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UNIQUENESS IN DISCRETE TOMOGRAPHY: THREE REMARKS AND A COROLLARY*

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Abstract. Discrete tomography is concerned with the retrieval of finite point sets in some $\mathbb{R}^d$ from their X-rays in a given number m of directions $u_1, \ldots, u_m$. In the present paper we focus on uniqueness issues. The first remark gives a uniform treatment and extension of known uniqueness results. In particular, we introduce the concept of J-additivity and give conditions when a subset J of possible positions is already determined by the given data. As a by-product, we settle a conjecture of Brunetti and Daurat on planar lattice convex sets. Remark 2 resolves a problem of Kuba posed in 1997 on the uniqueness in the case $d = m = 3$ with u_1, u_2, u_3 being the standard unit vectors. Remark 3 determines the computational complexity of finding a smallest set J of positions whose disclosure yields uniqueness. As a corollary, we obtain a hardness result for 0-1-polytopes.

Key words. discrete tomography, uniqueness, additivity, computational complexity, polytopes

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1. Introduction and main results. Discrete inverse problems play an important role in combinatorics and geometry. A particularly important and well-studied case is that of *discrete tomography*; see [16], [18], [15], [19] for surveys and [28], [26], [9], [20], [10] for various applications in and connections to areas like general image processing, graph theory, scheduling, statistical data security, game theory, etc.

The present paper reviews and settles various uniqueness issues for discrete inverse problems. While our results are proved in more generality later, we will introduce the problems in this introductory section solely in the language of discrete tomography (with line X-rays) to provide a most intuitive exposition.

Let $d \in \mathbb{N} \setminus \{1\}$, $u \in \mathbb{R}^d \setminus \{0\}$, and let $S := \mathrm{lin}\{u\}$ be the linear span of $\{u\}$. Further, let $\mathcal{F}^d$ denote the set of finite subsets of $\mathbb{R}^d$ and let $F \in \mathcal{F}^d$. Then the function $X_S(F) : \mathbb{R}^d \to \mathbb{N}_0$, defined by

$$X_S(F)(x) := |F \cap (x + S)|,$$

is called the *X-ray* of F in direction u (or, parallel to S).

Now, let $m \in \mathbb{N} \setminus \{1\}$, and for $i = 1, \ldots, m$, let $u_i \in \mathbb{R}^d \setminus \{0\}$, $S_i := \mathrm{lin}\{u_i\}$, and $X_i(F) := X_{S_i}(F)$. Let $F' \in \mathcal{F}^d$. Then F' is called *tomographically equivalent* to F if $X_i(F) = X_i(F')$ for all $i = 1, \ldots, m$. We say that F is *uniquely determined* by its X-rays $X_1(F), \ldots, X_m(F)$ if there does not exist a set F' different from but tomographically equivalent to F. We are interested in the question of just when F is uniquely determined by its X-rays $X_1(F), \ldots, X_m(F)$. We will deal with this problem in various ways.

First, we study *invariant sets*, i.e., sets each point of which either belongs to all solutions or does not belong to any solution. In particular, we extend the notion of

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additivity to that of J-additivity (for a definition see section 2) and give a unified (and more general) treatment of known concepts and results including those of Fishburn et al. [9] and Aharoni, Herman, and Kuba [1]. As it turns out, the seemingly "complex" concept of J-additivity boils down to the simple question of whether an associated polytope P is contained in a certain affine subspace. Therefore, in particular, J-additivity can be checked efficiently. As a by-product we disprove a conjecture of [6] about lattice convex sets and additivity.

Second, we answer in the negative a question of Kuba, who suggested that every subset F of $\mathbb{Z}^3$ might be uniquely determined by its X-rays in the three standard unit directions of $\mathbb{R}^3$ if and only if F is additive.

The third part deals with the computational complexity of enforcing uniqueness by prescribing or forbidding certain subsets of points.

Finally, we will draw some conclusions for a problem from computational convexity on fixing classes of 0-1-polytopes.

