Proceedings of the Edinburgh Mathematical Society (2016) **59**, 911–924
DOI:10.1017/S0013091515000322

ZERO-SEPARATING INVARIANTS FOR LINEAR ALGEBRAIC GROUPS

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(Received 27 February 2014)

Abstract Let G be a linear algebraic group over an algebraically closed field $\Bbbk$ acting rationally on a G-module V with $\mathcal{N}_{G,V}$ its null-cone. Let $\delta(G,V)$ and $\sigma(G,V)$ denote the minimal number d such that for every $v \in V^G \setminus \mathcal{N}_{G,V}$ and $v \in V \setminus \mathcal{N}_{G,V}$, respectively, there exists a homogeneous invariant f of positive degree at most d such that $f(v) \neq 0$. Then $\delta(G)$ and $\sigma(G)$ denote the supremum of these numbers taken over all G-modules V. For positive characteristics, we show that $\delta(G) = \infty$ for any subgroup G of $\mathrm{GL}_2(\Bbbk)$ that contains an infinite unipotent group, and $\sigma(G)$ is finite if and only if G is finite. In characteristic zero, $\delta(G) = 1$ for any group G, and we show that if $\sigma(G)$ is finite, then G^0 is unipotent. Our results also lead to a more elementary proof that $\beta_{\mathrm{sep}}(G)$ is finite if and only if G is finite.

Keywords: invariant theory; linear algebraic groups; geometrically reductive; prime characteristic; global degree bounds

2010 *Mathematics subject classification:* Primary 13A50

1. Introduction

In invariant theory the notion of geometric reductivity is of great importance. It implies finite generation of the invariants, the separability of disjoint orbit closures by invariants, and in characteristic zero even algebraic properties like the Cohen–Macaulayness of the invariant ring. It is defined to be the property that every non-zero fixed point of a finite-dimensional rational representation can be separated from zero by a homogeneous invariant of positive degree. Similarly, by definition every point outside the null-cone can be separated from zero by a homogeneous positive degree invariant. It is a natural question to ask what the maximum degree needed for a given representation is. While in our recent paper [**8**] we gave some (partial) answers to these questions for the case of finite groups, the current paper concentrates on the case of infinite groups. Before we go into more detail, we fix our setup.

Let G be a linear algebraic group over an algebraically closed field $\Bbbk$, let V be a finite-dimensional rational representation of G (which we will call a G-module), and denote by

$\Bbbk[V] \cong S(V^*)$ the ring of polynomial functions $V \to \Bbbk$. The action of G on V induces an action of G on $\Bbbk[V]$ via $(g \cdot f)(v) := f(g^{-1}v)$ for $g \in G$, $f \in \Bbbk[V]$ and $v \in V$. The set of G-invariant polynomial functions under this action is denoted by $\Bbbk[V]^G$, and inherits a natural grading from $\Bbbk[V]$, since the given action is degree preserving. We denote by $\Bbbk[V]^G_d$ the set of polynomial invariants of degree d and the zero-polynomial, and by $\Bbbk[V]^G_{\leqslant d}$ the set of polynomial invariants of degree at most d. For every subset S of $\Bbbk[V]$ we define S_+ as the set of elements in S with constant term zero. Then $\mathcal{N}_{G,V} := \mathcal{V}(\Bbbk[V]^G_+)$ denotes the null-cone of V. A linear algebraic group is said to be geometrically reductive if for every G-module V we have $V^G \cap \mathcal{N}_{G,V} = \{0\}$, i.e. for all non-zero $v \in V^G$ there exists $f \in \Bbbk[V]^G_+$ such that $f(v) \neq 0$. This inspires the definition of a δ-set: for a linear algebraic group G let us say a subset $S \subseteq \Bbbk[V]^G$ is a δ-*set* if, for all $v \in V^G \setminus \mathcal{N}_{G,V}$, there exists an $f \in S_+$ such that $f(v) \neq 0$. We shall call a subalgebra of $\Bbbk[V]^G$ a δ-*subalgebra* if it is a δ-set. The quantity $\delta(G, V)$ is then defined as

$$\delta(G, V) = \min\{d \geqslant 0 \mid \Bbbk[V]^G_{\leqslant d} \text{ is a } \delta\text{-set}\}.$$

Define, furthermore,

$$\delta(G) := \sup\{\delta(G, V) \mid V \text{ a } G\text{-module}\},$$

where we take the supremum of an unbounded set to be infinity. A reductive group is called linearly reductive if $\delta(G) = 1$. Note that in positive characteristics, due to Nagata (see [**12**, **14**]), a linear algebraic group G is linearly reductive if and only if its connected component G^0 is a torus such that its index $(G : G^0)$ is not divisible by the characteristic of the base field. Over a field of characteristic zero, Nagata and Miyata [**16**] have shown that reductive groups are linearly reductive. In fact their proof shows that in characteristic zero, for any linear algebraic group G and any G-module V, $\delta(G, V)$ equals 1 or 0 (the latter being the case when $V^G \subseteq \mathcal{N}_{G,V}$); see Proposition 2.1. A natural, but seemingly neglected, question is, for which geometrically reductive groups G is $\delta(G)$ strictly greater than 1, but still finite? For finite groups, Elmer and Kohls gave the following answer [**8**].

Theorem 1.1 (Elmer and Kohls [8, Theorem 1.1]). *Let G be a finite group, let $\Bbbk$ be an algebraically closed field of characteristic p and let P be a Sylow-p-subgroup of G. Then $\delta(G) = |P|$.*

Thus, $\delta(G)$ is finite for all finite groups, and strictly greater than 1 if and only if $|G|$ is divisible by p. In this paper we investigate $\delta(G)$ for infinite groups. In particular, we make and investigate the following conjecture.

Conjecture 1.2. *For G a linear algebraic group over a field of positive characteristic, we have that $\delta(G)$ is finite if and only if the connected component G^0 is a torus or trivial.*

We mention that the 'only if' part of the conjecture might only be generally true under the additional assumption that G is reductive. Our main results concerning the conjecture are that (1) the 'if' part of the conjecture holds and (2) the 'only if' part holds for G a closed subgroup of GL_2. Statement (1) follows from the following more precise theorem, which is a generalization of Theorem 1.1.

