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The two-dimensional Prouhet–Tarry–Escott problem

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Abstract

In this paper we generalize the Prouhet–Tarry–Escott problem (PTE) to any dimension. The one-dimensional PTE problem is the classical PTE problem. We concentrate on the two-dimensional version which asks, given parameters $n, k \in \mathbb{N}$, for two different multi-sets $\{(x_1, y_1), \dots, (x_n, y_n)\}$, $\{(x'_1, y'_1), \dots, (x'_n, y'_n)\}$ of points from $\mathbb{Z}^2$ such that $\sum_{i=1}^n x_i^j y_i^{d-j} = \sum_{i=1}^n x_i'^j y_i'^{d-j}$ for all $d, j \in \{0, \dots, k\}$ with $j \leqslant d$. We present parametric solutions for $n \in \{2, 3, 4, 6\}$ with optimal size, i.e., with $k = n - 1$. We show that these solutions come from convex $2n$-gons with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains an even number of vertices and prove that such convex $2n$-gons do not exist for other values of n. Furthermore we show that solutions to the two-dimensional PTE problem yield solutions to the one-dimensional PTE problem. Finally, we address the PTE problem over the Gaussian integers.

Keywords: Prouhet–Tarry–Escott problem; Tarry–Escott problem; Lattice polygon; Multigrade equations; Discrete tomography; Equal sum of like powers

1. Introduction

We introduce the following problem, which we call *the general Prouhet–Tarry–Escott problem*:

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Problem 1 *(PTE$_r$)*. Given $k, n, r \in \mathbb{N}$ find two different multi-sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of points from $\mathbb{Z}^r$ where $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir})$, $\mathbf{b}_i = (b_{i1}, \ldots, b_{ir})$ for $i = 1, \ldots, n$ such that

$$\sum_{i=1}^{n} a_{i1}^{j_1} a_{i2}^{j_2} \cdots a_{ir}^{j_r} = \sum_{i=1}^{n} b_{i1}^{j_1} b_{i2}^{j_2} \cdots b_{ir}^{j_r}$$

for all nonnegative integers $j_1, \ldots, j_r$ with $j_1 + j_2 + \cdots + j_r \leqslant k$.

In the sequel k is called the *degree*, n the *size*, and r the *dimension* of the solution. Solutions satisfying $n = k + 1$ are called *ideal solutions*, since nontrivial solutions with $n \leqslant k$ do not exist. For PTE$_1$ this is a well-known elementary result using symmetric functions and Newton's identities (see [3], [2]). Note that the result implies (by setting $j_{r+1} = 0$) that there cannot exist a solution to PTE$_{r+1}$ with $n \leqslant k$ and $r \geqslant 1$. We indicate Problem PTE$_r$ with $n = k + 1$ as PTE$_r(k)$. Let $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n) \in \mathbb{Z}$ be a solution to PTE$_1$. Then, $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, $(\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{Z}^r$ with $\mathbf{a}_i = (\alpha_i, \ldots, \alpha_i)$, $\mathbf{b}_i = (\beta_i, \ldots, \beta_i)$, for $i = 1, \ldots, n$, is trivially a solution to PTE$_r$. Sets that are not of this form will be called *proper sets*. In the sequel the considered solutions will always be proper sets.

An equivalent formulation of PTE$_r$ is as follows.

Problem 2. Given $k, n, r \in \mathbb{N}$ find two different multi-sets $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of points from $\mathbb{Z}^r$ such that

$$\sum_{i=1}^{n} P(\mathbf{a}_i) = \sum_{i=1}^{n} P(\mathbf{b}_i)$$

for every polynomial $P \in \mathbb{Z}[\mathbf{x}]$ of total degree $\leqslant k$, where $\mathbf{x} \in \mathbb{Z}^r$.

The main question is for which k, n, r proper solutions to PTE$_r$ exist, and in particular for which k, r proper solutions to PTE$_r(k)$ exist.