2. Remark 1: *J*-additivity vs. *J*-uniqueness.

2.1. *J*-additivity. Given $F \in \mathcal{F}^d$ and lines $S_1, \ldots, S_m$, let $\mathcal{T}_i(F)$ denote the set of all translates $x+S_i$ that intersect F, $i = 1, \ldots, m$, set $n := |F|$, $\mathcal{T}(F) := \bigcup_{i=1}^m \mathcal{T}_i(F)$, and let $b : \mathcal{T}(F) \to \mathbb{N}$ be defined by $b(T) := |F \cap T|$. Then, of course,

$$\sum_{T \in \mathcal{T}_1(F)} b(T) = \cdots = \sum_{T \in \mathcal{T}_m(F)} b(T) = n.$$

Also, F must be contained in its *tomographic grid*,

$$G := G(F) := \bigcap_{i=1}^{m} \left(\bigcup_{T \in \mathcal{T}_i(F)} T \right).$$

Now let $J \subset G$. Then the set F is called *J-additive* if there exist real numbers η_T for $T \in \mathcal{T}(F)$ such that the following conditions hold:

$$\begin{aligned}
&\sum_{T \in \mathcal{T}(F) \wedge g \in T} \eta_T \geq 1 && \text{if } g \in F \cap J;\\
&\sum_{T \in \mathcal{T}(F) \wedge g \in T} \eta_T \leq -1 && \text{if } g \in (G \setminus F) \cap J;\\
&\sum_{T \in \mathcal{T}(F) \wedge g \in T} \eta_T \geq 0 && \text{if } g \in F \setminus J;\\
&\sum_{T \in \mathcal{T}(F) \wedge g \in T} \eta_T \leq 0 && \text{if } g \in G \setminus (F \cup J).
\end{aligned}$$

Let us point out that our concept of J-additivity generalizes that of additivity introduced by Fishburn et al. in [9]; see also [30]. In fact, it is straightforward to see that their notion of additivity is equivalent to G-additivity. Hence we define a set F to be additive if F is G-additive. In particular, the following result generalizes a theorem of [9] and its extension of [1].

THEOREM 2.1. *Let $J \subset G$, let F be J-additive, and let $F' \in \mathcal{F}^d$ be tomographically equivalent to F. Then $F \cap J = F' \cap J$, i.e., F and F' coincide in all J-positions.*

Note that J-additivity has an intuitive geometric interpretation as a *weighted, filtered back-projection*. In particular, let F be G-additive and set $\mu_T := \eta_T / b(T)$ for

$T \in \mathcal{T}$. Then

$$g \in F \quad \Leftrightarrow \quad \sum_{T \in \mathcal{T}(F) \,\wedge\, g \in T} \mu_T b(T) \geq 1.$$

This means that if we add up the suitably weighted X-ray values on all lines of $\mathcal{T}$ through a point g and take all such points g for which this sum does not fall below the threshold 1, then we precisely recover F.

In the next subsection we will further generalize Theorem 2.1 in a spirit that is related to [1], give a simple geometric explanation of the underlying concept, and provide a short and concise proof of the result by means of linear programming duality.

2.2. Substructure uniqueness. Let $p, q \in \mathbb{N}$, $Q := \{1, \ldots, q\}$, and $b = (\beta_1, \ldots, \beta_p)^T \in \mathbb{R}^p$. Further, let $A = (\alpha_{i,j})$ be a real $p \times q$-matrix, and let $a_1, \ldots, a_q$ denote its column vectors. To exclude trivial cases we assume that $a_i \neq 0$ for all $i = 1, \ldots, q$, and that the rows of A are pairwise different.

We are interested in the solutions of

$$Ax = b \quad \wedge \quad 0 \leq x \leq \mathbb{1}_q \quad \wedge \quad x \in \mathbb{Z}^q,$$

where $\mathbb{1}_q$ denotes the all 1's vector of $\mathbb{R}^q$. Of course, $=$ and $\leq$ are meant componentwise. Note that, in fact, we are looking for a solution of a certain integer linear program. Its linear programming relaxation, which is obtained by abandoning the integrality constraint $x \in \mathbb{Z}^q$, leads to the polytope

$$P := \{x \in \mathbb{R}^q \,:\, Ax = b \wedge 0 \leq x \leq \mathbb{1}_q\}.$$

Clearly, with the ith row of A we can associate a *query set* $W_i := \{j \,:\, j \in Q \wedge \alpha_{i,j} \neq 0\}$. Now, let $x = (\xi_1, \ldots, \xi_q)^T$ be a solution of the above system, and set $Q_x := \{j \,:\, j \in Q \wedge \xi_j = 1\}$. Then, of course,

$$\sum_{j \in W_i \cap Q_x} \alpha_{i,j} = \beta_i$$

for each $i = 1, \ldots, p$. So, the set Q_x satisfies *weighted* query set constraints.

