus} \quad (x^*_i)^2 = \frac{q_i}{q_k} (x^*_k)^2,$$

which yields

$$r = \sum_{i=1}^{N+1} x_i^* = \sum_{i \in D(x^*)} x_i^* + \sum_{i \in R(x^*)} x_i^* = |D(x^*)|d + \frac{x_k^*}{\sqrt{q_k}} \sum_{i \in R(x^*)} \sqrt{q_i}\,,$$

proving the first part of (1).

On the other hand, for $j \in D(x^*)$ and $k \in R(x^*)$ we get $\lambda_k = 0$ and $\lambda_j \geq 0$, which yields

$$0 = \lambda_k = \lambda_0 - \frac{q_k}{\left(x_k^*\right)^2} \leq \lambda_j = \lambda_0 - \frac{q_j}{\left(x_j^*\right)^2}, \quad \text{thus} \quad \frac{q_j}{(x_j^*)^2} = \frac{q_j}{d^2} \leq \frac{q_k}{(x_k^*)^2},$$

completing the 'only if' part of the proof.

Now, let x^* satisfy (1). Then x^* is feasible since $x_i^* \geq d$ for all $i \in D(x^*) \cup R(x^*)$ and

$$\sum_{i=1}^{N+1} x_i^* = |D(x^*)|d + \sum_{j \in R(x^*)} \sqrt{q_j} \left(\sum_{i \in R(x^*)} \sqrt{q_i} \right)^{-1} (r - |D(x^*)|d) = r.$$

Further, we have

$$\frac{q_j}{\left(x_j^*\right)^2} = \frac{q_j}{d^2} \leq \frac{q_k}{\left(x_k^*\right)^2} \qquad \text{for all } j \in D(x^*) \text{ and } k \in R(x^*),$$

and

$$\frac{q_j}{\left(x_j^*\right)^2} = \frac{q_k}{\left(x_k^*\right)^2} \qquad \text{for all } j, k \in R(x^*).$$

Denoting this latter constant by λ_0, and setting

$$\lambda_i = \lambda_0 - \frac{q_i}{\left(x_i^*\right)^2} \qquad \text{for } i = 1, \ldots, N+1,$$

we see that $\lambda_0, \ldots, \lambda_{N+1}$ are non negative and satisfy the Karush-Kuhn-Tucker conditions for x^*, hence x^* is optimal.

Note that for sufficiently large r the minimum distance constraint is not binding i. e. ,

$$d < \frac{r \cdot \sqrt{q_k}}{\sum_{i=1}^{N+1} \sqrt{q_i}}$$

for all $k \in \{1, \ldots, N+1\}$, and the problem is solved completely. In general, Lemma 1 at least reduces the given instance of OWS to the determination of the *minimum distance set* D or, equivalently, its complement R. Of course, we may assume that $q_1 \leq \ldots \leq q_{N+1}$; then $x_1^* \leq \ldots \leq x_{N+1}^*$ for the optimal solution x^*, hence all that matters is the cardinality of D.

Therefore, to solve a given instance of OWS we first order the q_i's so that $q_1 \leq \ldots \leq q_{N+1}$. In the first step of the algorithm we begin with $D = \emptyset$ and compute x_1^* according to (1). If $x_1^* > d$, we compute the other components of the solution and terminate. Otherwise we replace x_1^* by d, augment D by $\{1\}$, compute x_2^*, etc. Algorithm 1 presents a structured form, formulated in real arithmetic.