The classical Prouhet–Tarry–Escott problem (PTE$_1$) is an old problem tracing back to works of Euler and Goldbach [4]. It is an open question [7] whether solutions to PTE$_1(k)$ exist for every $k \in \mathbb{N}$. At present, ideal solutions are only known for $k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$ (see [2,6]). No values of k are known for which no ideal solutions exist.

In the present paper we study the case $r = 2$. We give (proper) parametric solutions to PTE$_2(k)$ for $k \in \{1, 2, 3, 5\}$. Our approach is geometric. In fact, the terminology that we use originates from *Discrete Tomography* a relatively young field in discrete mathematics with applications ranging from electron microscopy to statistical data security (see [8] for a survey). Additional geometric examples for PTE$_1(k)$ solutions can be found in [1]. First we give some definitions.

A *lattice direction* $\operatorname{lin}\{(p, q)\}$ is a (nondegenerate) linear subspace which is generated by a vector $(p, q) \in \mathbb{Z}^2$. A (*discrete*) *X-ray* of a finite subset $F \subset \mathbb{Z}^n$ along $\operatorname{lin}\{(p, q)\}$ gives the number of points in F lying on a line parallel to $\operatorname{lin}\{(p, q)\}$. Two sets F_1, F_2 are said to have *equal* (*discrete*) *X-rays along direction* $\operatorname{lin}\{(p, q)\}$ if

$$\left|F_1 \cap \left(\mathbf{x} + \operatorname{lin}\{(p, q)\}\right)\right| = \left|F_2 \cap \left(\mathbf{x} + \operatorname{lin}\{(p, q)\}\right)\right| \quad \text{for all } \mathbf{x} \in \mathbb{Z}^2.$$

In other words, F_1 and F_2 have equal X-rays along $\operatorname{lin}\{(p, q)\}$ if they have an equal number of points on each line parallel to $\operatorname{lin}\{(p, q)\}$. Note that opposite directions are identified.

Problem 3 *(GP_2).* Given $k, n \in \mathbb{N}$ find a set of $k+1$ distinct directions and two proper sets of n points from $\mathbb{Z}^2$ such that the sets have equal X-rays along all $k+1$ directions.

Again we call a solution *ideal* if $n = k+1$ and indicate the problem in this case as $\mathrm{GP}_2(k)$. The following problem turns out to be equivalent with $\mathrm{GP}_2(k)$.

Problem 4. Given k find a convex $(2k+2)$-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains an even number of vertices.

Our results in Sections 2 and 3 show that this problem is solvable only for $k = 1, 2, 3$ and 5. In Section 4 we prove that every solution to $\mathrm{GP}_2(k)$ yields a solution to $\mathrm{PTE}_2(k)$ and we remark that solutions to $\mathrm{PTE}_2(k)$ lead to classes of solutions to $\mathrm{PTE}_1(k)$. Section 5 contains a result on the general PTE_r problem. The final section deals with the PTE problem for Gaussian integers.

2. Solutions to the geometric problems

We construct solutions to $\mathrm{GP}_2(k)$ for $k = 1, 2, 3$ and 5. See Fig. 2 for an illustration of a $\mathrm{GP}_2(5)$ solution.

It is clear from the geometry that the property of two sets having equal X-rays remains invariant under affine transformation ($f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ with $A \in \mathbb{Z}^{2\times 2}$ nonsingular and $\mathbf{b} \in \mathbb{Z}^2$). For PTE_1 this is known as *Frolov's theorem* [6].

The following results all have in common that the directions of the points

$$(p_0, q_0), \ldots, (p_k, q_k), (-p_0, -q_0), \ldots, (-p_k, -q_k)$$

viewed from the origin are listed in the order of increasing angle with the positive x-axis. By consecutive addition of the direction vectors $(p_0, q_0), \ldots, (p_k, q_k)$, $-(p_0, q_0), \ldots, -(p_k, q_k)$ we obtain alternately the points of F_1 and F_2. Such sets are called lattice *$\mathcal{L}$-gons*. As we will prove later we obtain PTE_2 solutions by taking as $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ the set of points of F_1 and as $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ the set of points of F_2.

