Range-limited, Distributed Algorithms on Higher-Order Voronoi Partitions in Multi-Robot Systems

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***Abstract*— This paper studies the problem of distributed computation of higher order Voronoi partition over a bounded region by a group of robots with both range-limited visibility sensors and communication devices. We model the sensing and communication capabilities by discs with limited radius. Motivated by the concept of dominating region in higher-order Voronoi partition, we propose a detecting ray based algorithm, which computes the boundary points of the dominating region of a robot in an omnidirectional manner, with local position information of its neighbors within the communication range. Simulations are provided to demonstrate the performance of our proposed algorithm by using a thirteen-robot group.**

I. Introduction

In the last two decades, a significant amount of interest has been raised in the research field of networked systems, see [1] for a recent overview. Among the various research directions, multi-robot coverage control is a particular problem of interest, which aims to coordinate a group of mobile robots with sensors (e.g. cameras) to provide coverage over a large region for better performance and efficiency than a single complex robot [2], [3], [4], [5], [6].

In the problem setting of coverage control, it is usually required that the algorithms are executed in a distributed manner. By distributed manner, we mean a robot is allowed to exchange information only with its neighbors, rather than all other robots in the network, to determine its coverage area or control input. We note that the Voronoi diagram [7] is a typical technique for computing the coverage region in the related coverage control literature [8], [9], [10], [11], [12]. The requirement of implementing the algorithm in a distributed manner gives rise to a new challenge, which aims to extend the commonly-used centralized Voronoi partition algorithm to a distributed algorithm. Some existing effort has been presented in [13], [14], [15], [16], [17] to address this challenge. However, they all concern the 1-order Voronoi partition.

In 1-order Voronoi partition, each robot is allocated to cover a convex area (Voronoi cell) whose shape is determined in part by position information from its neighboring agents, i.e. one robot dominates one cell. In higher-order Voronoi partition (or k-order Voronoi partition), the idea can be summarized as k ($k \geq 2$) robots dominates one single cell. We note that there lacks contribution to the problem of distributed computation of higher-order Voronoi partition, since its inner geometry relationship is highly nontrivial, for which the techniques developed in 1-order Voronoi partition are not applicable. To the best of our knowledge, the work in [18] is the only work that deals with the problem of distributed computation of higher-order Voronoi partition, in which the idea is to find the intersections of the dominating region of neighboring robots in the higher-order sense.

Our work is motivated by [18], in which we further introduce range constraints into consideration for both visibility sensors and communication devices equipped on robot. This definitely complicates the problem due to the following challenges: 1) the definition of dominating region introduced in [18] can not be directly applied since it may be restricted by the sensing range; 2) in a range-free, k-order Voronoi partition, any point within the covered region can be dominated by k robots. However, when considering the range limitation, there are more options. A point can be dominated by k robots, or $k-1$ robots, ..., or 1 robot, or even not dominated by any robot. To address the above challenges, in this paper, we develop a detecting ray based algorithm to compute each boundary point in an omnidirectional manner with respect to each robot.

The rest of this paper is structured as follows: Section II introduces the concepts and definitions of 1-order Voronoi partition, k-order Voronoi partition, dominating region in k-order Voronoi partition, models of sensing and communication discs and provides a formal definition of the problem. Section III presents the algorithm and the formal analysis. Simulations are provided in Section IV and the paper is concluded in Section V.

II. Preliminaries and Background

A. Order 1 Voronoi partition

Suppose there is a 2-D bounded area $\mathcal{A} \subset \mathbb{R}^2$ with n robots dispersed in this area. The set of n robots is defined as $N = \{1, 2, ..., n\}$. Let $p_i \in \mathbb{R}^2$ denote the position of robot i. The locations of n robots are represented by $P = (p_1, \dots, p_n)$. An arbitrary point in $\mathcal{A}$ is denoted as q. The 1-order Voronoi partition $V(P) = (v_1, \dots, v_n)$ of area $\mathcal{A}$ is defined as follows:

$$v_i = \{q \in \mathcal{A} | \ \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\} \tag{1}$$

