Technische Universität München
Fakultät für Mathematik

M14 Lehrstuhl für Wahrscheinlichkeitstheorie

Random polymers in disastrous environments

Stefan Florian Junk

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigten Dissertation.

Vorsitzende		Prof. Dr. Christina Kuttler
Prüfer der Dissertation	1.	Prof. Dr. Nina Gantert
	2.	Prof. Dr. Matthias Birkner
	3.	Prof. Dr. Ryoki Fukushima (schriftliche Beurteilung)

Die Dissertation wurde am 29.5.2019 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 31.7.2019 angenommen.

Acknowledgments

First of all, I want to thank my supervisor Nina Gantert for her support and encouragement, and for always having an open door for mathematical and non-mathematical discussions. I am grateful for her guidance since my time as a Bachelor student, for the opportunity to follow my research interests and for her help arranging several enlightening research stays.

I want to thank Noam Berger for many scientific and non-scientific stories to make our coffee-breaks entertaining, and for always having a helpful intuition about my problems.

I am also grateful to Ryoki Fukushima for hosting me in Kyoto and for having a lot of time to discuss mathematics while I was there. My time in Japan was very inspiring and I learned a lot.

I would like to thank all members of the examination committee for their time and effort in reading this thesis.

I want to thank all current and former members of M14 for a wonderful time, and for many nice coffee breaks to refresh our thoughts between mathematics.

Specifically, I want to thank my former flatmate David Criens, whose enthusiasm for mathematics I always found inspiring and who helped proofread this thesis.

I am also thankful to Lexuri Fernández, whom I consider an honorary member of M14. Thank you for an enjoyable time working together in Stochastic Analysis and for proofreading this thesis.

I also want to thank Dominik Schmid, without whom I certainly would have messed up some formality along the way, and who always has coffee prepared in the morning.

Finally, I am grateful to Felizitas Weidner for many tasty cakes, lasagnas and fondues, and for her great knowledge of the inner workings of the university.

Mein besonderer Dank gilt meiner Familie. In meiner Zeit als Mathematikstudent haben sie mich immer ermutigt und bestärkt, und hatten immer Zeit mir mit Rat und Tat zu helfen.

最後に、いつも応援してくれる島田優香さんに感謝します。君が支えてくれたからこの論文を終わらせることが出来ました。ありがとう！

Contents

Part I. Introduction

1. Non-technical explanation of "random media"

This thesis is in the wider research area of random media. Here the starting point is a well-understood but unrealistic model. To make it more realistic, one can introduce a random perturbation – the so-called "random environment" or "random medium". We then try to understand which properties of the original model are retained by the disturbed model.

A good analogy is to think of water that is being drained through the ground. In the homogeneous model, the earth is made up of fine sand without impurities, so that the flow of water is uniform everywhere. In the inhomogeneous case, there are large rocks buried in the ground, so that the water is forced to circumvent these obstacles and drain through gaps between the rocks. Intuitively, we expect that if the rocks are small, then the flow of water will still be roughly uniform everywhere, while large obstacles force the water to follow only certain paths prescribed by the rocks.

In this work, we mostly discuss the discrete-time random polymer model and some continuous-time generalizations. Those models have a parameter that adjusts the strength of the disturbance, with small values indicating weak disorder (small rocks) and large values indicating strong disorder (large rocks). The two regimes are naturally characterized by whether a certain limit is positive, which happens whenever the parameter is below a critical value. The main aim in the field is to show that the properties of the unperturbed model hold below the critical value and to understand the unusual behavior of the model in strong disorder. Many important questions are still open.

We study the "disastrous" case, where we allow environments that are more "degenerate" than what is typically assumed in the literature. Here some standard techniques break down, and we need to introduce new ideas to verify certain characteristics of the models. While this often leads to significantly longer and more technical proofs, we believe that the extra effort provides important insight into the core of the argument.

2. Overview of the results

This thesis is based on three publications: [30] (published in Electronic Journal of Probability, joint work with Nina Gantert), [28] (to appear in Annals of Applied Probability, joint work with Ryoki Fukushima) and [40] (submitted). The content is, for the most part, identical to the published versions, but we have significantly modified the presentation to improve readability and provide an easier understanding. Moreover, we have tried to highlight common aspects between the models by using similar structure and notation, whenever possible. Most figures are new, and we have added some minor results and examples.

In Part I, we introduce some important mathematical tools that will be used repeatedly throughout this thesis. This comprises several types of stochastic orders (Section 3.1), an improved version of the well-known superadditive lemma (Theorem H) as well as two concentration inequalities (Theorems I and J).

Moreover, we introduce the discrete-time random polymer model (Section 4), which is closely related to the continuous-time models from Parts II and IV. As motivation for studying those models, we discuss an open problem for the disastrous case of the discrete-time model (Section 4.4). Finally, we demonstrate the use of Strassen's theorem (Theorem C) by proving a stochastic comparison result for partition functions (Theorem 4.4).

Parts II and III are both based on [30]. More precisely, the main result (Theorem 8.1) from Part III builds on the preparation from Part II, specifically Proposition 5.6(iii). We decided to split the discussion into two parts, because we think that the preparation is of independent mathematical interest and because the structure of Part II closely resembles that of Part IV.

In Part II we discuss a single particle moving as a (continuous-time) random walk on the integer lattice $\mathbb{Z}^d$ in the presence of a random environment ω of space-time disasters. We say that the particle is killed if it hits a disaster, and study the behavior of its quenched survival probability. It is known (Theorem K) that the survival probability decays exponentially at a deterministic rate $\mathfrak{p}$, which can be regarded as the free energy of this model. As a technical preparation for the main results, we consider the probability that the random walk survives, conditioned on its endpoint, and prove a moment bound for this quantity, uniformly over all endpoints (Proposition 5.2). Using this, we then apply well-known techniques to show a concentration inequality (Proposition 5.3); continuity of the free energy $\mathfrak{p}$ in the jump rate of the random walk (Proposition 5.5); and the existence of the point-to-point free energy (Proposition 5.6). The final two results are not contained in the published version of [30].

