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Radii of Regular Polytopes*

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Abstract. For the first time complete lists of two pairs of inner and outer radii classes of the three types of regular polytopes which exist in all dimensions are presented. A new approach using isotropic polytopes provides better understanding of the underlying geometry and helps to unify the results.

1. Introduction

There are three types of regular polytopes which exist in every dimension d: regular simplices, (hyper-) cubes, and regular cross-polytopes. In this paper we investigate two pairs of inner and outer j-radii, (r_j, R_j) and $(\bar{r}_j, \bar{R}_j)$, of these polytopes (inner and outer radii classes are almost always considered in pairs, such that for a 0-symmetric body K and its dual K° the inner (outer) radii of K are the reciprocal values of the outer (inner) radii of K° [9]).

The inner j-radii r_j and $\bar{r}_j$ of a body K are defined as the radii of the largest j-balls contained in j-dimensional slices $K \cap F$ of K, whereby the value of r_j is obtained from maximizing and the value of $\bar{r}_j$ is obtained from minimizing over the possible directions of F. The outer j-radii R_j and $\bar{R}_j$ of a body K are defined as the radii of the smallest j-balls containing the projection of K onto j-dimensional subspaces F, whereby the value of R_j is obtained from minimizing and the value of $\bar{R}_j$ is obtained from maximizing over the possible directions of F. One should note that $r_d = \bar{r}_d$ is the usual inradius and $R_d = \bar{R}_d$ is the usual outer radius. Moreover, it is well known [3] that $R_1 = \bar{r}_1$ is half the width and $r_1 = \bar{R}_1$ is half the diameter. The inner radii r_j are also known as Bernstein diameters and the outer radii R_j as Kolmogorov diameters (or sometimes Kolmogorov width).

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The inner radii r_j of regular simplices were studied in [1], where Ball uses a well known result of John [10], which also plays an important role in our computation of the outer radii R_j of regular simplices.

Until recently we thought that besides the classical results of Steinhagen [15] and Jung [11] about the outer 1- and the outer d-radii, respectively, the R_j's of regular simplices were computed only in the case that $j = d - 1$ by Weißbach [16], [17]. The apparently open cases originally stimulated our work. However, on the one hand it turned out that in the Russian literature Pukhov [13] had already computed the R_j's of regular simplices in the remaining cases. On the other hand, in [4] it was shown that the proof of Weißbach for the $(d-1)$-case with even d contained a crucial error.

The $\bar{R}_j$'s and the $\bar{r}_j$'s of regular simplices were considered in [9] and [2], respectively. While the $\bar{R}_j$'s were completely listed, the $\bar{r}_j$'s could only be computed in several special cases. However, a lower bound was given and a criterion when this bound is attained. We show that this criterion is fulfilled in all remaining cases, which means that we can now complete this list.

As the last piece to complete the radii of regular simplices, the result about the $(d-1)$-case for even d could recently be reestablished in [5].

If we turn to the other two types of regular polytopes, it follows immediately from their (central) symmetry that $r_d = R_1$ and $r_1 = R_d$ and therefore that the $\bar{r}_j$'s and $\bar{R}_j$'s do not depend on j. Hence we concentrate our attention on (r_j, R_j) in case of symmetric bodies.

Pukhov gives references for papers in which the R_j's of regular cross-polytopes are computed, from which it is possible to deduce the r_j's of cubes via polarization [8]. Everett et al. [6] give a recursive formula for the inner radii of general d-dimensional boxes, which generalizes the cited result about cubes. However, Everett et al. obviously were not aware of the papers cited by Pukhov, since they thought that even the inner radii of cubes were not known previously, except from trivial cases and the inner 2-radius of a 3-cube, computed by Shklarsky et al. [14].

It seems the outer radii of boxes (and/or the inner radii of general cross-polytopes) are unknown. This gap is closed by showing that these radii are the circumradii of smallest j-faces of boxes. Table 1 summarizes the results about regular polytopes.

However, instead of just putting together the radii from all the authors cited above, we provide a unifying approach. It will be shown that all the (r_j, R_j) radii of regular polytopes, apart from the r_j's of regular simplices, can be obtained from the R_j's of regular simplices. Moreover, the $\bar{r}_j$'s of regular simplices can be obtained from the R_j's of regular simplices, in almost all cases.

Pukhov used the result about the R_j's of regular cross-polytopes in his computation of the R_j's of regular simplices, which would lead to an improper circular closing of our chain of proof. This is one reason to provide a completely new proof. Another is the strong connection to isotropic polytopes (Kawashima called them π-polytopes [12], but we prefer to call them isotropic as they are in an isotropic position in the sense of [7]). Specifically, we show that the existence of a $(j, d+1)$-isotropic polytope is equivalent to the existence of a j-dimensional projection of the regular d-simplex such that a previously computed general lower bound for the outer radii is attained. Afterwards we state a way of constructing $(j, d+1)$-isotropic polytopes for arbitrary pairs (j, d), except for the cases where d is even and $j \in \{1, d-1\}$, showing that the lower bound is tight in all but the two exceptional cases.

