Appeared in:

Risk and Reliability Analysis: Theory and Applications: In Honor of Prof. Armen Der Kiureghian. Edited by P. Gardoni, Springer, 2017

Reliability updating in the presence of spatial variability

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Abstract

During the production and operation of engineering systems, information on their properties and performance becomes available through monitoring and other means of data collection. Such information can be used to update predictions of a system's reliability through a Bayesian analysis. In this contribution, we focus on Bayesian analysis and updating of the reliability of engineering systems, which depend on physical quantities that vary randomly in space. These are modelled by means of random fields. The numerical treatment of random fields requires their discretization with a finite number of random variables. To this end, we employ the Expansion Optimal Linear Estimation (EOLE) method, which is shown to be especially efficient in obtaining an approximation of a second-order random field. This property is beneficial for Bayesian analysis in cases where the moment function depends on hyperparameters, such as the correlation length of a random field. We discuss the application of EOLE in the context of BUS, which is a recently proposed framework for Bayesian updating of parameters of engineering systems and the resulting system reliability. In BUS, monitoring data is expressed in terms of an equivalent limit state function such that Bayesian updating can be performed with structural reliability methods. We apply BUS with EOLE to update the reliability of the stability of a foundation resting on spatially variable soil with deformation measurements obtained at an intermediate construction stage.

1 Introduction

Structural reliability is commonly used to assess existing structures, which do not comply with codes and standards due to deterioration or changes in system properties, system demands or code requirements (Ellingwood 1996, Straub and Der Kiureghian 2011). Reliability methods have also been employed to appraise the effect of inspection on the reliability of structures and to optimize inspection efforts (Madsen et al. 1989, Faber et al. 2005, Goulet et al. 2015). Along the same lines, they have been utilized to assess the effect of structural health monitoring on a structural system (Pozzi and Der Kiureghian 2011) or site investigations in geotechnical engineering (Papaioannou and Straub 2012). With technological advances in monitoring technology, collecting additional data on structural performance becomes easier and cheaper, and methods to assess the effect of data on the system reliability are thus increasingly required for existing and new structures.

Data influences the system reliability by reducing uncertainty. Strictly, the reliability of an engineering system is not altered by the collection of information alone (Der Kiureghian 1989, Der Kiureghian and Ditlevsen 2009). However, it is altered by the actions that follow the collection of information. If the reduced uncertainty leads to more targeted actions, sufficient reliability is ensured with lower cost. Bayesian analysis is ideally suited to quantitatively assess the effect of data on the reliability, risk and cost. It allows to consistently combine the data with existing probabilistic models of structural systems, which is particularly relevant when dealing with (rare) failure events (Straub et al. 2016).

When combining data with probabilistic models for reliability analysis, it is often necessary to utilize more advanced mechanical models than for design purposes. Predictions made with simplified (empirical) models used in design processes may not match with the observations on the real structures. One prominent example is the treatment of spatially variable material or soil properties. While it is commonly possible and reasonable to represent such properties by an equivalent homogenous parameter in design calculations, such models can be oversimplifying and lead to erroneous predictions when including data (Papaioannou and Straub 2016). Therefore, there is a significant interest to enable a Bayesian updating of mechanical models with spatially distributed properties for the purpose of reliability assessment.

Bayesian analysis of spatially variable properties is an active field of research (e.g., Marzouk et al. 2007, Koutsourelakis 2009). However, only few investigations on reliability updating in the presence of spatial variability are documented in the literature. These include applications to geotechnical engineering (Papaioannou and Straub 2012), seismic risk (Bensi et al. 2015) and deteriorating structures (Straub 2011b).

This contribution focuses on the representation of random fields in the context of Bayesian analysis and reliability updating. In particular, we discuss the computational benefits of utilizing the Expansion Optimal Linear Estimation (EOLE) proposed in (Li and Der Kiureghian 1993) for discretizing the random field when the a-priori correlation length is uncertain. The recently proposed BUS approach to Bayesian analysis and reliability updating is shortly summarized and its application in the context of random fields is discussed. To demonstrate the approach, we consider an application from geotechnical engineering, where it is common to use observations during the construction phase to ensure the safety of the site; a process that is known as the observational method.

2 Methodology

2.1 Random field discretization

Random fields are used to describe spatially variable uncertain parameters. Common examples are soil parameters in geotechnical engineering, environmental loads in structural engineering and topology of multiphase materials in biomechanics. A random field $X(\mathbf{t})$ is defined as a collection of random variables indexed by a continuous spatial parameter $\mathbf{t} \in \Omega \subset \mathbb{R}^d$, with d denoting the spatial dimension. The random field is termed second-order if it has finite mean and variance functions. To completely define the random field $X(\mathbf{t})$, the joint distribution of the random variables $\{X(\mathbf{t}_1), X(\mathbf{t}_2), \dots, X(\mathbf{t}_n)\}$ for any $\{n, \mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n\}$ must be specified. This is straightforward when the random field is Gaussian (i.e. the random variables $\{X(\mathbf{t}_1), X(\mathbf{t}_2), \dots, X(\mathbf{t}_n)\}$ have the multivariate normal distribution) and the mean function $\mu(\mathbf{t})$ and the autocovariance function $C(\mathbf{t}_1, \mathbf{t}_2)$ are known. However, if the marginal distribution of $X(\mathbf{t})$ is not Gaussian, then a complete definition of the random field in terms of its first and second moment functions is not always possible. A class of non-Gaussian random fields, termed translation fields, can be defined by a nonlinear transformation of an underlying Gaussian field of the form

$$X(\mathbf{t}) = F_X^{-1}\left[\Phi\big(U(\mathbf{t})\big), \mathbf{t}\right] \tag{1}$$

where F_X^{-1} is the inverse of the non-Gaussian marginal distribution, Φ is the standard normal distribution and $U(\mathbf{t})$ is a Gaussian random field with zero mean and unit variance. The joint distribution for any selection of points in the spatial domain can then be modeled by a Gaussian copula, also known as the Nataf multivariate distribution (Nataf 1962, Der Kiureghian and Liu 1986). The specification of the auto-correlation coefficient function of $U(\mathbf{z})$ in terms of the one of $X(\mathbf{t})$ involves solving an integral equation.