The set of regions $v_1, \dots, v_n$ is called the Voronoi diagram for the generators $1, \dots, n$. Each Voronoi cell v_i represents

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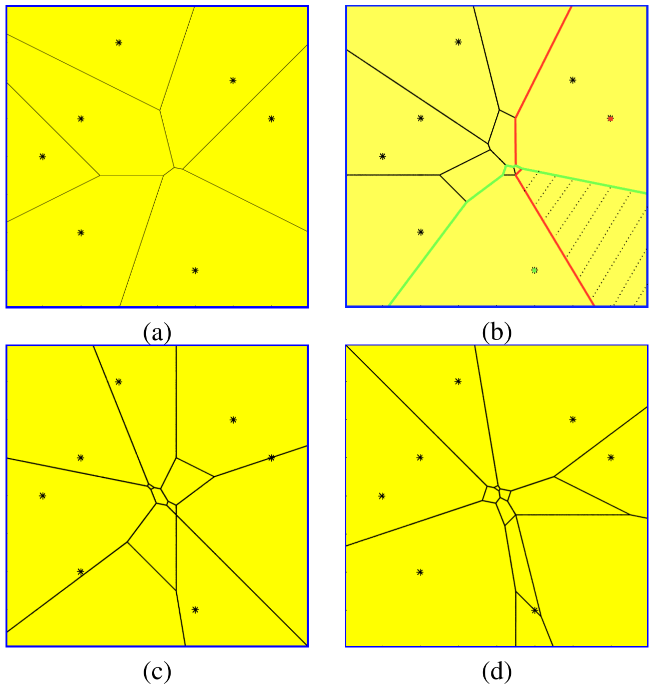

Fig. 1: Order k Voronoi partition for k = 1,2,3,4. (a) 1-order (b) 2-order (c) 3-order (d) 4-order

a enclosed region dominated by the i-th robot within which any point q inside v_i is closer to the robot i in terms of the Euclidean distance. Note that each 1-order Voronoi cell is convex. If the Voronoi cells of robot i and robot j are adjacent (v_i and v_j share a common boundary comprising an interval of nonzero length), the robot i and the robot j are defined as *Voronoi neighbors*. An example of 1-order Voronoi partition is shown in Fig. 1 (a).

B. Order k Voronoi partition

Suppose that $\mathcal{S}$ is a subset of N and there are k elements in $\mathcal{S}$. The definition of an order k Voronoi partition is given below:

$$\mathcal{V}_{\mathcal{S}} = \{q | \forall v \in \mathcal{S}, \forall \omega \in N \setminus \mathcal{S}, \|q - p_v\| < \|q - p_\omega\|, |\mathcal{S}| = k\}, \tag{2}$$

where $N \setminus \mathcal{S}$ denotes the relative complement set of $\mathcal{S}$ with respect to N. For any point q in $\mathcal{V}_{\mathcal{S}}$, q is closer to a robot in $\mathcal{S}$ than to any other robots not in $\mathcal{S}$ but in N. Generator of each cell in order 1 Voronoi partition changes to generator set $\mathcal{S}$ in k-order Voronoi partition and each Voronoi cell $\mathcal{V}_{\mathcal{S}}$ is also convex. And it is worth pointing out that not every $\mathcal{S} \in N$ necessarily defines a cell in the partition. We provide three examples to show 2-order, 3-order and 4-order Voronoi partitions in Fig. 1 (b) (c) (d), respectively.

C. Dominating region of robot i in a k-order Voronoi partition

The concept termed dominating region of robot i (introduced in [18]) in a k-order Voronoi partition serves as a basis of our algorithm. Across this paper, we term D_i^k as the dominating region of robot i in a k-order Voronoi partition. The mathematical definition of D_i^k is stated as:

$$D_i^k = \{q \in \mathcal{A} | |R_i^k(q)| \leqslant k - 1\}, \tag{3}$$

where

$$R_i^k(q) = \{j \in N | \|q - p_j\| \leqslant \|q - p_i\|, i \neq j\}. \tag{4}$$

The intuitive explanation of D_i^k can be described as follows: A point $q \in \mathcal{A}$ is said to belong to D_i^k if and only if there exist at most $k-1$ other generators such that their distance to q is less than $\|q - p_i\|$.

