Discretization of Structural Reliability Problems: An Application to Runway Overrun

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ABSTRACT: Bayesian networks (BNs) are a powerful tool for efficiently representing joint probability distributions and updating probabilities in near real-time. We combine BNs with structural reliability concepts to develop a warning system for runway overrun of a landing aircraft, one of the most critical accident types in civil aviation. This warning system allows to use currently available measurements of the aircraft weight, the head wind and the approach speed to update the probability of runway overrun in-flight. Based on the probability, the system informs the pilots whether or not it is save to land. One of the key challenges when treating structural reliability problems in a discrete BN framework is the discretization of the continuous outcome space of the reliability problem. We apply a heuristic developed in Zwirglmaier and Straub (2014) to discretize the reliability problem, such that the discretization error is kept small with only a moderate number of discretization intervals.

1. INTRODUCTION

In the field of structural reliability, one is interested in estimating the probability of failure $\Pr(F)$ of an engineering system. Failure is described through a limit state function (LSF) $g(\mathbf{X})$ as $F = \{g(\mathbf{X}) \leq 0\}$. Calculating the probability of failure corresponds to solving the following integral:

$$\Pr(F) = \Pr(g(\mathbf{X}) \leq 0) = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \qquad (1)$$

with $f_{\mathbf{X}}(\mathbf{x})$ being the probability density function of the random vector $\mathbf{X}$. Since there is no analytical solution for Equation 1, in general well-known structural reliability methods Rackwitz (2001) are used to approximate it. Due to performance issues of these methods in near real-time situations Straub and Der Kiureghian (2010a,b) proposed a framework for combining structural reliability with discrete Bayesian networks (BNs), for which exact inference algorithms exist. A major challenge is the discretization of the basic random variables, such that the problem can be treated in a discrete BN. In Zwirglmaier and Straub (2014) we propose a heuristic for efficiently discretizing the basic random variables of structural reliability problems. Here we apply this heuristic to develop a near real-time warning system, intended to prevent runway overrun (RWO). RWO of a landing aircraft, i.e. the event of a landing aircraft overrunning the end of the runway, is one of the most critical accident types in civil aviation IATA (2013). A physical model for the stopping distance of a landing aircraft was developed by Drees and Holzapfel (2012). By combining this physical model with a statistical model of its random input parameters, one can treat this problem as a structural reliability problem.

2. BAYESIAN NETWORKS FOR STRUCTURAL RELIABILITY PROBLEMS

For brevity only a short introduction to BNs will be given here. For a detailed introduction, the reader is referred to the standard textbooks e.g. Jensen and Nielsen (2007); Kjaerulff and Madsen (2013). BNs are an efficient representation of a joint probability distribution of a number of random variables $\mathbf{X}$. The qualitative dependence structure of BNs is represented through a directed acyclic graph (DAG). Each node in the DAG represents a random variable and links between the nodes represent dependencies. The dependencies are quantified by conditional probability tables (CPTs), which are attached to the nodes. Family terms are used to describe relationships between nodes e.g. in the BN of Figure 1 'Runway overrun' is among others a child of 'Head wind' and 'Landing weight', which in turn are its parents. A main feature of discrete BNs is their capability of updating probabilities in near real-time. For this reason Straub and Der Kiureghian (2010a,b) proposed a framework for combining BNs with structural reliability concepts. While structural reliability problems are typically applied to problems with a continuous outcome space, robust updating with exact inference algorithms is only possible for discrete BNs and some special continuous cases. These are BNs with Gaussian nodes, whose means are linear functions of their parents and BNs, whose nodes are defined as a mixture of truncated exponentials (MTE) Langseth et al. (2009). Discretization of the outcome space of a reliability problem is therefore a crucial element in the combination of structural reliability and BNs. Inevitably this discretization leads to an approximation, whose error should be minimized.