Algorithm 1 Solving the Wire Spacing Problem

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Input: An instance $(N, q_1, \dots, q_{N+1}, r, d)$ of OWP with $r > (N+1)d$ and
  $0 < q_1, \dots, q_{N+1}$.
Output: An optimal solution $x^*$.
Sort $(q_1, \dots, q_{N+1})$ to obtain $(q'_1, \dots, q'_{N+1})$ with $q'_1 \leq \dots \leq q'_{N+1}$.
$S \leftarrow \sum_{i=1}^{N+1} \sqrt{q'_i}$ and $\Delta \leftarrow 0$.
for $i = 1, \dots N+1$ do
    $x'_i \leftarrow \sqrt{q'_i} \cdot S^{-1}\,(r - \Delta \cdot d)$
    if $x'_i \leq d$ then
        $x'_i \leftarrow d$, $S \leftarrow S - \sqrt{q'_i}$, $\Delta \leftarrow \Delta + 1$
    end if
end for
Permute back $x'$ to obtain $x^*$, so that $x^*$ corresponds to the original order of
  $(q_1, \dots, q_{N+1})$.
```

Theorem 2 *Algorithm 1 is correct and requires at most $\mathcal{O}(N \log N)$ arithmetic operations in the real RAM model of computation.*

Proof For ease of notation we assume that $q'_1 = q_1, \dots, q'_{N+1} = q_{N+1}$ i. e. , the $q_1, \dots, q_{N+1}$ are already sorted and thus $x' = x^*$ in the algorithm. Let $S^{(i)}$ and $\Delta^{(i)}$ denote the values of S and Δ after the i-th pass through the 'for' loop. Denote by x^* the solution produced by the algorithm and let $S^* = S^{(N+1)}$, $\Delta^* = \Delta^{(N+1)}$ be the final values of S and Δ, respectively. We show that condition (1) of Lemma 1 holds for all $j \in D(x^*)$ and $k \in R(x^*)$.

The first inequality of (1) is clear. Let $m = \max\{l : l \in D(x^*)\}$, then $S^* = S^{(m)}$ and $\Delta^* = \Delta^{(m)} = m$. Monotonicity of $q_1, \dots, q_{N+1}$ implies that $x^*_1 \leq \dots \leq x^*_{N+1}$, thereby establishing the equality part of (1) for all $k \in R(x^*)$. To prove the second inequality of (1), first note that

$$x^*_j = d, \quad x^*_k = \frac{\sqrt{q_k}}{S^*}\,(r - \Delta^* \cdot d)\,.$$

Then the fact that $S^* = S^{(m-1)} - \sqrt{q_m}$ and $\Delta^* = (m-1) + 1 = \Delta^{(m-1)} + 1$ implies

$$\begin{aligned} x^*_k &= \frac{\sqrt{q_k}}{\sqrt{q_j}} \frac{\sqrt{q_j}}{S^*}(r - \Delta^* \cdot d) \leq \frac{\sqrt{q_k}}{\sqrt{q_j}} \frac{\sqrt{q_m}(r - \Delta^* \cdot d)}{S^*} \\ &= \frac{\sqrt{q_k}}{\sqrt{q_j}} \left(\frac{\sqrt{q_m}(r - \Delta^{(m-1)} \cdot d)}{S^{(m-1)} - \sqrt{q_m}} - \frac{d\sqrt{q_m}}{S^{(m-1)} - \sqrt{q_m}} \right) \\ &= \frac{\sqrt{q_k}}{\sqrt{q_j}} \left(\frac{\sqrt{q_m}(r - \Delta^{(m-1)} \cdot d)}{S^{(m-1)}} \cdot \frac{S^{(m-1)}}{S^{(m-1)} - \sqrt{q_m}} - \frac{d\sqrt{q_m}}{S^{(m-1)} - \sqrt{q_m}} \right). \end{aligned}$$

Since

$$\frac{\sqrt{q_m}(r - \Delta^{(m-1)} \cdot d)}{S^{(m-1)}} \leq d$$

we conclude

$$x^*_k \leq \frac{\sqrt{q_k}}{\sqrt{q_j}} d \left(\frac{S^{(m-1)}}{S^{(m-1)} - \sqrt{q_m}} - \frac{\sqrt{q_m}}{S^{(m-1)} - \sqrt{q_m}} \right) = \frac{\sqrt{q_k}}{\sqrt{q_j}} d.$$

Hence, by Lemma 1, x^* is the optimal solution.

The sorting step requires at most $\mathcal{O}(N \log N)$ arithmetic operations. The 'for' loop is executed $N+1$ times and each passage requires a constant number of arithmetic operations. Therefore the algorithm requires at most $\mathcal{O}(N \log N)$ arithmetic operations.

Note that it is indeed necessary to compute the optimal values for x_i one after the other. This can be illustrated by the following small example.

Example 1 Let $q_1 = 1$, $q_2 = \cdots = q_N = 4$ and $q_{N+1} = \alpha^2$. If we compute all x_i-values under the assumption of $D = \emptyset$, we obtain