In Part III we consider the branching random walk corresponding to the single-particle model from Part II – that is, in addition to the previous dynamics, each particle branches into two independent descendants at a fixed rate. All particles are affected by the same environment. We give a complete characterization of the regime of global survival (Theorem 8.1), i.e., of the set of parameters such that the branching process does not eventually die out. The free energy $\mathfrak{p}$ from Part II plays an important role in this characterization. We find three regimes of parameters (Definition 8.3), and in two of them the main result follows quickly with the help of our preparation from Part II (Section 9). The critical case takes the most effort, and here we adapt a technique originally developed for the critical contact process (Section 10).

Part IV is based on [28]. We study the natural generalization of the model from Part II to Euclidean space $\mathbb{R}^d$. The random walk is replaced with Brownian motion, which gets killed with probability p upon hitting a disaster of ω, for some fixed $p \in [0,1]$. Existence of the free energy $\mathfrak{p}$ is only known (Theorem M) for the "soft" version of the model, i.e. for $p < 1$. We show that it also exists in the case $p = 1$ (Theorem 11.6), and moreover that it is continuous at the transition from "soft" to "hard" disasters (Theorem 11.9).

As mentioned before, the approach here is somewhat parallel to that of Part II, in that most of the work is in a technical moment bound on the survival probability with some additional restrictions (Proposition 11.3). The main results then follow by well-established techniques.

Interestingly, while Propositions 5.2 and 11.3 are similar statements for their respective models, we find that the difficulties lie in different parts of the proof: On the lattice, the logarithmic survival probability (with constrained endpoint) is known to have exponential moments (Lemma 6.4), but it is hard to prove a bound uniformly in the constraint. In $\mathbb{R}^d$ on the other hand, we observe that both the Brownian bridge and the environment are invariant under affine scalings, and that therefore the survival probability (with constrained endpoint) has the same law for all endpoints. See the discussion at the beginning of Section 12.2. However, in continuous space the logarithmic survival probability is not integrable (Proposition 11.2), and much of the work is devoted to overcoming this.

Part V is based on [40]. We present a stochastic comparison result (Theorem 15.1) that has applications for all models described so far, and that we believe is a universal feature of processes in space-time random environments: that more randomness in the random walk driving the model implies less randomness in the partition function. For example, in the model from Part II, we show that a higher jump rate in the random walk implies a larger survival probability, see (16.10). That is, the survival probability has a slower decay rate. We prove the main result in an abstract setup (Section 15), and then discuss consequences for the respective models (Section 16) and limitations of this approach (Section 17).

Notational convention: In this thesis, we use c and C to denote positive constants whose values may change from line to line.

3. Mathematical background

3.1. Stochastic orders

We will use stochastic orders at various later points, so in this section we collect some important results. We refer to [50] for a survey on the topic.

3.1.1. The monotone stochastic order

The relation $X \preceq_{st} Y$ between random variables is a precise way of saying that X takes smaller values than Y.

Definition 3.1. Let X and Y be real random variables. We say $X \preceq_{st} Y$ (Y stochastically dominates X) if, for every $t \in \mathbb{R}$,

$$\mathbb{P}(X \leq t) \geq \mathbb{P}(Y \leq t). \tag{3.1}$$

Note that $\preceq_{st}$ depends only on the distributions of X and Y, and we also write $P \preceq_{st} Q$ if P and Q are the laws of X and Y. Several equivalent criteria are known:

Theorem A ([50, Theorem 1.2.4])**.** *The following are equivalent:*

(i) $X \preceq_{st} Y$.

(ii) $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ *for every increasing* $f \colon \mathbb{R} \to \mathbb{R}$.

(iii) *There exists a coupling* $(\widehat{X}, \widehat{Y})$ *such that* $X \overset{\mathrm{d}}{=} \widehat{X}$ *and* $Y \overset{\mathrm{d}}{=} \widehat{Y}$, *and such that almost surely* $\widehat{X} \leq \widehat{Y}$.

It can be hard to verify (3.1) in practice, because the distribution functions may be too complicated. We therefore introduce a stronger relation $\preceq_{lr}$, which has the advantage that it can be checked directly from the densities of X and Y. For simplicity, we only discuss discrete random variables.

Definition 3.2. Let $\Lambda \subseteq \mathbb{R}$ be countable, and X and Y real random variables supported on Λ. We say $X \preceq_{lr} Y$ (Y dominates X in the likelihood ratio order) if, for every $k, l \in \Lambda$ with $k \leq l$,

$$\mathbb{P}(X = l)\mathbb{P}(Y = k) \leq \mathbb{P}(X = k)\mathbb{P}(Y = l).$$

Taking $A = \mathbb{R}$ in the following theorem shows that $\preceq_{lr}$ is indeed stronger than $\preceq_{st}$.

Theorem B ([50, Theorem 1.4.6])**.** *Let* $A \in \mathcal{B}(\mathbb{R})$ *be such that* $\mathbb{P}(X \in A) > 0$ *and* $\mathbb{P}(Y \in A) > 0$. *Then* $X \preceq_{lr} Y$ *implies*

$$\mathbb{P}(X \in \cdot | X \in A) \preceq_{st} \mathbb{P}(Y \in \cdot | Y \in A).$$

3.1.2. The convex stochastic order

We discuss the convex order $X \preceq_{cx} Y$, which intuitively means that Y has more randomness than X.

Definition 3.3. Let X and Y be real random variables. We say $X \preceq_{cx} Y$ (X is smaller in convex order) if, for every $f \colon \mathbb{R} \to \mathbb{R}$ convex,

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]. \tag{3.2}$$

The concave order $\preceq_{cv}$ is obtained by replacing "convex" by "concave" in the previous definition. Clearly, $X \preceq_{cv} Y$ if and only if $Y \preceq_{cx} X$, so it is enough to discuss one of these relations.