Table 1. For the first time, a complete table of the radii of the three types of regular polytopes can be given. The polytopes are scaled such that their circumradius is 1.

Radii	Regular simplex	Cube	Regular cross-polytope
R_j	$\sqrt{\frac{j}{d}}$, $j \notin \{1, d-1\}$ or d odd $\frac{d+1}{d}\sqrt{\frac{1}{d+2}}$, $j = 1$ and d even $\frac{2d-1}{2d}$, $j = d-1$ and d even	$\sqrt{\frac{j}{d}}$	$\sqrt{\frac{j}{d}}$
r_j	$\sqrt{\frac{d+1}{j(j+1)d}}$	$\sqrt{\frac{1}{j(d+1)}}$	$\sqrt{\frac{1}{j(d+1)}}$
$\bar{R}_j$	$\sqrt{\frac{j(d+1)}{(j+1)d}}$	1	1
$\bar{r}_j$	$\sqrt{\frac{1}{j(d+1)}}$, $j \notin \{1, d-1\}$ or d odd $\frac{d+1}{d}\sqrt{\frac{1}{d+2}}$, $j = 1$ and d even $\frac{2}{\sqrt{d(d+2)}+\sqrt{d(d-2)}}$, $j = d-1$ and d even	$\sqrt{\frac{1}{d(d+1)}}$	$\sqrt{\frac{1}{d(d+1)}}$

We will then show that the lower bound criterion of [2] is fulfilled in almost all cases (all open cases), such that the $\bar{r}_j$'s of regular simplices can be completed.

Finally, it is shown how to deduce the radii of cubes and regular cross-polytopes from the results about the R_j's of regular simplices. Formulas for the radii of general boxes and cross-polytopes, as mentioned above, are stated.

2. Preliminaries

Let $\mathbb{E}^d = (\mathbb{R}^d, \|\cdot\|)$ denote the d-dimensional *Euclidean space*, let $\mathbb{B}^d$ and $\mathbb{S}^{d-1}$ be the *unit ball* and the *unit sphere* in $\mathbb{E}^d$, and let $\langle\cdot,\cdot\rangle$ be the usual *scalar product* $\langle x, y\rangle = x^T y$. Furthermore, we use $\{e_1, \dots, e_d\}$ for the *standard basis* of $\mathbb{E}^d$. A set $K \subset \mathbb{E}^d$ is called a *body* if it is bounded, closed, convex, and contains an inner point. For every body $K \subset \mathbb{E}^d$ let $K^\circ = \{y \in \mathbb{E}^d\colon \langle x, y\rangle \le 1 \text{ for all } x \in K\}$ denote the *polar* of K.

By $\mathcal{L}_{j,d}$ and $\mathcal{A}_{j,d}$ we denote the set of all j-dimensional *linear subspaces* and all j-dimensional *affine subspaces* of $\mathbb{E}^d$, respectively. For any $E \in \mathcal{L}_{j,d}$ let $E^\perp \in \mathcal{L}_{d-j,d}$ be the *orthogonal complement* of E. Let $\operatorname{lin}\{s_1, \dots, s_j\}$ denote the *linear span* $\{x \in \mathbb{E}^d\colon x = \sum_{k=1}^{j} \lambda_k s_k,\ \lambda \in \mathbb{E}^j\}$ of $s_1, \dots, s_j \in \mathbb{S}^{d-1}$. For any set $A \subset \mathbb{E}^d$ the (*orthogonal*) *projection* of A onto $E \in \mathcal{L}_{j,d}$ is denoted by $A|E$. For any $x \in \mathbb{E}^{d_1}$ and $y \in \mathbb{E}^{d_2}$

let $x \otimes y$ denote the rank 1 matrix with elements $x_i y_j$, $i = 1, \dots, d_1$, $j = 1, \dots, d_2$, and note that for any set of orthonormal vectors $\{s_1, \dots, s_j\}$ the projection P of $\mathbb{E}^d$ onto $\mathrm{lin}\{s_1, \dots, s_j\}$ can be represented by the matrix $\sum_{l=1}^{j} s_l \otimes s_l$. For any two sets $A, B \subset \mathbb{E}^d$ the *Minkowski sum* $A + B$ is defined as $A + B = \{a + b \in \mathbb{E}^d\colon\ a \in A, b \in B\}$ and for any $\lambda \in \mathbb{R}$ we use $\lambda A = \{\lambda a\colon\ a \in A\}$.