Theorem 1.3. *Let G be a linear algebraic group over a field of characteristic $p > 0$ such that G^0 is a torus or trivial. Let P be a Sylow-p-subgroup of the (finite) group G/G^0. Then $\delta(G) = |P|$.*

It is well known that an infinite linear algebraic group contains an infinite unipotent subgroup if and only if its connected component G^0 is not a torus (see, for example, [**12**, Lemmas 3.1 and 3.2]). Therefore, statement (2) follows from the following theorem.

Theorem 1.4. *Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$. Suppose that G is a closed subgroup of $\mathrm{GL}_2(\Bbbk)$ containing an infinite unipotent subgroup. Then $\delta(G) = \infty$.*

In particular, $\delta(\mathrm{SL}_2(\Bbbk)) = \delta(\mathrm{GL}_2(\Bbbk)) = \delta(\mathbb{G}_a) = \infty$ (where $\mathbb{G}_a = (\Bbbk, +)$ is the additive group of the ground field) in positive characteristics, supporting the conjecture.

In addition to $\delta(G)$, we study the closely related quantity $\sigma(G)$. We shall say a subset $S \subseteq \Bbbk[V]^G$ is a *σ-set* if, for all $v \in V \setminus \mathcal{N}_{G,V}$, there exists an $f \in S_+$ such that $f(v) \neq 0$. We shall call a subalgebra of $\Bbbk[V]^G$ a *σ-subalgebra* if it is a σ-set. Then the quantities $\sigma(G, V)$ and $\sigma(G)$ are defined along the same lines as $\delta(G, V)$ and $\delta(G)$. For a motivation of the importance of this number we content ourselves here by saying that, at least for linearly reductive groups in characteristic zero, the knowledge of $\sigma(G, V)$ gives upper bounds for the maximal degrees of generating sets (for example, in Derksen's famous bound [**4**]), and refer the reader to [**3**, **8**] for more details and some elementary properties of this number.

In the latter paper, Elmer and Kohls investigated $\sigma(G)$ for finite groups G, mainly for positive characteristic. In §§ 4 and 5 of this paper we investigate $\sigma(G)$ for infinite linear algebraic groups. Our main results are as follows.

Theorem 1.5. *Let G be a linear algebraic group over a field of characteristic $p > 0$. Then $\sigma(G)$ is finite if and only if $|G|$ is finite.*

Theorem 1.6. *Let G be a linear algebraic group over a field of characteristic 0. Then if $\sigma(G)$ is finite, G^0 is unipotent, i.e. either G is finite or G^0 is infinite unipotent.*

As reductive groups do not contain a non-trivial connected unipotent normal subgroup, we obtain the following as an immediate corollary.

Corollary 1.7. *Let G be a reductive group over a field of arbitrary characteristic. Then $\sigma(G)$ is finite if and only if G is finite.*

Somewhat surprisingly, for the (infinite) additive group $\mathbb{G}_a = (\Bbbk, +)$ of a field $\Bbbk$ of characteristic zero we will see that $\sigma(\mathbb{G}_a) = 2$. We do not know whether $\sigma(G)$ is finite for all unipotent groups in characteristic zero.

Another quantity associated with $\delta(G, V)$ and $\sigma(G, V)$, which has attracted some attention in recent years, is $\beta_{\mathrm{sep}}(G, V)$. It is defined as follows: a subset $S \subseteq \Bbbk[V]^G$ is called a *separating set* if, for every pair $v, w \in V$ such that there exists $f \in \Bbbk[V]^G$ with $f(v) \neq f(w)$, there exists $s \in S$ with $s(v) \neq s(w)$. Now again, $\beta_{\mathrm{sep}}(G, V)$ and $\beta_{\mathrm{sep}}(G)$

are defined along the same lines as $\sigma(G,V)$ and $\sigma(G)$. Our point of view is that δ- and σ-sets are 'zero-separating' sets. This leads to the inequalities [**8**, Proposition 1.4]

$$\delta(G,V) \leqslant \sigma(G,V) \leqslant \beta_{\mathrm{sep}}(G,V) \leqslant \beta(G,V)$$

for any linear algebraic group G and G-module V, and hence

$$\delta(G) \leqslant \sigma(G) \leqslant \beta_{\mathrm{sep}}(G) \leqslant \beta(G).$$

Here $\beta(G,V)$ is the classical local Noether number, i.e. the maximal degree of an invariant in a minimal generating set of $\Bbbk[V]^G$, which in a similar way to before leads to the definition of the global Noether number $\beta(G)$. It is worth remarking that, due to results of Bryant and Kemper [**2**] and Derksen and Kemper [**6**], for a linear algebraic group G we have that $\beta(G)$ is finite if and only if G is finite and the group order $|G|$ is not divisible by the characteristic of the base field.

Kohls and Kraft have shown [**13**] that $\beta_{\mathrm{sep}}(G)$ is finite if and only if G is finite (independently of the characteristic of $\Bbbk$). Some parts of the proof of this result required some deep results from geometric invariant theory. The results of our current paper allow one to replace these parts of the proof by more elementary arguments (see § 4).

2. General results on the δ-number

In this section we prove various general results on $\delta(G)$. For the convenience of the reader, we present the proof of the following result of Nagata and Miyata in language consistent with this paper.

Proposition 2.1 (Nagata and Miyata [16, Proof of Theorem 1]). *Let G be a linear algebraic group over a field $\Bbbk$ and let V be a G-module. Suppose that $v \in V^G$ and $f \in \Bbbk[V]_+^G$ is homogeneous such that $f(v) \neq 0$. If the characteristic of $\Bbbk$ does not divide the degree of f, then there exists a homogeneous invariant $\tilde{f} \in \Bbbk[V]_1^G$ of degree one satisfying $\tilde{f}(v) \neq 0$.*

