An Introduction to Model Order Reduction:
from linear to nonlinear dynamical systems

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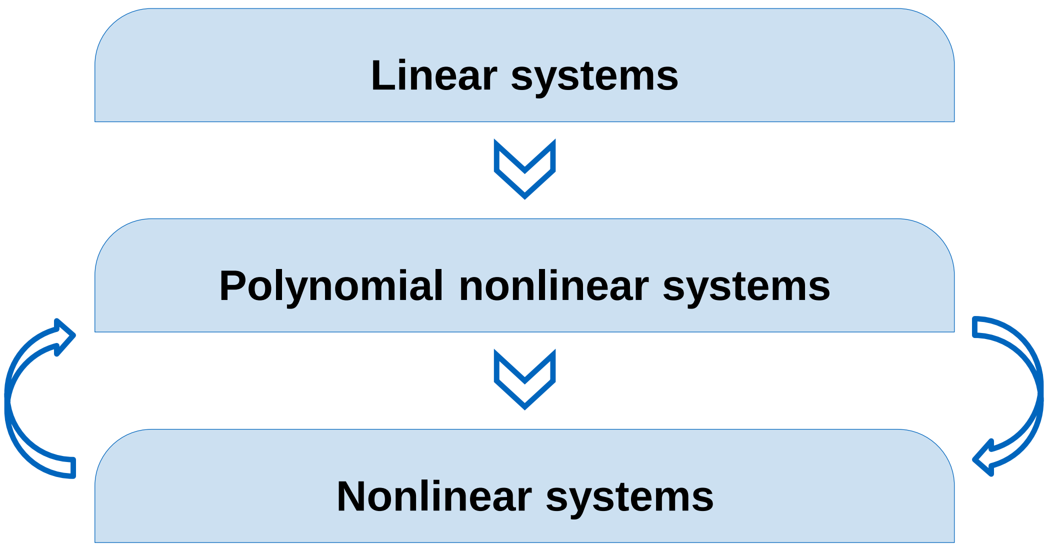

Brief personal introduction

Maria Cruz Varona
M.Sc. Electrical Engineering

University studies (10/08-03/14):

Electrical Engineering and Information Technology (KIT)

Study model 8: “Information and Automation”

Master thesis at IRS (group: “cooperative systems”)

Research assistant (since 08/14):

Chair of Automatic Control (Prof. Dr.-Ing. habil. B. Lohmann)

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Research interests:

Systems theory, model order reduction, nonlinear dynamical systems, Krylov subspace methods

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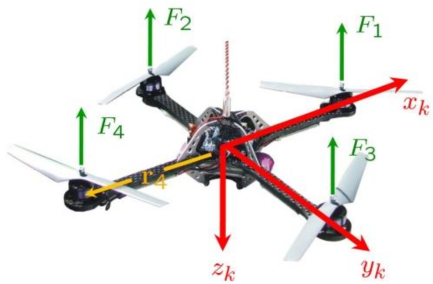

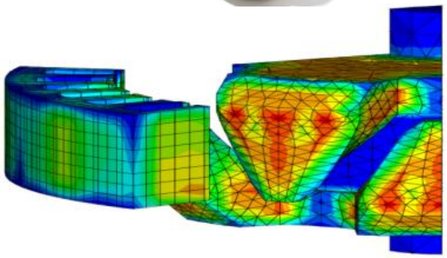

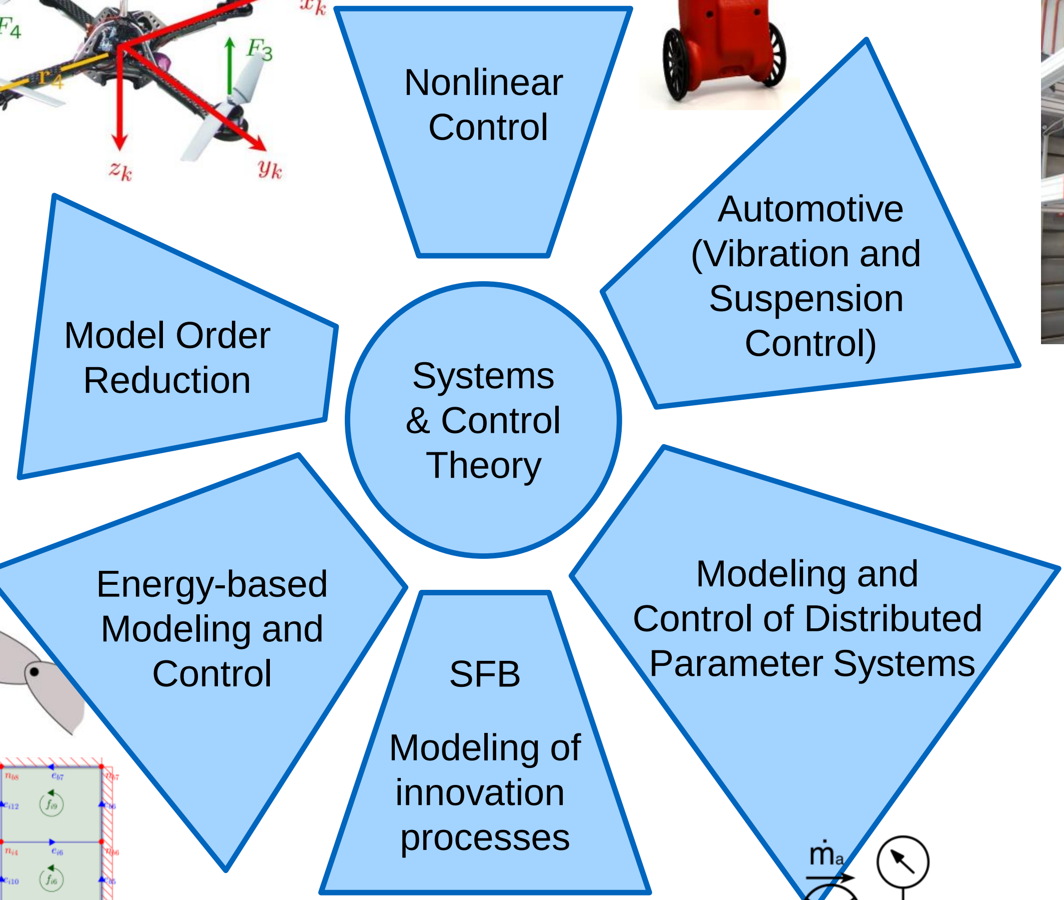

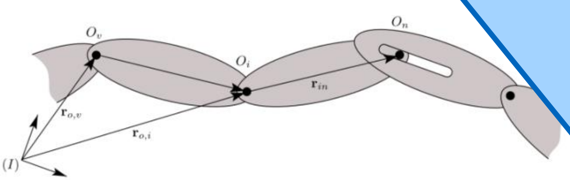

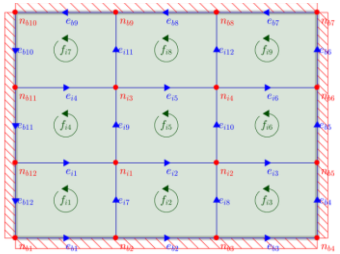

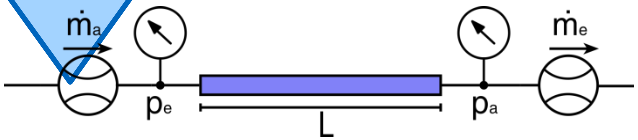

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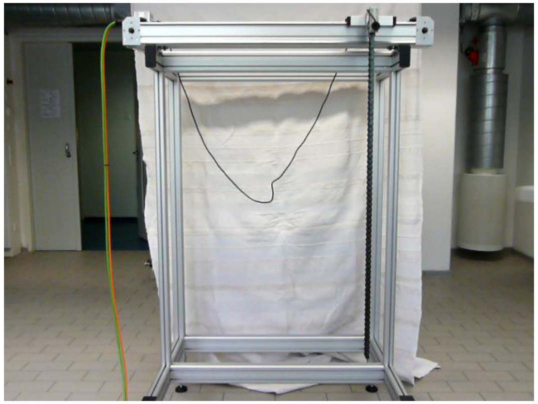
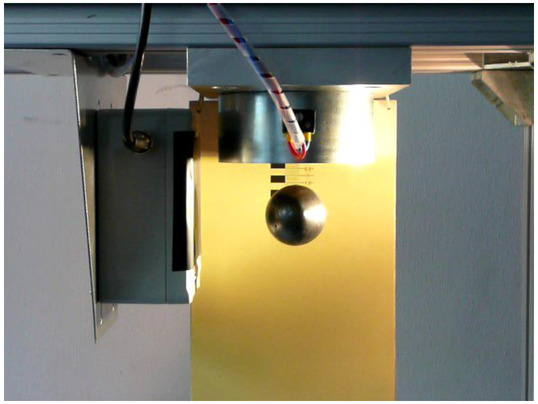
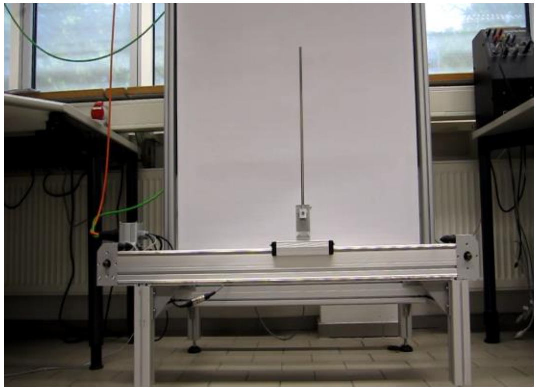
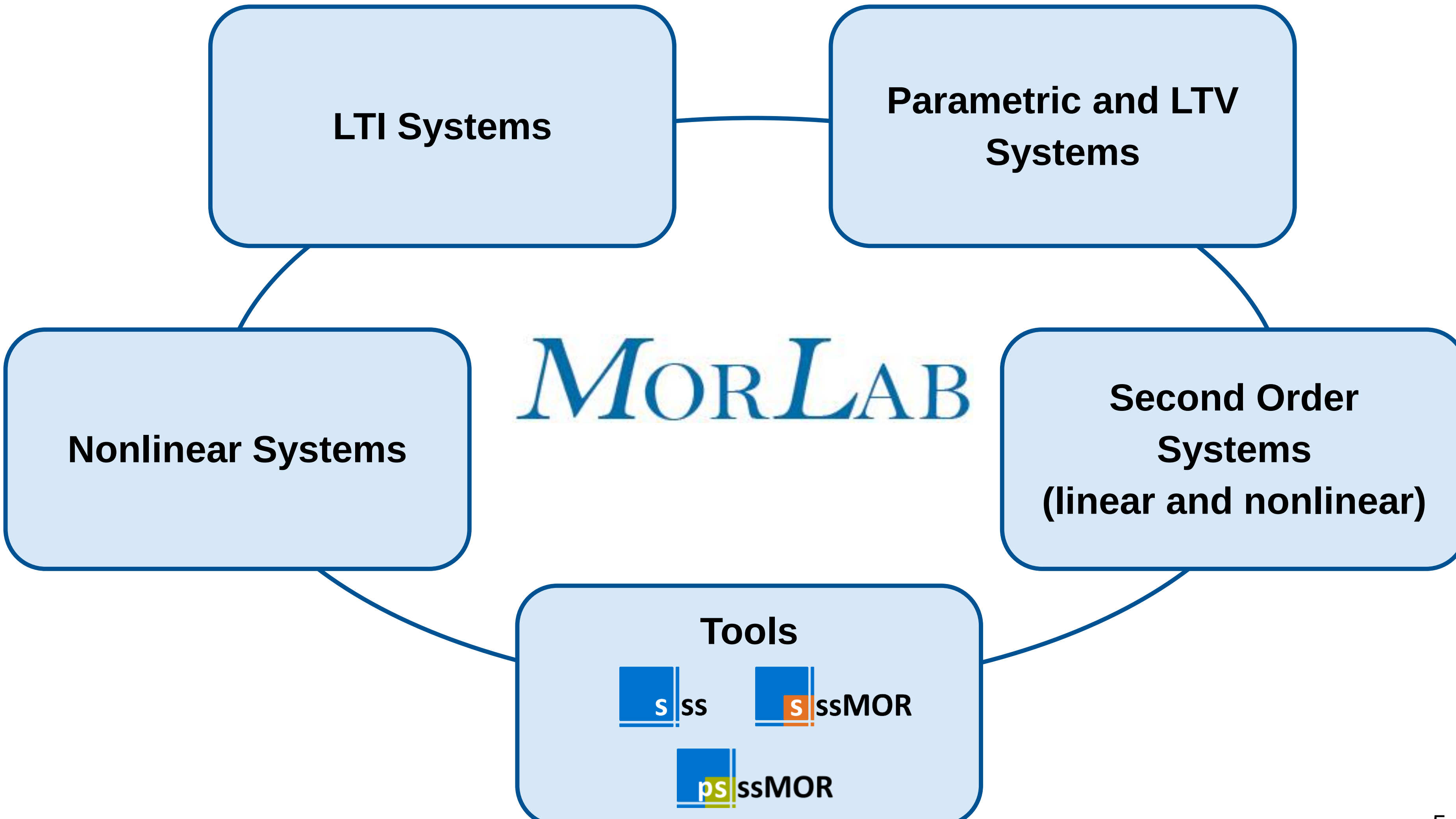