$$x_1 = \frac{r}{2N-1+\alpha}, \quad x_2 = \cdots = x_N = \frac{2r}{2N-1+\alpha}, \quad x_{N+1} = \frac{\alpha r}{2N-1+\alpha}.$$

Suppose now that d takes some value strictly between x_1 and x_2 i. e. ,

$$x_1 < d < x_2, \ldots, x_N.$$

Since this solution is not feasible, we set $x_1' = d$. Recomputation of the values for the remaining indices yields

$$x_1' = d, \quad x_2' = \cdots = x_N' = \frac{2(r-d)}{2N-2+\alpha}, \quad x_{N+1}' = \frac{\alpha(r-d)}{2N-2+\alpha}.$$

A simple calculation shows that for suitably chosen values of N, r and α the inequality

$$\frac{2(r-d)}{2N-2+\alpha} < \frac{2r}{2N-1+\alpha}$$

holds, so for d strictly in between theses values we have $x_2', \ldots, x_N' < d$.

Thus minimum distance indices cannot be determined in a 'one shot' manner, but one has to proceed iteratively instead.

Let us close this section by pointing out that adaptations are needed for working on rational input in the binary Turing machine model. This is due to the fact that, by Lemma 1, the optimal solutions x^* may be irrational, and the same is true for the optimum value

$$\sum_{i \in D(x^*)} \frac{q_i}{d} + \frac{1}{r - |D(x^*)|d} \left(\sum_{i \in R(x^*)} \sqrt{q_i} \right)^2 \tag{2}$$

of the objective function. It is, however, standard fare to replace the square root computation in Algorithm 1 by a suitable approximation in order to solve the problem to a given finite precision (that is regarded as part of the input).

5 The Wire Ordering Problem

As Theorem 2 shows, an optimal wire spacing can be found very efficiently. Allowing a reordering of the wires adds an additional potential for optimization, but also additional mathematical difficulties. Let $\pi \in \mathcal{S}_N$ be the permutation that assigns the switching frequency $s_{\pi(i)}$ to the ith position on the chip. Then by (2), the optimum of the objective function for this permutation is

$$F(\pi, x^\pi) = \sum_{i \in D^\pi} \frac{s_{\pi(i-1)} + s_{\pi(i)}}{d} + \frac{1}{r - |D^\pi| d} \left(\sum_{i \in R^\pi} \sqrt{s_{\pi(i-1)} + s_{\pi(i)}} \right)^2, \quad (3)$$

where x^π is the optimal wire spacing, and $D^\pi = D(x^\pi)$, and $R^\pi = R(x^\pi)$ according to Lemma 1. Here again, $s_0 = s_{N+1} = 0$ and $\pi(0) = 0$, $\pi(N+1) = N+1$ for a uniform notation, and we will identify $\mathcal{S}_N$ with the set of all permutations on $\{0, \ldots, N+1\}$ with fixed points 0 and $N+1$. Now, of course, we want to optimize over all such permutations π. Suppose for a moment, r was large enough to imply $D^\pi = \emptyset$ for each $\pi \in \mathcal{S}_N$. Then, in effect, we are asking for a permutation that minimizes

$$\sum_{i=1}^{N+1} \sqrt{s_{\pi(i-1)} + s_{\pi(i)}}. \quad (4)$$

Let $\omega_0, \omega_1, \ldots, \omega_{N+1}$ denote the switching activities, sorted in increasing order i. e. , $\omega_0 = s_0$, $\omega_1 = s_{N+1}$, $\{\omega_2, \ldots, \omega_{N+1}\} = \{s_1, \ldots s_N\}$ and $\omega_0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_{N+1}$. Then the wire ordering problem for $D^\pi = \emptyset$ becomes that of finding a Minimum Hamilton Path (MHP) in the complete graph on the vertices $\{0, \ldots, N+1\}$ with endpoints 0 and 1 whose edges $\{j, k\}$ carry the weights $\sqrt{\omega_j + \omega_k}$. While the general MHP problem is notoriously difficult, it turns out that this particular problem belongs to a class of easily solvable cases. In particular, for $i, j, k, l \in V$ with $i < k$ and $j < l$,