Proof. Write $d := \deg(f)$. Choose a basis $\{v =: v_0, v_1, \dots, v_n\}$ of V and let $\{x_0, x_1, \dots, x_n\}$ be the corresponding dual basis. Since $f(v) \neq 0$, we can write $f = \sum_{i=0}^{d} x_0^{d-i} c_i$ with $c_i \in \Bbbk[x_1, x_2, \dots, x_n]_i$ for each $i = 0, \dots, d$ and $c_0 \in \Bbbk \setminus \{0\}$. We may assume that $c_0 = 1$. Furthermore, since $v \in V^G$, note that $\langle x_1, x_2, \dots, x_n \rangle$ is a G-invariant space and we can write $g \cdot x_0 = x_0 + y(g)$ with $y(g) \in \langle x_1, x_2, \dots, x_n \rangle$ for each $g \in G$. For any $g \in G$ we have

$$\begin{aligned} g \cdot f &= (g \cdot x_0)^d + (g \cdot c_1)(g \cdot x_0)^{d-1} + (\text{terms of } x_0\text{-degree} \leqslant d-2) \\ &= (x_0 + y(g))^d + (g \cdot c_1)(x_0 + y(g))^{d-1} + (\text{terms of } x_0\text{-degree} \leqslant d-2) \\ &= x_0^d + (dy(g) + (g \cdot c_1))x_0^{d-1} + (\text{terms of } x_0\text{-degree} \leqslant d-2) \\ &= f, \end{aligned}$$

since f is invariant. Comparing coefficients of x_0^{d-1} tells us that for any $g \in G$ we have $c_1 = dy(g) + (g \cdot c_1)$. By assumption, the degree d is invertible in $\Bbbk$, and we now set $\tilde{f} := x_0 + d^{-1}c_1$. Notice that $\deg(\tilde{f}) = 1$ and for any $g \in G$ we have

$$g \cdot \tilde{f} = g \cdot x_0 + d^{-1}(g \cdot c_1) = x_0 + y(g) + d^{-1}(c_1 - dy(g)) = \tilde{f},$$

so $\tilde{f} \in \Bbbk[V]_1^G$. Furthermore, $\tilde{f}(v) = x_0(v) + d^{-1}c_1(v) = x_0(v) = 1 \neq 0$, completing the proof. □

Corollary 2.2. *Let G be a linear algebraic group and let V be a G-module. Then $\delta(G, V)$ equals either 0 or 1 or is divisible by the characteristic of $\Bbbk$. In particular, if $\Bbbk$ is a field of characteristic zero, then $\delta(G) = 1$.*

Proof. Note firstly that if $V^G \subseteq \mathcal{N}_{G,V}$, then $\delta(G, V) = 0$. Otherwise, $\delta(G, V) \geqslant 1$. Applying the above proposition shows that for any δ-set S consisting of homogeneous invariants, the set $\Bbbk[V]_1^G \cup \{f \in S \mid \deg(f) \text{ divisible by the characteristic}\}$ is also a δ-set. Finally, since $\delta(G, V) = 1$ when $V = \Bbbk$ is the trivial module, we must have $\delta(G) \geqslant 1$ for any linear algebraic group G. □

The proof of the following result is a slight adaption of [**15**, Lemma 3.1], where it is shown that if N is a closed normal subgroup of G such that N and G/N are reductive, then G is reductive.

Proposition 2.3. *Let N be a closed normal subgroup of G such that G/N is reductive. Then for any G-module V we have*

$$\delta(G, V) \leqslant \delta(N, V)\delta(G/N) \leqslant \delta(N)\delta(G/N),$$

and so, in particular, we have $\delta(G) \leqslant \delta(N)\delta(G/N)$.

Proof. Take a point $v \in V^G \setminus \mathcal{N}_{G,V}$. As a G-invariant separating v from zero is clearly also an N-invariant, we see that $v \in V^N \setminus \mathcal{N}_{N,V}$. Therefore, there is a homogeneous $f_0 \in \Bbbk[V]^N$ of positive degree $d \leqslant \delta(N, V)$ satisfying $f_0(v) \neq 0$. We may assume that $f_0(v) = 1$. Note that as N is a normal subgroup of G, we have that $U := \Bbbk[V]_d^N$ is a G-module on which N acts trivially, so it can be considered as a G/N-module. Furthermore, we define $U_0 := \{f \in U \mid f(v) = 0\}$. Note that U_0 is a G-invariant subspace of U, since $v \in V^G$. As $f_0 \notin U_0$, we have $U_0 \neq U$. For any $f \in U$ we have $f = (f - f(v)f_0) + f(v)f_0$ with $f - f(v)f_0 \in U_0$, and hence $U = U_0 \oplus \Bbbk f_0$ as a vector space. We can therefore define $\varphi \in U^*$ by $\varphi(u_0 + \lambda f_0) := \lambda$ for $u_0 \in U_0$ and $\lambda \in \Bbbk$. It is easily seen that φ is G-invariant. As mentioned, we can consider U as a G/N-module, and then we have $\varphi \in (U^*)^{G/N} \setminus \{0\}$. By assumption, G/N is a reductive group, so there exists a homogeneous $F \in \Bbbk[U^*]_{d'}^{G/N} = S^{d'}(U)^{G/N}$ of some positive degree $d' \leqslant \delta(G/N)$ such that $F(\varphi) \neq 0$. Let $\{f_1, \ldots, f_r\}$ denote a basis of U_0. Since $\varphi|_{U_0} = 0$, the fact that $F \in S^{d'}(\langle f_0, f_1, \ldots, f_r \rangle)^{G/N}$ such that $F(\varphi) \neq 0$ implies that $F = c \cdot f_0^{d'} + \tilde{F}$, where $c \in \Bbbk \setminus \{0\}$ and $\tilde{F}$ is an element of the ideal $(f_1, \ldots, f_r)S(U)$. Note that as $U = \Bbbk[V]_d^N$, there is a canonical map $S^{d'}(U)^{G/N} \to \Bbbk[V]_{dd'}^G$, so we can take F as an element of $\Bbbk[V]_{dd'}^G$. Clearly, $F(v) = cf_0(v)^{d'} \neq 0$ as $f_i(v) = 0$ for $i = 1, \ldots, r$ by the definition of U_0, showing that $\delta(G, V) \leqslant \delta(N, V)\delta(G/N)$. □

Corollary 2.4. *Let G be a linear algebraic group and let G^0 denote the connected component of the identity. Then we have*

$$\delta(G) \leqslant \delta(G/G^0)\delta(G^0).$$

In particular, $\delta(G)$ is finite if $\delta(G^0)$ is finite.

Remark 2.5. If N is a normal subgroup of G, then $\delta(G/N) \leqslant \delta(G)$, since any G/N-module becomes a G-module via the map $G \to G/N$.