2.1. *Proper ideal solutions of degree $k = 1$*

We choose parameters $a, b, c, d \in \mathbb{Z}$ such that $\mathrm{lin}\{(a, b)\}$ and $\mathrm{lin}\{(c, d)\}$ are different directions. Then,

$$F_1 = \{(0,0), (a+c, b+d)\}, \qquad F_2 = \{(a,b), (c,d)\}$$

have equal X-rays along both directions and yield solutions to $\mathrm{PTE}_2(1)$. See Fig. 1 for an illustration. Every solution yields a parallelogram with $(\frac{a+c}{2}, \frac{b+d}{2})$ as point of symmetry.

2.2. *Proper ideal solutions of degree $k = 2$*

We choose parameters $a, b, c \in \mathbb{Z}$ such that $\mathrm{lin}\{(a, 0)\}$, $\mathrm{lin}\{(b, c)\}$ and $\mathrm{lin}\{(b-a, c)\}$ are different directions. It is clear, that

$$F_1 = \{(0,0), (a+b, c), (2b-a, 2c)\}, \qquad F_2 = \{(a,0), (2b, 2c), (b-a, c)\}$$

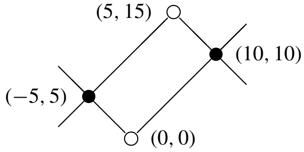

Fig. 1. Solution of $\mathrm{GP}_2(1)$ with $a=b=10$, $c=-5$ and $d=5$. White points indicate the points of F_1; black points indicate the points of F_2. The sets have equal X-rays along $\mathrm{lin}\{(10,10)\}$ and $\mathrm{lin}\{(-5,5)\}$ as indicated. Clearly, $\{\mathbf{a}_1,\mathbf{a}_2\}=\{(0,0),(5,15)\}=F_1$, $\{\mathbf{b}_1,\mathbf{b}_2\}=\{(10,10),(-5,5)\}=F_2$ is a solution to $\mathrm{PTE}_2(1)$.

have equal X-rays along all three directions and yield solutions to $\mathrm{PTE}_2(2)$. Note that (b,c) is a point of symmetry. A convex solution is, for example, obtained by taking $a=b=c=1$.

2.3. *Proper ideal solutions of degree $k=3$*

We choose parameters $a,b,c\in\mathbb{N}$, where $b\mid ac$, and such that $\mathrm{lin}\{(a,0)\}$, $\mathrm{lin}\{(b,c)\}$, $\mathrm{lin}\{(0,ac/b)\}$ and $\mathrm{lin}\{(-b,c)\}$ are different directions. Clearly,

$$F_1=\big\{(0,0),(a+b,c),(a,2c+ac/b),(-b,c+ac/b)\big\},$$
$$F_2=\big\{(a,0),(a+b,c+ac/b),(0,2c+ac/b),(-b,c)\big\}$$

have equal X-rays along all four directions and yield solutions to $\mathrm{PTE}_2(3)$. Note that $(\frac{a}{2},c+\frac{ac}{2b})$ is a point of symmetry. A convex solution is, for example, obtained by taking $a=b=c=1$. It is easy to verify that the solutions are not all equivalent under affine transformations.