By observing (1) and (3), it is straightforward to conclude that D_i^k and V_i shares the same definition in 1-order Voronoi partition. In k-order Voronoi partition, D_i^k can be regarded as the union of the Voronoi cells for which robot i is one generator of this cell.

We take 2-order Voronoi partition as an example to further clarify the dominating region. As shown in the Fig. 1(b), the region enclosed by green solid line (resp. red) is the dominating region of the green highlight robot (resp. red highlight robot). The hatched area by green dominating region and red dominating region is the Voronoi cell defined by the red robot and green robot. i.e., the points in the hatched area are closer to these two highlighted robots than any other robots.

D. Sensing and communication models

In real-world applications, robots are usually equipped with omnidirectional visibility sensors, such as cameras or laser radars. The sensing model of an omnidirectional visibility sensor is usually described by a disc with radius R, centered at robot's position (we assume all robots share a homogeneous sensing radius R in this paper). We follow the definition of visibility disc in [19], which is presented as follows:

Definition *1:* The visibility disc C_i of a robot is defined as the set of points q being in distance less than or equal to the sensing radius of the i-th robot, i.e.:

$$C_i := \{q \in \mathcal{A} | \|q - p_i\| \leq R\}.$$

Our distributed algorithm requires the robots to receive the position information from its neighbors within their communication ranges. In this paper, we assume that the communication range is twice as much that of the robot's sensing range R, i.e., $2R$. Thus we have the following definition.

Definition *2:* The definition of the neighbors of robot i within its communication range is denoted as

$$N_i := \{j \in N \mid \|p_i - p_j\| \leq 2R, j \neq i\}.$$

E. Problem formulation

Suppose a bounded region $\mathcal{A} \subset \mathbb{R}^2$ with n robots dispersed. We assume all robots are stationary. We further assume that the sensing model of each robot is described by Definition 1 and the communication range is the double of the sensing range. All robots have a preliminary knowledge

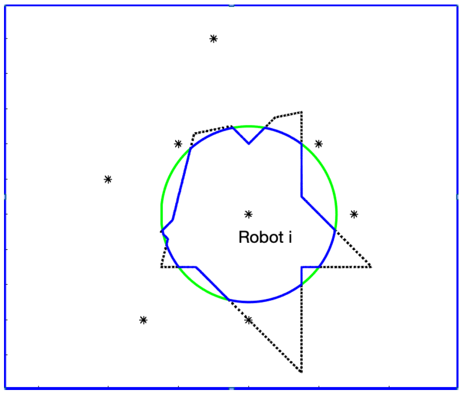

Fig. 2: The dominating region in order 2 Voronoi partition of robot i with limited sensing range. The robots are denoted by stars. The dashed line is the dominating region of n_i in order 2 Voronoi partition without range limitation. The green circle indicates the sensing range. The blue solid line is the final boundary of the dominating region D_i^k.

of boundaries of the region $\mathcal{A}$, i.e. the boundaries of region also determine the coverage region.

The objective of our algorithm is to let robot i compute its own dominating region D_i^k in the sense of order k Voronoi partition within its sensing range (an graph illustration of D_i^k is shown in Fig. 4), by using the position information received from its neighbors in set N_i.

III. The algorithm and its analysis

The idea of our proposed distributed algorithm on computing the k-order Voronoi partition can be divided into two steps: 1) for each robot i, determines its dominating region D_i^k in terms of a selected group of neighbors within its communication range, meanwhile with respect to the robot's sensing range; 2) number the Voronoi cell by computing the intersection of the dominating regions of the neighboring robots. We now present and illustrate our proposed algorithm to distributedly compute the dominating region D_i^k.