$$\sqrt{\omega_i + \omega_j} + \sqrt{\omega_k + \omega_l} \leq \sqrt{\omega_i + \omega_l} + \sqrt{\omega_j + \omega_k}, \quad (5)$$

an inequality known as the *Monge property* in the context of Traveling Salesman Problems. Clearly, (5) implies the *Supnick property* i. e. ,

$$\sqrt{\omega_i + \omega_j} + \sqrt{\omega_k + \omega_l} \leq \sqrt{\omega_i + \omega_k} + \sqrt{\omega_j + \omega_l} \leq \sqrt{\omega_i + \omega_l} + \sqrt{\omega_j + \omega_k}$$

for all $i, j, k, l \in V$ with $i < j < k < l$; see [1] and [3] for other results on these and related properties. Hence, by [13], the permutation τ_N defined by

$$\tau_N(i) = \begin{cases} 2i & \text{for } 0 \leq i \leq \frac{N+1}{2}, \\ 2(N-i)+3 & \text{for } \frac{N}{2}+1 \leq i \leq N+1 \end{cases} \quad (6)$$

always gives an optimal TSP-tour. We refer to [3] and [5] for a comprehensive survey on this and other easily solvable cases of the TSP and to [16] for an additional account of classes of TSPs solved by the same permutation τ_N; [2] studies related questions for the quadratic assignment problem.

Noting that our instances of MHP with end nodes 0 and 1 of minimum weight are equivalent to the corresponding instances of TSP, as there always exists an optimal TSP tour that contains the edge between 0 and 1, it follows that τ_N solves our given instance of MHP i.e., the specific pyramidal permutation

$$(0, 2, 4, \ldots, 5, 3, 1)$$

is optimal.

For the optimal wire placement problem, however, D is not empty, not known, and there are exponentially many possible such sets. But, still worse, even if we could check each such set one by one, an optimal permutation could not be determined with the above results. In fact, the general objective $F(\pi, x^\pi)$ is not a sum of edge weights i.e., it does not have the structure of an admissible MHP or TSP objective function, let alone the Monge property. Actually, as the quadratic term indicates, the objective function value does not only depend on edge weights, but also on the selected Hamiltonian path as a whole. Hence it is not directly clear at all how to optimize (3).

In the present section we derive some structural results to be able to handle the general wire ordering problem. We will show that there is always an optimal wire placement where π is *evenly separated*; see Definition 3. For such permutations the set D^π is already determined by its cardinality. This combinatorial property will enable us to derive the final $\mathcal{O}(N \log N)$ algorithm for the optimal wire placement problem.

The following proofs utilize certain exchange techniques that will be introduced now.

Definition 1 Let (π, x) be a feasible wire placement and let $j, k \in \{1, \ldots, N\}$ with $j < k$. Define $S_{jk}(\pi), \overline{S}_{jk}(\pi) \in \mathcal{S}_N$ and $T_{jk}(x), \overline{T}_{jk}(x) \in \mathbb{R}^{N+1}$ by

$$S_{jk}(\pi)(i) = \begin{cases} \pi(k+j-i) & \text{for } j < i < k \\ \pi(i) & \text{else,} \end{cases}$$

$$\overline{S}_{jk}(\pi)(i) = \begin{cases} \pi(k+j-i) & \text{for } j \le i \le k \\ \pi(i) & \text{else,} \end{cases}$$

$$(T_{jk}(x))_i = (\overline{T}_{jk}(x))_i = \begin{cases} x_{k+j-i+1} & \text{for } j < i \le k \\ x_i & \text{else,} \end{cases}$$

and set

$$\mathscr{R}_{jk}(\pi, x) = \big(S_{jk}(\pi), T_{jk}(x)\big), \qquad \overline{\mathscr{R}}_{jk}(\pi, x) = \big(\overline{S}_{jk}(\pi), \overline{T}_{jk}(x)\big).$$