2.4. *Proper ideal solutions of degree $k=5$*

We choose parameters $a,b\in\mathbb{N}$ such that $\mathrm{lin}\{(2a,0)\}$, $\mathrm{lin}\{(b,b)\}$, $\mathrm{lin}\{(a,3a)\}$, $\mathrm{lin}\{(0,2b)\}$, $\mathrm{lin}\{(-a,3a)\}$ and $\mathrm{lin}\{(-b,b)\}$ are different directions. It can be easily verified that

$$F_1=\big\{(0,0),(2a+b,b),(3a+b,3a+3b),(2a,6a+4b),(-b,6a+3b),(-a-b,3a+b)\big\},$$
$$F_2=\big\{(2a,0),(3a+b,3a+b),(2a+b,6a+3b),(0,6a+4b),(-a-b,3a+3b),(-b,b)\big\}$$

have equal X-rays along all six directions and yield solutions to $\mathrm{PTE}_2(5)$. Note that $(a,3a+2b)$ is a point of symmetry. A convex solution is, for example, obtained by taking $a=b=1$. It is easy to verify that the solutions are not all equivalent under affine transformations.

3. Nonexistence of solutions to $\mathrm{GP}_2(k)$

In this section we prove that Problem $\mathrm{GP}_2(k)$ and Problem 1 are equivalent. Subsequently we prove that these problems do not admit solutions for $k\notin\{1,2,3,5\}$. Because of a result of Gardner and Gritzmann [5] only the case $k=4$ has to be treated here.

For the next lemma it is irrelevant that the vertices are in $\mathbb{Z}^2$.

Lemma 5. *Let $k\in\mathbb{N}$. The solutions to $GP_2(k)$ are precisely the solutions to Problem* 4.

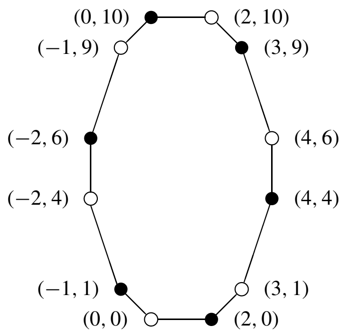

Fig. 2. Solution of $\mathrm{GP}_2(5)$ with $a = b = 1$. White points indicate the points of F_1; black points indicate the points of F_2.

Proof. ($\Rightarrow$) Suppose there are a set of $k + 1$ distinct directions and two sets of $k + 1$ points such that the sets have equal X-rays along all the directions. Consider the convex hull of the $2(k + 1)$ points. In each of the $k + 1$ directions there are two lines such that each line contains a point from each set and there are no points outside the strip between the lines. Hence the $2(k + 1)$ points form a convex $(2k + 2)$-gon where each point is a proper vertex and two opposite sides are parallel. Moreover, because of the tomographic property, every line parallel to a side contains as many vertices from one set as from the other, hence an even number of vertices in total.

($\Leftarrow$) Let a convex $(2k + 2)$-gon be given such that every line parallel to a side contains an even number of vertices. Go around the polygon and put the vertices alternately in set F_1 and in set F_2. So we get two sets of $k + 1$ points each such that every side of the polygon contains a point from F_1 and one from F_2. The sides of the polygon define $k + 1$ distinct directions. Start with any two adjacent vertices. They belong to different sets and are connected by a line in one of the $k + 1$ directions. Move the line keeping its direction towards the other vertices. If the next vertex is met by the shifting line, another vertex is met simultaneously by the condition of Problem 4. Because of the convexity there are no more than two vertices on the line. By an induction argument one vertex is adjacent to a point in F_1 and therefore in F_2 and the other is adjacent to a point in F_2 and therefore in F_1. We conclude that F_1 and F_2 have equal X-rays along the $k + 1$ directions determined by the sides of the polygon. □

By the result of Gardner and Gritzmann [5, Theorem 4.5], there do not exist lattice $\mathcal{L}$-gons for more than six directions, i.e., the solution for $k = 5$ with 6 points is the solution to $\mathrm{GP}_2(k)$ with maximal k. After Section 2 the only remaining value to be considered is $k = 4$. In view of Lemma 5 it suffices to establish the following result to show that there are no solutions in that case.