For any convex set K let $r(K)$ and $R(K)$ denote the inner and outer radius of K, respectively. Now, for any $j \in \{1, \dots, d\}$ the *inner j-radii* of K are defined by

$$r_j(K) = \max_{E \in \mathcal{L}_{j,d}} \max_{q \in \mathbb{E}^d} r(K \cap (E + q)),$$

$$\bar{r}_j(K) = \min_{E \in \mathcal{L}_{j,d}} \max_{q \in \mathbb{E}^d} r(K \cap (E + q)),$$

and the *outer j-radii* by

$$R_j(K) = \min_{E \in \mathcal{L}_{j,d}} R(K \mid E),$$

$$\bar{R}_j(K) = \max_{E \in \mathcal{L}_{j,d}} R(K \mid E).$$

Equivalently, the outer radii can be defined in terms of enclosing cylinders. That means, defining a *j-cylinder* as the set $F + q + \rho(\mathbb{B} \cap F^{\perp})$, for $F \in \mathcal{L}_{d-j,d}$, $q \in \mathbb{E}^d$, and radius $\rho > 0$, then, e.g., $R_j(K)$ is the minimal radius of a K enclosing j-cylinder.

Surely, for all $1 \le j \le d$ the inner and outer j-radii are invariant under translation and rotation. Furthermore, if the convex body is scaled by a factor ρ, so are its radii. For this reason, we use the term "ball" to signify any body similar (in the above sense) to $\mathbb{B}^d$, and we do the same for simplices, cross-polytopes, and boxes.

Let T^d denote the *regular d-simplex* of circumradius $R(T^d) = 1$, which we assume to be embedded in $\mathbb{E}^{d+1}$ as $T^d = \sqrt{((d+1)/d)}\,\mathrm{conv}\{e_1, \dots, e_{d+1}\}$. By $B_{a_1,\dots,a_d}$ we denote a d-dimensional *box* of the form $\left\{x \in \mathbb{E}^d\colon\ -a_i \le x_i \le a_i, i \in \{1, \dots, d\}\right\}$ and the *cube* $\sqrt{(1/d)}B_{1,\dots,1}$ is denoted by C^d. Finally, a general *cross-polytope* $X_{a_1,\dots,a_d} = \mathrm{conv}\{\pm a_1 e_1, \dots, \pm a_d e_d\}$ is just the polar of $B_{1/a_1,\dots,1/a_d}$ and especially the *regular cross-polytope* $X^d = \mathrm{conv}\{\pm e_1, \dots, \pm e_d\}$ is the polar of $\sqrt{d}\,C^d$.

3. Regular Simplices

The following results about the r_j's and $\bar{R}_j$'s of regular simplices are taken from [1] and [9], respectively.

Proposition 3.1. *For all* $1 \le j \le d$,

(i) $r_j(T^d) = \sqrt{(d+1)/j(j+1)d}$ *and*
(ii) $\bar{R}_j(T^d) = \sqrt{j(d+1)/(j+1)d}$.

In both cases the extreme j-spaces are parallel to the j-faces of T^d.

Definition 3.2. Any set of orthonormal vectors $\{s_1, \ldots, s_j\} \subset \mathbb{E}^{d+1}$, $1 \le j \le d$, is called a

(i) *valid subset basis* (*vsb* for short) if $\sum_{k=1}^{d+1} s_{lk} = 0$, $1 \le l \le j$, and
(ii) *good subset basis* (*gsb*) if it is a vsb and $\sum_{l=1}^{j} s_{lk}^2 = j/(d+1)$, $1 \le k \le d+1$.

Note that any set of orthonormal vectors $\{s_1, \ldots, s_j\}$ is a vsb if it spans a j-dimensional subspace of $\mathbb{E}_0^d = \{x \in \mathbb{E}^{d+1}\colon \sum_{k=1}^{d+1} x_k = 0\}$, the d-dimensional linear subspace of $\mathbb{E}^{d+1}$ parallel to the hyperplane in which T^d is embedded.

The projection of T^d onto $\mathbb{E}_0^d$ can be written as $I^{d+1} - (1/(d+1))\mathbf{1}^{d+1}$, where I^{d+1} denotes the identity matrix in $\mathbb{E}^{(d+1)\times(d+1)}$ and $\mathbf{1}^{d+1}$ the matrix in $\mathbb{E}^{(d+1)\times(d+1)}$ consisting only of ones. Hence $\sum_{l=1}^{d} s_l \otimes s_l = I^{d+1} - (1/(d+1))\mathbf{1}^{d+1}$, for every vsb of d elements. This implies the important fact that each vsb is a gsb if $j = d$, which will be used in Corollary 3.4.

Now we start computing the outer radii of regular simplices by giving a general lower bound, which turns out to be tight in almost all cases. This lemma also motivates why we call a vsb good if it fulfills condition (ii) in Definition 3.2.

Lemma 3.3. $R_j(T^d) \ge \sqrt{j/d}$ *for all* $1 \le j \le d$, *and equality holds if and only if there exists a gsb* $\{s_1, \ldots, s_j\} \subset \mathbb{E}^{d+1}$.