A random field is said to be second-order (or weakly) homogeneous if its probabilistic structure is invariant to a shift in the spatial parameter up to a second order. A second-order homogeneous random field has constant mean and variance functions and its autocovariance function depends on the difference in location, i.e. $C(\mathbf{t}_1, \mathbf{t}_2) \rightarrow C(\Delta\mathbf{t})$ where $\Delta\mathbf{t} = \mathbf{t}_1 - \mathbf{t}_2$. The spatial variability of a second-order homogeneous random field is defined by the autocorrelation coefficient function $\rho(\Delta\mathbf{t}) = \frac{C(\Delta\mathbf{t})}{\sigma^2}$, with σ^2 being the variance of the random field. A common measure of the spatial variability is the scale of fluctuation θ, defined as the integral of $\rho(\Delta\mathbf{t})$ (Vanmarcke 2010). The smaller θ is, the higher is the variability in the sample functions of the random field. An alternative measure for the spatial variability is the correlation length l, defined as the distance $l = \|\Delta\mathbf{t}\|$ for which $\rho(\Delta\mathbf{t}) = e^{-1}$ (Li and Der Kiureghian 1993).

In order to numerically represent the random field $X(\mathbf{t})$, it is necessary to discretize it with a finite number of random variables. Several methods have been proposed for the discretization of random fields, see (Sudret and Der Kiureghian 2000) for a comprehensive review. The efficiency of random field discretization methods depends on their ability to approximate accurately the random field with as few random variables as possible. Accuracy is defined in terms of an error measure such as the mean-square error or the variance error. Efficient random field representations involving small numbers of random variables are beneficial for most numerical methods for uncertainty quantification and reliability analysis.

Random field discretization methods include point methods, spatial average methods and series expansion methods. In point and spatial average methods, the random field is expressed in terms of random variables that correspond to spatial points or averages of discrete parts of the spatial domain. Series expansion methods express the random field as a superposition of products of deterministic spatial functions and random variables, such that each random variable in the expansion has a global influence in the approximation of the random field. Through a proper choice of the spatial functions, series expansion methods are able to describe the spatial variability accurately with much fewer random variables as compared to point or average methods (Sudret and Der Kiureghian 2000). Popular series expansion methods include orthogonal series expansions (Zhang and Ellingwood 1994), the Karhunen-Loève (KL) expansion (e.g., Ghanem and Spanos 1991) and the expansion optimal linear estimation (EOLE) method (Li and Der Kiureghian 1993).

The spatial functions in orthogonal series expansions are chosen as orthogonal functions on the domain of definition of the random field (Zhang and Ellingwood 1994). In particular, the KL expansion employs the eigenfunctions of the autocovariance function of the random field, which are shown to be the optimal choice among all sets of orthogonal functions in the sense that they minimize the global mean square error of the discretization (Ghanem and Spanos 1991). Determination of the eigenfunctions in the KL expansion requires the solution of an integral eigenvalue problem. Aside from a few specific cases, the integral eigenvalue problem needs to be solved numerically resulting in an approximation of the KL expansion. Application of Galerkin-based methods for the solution of the integral eigenvalue problem is time consuming, as they require the assembly of a matrix eigenvalue problem through performing a two-folded integration over the spatial domain (Betz et al. 2014) .

The EOLE method proposed in (Li and Der Kiureghian 1993) combines concepts from linear estimation theory and principle component analysis to derive the spatial functions in the representation of the random field. The method can be understood as a numerical KL expansion for the case where the KL eigenvalue problem is solved by the Nyström method (Betz et al. 2014). EOLE has the advantage over the Galerkin-based KL expansion that integration is not required to assemble the matrix eigenvalue problem. Hence, the method is particularly efficient in determining the spatial functions in the random field representation. This can be of advantage if the random field description changes throughout the computation, as is the case in Bayesian analysis when the autocovariance function of the random field is described by uncertain (hyper-) parameters.

In the EOLE method, the discrete representation $\hat{X}(\mathbf{t})$ of the random field $X(\mathbf{t})$ reads

$$\hat{X}(\mathbf{t}) = \mu(\mathbf{t}) + \sum_{i=1}^{m} \frac{\xi_i}{\sqrt{\lambda_i}} \mathbf{C}_q(\mathbf{t})^T \mathbf{v}_i \tag{2}$$

Here, $\{\lambda_i, \mathbf{v}_i\}$ are the eigenpairs of the covariance matrix of the random variables $X(\mathbf{z}_j)$ corresponding to a set of spatial points $\{\mathbf{t}_j, j = 1, \dots, q\}$ and $\mathbf{C}_q(\mathbf{t})$ is a $q \times 1$ vector function with j element $C(\mathbf{t}, \mathbf{t}_j)$ (Li and Der Kiureghian 1993) . The eigenpairs $\{\lambda_i, \mathbf{v}_i\}$ are evaluated through solving the following matrix eigenvalue problem

$$\mathbf{C}\mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{3}$$

where $\mathbf{C}$ is the covariance matrix of the spatial points with (i,j) element $C(\mathbf{t}_i, \mathbf{t}_j)$. The eigenpairs are arranged in decreasing order of magnitude of the eigenvalues and the first m terms are retained in the EOLE representation. The variables $\{\xi_i, i = 1, \dots, m\}$ are zero mean orthonormal random variables. If the random field is Gaussian, then $\{\xi_i, i = 1, \dots, m\}$ are independent standard normal random variables. If it is non-Gaussian then it is not always possible to determine the joint distribution of $\{\xi_i, i = 1, \dots, m\}$. Therefore, if the random field is defined by a transformation of the type of Eq. (1), it is convenient to apply the EOLE method to discretize the underlying Gaussian field.

The point-wise variance of the EOLE truncation error of $X(\mathbf{t})$ is given by (Li and Der Kiureghian 1993) :

$$\mathrm{Var}\big[X(\mathbf{t}) - \hat{X}(\mathbf{t})\big] = \mathrm{Var}[X(\mathbf{t})] - \sum_{i=1}^{m} \frac{1}{\lambda_i} \left(\mathbf{C}_q(\mathbf{t})^T \mathbf{v}_i\right)^2 \tag{4}$$

This equation can be used to determine the number of terms m in the EOLE representation for a desired accuracy in the approximation of the random field.