Theorem 6. *There does not exist a proper convex* 10*-gon with all vertices in* $\mathbb{Z}^2$ *such that every line parallel to a side contains an even number of vertices.*

Proof. Suppose there exists a convex 10-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains 0 or 2 vertices. Arrange the five directions of the sides in such a way that when going around counterclockwise the directions are $(p_0, q_0), \ldots, (p_4, q_4)$, $-(p_0, q_0), \ldots, -(p_4, q_4)$. We apply a rational transformation such that the first direction becomes $\mathrm{lin}\{(1, 0)\}$ and the fourth $\mathrm{lin}\{(0, 1)\}$. The lattice need no longer consist of integer points, but there exists a positive integer D such that the coordinates of the images of all the lattice

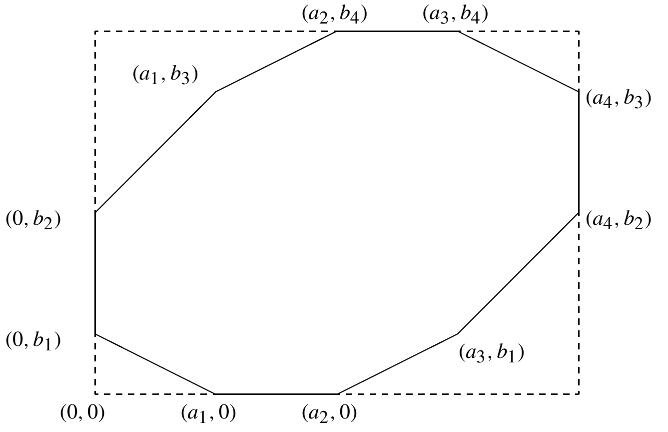

Fig. 3. Location of the 10-gon after the affine transformation. There are horizontal and vertical sides.

points are multiples of D^{-1}. Thus, by replacing all coordinates by their D-multiple, we obtain a convex 10-gon with all vertices in $\mathbb{Z}^2$ such that every line parallel to a side contains 0 or 2 vertices, and the first direction is the direction of the positive x-axis and the fourth direction that of the positive y-axis.

Without loss of generality we may assume that the ten vertices of the 10-gon in counterclockwise order are given by

$$(a_1, 0), (a_2, 0), (a_3, b_1), (a_4, b_2), (a_4, b_3), (a_3, b_4), (a_2, b_4), (a_1, b_3), (0, b_2), (0, b_1).$$

(See Fig. 3.) Here we have already used that on every horizontal and every vertical line there are 0 or 2 vertices.

The tomography condition in the fifth and the second direction imply that

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3 - b_1}{a_3 - a_1} = \frac{b_4 - b_2}{a_4 - a_2} = \frac{b_4 - b_3}{a_4 - a_3}$$

and

$$\frac{b_2}{a_4 - a_1} = \frac{b_1}{a_3 - a_2} = \frac{b_3 - b_1}{a_4} = \frac{b_4 - b_2}{a_3} = \frac{b_4 - b_3}{a_2 - a_1},$$

respectively. Hence

$$\frac{a_4 - a_1}{a_2} = \frac{a_4}{a_3 - a_1} = \frac{a_3}{a_4 - a_2} = \frac{a_2 - a_1}{a_4 - a_3} = \frac{a_3 - a_2}{a_1}.$$

Put $a := a_1$, $b := a_2 - a_1$, $c := a_3 - a_2$, $d := a_4 - a_3$. Then we obtain

$$\frac{b + c + d}{a + b} = \frac{a + b + c + d}{b + c} = \frac{a + b + c}{c + d} = \frac{b}{d} = \frac{c}{a}.$$

Successively we find $ab = cd$, $ad + cd = bc$, $db + d^2 = b^2$. However, the latter equation is not solvable in nonzero integers. $\square$

4. The relation between GP_2 and PTE_2

In this section we show that every proper solution of GP_2 leads to a proper solution of PTE_2. However, it is not clear whether the converse is true. For notational convenience, we write

$$\{(x_1, y_1), \dots, (x_n, y_n)\} \stackrel{k}{=} \{(x'_1, y'_1), \dots, (x'_n, y'_n)\}$$

if the sets $\{(x_1, y_1), \dots, (x_n, y_n)\}$ and $\{(x'_1, y'_1), \dots, (x'_n, y'_n)\}$ are solutions to PTE_2 of degree k.