Then $\mathscr{R}_{jk}(\pi, x)$ is called the *open j-k reversal* while $\overline{\mathscr{R}}_{jk}(\pi, x)$ is the *closed j–k reversal* of (π, x).

Lemma 2 *Let (π, x) be a feasible wire placement and let $j, k \in \{1, \ldots, N\}$ with $j < k$. Then both, $\mathscr{R}_{jk}(\pi, x)$ and $\overline{\mathscr{R}}_{jk}(\pi, x)$, are feasible wire placements,* *and their objective values are*

$$F\big(\mathscr{R}_{jk}(\pi,x)\big) = F(\pi,x) + \big(s_{\pi(j)} - s_{\pi(k)}\big)\left(\frac{1}{x_k} - \frac{1}{x_{j+1}}\right),$$
$$F\big(\overline{\mathscr{R}}_{jk}(\pi,x)\big) = F(\pi,x) + \big(s_{\pi(j)} - s_{\pi(k)}\big)\left(\frac{1}{x_{k+1}} - \frac{1}{x_j}\right).$$

Proof Feasibility of both open and closed j-k reversal is clear as the distance vector is just permuted, so the overall sum stays constant and no distance can fall below d. The new objective values are

$$\begin{aligned} F\big(\mathscr{R}_{jk}(\pi,x)\big) &= F(\pi,x) - \frac{s_{\pi(j)} + s_{\pi(j+1)}}{x_{j+1}} - \frac{s_{\pi(k-1)} + s_{\pi(k)}}{x_k} \\ &\quad + \frac{s_{\pi(j)} + s_{\pi(k-1)}}{x_k} + \frac{s_{\pi(j+1)} + s_{\pi(k)}}{x_{j+1}} \\ &= F(\pi,x) + \big(s_{\pi(j)} - s_{\pi(k)}\big)\left(\frac{1}{x_k} - \frac{1}{x_{j+1}}\right) \\ \text{and} \quad F\big(\overline{\mathscr{R}}_{jk}(\pi,x)\big) &= F(\pi,x) - \frac{s_{\pi(j-1)} + s_{\pi(j)}}{x_j} - \frac{s_{\pi(k)} + s_{\pi(k+1)}}{x_{k+1}} \\ &\quad + \frac{s_{\pi(j-1)} + s_{\pi(k)}}{x_j} + \frac{s_{\pi(j)} + s_{\pi(k+1)}}{x_{k+1}} \\ &= F(\pi,x) + \big(s_{\pi(j)} - s_{\pi(k)}\big)\left(\frac{1}{x_{k+1}} - \frac{1}{x_j}\right), \end{aligned}$$

respectively, completing the proof.

Definition 2 Let (π, x) be a feasible wire placement, and set

$$l(x) = \min\{i \in \{0,\dots,N\} : x_{i+1} > d\},$$
$$u(x) = \max\{i \in \{1,\dots,N+1\} : x_i > d\}.$$

Then $l(x)$, $u(x)$ are called *lower* and *upper separation point*, respectively. For $x = x^\pi$ we use the abbreviations $l^\pi = l(x^\pi)$ and $u^\pi = u(x^\pi)$ (or, when there is no risk of confusion, simply l and u, respectively). An optimal wire placement (π, x) is called *separated*, if it has the following properties:

1. $x_{l+1}, x_{l+2}, \dots, x_u > d$;
2. $\max\left\{s_{\pi(0)}, \dots, s_{\pi(l-1)}, s_{\pi(u+1)}, \dots, s_{\pi(N+1)}\right\} \le \min\left\{s_{\pi(l)}, s_{\pi(u)}\right\}$;
3. $\max\left\{s_{\pi(l)}, s_{\pi(u)}\right\} \le \min\left\{s_{\pi(l+1)}, \dots, s_{\pi(u-1)}\right\}$.

Lemma 3 *For any feasible instance of the optimal wire placement problem there is an optimal solution (π, x^π) that is separated.*

Proof Suppose there is no optimal solution with Property 1. Let (π, x^π) be an optimal solution for which $l = l^\pi$ is maximal. Then there is some $k \in \{1,\dots,N\}$ with $l+1 < k < u$ such that $x_k^\pi = d$, and we choose the maximal such k, i.e. , $x_{k+1}^\pi, \dots, x_u^\pi > d$. Suppose, first, that $s_{\pi(l)} > s_{\pi(k)}$

(note that this implies $s_{\pi(l)} > 0$ and thus $l > 0$). Then, by Lemma 2, the closed l-k reversal $\overline{\mathscr{R}}_{lk}(\pi, x^\pi)$ is feasible and has objective value

$$F\big(\overline{\mathscr{R}}_{lk}(\pi, x^\pi)\big) = F(\pi, x^\pi) + \underbrace{\big(s_{\pi(l)} - s_{\pi(k)}\big)}_{>0} \underbrace{\left(\frac{1}{x^\pi_{k+1}} - \frac{1}{x^\pi_l}\right)}_{<0} < F(\pi, x^\pi),$$

a contradiction to the optimality of (π, x^π). If, on the other hand, $s_{\pi(l)} \leq s_{\pi(k)}$, then