Proof. Let P denote the projection onto a subspace spanned by a vsb $\{s_1, \ldots, s_j\}$. It follows that

$$\|Pe_k\|^2 = \langle Pe_k, e_k\rangle = \left\langle \sum_{l=1}^{j} s_{lk}s_l, e_k \right\rangle = \sum_{l=1}^{j} s_{lk}^2.$$

Now assume there exists an $x \in \mathbb{E}^{d+1}$ such that $\|x - Pe_k\|^2 < j/(d+1)$ for all $k = 1, \ldots, d+1$. Summing over the k's it follows that

$$\begin{aligned} j &> \sum_{k=1}^{d+1} \|x - Pe_k\|^2 \\ &= \sum_{k=1}^{d+1} (\|x\|^2 - 2\langle x, Pe_k\rangle + \|Pe_k\|^2) \\ &= (d+1)\|x\|^2 - 2\left\langle x, \sum_{k=1}^{d+1}\sum_{l=1}^{j} s_{lk}s_l \right\rangle + \sum_{k=1}^{d+1}\sum_{l=1}^{j} s_{lk}^2 \end{aligned}$$

and since $\sum_{k=1}^{d+1} s_{lk} = 0$ and $\sum_{k=1}^{d+1} s_{lk}^2 = 1$ the last expression can be simplified to $(d+1)\|x\|^2 + j \ge j$, which is a contradiction. This proves the first part of the lemma. In order to show the second part we note that equality in $\|x - Pe_k\|^2 \le j/(d+1)$ for all k can only be obtained if $x = 0$ and $\sum_{l=1}^{j} s_{lk}^2 = j/(d+1)$ for each k. $\square$

As every vsb of d vectors is already a gsb, the following corollary can be obtained from Lemma 3.3 by the basis extension property (used on $\mathbb{E}_0^d$).

Corollary 3.4. *For any $1 \leq j \leq d-1$ it holds that $R_j(T^d) = \sqrt{j/d}$ if and only if $R_{d-j}(T^d) = \sqrt{(d-j)/d}$. Moreover, there always exists a pair of corresponding optimal projections which take place in orthogonal subspaces.*

Since Steinhagen [15] showed that

$$R_1(T^d) = \begin{cases} \sqrt{\dfrac{1}{d}} & \text{if } d \text{ odd}, \\ \dfrac{d+1}{d}\sqrt{\dfrac{1}{d+2}} & \text{if } d \text{ even}, \end{cases} \tag{1}$$

Corollary 3.4 implies for the outer $(d-1)$-radius that the lower bound of Lemma 3.3 is attained for odd dimensions, but that it is not attained for even dimensions. The following formula was claimed in [13] and first shown in [17]. Although the proof in [17] contained a crucial error, the correctness of the formula reestablished recently [5].

Proposition 3.5.

$$R_{d-1}(T^d) = \frac{2d-1}{2d} \qquad \textit{if } d \textit{ is even.}$$

We will soon see that $j \in \{1, d-1\}$ for even d are the only cases where the lower bound is not attained.

The following proposition was first shown in [10] (see also [1]).

Proposition 3.6. *$\mathbb{B}^j$ is the ellipsoid of minimal volume containing some body $K \subset \mathbb{E}^j$ if and only if $K \subset \mathbb{B}^j$ and for some $m > j$ there are unit vectors $u_1, \ldots, u_m$ on the boundary of K, and positive numbers $c_1, \ldots, c_m$ summing to j such that*

(i) $\sum_{i=1}^m c_i u_i = 0$ *and*
(ii) $\sum_{i=1}^m c_i u_i \otimes u_i = I^j$.

It is obvious that if K is a regular polytope all c_i can be chosen as j/m where m is the number of vertices of K. However, it is not obvious which other polytopes fulfill this property. Nevertheless, according to [7] these polytopes are in an isotropic position, corresponding to the discrete measure μ^* on $\mathbb{S}^{d-1}$ that gives mass j/m to all vertices u_i (see Section 5 of [7] for more details). This is the source for the following definition.

Definition 3.7. Let $u_1, \ldots, u_m \in \mathbb{S}^{j-1}$ (not necessarily different) and $K = \text{conv}\{u_1, \ldots, u_m\}$. K is called *(j, m)-isotropic* if all the c_i's in Proposition 3.6 can be taken as j/m.

Lemma 3.8. *There exists a gsb $\{s_1, \ldots, s_j\}$ of $\mathbb{E}^{d+1}$ if and only if there exists a $(j, d+1)$-isotropic polytope $K = \text{conv}\{u_1, \ldots, u_{d+1}\} \subset \mathbb{E}^j$, $j \leq d$. Moreover, if T^d is projected onto $\text{lin}\{s_1, \ldots, s_j\}$ the projection equals the corresponding K up to rotation and dilatation.*

Proof. If $K = \text{conv}\{u_1, \ldots, u_{d+1}\}$ is a $(j, d+1)$-isotropic polytope then

(i) $\|u_k\| = 1$,
(ii) $\sum_{k=1}^{d+1} u_k = 0$, and
(iii) $\sum_{k=1}^{d+1} u_k \otimes u_k = ((d+1)/j) I^j$.

