Three-dimensional stochastic free vibration of the tire using generalized polynomial chaos expansion method

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Introduction

Tire plays an important role on the NVH analysis. In the vehicle-road dynamic response analysis, uncertainty stems from two aspects: the uncertainty of the system itself and the randomness of the action load. Considering the complicated production process of tires which is inevitable to cause uncertainty of the tire structure, the dispersion of material and structural parameters will result in uncertainty of the system responses. However, it is not easy to accurately describe the dynamic response of tires because of its complex structure and the nonlinear mechanic properties. In the literature, tires have been modeled as a ring on elastic foundation (REF) because of the completeness and simplicity of ring theory without sacrificing result accuracy. In previous studies, most of the ring models only focused on the in-plane vibration[1-4]. In this paper, the ring model which can describe the three-dimensional deformation of tires was introduced to describe tire vibration [5]. In addition, orthogonal expansion method is based on the homogeneous chaos theory proposed by Wiener[6]. In the field of structural dynamics, Ghanem[7] firstly introduced Hermite chaos in the spectrum method for solving stochastic mechanical problems by combining the finite element method. Xiu[8] developed the generalized polynomial chaos (gPC) expansion method in which a group of the chaotic functions with optimal convergence rate were adopted according to different distribution. The effect of the input uncertainty on the dynamic responses also can be analyzed by probabilistic collocation method[9]. The purpose of this paper is to analyze the three-dimensional stochastic free vibration of tires using gPC expansion method.

Ring Model of Tire Free Vibration

A method of transforming a pneumatic tire into a ring model holds a significant position in vehicle dynamics analysis. The tread of a tire is modeled as a three-dimensional deformable ring. The sidewall is treated as an elastic foundation. The elastic properties of the foundation are modeled by distributed springs in the radial, circumferential and axial directions (k_u, k_v and k_w). Figure 1 shows a ring with a rectangular cross section on an elastic foundation.

In this paper, the deformations of ring include radial, circumferential and axial displacements and the torsional angle. The torsional angle ϕ_θ that represents the rotational angle around the axis of the ring is shown in Figure 2. However, considering the Euler-Bernoulli beam theory, the plain section assumption was still applied. In Table 1 some parameters of the REF model of an actual tire structure 385/65R22.5 are given [5].

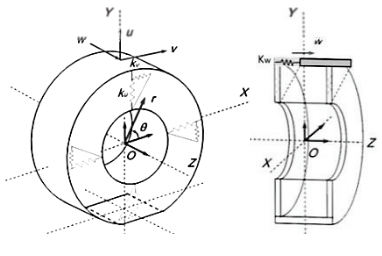


Figure 1: Schematic of the three-dimensional ring model.

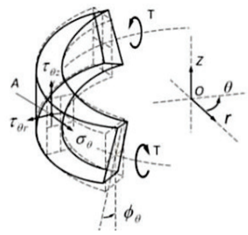


Figure 2: Torsional angle of the ring.

Table 1: Geometrical and structural parameters of a 385/65R22.5 radial tire

Parameter type	Unit	Value
Ring width b	m	0.273
Ring thickness h	m	0.0375
Effective density ρ	kg/m^3	1.77×10^3
Mean radius r	m	0.53
Line density of tread band ρA	kg/m	18.12

The Hamilton principle is used to derive the governing equations. Here we use the solutions as stated in [5]. The in-plane natural frequency can be expressed as follows,

$$\omega^2 = \frac{B_1 \pm \sqrt{B_1^{\,2} - 4C_1}}{2\rho A} \tag{1}$$

where,

$$\begin{aligned} B_1 &= \left(n^2 \frac{EI_z}{r^4} + \frac{EA}{r^2}\right)\left(1+n^2\right) + 2p_0 b \frac{n^2}{r} + k_v + k_u \\ C_1 &= \frac{p_0 bEI_z}{r^5}\left(n^6 - n^4\right) + \frac{p_0 bEA}{r^3}\left(n^4 - n^2\right) + \frac{k_v EI_z}{r^4} n^4 \\ &+ \frac{EI_z EA}{r^6}\left(n^6 - 2n^4 + n^2\right) + k_u p_0 b \frac{n^2}{r} + \frac{k_v EA}{r^2} + k_u k_v \\ &+ \frac{k_u EI_z}{r^4} n^2 + \frac{k_u EA}{r^2} n^2 + k_v p_0 b \frac{n^2}{r} + p_0^{\,2} b^2 \frac{n^4 - n^2}{r^2} \end{aligned} \tag{2}$$