Note that if Y is any random variable and $X := \mathbb{E}[Y]$, then $X \preceq_{cx} Y$ follows from Jensen's inequality. This is an elementary demonstration of the statement "Y has more randomness than X" mentioned before. In general, this becomes precise with the following result:

Theorem C ([50, Theorem 1.5.20])**.** *The following are equivalent:*

(i) $X \preceq_{cx} Y$.

(ii) *There exists a coupling* $(\widehat{X}, \widehat{Y})$ *such that* $\widehat{X} \overset{\mathrm{d}}{=} X$ *and* $\widehat{Y} \overset{\mathrm{d}}{=} Y$, *and such that almost surely* $\mathbb{E}[\widehat{Y} | \widehat{X}] = \widehat{X}$.

Here the implication (ii) $\Longrightarrow$ (i) follows almost immediately from Jensen's inequality, see also the proof of Corollary 15.2 below.

Note that in the previous example, the coupling can be constructed as $(\widehat{X}, \widehat{Y}) := (\mathbb{E}[Y], Y)$. In practice, however, the coupling can be quite complicated and it is often easier to verify (3.2) using analytic techniques (see for example [55, Lemma 2.2]). In this thesis, we generally prefer to prove convex comparison using the implication (ii) $\Longrightarrow$ (i) (see Theorem 4.4, Lemma 6.3 and Theorem 15.1).

Next, we introduce tools for proving $\preceq_{cx}$ without having to construct the coupling.

3.1.3. The majorization order

The relation presented in this section is not a stochastic ordering in the usual sense. We refer to [47] for a survey on the topic.

Consider a finite set Λ with K elements, and let p, $q \in \mathcal{M}_{<\infty}(\Lambda)$ be finite measures on Λ. Let moreover π and σ be bijections $\pi, \sigma \colon \{1, \dots, K\} \to \Lambda$, chosen in such a way that $i \mapsto p(\pi(i))$ and $i \mapsto q(\sigma(i))$ are decreasing. In other words, π and σ encode the relative orders of the weights of p and q.

Definition 3.4. We say $p \preceq_M q$ (p is majorized by q) if, for all $k = 1, \dots, K$,

$$\textstyle\sum_{i=1}^{k} p(\pi(i)) \leq \sum_{i=1}^{k} q(\sigma(i)) \tag{3.3}$$

and if equality holds for $k = K$.

Intuitively, $p \preceq_M q$ means that the mass of q is distributed more unevenly than in p. To illustrate this, let us restrict to probability measures for a moment. Then we observe that the minimal element with respect to $\preceq_M$ is the uniform distribution on Λ (where the mass is spread evenly among all sites), while all Dirac measures (where all mass is concentrated on one site) are maximal. The relation $\preceq_M$ is used, for example, to compare how "fair" wealth is distributed within societies.

Theorem D ([47, Theorems 1.A.4 and 4.B.1])**.** *Let $p, q \in \mathcal{M}_1(\Lambda)$. The following are equivalent:*

(i) $p \preceq_M q$*.*

(ii) There exists a doubly stochastic matrix $\Pi \in \mathbb{R}^{\Lambda \times \Lambda}$ such that $p = q \cdot \Pi$ (where q and p are interpreted as row-vectors).

(iii) For all continuous, convex functions $f \colon [0,1] \to \mathbb{R}$

$$\textstyle\sum_{a \in \Lambda} f(p(a)) \leq \sum_{a \in \Lambda} f(q(a)).$$

The equivalence (i) $\Longleftrightarrow$ (iii) relates the majorization order to the convex order discussed before. This is not necessarily helpful in itself, because verifying $\mu \preceq_M \nu$ can be equally hard in practice. We recall a criterion for preservation of $\preceq_M$:

Theorem E ([47, Theorems 5.A.17 and 5.A.17a])**.** *Let $\Lambda = \{1, \dots, K\}$, $\Lambda' = \{1, \dots, K'\}$. Let $A \in \mathbb{R}^{K \times K'}$ be such that every permutation of a column of A is also a column of A. That is, for every $k' \in \{1, \dots, K'\}$ and every permutation $\pi \colon \{1, \dots, K\} \to \{1, \dots, K\}$ there exists $l' \in \{1, \dots, K'\}$ such that*

$$A(\pi(k), k') = A(k, l') \quad \text{for all } k \in \{1, \dots, K\}.$$

Then $p \preceq_M q$ implies $(p \cdot A) \preceq_M (q \cdot A)$.

Next, we discuss the situation where Λ itself is a partially ordered set, i.e., there exists a partial order $\preceq_\Lambda$ on Λ.

Definition 3.5. A function $f \colon \Lambda \to \mathbb{R}$ is $\preceq_\Lambda$-decreasing if $a \preceq_\Lambda b$ implies $f(a) \leq f(b)$. A set $A \subseteq \Lambda$ is $\preceq_\Lambda$-decreasing if its indicator function $\mathbb{1}_A$ is $\preceq_\Lambda$-decreasing, i.e. if $a \preceq_\Lambda b$ and $b \in A$ imply $a \in A$.

We state a sufficient criterion for $\preceq_M$ based on $\preceq_\Lambda$-decreasing sets:

Theorem F ([7, Theorem 3])**.** *Let $p, q \in \mathcal{M}_1(\Lambda)$. Assume that p and q are $\preceq_\Lambda$-decreasing, and that for every $\preceq_\Lambda$-decreasing set $A \subseteq \Lambda$*

$$p(A) \leq q(A). \tag{3.4}$$

Then $p \preceq_M q$.