Proof of Theorem 1.3. As tori are linearly reductive, $\delta(G^0) = 1$. Hence, we obtain $\delta(G/G^0) \leqslant \delta(G) \leqslant \delta(G^0)\delta(G/G^0) = \delta(G/G^0)$, so $\delta(G) = \delta(G/G^0)$. As G/G^0 is a finite group, the value of $\delta(G/G^0)$ is the size of a Sylow-p-subgroup by Theorem 1.1. □

Theorem 1.3 shows that there are many examples of infinite groups G with finite $\delta(G) > 1$; simply define $G = P \times T$, where P is a finite p-group and T is a non-trivial torus, then $\delta(G) = |P|$. For a more interesting example, consider $G = \mathrm{O}_2(\Bbbk)$ with $\Bbbk$ an algebraically closed field of characteristic 2. It is well known that $G \cong \Bbbk^* \rtimes Z_2$, where Z_2 denotes the cyclic group of order 2. Therefore, $G^0 \cong \Bbbk^*$ is a torus, and $G/G^0 \cong Z_2$. By Theorem 1.3, $\delta(\mathrm{O}_2(\Bbbk)) = 2$.

3. The δ-number for subgroups of $\mathrm{GL}_2(\Bbbk)$

The goal of this section is to prove Theorem 1.4. Throughout we assume that $\Bbbk$ is a field of characteristic $p > 0$. We begin by introducing another number associated with a representation of a group, which is useful for finding lower bounds for both the δ-number and the σ-number. Let G be a linear algebraic group and let V be a G-module. Let $v \in V$. Then we set

$$\varepsilon(G, v) := \inf\{d \in \mathbb{N}_{>0} \mid \text{there exists } f \in \Bbbk[V]^G_d \text{ such that } f(v) \neq 0\},$$

where the infimum of an empty set is infinity. Notice that if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$, then

$$\delta(G, V) = \sup\{\varepsilon(G, v) \mid v \in V^G \setminus \mathcal{N}_{G,V}\},$$

and if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$, then

$$\sigma(G, V) = \sup\{\varepsilon(G, v) \mid v \in V \setminus \mathcal{N}_{G,V}\}.$$

For a submodule $W \subseteq V$ we define

$$\varepsilon(G, W, V) := \inf\{\varepsilon(G, v) \mid v \in W \setminus \mathcal{N}_{G,V}\}$$

and we set

$$\varepsilon(G, V) := \varepsilon(G, V^G, V) \quad \text{and} \quad \tau(G, V) := \varepsilon(G, V, V).$$

It is immediately clear that for any linear algebraic group G we have $\delta(G, V) \geqslant \varepsilon(G, V)$ if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$, and $\sigma(G, V) \geqslant \tau(G, V)$ if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$. In fact we have the following slightly stronger result, which we mainly use for H a *finite* subgroup of G (the second inequality is not used and is only stated for completeness).

Lemma 3.1. *Let G be a linear algebraic group, let V be a G-module and let H be a subgroup of G. Then $\delta(G,V) \geqslant \varepsilon(H,V)$ if $V^G \setminus \mathcal{N}_{G,V} \neq \emptyset$, and $\sigma(G,V) \geqslant \tau(H,V)$ if $V \setminus \mathcal{N}_{G,V} \neq \emptyset$.*

Proof. Choose a $v \in V^G \setminus \mathcal{N}_{G,V}$ such that $\delta(G,V) = \varepsilon(G,v)$. Clearly, $v \in V^H \setminus \mathcal{N}_{H,V}$, and hence $\delta(G,V) = \varepsilon(G,v) \geqslant \varepsilon(H,v) \geqslant \varepsilon(H,V^H,V) = \varepsilon(H,V)$. For the second inequality choose $v \in V \setminus \mathcal{N}_{G,V}$ such that $\sigma(G,V) = \varepsilon(G,v)$. As also $v \in V \setminus \mathcal{N}_{H,V}$, $\sigma(G,V) = \varepsilon(G,v) \geqslant \varepsilon(H,v) \geqslant \varepsilon(H,V,V) = \tau(H,V)$. □

We believe a thorough investigation of the numbers $\varepsilon(G,V)$ when G is a finite group may hold the key to proving Conjecture 1.2. In order to prove Theorem 1.4, we require the following lemma, whose proof is very similar to the proof of [**8**, Proposition 2.5], but the point of view is different. For any finite group G, let $V_{\mathrm{reg},G} := \Bbbk G$ denote its regular representation.

Lemma 3.2. *Suppose that G is a finite group and that P is a Sylow-p-subgroup of G. If $V = V_{\mathrm{reg},G}^n$ is a free G-module over $\Bbbk$, then $\varepsilon(G,v) = |P|$ for any $v \in V^G \setminus \{0\}$.*

Proof. For each $i = 1,\dots,n$ choose a permutation basis $\{v_{g,i} \mid g \in G\}$ of the ith summand (which is isomorphic to $V_{\mathrm{reg},G}$), so that $\{v_{g,i} \mid g \in G,\ i = 1,\dots,n\}$ is a basis of V. Let $\{x_{g,i} \mid g \in G,\ i = 1,\dots,n\}$ be the basis dual to our chosen basis of V so that $\Bbbk[V] = \Bbbk[x_{g,i} \colon g \in G,\ i = 1,\dots,n]$. The fixed-point space of the ith summand is spanned by $v_i := \sum_{g\in G} v_{g,i}$, and therefore $V^G = \langle v_1,\dots,v_n\rangle$. For a point $v = \sum_{i=1}^n \lambda_i v_i \in V^G \setminus \{0\}$ with scalars $\lambda_i \in \Bbbk$ (not all of them zero), we will show that $\varepsilon(G,v) = |P|$. We show first $\varepsilon(G,v) \geqslant |P|$, i.e. $\deg(f) \geqslant |P|$ for any homogeneous $f \in \Bbbk[V]_+^G$ such that $f(v) \neq 0$. Since V is a permutation module, such an f is a linear combination of orbit sums of monomials

$$O_G(m) := \sum_{m' \in G\cdot m} m',$$

where m is a monomial in $\Bbbk[V]_+$. It follows that there exists a monomial $m \in \Bbbk[V]_+$, whose degree is the same as $\deg(f)$, such that $O_G(m)(v) \neq 0$. Now, if $m' \in G\cdot m$, then $m' = g\cdot m$ for some $g \in G$, and $m'(v) = (g\cdot m)(v) = m(g^{-1}v) = m(v)$ since $v \in V^G$. Therefore,