In particular, if A is a 0-1-matrix, then Q_x must contain precisely β_i points of the query set W_i for each $i = 1, \ldots, p$, and we are back in the "combinatorial world." In the realm of discrete tomography with line X-rays, Q corresponds to G, the query sets to lines parallel to $S_1, \ldots, S_m$, and x is the incidence vector of F (i.e., the components of x are 1 exactly at the coordinates that correspond to F-positions and 0 otherwise). Note that in this case b is the *X-ray vector* containing all X-ray measurements.

Hence, the present setting generalizes that of subsection 2.1. In particular, the discrete Radon transform (where the query sets are induced by hyperplanes), general k-dimensional X-rays, and "box window queries" are included.

Let us now turn to the question of whether, for some given subset $J \subset Q$, the corresponding coordinates ξ_j of any solution x of the system are already uniquely determined. So, let $J \subset Q$, $x^* = (\xi_1^*, \ldots, \xi_q^*)^T \in P \cap \mathbb{Z}^q$ and set

$$\begin{aligned} L_J &:= L_J(x^*) := \big\{(\xi_1, \ldots, \xi_q)^T \in \mathbb{R}^q \,:\, (\xi_j^*)_{j \in J} = (\xi_j)_{j \in J}\big\}, \\ P_J &:= P_J(x^*) := P \cap L_J. \end{aligned}$$

Then x^* is called *J-unique* if $P \cap \mathbb{Z}^q = P_J \cap \mathbb{Z}^q$ and *strongly J-unique* if $P = P_J$. Hence x^* is J-unique if all 0-1-points of P have the same J-coordinates as x^*. If this is true for all points of P, then x^* is strongly J-unique. So, J-uniqueness is the appropriate notion in the context of combinatorics, while strong J-uniqueness restricts even all fractional solutions. Clearly, strong J-uniqueness implies J-uniqueness, while, as we will demonstrate later in Theorem 3.2, the converse is not even true in very restricted cases.

We will now extend the notion of J-additivity introduced in subsection 2.1 to the more general framework of the present subsection. A vector $x^* = (\xi_1^*, \dots, \xi_q^*)^T \in P \cap \mathbb{Z}^q$ is called *J-additive* if there exists a *weight vector* $y \in \mathbb{R}^p$, such that for each $j = 1, \dots, q$ we have

$$\begin{array}{ll} a_j^T y \geq 1 & \text{if } \xi_j^* = 1 \text{ and } j \in J; \\ a_j^T y \leq -1 & \text{if } \xi_j^* = 0 \text{ and } j \in J; \\ a_j^T y \geq 0 & \text{if } \xi_j^* = 1 \text{ and } j \notin J; \\ a_j^T y \leq 0 & \text{if } \xi_j^* = 0 \text{ and } j \notin J. \end{array}$$

Clearly, in the context of discrete tomography, this notion coincides with that of subsection 2.1 via the natural identification of sets with their incidence vectors.

Now we can give a characterization of strong J-uniqueness of a point x^* via J-additivity and thus a sufficient criterion for J-uniqueness.

THEOREM 2.2. *Let $x^* = (\xi_1^*, \dots, \xi_q^*)^T \in P \cap \mathbb{Z}^q$ and $J \subset Q$. Then x^* is strongly J-unique if and only if x^* is J-additive. Consequently, J-additivity implies J-uniqueness.*

Proof. Let $c := c(x^*) = (\gamma_1, \dots, \gamma_q)^T \in \{-1, 0, 1\}^q$ be defined by

$$\gamma_j := \begin{cases} 2\xi_j^* - 1 & \text{if } j \in J; \\ 0 & \text{if } j \notin J \end{cases} \qquad (j \in Q).$$

Suppose $x = (\xi_1, \dots, \xi_q)^T \in P$ and $\xi_j^* \neq \xi_j$ for some $j \in J$. Then $\gamma_j \xi_j < \gamma_j \xi_j^*$; thus $c^T x < c^T x^*$. Therefore, $(\xi_j^*)_{j \in J} = (\xi_j)_{j \in J}$ for all points $x \in P$ with $c^T x = c^T x^*$. In particular, this means that strong J-uniqueness is equivalent to

$$\min_{x \in P} c^T x = c^T x^*,$$

and it suffices to show that J-additivity is equivalent to the same condition.

We will now apply linear programming duality including the complementary slackness condition; see, e.g., [27] or [22] for the basic background on this topic.