Chair of Automatic Control – MORLab



Agenda

I. Motivation & Linear Model Order Reduction

- Modeling, Modeling Strategies
- Large-scale models, Sparsity
- Reduced order models, Applications
- Projective MOR, Linear MOR methods
- Numerical Examples, FEM & MOR software

II. Polynomial & Nonlinear Model Order Reduction

- Projective NLMOR, Overview NLMOR methods
- Polynomial Nonlinear Systems, Volterra series representation
- Nonlinear Systems, Proper Orthogonal Decomposition

III. Summary & Outlook

Motivation & Linear Model Order Reduction

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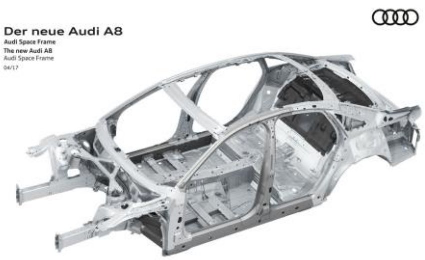
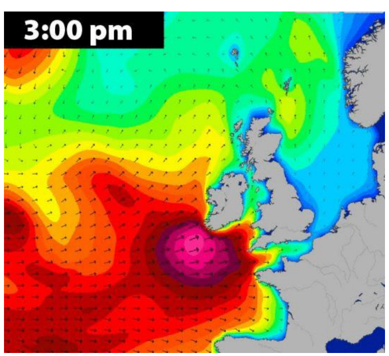
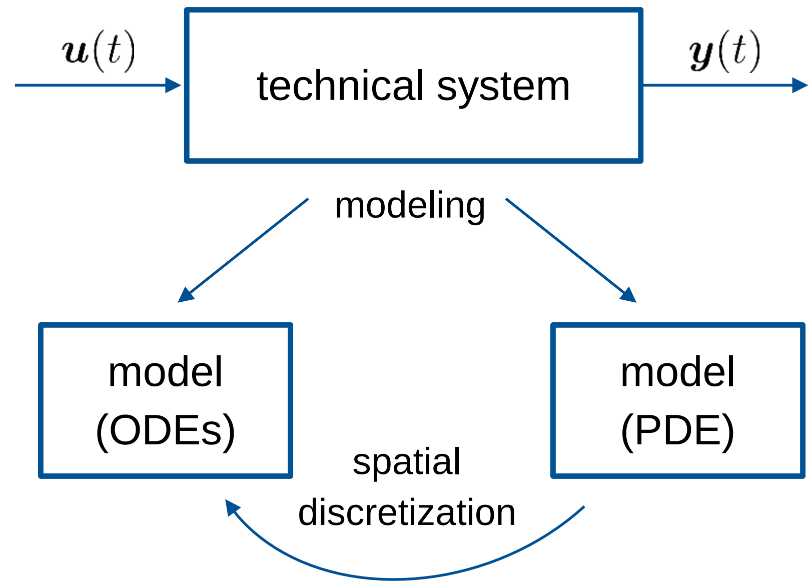
Modeling of complex dynamical systems







- Models described by ODEs:

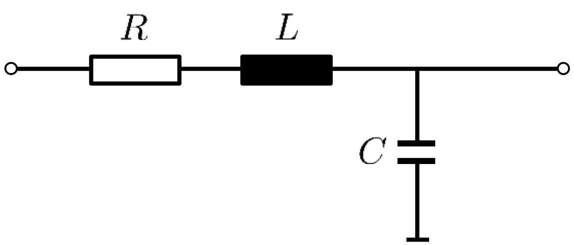
$$\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

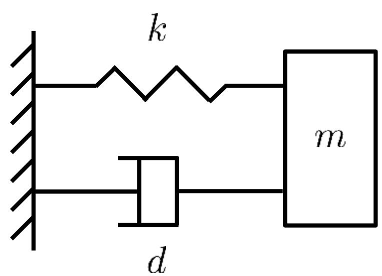
- Models described by PDEs:

$$\frac{\partial T(z,t)}{\partial t} = \frac{\partial^2 T(z,t)}{\partial z^2} + u(z,t)$$

Modeling – Strategies

0D modeling: lumped-parameter model





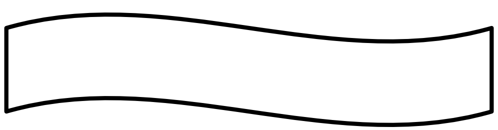
u : voltage ⇔ v : velocity

i : current ⇔ F : force

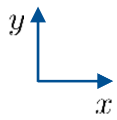
p : pressure ⇔ e : effort

q : flow rate ⇔ f : flow

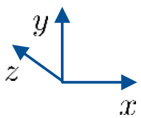
1D, 2D, 3D modeling: distributed-parameter model



$p(x,t),\ q(x,t)$



$p(x,y,t),\ q(x,y,t)$



$p(x,y,z,t),\ q(x,y,z,t)$

Data-driven modeling: identification of model using experimental data

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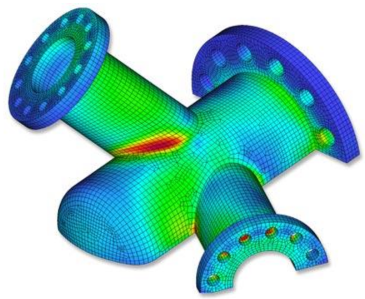
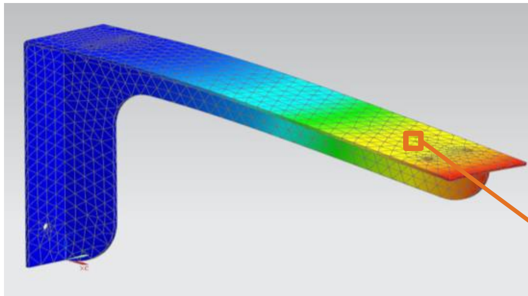
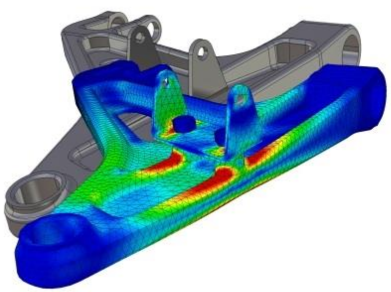
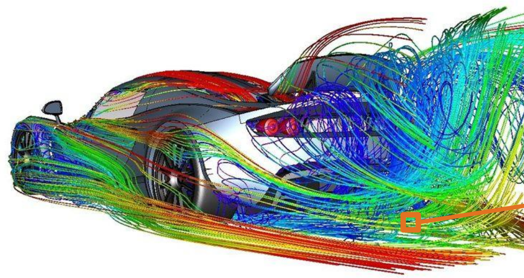
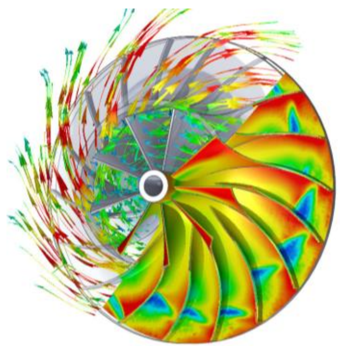
Large-scale models from spatial discretization

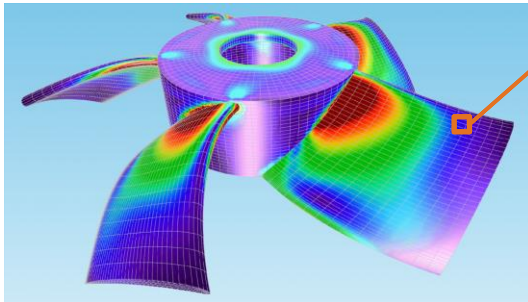












Spatial discretization using:

- Finite-Difference-Method (FDM)
- Finite-Element-Method (FEM)
- Finite-Volume-Method (FVM)

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_k(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad n = 10^3, \ldots, 10^6$$

High dimension complicates:

- numerical simulation
- design optimization
- estimation, prediction & control

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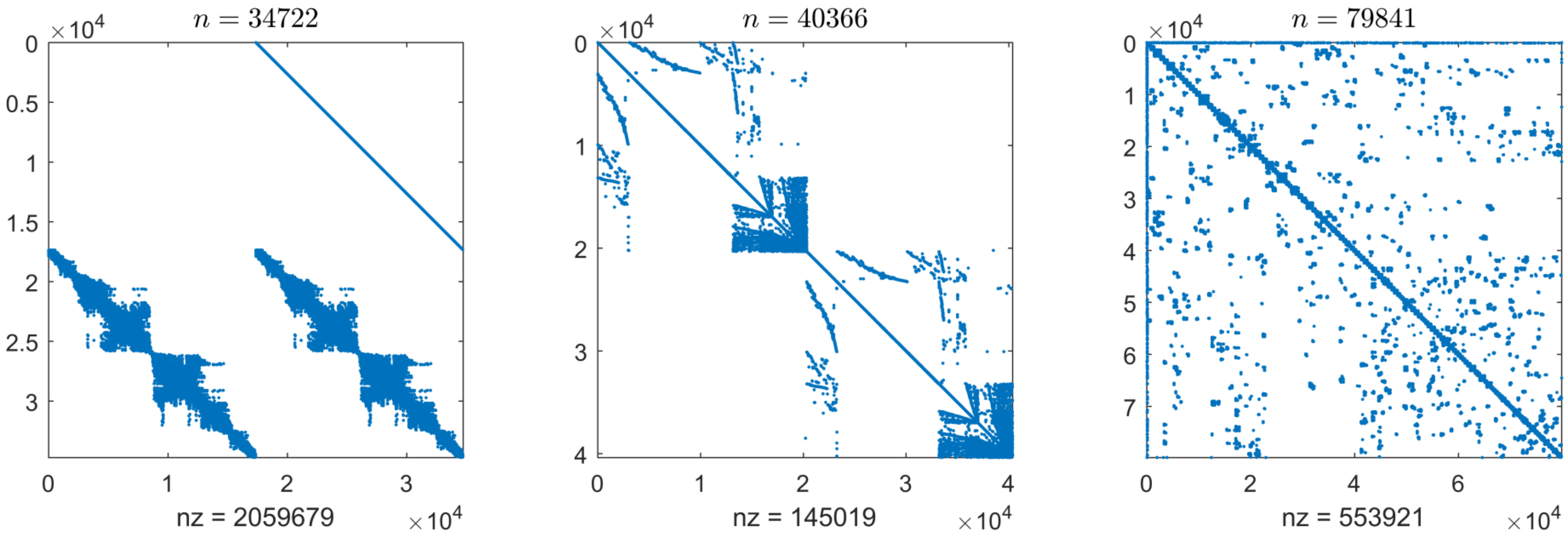
Sparsity of matrices

Matrices coming from FEM/FVM discretization are generally *sparse*



gyroscope   power system Bauru   steel profile rail_79841

Storage requirement: $\boldsymbol{A} \in \mathbb{R}^{34722 \times 34722}$

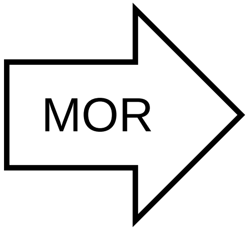
- Sparse: ~33.2 MB
- Full / Dense: 9.0 GB required!

Goal of Model Order Reduction (MOR)

Large-scale full order model (FOM)

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \qquad \det(\boldsymbol{E}) \neq 0$$

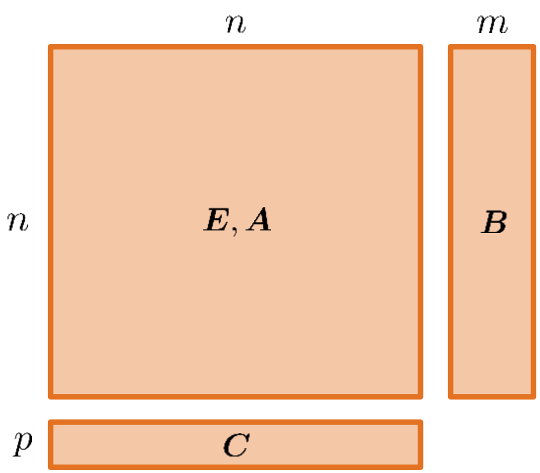


Reduced order model (ROM)

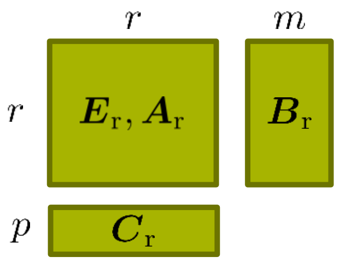
$$\boldsymbol{E}_{\mathrm{r}}\dot{\boldsymbol{x}}_{\mathrm{r}} = \boldsymbol{A}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}} + \boldsymbol{B}_{\mathrm{r}}\boldsymbol{u}$$

$$\boldsymbol{y}_{\mathrm{r}} = \boldsymbol{C}_{\mathrm{r}}\boldsymbol{x}_{\mathrm{r}}$$

$$\boldsymbol{x}_{\mathrm{r}} \in \mathbb{R}^r, \; r \ll n$$



- ✓ good approximation
- ✓ preservation of properties
- ✓ numerically efficient



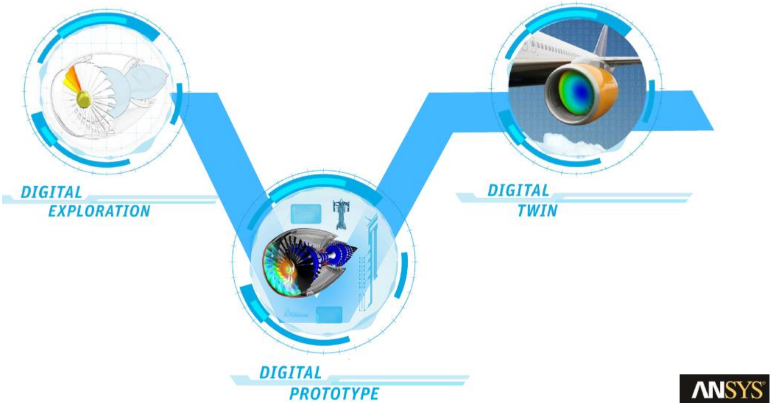
Applications of ROMs

Off-line applications:

- Efficient numerical simulation – "solves in seconds vs. hours"
- Design optimization – analysis for different parameters and "what if" scenarios
- Computer-aided failure mode and effects analysis (FMEA) – validation

On-line applications:

- Parameter estimation, Uncertainty Quantification
- Real-time optimization and control
- Digital Twin, Predictive Maintenance



Physical domains:

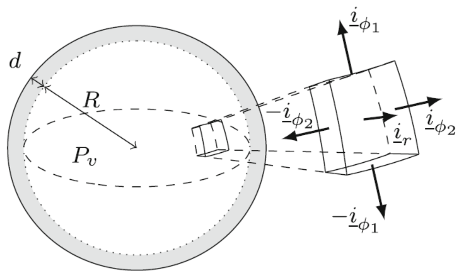
mechanical, electrical, thermal, fluid, acoustics, electromagnetism, …

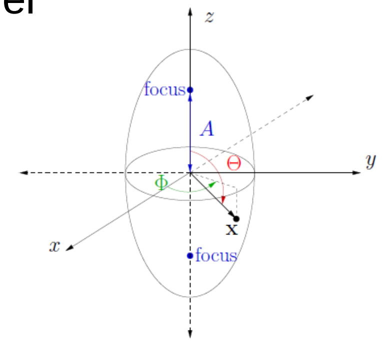
Application areas:

CSD, CFD, FSI, EMBS, MEMS, crash simulation, vibroacoustics, civil & geo, biomedical, …

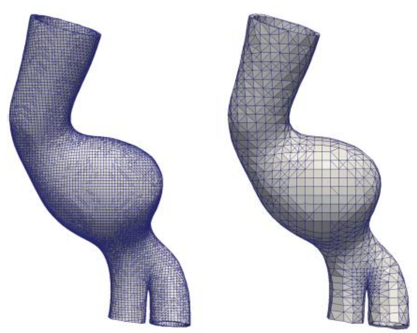
Reduced Order Modeling – Strategies

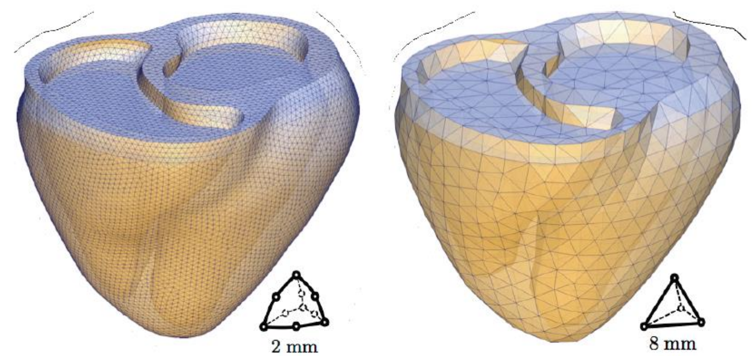
0D modeling: lumped-parameter / simplified model



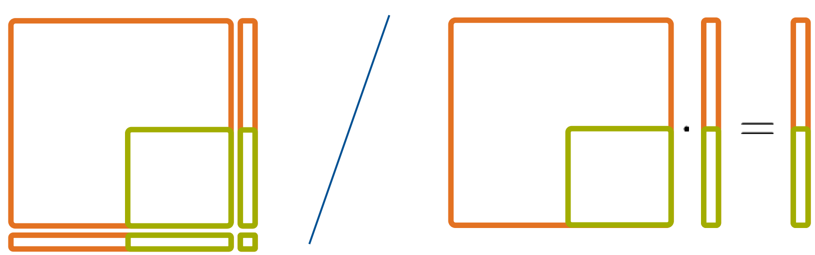


Coarse mesh:



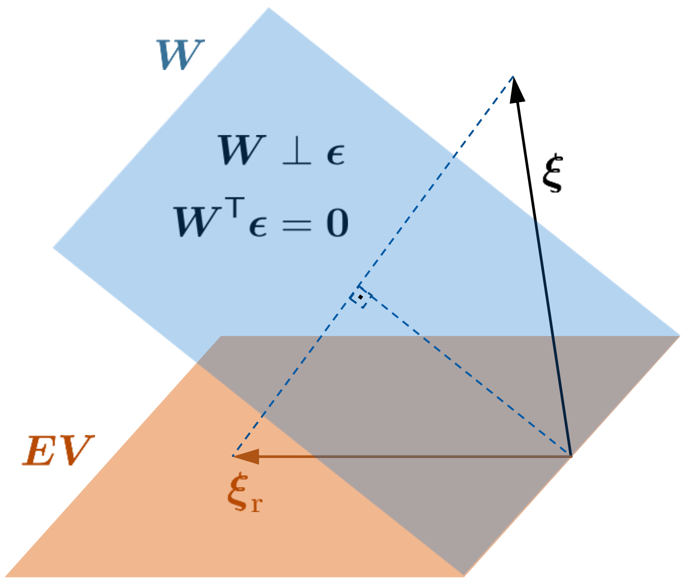


Fine mesh & Projection-based MOR:



Projective MOR

Assumption: Dynamical system does not transit all regions of the state-space equally often, but mainly stays and evolves in a subspace of lower dimension



Approximation of the state vector:

$$x = V x_r + e, \quad V \in \mathbb{R}^{n \times r}$$

Petrov-Galerkin projection: $\Pi = EV(W^\top EV)^{-1}W^\top$

$$\underbrace{W^\top E V}_{E_r} \dot{x}_r = \underbrace{W^\top A V}_{A_r} x_r + \underbrace{W^\top B}_{B_r} u$$

$$y \approx y_r = \underbrace{C V}_{C_r} x_r$$

How to choose **V** and **W**?

Linear MOR methods – Overview

1. Modal Reduction
   - Preservation of dominant eigenmodes
   - Frequently used in structural dynamics / second order systems
2. Truncated Balanced Realization / Balanced Truncation
   - Retention of state-space directions with highest energy transfer
   - Requires solution of Lyapunov equations, i.e. linear matrix equations (LMEs)
   - Applicable for medium-scale models: $n \approx 5000$
3. Rational Krylov subspaces
   - "Moment Matching": matching some Taylor-series coefficients of the transfer function
   - Requires solution of linear systems of equations (LSEs) – applicable for $n \approx 10^6$
   - Also employed for: approximate solution of eigenvalue problems, LSEs, LMEs,…
4. Iterative Krylov algorithm IRKA
   - H2-optimal reduction
   - Adaptive choice of Krylov reduction parameters (e.g. shifts)

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Modal Reduction

Goal: Preserve dominant eigenmodes of the system

Procedure:

1 *Modal transformation:* Bring system into modal coordinates through state-transformation

$$[\boldsymbol{W}, \boldsymbol{\Lambda}, \boldsymbol{V}] = \lambda(\boldsymbol{A}, \boldsymbol{E})$$

$$\boldsymbol{W}^\top \boldsymbol{A} \boldsymbol{V} = \boldsymbol{\Lambda}, \quad \boldsymbol{W}^\top \boldsymbol{E} \boldsymbol{V} = \mathbf{I}_r$$

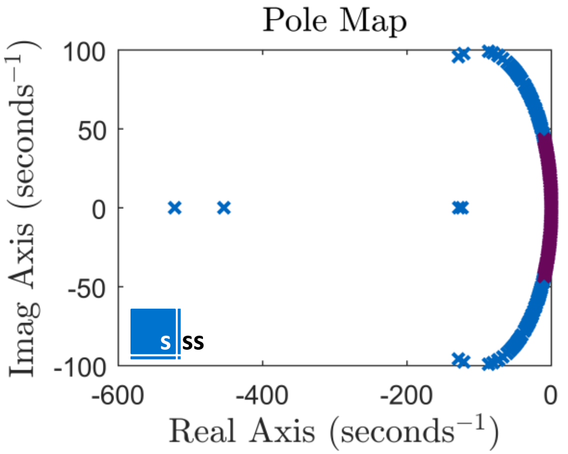
$$\hat{\boldsymbol{B}} = \boldsymbol{W}^\top \boldsymbol{B}, \qquad \hat{\boldsymbol{C}} = \boldsymbol{C} \boldsymbol{V}$$

$$\boldsymbol{x} = \boldsymbol{V} \boldsymbol{z} \quad \Rightarrow$$

$$\dot{\boldsymbol{z}} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \\ & \boldsymbol{\Lambda}_2 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} \hat{\boldsymbol{B}}_1 \\ \hat{\boldsymbol{B}}_2 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} \hat{\boldsymbol{C}}_1 & \hat{\boldsymbol{C}}_2 \end{bmatrix} \boldsymbol{z}$$

2 *Truncation:* $\boldsymbol{A}_\mathrm{r} = \boldsymbol{\Lambda}_1, \ \boldsymbol{B}_\mathrm{r} = \hat{\boldsymbol{B}}_1, \ \boldsymbol{C}_\mathrm{r} = \hat{\boldsymbol{C}}_1, \ \boldsymbol{E}_\mathrm{r} = \mathbf{I}_r$



Practical implementation:

Entire modal transformation of FOM is expensive!

→ Only a few eigenvalues and left and right eigenvectors are computed via `eigs`

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Truncated Balanced Realization (TBR)

Goal: Preserve state-space directions with highest enery transfer

Controllability and Observability Gramians:

$$\boldsymbol{A}\,\boldsymbol{P}\,\boldsymbol{E}^\mathsf{T} + \boldsymbol{E}\,\boldsymbol{P}\,\boldsymbol{A}^\mathsf{T} + \boldsymbol{B}\,\boldsymbol{B}^\mathsf{T} = \boldsymbol{0}$$

$$\boldsymbol{A}^\mathsf{T}\,\boldsymbol{Q}\,\boldsymbol{E} + \boldsymbol{E}^\mathsf{T}\,\boldsymbol{Q}\,\boldsymbol{A} + \boldsymbol{C}^\mathsf{T}\,\boldsymbol{C} = \boldsymbol{0}$$

Energy interpretation:

$$\min_{\boldsymbol{x}(0)=\boldsymbol{0},\ \boldsymbol{x}(\infty)=\boldsymbol{x}_\mathrm{e}} \int_0^\infty |\boldsymbol{u}(t)|^2\,\mathrm{d}t = \boldsymbol{x}_\mathrm{e}^\mathsf{T}\,\boldsymbol{P}^{-1}\,\boldsymbol{x}_\mathrm{e}$$

$$\|\boldsymbol{y}(t)\|_2^2 = \boldsymbol{x}_0^\mathsf{T}\,\boldsymbol{Q}\,\boldsymbol{x}_0$$

Procedure:

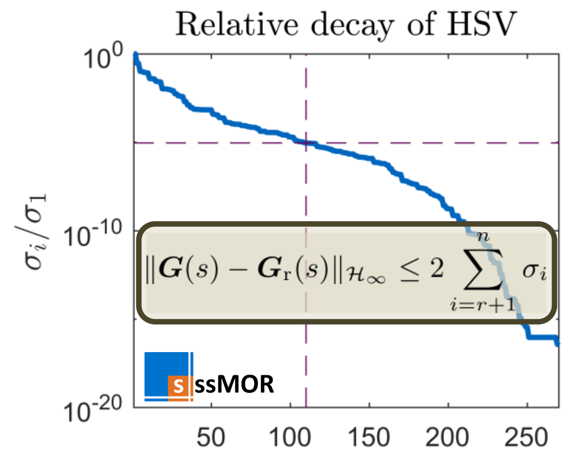
1. *Balancing step:* Compute balanced realization, where $\boldsymbol{P} = \boldsymbol{E}^\mathsf{T}\boldsymbol{Q}\boldsymbol{E} = \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$

$$\boldsymbol{P} = \boldsymbol{R}\boldsymbol{R}^\mathsf{T}, \quad \boldsymbol{Q} = \boldsymbol{S}\boldsymbol{S}^\mathsf{T}$$

$$\boldsymbol{S}^\mathsf{T}\boldsymbol{E}\boldsymbol{R} = \begin{bmatrix}\boldsymbol{U}_1 & \boldsymbol{U}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{\Sigma}_1 & \\ & \boldsymbol{\Sigma}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{N}_1^\mathsf{T} \\ \boldsymbol{N}_2^\mathsf{T}\end{bmatrix}$$

2. *Truncation step:* $\sigma_i \gg \sigma_j,\ i = 1, \dots, r,\ j = r+1, \dots, n$

$$\boldsymbol{W}^\mathsf{T} = \boldsymbol{\Sigma}_1^{-1/2}\,\boldsymbol{U}_1^\mathsf{T}\,\boldsymbol{S}^\mathsf{T}, \qquad \boldsymbol{V} = \boldsymbol{R}\,\boldsymbol{N}_1\,\boldsymbol{\Sigma}_1^{-1/2}$$



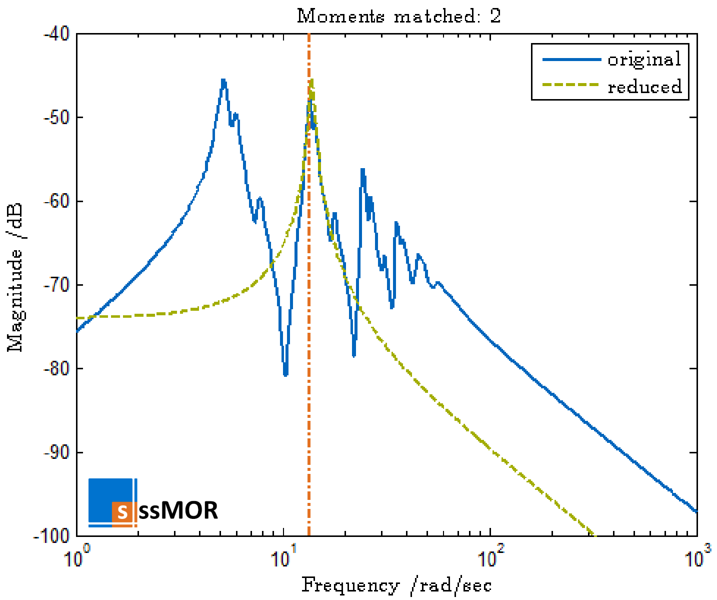
Rational Interpolation by Krylov subspace methods