σ_θ^0 is the initial stress, A is the ring section, ρ is the density, b is the belt effective width and p_0 is the internal pressure. EI_z is the in-plane bending stiffness, EA is the membrane stiffness. Similarly, the expressions of out-of-plane natural frequency are shown.

$$\omega^2 = \frac{B_2 \pm \sqrt{B_2^2 - 4C_2}}{2r^2} \tag{3}$$

where,

$$\begin{aligned} B_2 &= \frac{1}{\rho I_p}\left(GI_p n^2 + EI_r + \frac{k_u b_{op}^2 r^2}{4}\right) \\ &+ \frac{1}{\rho A}\left(\frac{GI_p n^2}{r^2} + \frac{EI_r n^4}{r^2} + k_w r^2\right) \\ C_2 &= \frac{GI_p EI_r n^2\left(n^2-1\right)^2}{\rho I_p \rho A r^2} + \frac{(GI_p n^2 + EI_r n^4)k_u b_{op}^2}{4\rho I_p \rho A} \\ &+ \frac{\left(GI_p r^2 n^2 + EI_r r^2\right)k_w}{\rho I_p \rho A} + \frac{k_w k_u b_{op}^2 r^4}{4\rho I_p \rho A} \end{aligned} \tag{4}$$

where b_{op} is the nominal width of tire. EI_r is the out-of-plane bending stiffness, GI_p is the torsional stiffness, ρI_p is the torsional inertia.

Generalized Polynomial Chaos Expansion

In order to analyze the influence of many uncertain factors on the manufacture process of the tires, the generalized polynomial chaos (gPC) method will be applied in the uncertainty analysis of the natural frequency prediction. The first step is to use gPC expansion to represent the input and the output variables. With the polynomial chaos expansion, each random field or variable of interest can be expressed as,

$$\begin{aligned} \chi &= x_0\Phi_0 + \sum_{i_1=1}^{\infty} x_{i_1}\Phi_1\left(\xi_{i_1}\right) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} x_{i_1 i_2}\Phi_2\left(\xi_{i_1},\xi_{i_2}\right) \\ &+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} x_{i_1 i_2 i_3}\Phi_3\left(\xi_{i_1},\xi_{i_2},\xi_{i_3}\right) + \cdots \end{aligned} \tag{5}$$

where x_0 and $x_{i_1 i_2 \ldots}$ are deterministic coefficients. $\Phi_i\left(\boldsymbol{\xi}\right)$ is a set of orthogonal polynomials with respect to the vector consisting of the input variables $\boldsymbol{\xi} = \left(\xi_1, \xi_2, \xi_3 \ldots\right)$. The type of the polynomials is determined by the distribution of random variables. For notational simplicity, equation (5) is usually truncated by finite terms, and it can be rewritten as

$$\chi = \sum_{i=0}^{N} x_i \Phi_i\left(\boldsymbol{\xi}\right) \tag{6}$$

Usually the distribution types of the input parameters are known. Considering the property of the inner product and the orthogonality of polynomials, the Galerkin projection method can be used to calculate the coefficients of the truncated gPC representation x_i.

$$x_i = \frac{\left\langle \chi, \Phi_k\left(\boldsymbol{\xi}\right)\right\rangle}{\left\langle \Phi_i^2 \right\rangle}, \quad i,k = 0,1,2 \ldots N \tag{7}$$

However, the probability distributions of the output variables are unknown. The gPC coefficients cannot be obtained by directly calculating the inner product. Intrusive Polynomial Chaos Expansion (IPCE) and Probabilistic Collocation Method (PCM) are two kinds of method to determinate the unknown coefficients. Here we choose the PCM, which is capable of easily dealing with complex nonlinear equations.