Let us consider the *primal* linear program

$$\text{(PLP)} \qquad \min c^T x \quad \text{s.t.} \quad Ax = b \wedge 0 \leq x \leq \mathbb{1}_q$$

and its *dual*

$$\text{(DLP)} \qquad \max\,(b^T y - \mathbb{1}_q^T z) \quad \text{s.t.} \quad A^T y - z \leq c \wedge z \geq 0.$$

Since (PLP) has a finite optimum, so does (DLP). Let

$$W := \left\{ w = \begin{pmatrix} y \\ z \end{pmatrix} : A^T y - z \leq c \wedge z \geq 0 \right\}.$$

Then (x, w) is called a *primal-dual pair* if $x \in P$ is optimal for (PLP) and $w \in W$ is optimal for (DLP). By complementary slackness, (x, w) is a primal-dual pair if and only if $x \in P$, $w \in W$, and

$$\xi_j(\gamma_j - a_j^T y + \zeta_j) = 0 \qquad \wedge \qquad (1 - \xi_j)\zeta_j = 0$$

for each $j = 1, \ldots, q$, where $x = (\xi_1, \ldots, \xi_q)^T$ and $z = (\zeta_1, \ldots, \zeta_q)^T$.

Suppose now that x^* is J-additive and let $y \in \mathbb{R}^p$ be the corresponding weight vector. We define $z = (\zeta_1, \ldots, \zeta_q)^T$ by

$$\zeta_j := \begin{cases} a_j^T y - \gamma_i & \text{if } \xi_j^* = 1; \\ 0 & \text{if } \xi_j^* = 0 \end{cases} \qquad (j \in Q),$$

and set $w^* := (y^T, z^T)^T$. Clearly $w^* \in W$, and since $x^* \in \{0, 1\}^q$, the pair (x^*, w^*) satisfies the complementary slackness conditions and is hence a primal-dual pair of optimal solutions. In particular, x^* is a minimizer of the primal program.

Conversely, suppose that x^* is a minimizer of the primal linear program, and let $w^* = (y^T, z^T)^T \in W$ such that (x^*, w^*) is a primal-dual pair. Then, by complementary slackness, the conditions for J-additivity of x^* are satisfied with weight vector y. $\square$

So, the seemingly "complex" concept of additivity boils down to the simple question of whether P lies in the affine subspace L_J or, equivalently, whether the orthogonal projection of P parallel to L_J is a singleton. Therefore the sufficient condition for strong (substructure) uniqueness can be easily checked with any linear programming algorithm. On the other hand, J-uniqueness is computationally much harder in general. In fact, [10, Theorem 4.3] shows that, even when restricted to instances that correspond to planar discrete tomography with line X-rays in three directions, deciding G-uniqueness is $\mathbb{NP}$-hard. For background information on computational complexity theory, see, e.g., [13].

As a by-product, our approach can be utilized to perform a simple search to settle Conjecture 11 of [6] in the negative. In fact, we show that even a weaker conjecture is false. A set $F \subseteq \mathbb{Z}^2$ is called *lattice convex* if $\mathbb{Z}^2 \cap \operatorname{conv} F = F$, where conv denotes the convex hull operator. As a special case of a more general result of [11], any planar lattice convex set is uniquely determined (within this class) by its X-rays in the directions of $\mathscr{S}^* := \{(1,0)^T, (2,1)^T, (0,1)^T, (-1,2)^T\}$. It was conjectured by Brunetti and Daurat [6] that, in particular, any lattice convex set is additive w.r.t. $\mathscr{S}^*$. However, Figure 1 gives two tomographically equivalent lattice set w.r.t. $\mathscr{S}^*$, one being lattice convex. Hence Theorem 2.2 implies the following remark.

Remark 2.3. For $\alpha, \beta \in \mathbb{N}$ let $F_{\alpha,\beta} := \mathbb{Z}^2 \cap \operatorname{conv}\{(\alpha,\beta)^T, (-\beta,\alpha)^T, (-\alpha,-\beta)^T, (\beta,-\alpha)^T\}$. Then $F_{2,4}$ is not additive w.r.t. $\mathscr{S}^*$ (see Figure 1). $F_{3,5}$, $F_{4,6}$, and $F_{5,7}$ are other lattice convex sets that are not additive w.r.t. $\mathscr{S}^*$.