Let $s_l = \sqrt{j/(d+1)}(u_{1,l}, \ldots, u_{d+1,l})^T$, $1 \leq l \leq j$. This defines a gsb, since the s_l form an orthonormal set because of (iii), $\sum_{k=1}^{d+1} s_{lk} = 0$ because of (ii), and $\sum_{l=1}^{j} s_{lk}^2 = j/(d+1)$, $1 \leq k \leq d+1$, because of (i). The other direction can be shown by a similar reasoning. Now the projections of the vertices of T^d onto $\text{lin}\{s_1, \ldots, s_j\}$ are

$$P\left(\sqrt{\frac{d+1}{d}} e_k\right) = \sum_{l=1}^{j} \sqrt{\frac{d+1}{d}} s_{lk} s_l = \sum_{l=1}^{j} \sqrt{\frac{j}{d}} u_{kl} s_l.$$

Hence the values $\sqrt{j/d}\, u_{kl}$ are just the coordinates of the projected vertices in terms of the basis $s_1, \ldots, s_j$. □

Lemma 3.8 can be used in two ways:

(i) We know that $R_j(T^d) = \sqrt{j/d}$ whenever we find a $(j, d+1)$-isotropic polytope and vice versa. For example, it follows that there cannot exist $(1, d+1)$-isotropic polytopes nor $(d-1, d+1)$-isotropic polytopes if d is even.
(ii) We know that $R_k(K) \geq \sqrt{k/j}$ for any $(j, d+1)$-isotropic polytope K and any $k \leq j$ and equality holds if and only if the gsb $\{s_1, \ldots, s_j\}$ corresponding to K can be split into two gsb's $\{s_1, \ldots, s_k\}$ and $\{s_{k+1}, \ldots, s_j\}$.

We first concentrate our attention on (i) but come back to (ii) later. The following lemma states a rule on how to construct higher-dimensional isotropic polytopes from lower-dimensional ones. It is called the *additive rule*.

Lemma 3.9. *Let $0 \leq j_i < m_i$, $i = 1, 2$, such that $m_2 j_1 > m_1 j_2$. Let $j = j_1 + j_2$, $m = m_1 + m_2$, $\alpha = \sqrt{(m_2 j_1 - m_1 j_2)/m_2 j}$, and $\beta = \sqrt{m j_2 / m_2 j}$, and suppose there exists a (j_1, m_1)-isotropic polytope $K_1 = \text{conv}\{u_1, \ldots, u_{m_1}\}$, a (j_1, m_2)-isotropic polytope $K_2 = \text{conv}\{v_1, \ldots, v_{m_2}\}$, and a (j_2, m_2)-isotropic polytope $K_3 = \text{conv}\{w_1, \ldots, w_{m_2}\}$, such that*

$$K' = \text{conv}\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}}\begin{pmatrix} v_{m_2} \\ w_{m_2} \end{pmatrix}\right\}$$

is a (j, m_2)-isotropic polytope. Then

$$K = \text{conv}\left\{\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_{m_1} \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta w_1 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha v_{m_2} \\ \beta w_{m_2} \end{pmatrix}\right\}$$

is a (j, m)-isotropic polytope.

Proof. Since $\alpha^2 + \beta^2 = 1$ all vertices of K are situated on $\mathbb{S}^{j-1}$ and obviously the origin is the centroid of K. Hence we only have to show that condition (ii) from Proposition 3.6

holds with $c_i = j/m, i = 1, \ldots, m$:

$$\sum_{i=1}^{m_1} \begin{pmatrix} u_i \\ 0 \end{pmatrix} \begin{pmatrix} u_i \\ 0 \end{pmatrix}^T + \sum_{i=1}^{m_2} \begin{pmatrix} \alpha v_i \\ \beta w_i \end{pmatrix} \begin{pmatrix} \alpha v_i \\ \beta w_i \end{pmatrix}^T$$

$$= \begin{pmatrix} (m_1/j_1) I^{j_1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (m_2/j_1)\alpha^2 I^{j_1} & 0 \\ 0 & (m_2/j_2)\beta^2 I^{j_2} \end{pmatrix}$$

$$= \begin{pmatrix} ((m_1 j + m_2 j_1 - m_1 j_2)/j_1 j_2) I^{j_1} & 0 \\ 0 & (m/j) I^{j_2} \end{pmatrix}$$

$$= \frac{m}{j} I^j. \qquad \square$$

The reader may convince himself that it is neither possible to construct a $(1, d+1)$-isotropic polytope by the additive rule if d is even, nor is it possible to construct a $(d-1, d+1)$-isotropic polytope by the additive rule at all.