$$O_G(m)(v) = |G\cdot m| m(v) = (G : \mathrm{Stab}_G(m)) m(v) \neq 0.$$

This implies that $\mathrm{Stab}_G(m)$ contains a Sylow-p-subgroup of G, which without loss of generality we can assume to be P. Therefore, if $x_{g,i}$ is any variable dividing m, then m is also divisible by $x_{g'g,i}$ for every $g' \in P$. In particular, since m is not constant, we obtain $\deg(f) = \deg(m) \geqslant |P|$ as required. Secondly, choose an i such that $\lambda_i \neq 0$ and define $m := \prod_{g\in P} x_{g,i}$. Then $O_G(m)$ is an invariant of degree $|P|$ satisfying

$$O_G(m)(v) = (G : \mathrm{Stab}_G(m)) m(v) = (G : P)\lambda_i^{|P|} \neq 0,$$

showing that $\varepsilon(G,v) \leqslant |P|$. □

Proposition 3.3. *Let $p > 0$ be a prime and let $\Bbbk$ be an algebraically closed field of characteristic p. Let $G_n = (\mathbb{F}_{p^n}, +)$ be the additive group of the finite subfield $\mathbb{F}_{p^n}$ of $\Bbbk$. Let V be the G_n-module spanned by vectors X and Y such that the action $*$ of G_n on V is given by*

$$t * X = X \quad \text{and} \quad t * Y = Y + tX \quad \text{for all } t \in G_n.$$

Then $S^{p^n-1}(V)$ is isomorphic to the regular representation of G_n.

Proof. We will show that $S := \{t * Y^{p^n-1} \mid t \in G_n\}$ is a basis of $S^{p^n-1}(V)$, which clearly implies that $S^{p^n-1}(V) \cong V_{\mathrm{reg},G_n}$. As $|G_n| = p^n$ equals the dimension of $S^{p^n-1}(V)$, it is enough to show that the $p^n \times p^n$ matrix A with columns formed by the coordinate vectors of the elements $t*Y^{p^n-1}$, $t \in G_n$, with respect to the standard basis $\{Y^{p^n-1-i}X^i \mid i \in \{0, \dots, p^n-1\}\}$ of $S^{p^n-1}(V)$ has a non-zero determinant. Using the binomial theorem and Lemma 3.4 we compute

$$\begin{aligned} t * Y^{p^n-1} = (Y + tX)^{p^n-1} &= \sum_{i=0}^{p^n-1} \binom{p^n-1}{i} Y^{p^n-1-i}(tX)^i \\ &= \sum_{i=0}^{p^n-1} (-1)^i Y^{p^n-1-i}(tX)^i \quad \text{(by Lemma 3.4)} \\ &= \sum_{i=0}^{p^n-1} (-t)^i Y^{p^n-1-i} X^i. \end{aligned}$$

Thus, $A = ((-t)^i)_{i\in\{0,\dots,p^n-1\},\, t\in G_n} \in \Bbbk^{p^n\times p^n}$, where we enumerated the p^n columns of A by the set G_n, which is harmless as the order of the columns only affects the sign of the determinant of A. Note that A is the $p^n \times p^n$ Vandermonde matrix of the p^n *different* elements of $-G_n (= G_n)$, and hence $\det(A) \neq 0$, which proves the claim. $\square$

In the preceding proof we used the following number-theoretic lemma, of which we provide a proof for the convenience of the reader.

Lemma 3.4. *Let p be a prime number and let $0 \leqslant k \leqslant p^n - 1$. Then*

$$\binom{p^n-1}{k} \equiv (-1)^k \mod p.$$

Proof. We have

$$\binom{p^n-1}{k} = \prod_{m=1}^{k} \frac{p^n - m}{m}.$$

We show that the reduced fraction of each factor has a denominator coprime to p, and equals $-1+p\mathbb{Z}$ if computed in the field $\mathbb{Z}/p\mathbb{Z}$. Because of this, for $1 \leqslant m \leqslant k \leqslant p^n-1$ write $m = p^r s$, where s and p are coprime. Then $r < n$ and $(p^n - m)/m = (p^n - p^r s)/p^r s = (p^{n-r} - s)/s$. In the field $\mathbb{Z}/p\mathbb{Z}$, the last fraction equals -1. $\square$

Having set up all the necessary machinery, we are now in a position to prove Theorem 1.4. Let G be a subgroup of $\mathrm{GL}_2(\Bbbk)$ containing an infinite unipotent subgroup U. As U is conjugate in $\mathrm{GL}_2(\Bbbk)$ to the subgroup of unipotent upper triangular 2×2 matrices (see [**11**, Corollary 17.5]), we can replace G by a conjugate subgroup and assume that $U = \{u_t \mid t \in \Bbbk\}$, where $u_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in G$. Note that U is isomorphic to the additive group of the ground field $\mathbb{G}_a = (\Bbbk, +)$. Let V denote the restriction of the natural two-dimensional $\mathrm{GL}_2(\Bbbk)$-module to G. We may choose a basis $\{X, Y\}$ of V such that

$$u_t * X = X \quad \text{and} \quad u_t * Y = Y + tX \quad \text{for all } t \in \mathbb{G}_a.$$

Theorem 1.4 follows immediately from the following proposition.

Proposition 3.5. *For any integer n set $V_n := \mathrm{Hom}_\Bbbk(S^{p^n-1}(V), S^{p^n-1}(V))$. Then $\delta(G, V_n) \geqslant p^n$.*

Proof. First note that $V_n^G \setminus \mathcal{N}_{G,V_n} \neq \emptyset$. To see this consider the identity homomorphism $\mathrm{id}\colon S^{p^n-1}(V) \to S^{p^n-1}(V)$, which is an element of V_n^G. The determinant map $\det\colon V_n \to \Bbbk$ is an element of $\Bbbk[V_n]^G$, and $\det(\mathrm{id}) = 1 \neq 0$, so $\mathrm{id} \in V_n^G \setminus \mathcal{N}_{G,V_n}$. Therefore, we can apply Lemma 3.1 to G and its finite subgroup $U_n := \{u_t \mid t \in \mathbb{F}_{p^n}\}$, and hence $\delta(G, V_n) \geqslant \varepsilon(U_n, V_n)$. Note that $U_n \cong (\mathbb{F}_{p^n}, +)$. By Proposition 3.3, $S^{p^n-1}(V)$ is a free U_n-module. Recall that tensoring a free/projective module with any other module again yields a free/projective module (see [**1**, p. 47, proof of Lemma 7.4]), and hence $V_n = \mathrm{Hom}_\Bbbk(S^{p^n-1}(V), S^{p^n-1}(V)) \cong S^{p^n-1}(V) \otimes (S^{p^n-1}(V))^*$ is also a free U_n-module. Using Lemma 3.2 we obtain

$$\delta(G, V_n) \geqslant \varepsilon(U_n, V_n) = |U_n| = p^n,$$

as required. □

We record the following observation for later use.