Lemma 7. *Given $k+1$ different directions $\operatorname{lin}\{(p_i, q_i)\}$ $(i = 0, \dots, k)$, the polynomials*

$$(q_0 x - p_0 y)^k, \dots, (q_k x - p_k y)^k \in \mathbb{R}[x, y]$$

form a basis of the vector space V_k, which is generated by the monomials $y^k, x^1 y^{k-1}, \dots, x^{k-1} y^1, x^k$.

Proof. Every polynomial $(q_i x - p_i y)^k$ can be expressed in the monomial basis $\mathcal{B} = (y^k, x^1 y^{k-1}, \dots, x^{k-1} y^1, x^k)$ as $(\binom{k}{0}(q_i)^0(-p_i)^k, \dots, \binom{k}{k}(q_i)^k(-p_i)^0)$. Thus we have to show only that these $k+1$ vectors are linearly independent, i.e., we have to show that the matrix

$$C = \left(\binom{k}{j}(q_i)^j(-p_i)^{k-j}\right)_{i,j=0,1,\dots,k} \in \mathbb{R}^{(k+1)\times(k+1)}$$

is nonsingular. Suppose first, that $p_0 \cdots p_k \neq 0$. By setting $t_i = -q_i/p_i$, and by denoting the Vandermonde matrix $(t_i^j)_{i,j=0,\dots,k}$ by C', we obtain

$$\det(C) = \det(C') \cdot \prod_{i=0}^{k} \binom{k}{i}(-p_i)^k = \prod_{i>j}(t_i - t_j) \cdot \prod_{i=0}^{k} \binom{k}{i}(-p_i)^k \neq 0,$$

since $t_{i_0} = t_{j_0}$, that is $\frac{q_{i_0}}{p_{i_0}} = \frac{q_{j_0}}{p_{j_0}}$, contradicts that the $k+1$ directions are different. Now suppose that one of the p_i is zero. Without loss of generality we may assume that $p_0 = 0$. Note that then $p_i \neq 0$ for $i > 0$. The first row of C is now a nonzero multiple of $(0, \dots, 0, 1)$. By developing $\det(C)$ with respect to the first row, we see that the same argument as in the first case applies again. □

Theorem 8. *Let two different sets $F_1 = \{(x_1, y_1), \dots, (x_n, y_n)\}$, and $F_2 = \{(x'_1, y'_1), \dots, (x'_n, y'_n)\}$ of points from $\mathbb{Z}^2$, which have equal X-rays along $k+1$ different directions, be given. Then F_1 and F_2 are proper solutions to PTE_2 of degree k, i.e.,*

$$\sum_{i=1}^{n} x_i^j y_i^d = \sum_{i=1}^{n} x_i'^j y_i'^d, \quad \textit{for all integers } d, j \geqslant 0 \textit{ with } d + j \leqslant k.$$

Proof. Let us denote the directions by $\operatorname{lin}\{(p_i, q_i)\}$, $i = 0, 1, \dots, k$. These directions are lattice directions since $F_1, F_2 \subset \mathbb{Z}^2$.