Moments of a transfer function

$$G(s) = C(sE - A)^{-1}B$$
$$= G(\Delta s + \sigma) = \sum_{i=0}^{\infty} M_i(\sigma)\,(s-\sigma)^i$$

σ : interpolation point (shift)

$M_i(\sigma)$: i-th moment around σ



(Multi)-Moment Matching by Rational Krylov (RK) subspaces

Bases for input and output Krylov-subspaces:

$$\mathrm{Ran}(V) \supseteq \mathrm{span}\left\{A_\sigma^{-1}B,\; A_\sigma^{-1}EA_\sigma^{-1}B,\; \ldots,\; (A_\sigma^{-1}E)^{r-1}A_\sigma^{-1}B\right\}$$

$$\mathrm{Ran}(W) \supseteq \mathrm{span}\left\{A_\sigma^{-\mathsf{T}}C^\mathsf{T},\; A_\sigma^{-\mathsf{T}}E^\mathsf{T}A_\sigma^{-\mathsf{T}}C^\mathsf{T},\; \ldots,\; (A_\sigma^{-\mathsf{T}}E^\mathsf{T})^{r-1}A_\sigma^{-\mathsf{T}}C^\mathsf{T}\right\}$$

$\Rightarrow$ $M_i(\sigma) = M_{\mathrm{r},i}(\sigma)$

for $i = 0, \ldots, 2r-1$

Moments from full and reduced order model around certain shifts **match!**

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$\mathcal{H}_2$-optimal model order reduction

Goal: Find ROM that minimizes the $\mathcal{H}_2$-error

$$\|G - G_\mathrm{r}\|_{\mathcal{H}_2} = \min_{\dim(\widetilde{G}_\mathrm{r})=r} \left\|G - \widetilde{G}_\mathrm{r}\right\|_{\mathcal{H}_2}$$

H2-norm: $\|G(s)\|^2_{\mathcal{H}_2} = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |G(\mathrm{i}\omega)|^2 \mathrm{d}\omega$

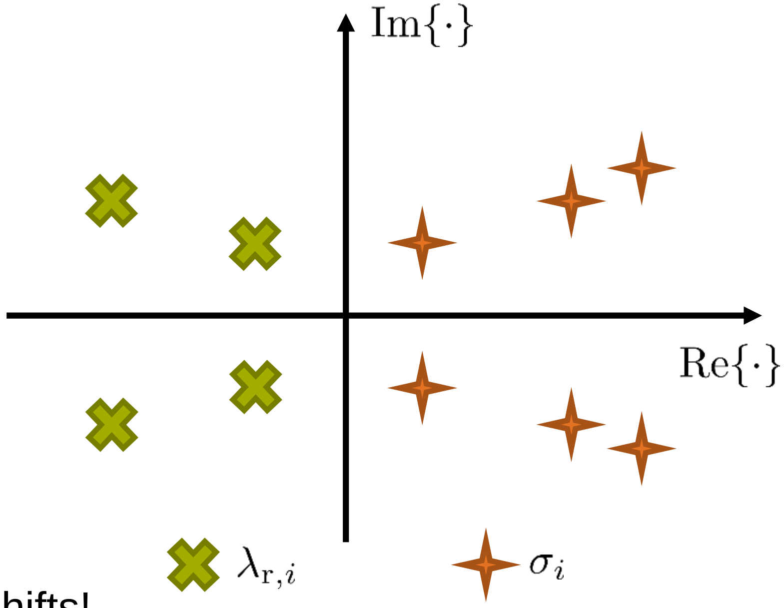
Necessary optimality conditions:

$$G(-\overline{\lambda}_{\mathrm{r},i}) = G_\mathrm{r}(-\overline{\lambda}_{\mathrm{r},i})$$
$$G'(-\overline{\lambda}_{\mathrm{r},i}) = G'_\mathrm{r}(-\overline{\lambda}_{\mathrm{r},i})$$
$$i = 1, \dots, r$$

Algorithm 1 Iterative Rational Krylov Algorithm (SISO)

Input: $\Sigma := (\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}^\mathsf{T})$, σ_i, tol
Output: locally $\mathcal{H}_2$-optimal ROM $\Sigma_\mathrm{r}^\mathrm{opt}$, σ_i^opt

while (relative change of $\sigma_i >$ tol) **do**
  $\Sigma_\mathrm{r} \leftarrow \mathrm{RK}(\Sigma, \sigma_i)$ ▷ Rational Krylov reduction
  $\boldsymbol{\Lambda}_\mathrm{r} = \lambda(\boldsymbol{A}_\mathrm{r}, \boldsymbol{E}_\mathrm{r})$ ▷ Compute eigenvalues of ROM
  $\sigma_i \leftarrow -\overline{\lambda}_{\mathrm{r},i}$ ▷ Mirror reduced eigenvalues
end while
$\Sigma_\mathrm{r}^\mathrm{opt} \leftarrow \Sigma_\mathrm{r}$, $\sigma_i^\mathrm{opt} \leftarrow \sigma_i$ ▷ Return optimal ROM and shifts



IRKA achieves multipoint moment matching at optimal shifts!

Comparison: BT vs. Krylov subspace methods

Balanced Truncation (BT)

- \+ stability preservation
- \+ automatable
- \+ error bound (a priori)

- – computing-intensive
- – storage-intensive
- – n < 5000

Rational Krylov (RK) subspaces

- \+ numerically efficient
- \+ $n \approx 10^6$
- \+ H_2-optimal (IRKA)
- \+ many degrees of freedom

- – many degrees of freedom
- – stability gen. not preserved
- – no error bounds

Subject of research

- Numerically efficient solution of large-scale Lyapunov equations

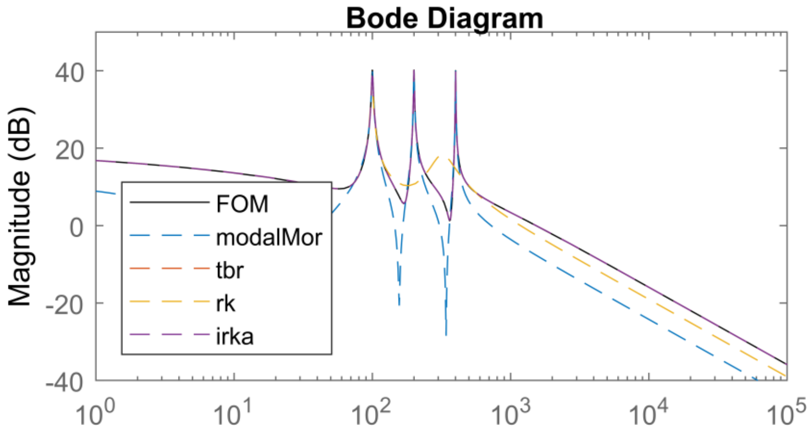
⇨ Krylov-based Low-Rank Approximation
  - ADI (Alternating Directions Implicit)
  - RKSM (Rational Krylov Subspace Method)

Subject of research

- Adaptive choice of reduction parameters
  - Reduced order
  - Interpolation data (shifts, etc.)
- Stability preservation
- Numerically efficient computation of rigorous error bounds

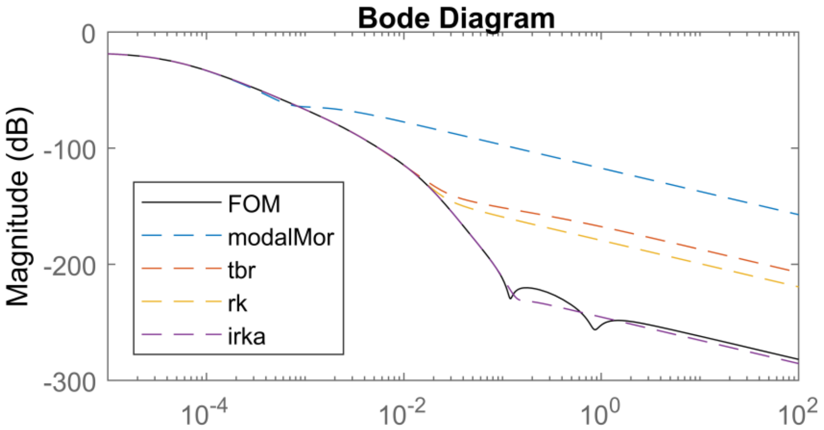
Numerical comparison

fom: $n = 1006, r = 20$



	red. time [s]	$\frac{\|G-G_\mathrm{r}\|_{\mathcal{H}_2}}{\|G\|_{\mathcal{H}_2}}$	$\frac{\|G-G_\mathrm{r}\|_{\mathcal{H}_\infty}}{\|G\|_{\mathcal{H}_\infty}}$
modalMor (lr)	0.40	19.40e-02	4.16e-02
tbr	0.20	1.18e-09	5.78e-09
rk	0.09	81.47e-02	96.73e-02
irka	0.60	8.56e-08	5.80e-09

steel profile rail_1357: $n = 1357, r = 20$



	red. time [s]	$\frac{\|G-G_\mathrm{r}\|_{\mathcal{H}_2}}{\|G\|_{\mathcal{H}_2}}$	$\frac{\|G-G_\mathrm{r}\|_{\mathcal{H}_\infty}}{\|G\|_{\mathcal{H}_\infty}}$
modalMor (lr)	1.21	4.61e-02	3.76e-03
tbr	0.49	3.47e-05	2.65e-06
rk	0.10	1.34e-07	3.36e-07
irka	1.32	2.38e-12	9.61e-11