3. DISCRETIZATION

The random variable of interest in structural reliability problems is the performance of the component or system. In the scope of this paper, the interest is in whether a landing aircraft runs over the end of a runway or not. A binary node 'Runway overrun' is introduced as the target node (Figure 1). Each basic random variable of the reliability problem is modeled as a parent of the target node. Measurements can be performed on some of the quantities represented by the basic random variables. To preserve causality in a BN, measurement variables are generally included as children of the corresponding basic random variables. The computational demand for computing the parameters of the BN and for inference in the BN is governed by the size of the target node's CPT. To keep this CPT at a feasible size, it is necessary to limit the number of

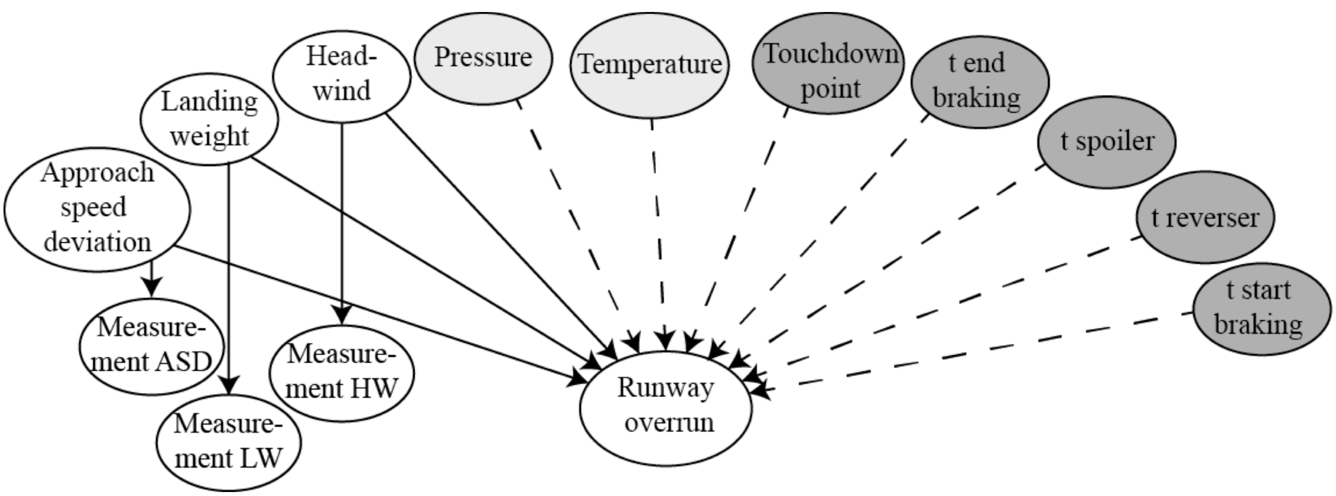

Figure 1. A Bayesian network for assessing RWO risks. The light grey nodes are not included explicitly in the BN because the uncertainty associated with them can be neglected. The dark grey nodes are excluded because the BN is intended as a tool for decision support at a point in time when no information on the state of these nodes can be obtained. The explicitly modeled nodes are shown in white.

discretization intervals of the basic random variables. The objective of using only a moderate number of intervals is however opposed to the objective of minimizing the discretization error. As long as the nodes representing basic random variables have their prior distribution, there is no discretization error. However, once evidence updates the distributions of the basic random variables, a discretization error occurs. The reason for that is, that the discrete BN is not capable of representing the dependency between the measurement variables and the target variable exactly. This has been recognized by Straub and Der Kiureghian (2010a,b) and is discussed in detail in Zwirglmaier and Straub (2014). The magnitude of the discretization error is determined by the discretization scheme used. An efficient discretization scheme should minimize the posterior discretization error while using only a feasible number of intervals to discretize the outcome space of the basic random variables.