We close this section with a remark on an even stronger uniqueness condition. By Theorem 2.2, G-uniqueness of a solution x^* is guaranteed when P is a singleton, i.e., the system

$$Ax = b \wedge 0 \leq x \leq \mathbb{1}_q$$

has a unique solution. This is certainly the case if not even the system $Ax = b$ of linear equations has a solution different from x^*. As it turns out, this is underlying Rényi's condition [24], which states that a set F is always uniquely determined by

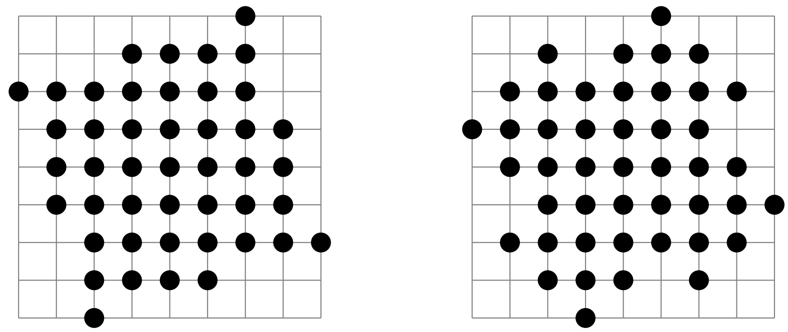

Fig. 1. *The lattice convex set $F_{2,4}$ (left) and the lattice set on the right are tomographically equivalent w.r.t. $\mathscr{S}^*$.*

its X-rays in any $|F|+1$ different directions. In fact, we have the following slightly stronger result, which is in essence just a reinterpretation of a certain classical proof of Rényi's result; see [12, Theorem 4.3.3].

Remark 2.4. Let $F \in \mathcal{F}^d$, let $S_1, \dots, S_m$ be different lines in $\mathbb{R}^d$, and set $p := |\mathcal{T}(F)|$, $q := |G|$. Further, denote the entries of $\mathcal{T}(F)$ and G by $T_1, \dots, T_p$ and $g_1, \dots, g_q$, respectively, and let $A := (|T_i \cap \{g_j\}|)_{i\in\{1,\dots,p\},j\in\{1,\dots,q\}} \in \{0,1\}^{p\times q}$ be the corresponding *incidence matrix*. If $m \geq |F|+1$, then A contains a $q \times q$ permutation matrix; hence $\operatorname{rank}(\hat{A}) = q$ for any $\hat{A} \in \mathbb{R}^{p\times q}$ with nonzero entries exactly at the 1-entries of A.

3. Remark 2: Solution of a problem of Kuba. It is well known that a lattice set $F \subset \mathbb{Z}^2$ is uniquely determined by its X-rays in the two standard coordinate directions if and only if F is additive. At a Dagstuhl conference on discrete tomography in 1997, Attila Kuba asked the following question:

Question 3.1. *Is it true that a lattice set $F \subset \mathbb{Z}^3$ is uniquely determined by its X-rays in the three standard coordinate directions if and only if F is additive?*

As it turns out, our approach of the previous section allows us to resolve this problem in the negative. (We will, however, not describe the solution in polytopal terms.)

Theorem 3.2. *There are nonadditive lattice sets $F \subset \mathbb{Z}^3$ that are uniquely determined by their X-rays in the three standard coordinate directions.*

Proof. Let F be the subset of $\mathbb{Z}^3$ depicted on the left side of Figure 2. Here, gray points lie in the first layer ($\xi_3 = 1$), while the white and black points lie in the second and third layers, $\xi_3 = 2$, $\xi_3 = 3$, respectively. The small crosses indicate empty lattice positions.

The picture on the right represents a fractional solution satisfying the same X-ray constraints as F. Semicircles indicate a value of 1/2 in the according layer and position. Hence, by Theorem 2.2, F is not additive.

Now we show that, nevertheless, F is uniquely determined by its X-rays in the three standard coordinate directions. Let us first consider the first layer. Figure 3 (left) shows all possible positions for points in the first layer.

It is straightforward to check that the gray points can only occur in two possible configurations. The first is depicted in Figure 2 (left), the second in Figure 3 (right). Suppose the first layer contains the second configuration. Then the point $(7,4)^T$ must be black. This implies that $(4,4)^T$ is white; then $(4,5)^T$ is black, and hence $(1,5)^T$ is white. But the vertical X-rays show that there is no white point in the first column, a