If m_2 is even, a good choice for K' is often a prism

$$\operatorname{conv} \left\{ \sqrt{\frac{1}{2}} \begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}} \begin{pmatrix} v_{m_2/2} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v_1 \\ -1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}} \begin{pmatrix} v_{m_2/2} \\ -1 \end{pmatrix} \right\}$$

or antiprism

$$\operatorname{conv} \left\{ \sqrt{\frac{1}{2}} \begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}} \begin{pmatrix} v_{m_2/2} \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} -v_1 \\ -1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}} \begin{pmatrix} -v_{m_2/2} \\ -1 \end{pmatrix} \right\}$$

built from a $(j-1, m_2/2)$-isotropic base $K_2 = \operatorname{conv}\{v_1, \ldots, v_{m_2/2}\}$.

Lemma 3.10. *For every odd d and $j \in \{1, \ldots, d\}$ as well as for every even d and $j \in \{2, \ldots, d-2, d\}$ there exists a $(j, d+1)$-isotropic polytope.*

Proof. We do an inductive proof over j and d. From (1) and since every regular $(d+1)$-gon with vertices on $\mathbb{S}^1$ is $(2, d+1)$-isotropic it follows that the claim is true for pairs (j, d) with $j \leq 2$. Moreover, the claim is true for $j \geq d-2$ because of Corollary 3.4.

Now assume that the claim is true for every pair (j', d') with $j' < j$, $d' \leq d$ or $j' \leq j$, $d' < d$. Regarding the initial statements we can assume $j \geq 3$ and because of Corollary 3.4 that $j < (d+1)/2$. We start with the case $(j, d+1) = (3, 9)$ and choose $j_1 = 2, j_2 = 1, m_1 = 3, m_2 = 6$. For sure $K_1 = K_2 = T^2$ are $(2, 3)$-isotropic and also $(2, 6)$-isotropic by duplicating every vertex. Moreover $K_3 = T^1 = [-1, 1]$ is $(1, 6)$-isotropic (triplicating the two vertices) and

$$K' = \operatorname{conv} \left\{ \sqrt{\frac{1}{2}} \begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v_2 \\ 1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v_3 \\ 1 \end{pmatrix}, \right.$$

$$\left. \sqrt{\frac{1}{2}} \begin{pmatrix} v_1 \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v_2 \\ -1 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} v_3 \\ -1 \end{pmatrix} \right\}$$

is (3, 6)-isotropic. Hence K_1, K_2, and K_3 fulfill the conditions of the additive rule and therefore there exists a (3, 9)-isotropic polytope.

Next we assume that $j \geq 5$ is odd and that (as in the case before) $m = d+1 = 2j+3$. Then let $j_1 = j-2$, $j_2 = 2$, $m_1 = m-j-1$, and $m_2 = j+1$. Since $j < m/2$ it holds that $j_1 < m_1$ and since $j_1 = j-2 \neq j = m-j-3 = m_1-2$ there exists a (j_1, m_1)-isotropic polytope K_1. Completing the conditions of the additive rule we choose an m_2-gon for K_3 and the projection of T^j onto $(\operatorname{lin} K_3)^\perp$ as K_2 (thus $K' = T^j$). One should notice that $m_2 j_1 = j^2 + j > 2m = m_1 j_2$ since $j \geq 5$.

Finally, let j be even or $m \neq 2j+3$. Then let $j_1 = j$, $j_2 = 0$, $m_1 = j+1$, and $m_2 = m-j-1$. Since $j < m/2$ it holds that $m_2 > j$ and if $j+2$ is odd then $m_2 \neq j+2$ since $m \neq 2j+3$. Hence there exists a (j, m_2)-isotropic polytope K_2 by the induction hypothesis and $K_1 = T^{j+1}$ is a (j, m_1)-isotropic polytope, which obviously fulfills the conditions of the additive rule. $\square$

The following proposition is taken from [2]. It gives a lower bound for $\bar{r}_j(T^d)$ and a criterion when this lower bound is attained. For the purpose of the proposition let $a_1, \ldots, a_{d+1} \in \mathbb{S}^{d-1}$ be such that

$$T^d = \sqrt{\frac{1}{d(d+1)}}\{x \in \mathbb{E}^d\colon \langle x, a_i\rangle \leq 1, i = 1, \ldots, d+1\}.$$

Proposition 3.11. *$\bar{r}_j(T^d) \geq \sqrt{1/j(d+1)}$ for all $1 \leq j \leq d$, and equality holds if and only if there exists an $E \in \mathcal{L}_{j,d}$ such that $\|a_i|E\| = \sqrt{j/d}$ for all $i = 1, \ldots, d+1$.*

It follows from the self-duality of the regular simplex (that is, if 0 is the centroid, it holds that $(T^d)^\circ = \sqrt{d(d+1)}T^d$) that the criterion for equality in Proposition 3.11 is fulfilled if and only if $R_j(T^d) = \sqrt{j/d}$.