2.2 Reliability analysis

Structural Reliability Methods (SRM) have been developed since the 1970s for computing the (small) probability of failure of engineering systems (e.g., Rackwitz and Fiessler 1978, Der Kiureghian and Liu 1986). The failure event F is described in terms of a limit state function $g(\mathbf{X})$, where $\mathbf{X} = [X_1; X_2; \dots; X_n]$ is the vector of the n input random variables. By definition, the event F corresponds to

$$F = \{g(\mathbf{X}) \leq 0\} \tag{5}$$

It is helpful to interpret the SRM geometrically: Ω_F corresponds to the domain in the outcome space of $\mathbf{X}$ for which $g(\mathbf{x}) \leq 0$. The probability of the event F is the probability of $\mathbf{X}$ taking a value within Ω_F. It can be computed by integrating the joint probability density function of $\mathbf{X}$, denoted by $f(\mathbf{x})$, over Ω_F:

$$\Pr(F) = \int_{\Omega_F} f(\mathbf{x}) \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n \tag{6}$$

More generally, the event of failure can be defined through a system formulation, as a function of multiple component failure events F_i with corresponding limit state functions g_i. The system failure domain is (Der Kiureghian 2005):

$$\Omega_F = \left\{ \min_{1 \le k \le K} \left[\max_{i \in I_1} g_i(\mathbf{x}), \dots, \max_{i \in I_K} g_i(\mathbf{x}) \right] \le 0 \right\}, \tag{7}$$

where I_k is an index set corresponding to the k-th cut set of the system. For $K = 1$, this reduces to a parallel system reliability problem; for the case that each cut set contains only one index, this reduces to a series system.

To solve Eq. (6), most SRM involve a transformation of the problem from the original space of the random variables $\mathbf{X}$ to the space of independent standard normal random variables $\mathbf{U}$ by a suitable transformation $\mathbf{U} = \mathbf{T}(\mathbf{X})$. If the joint distribution of $\mathbf{X}$ is of the Gaussian copula class, the transformation of (Der Kiureghian and Liu 1986) can be applied; if the joint distribution of $\mathbf{X}$ is of any arbitrary form, the Rosenblatt transformation can be used (Hohenbichler and Rackwitz 1981). Let G denote the transformed limit state function in standard normal space:

$$G(\mathbf{U}) = g(\mathbf{T}^{-1}(\mathbf{U})) \tag{8}$$

where $\mathbf{T}^{-1}(\mathbf{U}) = \mathbf{X}$ is the inverse transformation from standard normal space to the original outcome space of the random variables. The transformation $\mathbf{T}$ is probability conserving, therefore $\Pr(F) = \Pr(g(\mathbf{X}) \le 0) = \Pr(G(\boldsymbol{U}) \le 0)$. In analogy to Eq. (6), the probability of the failure event F is computed by

$$\Pr(F) = \int_{G(\mathbf{u}) \le 0} \varphi(\mathbf{u}) \mathrm{d}u_1 \mathrm{d}u_2 \dots \mathrm{d}u_n, \tag{9}$$

where φ is the independent standard multivariate normal probability density function (PDF).

A potentially highly efficient method for approximating the probability of failure is the First-Order Reliability Method (FORM). It approximates the limit state function $G(\mathbf{U})$ by a first-order Taylor expansion at the expansion point $\mathbf{u}^*$, denoted by $G'(\mathbf{U})$. To limit the approximation error, $\mathbf{u}^*$ is selected as the point in the failure domain with the highest probability density value, the most likely failure point (MLFP). The MLFP is found as the point in the domain $\{G'(\mathbf{U}) \le 0\}$ closest to the origin (Der Kiureghian 2005). The probability $\Pr(G'(\mathbf{U}) \le 0)$ is equal to the standard normal CDF Φ evaluated at $-\beta_{\mathrm{FORM}}$, where $\beta_{\mathrm{FORM}} = ||\mathbf{u}^*||$ is the distance of $\mathbf{u}^*$ from the origin (for values of $\Pr(G'(\mathbf{U}) \leq 0) < 0.5$). The FROM approximation is therefore

$$\Pr(F) \approx \Pr(G'(\mathbf{U}) \leq 0) = \Phi(-\beta_{\text{FORM}}). \tag{10}$$

The computational bottleneck of FORM is the identification of the design point $\mathbf{u}^*$ through the solution of a constrained geometrical optimization problem. Tailored algorithms exist for this purpose (Liu and Der Kiureghian 1991). Since these are gradient-based methods, the computational cost of the optimization increases with increasing number of dimensions n. FORM-based methods can nevertheless be applied for higher-dimensional problems (Rackwitz 2001, Allaix and Carbone 2015), and have also been shown to perform well for Bayesian updating problems (Straub et al. 2016).

As an alternative to FORM, in this contribution we apply subset simulation (SuS), a SRM that is tailored to work well for problems with larger numbers of random variables. SuS, originally developed in (Au and Beck 2001), is an adaptive simulation method. It is based on expressing $\Pr(F)$ as a product of larger conditional probabilities. The conditional probabilities are defined in terms of a set of nested intermediate failure events $F_0 \supset F_1 \supset \cdots \supset F_M = F$, where F_0 denotes the certain event. The probability $\Pr(F)$ can be expressed as:

$$\Pr(F) = \Pr\left(\bigcap_{i=1}^{M} F_i\right) = \prod_{i=1}^{M} \Pr(F_i | F_{i-1}) \tag{10}$$

The intermediate events are defined in standard normal space as $F_i = \{G(\mathbf{U}) \leq b_i\}$, where $b_1 > b_2 > \cdots > b_M = 0$. The values of b_i are chosen adaptively such that the estimates of the conditional probabilities correspond to a chosen value p_0, where typically p_0 is chosen as 0.1. To find b_i, samples of $\mathbf{U}$ conditional on the intermediate failure event F_{i-1} are generated through a Markov Chain Monte Carlo (MCMC) sampling. While SuS can also work in the outcome space of $\mathbf{X}$, the definition of the problem in standard normal space is advantageous as it enables a simpler and efficient MCMC sampling, as shown in (Papaioannou et al. 2015).

2.3 Bayesian analysis and reliability updating

With monitoring or inspections, data $\mathbf{d}$ becomes available, providing information directly or indirectly on the random variables $\mathbf{X}$ in the limit state function. This data can be used to update the prior probability distribution $f_{\mathbf{X}}$ to a posterior distribution $f_{\mathbf{X}|\mathbf{d}}$ following Bayes' rule:

$$f_{\mathbf{X}|\mathbf{d}}(\mathbf{x}) \propto L(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \tag{11}$$

$L(\mathbf{x}) = f_{\mathbf{D}|\mathbf{X}}(\mathbf{d}|\mathbf{x})$ is the likelihood function describing the data. The difficulty lies in the computation of the proportionality constant in Eq. (11). In most cases, analytical solutions are not available and sampling-based approaches are necessary instead (Gelman et al. 2014). MCMC methods are commonly applied for this task, resulting in (correlated) samples from $f_{\mathbf{X}|\mathbf{d}}$ (Beck and Au 2002, Ching and Chen 2007, Straub and Kiureghian 2008, Betz et al. 2016).