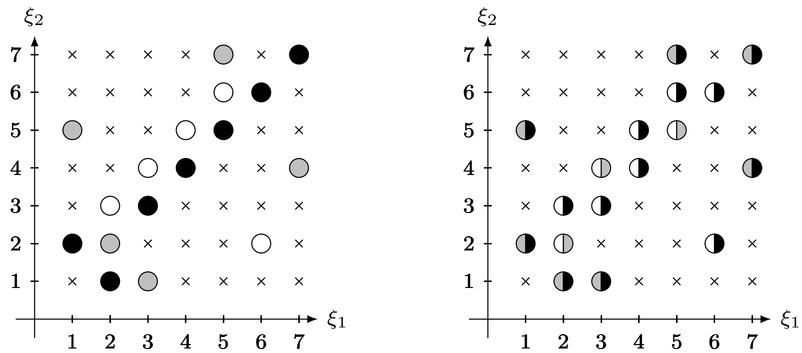

FIG. 2. *A* 0-1- *and a fractional solution.*

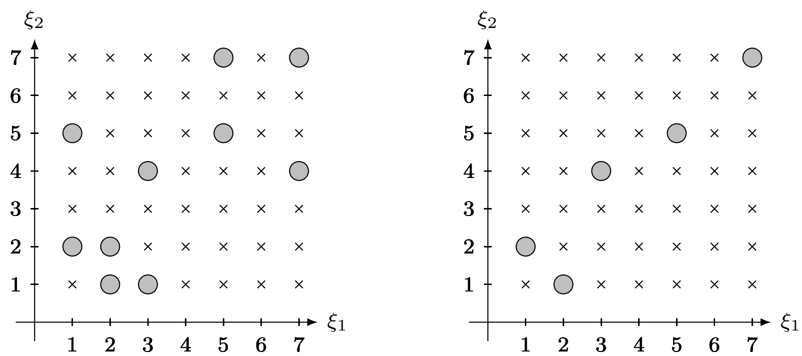

FIG. 3. *Possible positions for gray points; second solution.*

contradiction. Hence the gray layer must be that of Figure 2 (left). It is now easy to verify that the gray configuration forces all other positions of F. Hence F is uniquely determined by its X-rays. □

4. Remark 3: On the complexity of forcing uniqueness. Let us now shift our point of view from substructure uniqueness to *uniqueness induced by substructures.*

Let $S_1, \dots, S_m$ be different lines in $\mathbb{R}^d$ through 0 as before. A set $F \in \mathcal{F}^d$ might not be uniquely determined by its X-rays $X_1(F), \dots, X_m(F)$, but if we knew for some positions of G whether or not a point of F is present, we might have enough information to guarantee uniqueness. This gives rise to the following decision problem.

UNIQUENESSFORCINGSUBSET$_{S_1,\dots,S_m}$.

> Given $F \in \mathcal{F}^d$ and $\tau \in \mathbb{N}_0$, decide whether there exists $C \subset G$ with $|C| \leq \tau$ which *forces uniqueness* for F; i.e., if $F' \in \mathcal{F}^d$ is tomographically equivalent to F and $F' \cap C = F \cap C$, then $F' = F$.

In particular we show that this problem is computationally difficult, even for just the two coordinate directions in the plane. Again, the algorithmic difficulty is specified within the realm of the theory of computational complexity; see [13].

THEOREM 4.1. *Let $d = m = 2$, $S_1 := \text{lin}\{(1,0)^T\}$, and $S_2 := \text{lin}\{(0,1)^T\}$. Then the problem* UNIQUENESSFORCINGSUBSET$_{S_1,S_2}$ *is* $\mathbb{NP}$*-complete.*

This result is somewhat surprising, because checking uniqueness is easy, with or without prescribed positions, for $m = 2$; see, [8], [28], [25], [26], or [4]. Typically, hardness results in discrete tomography require $m \geq 3$; see, e.g., [10], [12], [14], [3], [7].

In the proof we show that a specific $\mathbb{NP}$-complete graph theoretical problem can be polynomially reduced to $\text{UniquenessForcingSubset}_{S_1,\ldots,S_m}$, showing that the latter is no easier than the former.

In order to formulate this problem we need some notions from graph theory. See, e.g., [5] for a general treatment of graph theory and more information on the concepts needed here.

In the following we need the notion of feedback arc sets and of bipartite tournaments. So, let $D = (U, E)$ be a directed graph and $B \subset E$. Recall that B is a *feedback arc set* if B contains at least one arc from each directed cycle in D. The directed graph D is called a *bipartite tournament* if there is a partition of U into two sets V and W such that $E \subset (V \times W) \cup (W \times V)$ and for each $v \in V$ and $w \in W$ precisely one of the arcs (v, w) or (w, v) is in E.