Probabilistic Collocation Method

Implementation of Probabilistic Collocation Method

Based on the regression method, Probabilistic Collocation Method, which is the extension of Response Surface Method in probability space, is also known as Stochastic Response Surface Method (SRSM) [9]. Generally, N_c available sets of sample points are selected. In order to obtain an accurate result of the unknown coefficients, oversampling technique is adopted. It should be noted that the sample points $\boldsymbol{\xi}^s = \left[\boldsymbol{\xi}_1^s, \cdots \boldsymbol{\xi}_j^s, \cdots \boldsymbol{\xi}_{N_c}^s\right]^{\mathrm{T}}$ where, $\boldsymbol{\xi}_j^s = \left[\xi_{j1}^s, \cdots \xi_{jk}^s, \cdots \xi_{jd}^s\right]^{\mathrm{T}}$ are selected form the standard random space (ξ space). To obtain the corresponding output responses, $\boldsymbol{\xi}^s$ should be transformed into random space (X space) and then substitute each set of the sample point into the deterministic equation $g(x)$. The PCM forces the residual error to be deterministically zero at those sample points. It is given by

$$\begin{bmatrix} \Phi_0\left(\boldsymbol{\xi}_1^s\right) & \Phi_1\left(\boldsymbol{\xi}_1^s\right) & \cdots & \Phi_N\left(\boldsymbol{\xi}_1^s\right) \\ \Phi_0\left(\boldsymbol{\xi}_2^s\right) & \Phi_1\left(\boldsymbol{\xi}_2^s\right) & \cdots & \Phi_N\left(\boldsymbol{\xi}_2^s\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_0\left(\boldsymbol{\xi}_{N_c}^s\right) & \Phi_1\left(\boldsymbol{\xi}_{N_c}^s\right) & \cdots & \Phi_N\left(\boldsymbol{\xi}_{N_c}^s\right) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} g\left(\mathbf{X}_1^s\right) \\ g\left(\mathbf{X}_2^s\right) \\ \vdots \\ g\left(\mathbf{X}_{N_c}^s\right) \end{bmatrix} \tag{8}$$

equation (8) can be expressed in a matrix form.

$$\mathbf{Ax} = \mathbf{G} \tag{9}$$

In which $\mathbf{x}$ is the vector of the gPC coefficients; $\mathbf{A}$ is the polynomial information matrix with dimension $N_c \times (N+1)$; $\mathbf{G}$ is the vector of the corresponding output responses. The coefficients can be calculated by least squares regression.

$$\mathbf{x} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{G} \tag{10}$$

Once obtaining the gPC coefficients, the statistical properties of the output response can be easily estimated by applying the Monte Carlo Simulation (MCS) on the gPC model. In short words, the PCM provides a surrogate model. It should be noted that the sample points applied to MCS should be generated according to the probability density function of $\boldsymbol{\xi}$.

Selection of the Collocation Points

One key point in the probabilistic collocation method is how to select the appropriate collocation points. Generally, if the order of a PCE approximation is given, the collocation points should be selected from the roots of the next order orthogonal polynomial. This method will obtain more accurate results than just using some randomly selected points. For a p-order gPC expansion involving d dimensional random vector $\boldsymbol{\xi}$, the number of the unknown coefficients (N+1) can be calculated by equation (11).

$$N+1=\frac{(d+p)!}{d!\,p!} \tag{11}$$

However, the combination of the (p+1) order roots can generate some sample points. If the origin is added as a root of the even order polynomial, the total number of the available points N_c can be calculated as follow,

$$N_c=\begin{cases}(p+1)^n, & p=odd\ number\\(p+2)^n, & p=even\ number\end{cases} \tag{12}$$

With the increase of the p value, the number of the sample points N_c is much larger than (N+1). Therefore, it is possible to select only N sets from the N_c combinations of the roots [10]. Considering the unique-solution condition of the linear equations, the coefficient matrix **A** in equation (12) should have a full rank. Another consideration for selecting collocation points is that keeping as many points as possible in the area with a high probability density. Thus the N_c sample points firstly should be sorted in an order of decreasing probability density. Then, calculate the coefficients vector corresponding to the $(i+1)^{th}$ collocation point. The $(i+1)^{th}$ row of matrix **A** must be linearly independent with the previous i rows. Otherwise, the point with the next highest probability density should be tested. Fortunately, the required linearly independent collocation points can be selected once and used directly for next calculation. The feasibility of this method is illustrated with the natural frequency of the tire in-plane free vibration.