The updated probability of failure conditional on the data $\Pr(F|\mathbf{d})$ can – in principle – be calculated by evaluating Eq. (6) with $f_{\mathbf{X}}$ replaced by $f_{\mathbf{X}|\mathbf{d}}$. However, if $f_{\mathbf{X}|\mathbf{d}}$ is known only approximately through samples of the distribution, most structural reliability methods are not directly applicable to evaluate Eq. (6). Hence this approach, albeit seemingly straightforward, does not generally lead to efficient or simple solutions for computing $\Pr(F|\mathbf{d})$.

As an alternative to the direct method, (Straub and Papaioannou 2015), based on earlier ideas from (Straub 2011a), proposed the BUS (Bayesian Updating with SRM) method. It circumvents the problem by formulating an equivalent observation event Z describing the data, which can be used to perform Bayesian analysis within the framework of SRM. The observation event is defined as

$$Z = \{P \leq cL(\mathbf{X})\}, \tag{12}$$

where P is a random variable with standard uniform distribution, and c is a constant that is chosen to ensure that $cL(\mathbf{x}) \leq 1$ for any $\mathbf{x}$.

The observation event Z is equivalent to $\mathbf{d}$ in the sense that updating $\mathbf{X}$ with the event Z leads to the same posterior distribution as updating with $\mathbf{d}$, $f_{\mathbf{X}|Z} = f_{\mathbf{X}|\mathbf{d}}$, as shown in (Straub and Papaioannou 2015). Hence it follows that also $\Pr(F|Z) = \Pr(F|\mathbf{d})$.

The advantage of defining the data through Z is that this event can be represented by a limit state function

$$h(p, \mathbf{x}) = p - cL(\mathbf{x}), \tag{13}$$

such that $Z = \{h(P, \mathbf{X}) \leq 0\}$, in analogy to the definition of the failure event, Eq. (5).

The conditional probability of failure is (Madsen 1987)

$$\Pr(F|Z) = \frac{\Pr(F \cap Z)}{\Pr(Z)}, \tag{14}$$

which corresponds to solving two structural reliability problems:

$$\begin{aligned}\Pr(F|Z) &= \frac{\Pr[g(\mathbf{X}) \leq 0 \cap h(P, \mathbf{X}) \leq 0]}{\Pr[h(P, \mathbf{X}) \leq 0]} \\ &= \frac{\int_{g(\mathbf{x}) \leq 0 \cap h(p,\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}p \, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n}{\int_{h(p,\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) \mathrm{d}p \, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_n}\end{aligned} \tag{15}$$

The numerator is a system reliability problem, whereas the denominator is a component reliability problem whose limit state function is Eq. (13).

In most cases, it is beneficial to solve Eq. (15) in standard normal space. The observation limit state function H in standard normal space is

$$H(\mathbf{u}) = u_0 - \Phi^{-1}\left(cL\left(\mathbf{T^{-1}}(u_1, \dots, u_n)\right)\right). \tag{16}$$

with Φ^{-1} being the inverse standard normal CDF. U_0 is the standard normal random variable corresponding to P. The probability of the failure event F conditional on the data can now be expressed in terms of the standard normal $\mathbf{U}$:

$$\begin{aligned}\Pr(F|Z) &= \frac{\Pr[G(\mathbf{U}) \leq 0 \cap H(\mathbf{U}) \leq 0]}{\Pr[H(\mathbf{U}) \leq 0]} \\ &= \frac{\int_{G(\mathbf{u}) \leq 0 \cap H(\mathbf{u}) \leq 0} \varphi(\mathbf{u}) \mathrm{d}u_0 \mathrm{d}u_1 \dots \mathrm{d}u_n}{\int_{H(\mathbf{u}) \leq 0} \varphi(\mathbf{u}) \mathrm{d}u_0 \mathrm{d}u_1 \dots \mathrm{d}u_n}\end{aligned} \tag{17}$$

Any SRM is applicable to solve Eq. (15) or (17). Here, SuS is employed as it is efficient in high dimensions. In Eq. (17), the numerator is a subset of the denominator. This allows to reuse the samples generated in the last step of the computation of the denominator in the SuS run for computing the numerator, as described in (Straub et al. 2016).

The BUS algorithm implemented with SuS is illustrated in Figure 1.

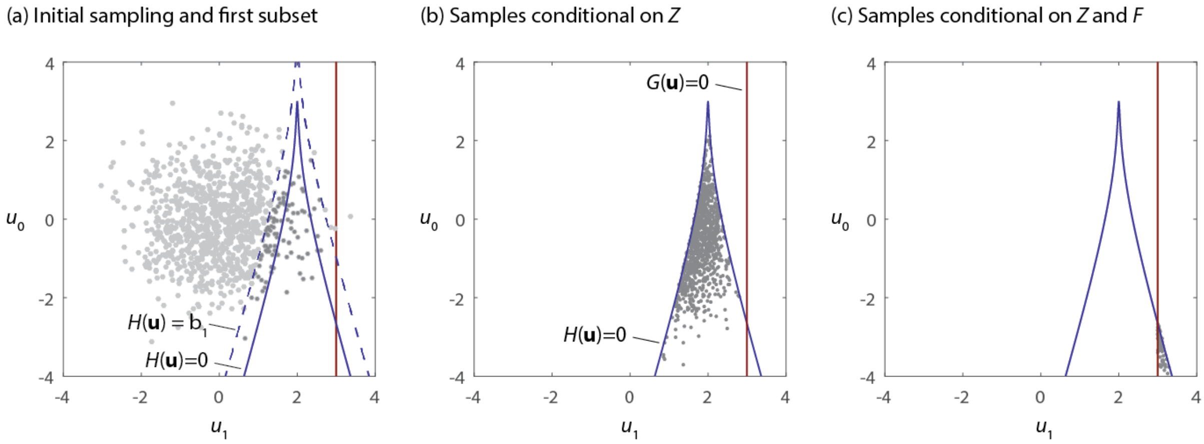

Figure 1. Illustration of BUS with subset simulation: (a) Initial (Monte Carlo) sampling. The surface of the first subset is indicated with dashed lines. The darker dots correspond to the samples that fall into this subset and are used as seeds in the MCMC algorithm to generate the samples conditional on the first subset. (b) Samples generated conditional on the observation, $Z = \{H(\boldsymbol{U}) \leq 0\}$*. (c) Samples generated conditional on the observation and the failure event,* $Z \cap F = \{H(\boldsymbol{U}) \leq 0 \cap G(\boldsymbol{U}) \leq 0\}$*. These are the final samples.*