Theorem 3.12. *For every $1 \leq j \leq d$, such that d is odd or $j \notin \{1, d-1\}$,*

(i) $R_j(T^d) = \sqrt{j/d}$ *and*
(ii) $\bar{r}_j(T^d) = \sqrt{1/j(d+1)}$.

Proof. Statement (i) follows directly from Lemmas 3.3, 3.8, and 3.10. Statement (ii) follows from (i) and Proposition 3.11. $\square$

One should mention that $R_j(T^d)\bar{r}_j((T^d)^\circ) = 1$ for every pair (j, d) which fulfills the conditions of Theorem 3.12. Hence it follows from the self-duality of T^d that the result above combined with the results in [2] show that the minimal j-balls of T^d in the sense of $\bar{r}_j$ are at the same time the maximal j-balls contained in T^d and centered in the centroid of T^d.

For completeness we state the two remaining inner radii of regular simplices [2, Theorem 3].

Proposition 3.13. *For even d,*

$$\bar{r}_1(T^d) = R_1(T^d) = \frac{d+1}{d}\sqrt{\frac{1}{d+2}} \quad \textit{and}$$

$$\bar{r}_{d-1}(T^d) = \frac{2}{\sqrt{d(d+2)} + \sqrt{d(d-2)}}.$$

4. Boxes and Cross-Polytopes

As already mentioned in the Introduction, for all symmetric bodies K it holds that $\bar{r}_1(K) = \cdots = \bar{r}_d(K)$ and $\bar{R}_1(K) = \cdots = \bar{R}_d(K)$. Hence, we can draw our attention in this section to (r_j, R_j).

A proof of the following proposition can be found in [8].

Proposition 4.1. *Let $1 \leq j \leq d$ and let K be a 0-symmetric body. Then $r_j(K)R_j(K^\circ) = 1$ and $R_j(K)r_j(K^\circ) = 1$.*

Now we come back to the second statement after Lemma 3.8, saying that the k-radius of any (j, m)-isotropic polytope K is $\sqrt{k/j}$ if the gsb $\{s_1, \ldots, s_j\}$ corresponding to K can be partioned into a gsb $\{s_1, \ldots, s_k\}$ and a gsb $\{s_{k+1}, \ldots, s_j\}$—in other words, if K is the cross-product of a (k, m)-isotropic polytope K_1 and a $(j-k, m)$-isotropic polytope K_2. Applied to cubes and regular cross-polytopes this leads to the following corollary.

Corollary 4.2. *For all $1 \leq j \leq d$,*

(i) $R_j(C^d) = R_j(X^d) = \sqrt{j/d}$ *and*
(ii) $r_j(C^d) = r_j(X^d) = \sqrt{1/j(d+1)}$.

Proof. It suffices to show that for every $1 \leq j \leq d$ both C^d and X^d have (j, m)-isotropic projections (up to dilatation), since then (i) follows from the argument before the corollary and (ii) from Proposition 4.1.

For C^d every j-tuple of coordinate rows of its vertices describes a j-cube which is parallel to a j-face of C^d and surely isotropic.

Now consider X^d. First project T^{2d-1} onto a d-flat by using the gsb

$$\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_{d-1} \\ -s_{d-1} \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix}\right\},$$

where $\{s_1, \ldots, s_{d-1}\}$ is an arbitrary gsb for T^{d-1} and $\mathbf{1}_{d-1} = (1, \ldots, 1)^T \in \mathbb{E}^{d-1}$. The projection is $\sqrt{d/(2d-1)}X^d$. It follows from Lemma 3.10 that for every $j \notin \{1, d-2\}$ there exists a subset of size j of $\{s_1, \ldots, s_{d-1}\}$ that is again a gsb; without loss of generality $\{s_1, \ldots, s_j\}$. Also, if $j = d-2$ we can assume that $\{s_1, \ldots, s_{j-1}\}$ is a gsb. Hence the set

$$\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \ldots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_j \\ -s_j \end{pmatrix}\right\}$$

or, if $j \in \{1, d-2\}$, the set

$$\left\{\sqrt{\frac{1}{2}}\begin{pmatrix} s_1 \\ -s_1 \end{pmatrix}, \dots, \sqrt{\frac{1}{2}}\begin{pmatrix} s_{j-1} \\ -s_{j-1} \end{pmatrix}, \sqrt{\frac{1}{2}}\begin{pmatrix} \mathbf{1}_{d-1} \\ -\mathbf{1}_{d-1} \end{pmatrix}\right\}$$

is a gsb in $\mathbb{E}^{2d}$ such that the projection of T^{2d-1} is a $(j, 2d)$-isotropic polytope K. Since this gsb is a subset of the one for which $\sqrt{d/(2d-1)}X^d$ was the projection of T^{2d-1}, we obtain that $\sqrt{(2d-1)/d}\,K$ is a projection of X^d. $\square$

Corollary 4.2 can be generalized to obtain the inner and outer radii of general cross-polytopes and boxes.