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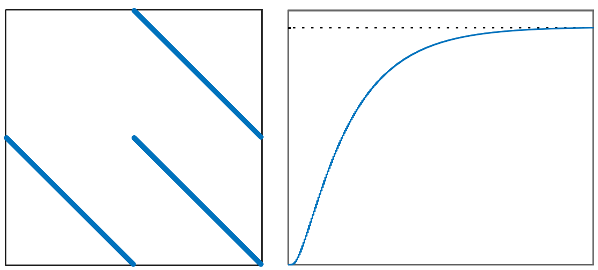
Toolboxes for sparse, large-scale models in

sss

```
sys = sss(A,B,C,D,E);
```

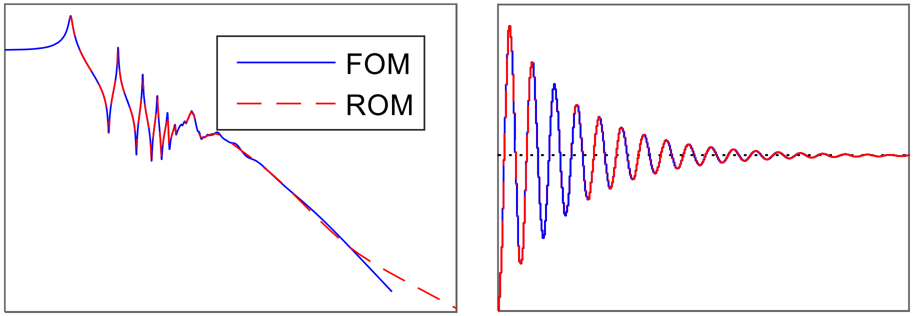


```
bode(sys), sigma(sys)
step(sys), impulse(sys)
norm(sys,2), norm(sys,inf)

c2d, lsim, eigs, connect,…
```

sssMOR

```
sysr = tbr(sys,r)
sysr = rk(sys,s0)
sysr = irka(sys,s0)
sysr = cure(sys)
sysr = cirka(sys,s0)
   ⋮
```



Powered by: **M-M.E.S.S. toolbox** [Saak, Köhler, Benner] for Lyapunov equations
Available at www.rt.mw.tum.de/?sssMOR and https://github.com/MORLab.
[Castagnotto/Cruz Varona/Jeschek/Lohmann '17]: **„sss & sssMOR: Analysis and Reduction of Large-Scale Dynamic Systems in MATLAB“,** at-Automatisierungstechnik]

GitHub

Main characteristics

sss

- **State-space models** of very high order on a standard computer $(n \approx 10^8)$
- Many Control System Toolbox functions, revisited to **exploit sparsity**
- Allows system analysis in **frequency** (`bode, sigma`, …) and **time domain** (`step,norm,lsim`,…), as well as **manipulations** (`connect,truncate`,…)
- Is **compatible** with the built-in syntax
- **New functionality:** `eigs, residue, pzmap`,…

sssMOR

- **Classical** (`modalMor, tbr, rk`,…) and **state-of-the-art** (`isrk, irka, cirka, cure`,…) reduction methods
- Both **highly-automatized**

```
sysr = irka(sys,n)
```

  and **highly-customizable**

```
Opts.maxiter = 100
Opts.tol = 1e-6
Opts.stopcrit = 'combAll'
Opts.verbose = true
sysr = irka(sys,n,Opts)
```

- `solveLse` and `lyapchol` as **core functions**

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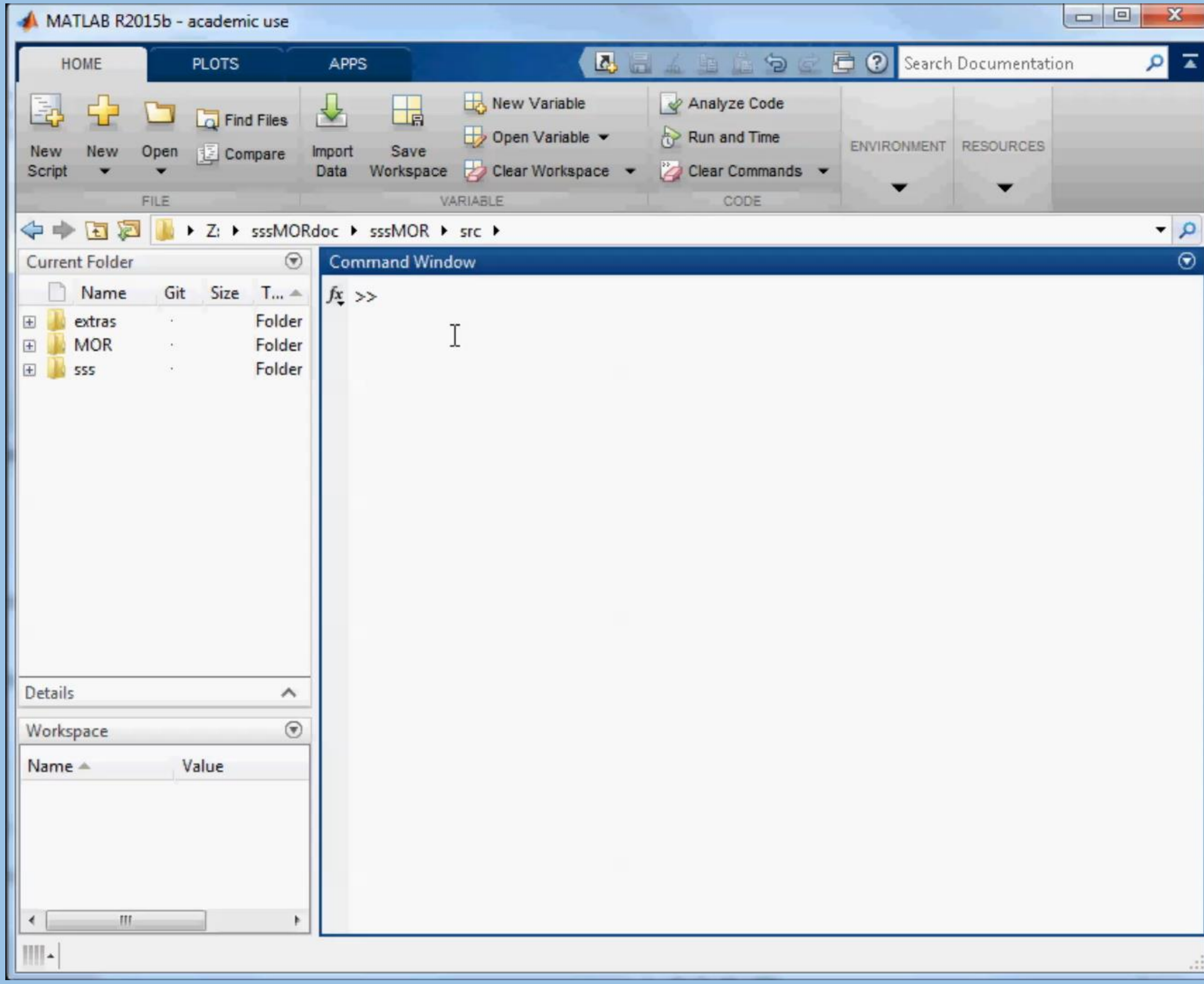
sss sssMOR

Comprehensive **documentation** with examples and references

sssMOR App
graphical user interface

completely **free**
and **open source**
(contributions welcome)



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sss sssMOR

Comprehensive **documentation** with examples and references

sssMOR App
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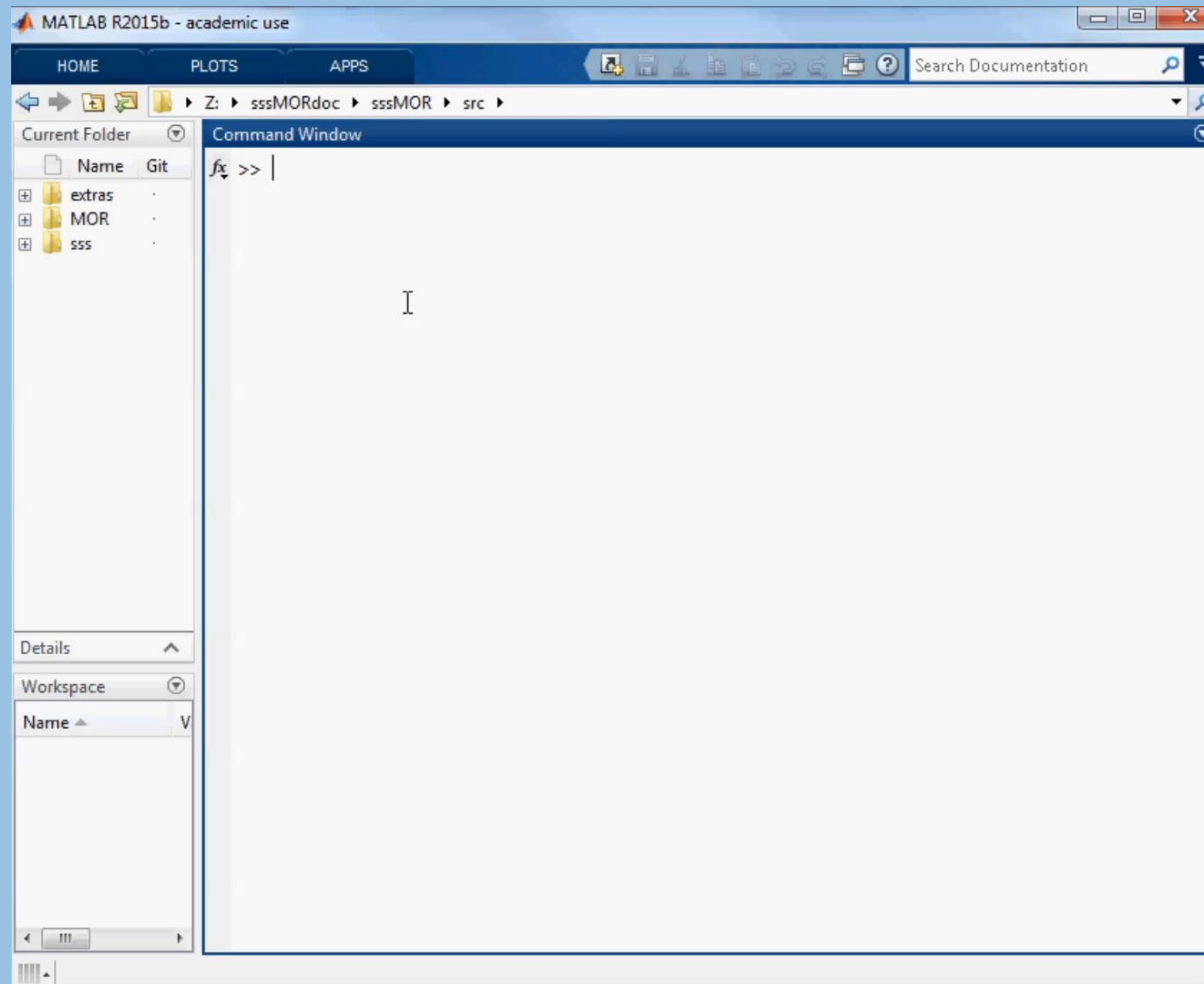