Corollary 3.6. *Let G be an infinite connected unipotent algebraic group over an algebraically closed field of positive characteristic. Then $\delta(G) = \infty$.*

Proof. It is well known that such a group G contains a closed normal subgroup N such that $G/N \cong \mathbb{G}_a$. We can embed $\mathbb{G}_a$ in $\mathrm{GL}_2(\Bbbk)$ as above. Now, using Remark 2.5 and Theorem 1.4, we have $\delta(G) \geqslant \delta(G/N) = \delta(\mathbb{G}_a) = \infty$. □

Combining Theorem 1.4 and Proposition 2.3 leads to more examples of groups with infinite δ-value: whenever $\delta(G) = \infty$ and N is a closed normal subgroup of G such that G/N is reductive, either $\delta(N) = \infty$ or $\delta(G/N) = \infty$.

Example 3.7. Take $G = \mathrm{GL}_2(\Bbbk)$ and consider its centre $Z(G) = \{aI_2 \mid a \in \Bbbk \setminus \{0\}\}$. As a torus, $Z(G)$ is linearly reductive, and hence $\delta(Z(G)) = 1$. Therefore, $\delta(\mathrm{PGL}_2(\Bbbk)) = \delta(G/Z(G)) = \infty$.

4. The σ-number of infinite groups

In this section we prove Theorems 1.5 and 1.6. Some of the groundwork was done in [**8**]. In particular, we recall the following result.

Proposition 4.1 (Elmer and Kohls [8, Corollary 3.13]). *Let G be a linear algebraic group with G^0 the connected component of G containing the identity. We have the inequalities*

$$\sigma(G^0) \leqslant \sigma(G) \leqslant (G : G^0)\sigma(G^0).$$

In particular, $\sigma(G)$ and $\sigma(G^0)$ are either both finite or both infinite.

The following proposition is key to the proofs.

Proposition 4.2. *Let G be a linear algebraic group over a field $\Bbbk$ of arbitrary characteristic. Suppose that G contains a non-trivial torus. Then $\sigma(G) = \infty$.*

Proof. We exhibit a sequence of G-modules $\{U_m \mid m \in \mathbb{N}\}$ such that $\sigma(G, U_m) \geqslant m+1$ for all $m \in \mathbb{N}$. By assumption, G contains a subgroup $T \cong \Bbbk^*$, so there is an isomorphism $\Bbbk^* \to T$, $t \mapsto a_t$. As a linear algebraic group, G can be considered as a closed subgroup of some $\mathrm{GL}_{n+1}(\Bbbk)$, and then $V = \Bbbk^{n+1}$ becomes a faithful G-module. We can choose a basis $\{v_0, v_1, \ldots, v_n\}$ of V on which T acts diagonally, and as T acts faithfully, it acts non-trivially on at least one basis vector, say v_0. Therefore, for some $r \in \mathbb{Z} \setminus \{0\}$, we have $a_t * v_0 = t^r v_0$ for all $t \in \Bbbk^*$. Write $\{y_0, y_1, \ldots, y_n\}$ for the basis of V^* dual to $\{v_0, v_1, \ldots, v_n\}$. A basis for $S^m(V^*)$ is then given by the set of monomials

$$\left\{ \boldsymbol{y}^{\boldsymbol{e}} := \prod_{i=0}^{n} y_i^{e_i} \in S^m(V^*) \mid \boldsymbol{e} \in \mathbb{N}_0^{n+1},\ |\boldsymbol{e}| := \sum_{i=0}^{n} e_i = m \right\}$$

of degree m. Furthermore, let

$$\{Z_{\boldsymbol{e}} \in S^m(V^*)^* \mid \boldsymbol{e} \in \mathbb{N}_0^{n+1},\ |\boldsymbol{e}| = m\}$$

denote the corresponding dual basis of $S^m(V^*)^*$, i.e. $Z_{\boldsymbol{e}}(y^{\boldsymbol{e}'}) = \delta_{\boldsymbol{e},\boldsymbol{e}'}$ (the Kronecker-delta). Now we set $U_m := V \oplus S^m(V^*)$. We may identify $\Bbbk[U_m]$ with

$$S(U_m^*) = S(V^* \oplus S^m(V^*)^*) = \Bbbk[y_0, y_1, \ldots, y_n][Z_{\boldsymbol{e}} \colon \boldsymbol{e} \in \mathbb{N}_0^{n+1}, |\boldsymbol{e}| = m].$$

Consider the point $v := v_0 + y_0^m \in U_m$. We claim that $v \notin \mathcal{N}_{G,U_m}$, and we will show that $\varepsilon(G, v) = m+1$. As a consequence, $\sigma(G, U_m) \geqslant \varepsilon(G, v) = m+1$, finishing the proof. To see this, we define the polynomial

$$f := \sum_{\boldsymbol{e} \in \mathbb{N}_0^{n+1},\, |\boldsymbol{e}|=m} \boldsymbol{y}^{\boldsymbol{e}} Z_{\boldsymbol{e}} \in \Bbbk[U_m],$$

which can be interpreted as the identity map $\mathrm{id} \colon S^m(V^*) \to S^m(V^*)$, and is hence an invariant, i.e. $f \in \Bbbk[U_m]^G$. Note that here we used the isomorphism

$$\mathrm{Hom}_{\Bbbk}(S^m(V^*), S^m(V^*)) \cong S^m(V^*) \otimes S^m(V^*)^*$$

and that

$$\Bbbk[U_m] \cong S(V^* \oplus S^m(V^*)^*) \cong S(V^*) \otimes S(S^m(V^*)^*)$$

contains a direct summand isomorphic to $S^m(V^*) \otimes S^m(V^*)^*$. Clearly, $f(v) = 1 \neq 0$, which shows that $v \notin \mathcal{N}_{G,U_m}$. Furthermore, we have $\deg(f) = m+1$, so we have $\varepsilon(G,v) \leqslant m+1$. It remains to show that $\varepsilon(G,v) \geqslant m+1$. Suppose that a homogeneous $f' \in \Bbbk[U_m]_+^G$ also satisfies $f'(v) \neq 0$; we will show that $\deg(f') \geqslant m+1$. Observe that *a fortiori* we have $f' \in \Bbbk[U_m]_+^T$. Therefore, f' can be written as a sum of T invariant monomials, so in particular there exists a T-invariant monomial h (of the same degree as f') satisfying $h(v) \neq 0$. As $v = v_0 + y_0^m$, the only variables that can appear in h are those dual to v_0 and y_0^m, i.e. the variables y_0 and $Z_{\boldsymbol{e}_0}$ with $\boldsymbol{e}_0 := (m, 0, 0, \ldots, 0)$. We thus have $h = y_0^k Z_{\boldsymbol{e}_0}^l$ with $k, l \in \mathbb{N}_0$ and $\deg(h) = k + l > 0$. On the other hand, since $h \in \Bbbk[U_m]^T$, we have