Note that if $\Lambda \subseteq \mathbb{R}$ and $\preceq_\Lambda$ is the usual relation "$\leq$", then p and q are $\leq$-decreasing if $i \mapsto p(i)$ and $i \mapsto q(i)$ are decreasing on Λ, and (3.4) is the same as (3.3) in the definition.

3.2. Superadditivity

The superadditive lemma is well-known:

Theorem G (Superadditive lemma)**.** *Let $(a(t))_{t \geq 1}$ be a real sequence such that, for all $s, t \geq 1$,*

$$a(s + t) \geq a(s) + a(t). \tag{3.5}$$

Then $\lim_{t\to\infty} \frac{a(t)}{t} = A$, where $A := \sup_{t\geq 1} \frac{a(t)}{t} \in \mathbb{R} \cup \{\infty\}$.

A similar statement holds if a small error term is introduced in (3.5):

Theorem H ([38, Theorem 2])**.** *Let $(a(t))_{t \geq 1}$ be a sequence such that, for all $s, t \geq 1$,*

$$a(s + t) \geq a(s) + a(a) - b(s + t). \tag{3.6}$$

Assume that $t \mapsto b(t)$ is increasing and that $\int_1^\infty \frac{b(t)}{t^2} \mathrm{d}t < \infty$. Then the limit $A := \lim_{t\to\infty} \frac{a(t)}{t}$ exists in $\mathbb{R} \cup \{\infty\}$, and for all $t \geq 1$

$$a(t) \leq tA - b(t) + 4t \int_{2t}^\infty \frac{b(s)}{s^2} \mathrm{d}s. \tag{3.7}$$

Note that if $(a(t))_{t \geq 1}$ is superadditive, then $A = \sup_{t \geq 1} \frac{a_t}{t}$ implies that the upper bound

$$a(t) \leq tA$$

is valid for every $t \geq 1$. If the sequence is only almost-superadditive in the sense of (3.6), then the corresponding bound is provided by (3.7).

3.3. Concentration inequalities

We present two concentration inequalities that will be useful to control the fluctuations of the logarithmic partition function $\log Z_t(\omega)$ around its expectation $\mathbb{E}[\log Z_t]$.

Let us briefly explain the approach, for simplicity in the case $t \in \mathbb{N}$. In both cases, the idea is to divide the environment until time t into time-slices (of equal size) and to show that no time-slice has a large influence on $\log Z_t$. More precisely, let ω be an environment. For $k = 1, \ldots, t$ we let $\omega^{(k)}$ denote an environment that agrees with ω, except that the k^{th} time-slice is independently resampled. For both concentration inequalities, we need to show that

$$\big| \log Z_t(\omega) - \log Z_t(\omega^{(k)}) \big|$$

cannot be too large (in some sense). The actual statements are quite technical, so we do not go into any more detail.

Theorem I ([46, Theorem 2.1])**.** *Let $X_1, X_2, \ldots$ be a sequence of super-martingale differences with respect to some filtration $\mathcal{F} = (\mathcal{F}_t \colon t \in \mathbb{N})$, and assume that there exists $K > 0$ such that for all $k = 1, \ldots, t$, almost surely,*

$$\mathbb{E}\big[e^{|X_k|} \big| \mathcal{F}_{k-1}\big] \leq K. \tag{3.8}$$

Then, for any $u \geq 0$,

$$\mathbb{P}\Big(\Big| \sum_{k=1}^{t} X_k \Big| \geq tu \Big) \leq \begin{cases} 2e^{-\frac{tu^2}{K(1+\sqrt{2})^2}} & \text{if } u \in [0, K) \\ 2e^{-\frac{tu}{(1+\sqrt{2})^2}} & \text{if } u \geq K \end{cases}. \tag{3.9}$$

Theorem J ([6, Theorem 15.5])**.** *Let $\xi = \{\xi_k \colon k = 1, \ldots, t\}$ be an independent sequence of random variables (not necessarily real), and let $\xi' = \{\xi'_k \colon k = 1, \ldots, t\}$ be an independent copy of ξ. We write $\widehat{\xi}^{(k)}$ for the sequence*

$$\widehat{\xi}^{(k)} := \big(\xi_1, \ldots, \xi_{k-1}, \xi'_k, \xi_{k+1}, \ldots, \xi_t\big).$$

Let f be a measurable, real function of ξ, and define

$$\begin{aligned} V_+(\xi) &:= \sum_{k=1}^{t} \mathbb{E}\big[\big(f(\xi) - f(\widehat{\xi}^{(k)})\big)_+^2 \big| \sigma(\xi)\big] \\ V_-(\xi) &:= \sum_{k=1}^{t} \mathbb{E}\big[\big(f(\xi) - f(\widehat{\xi}^{(k)})\big)_-^2 \big| \sigma(\xi)\big]. \end{aligned}$$

Then, for any $q \in \mathbb{N}$, there exists a universal constant $C(q)$ such that

$$\begin{aligned} \mathbb{E}\big[\big(f(\xi) - \mathbb{E}[f(\xi)]\big)_+^{2q}\big] &\leq C(q)\mathbb{E}\big[V_+^q\big] \\ \mathbb{E}\big[\big(f(\xi) - \mathbb{E}[f(\xi)]\big)_-^{2q}\big] &\leq C(q)\mathbb{E}\big[V_-^q\big]. \end{aligned}$$

4. The discrete-time random polymer model

The random polymer model has been studied extensively, see for example the surveys [10], [23] or [14]. Since it is the discrete-time version of the models from Parts II and IV, we will use it as a reference point in our discussion in the remainder of this thesis.

After introducing the model, we explain the standard technique for proving the existence and continuity of the so-called free energy (Section 4.3), which is an important observable of such models. We also discuss why this approach fails in the disastrous case and how this motivated our research on disastrous models in continuous time (Section 4.4).