2.4 Reliability updating with random fields

Application of BUS for reliability updating with random fields requires the solution of the integrals in Eq. (15), whereby both integrals are infinite dimensional because a random field consist of an infinite number of random variables. In practice, the random field is represented by a finite number of random variables through the random field discretization, e.g. according to Eq. (2). If the random field is a translation field and the EOLE method is applied to discretize the underlying Gaussian random field, then the random variables in the discretization are already independent standard normal and a transformation to a standard normal space is not required.

If the variability of the random field is high, reflected by a small scale of fluctuation, then a large number of random variables will be needed to obtain a small variance error in the EOLE representation. In such case, it is beneficial to apply a SRM that is able to handle efficiently high dimensional problems. In this study, SuS is employed for this purpose.

A main challenge when dealing with random fields in the context of Bayesian updating is the choice of the prior distribution. Prior distributions should reflect the knowledge available before the observations. Often there is uncertainty on the prior knowledge of the parameters of the marginal distribution F_X and/or covariance structure of the random field. This uncertainty can be included by introducing additional random variables. In Bayesian analysis, such random variables are termed hyperparameters to distinguish them from the parameters of the model of the underlying system. Once hyperparametes are included, the probabilistic description of the random field becomes conditional on outcomes of the hyperparameters. This implies that for each realization of the hyperparameters, the spatial functions in the EOLE representation will change. However, as discussed in Section 2.1, EOLE is particularly efficient in obtaining the spatial functions, as the assembly of the matrix eigenvalue problem of Eq. (3) can be done in an efficient manner.

If the spatial covariance structure of the random field is represented by means of hyperparameters, the error measure in Eq. (4) depends on the realization of the hyperparameters. A constant (small) error that is independent of the realization of the hyperparameters can only be achieved if the number of EOLE-terms (and, thus, the number of random variables) is selected conditional on the hyperparameters. However, inference algorithms typically require the number of random variables to be constant throughout the analysis. To avoid this problem, the number of EOLE-terms is selected conservatively in this study, by choosing a number of EOLE terms that leads to an acceptable discretization error for most realizations of the hyperparameters. Thereby, it is necessary to verify that for the prior covariance structure with high posterior density the error term is still acceptable.

It is noted that the a-posteriori first- and second-order statistics of the random field will differ from the ones prior to the measurements. For instance, a random field that is second-order homogeneous a-priori will no longer be homogeneous a-posteriori, because of the influence of the locality of the measurements. If the spatial variability of the random field increases considerably conditional on the measurements, the number of random variables in the random field representation of the prior random field might not be sufficient for representing the posterior field. The posterior representation of the random field should therefore be carefully checked.

3 Numerical investigations

3.1 Problem description

We consider a problem from geotechnical engineering: The stability of an eccentrically loaded foundation is investigated. Serviceability of the foundation is ensured if the inclination α of the foundation under the final loading P is smaller than 4 degrees; i.e. the limit-state function describing the associated reliability problem is: $g(\mathbf{X}) = 4° - \alpha(\mathbf{X})$. The foundation is bedded on a soil layer that has a depth of 8m. The soil is modeled as linear elastic, with a Young's modulus *E* that is spatially variable, hence described by random field, and a fixed Poisson ratio of 0.35. Beneath the soil layer, a sandstone layer is located, whose influence on the analysis is negligible.

The investigated foundation has a width of 1.5m; after construction it is loaded eccentrically with load *P* (Figure 2). The lever arm of the load is 0.5m. The load *P* is uncertain: it follows a Gumbel distribution with mean 1MN and 10% coefficient of variation. At an intermediate construction stage, a centric load *F* of 0.4MN is applied. The displacements under *F* at the left and right ending of the foundation are measured as $\hat{x}_l = 1$cm and $\hat{x}_r = 1.5$cm, see Figure 2. We utilize an additive model for the combined measurement/modeling errors; the error follows a normal distribution with zero mean and a standard deviation of $\sigma_\varepsilon = 0.5$cm. The errors associated with $\hat{x}_l$ and $\hat{x}_r$ are correlated with a correlation coefficient of $\rho = 0.9$. It is therefore implied that modeling errors are dominating over measurement errors.

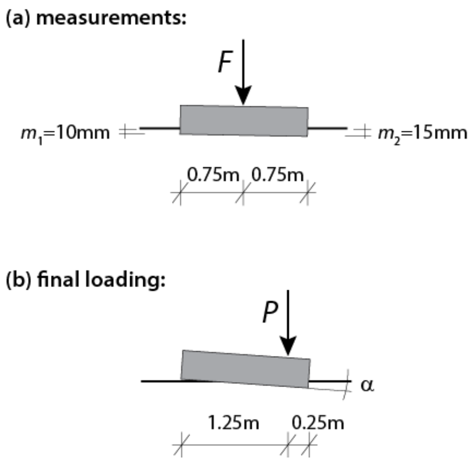

Figure 2. Loading of the foundation (a) during the measurements and (b) in the final state.

The Young's modulus *E* of the soil layer is modeled as lognormal random field with a mean of 40MPa and a coefficient of variation of 50%. The correlation coefficient function between points $\mathbf{t}_1$ and $\mathbf{t}_2$ of the underlying Gaussian random field is $\rho(\Delta\mathbf{t}) = \exp\left(-\sqrt{\Delta t_x^2 + \Delta t_z^2}/l\right)$, where Δt_x and Δt_y are the horizontal and vertical distance between points $\mathbf{t}_1$ and $\mathbf{t}_2$, and l is the correlation length. The correlation length l is modeled as uncertain. As prior distribution for l we select a lognormal distribution that has mean 4m and a coefficient of variation of 100%.