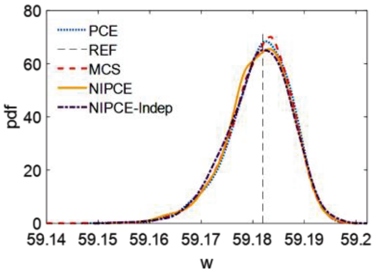
In-plane Free Vibration

As in the case study, three parameters of a tire, membrane stiffness EA, in-plane bending stiffness EI_z, and internal pressure p_0 are considered as random parameters. The mean and the distribution of those parameters are listed in Table 2.

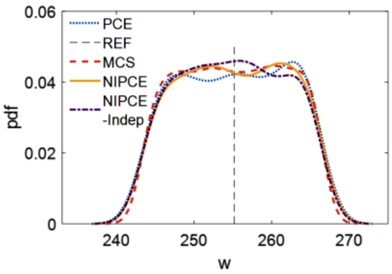
Table 2: Mean and distribution of the input variables for in-plane vibration analysis

Parameter	Distribution Type	Mean
p_0 ($\times10^5$ Pa)	U (7.47, 9.13)	8.3
EI_z (N·m^2)	N (7.401, 0.85)	7.401
EA (N)	N (4.603×10^7, 5×10^5)	4.603×10^7

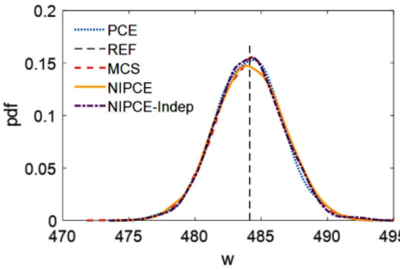
In this section, four methods are used to analyze how the proposed random parameters exert an influence over the in-plane free vibration of the tire. Firstly, the random variables expanded by equation (6), are directly substituted into the analytical expression of the natural frequency. Then the traditional PCM and the linear-independent-based PCM which do not need to expand input parameters are applied. The output responses are approximated by using 3rd-order gPC expansion. Figure 3 shows the similar result using 10000 MCSs to verify the effectiveness of the gPC methods. The error of different methods is shown in Table 3. Even if 125 collocation points are selected in the traditional PCM, the accuracy of the calculations has not been improved. When 76 collocation points are tested based on the criteria as discussed previously, the matrix **A** has a full rank of 20 equaling the number of unknown coefficients in the gPC expansion. The linear-independent PCM are much more efficient than the traditional PCM.



(a) 1st-order radial mode



(b) 10th-order radial mode



(c) 0th-order circumferential mode (breathing mode)

Figure 3: Distribution of in-plane natural frequency.

Table 3: Comparison between the different collocation point methods and MCS

Analysis Method	10th order Radial Mode		
	Sample Number	$\bar{\omega}$ (Hz)	Error
MCS	10000	255.18	0.001%
NIPCE	125	254.86	0.126%
NIPCE-INDEP	20	255.08	0.039%

Furthermore, the effect of the level of uncertainty on the in-plane natural frequency is investigated. The variances of all the three input random parameters are verified from 5% to 20%. The results are plotted in Figure 4. It is observed that as the uncertainty level is increased, the quantities of the frequency are more dispersed.