We introduce the random polymer model in somewhat greater generality than what is common in the literature – this does not change the behavior of the model in a substantive way, but it will be useful later on to illustrate some of our results.

4.1. Definition of the model

Let $\Omega := [0,\infty)^{\mathbb{N}\times\mathbb{Z}^d}$, $\mathcal{F}$ the sigma-field generated by the cylinder functions on Ω, and $\mathbb{P}$ a probability measure on $(\Omega,\mathcal{F})$. We assume that $\mathbb{P}(\omega(0,0)>0)>0$ and that

$$\omega = \{\omega(t,i)\colon t\in\mathbb{N}, i\in\mathbb{Z}^d\} \quad \text{is i.i.d..}$$

Let $\mathcal{M}_1(\mathbb{Z}^d)$ denote the set of probability measures on $\mathbb{Z}^d$, and for $p\in\mathcal{M}_1(\mathbb{Z}^d)$ write P^p for the law of a random walk with increment distribution p. More precisely, let $\mathcal{I}$ denote the set of paths $x\colon\mathbb{N}\to\mathbb{Z}^d$ and $\mathcal{G}$ the sigma-field generated by the coordinate processes. Then P^p is a probability measure on $(\mathcal{I},\mathcal{G})$ such that $X(0)=0$ and $(X(t+1)-X(t))_{t\in\mathbb{N}}$ is i.i.d. with common law p.

Fix a parameter $\beta\geq 0$ called the **inverse temperature**. The interaction between environment and random walk is described by a family $(F_t^\beta)_{t\in\mathbb{N}}$, defined by

$$F_t^\beta\colon \Omega\times\mathcal{I}\to\mathbb{R}_+, \quad (\omega,x)\mapsto\prod_{s=1}^{t}\omega(t,x(t))^\beta. \tag{4.1}$$

For $\omega\in\Omega$, we consider the **polymer measure** $\mu_{\omega,t}^{\beta,p}\in\mathcal{M}_1(\mathcal{I})$, that is the (random) probability measure on $(\mathcal{I},\mathcal{G})$ with weights proportional to F_t^β:

$$\mu_{\omega,t}^{\beta,p}(\mathrm{d}x) = \big(Z_t^{\beta,p}(\omega)\big)^{-1}F_t^\beta(\omega,x)P^p(\mathrm{d}x). \tag{4.2}$$

Here $Z_t^{\beta,p}(\omega)$ is the normalizing factor, called the **partition function**, with

$$Z_t^{\beta,p}\colon\Omega\to\mathbb{R}_+, \quad \omega\mapsto E^p\big[F_t^\beta(\omega,X)\big]. \tag{4.3}$$

The term "polymer", used for example in the title of this thesis, always refers to a path sampled according to the polymer measure.

4.2. Motivation

By definition, $\mu_{\omega,t}^{\beta,p}$ is a random perturbation of P^p, where the (exponential) weight of a path is proportional to the environment "collected" up to time t. The polymer is attracted by large values of ω and repelled by small values. What makes this model challenging is that this "maximization" of $F_t^\beta(\omega, x)$ is counteracted by the entropic cost $P^p(\mathrm{d}x)$ of the path x.

The inverse temperature β controls the strength of the perturbation. Consequently, we expect that $\mu_{\omega,t}^{\beta,p}$ behaves like P^p for β small (if the disorder is not strong enough to overcome the diffusive behavior of P^p), and localizes at an optimal path for β large (if the cost $P^p(\mathrm{d}x)$ is dominated by the $F_t^\beta(\omega, x)$-term).

This picture is by now mostly confirmed: If we assume $\mathbb{E}[\omega(0,0)^\beta + \omega(0,0)^{-\beta}] < \infty$ and restrict to simple random walk, then it is known that there is a phase transition between the two regimes, at least in dimension $d \geq 3$. More precisely, there exists a critical $\beta_c \in (0,\infty)$ such that $\mu_{\omega,t}^\beta$ satisfies a central limit theorem for $\beta < \beta_c$ (see [17, Theorem 1.2]), and localizes for $\beta > \beta_c$ (see [13, Theorem 1.1] or [10, Theorem 6.1]). In low dimensions ($d = 1$ and $d = 2$) the critical inverse temperature is $\beta_c = 0$ (see [43, Theorems 1.4 and 1.6]), and localization occurs for all $\beta > 0$. Here we say that localization occurs if $\max_{i\in\mathbb{Z}^d} \mu_{\omega,t}^\beta(X_t = i)$ does not decay to zero.

It turns out that to understand the behavior of $\mu_{\omega,t}^{\beta,p}$, it is helpful to first study the normalizing constant, $Z_t^{\beta,p}(\omega)$. Roughly speaking, $\mu_{\omega,t}^{\beta,p}$ behaves similar to the unperturbed measure P^p if and only if the martingale

$$W_t^{\beta,p}(\omega) := \frac{Z_t^{\beta,p}(\omega)}{\mathbb{E}[\omega(0,0)^\beta]^t} \tag{4.4}$$

converges in L^1. Since F_t^β has product form, a natural approach is to analyze whether the partition function $Z_t^{\beta,p}$ has an exponential growth rate. Clearly, a necessary criterion for $W_\infty^{\beta,p}(\omega) := \lim_{t\to\infty} W_n^{\beta,p}(\omega) > 0$ is

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\beta,p}(\omega) = \log \mathbb{E}[\omega(0,0)^\beta], \tag{4.5}$$

and in fact it is conjectured that this is also sufficient, see [10, Open Problem 3.2(i)]. The limit on the l.h.s. exists almost surely under mild assumptions, which we discuss in the next section, and it is equal to a deterministic constant $\mathfrak{p}(\beta, p)$, called the **free energy**.

In summary, analyzing the free energy is a first step towards understanding the martingale limit $W_\infty^{\beta,p}$, which is itself a step towards understanding the long-term behavior of the polymer measure $\mu_{t,\omega}^{\beta,p}$.