$$F\big(\mathscr{R}_{lk}(\pi, x^\pi)\big) = F(\pi, x^\pi) + \underbrace{\big(s_{\pi(l)} - s_{\pi(k)}\big)}_{\leq 0} \underbrace{\left(\frac{1}{x^\pi_k} - \frac{1}{x^\pi_{l+1}}\right)}_{>0} \leq F(\pi, x^\pi).$$

Hence the open l-k reversal $\mathscr{R}_{lk}(\pi, x^\pi)$ is an optimal solution, however, with a strictly greater lower separation point. This contradiction shows that there is an optimal solution with Property 1.

In the following, let (π, x^π) be an optimal wire placement with Property 1, and let l and u be its lower and upper separation points. We prove by contradiction that (π, x^π) has Properties 2 and 3. So, suppose there was some $m \in \{1, \dots, l-1\} \cup \{u+1, \dots, N\}$ such that $s_{\pi(m)} > \min\big\{s_{\pi(l)}, s_{\pi(u)}\big\}$. By reversing the wire placement if necessary, we may assume that $m \in \{1, \dots, l-1\}$.

Let us first consider the case $u = N + 1$; then $s_{\pi(m)} > s_{\pi(u)} = 0$ and application of an open m-u reversal yields

$$F\big(\mathscr{R}_{m,u}(\pi, x^\pi)\big) = F(\pi, x^\pi) + \underbrace{\big(s_{\pi(m)} - s_{\pi(u)}\big)}_{>0} \underbrace{\left(\frac{1}{x_u} - \frac{1}{x_{m+1}}\right)}_{<0} < F(\pi, x^\pi)$$

contradicting optimality of (π, x^π). For $u \neq N + 1$, if $s_{\pi(l)} > s_{\pi(u)}$, we apply a closed l-u-reversal. Since $x_l = x_{u+1} = d$, we get

$$F(\overline{\mathscr{R}}_{l,u}(\pi, x^\pi)) = F(\pi, x^\pi)$$

and may hence assume that $s_{\pi(m)} > s_{\pi(l)}$. But then, since $x^\pi_m = d < x^\pi_{l+1}$, we have for the closed m-l reversal

$$F(\overline{\mathscr{R}}_{ml}(\pi, x^\pi)) = F(\pi, x^\pi) + \underbrace{\big(s_{\pi(m)} - s_{\pi(l)}\big)}_{>0} \cdot \underbrace{\left(\frac{1}{x^\pi_{l+1}} - \frac{1}{x^\pi_m}\right)}_{<0} < F(\pi, x^\pi),$$

contradicting optimality of (π, x^π).

For Property 3, suppose that there was some $m \in \{l+1, \dots, u-1\}$ such that $s_{\pi(m)} < \max\big\{s_{\pi(l)}, s_{\pi(u)}\big\}$. We may again assume without loss of generality that $s_{\pi(l)} \leq s_{\pi(u)}$. Note that $\pi(N+1) = N+1$ and $s_{N+1} = 0$, hence $u \leq N$. Further, $x^\pi_{u+1} = d < x^\pi_m$ by Property 1, thus

$$F\big(\overline{\mathscr{R}}_{mu}(\pi, x^\pi)\big) = F(\pi, x^\pi) + \underbrace{\big(s_{\pi(m)} - s_{\pi(u)}\big)}_{<0} \cdot \underbrace{\left(\frac{1}{x^\pi_{u+1}} - \frac{1}{x^\pi_m}\right)}_{>0} < F(\pi, x^\pi),$$

again a contradiction. This completes the proof.

By Lemma 3 we can restrict our search for an optimal wire placement to separated solutions. Since the first part of the objective function $F(\pi, x^\pi)$ is just $1/d$ times $\sum_{i\in D^\pi}\bigl(s_{\pi(i-1)}+s_{\pi(i)}\bigr)$, we can 'normalize' the set D^π even further. We will do this in a way that is most suitable for our purposes. For simplicity of exposition we will assume from now on that

$$0 = s_{N+1} = s_0 \le s_1 \le \cdots \le s_N.$$

Definition 3 Let (π, x) be a separated wire placement with separation points l and u, and let $\Delta = |D(x)|$. Then (π, x) is called *evenly separated*, if the following conditions hold:

1. $D(x) = \bigl\{1, \ldots, \lfloor\frac{\Delta}{2}\rfloor\bigr\} \cup \bigl\{N+2-\lceil\frac{\Delta}{2}\rceil, \ldots, N+1\bigr\}$;
2. $l = \lfloor\frac{\Delta}{2}\rfloor$ and $\pi(l) = 2\lfloor\frac{\Delta}{2}\rfloor$;
3. $u = N+1-\lceil\frac{\Delta}{2}\rceil$ and $\pi(u) = 2\lceil\frac{\Delta}{2}\rceil - 1$;