The inner radii of boxes were computed in [6]. The part about outer radii of cross-polytopes follows from Proposition 4.1.

Proposition 4.3. *Let $1 \leq j \leq d$ and $0 < a_1 \leq \dots \leq a_d$. Then*

(i) $$r_j(B_{a_1,\dots,a_d}) = \sqrt{\frac{a_1^2 + \dots + a_{d-k}^2}{j-k}},$$

where $k \in \{0, \dots, j-1\}$ is the smallest integer satisfying

$$a_{d-k} \leq \sqrt{\frac{a_1^2 + \dots + a_{d-k-1}^2}{j-k-1}},$$

and

(ii) $$R_j(X_{a_1,\dots,a_d}) = \sqrt{\frac{(j-k)\prod_{i=k}^{d} a_i^2}{\sum_{i=k}^{d}\prod_{l\neq i} a_l^2}},$$

where $k \in \{0, \dots, j-1\}$ is the smallest integer satisfying

$$a_k \geq \sqrt{\frac{(j-k-1)\prod_{i=k+1}^{d} a_i^2}{\sum_{i=k+1}^{d}\prod_{l\neq i} a_l^2}}.$$

The corresponding result about the outer radii of boxes is very intuitive. It says that the minimal projection of a box is the one onto the subspace parallel to one of its smallest faces.

Theorem 4.4. *Let $1 \leq j \leq d$ and $0 < a_1 \leq \dots \leq a_d$. Then*

(i) $$R_j(B_{a_1,\dots,a_d}) = \sqrt{a_1^2 + \dots + a_j^2}$$

and

(ii) $$r_j(X_{a_1,\dots,a_d}) = \left(\prod_{i=d-j+1}^{d} a_i\right) \Big/ \sqrt{\sum_{i=d-j+1}^{d}\prod_{l\neq i} a_l^2}.$$

Proof. It suffices to show (i), since then (ii) follows from Proposition 4.1. Moreover, as the result is obvious if $d = 1$, we assume that $d \geq 2$. Any vertex v of $B_{a_1,\dots,a_d}$ can be written in the form $v = \sum_{k=1}^{d} \pm a_k e_k$ and all possible choices of pluses or minuses in this formula lead to a vertex of $B_{a_1,\dots,a_d}$. Hence, for every projection $P = \sum_{l=1}^{j} s_l \otimes s_l$ with pairwise orthogonal unit vectors $s_l \in \mathbb{E}^d$, it holds that $\|Pv\|^2 = \sum_{l=1}^{j} \langle v, s_l \rangle^2 = \sum_{l=1}^{j} (\sum_{k=1}^{d} \pm a_k s_{lk})^2$. However, since the average value of $\|Pv\|^2$ over all vertices v is $\sum_{k=1}^{d} a_k^2 \sum_{l=1}^{j} s_{lk}^2$, there exists a vertex of $B_{a_1,\dots,a_d}$ such that $\|Pv\|^2 \geq \sum_{k=1}^{d} a_k^2 \sum_{l=1}^{j} s_{lk}^2$.

Now extend the set $\{s_1, \dots, s_j\}$ to an orthonormal basis of $\mathbb{E}^d$. Since $\sum_{l=1}^{d} s_l \otimes s_l = I^d$ it follows that $\sum_{k=1}^{d} s_{lk}^2 = \sum_{l=1}^{d} s_{lk}^2 = 1$, for all $k = 1, \dots, d$ and all $l = 1, \dots, d$, respectively. Hence $t_k := \sum_{l=1}^{j} s_{lk}^2 \in [0, 1]$ and since $\sum_{k=1}^{d} t_k = \sum_{l=1}^{j} \sum_{k=1}^{d} s_{lk}^2$ has to equal j, the minimum value of $\sum_{k=1}^{d} t_k a_k^2$ is obtained for $t_1 = \cdots = t_j = 1$ and $t_{j+1} = \cdots = t_d = 0$. It follows that $R_j(B_{a_1,\dots,a_d}) \geq \sqrt{a_1^2 + \cdots + a_j^2}$. Finally, since the projection of $B_{a_1,\dots,a_d}$ through its j-face $B_{a_1,\dots,a_j}$ achieves this value, we obtain the desired result. $\square$

Compared with the radii of boxes and general cross-polytopes very little can be stated about general simplices. As Gritzmann and Klee [8] showed that the computation of $R_j(S)$ is $\mathbb{N}P$-hard for general simplices S and many j, a general formula cannot be expected. However, in [5] it could be shown that in "typical" configurations all vertices of the simplex are projected onto the minimal enclosing sphere in an optimal projection, and in [4] solution methods and a formula for a special case are given for $j = 2$ and $d = 3$.

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