4.3. The free energy

We explain a well-known technique to show the existence of the limit in (4.5). The same approach will be useful later on in our discussion of related models, where we have to deal with additional issues caused, for example, by having continuous time/space or less integrability.

Proposition 4.1. *Assume* $\mathbb{E}[\omega(0,0)^\beta] < \infty$.

(i) Assume in addition that $\mathbb{E}[\log \omega(0,0)] > -\infty$. *There exists* $\mathfrak{p}(\beta,p) \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log Z_t^{\beta,p}\big] = \mathfrak{p}(\beta,p).$$

(ii) Assume in addition that $\mathbb{E}[\omega(0,0)^{-\beta}] < \infty$ *and let* $\varepsilon \in (0,1/2)$. *There exists* $C(\beta) > 0$ *such that, for all* $t \in \mathbb{N}$ *and* $p \in \mathcal{M}_1(\mathbb{Z}^d)$,

$$\mathbb{P}\Big(\big|\log Z_t^{\beta,p} - \mathbb{E}[\log Z_t^{\beta,p}]\big| \geq t^{1/2+\varepsilon}\Big) \leq e^{-C(\beta)t^{2\varepsilon}}$$

Moreover, $0 < \inf_{\beta\in[\beta_1,\beta_2]} C(\beta) \leq \sup_{\beta\in[\beta_1,\beta_2]} C(\beta) < \infty$ *for all* $\beta_1 < \beta_2$.

(iii) Under the assumptions of (ii), almost surely

$$\lim_{t\to\infty} \frac{1}{t}\log Z_t^{\beta,p} = \mathfrak{p}(\beta,p).$$

Sketch of the proof. **Part (i)**: The space-time shift $\theta^{t,i}\colon \Omega \to \Omega$ is defined by

$$(\theta^{t,i}\omega)(s,j) \coloneqq \omega(s+t, i+j).$$

The Markov property of P^p implies

$$\begin{aligned} Z_{s+t}^{\beta,p}(\omega) &= \sum_{i\in\mathbb{Z}^d} E^p\big[F_s^\beta(\omega,X)\mathbb{1}\{X(s)=i\}\big] Z_t^{\beta,p}(\theta^{s,i}\omega) \\ &= Z_s^{\beta,p}(\omega)\sum_{i\in\mathbb{Z}^d} \mu_{\omega,s}^{\beta,p}(X(s)=i) Z_t^{\beta,p}(\theta^{s,i}\omega). \end{aligned}$$

Here $\mu_{\omega,s}^{\beta,p}$ is the polymer measure defined in (4.2). After taking logarithms, we can apply Jensen's inequality to get

$$\log Z_{s+t}^{\beta,p}(\omega) \geq \log Z_s^{\beta,p}(\omega) + \sum_{i\in\mathbb{Z}^d} \mu_{\omega,s}^{\beta,p}(X(s)=i)\log Z_t^{\beta,p}(\theta^{s,i}\omega).$$

It is clear that $\log Z_t^{\beta,p}(\theta^{s,i}\omega)$ has the same law as $\log Z_t^{\beta,p}(\omega)$, and that it is independent of $\mu_{\omega,s}^{\beta,p}(X(s)=i)$, for any $i \in \mathbb{Z}^d$. Taking expectation therefore shows that

$$a_{\beta,p}(t) \coloneqq \mathbb{E}[\log Z_t^{\beta,p}]$$

is superadditive, i.e. satisfies (3.5). The first conclusion then follows from Theorem G. To see that the limit is finite, we again use Jensen's inequality:

$$\mathbb{E}\big[\log Z_t^{\beta,p}(\omega)\big] \leq \log \mathbb{E}\big[Z_t^{\beta,p}\big] = \log E\Big[\prod_{s=1}^t \mathbb{E}\big[\omega(s,X(s))^\beta\big]\Big] = t\log \mathbb{E}\big[\omega(0,0)^\beta\big].$$

Part (ii): Let $\mathcal{F}_s \coloneqq \sigma(\omega(r,i)\colon r \le s, i \in \mathbb{Z}^d)$ and observe that

$$\log Z_t^{\beta,p} - \mathbb{E}[\log Z_t^{\beta,p}] = \sum_{s=1}^{t} \mathbb{E}\big[\log Z_t^{\beta,p} \big| \mathcal{F}_s\big] - \mathbb{E}\big[\log Z_t^{\beta,p} \big| \mathcal{F}_{s-1}\big] \eqqcolon \sum_{s=1}^{t} V_s.$$

In order to apply Theorem I, we need to show that there exists K, independent of t, such that almost surely for all $s = 1, \dots, t$

$$\mathbb{E}\big[e^{|V_s|} \big| \mathcal{F}_{s-1}\big] < K.$$

This is shown in [46, Lemma 6.4] for the case of simple random walk, and the same proof works without change for general increment distributions.
Part (iii): Let

$$A_t \coloneqq \big\{ \big| \log Z_t^{\beta,p}(\omega) - \mathbb{E}[\log Z_t^{\beta,p}] \big| \ge t^{3/4} \big\}$$

and note that $\mathbb{P}(A_t) \le e^{-C(\beta)t^{1/2}}$ by part (ii). The Borel–Cantelli lemma shows that $\mathbb{P}(A_t \text{ infinitely often}) = 0$, so there exists $T(\omega)$ such that for all $t \ge T(\omega)$

$$\Big| \frac{1}{t} \log Z_t^{\beta,p}(\omega) - \frac{1}{t}\mathbb{E}[\log Z_t^{\beta,p}] \Big| \le t^{-1/4}.$$

The conclusion then follows by part (i). $\square$

Remark 4.2. In fact, the assumption $\mathbb{E}\big[\omega(0,0)^{\beta}\big]$ is not necessary to guarantee $\mathfrak{p}(\beta,p) < \infty$. In [48, Theorem 1.1] they show that (in the case of simple random walk) it is enough that

$$\int_0^t \mathbb{P}\big(\omega(0,0) > e^t\big)^{1/d} \mathrm{d}t < \infty.$$

In this thesis we are mostly concerned with repulsive environments, so we do not ask for optimal conditions on the upper tail of the environment.