Comprehensive **documentation** with examples and references

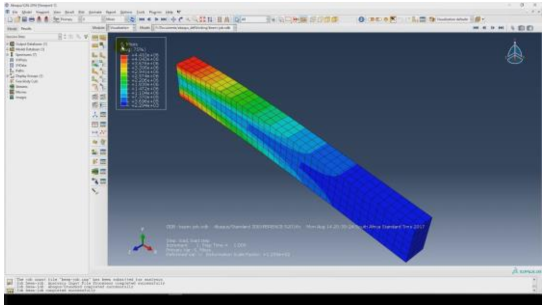
sssMOR App
graphical user interface

completely **free** and **open source** (contributions welcome)

FEM & MOR software

Commercial FEM software:

ANSYS, Abaqus, COMSOL Multiphysics, LS-DYNA, Nastran, …

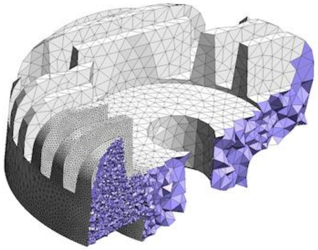


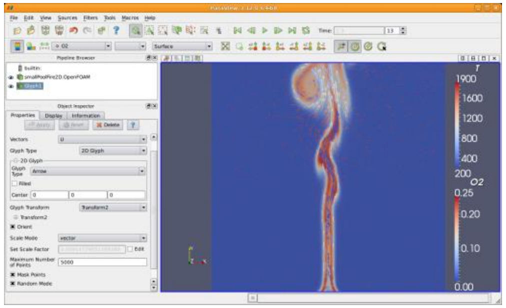
Open-source FEM software:

AMfe, CalculiX, FEniCS Project, FreeFEM++, JuliaFEM, KRATOS, OOFEM, OpenFOAM, …

Open-source Pre-/Post-Processing tools:

Gmsh, ParaView, …





Open-source MOR software:

pyMOR, sss, sssMOR, psssMOR, emgr, M.E.S.S., MOREMBS, MORE, RBmatlab, …

Polynomial & Nonlinear Model Order Reduction

Projective MOR for Nonlinear Systems

Given a large-scale nonlinear control system of the form

$$\det(\boldsymbol{E}) \neq 0$$

$$\boldsymbol{E}\,\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$$
$$y = h(\boldsymbol{x})$$

$$\boldsymbol{x}(t) \in \mathbb{R}^n$$

with $\boldsymbol{f}(\boldsymbol{x},\, u) : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ and $h(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^1$

Simulation, design, control and optimization cannot be done efficiently!

MOR

$$\boldsymbol{x} \approx \boldsymbol{V}\,\boldsymbol{x}_{\mathrm{r}}\,, \quad \boldsymbol{V} \in \mathbb{R}^{n \times r}$$

Reduced order model (ROM)

$$\underbrace{\boldsymbol{V}^{\mathsf{T}}\;\boldsymbol{E}\;\boldsymbol{V}}_{\boldsymbol{E}_{\mathrm{r}}}\;\dot{\boldsymbol{x}}_{\mathrm{r}} = \underbrace{\boldsymbol{V}^{\mathsf{T}}\;\boldsymbol{f}(\boldsymbol{V}\,\boldsymbol{x}_{\mathrm{r}},\, u)}_{\boldsymbol{f}_{\mathrm{r}}(\boldsymbol{x}_{\mathrm{r}},\, u)}$$
$$y_{\mathrm{r}} = \underbrace{h(\boldsymbol{V}\,\boldsymbol{x}_{\mathrm{r}})}_{h(\boldsymbol{x}_{\mathrm{r}})}$$

$$\boldsymbol{x}_{\mathrm{r}}(t) \in \mathbb{R}^r,\; r \ll n$$

Goal:
$$y_{\mathrm{r}}(t) \approx y(t)$$

with $\boldsymbol{f}_{\mathrm{r}}(\boldsymbol{x}_{\mathrm{r}},\, u) : \mathbb{R}^r \times \mathbb{R}^1 \to \mathbb{R}^r$ and $h(\boldsymbol{x}_{\mathrm{r}}) : \mathbb{R}^r \to \mathbb{R}^1$

Challenges of Nonlinear MOR

Nonlinear systems can exhibit complex behaviours

- Strong nonlinearities
- Multiple equilibrium points
- Limit cycles
- Chaotic behaviours

Input-output behaviour of nonlinear systems **cannot** be described with transfer functions, the state-transition matrix or the convolution integral (only possible for special cases)

Choice of the reduced order basis

- Projection basis should comprise most dominant directions of the state-space
- Different existing approaches:
  - Simulation-based methods
  - System-theoretic techniques

Expensive evaluation of $\boldsymbol{f}(\boldsymbol{V}\boldsymbol{x}_\mathrm{r})$

- Vector of nonlinearities **f** still has to be evaluated in full dimension
- Approximation of **f** by so-called hyper-reduction techniques:
  → EIM, DEIM, GNAT, ECSW…

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Nonlinear MOR methods – Overview

Polynomial nonlinear systems

Reduction of bilinear systems

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{N}\boldsymbol{x}u + \boldsymbol{b}u$$
$$y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

- ☑ Transfer of system-theoretic concepts
- ☑ Generalization of linear MOR methods:
  - Balanced truncation
  - **Krylov / $\mathcal{H}_2$-optimal approach**

Reduction of quadratic-bilinear systems

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{H}(\boldsymbol{x} \otimes \boldsymbol{x}) + \boldsymbol{N}\boldsymbol{x}u + \boldsymbol{b}u$$
$$y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

- ☑ Reduction methods for **MIMO** models
- ➢ Input-awareness:
  - signal generators
  - eigenfunctions

Nonlinear systems

Reduction of nonlinear (parametric) systems

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u) \qquad \boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u$$
$$y = h(\boldsymbol{x}) \qquad y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

- ☑ Simulation-based:
  - POD, TPWL
  - Reduced Basis, Empirical Gramians
- ➢ Simulation-free / System-theoretic

Polynomial Nonlinear Systems

Polynomialization / Carleman linearization

Starting point: $\boldsymbol{E}\,\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\,u$

$$y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

Power series

Assumptions:

- $\boldsymbol{x}_{\mathrm{S}} = \boldsymbol{0}$
- $\boldsymbol{f}(\boldsymbol{x}_{\mathrm{S}}) = \boldsymbol{0}$

$$\boldsymbol{A}^{(1)} \in \mathbb{R}^{n \times n}$$
$$\boldsymbol{A}^{(2)} \in \mathbb{R}^{n \times n^2}$$
$$\boldsymbol{A}^{(3)} \in \mathbb{R}^{n \times n^3}$$
$$\vdots$$

$$\boldsymbol{x}^{(1)} = \boldsymbol{x} \in \mathbb{R}^{n}$$
$$\boldsymbol{x}^{(2)} = \boldsymbol{x} \otimes \boldsymbol{x} \in \mathbb{R}^{n^2}$$
$$\boldsymbol{x}^{(3)} = \boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x} \in \mathbb{R}^{n^3}$$
$$\vdots$$

$$\boldsymbol{E}\,\dot{\boldsymbol{x}} = \boldsymbol{A}^{(1)}\,\boldsymbol{x} + \boldsymbol{A}^{(2)}\,(\boldsymbol{x} \otimes \boldsymbol{x}) + \cdots + \boldsymbol{N}^{(1)}\,\boldsymbol{x}\,u + \cdots + \boldsymbol{b}\,u$$
$$y = \boldsymbol{c}^{\mathsf{T}}\,\boldsymbol{x}$$

Bilinear dynamical systems

- Result from direct modeling or Carleman (bi)linearization
- Linear in input and linear in state, but not jointly linear in both
- Interface between fully nonlinear and linear systems

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{N}\boldsymbol{x}u + \boldsymbol{b}u$$
$$y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

Volterra series representation

[Rugh '81]

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{N}\,\boldsymbol{x}(t)\,u(t) + \boldsymbol{b}\,u(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$
$$y(t) = \boldsymbol{c}^{\mathsf{T}}\,\boldsymbol{x}(t).$$

Picard fixed-point iteration (successive approximation)

Approximate solution of the bilinear system

$$\boldsymbol{x}_1(t) = \int_{\tau=0}^{t} \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)\,\mathrm{d}\tau + \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}_0\,,$$

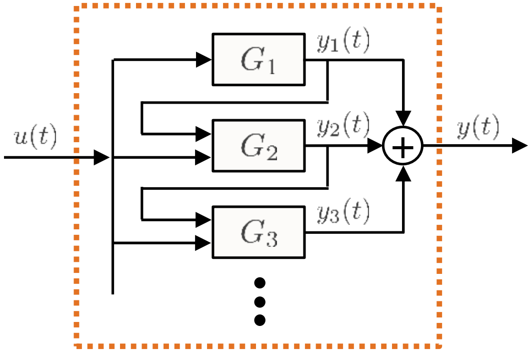
$$\boldsymbol{x}_k(t) = \int_{\tau=0}^{t} \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{N}u(\tau)\boldsymbol{x}_{k-1}(\tau)\,\mathrm{d}\tau\,, \quad k \geq 2\,.$$

$$\boldsymbol{x}(t) = \sum_{k=1}^{\infty} \boldsymbol{x}_k(t)$$

Variational equations (subsystems)

Interpretation as a series of homogenous, cascaded subsystems:

$$\dot{\boldsymbol{x}}_1(t) = \boldsymbol{A}\,\boldsymbol{x}_1(t) + \boldsymbol{b}\,u(t)\,, \qquad \boldsymbol{x}_1(0) = \boldsymbol{x}_0\,,$$
$$\dot{\boldsymbol{x}}_k(t) = \boldsymbol{A}\,\boldsymbol{x}_k(t) + \boldsymbol{N}\,\boldsymbol{x}_{k-1}(t)\,u(t)\,, \quad \boldsymbol{x}_k(0) = \boldsymbol{0}\,, \quad k \geq 2$$



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Systems Theory for Volterra systems (1)

[Rugh '81]