4. $\bigl\{\pi(i) : i \in \{0, \ldots, l\}\bigr\} = \bigl\{0, 2, \ldots, 2\lfloor\frac{\Delta}{2}\rfloor\bigr\}$;
5. $\bigl\{\pi(i) : i \in \{u, \ldots, N+1\}\bigr\} = \bigl\{1, 3, \ldots, 2\lceil\frac{\Delta}{2}\rceil - 1\bigr\} \cup \{N+1\}$.

Note that these conditions, in particular the last two requirements, aim at 'compatibility' of the set $D(x)$ with the Minimum Hamiltonian Path τ_N introduced in (6).

Theorem 3 *Each feasible instance of OWP admits an optimal solution that is evenly separated.*

Proof Let (π, x^π) be an optimal wire placement and set $\Delta = |D^\pi|$. By Lemma 3, we can assume that (π, x^π) is separated. Hence by Definition 2, Properties 2 and 3, we may further assume that

$$\bigl\{\pi(l), \pi(u)\bigr\} = \left\{2\left\lfloor\frac{\Delta}{2}\right\rfloor, 2\left\lceil\frac{\Delta}{2}\right\rceil - 1\right\} = \{\Delta - 1, \Delta\}.$$

(Here we use that $0 = s_{N+1} = s_0 \le s_1 \le \cdots \le s_N$.) By reversing the wire placement if necessary we obtain $\pi(l) = 2\lfloor\frac{\Delta}{2}\rfloor$. Similarly, we may assume

$$\bigl\{\pi(i) : i \in \{0, \ldots, l\} \cup \{u, \ldots, N+1\}\bigr\} = \{0, 1, \ldots, \Delta\} \cup \{N+1\}.$$

The objective function then evaluates to

$$F(\pi, x^\pi) = \frac{2}{d}\sum_{i=1}^{\Delta-2} s_i + \frac{s_{\Delta-1}+s_\Delta}{d} + \frac{1}{r - d\Delta}\left(\sum_{i=l^\pi+1}^{u^\pi}\sqrt{s_{\pi(i-1)}+s_{\pi(i)}}\right)^2. \quad (7)$$

Clearly, the positions $\{0, \ldots, l-1\} \cup \{u+1, \ldots, N+1\}$ can be reordered freely without changing the objective function value as long as they are placed before $s_{2\lfloor\Delta/2\rfloor}$ or after $s_{2\lceil\Delta/2\rceil-1}$. Hence there exists an optimal solution that is evenly separated.

We have now reached the following principle procedure for finding optimal wire placements. If $r < (N+1)d$, the problem is infeasible. Otherwise, we compute for each $\Delta \in \{0, \dots, N\}$ a set D of cardinality Δ according to Theorem 3. Then, we are in effect confronted with the Minimum Hamilton Path problem on the node set $V(\Delta) = \{\Delta - 1, \dots, N\}$ with edge weights $\sqrt{s_i + s_j}$ for $i, j \in V(\Delta)$ and $i \neq j$ and endpoints $\Delta - 1$ and Δ that is obtained by ignoring the square in the second component of (3). We solve this MHP problem by utilizing the underlying Supnick property.

The two parts can, however, be closely interwoven. In fact, the procedure for computing an optimal order for the wires between lower and upper separation point is in full accordance with the property of a wire placement to be evenly separated; cf. Definition 3. Hence when the wires at minimum distance (determined by Δ) are added, the optimal tour τ_N coincides with the tour produced by adding to both sides of $\tau_{N-\Delta}$ the remaining wires at minimum distance in a way that yields an evenly separated wire placement. This means we obtain the same wire ordering for every Δ. So, rather than actually going through all different values for Δ we can just compute an optimal wire ordering as if there was no minimum distance requirement (or, if one prefers, for sufficiently large r) and subsequently determine the optimal distances (and the correct set of minimum distance wires) along the lines of Algorithm 1 to obtain an optimal wire placement. The complete procedure is formalized in Algorithm 2.

Algorithm 2 Construction of an optimal wire placement.