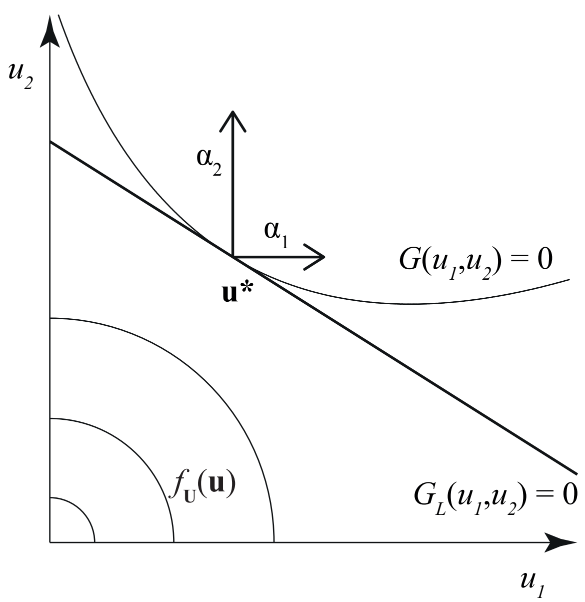

Figure 2. Linear (FORM) approximation $G_L(u_1, u_2)$ of the LSF $G(u_1, u_2)$ at the design point $\mathbf{u}^*$ in standard normal space. The elements of the normal vector on $G_L(u_1, u_2)$, α_i can be viewed as importance measured for the random variables U_i.

In Zwirglmaier and Straub (2014) we propose a heuristic for an efficient discretization of a reliability problem's outcome space, such that it can be treated in a discrete BN framework. To determine the discretization scheme for each basic random variable X_i, the heuristic uses the FORM importance measure α_i (Fig. 2). α_i is the i-th element of the normal vector on the linear approximation $G_L(\mathbf{u})$ of the LSF $G(\mathbf{u})$ in standard normal space. If the uncertainty associated to a random variable U_i does not have any influence on $\Pr(F)$, α_i is 0. If $\Pr(F)$ does only depend on the uncertainty associated with U_i, $|\alpha_i|$ is 1. In the following, we summarize the derivation of the discretization heuristic, which starts out in standard normal space. The resulting discretization scheme is then transformed back to the original space.

3.1. Optimal discretization schemes for linear LSFs in U-space

We consider linear LSFs in U-Space, i.e. the FORM approximations. For each of the random variables in U-space, a hypothetical measurement is considered, to which an additive normal distributed error ε_i is associated. For this case, there exists an analytical solution to computing the posterior probability of failure given the measurements. For this reason, it is feasible to find an optimal discretization scheme through optimization. We parameterize discretization schemes as illustratively shown in Figure 3 for a linear two-dimensional reliability problem in U-space. In each dimension, there is a first and a last interval boundary. All outer interval boundaries combined form the discretization frame. The cells inside of this discretization frame are fine while coarse cells capture the remaining outcome space. The discretization frame has a midpoint with a position $\mathbf{v}$, which is measured relative to the design point. n_i intervals are used to discretize the basic random variable U_i, this includes the two intervals outside of the discretization frame. The width of the discretization frame w_i in dimension i is the distance between the first and the last interval boundary in this dimension. Let $\mathbf{d}$ be a vector containing the discretization parameters for the m basic random variables i.e.

$\mathbf{d} = [n_1, \dots, n_m, w_1, \dots, w_m, v_1, \dots, v_m]$. The optimal discretization is then defined as:

$$\mathbf{d}^{opt} = \arg\min_{\mathbf{d}} \mathrm{E}_M\left[err_{post}(\mathbf{d},\mathbf{m})\right] = \arg\min_{\mathbf{d}} \int_{\mathbf{M}} err_{post}(\mathbf{d},\mathbf{m}) f(\mathbf{m})\, d\mathbf{m} \qquad (2)$$

subject to:

$$c_{max} \geq \prod_m n_i \qquad (3)$$

c_{max} is the maximum number of cells that may be used to discretize the problem. $\mathrm{E}_M\left[err_{post}(\mathbf{d},\mathbf{m})\right]$ in Eq. 2 denotes the expected posterior error, where E_M is the expectation with respect to the measurement outcomes. The error $err_{post}(\mathbf{d},\mathbf{m})$ is here defined as:

$$err_{post}(\mathbf{d},\mathbf{m}) = \left| \frac{\log_{10}\left(\widehat{\mathrm{P}}_{\mathrm{F}|\boldsymbol{M}}(\mathbf{d},\mathbf{m})\right) - \log_{10}\left(\mathrm{P}_{\mathrm{F}|\boldsymbol{M}}(\mathbf{m})\right)}{\log_{10}\left(\mathrm{P}_{\mathrm{F}|\boldsymbol{M}}(\mathbf{m})\right)} \right| \qquad (4)$$