We prove the $\mathbb{NP}$-hardness of $\text{UniquenessForcingSubset}_{S_1,S_2}$ by giving a reduction from FeedbackArcSet for bipartite tournaments, which is known to be $\mathbb{NP}$-complete; see [2] (for tournaments), [17], [23]. For the purpose of an easy exposition (and since this direction is the basis of a later comment on the approximability of $\text{UniquenessForcingSubset}_{S_1,S_2}$) we start from $\text{UniquenessForcingSubset}_{S_1,S_2}$ and show later that the construction can be reversed to obtain the required reduction from FeedbackArcSet.

Proof of Theorem 4.1. Given $F \in \mathcal{F}^2$ and $C \subset G$ it is easy to check whether there is a set F' that is tomographically equivalent to F w.r.t. S_1 and S_2 with $F' \cap C \neq F \cap C$. In fact, one can use the network flow approach from [28] or the algorithms given in [8], [25], [26], or [4]. Hence $\text{UniquenessForcingSubset}_{S_1,S_2}$ is in $\mathbb{NP}$.

Let $r_1, r_2 \in \mathbb{N}$, $F \subset R := \{1, \ldots, r_1\} \times \{1, \ldots, r_2\}$. Suppose without loss of generality that $G = R$, and let $b_1 \in \mathbb{N}^{r_1}$, $b_2 \in \mathbb{N}^{r_2}$ be the corresponding X-ray vectors. We now construct a bipartite tournament $D = (U, E)$. Let $V := \{v_1, \ldots, v_{r_1}\}$ and $W := \{w_1, \ldots, w_{r_2}\}$ denote two disjoint sets of vertices and set $U := V \cup W$. Further, let $(v_i, w_j) \in E$ if and only if $(i, j)^T \in F$ and $(w_j, v_i) \in E$ if and only if $(i, j)^T \in R \setminus F$. Hence the arcs in D encode the points of F in R. Then, of course,

$$(\text{out-deg}(v_i))_{i=1,\ldots,r_1} = b_1 \quad \wedge \quad (\text{in-deg}(w_i))_{i=1,\ldots,r_2} = b_2,$$

where out-deg(v) and in-deg(w) denote the out-degree and the in-degree of a vertex v and w, respectively, i.e., the number of arcs leaving v and entering w, respectively.

Clearly, this construction can be reversed. (Note that the above assumption that $G = R$ incurs no loss of generality. It fact, it corresponds to the deletion of all vertices $v \in V$ with out-deg$(v) = 0$, all vertices $w \in W$ with in-deg$(v) = 0$, and all incident arcs. Such arcs are, of course, never needed in any feedback arc set.) Finally, we observe that a set $C \subset G$ forces uniqueness for F if and only if the corresponding set B of arcs of D forms a feedback arc set. In fact, suppose that there is a directed cycle in D. A reversal of all arcs preserves the in- and out-degrees of the vertices and produces a bipartite tournament D' different from D, and hence a set F' different from but tomographically equivalent to F. Again, the argument can be reversed: If F' is a second solution, then the *symmetric difference* $F \Delta F' := (F \setminus F') \cup (F' \setminus F)$ corresponds to a subgraph of D with equal in- and out-degree at each vertex; such a subgraph decomposes into cycles.

Of course, the transformations run in polynomial time. ∎

Note, in passing, that despite the hardness result of Theorem 4.1, our construction allows us to evoke an approximation result of [29] to obtain a polynomial-time approximation for UniquenessForcingSubset$_{S_1,S_2}$ which produces a subset C which forces uniqueness and is at most four times larger than such a set C^* of minimum size.

5. A corollary: Fixing classes of 0-1-polytopes. The final section deals with 0-1-*polytopes*, i.e., polytopes having only vertices with coordinates in $\{0,1\}$, which are particularly important in combinatorial optimization. For background information on polytopes see [27] or [31].

Let V be a nonempty finite subset of $\{0,1\}^d$, let $P := \mathrm{conv}(V)$, the convex hull of V, and let $\tau \in \mathbb{N}_0$. We say that a vertex v of P is τ-*fixed* in P if there exists a $(d-\tau)$-dimensional coordinate subspace Y such that $P \cap (v+Y) = \{v\}$. If every vertex v of P is τ-fixed, P is called τ-fixed. Clearly, each 0-1-polytope is d-fixed, and only singletons are 0-fixed. If τ is the smallest nonnegative integer so that P is τ-fixed, we say that P belongs to the *fixing class* τ.