On each side of the foundation, a soil-stripe of 15m is modeled explicitly. The random field is discretized using the EOLE method. The EOLE discretization points are distributed uniformly over the domain with 25 points per square meter. The 200 most important terms in EOLE are used to represent the random field.

The mechanical model is discretized and solved by means of higher-order finite elements (Szabó et al. 2004). The finite element mesh of the mechanical model is depicted in Figure 3. The order of the shape functions of the mechanical model is 4.

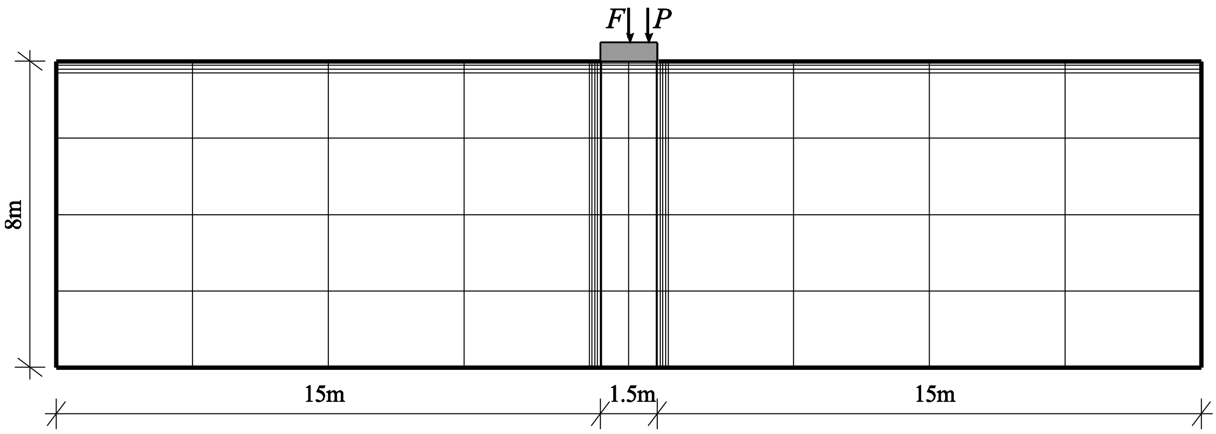

Figure 3. Finite element mesh used to discretize the soil in the mechanical model: the order of the shape functions is 4.

Samples of the posterior are obtained by means of BUS-SuS with 1000 samples per level. The posterior probability of failure is obtained by an additional run of SuS with 1000 samples per level using the generated posterior samples as basis. Samples of the prior are generated directly. The two computationally challenging tasks are (i) the discretization of the prior random field conditional on the current value of the hyperparameters, and (ii) the computation of the finite element model. In the example at hand, the computational costs of the random field discretization dominate the overall computational costs.

3.2 Results and discussion

The prior probability of failure is 10^{-2}. The posterior probability of failure conditional on the settlement measurements $\hat{x}_l$ and $\hat{x}_r$ is $3 \cdot 10^{-4}$. The probability of failure estimate is therefore significantly reduced by the measurements performed at the intermediate construction stage.

To further investigate the effect of the measurement on the random variables and the failure event, Table 1 provides mean values and standard deviations of the final load P, correlation length[1] l, and inclination and center settlement of the foundation at final loading under different information. The prior case is compared to different posterior cases: (m) conditional on the measurement, (F) conditional on failure of the foundation (without measurement), and (m&F) conditional on measurement and failure. The statistics conditional on failure provide an indication of the parameter values in a failure case, similar to the design point in a FORM analysis.

As expected, the distribution of the final load P is not influenced by measuring displacements in the intermediate construction stage (the m case). However, the mean of P in the case of failure without measurement is lower than in the case with measurements. Since the measurements reduce the probability of failure, a larger value of P is required in this case to lead to failure.

Table 1. Mean values and standard deviations of final load P, correlation length l, and inclination and center settlement of the foundation at final loading. Values are provided for the prior case and different posterior cases: conditional on measurements (m), conditional on failure of the foundation (F) and conditional on a combination of both.

	mean				standard deviation			
	prior	posterior			prior	posterior		
		m	F	m&F		m	F	m&F
final load P	1	1	1.07	1.22	0.1	0.1	0.13	0.16
corr. length l	4	3.4	6	2.7	4	3.1	4.9	1.5
inclination (final stage)	1.7	1.7	4.7	4.3	0.7	0.5	0.8	0.3
final center displ. (final)	$4.3 \cdot 10^{-2}$	$4.0 \cdot 10^{-2}$	$9.8 \cdot 10^{-2}$	$7.6 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$	$8.9 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$	$7.8 \cdot 10^{-3}$

The expected correlation length l is decreased by the performed measurements. Interestingly, without measurements, larger correlation lengths are associated with failure, whereas smaller correlation lengths are associated with failure once measurements are considered. This indicates that without the measurements, the foundation is expected to fail due to a globally reduced soil stiffness. In contrast, after the measurements are performed, failure is expected as a consequence of a locally reduced soil stiffness.

Even though the measurements clearly reduce the probability of failure, the expected inclination and the expected center settlement of the foundation under final loading are only marginally

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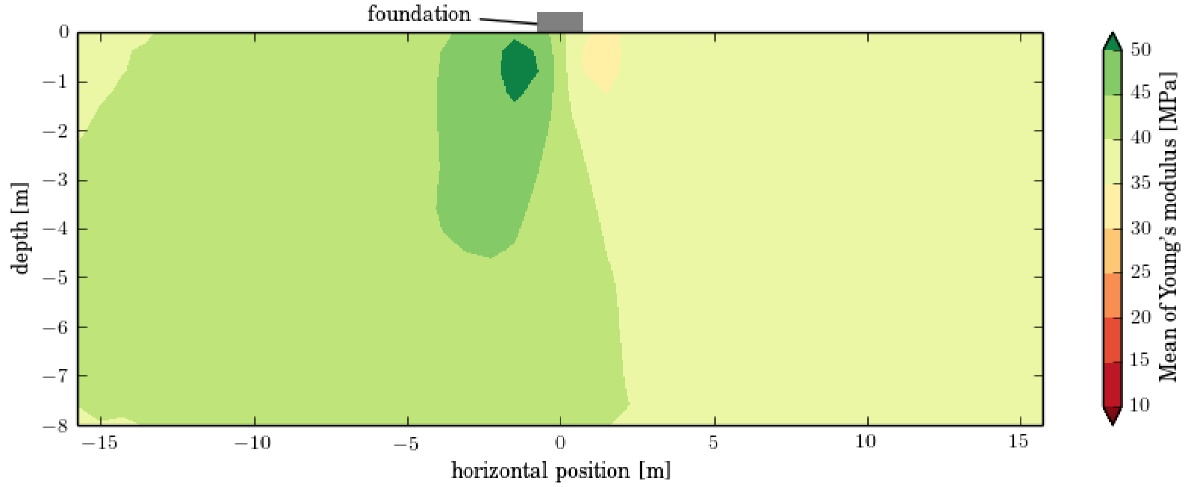
[1] Note that the correlation length is defined only for the a-priori case, when the random field is stationary. Strictly, l is the hyperparameter representing the correlation length of the prior random field.