Remark 4.3. Adapting a technique from [58] for directed first-passage percolation, one can show that, for any $K > 0$,

$$[0,\infty) \times \mathcal{M}_1^K(\mathbb{Z}^d) \to \mathbb{R}, \quad (\beta,p) \mapsto \mathfrak{p}(\beta,p)$$

is continuous. Here $\mathcal{M}_1^K(\mathbb{Z}^d)$ denotes the set of all probability measures supported on $\mathbb{Z}^d \cap [-K,K]^d$. We refer to Proposition 5.5 and Theorem 11.9 below for details (in related, continuous-time models).

4.4. The disastrous case

We have deviated from the notation commonly used in the literature by including the increment distribution p as a parameter, and also by defining the partition function in a different (but equivalent) way – the standard notation can be recovered by considering environment $\eta(t,i) := \log \omega(t,i)$ and partition function

$$Z_t^{\beta,p}(\eta) = E^p\big[e^{\beta \sum_{s=1}^t \eta(s,X(s))}\big]. \tag{4.6}$$

We are using (4.3) instead of (4.6) because in this way it is easier to include the disastrous case, which we are most interested in. Here and in the following, we will use the term "disastrous" if

$$\mathbb{P}\big(P(F_t^\beta(\omega,X) = 0) > 0\big). \tag{4.7}$$

It is easy to see that, if $\beta > 0$, then the random polymer model is disastrous if and only if $\mathbb{P}(\omega(0,0) = 0) > 0$. If p has compact support, then (4.7) implies that also

$$\mathbb{P}\big(Z_t^{\beta,p}(\omega) = 0\big) > 0.$$

This clearly violates both the assumptions and the spirit of Proposition 4.1, since in particular $\log Z_t^{\beta,p}$ is not integrable.

The models discussed later (Parts II and IV) are also disastrous in the sense of (4.7), but because they are in continuous time they still satisfy $Z_t(\omega) > 0$ almost surely – note that this takes care of the most immediate obstacle to repeating the proof of Proposition 4.1. In fact, we find that after overcoming substantial additional problems, the same basic approach can be used to show the existence of the free energy for those models.

In the remainder of this section, we discuss the disastrous case of the discrete-time random polymer model. For simplicity, we fix $\beta = 1$ and P^p the law of simple random walk. We consider a Bernoulli environment $\mathbb{P}_\infty$, with

$$\mathbb{P}_\infty(\omega(0,0) = 0) = 1 - \mathbb{P}_\infty(\omega(0,0) = 1) = q \in (0,1). \tag{4.8}$$

The reason for writing $\mathbb{P}_\infty$ instead of $\mathbb{P}$ will become clear later. Let $\mathcal{O}(\omega) := \{(t,i) \in \mathbb{N} \times \mathbb{Z}^d \colon \omega(t,i) = 1\}$ denote the set of admissible space-time sites and

$$\mathcal{N}_t(\omega) := \big\{x \colon \omega(s, x(s)) \in \mathcal{O}(\omega) \text{ for all } s = 1, \dots, t\big\}$$

the set of nearest-neighbor paths x of length t in $\mathcal{O}$, starting in the origin. Note that $|\mathcal{N}_t|$ is the number of open paths in directed site percolation, and that the partition function can be written as

$$Z_t(\omega) = E[F_t(\omega,X)] = \frac{|\mathcal{N}_t(\omega)|}{2^d}.$$

That is, if the free energy (4.5) exists, it corresponds to the growth rate of the number of open paths in directed site percolation. However, observe that

$$\mathbb{P}_\infty\big(\exists t \in \mathbb{N} \text{ s.t. } \mathcal{O} \cap (\{t\} \times \{-t, \dots, t\}) = \varnothing\big) > 0,$$

and in this case $|\mathcal{N}_t(\omega)| = 0$ for all t large enough. We therefore consider

$$\{\mathcal{O} \leftrightarrow \infty\} := \big\{\exists x \colon (t, x(t)) \in \mathcal{O} \text{ for all } t \in \mathbb{N}\big\}$$

the event that there exists an infinite nearest-neighbor path x in $\mathcal{O}$, starting in the origin. This event has positive probability for all $q > q_c$, where q_c denotes the critical probability in oriented site percolation. Clearly $Z_t(\omega) > 0$ for all t on $\{\mathcal{O} \leftrightarrow \infty\}$, and it was recently shown that the free energy exists on this event:

Theorem ([31, Theorem 1.1])**.** *For every $q > q_c$ there exists $\mathfrak{p}(q, \infty) \in \mathbb{R}$ such that $\mathbb{P}_\infty(\cdot|\mathcal{O} \leftrightarrow \infty)$-almost surely*

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t(\omega) = \mathfrak{p}(q, \infty).$$

Observe that the approach used in Proposition 4.1 does not seem to be well-suited for this type of result, because $\log Z_t$ is not integrable with respect to $\mathbb{P}_\infty$. Moreover, if we replace $\mathbb{P}_\infty$ by $\mathbb{P}_\infty(\cdot|\mathcal{O} \leftrightarrow \infty)$ to fix this problem, we lose the i.i.d. structure of the environment, which is also an essential ingredient. The proof in [31] instead relies on an intricate combination of various techniques developed in the study of the contact process and oriented percolation.