Naturally, basic combinatorial properties of 0-1-polytopes (including their f-vectors whose kth coordinate gives the number of k-dimensional faces, and their combinatorial diameter) depend on their fixing classes, and it is an interesting problem to characterize all polytopes of a given fixing class. However, determining whether a vertex v is τ-fixed is computationally hard.

Theorem 5.1. *The task of determining, for a given nonempty finite subset V of $\{0,1\}^d$ of cardinality at most $1+d(d-1)/2$, $v \in V$, and $\tau \in \mathbb{N}_0$, whether v is τ-fixed in $P := \mathrm{conv}(V)$ is* $\mathbb{NP}$*-complete.*

Proof. Given a subspace Y, one can check whether $|P \cap Y| = 1$ by linear programming. Hence, membership in $\mathbb{NP}$ is clear. To prove $\mathbb{NP}$-hardness we give a reduction from VertexCover. Here, the instances consist of a finite set S, a collection $\mathscr{C}$ of 2-element subsets of S, and a number $\tau \in \mathbb{N}$. The task is to find a *vertex cover* of cardinality at most τ, i.e., a set $H \subset S$ with $|H| \le \tau$ and $H \cap C \ne \emptyset$ for each $C \in \mathscr{C}$. This is a classical $\mathbb{NP}$-complete problem; see [21] and also [13].

Now, let $(S, \mathscr{C}, \tau)$ be an instance of VertexCover. We suppose without loss of generality that $S = \{1,\ldots,d\}$. Then, of course, $k := |\mathscr{C}| \le \binom{|S|}{2} = d(d-1)/2$.

Let $\mathscr{C} = \{C_1,\ldots,C_k\}$, let $v_0 := 0 \in \mathbb{R}^d$, and for $i = 1,\ldots,k$ let v_i be the incidence vector of C_i; i.e., all components of v_i are 0 except for the entry 1 at the two positions j with $j \in C_i$. Finally, set $P = \mathrm{conv}\big(\{v_0, v_1, \ldots, v_k\}\big)$. Then there exists a vertex cover $H \subset S$ with $|H| \le \tau$ if and only if v_0 is τ-fixed in P. In fact, any index set of coordinates of v_0 whose disclosure fixes v_0 in P is a vertex cover for $(S, \mathscr{C})$, and vice versa. Of course, the transformation runs in polynomial time. ∎

Theorem 5.1 involves polytopes that are given as the convex hull of an explicit set V of points, so-called $\mathcal{V}$-*polytopes*. Not surprisingly, the fixing problem is no easier for $\mathcal{H}$-*polytopes* which are given as the set of feasible points of an explicitly given system of linear inequalities. The following result is a corollary to Theorem 4.1. It involves *totally unimodular* matrices, i.e., 0-1-matrices all of whose quadratic submatrices have determinants in $\{-1,0,1\}$.

Corollary 5.2. *The task of determining, for given $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, $p := 2r$, $q := r^2$, a totally unimodular 0-1-matrix $A \in \mathbb{R}^{p\times q}$, a vector $b \in \mathbb{N}_0^p$, and a vertex v of the polytope $P := \{x \in \mathbb{R}^q : Ax = b \wedge 0 \le x \le \mathbb{1}_q\}$, whether v is τ-fixed in P is* $\mathbb{NP}$*-complete.*

Proof. Again, membership in $\mathbb{NP}$ is clear. $\mathbb{NP}$-hardness follows directly from Theorem 4.1. In fact, let $F \subset R := \{1,\ldots,r\}^2$, let $b_1, b_2 \in \mathbb{N}_0^r$ encode the X-ray data with respect to the two standard coordinate directions in the plane, and set $b := (b_1^T, b_2^T)^T$. Then the X-ray conditions can be expressed in matrix form $Ax = b$ with

$$A := \begin{bmatrix} \mathbb{1}_r^T & & \\ & \ddots & \\ & & \mathbb{1}_r^T \\ I_r & \cdots & I_r \end{bmatrix} \in \{0,1\}^{(2r)\times r^2},$$

where I_r denotes the $r \times r$ unit matrix. Note that A is totally unimodular; see [27, Theorem 19.3]. Hence, $P := \{x \in \mathbb{R}^q : Ax = b \wedge 0 \leq x \leq \mathbb{1}_q\}$ is a 0-1-polytope which encodes all sets that are tomographically equivalent to F as its vertices. Let v denote the vertex corresponding to F. Then fixing the points of a subset C of G as belonging or not belonging to F corresponds to selecting a coordinate subspace Y_C. Hence C forces uniqueness if and only if $(v + Y_C) \cap P = \{v\}$. ∎

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