The implementation of the algorithm is stated in the Algorithm 1 block.

To compute the dominating region D_i^k, robot i needs to firstly identify its neighbor set N_i (defined in Definition 2) within its communication range and number them from small to large according to their distances to p_i. If N_i is an empty set, it is obvious that the dominating region D_i^k of robot i is equal to its visibility disc C_i, since there does not exist intersections between the visibility discs of robot i and other robots in the area $\mathcal{A}$. Furthermore, if the neighbor set N_i is not empty and its size is less than k, the dominating region D_i^k of robot i is equal to its visibility disc C_i. Otherwise the following process is applied to find the boundary points. The neighbor set identification process is illustrated in Fig. 3.

After the identification of the neighboring set, we turn to explain the detecting ray based algorithm to compute the boundary points of the dominating region D_i^k. The boundaries of D_i^k are denoted as B_i^k. Our idea is to compute the

Algorithm 1: A distributed algorithm for computing dominating region D_i^k

Input: the position information p_i of each robot $i \in N$; robot's sensing range R, communication range 2R
Output: Dominating region of robot i
foreach $j \in N \setminus i$ **do**
  $N_i \leftarrow \{j|\ \|p_j - p_i\| \leqslant 2R\}$
  Numbering $j \in$N$_i$ from small to large according to its distance to p_i, p_i^k is the position of robots in N_i, where k means the k-th nearest neighbour of robot i, also means the order of the Voronoi partition we would like to compute.
end
if $\|N_i\| < k$ **then**
  $B_i^k \leftarrow \{q|\|q - p_i\| = R\}$
  Go to **step 0**
end
foreach $d \in \mathcal{A}, s.t. \|p_d - p_i\| = \frac{\|p_i - p_i^k\|}{2}$ **do**
  $\gamma = \frac{\|p_i - p_i^k\|}{2}$, *bound* $\leftarrow$ ***False***
  while $\gamma \leqslant R$ **do**
    $R_i^k \leftarrow$ ***empty***
    $q \leftarrow \{(\gamma, p_d)|\ \|q - p_i\| = \gamma, q \in \overrightarrow{p_i p_d}\ \}$
    foreach $n_j \in N_i$ **do**
      $R_i^k \leftarrow \{n_j \in N_i|\ \|p_i^j - q\| < \|p_i - q\|\}$
    end
    if $|R_i^k| \geqslant k$ **then**
      $B_i^k \leftarrow \{q|\ \|q - p_i\| = \lambda, q \in \overrightarrow{p_i p_d}\}$,
      bound $\leftarrow$ ***True***,
      break
    else
      $\gamma = \gamma + \lambda$
    end
  end
  if *bound* = ***False*** **then**
    $B_i^k \leftarrow \{q|q = (R, d)\}$
  end
end
Use B_i^k*to define* D_i^k ***(step 0)***

boundaries (B_i^k) of the dominating region D_i^k distributedly. The starting point of the detecting ray is p_i, the other point that jointly define the detecting ray with the starting point, is an arbitrary point d on an initial circle centered at p_i with the radius γ_0. In the following, we use the symbols $\overrightarrow{p_i d}$ and $\mathcal{C}_i$ to represent the detecting ray and the initial circle, respectively. Now we use the following proposition to illustrate the selection criteria for the magnitude of the radius γ_0 of the initial circle.

Proposition 1: Assume that robot i has at least k neighbors within its communication range. For any point d with the position p_d on the ray $\overrightarrow{p_i d}$, if the distance $\|p_i - p_d\|$ between the robot i and d, is less than or equal to $\frac{\|p_i - p_i^k\|}{2}$, where p_i^k is the position of the k-th nearest neighbor of robot

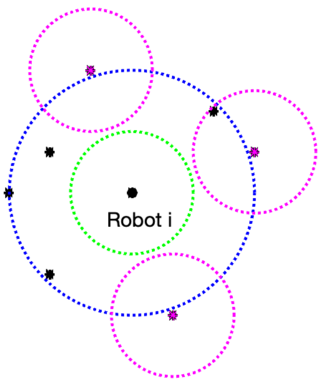

Fig. 3: The identification of neighbor set N_i. The visibility range and the communication range of robot i are denoted by the green and blue circles, respectively. The robots marked in magenta are identified not the neighbours of robot i.