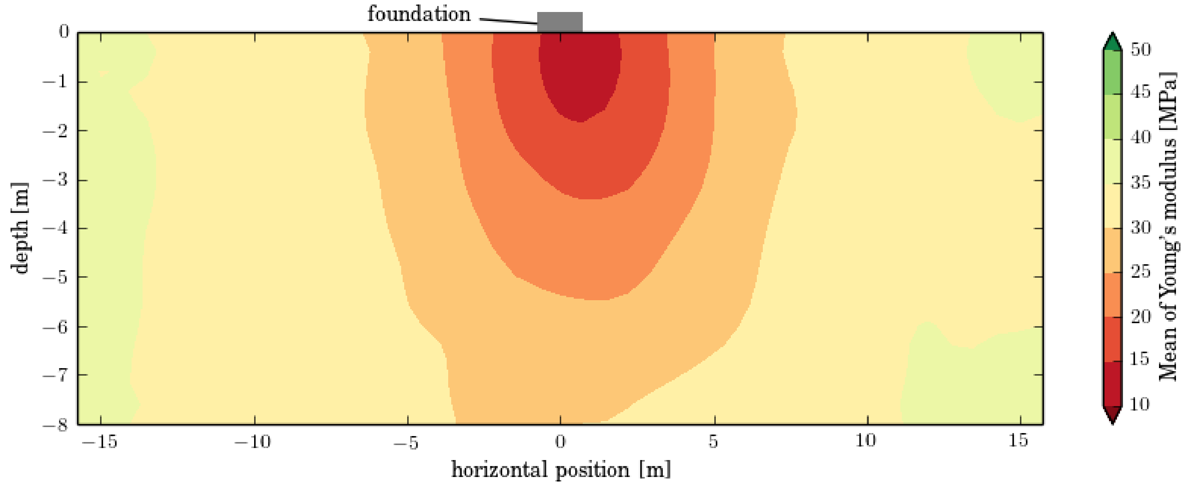
influenced by the observations. This shows that the main reason for a reduction in the probability of failure is the reduced uncertainty on the soil parameters.

The mean value of the soil stiffness is plotted in Figure 4 for the three posterior cases. The corresponding standard deviations are illustrated in Figure 5. The measurements performed at the intermediate construction stage suggests that the stiffness of the soil under the foundation is slightly larger on the left-hand side of the foundation than on the right-hand side (Figure 4a). Conditioning on failure without the measurements, the expected stiffness of the soil is reduced beneath the foundation with a small shift to the right that increases the inclination of the foundation (Figure 4b). When conditioning on failure including the measurements, the expected stiffness of the soil suggests a weak spot below the right side of the foundation (Figure 4c).

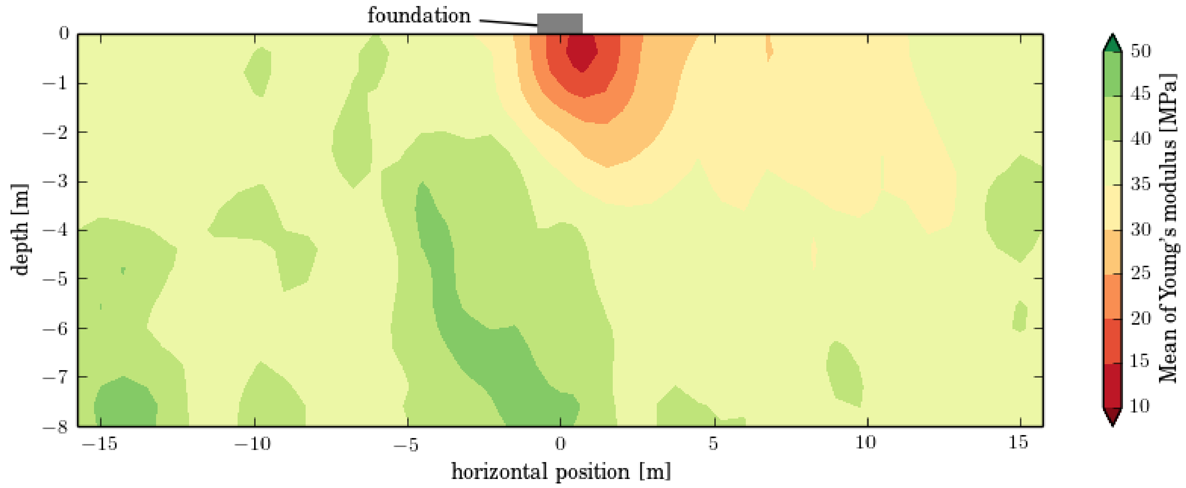
The plots of the standard deviations (Figure 5) show that the standard deviation is primarily reduced in the areas where the mean stiffness changes from the prior to the posterior. These are the areas for which the conditioning on the deformation measurements provides information.