$\widehat{\mathrm{P}}_{\mathrm{F}|\boldsymbol{M}}(\mathbf{d},\mathbf{m})$ is the estimate of the posterior failure probability obtained from the discrete BN and $\mathrm{P}_{\mathrm{F}|\boldsymbol{M}}(\mathbf{m})$ is the exact posterior failure probability. This definition of the error can be seen as a relative error with a weighting term that ensures that a relative error is considered to be worse if the posterior failure probability is large. Calculating the expected value in Eq. 2 is essentially a preposterior analysis, meaning that before any measurements have been made one integrates over all possible measurements outcomes Benjamin and Cornell (1970); Straub (2014).

3.2. Heuristic

The most important findings from the optimization described above are briefly summarized. They are based on optimizations for different linear problems in standard normal space with two or three basic random variables.

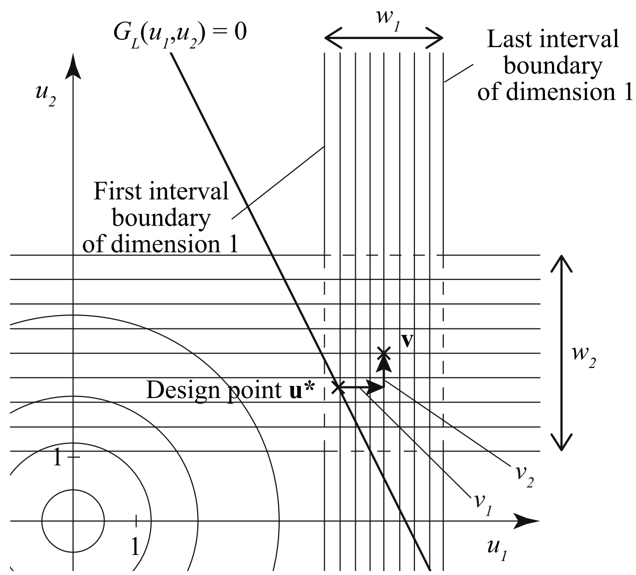

Figure 3. Parameterization of a discretization scheme. The last and the first interval in each dimension build a frame with width w_i, which encloses a region that is discretized by fine intervals. The midpoint of this frame has a position v_i relative to the design point in each dimension. Furthermore in each dimension there is am integer number n_i of intervals used to discretize the outcome space.

The considered problems differ in terms of:

- The importance of the basic random variables (expressed through the FORM importance measures α_i)
- The prior failure probabilities $\Pr(F)$ of the problem
- The maximal number of cells c_{max}
- The standard deviation of the measurement error ε_i

The results show that the optimal discretization is independent of the standard deviation of the measurement error. Furthermore, when fixing an integer number d and choosing c_{max} as $c_{max} = d^m$, the discretization parameters are independent of the number of basic random variables m. Independent of all other factors, the discretization frame is always centered at the design point, i.e. the shift $\mathbf{v}$ of the discretization frame is negligible. The optimal number of intervals n_i used to discretize U_i is equal, or at least almost equal, for all basic random variables in a problem, independent of the investigated factors.