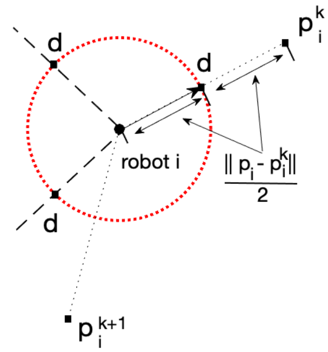

Fig. 4: The initial circle. Any points within this circle should be dominated by robot i.

i, the point d is dominated by robot i.

Proof: We number the robots in robot i's neighbor set N_i according to the magnitude of their distances to robot i, from small to large. We use p_i^k to denote the k-th nearest neighbor, whose distances to robot i is $\|p_i - p_i^k\|$. What we want to prove is that all the points on the circle centered at p_i with radius $\frac{\|p_i - p_i^k\|}{2}$ are dominated by robot i. For notation simplicity, we still use d to represent an arbitrary point on this circle. By observing the definition of the dominating region D_i^k in (3), we know that we can judge if point d belongs to D_i^k by counting the number of robots in the set $R_i^k(d)$, defined in (4). Since point d is on the circle centered at p_i with radius $\frac{\|p_i - p_i^k\|}{2}$, we have that the distance $\|p_d - p_i^k\|$ between the point d and the k-th nearest neighbor is equal to or larger than $\frac{\|p_i - p_i^k\|}{2}$. For those neighbors whose distances to robot i is larger than $\|p_i - p_i^k\|$, we have that its distance to d is always larger than $\frac{\|p_i - p_i^k\|}{2}$. The above analysis indicates the fact that the set $R_i^k(d)$ contains at most $k-1$ robots, which in turn shows that d is always dominated by robot i.

Now we have that a point q on the ray $\overrightarrow{p_i d}$, whose distance $\gamma = \|q - p_i\|$ to robot i is equal to $\frac{\|p_i - p_i^k\|}{2}$ (k represents the order of the partition), can be selected as the initial point of the detecting process for each d. We note that the detecting process aims to determine if a point q on $\overrightarrow{p_i d}$ with distance γ to robot i, which is a boundary point of the dominating region D_i^k. In the detecting process, the algorithm will check the eligibility of each point on $\overrightarrow{p_i d}$, whose distance γ to p_i is larger than $\frac{\|p_i - p_i^k\|}{2}$, if it fits the termination conditions. The termination conditions comprise of three parts: 1) the point d reaches the boundaries of the closed region $\mathcal{A}$; 2) the point d reaches the boundaries of the visibility disc C_i; and 3) the point d satisfies the condition $R_i^k(q) \geq k$. The first two termination conditions are straightforward to understand thus we omit the explanations. The idea behind the third termination condition directly arises from the definition of the dominating region D_i^k in (3), for which if there exist at least k other generators such that their distances to q is less than $\|q - p_i\|$, then point q can be regarded as a boundary point. The above detecting process determines a boundary point in one direction, then we need to repeat this process in an omnidirectional manner to compute the boundary set B_i^k for each robot, which completes our Algorithm 1. Applying Algorithm 1 for each robot, we will finally obtain a range-limited, k-order Voronoi partition for region $\mathcal{A}$.

A graph illustration of the above process is shown in Fig. 5.

***Remark** 1:* The idea of Algorithm 1 can be concluded as checking the points that outside the initial circle $\mathcal{C}_i$ to determine if they are the boundary points of $R_{n_i}^k$. The detecting ray actually serves as a "direction" to help select points. Thus the resolution of the points selection process actually determines the accuracy of the boundaries of $R_{n_i}^k$. In real implementations, the number of detecting rays and the magnitude of λ (the step size proposed in Algorithm 1) determines the resolution of the boundaries.