(a) conditional on measurements

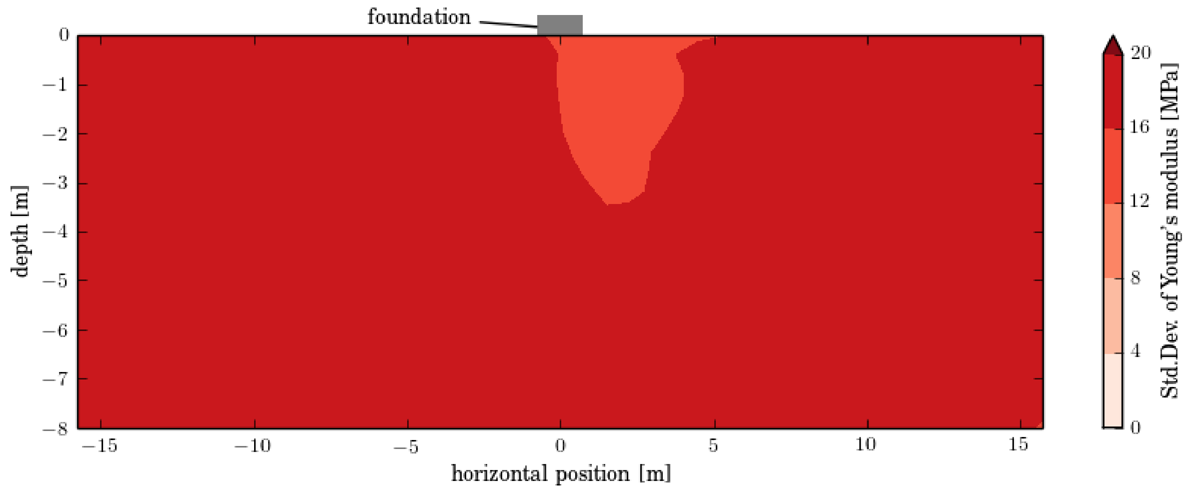


(b) conditional on foundation failure

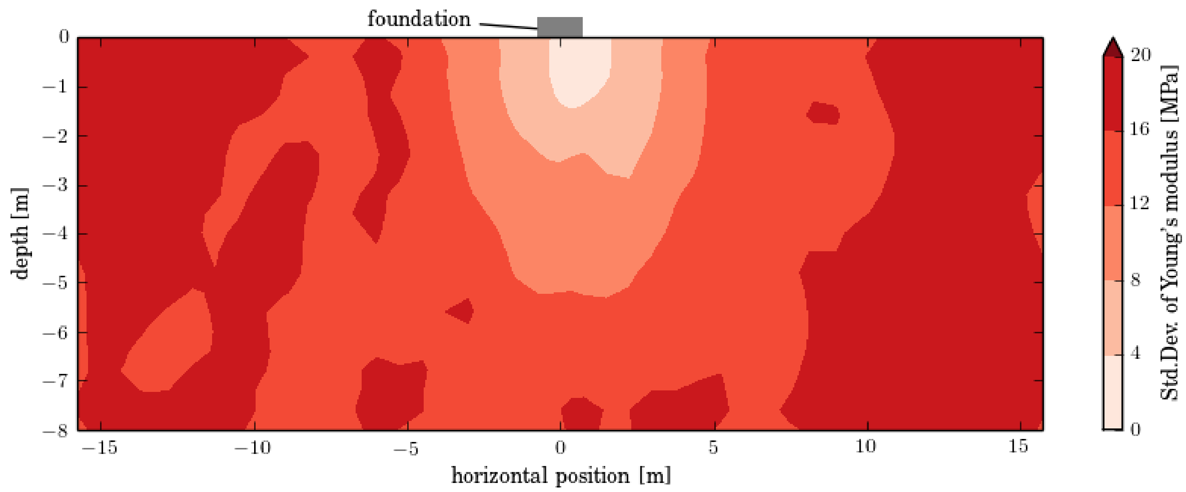


(c) conditional on measurements and foundation failure

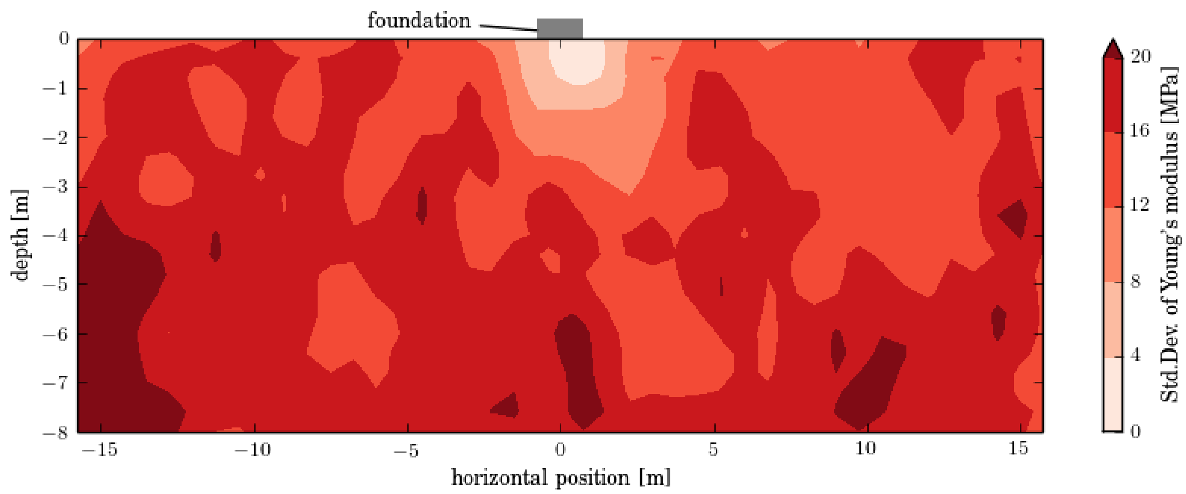
Figure 4. Expected values of the soil stiffness for the three different posterior cases.



(a) conditional on measurements



(b) conditional on foundation failure



(c) conditional on measurements and foundation failure

Figure 5. Standard deviation of the soil stiffness for the three different posterior cases.

4 Concluding remarks

Bayesian analysis and reliability updating is a consistent and potentially powerful method to incorporate and combine different information sources for predicting the performance of an engineering system. In many instances, spatially varying properties in these systems should be explicitly represented through random fields. This can lead to demanding computations, and efficient methods for random field discretization are required. In this contribution, we discuss that the EOLE method is beneficial if the analysis includes a random field with an uncertain covariance structure. In these cases, the random field representation will alter in function of the covariance structure, and the efficiency of EOLE in obtaining an approximation of the random field leads to overall reduced computational effort. For Bayesian analysis we apply the BUS approach, which enables the use of classical structural reliability methods to compute the posterior distribution and reliability. The procedure is applied to the updating of the reliability of a shallow foundation with results from displacement measurements. Numerical results show that the reliability estimate can be significantly lowered by including such measurements even if the posterior mean estimates of the soil properties are not more favourable than a-priori, due to the reduction of the uncertainty. Closer investigation showed that the consideration of the uncertainty in the covariance structure of the random field leads to significant changes in the likely failure modes from the a-priori to the posterior case.

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