```
Input: An instance of OWP i. e. ,
    $N \in \mathbb{N}$, $s_1, \dots, s_N \in [0, \infty[$ and $d, r \in ]0, \infty[$ with $r > (N+1)d$.
Output: An optimal wire placement $(\pi, x^\pi)$.
Set $s_0 = s_{N+1} = 0$ and sort $s_1, \dots, s_N$ in increasing order, i. e. , let $\rho \in \mathcal{S}_N$ be
    a permutation such that $0 = s_{\rho(0)} = s_{\rho(1)} \leq s_{\rho(2)} \cdots \leq s_{\rho(N+1)}$.
Define the permutation $\pi \in \mathcal{S}_N$ by setting
    $\pi(i) = \begin{cases} 2\rho(i) & \text{for } 0 \leq i \leq \frac{N+1}{2}, \\ 2\big(N - \rho(i)\big) + 3 & \text{for } \frac{N}{2} + 1 \leq i \leq N + 1. \end{cases}$
Set $q_i = s_{\pi(i-1)} + s_{\pi(i)}$ for $i = 1, \dots, N+1$.
Compute optimal distances $x^\pi$ for the permutation $\pi$ and input parameters
    $(N, q_1, \dots, q_{N+1}, r, d)$ using Algorithm 1.
```

Theorem 4 *OWP can be solved in time $\mathcal{O}(N \log N)$.*

Proof The correctness and optimality of the solution (π, x^π) produced by Algorithm 2 is a direct consequence of Theorems 2 and 3, and optimality of the tour τ_N.

The sorting step can, of course, be implemented using no more than $\mathcal{O}(N \log N)$ arithmetic operations, π can subsequently be computed in $\mathcal{O}(N)$ steps and Algorithm 1 needs at most $\mathcal{O}(N \log N)$ arithmetic operations in the real RAM model.

6 Final Remarks

In this final section, let us briefly comment on the electrotechnical significance of results derived in this paper.

In comparison with measurements performed in experiments and simulations, the mathematical model given and justified in Section 2 turned out to be very realistic when dealing with parallel wires. Naturally, in order to fully exploit the potential of wire placement one would need to fully integrate spacing and ordering into the complete logical and physical design and layout process.

However, even post optimization already leads to significant power reduction. In fact, optimal spacing within local patches of a large real-world semiconductor design (identified by a simple search procedure) lead to an overall reduction in power consumption of 3 – 5%. The effect was even greater when optimal wire spacing was applied to a broad range of benchmark circuits produced by state-of-the-art commercial layout tools as a post layout optimization step. This way, the potential of existing routing tools was utilized before the results of this paper were exploited for further optimization. See the extended abstract [19] for some experimental results and [18] for a comprehensive study.

While wire spacing can easily be implemented as post-processing part of the design process (possibly with the aid of some 'detours' to connect differently spaced parts of wire), this is not so obvious for wire ordering. One possible solution is to use small permutation networks in order to connect re-ordered parts of a design to the remaining wires. The effect of such permutation networks has been investigated in [7]; it is concluded that in realistic circuits the overhead in terms of area and power is much more than compensated by the benefit of wire ordering. A different approach exploits the fact that today's circuit designs are mostly IP-based (i.e., 'Intellectual Property' based), hence pre-developed modules are frequently re-used and combined for new designs. The ordering of such an IP module's pads is generally fixed arbitrarily and is not subject to optimization during the design process of the chip. Thus the designers of the IP modules may apply optimal wire ordering as proposed in this paper to the initial design of the modules. This methodology is all the more important as switching activities are usually module specific.

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