***Remark** 2:* The sensing range constraints of the mobile robots introduce an interesting property: not all regions in area $\mathcal{S}$ are dominated by k sensor nodes. Moreover, we note that the normal k-order Voronoi partition without range constraints can be regarded as a special case of our algorithm, for which we only need to select large enough sensing range and communication range.

***Remark** 3:* Our proposed algorithm is able to compute n-order Voronoi partition with limited range in a distributed manner, which can be directly applied in distributed coverage tasks, e.g. [20].

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IV. Simulation

In this section, we provide two simulation experiments to illustrate the results of our algorithm in computing 2-order and 3-order Voronoi partition with limited sensing range in Fig. 6 and Fig. 7, respectively. For the simulation presented

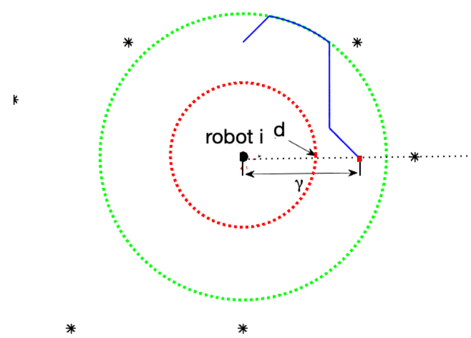

Fig. 5: The detecting process. Green dashed circle is robot i's sensor range. Red dashed circle is direction ring of robot i. Black dot represents the robot i. Star dot represent other robots in N_i. The black dashed line is the ray $\overrightarrow{p_i p_d}$. Red point is the q who satisfies the stop criterion. The blue solid line is part of the dominating region boundary.

in Fig. 6, we select the sensing range for each robot as 2.5. For the simulation presented in Fig. 7, the radius of the sensing range is selected as 2.5. It can be directly observed from the figures that not every point inside $\mathcal{A}$ can be dominated by k robots.

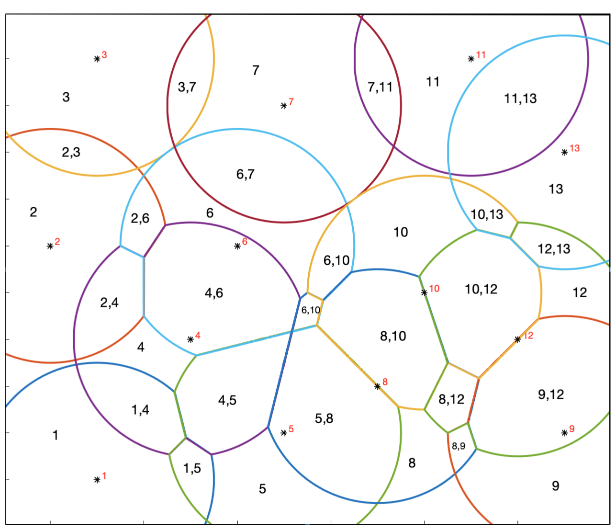

Fig. 6: A 2-order Voronoi partition with limited sensing range. The robots are represented by the stars. The red numbers represent the indexes of the robots. The black number indicates the dominating robots for a closed area.

We further present a special case in Fig. 8. The simulation in Fig. 8 aims to compute a 3-order Voronoi partition. However, because of the sensing range is quite small, no point can be dominated by three robots simultaneously.