For the width of the discretization frame w_i in dimension i a clear dependence was found on:

- The importance of the basic random variables α_i
- The prior failure probabilities $\Pr(F)$ of the problem
- The maximum number of cells c_{max}

The dependency of w_i on α_i is shown in Figure 4 (grey curve) for $\Pr(F) = 10^{-7}$ and $c_{max} = 10^m$. When considering not directly the width w_i as a function of α_i, but instead the probability mass enclosed by the discretization frame, $\log\left(\Phi(u_i^* + w_i/2) - \Phi(u_i^* - w_i/2)\right)$, the relation to α_i can be described by an exponential function. Here u_i^* is the i-th component of the design point. The grey curve representing the dependency between α_i and w_i in Figure 4 is derived from the exponential function representing the relationship between α_i and the enclosed probability mass. For $\Pr(F) = 10^{-7}$ and $c_{max} = 10^m$ the exponential function fitted to the points obtained through optimization is:

$$\log\left(\Phi\left(u_i^* + \frac{w_i}{2}\right) - \Phi\left(u_i^* - \frac{w_i}{2}\right)\right) = -0.14\exp(4.7\,|\alpha_i|) \quad (5)$$

3.3. Application to general reliability problems

The proposed heuristic is derived in standard normal space for linear LSFs corresponding to the FORM solution. It is known that in most practical applications, the FORM estimate is quite accurate, meaning that most practical LSFs are not strongly non-linear in U-space Rackwitz (2001). Therefore, it is reasonable to assume that a discretization scheme, which is optimal for a linear surrogate LSF, is also efficient for the corresponding non-linear LSF. Since the final BN is intended as a model of the problem in its original space, the discretization scheme needs to be transformed back to the original space. If the basic random variables are statistically independent, the interval boundaries of each

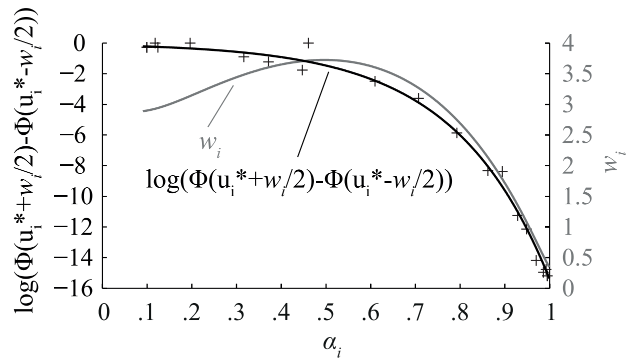

Figure 4. The optimal width of the discretization frame respectively the logarithm of the probability mass enclosed between the first and the last interval of the discretization frame in dimension i as a function of α_i. The crosses mark the points obtained through optimization, which are used to fit the exponential function.

basic random variable can be transformed independently. If the basic random variables are dependent, the interval boundaries of one dimension cannot be transformed independently of the other dimensions. A discretization scheme, which is orthogonal in U-space, will thus not be orthogonal in X-space. We can overcome this problem as follows. Consider transforming u_i^k, the k-th interval boundary of the discretization scheme of U_i. Orthogonality in X-space can be preserved by transforming a point $[u_1^*, \dots, u_{i-1}^*, u_i^k, u_{i+1}^*, \dots, u_m^*]$, i.e. the point which equals the design point except for the k-th dimension, to obtain the interval boundary x_i^k in X-space. The resulting discretization scheme will only approximately correspond to the discretization scheme in U-space. Finally, the conditional probabilities forming the target node's CPT are calculated based on the original LSF $g(\mathbf{x})$ by Monte Carlo simulation or latin hypercube sampling.

4. APPLICATION TO RUNWAY OVERRUN

Runway overrun (RWO) is one of the most critical accidents types in civil aviation. RWO of a landing aircraft corresponds to the event of the operational landing distance exceeding the available runway length (Fig. 5). Drees and Holzapfel (2012) proposed a model for the operational landing distance required by a landing aircraft. We use this model to define a LSF for runway overrun, which using the definitions from Fig. 5 can be written as:

$$g(\mathbf{x}) = \text{Stop margin}(\mathbf{x}) \tag{6}$$

with **X** representing the basic random variables of the problem i.e. the random variables summarized in Table 1. In Zwirglmaier et al. (2014) subset simulation Au and Beck (2001) is applied to calculate RWO probabilities for this problem. Here we develop a BN that allows to update RWO probabilities of an approaching aircraft in near real-time. While approaching an airport observations and measurements related to the factors influencing RWO can be made. These measurements can be used to reduce the uncertainty associated with the occurrence of RWO and to make a decision on whether landing is save or not. Which variables can be observed is closely related to the question, which variables should be modeled explicitly in the BN. The pilots make a decision before the aircraft touches down, before spoilers and reversers are deployed and before starting to break. Since it is not possible to make measurements related to the touchdown point, the spoiler- or reverser-deployment time and to the braking time before the decision is made, these random variables are not modeled explicitly as nodes in the BN. In Figure 1 these variables are shown in dark grey and with dashed links only for the purpose of illustration. Also the nodes shown in light grey (Fig. 1) are included only implicitly in the BN. The reason for that is that their influence on $\Pr(RWO)$ is negligible; this becomes evident from the FORM importance measures in Table 1. The remaining variables are thus 'Approach speed deviation', 'Landing weight' and 'Head wind'. For these variables, measurements of their current states can be used to reduce the uncertainty about their state at the time, when the aircraft is approaching the runway. We assume the measurement m_i (i.e. the quantity at the current point in time) equals the state of the random variable X_i at landing plus a random term ε_i.

$$m_i = x_i + \varepsilon_i \tag{7}$$

ε_i is modeled by a normal distribution with zero mean and standard deviation σ_{ε_i}. For the random variable landing weight (at landing time) we assume the standard deviation of the corresponding measurement error to be $\sigma_{\varepsilon_{LW}} = 0.2 \cdot \sigma_{LW}$; where σ_{LW} is the standard deviation of the basic random variable landing weight. Due to turbulences governing wind speeds, the measurement of the current head wind speed is a less reliable indicator for the head wind speed at landing time we model the measurement error with a standard deviation $\sigma_{\varepsilon_{HW}} = 0.5 \cdot \sigma_{HW}$. An even higher uncertainty is assumed for the approach speed deviation at landing, given a current speed deviation: $\sigma_{\varepsilon_{ASD}} = \sigma_{ASD}$.

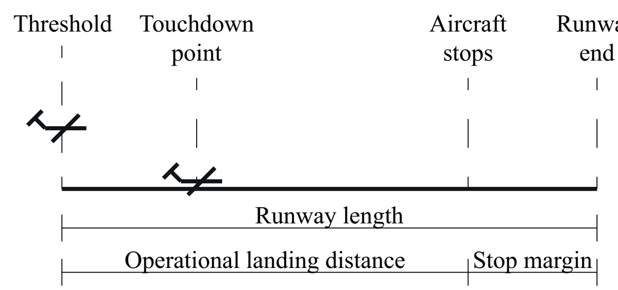

Figure 5. Term definitions for runway overrun of a landing aircraft.

5. RESULTS

We base the discretization for the problem at hand on the reliability analysis in Zwirglmaier et al. (2014). The FORM importance measures α_i obtained there are shown in Table 1. Applying Eq. 5 gives us the widths w_i of the discretization frame (Table 2). In the considered problem, the widths w_i are similar for all basic random variables. In Figure *4* it can be seen that smaller w_i's are only optimal for random variables with higher importance i.e. above $|\alpha_i| = 0.8$. A discretization frame with the obtained widths is centered at the design point. The edges of this discretization frame represent the first and the last interval boundary in each dimension. The remaining interval boundaries are then introduced between those outer intervals.

Transforming the boundaries back to the original space yields the interval boundaries shown I Table 2. For each cell in the discretization scheme, conditional failure probabilities are calculated. To this end samples from each cell as well as samples from the remaining basic random variables, not modeled explicitly as nodes, are generated. The samples from the remaining, implicitly modeled basic random variables are thereby generated from an importance sampling density centered at the design point. To calculate the conditional probabilities, the samples are weighted accordingly. For the fine cells inside the discretization frame, 300 samples are used per cell, while for the boundary cells 500 samples are used to calculate the conditional probabilities. The CPTs of the measurement nodes are calculated by Monte Carlo Simulation. This is computationally inexpensive compared to the computation of the target node's CPT. Without entering any evidence, the final BN gives a RWO probability of $1.7 \cdot 10^{-7}$. An exact benchmark solution does not exist, however this value is in the range of the estimate obtained through subset simulation in Zwirglmaier et al. (2014), where a probability of $1.1 \cdot 10^{-7}$ was computed. Observing no head wind at the current point in time and entering this as evidence to the BN reduces the RWO probability to $1.9 \cdot 10^{-10}$. Having on the other hand currently a speed deviation of $+30$ knots, will increase the probability of a RWO to $2.9 \cdot 10^{-7}$ according to the BN. Observing both events at the same time yields a RWO probability of $4.8 \cdot 10^{-10}$.