$$\begin{aligned} y_0^k Z_{\boldsymbol{e}_0}^l = h = a_t * h = (a_t * y_0)^k (a_t * Z_{\boldsymbol{e}_0})^l &= (t^{-r} y_0)^k (t^{mr} Z_{\boldsymbol{e}_0})^l \\ &= t^{mrl-kr} y_0^k Z_{\boldsymbol{e}_0}^l \quad \text{for all } t \in \Bbbk^*, \end{aligned}$$

i.e. $r(ml - k) = 0$. Since $r \neq 0$ and $k + l > 0$, it must be the case that $k = ml \geqslant m$ and $l \geqslant 1$. Therefore, $\deg(f') = \deg(h) = ml + l \geqslant m + 1$, as required. □

Corollary 4.3. *Suppose that G is a linear algebraic group such that $\sigma(G)$ is finite. Then G^0 is unipotent, i.e. either G is finite or G^0 is infinite unipotent.*

Proof. If $\sigma(G)$ is finite, $\sigma(G^0)$ is finite by Proposition 4.1. It follows from Proposition 4.2 that G^0 does not contain any non-trivial torus, i.e. the rank of the connected group G^0 (the dimension of a maximal torus) is zero, and hence G^0 is unipotent by [**11**, Exercise 21.4.1]. □

Specializing to the case in which $\Bbbk$ is a field of characteristic zero, this completes the proof of Theorem 1.6. To finish the proof of Theorem 1.5, it remains to show that over a field of positive characteristic, if G^0 is infinite unipotent, we have $\sigma(G) = \infty$. This follows from $\sigma(G^0) \leqslant \sigma(G)$ (Proposition 4.1), the inequality $\delta(G^0) \leqslant \sigma(G^0)$ and from $\delta(G^0) = \infty$ (Corollary 3.6). The following proposition, which provides some examples of their own interest, gives a more direct proof that $\delta(\mathbb{G}_a) = \sigma(\mathbb{G}_a) = \infty$ for a field of positive characteristic. Additionally, it gives another proof of $\beta_{\text{sep}}(\mathbb{G}_a) = \infty$ for such a field, which is also shown in [**13**, Proposition 4] (see also the following remark for more details). As before, it follows that $\delta(G) = \sigma(G) = \infty$ for any infinite unipotent connected group with a normal subgroup N such that $G/N \cong \mathbb{G}_a$. We mention that $\mathbb{G}_a$-modules of the type as in the proposition are also investigated in [**9**, **17**]. The generators of the considered invariant ring would also follow from the latter paper, but we give a self-contained argument.

Proposition 4.4. *Assume that $\Bbbk$ is a field of characteristic $p > 0$ and let $V_n = \Bbbk^3$ ($n \geqslant 1$) be the $\mathbb{G}_a = (\Bbbk, +)$-module given by the representation*

$$\mathbb{G}_a \mapsto \mathrm{GL}_3(\Bbbk), \qquad t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^{p^n} & 0 & 1 \end{pmatrix}.$$

If we write $\Bbbk[V_n] = \Bbbk[x_0, x_1, x_2]$, *then we have*

$$\Bbbk[V_n]^{\mathbb{G}_a} = \Bbbk[x_0, x_2x_0^{p^n-1} - x_1^{p^n}] \quad \text{and} \quad \delta(\mathbb{G}_a, V_n) = \sigma(\mathbb{G}_a, V_n) = p^n.$$

Consequently, $\delta(\mathbb{G}_a) = \sigma(\mathbb{G}_a) = \infty$.

Proof. The action $*$ of $\mathbb{G}_a$ on $\Bbbk[V_n]$ is given by

$$t * f(x_0, x_1, x_2) = f(x_0, x_1 + tx_0, x_2 + t^{p^n}x_0) \quad \text{for } t \in \mathbb{G}_a,\ f(x_0, x_1, x_2) \in \Bbbk[V_n].$$

If f is an invariant, the equation $t * f = f$ for all $t \in \mathbb{G}_a$ implies that for an additional independent variable t the equation

$$f(x_0, x_1, x_2) = f(x_0, x_1 + tx_0, x_2 + t^{p^n}x_0)$$

holds in the polynomial ring $\Bbbk[V_n][t]$. Substituting $t := -x_1/x_0$ leads to

$$f(x_0, x_1, x_2) = f\left(x_0, 0, x_2 - \frac{x_1^{p^n}}{x_0^{p^n}}x_0\right) = f\left(x_0, 0, \frac{x_2x_0^{p^n-1} - x_1^{p^n}}{x_0^{p^{n-1}}}\right). \tag{4.1}$$

We have to show that $\Bbbk[V_n]^{\mathbb{G}_a} \subseteq \Bbbk[x_0, x_2x_0^{p^n-1} - x_1^{p^n}]$, as the reverse inclusion is checked immediately. For an $f \in \Bbbk[V_n]^{\mathbb{G}_a}$ write $f = \sum_{k=0}^m a_k(x_0, x_1)x_2^k$ with polynomials $a_k \in \Bbbk[x_0, x_1]$. Equation (4.1) implies that

$$f = \sum_{k=0}^m a_k(x_0, 0)\left(\frac{x_2x_0^{p^n-1} - x_1^{p^n}}{x_0^{p^n-1}}\right)^k = \sum_{k=0}^m \frac{b_k(x_0)}{(x_0^{p^n-1})^k}(x_2x_0^{p^n-1} - x_1^{p^n})^k \tag{4.2}$$

with polynomials $b_k(x_0) := a_k(x_0, 0) \in \Bbbk[x_0]$. Substituting $x_2 := 0$ leads to