At last, we show the distributed implementation steps

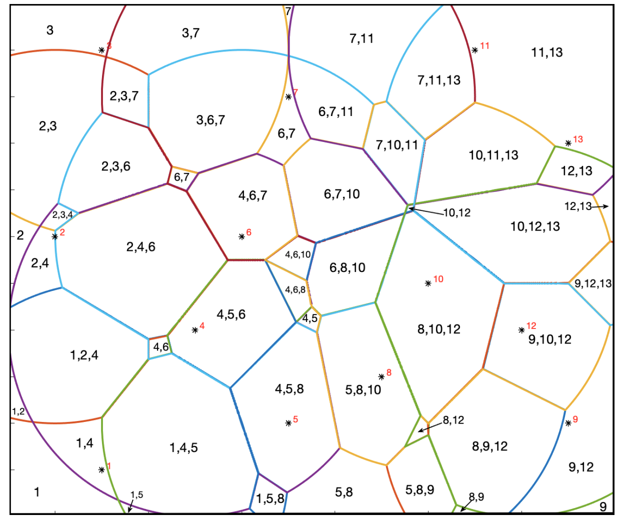

Fig. 7: A 3-order Voronoi partition with limited sensing range. The robots are represented by the stars. The red numbers represent the indexes of the robots. The black numbers indicate the dominating robots for a closed area.

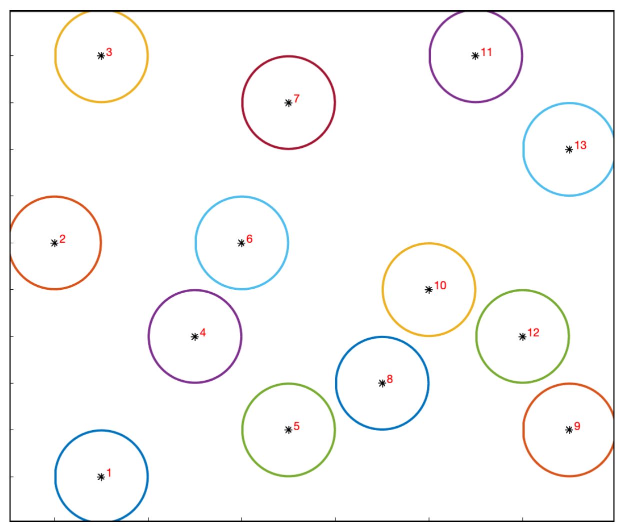

Fig. 8: A 3-order Voronoi partition with a limited sensing range. No point can be dominated by three robots simultaneously.

in Fig. 9. The four steps to distributedly partitioning the bounded region are presented step by step.

V. Conclusion

In this paper, we proposed a distributed algorithm for computing order k Voronoi partition with limited sensing range in a multi-robot system. The sensing and communication capabilities are both modelled by discs. We propose a detecting ray based algorithm, to determine the points outside a initial circle to if they are the boundary points. Only local position information of the robot's neighbours are required in the algorithm.

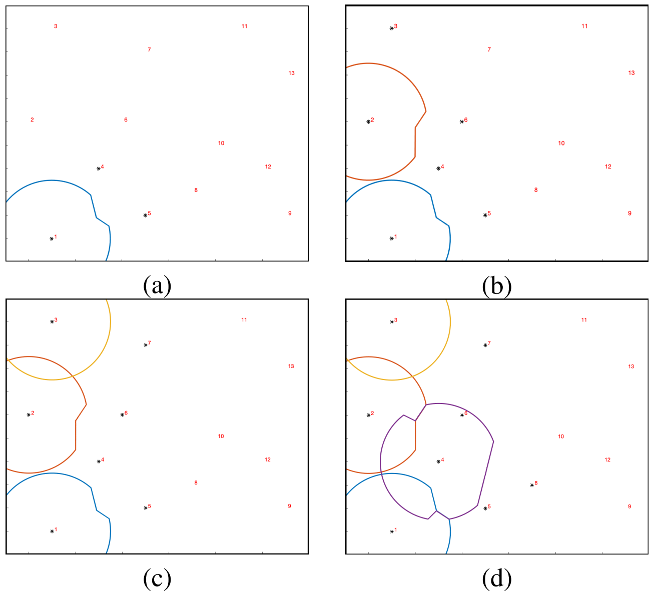

Fig. 9: Dominating regions in computing process of Order 2 Voronoi partition for each robot. a) Robot 1 finishes computation; b) Robot 2 finishes computation; c) Robot 3 finishes computation; and d) Robot 4 finishes computation
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