Input-Output behavior

$$y(t) = \sum_{k=1}^{\infty} y_k(t)$$

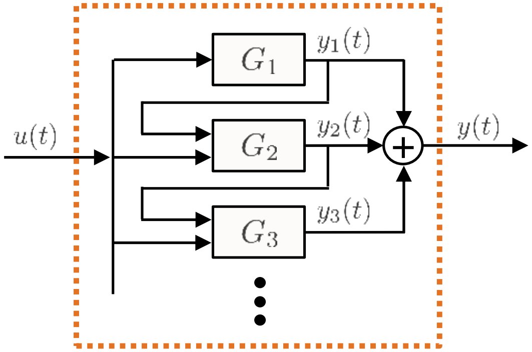
$$y(t) = \sum_{k=1}^{\infty} \int_{\tau_1=-\infty}^{\infty} \cdots \int_{\tau_k=-\infty}^{\infty} \underbrace{\boldsymbol{c}^\mathsf{T} \mathrm{e}^{\boldsymbol{A}\tau_k} \boldsymbol{N} \cdots \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_2} \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_1} \boldsymbol{b}}_{g_k(\tau_1,\ldots,\tau_k)} \times u(t-\tau_k) \cdots u(t-\tau_k-\ldots-\tau_1)\, \mathrm{d}\tau_k \cdots \mathrm{d}\tau_1$$

Kernels

$$k=1: \quad g_1(\tau_1) = \boldsymbol{c}^\mathsf{T} \mathrm{e}^{\boldsymbol{A}\tau_1} \boldsymbol{b}$$

$$k=2: \quad g_2(\tau_1,\tau_2) = \boldsymbol{c}^\mathsf{T} \mathrm{e}^{\boldsymbol{A}\tau_2} \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_1} \boldsymbol{b}$$

$$k=3: \quad g_3(\tau_1,\tau_2,\tau_3) = \boldsymbol{c}^\mathsf{T} \mathrm{e}^{\boldsymbol{A}\tau_3} \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_2} \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_1} \boldsymbol{b}$$



Transfer functions

$$k=1: \quad G_1(s_1) = \boldsymbol{c}^\mathsf{T} (s_1 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{b}$$

$$k=2: \quad G_2(s_1,s_2) = \boldsymbol{c}^\mathsf{T} (s_2 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{N} (s_1 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{b}$$

$$k=3: \quad G_3(s_1,s_2,s_3) = \boldsymbol{c}^\mathsf{T} (s_3 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{N} (s_2 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{N} (s_1 \mathbf{I} - \boldsymbol{A})^{-1} \boldsymbol{b}$$

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Systems Theory for Volterra systems (2)

[Rugh ’81]

Gramians

$$\boldsymbol{P} = \sum_{k=1}^{\infty} \boldsymbol{P}_k\,, \quad \boldsymbol{Q} = \sum_{k=1}^{\infty} \boldsymbol{Q}_k\,,$$

$$g_k(\tau_1,\ldots,\tau_k) = \underbrace{\boldsymbol{c}^\mathsf{T} \mathrm{e}^{\boldsymbol{A}\tau_k} \boldsymbol{N} \cdots \overbrace{\boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_2} \boldsymbol{N} \mathrm{e}^{\boldsymbol{A}\tau_1} \boldsymbol{b}}^{\overline{\boldsymbol{p}}_k(\tau_1,\ldots,\tau_k)}}_{\overline{\boldsymbol{q}}_k(\tau_1,\ldots,\tau_k)^\mathsf{T}}$$

$$\boldsymbol{P}_k = \int_{\tau_1=0}^{\infty} \cdots \int_{\tau_k=0}^{\infty} \overline{\boldsymbol{p}}_k(\tau_1,\ldots,\tau_k)\overline{\boldsymbol{p}}_k(\tau_1,\ldots,\tau_k)^\mathsf{T}\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\mathrm{k}$$

$$\boldsymbol{Q}_k = \int_{\tau_1=0}^{\infty} \cdots \int_{\tau_k=0}^{\infty} \overline{\boldsymbol{q}}_k(\tau_1,\ldots,\tau_k)\overline{\boldsymbol{q}}_k(\tau_1,\ldots,\tau_k)^\mathsf{T}\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_\mathrm{k}$$

$$\boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^\mathsf{T} + \boldsymbol{N}\boldsymbol{P}\boldsymbol{N}^\mathsf{T} + \boldsymbol{b}\boldsymbol{b}^\mathsf{T} = \boldsymbol{0}$$

$$\boldsymbol{A}^\mathsf{T}\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A} + \boldsymbol{N}^\mathsf{T}\boldsymbol{Q}\boldsymbol{N} + \boldsymbol{c}\boldsymbol{c}^\mathsf{T} = \boldsymbol{0}$$

H2-norm

$$\|\zeta\|_{\mathcal{H}_2}^2 = \sum_{k=1}^{\infty} \int_{\tau_1=0}^{\infty} \cdots \int_{\tau_k=0}^{\infty} g_k(\tau_1,\ldots,\tau_k) g_k(\tau_1,\ldots,\tau_k)^\mathsf{T} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_k$$

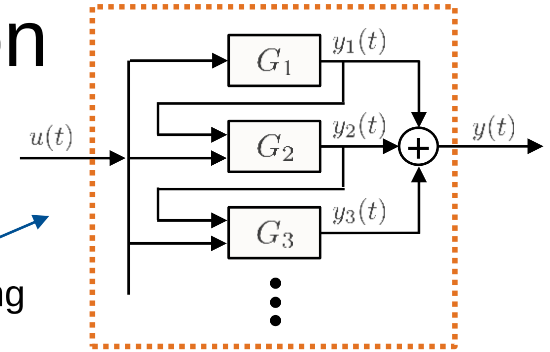
$$= \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} G_k(\mathrm{i}\omega_1,\ldots,\mathrm{i}\omega_k)\, G_k(-\mathrm{i}\omega_1,\ldots,-\mathrm{i}\omega_k)^\mathsf{T}\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_k$$

$$= \boldsymbol{c}^\mathsf{T}\, \boldsymbol{P}\, \boldsymbol{c} = \boldsymbol{b}^\mathsf{T}\, \boldsymbol{Q}\, \boldsymbol{b}$$

Multipoint Volterra Series Interpolation

Goal: Enforcing multipoint interpolation of the underlying Volterra series

Moment matching



Multipoint Volterra series interpolation [Flagg/Gugercin '15]

Set of interpolation points: $S = \{\sigma_1, \dots, \sigma_r\}, \quad i = 1, \dots, r$

$$\sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\dots,l_{k-1},i} \, G_k(\sigma_{l_1}, \dots, \sigma_{l_{k-1}}, \sigma_i) = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\dots,l_{k-1},i} \, G_{k,\mathrm{r}}(\sigma_{l_1}, \dots, \sigma_{l_{k-1}}, \sigma_i)$$

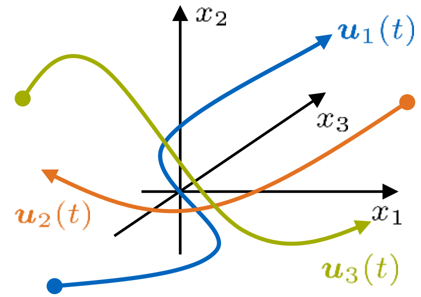
This approach interpolates the weighted series at the interpolation points $\sigma_1, \dots, \sigma_r$

Projection matrices for Volterra series interpolation

$$\boldsymbol{v}_i = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\dots,l_{k-1},i} (\sigma_i \boldsymbol{E} - \boldsymbol{A})^{-1} \boldsymbol{N} (\sigma_{l_{k-1}} \boldsymbol{E} - \boldsymbol{A})^{-1} \boldsymbol{N} \cdots \boldsymbol{N} (\sigma_{l_1} \boldsymbol{E} - \boldsymbol{A})^{-1} \boldsymbol{b}$$

$$\boldsymbol{w}_i = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \vartheta_{l_1,\dots,l_{k-1},i} (\mu_{l_1} \boldsymbol{E} - \boldsymbol{A})^{-\mathsf{T}} \boldsymbol{N}^{\mathsf{T}} (\mu_{l_2} \boldsymbol{E} - \boldsymbol{A})^{-\mathsf{T}} \boldsymbol{N}^{\mathsf{T}} \cdots \boldsymbol{N}^{\mathsf{T}} (\mu_i \boldsymbol{E} - \boldsymbol{A})^{-\mathsf{T}} \boldsymbol{c}$$

Proper Orthogonal Decomposition (POD)



Starting point: $\boldsymbol{E}\,\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$

$$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$$

1. Choose suitable training input signals $\boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \ldots, \boldsymbol{u}_t(t)$
2. Take snapshots from simulated full order state trajectories

$$\underset{(n,n_\mathrm{s})}{\boldsymbol{X}} = \left[\boldsymbol{x}^{\boldsymbol{u}_1}(t_1),\, \boldsymbol{x}^{\boldsymbol{u}_1}(t_2),\, \cdots,\, \boldsymbol{x}^{\boldsymbol{u}_1}(t_N) \quad \boldsymbol{x}^{\boldsymbol{u}_2}(t_1),\, \boldsymbol{x}^{\boldsymbol{u}_2}(t_2),\, \cdots\right]$$

3. Perform singular value decomposition (SVD) of snapshot matrix $\boldsymbol{X}$

$$\boldsymbol{X} = \underset{(n,n)}{\boldsymbol{M}}\ \underset{(n,n_\mathrm{s})}{\boldsymbol{\Sigma}}\ \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}^\top} \approx \underset{(n,r)}{\boldsymbol{M}_\mathrm{r}}\ \underset{(r,n_\mathrm{s})}{\boldsymbol{\Sigma}_\mathrm{r}}\ \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}_\mathrm{r}^\top}$$

4. Reduced order basis: $\boldsymbol{V} = \boldsymbol{M}_\mathrm{r} \in \mathbb{R}^{n\times r}$

Advantages:

- ☑ Straightforward data-driven method
- ☑ Choice of reduced order from singular values / error bound for approx. error
- ☑ Optimal in least squares sense:

$$\min_{\mathrm{rank}(\boldsymbol{X}_\mathrm{r})=r} ||\boldsymbol{X} - \boldsymbol{X}_\mathrm{r}||_2$$

Disadvantages:

- ! Simulation of full order model for different input signals required
- ! SVD of large snapshot matrix required
- ! Training input dependency

Summary & Outlook

Take-Home Messages:

- Modeling via FEM/FVM is becoming more and more important!
- Applicable for several physical domains and many technical applications!
- Model Order Reduction is indispensable to reduce the computational effort
- Reduction is done via projection
- Linear MOR is well developed
- Generalization of system-theoretic concepts and MOR methods to polynomial systems
- POD is still the most employed nonlinear MOR method
- Simulation-free / Sytem-theoretic nonlinear MOR techniques are aimed

Ongoing work:

- Polynomial nonlinear systems
- Simulation-free / System-theoretic NLMOR

Thank you for your attention!

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