6. CONCLUDING REMARKS

We model a reliability problem with a physically-based performance model through a discrete BN. The advantage of this lies in the capability of discrete BNs to rapidly update probabilities, once new information becomes available. Such a feature is especially of interest in near real-time applications. Treating continuous reliability problems in a discrete BN framework requires the discretization of the continuous outcome space of the reliability problem. This leads inevitably to a discretization error. In order to keep this error small, a heuristic is applied. In this paper we use 10 intervals for each basic random variable, that is modeled explicitly as a node in the BN. Of the 10 basic random variables in the original problem, we model 3 explicitly as nodes. The number of free parameters of the target variable's CPT is thus 1000. Computing the parameters for this setting

Table 1. Basic random variables of the problem. The FORM importance measures indicate the influence of the uncertainty associated with the respective random variable on the probability of failure (RWO). Note: The mean value of the random variable 'Touchdown point' is a deterministic function of other basic random variables Zwirglmaier et al. (2014).

Random variable	Distribution model	Mean	Std. dev.	α_i
Landing weight [t]	Weibull(60.0,44.3)	59.3	1.69	0.116
Head wind [kts]	Normal(5.4,5.8)	5.4	5.8	−0.693
Temperature [°C]	GEV(−0.26,7.9,6.5)	9.4	8.0	0.020
Pressure [hPa]	Normal(1016,8.1)	1016	8.1	0.010
Approach speed deviation [kts]	GEV(−0.20,4.0,3.0)	4.7	4.2	0.202
t reverser [s]	Gamma(9.0,0.55)	5.0	1.7	0.037
t start braking [s]	GEV(0.15,3.9,10.2)	13.1	6.4	0.650
t spoiler [s]	GEV(0.11,0.89,3.7)	4.3	1.3	0.019
t end braking [s]	Normal(25,5)	25	5	0.004
Touchdown point [m]	Normal(μ_{TDP}, 121.9)	μ_{TDP}	121.9	0.204

Table 2. The width of the discretization frame in U-space and the interval boundaries in the original space for each of the basic random variables derived following Equation 5.

Random variable	Width U-space	Interval boundaries (X-space)
Landing weight	2.92	[58.0; 58.7; 59.3; 59.9; 60.4; 60.8; 61.2; 61.6; 61.9]
Head wind	3.32	−[25.1; 22.7; 20.3; 17.9; 15.5; 13.1; 10.7; 8.2; 5.8]
Appr. speed dev.	3.15	[2.2; 3.9; 5.6; 7.3; 9.2; 11.0; 12.8; 14.6; 16.25]

is feasible for the LSF considered in the scope of this paper. However if the number of cells c_{max}, used for discretization is increased by some orders of magnitude this may lead to considerable computation cost. It is therefore necessary to decide carefully, which random variables should be modeled explicitly as nodes. For computationally more demanding LSFs it may be necessary to reduce the number of intervals per dimension and accept a larger discretization error. By applying an importance sampling approach to sample from the distributions of the implicitly modeled random variables, we can reduce the number of samples required to populate the CPT of the target variable to some extend. Furthermore while a computational demanding LSF may cause a large computational effort in the process of establishing the model, it does not have any effect on the computational effort in the application of the model. Finally it should be noted that while the final BN developed in this paper is quite simple, such BNs can be incorporated into more complex BN models, fully exploiting the advantages of the modeling framework.

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