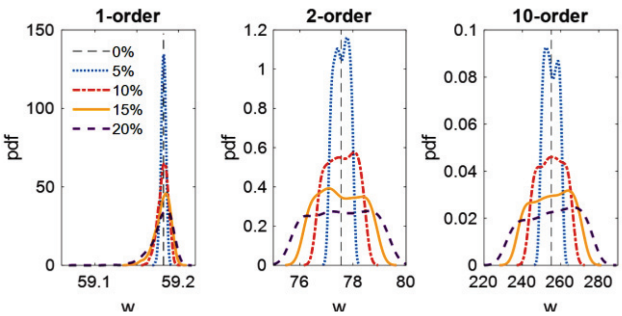
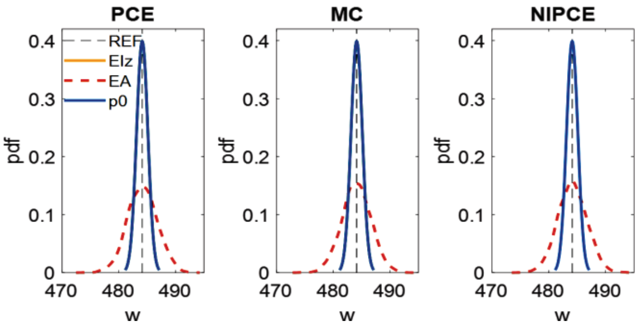
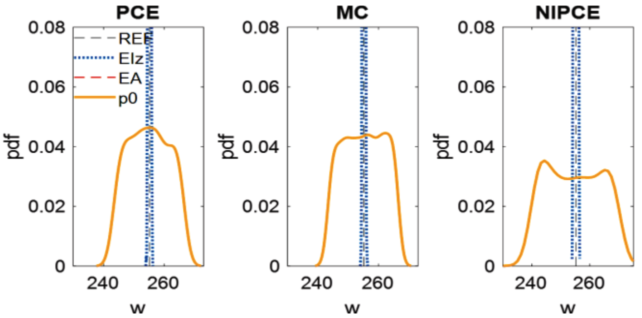


Figure 4: Distribution of 1st-order radial natural frequency under different variances of input parameters.



(a) 0th-order circumferential mode (breathing mode)



(b) 10th-order radial mode

Figure 5: Effect of individual parameter on the in-plane natural frequency.

Considering uncertainty in only one parameter, meanwhile the other input variables are keeping deterministic, the distributions are calculated. In Figure 5, it can note that the membrane stiffness *EA* has more influence on breathing mode. However, the dispersion of high-order radial modes caused by the internal pressure p_0 is greater than the other two variables.

Out-Of-Plane Free Vibration

To analyze the influence of the model parameters, torsional inertia ρI_p, torsional rigidity GI_p and out-of-plane bending stiffness EI_r, on the out-of-plane frequency, a similar analysis method was applied.

Table 4: Mean and distribution of the input variables for out-of-plane vibration analysis

Parameter	Distribution Type	Mean
ρI_p (kg/m)	U (0.23, 0.29)	0.26
GI_p (N·m^2)	N (1.61×10^3, 1.61×10^2)	1.61×10^3
EIr (N·m^2)	N (1.06×10^5, 1.06×10^4)	1.06×10^5

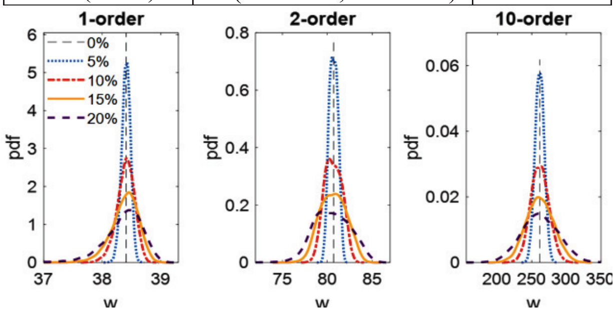


Figure 6: Distribution of 1st-order lateral natural frequency under different variances of input parameters.

Conclusion

In this paper, considering uncertainty in the structural parameters, the three-dimensional free vibration of the tire based on REF model was investigated. The gPC expansion was applied to describe the input and output parameters. The traditional PCM and another procedure for the selection collocation points were used to obtain the probability distribution of the natural frequency. The results were compared with the results calculated by 10000 MCS. It shows that the effectiveness of the selection method which can improve the computational efficiency without reducing the accuracy. Furthermore, the effect of each individual parameter and the distributions of the in-plane and out-of-plane natural frequency under different variances of input parameters are studied. It is observed that as the uncertainty level is increased, the quantities of the frequency are more dispersed. For different orders, the factors affecting the dispersion degree of the natural frequencies are different.

Acknowledgments

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