For every $d, j \in \mathbb{N}$ with $d + j \leqslant k$, we know by Lemma 7 that there are $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that

$$x^j y^d = \sum_{i=0}^{k} \alpha_i (q_i x - p_i y)^{j+d}. \tag{1}$$

For $i = 1, \ldots, k$ on every line g in direction $\mathrm{lin}\{(p_i, q_i)\}$, there is a one-to-one correspondence between points of $F_1 \cap g$ and $F_2 \cap g$, thus

$$\big\{(q_i x_1 - p_i y_1), \ldots, (q_i x_n - p_i y_n)\big\} = \big\{\big(q_i x_1' - p_i y_1'\big), \ldots, \big(q_i x_n' - p_i y_n'\big)\big\}.$$

Thus, if we evaluate (1) at the points of F_1 and F_2 we obtain:

$$\begin{aligned} \sum_{l=1}^{n} x_l^j y_l^d - \sum_{l=1}^{n} x_l'^j y_l'^d &= \sum_{l=1}^{n} \sum_{i=0}^{k} \alpha_i (q_i x_l - p_i y_l)^{d+j} - \sum_{l=1}^{n} \sum_{i=0}^{k} \alpha_i \big(q_i x_l' - p_i y_l'\big)^{d+j} \\ &= 0 \end{aligned} \tag{2}$$

for all $d, j \in \mathbb{N}$ with $1 \leqslant d + j \leqslant k$. Because of $|F_1| = |F_2| = n$ we obtain that (2) also holds if $d + j = 0$. This means that we obtain a solution to PTE_2:

$$\big\{(x_1, y_1), \ldots, (x_n, y_n)\big\} \stackrel{k}{=} \big\{\big(x_1', y_1'\big), \ldots, \big(x_n', y_n'\big)\big\}.$$

Note that the solution is proper because F_1 and F_2 have equal X-rays along $k + 1 \geqslant 2$ different lattice directions. □

Corollary 9. *The parametric solutions given in Section* 2 *provide solutions to* $\mathit{PTE}_2(k)$ *for values* $k \in \{1, 2, 3, 5\}$.

Remark 10. Solutions of PTE_2 lead to solutions of PTE_1, e.g., by taking only the x-components of the solution. It may happen that $\{x_1, \ldots, x_n\} = \{x_1', \ldots, x_n'\}$, hence they provide trivial solutions to PTE_1. But it is always possible to rotate two distinct proper solutions to $\mathrm{PTE}_2(k)$ (having equal X-rays) in such a way that they do not have equal X-rays along the direction $\mathrm{lin}\{(1, 0)\}$. Since $\{x_1, \ldots, x_n\} \neq \{x_1', \ldots, x_n'\}$, they lead to a nontrivial $\mathrm{PTE}_1(k)$ solution. Consequently, the stated $\mathrm{PTE}_2(k)$ solutions all lead to parametric $\mathrm{PTE}_1(k)$ solutions by rotating (or by affine transforming) the sets F_1 and F_2.

Remark 11. A solution to $\mathrm{PTE}_2(k)$ may, by rotating in different ways and projecting it afterwards on the x-axis, lead to different solutions to $\mathrm{PTE}_1(k)$ which are not affinely equivalent. For example, by applying the affine transformation $x \mapsto Ax$ with $A = \left(\begin{smallmatrix} -2 & 8 \\ -1 & -2 \end{smallmatrix}\right)$ to the sets in Fig. 2 we obtain the following degree 5 solutions to $\mathrm{PTE}_2(5)$ (see Fig. 4):

$$\begin{aligned} F_1' &= \big\{(0,0), (2,-5), (40,-16), (76,-22), (74,-17), (36,-6)\big\}, \\ F_2' &= \big\{(-4,-2), (24,-12), (66,-21), (80,-20), (52,-10), (10,-1)\big\}. \end{aligned}$$

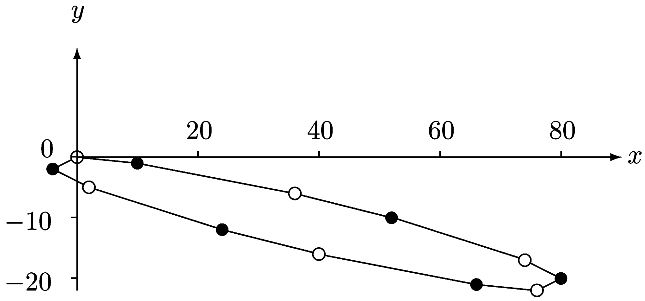

Fig. 4. Affine image of the sets from Fig. 2 representing two nonequivalent $\mathrm{PTE}_1(5)$ solutions (see Remark 11). The white points represent the points of F_1'; the black points represent the points of F_2'.