$$f(x_0, x_1, 0) = \sum_{k=0}^m \frac{b_k(x_0)}{(x_0^{p^n-1})^k}(-x_1^{p^n})^k \in \Bbbk[x_0, x_1],$$

which implies that $c_k(x_0) := b_k(x_0)/(x_0^{p^n-1})^k$ is actually a polynomial, i.e. an element of $\Bbbk[x_0]$. Resubstituting in (4.2) implies that

$$f = \sum_{k=0}^m c_k(x_0)(x_2x_0^{p^n-1} - x_1^{p^n})^k \in \Bbbk[x_0, x_2x_0^{p^n-1} - x_1^{p^n}],$$

as desired. It follows that $\sigma(\mathbb{G}_a, V_n) \leqslant p^n$ and $\mathcal{N}_{\mathbb{G}_a,V_n} = \{(0, 0, a_2) \in V_n \mid a_2 \in \Bbbk\}$, and clearly we have $V_n^{\mathbb{G}_a} = \{(0, a_1, a_2) \in V_n \mid a_1, a_2 \in \Bbbk\}$. Now the point $v := (0, 1, 0) \in V_n^{\mathbb{G}_a} \setminus \mathcal{N}_{\mathbb{G}_a,V_n}$ satisfies $x_0(v) = 0$ and $(x_2x_0^{p^n-1} - x_1^{p^n})(v) = -1$, which shows that $\delta(\mathbb{G}_a, V_n) = \sigma(\mathbb{G}_a, V_n) = p^n$. □

Remark 4.5. Theorems 1.5 and 1.6 were proved by 'elementary' means, in the sense that we did not use any geometric invariant theory. We can use these results to give an elementary proof of [**13**, Theorem A], which states that $\beta_{\mathrm{sep}}(G)$ is finite if and only if G is finite. That $\beta_{\mathrm{sep}}(G)$ is finite for a finite group G is well known (see [**5**, Corollary 3.9.14]), so it remains to prove the converse. Suppose that $\beta_{\mathrm{sep}}(G)$ is finite. The inequality $\sigma(G) \leqslant \beta_{\mathrm{sep}}(G)$ implies in particular that $\sigma(G)$ is finite, so if $\Bbbk$ has characteristic $p > 0$, we are done, by Theorem 1.5. Otherwise we conclude that G^0 is unipotent from Theorem 1.6. Now the results $\beta_{\mathrm{sep}}(\mathbb{G}_a) = \infty$ and $\beta_{\mathrm{sep}}(G^0) \leqslant \beta_{\mathrm{sep}}(G)$, which are both proven in elementary fashion in [**13**, Proposition 5 and Theorem B], imply that $\beta_{\mathrm{sep}}(G) = \infty$ when G^0 is an infinite unipotent group. Hence, if $\beta_{\mathrm{sep}}(G) < \infty$, G^0 and G are finite.

We do not know very much about $\sigma(G)$ when G is an infinite unipotent group over a field of characteristic zero. Unlike $\beta_{\mathrm{sep}}(G)$, it is not always infinite, as the following surprising result shows.

Proposition 4.6. *Assume that $\Bbbk$ is a field of characteristic 0. Then $\sigma(\mathbb{G}_a) = 2$.*

Proof. In [**7**, § 3] we give for any $\mathbb{G}_a$-module V an explicit set of invariants of degree at most 2 that cuts out the null-cone. It follows that $\sigma(\mathbb{G}_a) = 2$. □

We conclude with an example that shows that $\sigma(\mathbb{G}_a \times \mathbb{G}_a) \geqslant 3$.

Example 4.7. Let $\Bbbk$ be an algebraically closed field of characteristic zero and let $V = \Bbbk^4$. Consider an action of $G := \mathbb{G}_a \times \mathbb{G}_a$ defined as follows: $(s, t) \in \Bbbk \times \Bbbk$ acts on V as multiplication by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 \\ \frac{1}{2}s^2 - t & -s & 1 & 0 \\ -\frac{1}{6}s^3 + st & \frac{1}{2}s^2 - t & -s & 1 \end{pmatrix}.$$

Let $\{x_0, x_1, x_2, x_3\}$ denote the basis of V^* dual to the standard basis of V. Then we claim that the ring of invariants $\Bbbk[V]^G$ is generated by the invariants x_0 and $f := x_1^3 - 3x_0x_1x_2 + 3x_0^2x_3$. Under this assumption we have that the point $v = (0, 1, 0, 0) \in V$ is not contained in the null-cone, since $f(v) = 1 \neq 0$, and is not separated from zero by any invariant of degree less than 3, which shows that $\sigma(G, V) = 3$, and hence $\sigma(G) \geqslant 3$.

To prove the claim, consider the subgroup $H := \{(0, t) \in G \mid t \in \Bbbk\}$ of G. The action of H on $\Bbbk[V]$ is given by

$$\begin{aligned} (0, t) * x_0 &= x_0, \\ (0, t) * x_1 &= x_1, \\ (0, t) * x_2 &= x_2 + tx_0, \\ (0, t) * x_3 &= x_3 + tx_1 \quad \text{for all } t \in \Bbbk. \end{aligned}$$

This $\mathbb{G}_a$-action corresponds to the direct sum of two copies of the natural representation of $\mathbb{G}_a$, and the invariant ring is well known to be given by $\Bbbk[V]^H = \Bbbk[x_0, x_1, x_0x_3 - x_2x_1]$. Crucially, this is a polynomial ring in three variables. Now, $\Bbbk[V]^G = \Bbbk[x_0, x_1, x_0x_3 -$

$x_2x_1]^{G/H}$ is isomorphic to the ring of invariants of a nonlinear action of $\mathbb{G}_a$ on a polynomial ring in three variables; by a theorem of Miyanishi (see [**10**, Theorem 5.1]) this ring of invariants is again polynomial, with two generators. Therefore, $\Bbbk[V]^G$ is a graded polynomial ring with two generators. One may readily check that x_0 is the only invariant of degree one, and as f is an invariant of smallest possible degree not contained in $\Bbbk[x_0]$, we see that $\Bbbk[V]^G = \Bbbk[x_0, f]$ as claimed.

Acknowledgements. This paper was prepared during visits of M.K. to the University of Aberdeen and of J.E. to T. U. München. The first of these visits was supported by the Edinburgh Mathematical Society's Research Support Fund. We want to thank Gregor Kemper and the Edinburgh Mathematical Society for making these visits possible. We also thank the anonymous referee for useful suggestions.

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