It can be easily checked that the projection on the x-axis, i.e.

$$\{0, 2, 40, 76, 74, 36\} \stackrel{5}{=} \{-4, 24, 66, 80, 52, 10\}$$

and the projection on the y-axis

$$\{0, -5, -16, -22, -17, -6\} \stackrel{5}{=} \{-2, -12, -21, -20, -10, -1\},$$

provide not affinely equivalent solutions to $\mathrm{PTE}_1(5)$.

5. A generalization of a theorem of Prouhet

If one cannot prove the existence of ideal solutions to PTE_2 for given k, one may wonder for which k and n PTE_2 is solvable. We prove a theorem about the existence of large solutions to PTE_2, which is in the spirit of the theorem proved by Prouhet (see [9,10]).

Theorem 12. *For every degree $k \in \mathbb{N}$ there exist proper solutions*

$$\{(x_1, y_1), \ldots, (x_n, y_n)\} \stackrel{k}{=} \{(x_1', y_1'), \ldots, (x_n', y_n')\}$$

to PTE_2, where $n = 2^k$.

Proof. Given $k + 1$ different lattice directions $\mathrm{lin}\{(p_0, q_0)\}, \ldots, \mathrm{lin}\{(p_k, q_k)\}$, we have to construct sets of cardinality 2^k having equal X-rays along these directions, thus leading to PTE_2 solutions by Theorem 8. They can be obtained by taking $U_2, V_2 \subset \mathbb{Z}^2$, where $U_2 = \{(0, 0), (p_1 + p_2, q_1 + q_2)\}$, $V_2 = \{(p_1, q_1), (p_2, q_2)\}$, and by recursively defining

$$U_{i+1} = V_i \cup (\theta_{i+1}(p_{i+1}, q_{i+1}) + U_i), \qquad V_{i+1} = U_i \cup (\theta_{i+1}(p_{i+1}, q_{i+1}) + V_i),$$

for $i = 2, \ldots, k + 1$. The $\theta_i \in \mathbb{Z}$ have to be chosen such that $V_i \cap (\theta_{i+1}(p_{i+1}, q_{i+1}) + U_i) = \emptyset$ and $U_i \cap (\theta_{i+1}(p_{i+1}, q_{i+1}) + V_i) = \emptyset$. It is clear that if the θ_i are chosen sufficiently large, then this can be achieved. The sets $F_1 = U_{k+1}$ and $F_2 = V_{k+1}$ have the desired properties by definition. □

6. The Prouhet–Tarry–Escott problem over the Gaussian integers

To our knowledge, PTE_1 over the Gaussian integers has not been investigated yet. Thus, we ask for two distinct sets $\{\xi_1,\dots,\xi_n\}$, $\{\xi'_1,\dots,\xi'_n\}$ of Gaussian integers such that $\sum_{i=1}^n \xi_i^d = \sum_{i=1}^n \xi_i'^d$ for $d=0,\dots,k$. Of course, we obtain solutions of points lying on a straight line from integer solutions of PTE_1. But proper Gaussian solutions can be obtained from proper solutions of PTE_2, setting $\xi_i = x_i + y_i i$, $\xi'_i = x'_i + y'_i i$.

Besides, there are proper Gaussian solutions to PTE which do not arise as solutions of the mentioned form. This can been seen, for example, by considering

$$\{0, 2i, 2+i\} \stackrel{2}{=} \{i, 1+i, 1+i\}.$$

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