We close this discussion by mentioning an open problem: It would be desirable to know whether the disastrous model (4.8) can be approximated by the more tractable soft-obstacle model. That is, fix $q > q_c$ and write $\mathbb{P}_\gamma$ for the environment with $\mathbb{P}_\gamma(\omega(0,0) = 1) = q$ and

$$\mathbb{P}_\gamma(\omega(0,0) = e^{-\gamma}) = 1 - q.$$

We can apply Proposition 4.1 for every $\gamma < \infty$, and thus find a constant $\mathfrak{p}(q, \gamma)$ such that $\mathbb{P}_\gamma$-almost surely

$$\lim_{t\to\infty} \frac{1}{t} \log Z_t(\omega) = \mathfrak{p}(q, \gamma).$$

To the best of our knowledge, it is not known whether $\mathfrak{p}(q, \infty) = \lim_{\gamma\to\infty} \mathfrak{p}(q, \gamma)$.

Continuity in the zero-temperature limit was a partial motivation in our study of the Brownian polymer model in Part IV, where we follow the superadditive approach from Proposition 4.1. In that model, we find that the additional randomness from continuous time is enough to overcome the non-integrability of the logarithmic partition function. We will discuss further results on the zero-temperature limit in Section 11.3.

Finally, we mention that even though we do not carry out the proof here, our results also show continuity in the zero-temperature limit for the lattice-model from Part II.

4.5. Concave comparison of the partition function

We provide an example that illustrates the power of Theorem C. In Part V we show a second comparison result (Theorem 15.1) for partition functions, see also the discussion in Section 16.1. More results about stochastic orders for the random polymer model can be found in [52].

The theorem below is a discrete-time version of [29, Lemma 2.2] for the continuous-time random walk from Part II, see also the discussion before Lemma 6.3.

Let $\mathcal{J} = \{J_1, J_2, \dots\}$ denote a partition of $\mathbb{Z}^d$. We call an element $J \in \mathcal{J}$ an equivalence class. We write $\mathbb{P}_{\mathcal{J}}$ for the environment which has the same marginals as $\mathbb{P}$, and which is constant over each equivalence class and independent between different equivalence classes.

Let $\mathcal{J}$ and $\mathcal{K}$ be two partitions. We say that $\mathcal{K}$ is finer than $\mathcal{J}$ if for every $K \in \mathcal{K}$ there exists $J \in \mathcal{J}$ such that $K \subseteq J$. In this way, the coarsest partition is $\mathcal{J}_{\max} := \{\mathbb{Z}^d\}$, where all sites are in the same equivalence class, while the finest partition is $\mathcal{J}_{\min} := \{\{i\} : i \in \mathbb{Z}^d\}$, where each site forms its own equivalence class. Observe that $\mathbb{P}_{\mathcal{J}_{\min}}$ agrees with the i.i.d. environment from the previous sections, while under $\mathbb{P}_{\mathcal{J}_{\max}}$ the environment is the same at every site. Thus, it makes sense to say that $\mathbb{P}_{\mathcal{J}_{\min}}$ has more randomness than $\mathbb{P}_{\mathcal{J}_{\max}}$.

The following result makes this precise by using the convex stochastic order. Moreover, we show that we can compare more partitions than just the extremal elements $\mathcal{J}_{\max}$ and $\mathcal{J}_{\min}$.

Theorem 4.4. *Let $\mathcal{J}$ and $\mathcal{K}$ be two partitions of $\mathbb{Z}^d$, and assume that $\mathcal{K}$ is finer than $\mathcal{J}$. Moreover, for $A \subseteq \mathbb{Z}^d$, let $Z_t^{\beta,p,A}$ denote the restricted partition function, where F_t^β has been replaced by*

$$F_t^{\beta,A}(\omega, x) := F_t^\beta(\omega, x)\mathbb{1}\{x_t \in A\}.$$

Assume $\mathbb{E}[\omega(0,0)^\beta] < \infty$. For $f : \mathbb{R}_+ \to (-\infty, \infty]$ convex and $t \in \mathbb{N}$,

$$\mathbb{E}_{\mathcal{K}}[f(Z_t^{\beta,p,A})] \le \mathbb{E}_{\mathcal{J}}[f(Z_t^{\beta,p,A})].$$

Proof. Let us write $\mathcal{J} = \{J_1, J_2, \dots\}$ and $\mathcal{K} = \{K_0, K_1, \dots\}$. By considering a sequence of intermediate partitions, it is enough to assume $J_i = K_i$ for every $i \ge 2$, and $J_1 = K_0 \cup K_1$. We construct a coupling $(\omega, \widehat{\omega})$ with marginals $\mathbb{P}_{\mathcal{K}}$ and $\mathbb{P}_{\mathcal{J}}$, and such that almost surely

$$\mathbb{E}\big[Z_t^{\beta,p,A}(\widehat{\omega}) \big| \sigma(Z_t^{\beta,p,A}(\omega))\big] = Z_t^{\beta,p,A}(\omega). \tag{4.9}$$

The claim then follows from the implication (ii) $\Longrightarrow$ (i) in Theorem C. Let ω be an environment with law $\mathbb{P}_{\mathcal{K}}$. Observe that we could obtain $\widehat{\omega}$ by deciding, for every time s, whether to take the value $\omega(s, K_0)$ or $\omega(s, K_1)$ for $\widehat{\omega}(s+1, J_1)$. That is, we can obtain an environment with law $\mathbb{P}_{\mathcal{J}}$ by setting $\widehat{\omega}$ equal to ω on $\mathbb{N} \times \mathbb{Z}^d \setminus \{(s, J_1)\}$, and

$$\mathbb{P}\big(\widehat{\omega}(s, J_1) = \omega(s, K_0)\big) = \mathbb{P}\big(\widehat{\omega}(s, J_1) = \omega(s, K_1)\big) = 1/2.$$

This does not result in a coupling with the desired property (4.9), but we will use a similar idea. Let us introduce some notation. For $s = 0, \dots, t$ and $(u_1, \dots, u_s) \in \{0,1\}^s$, let $\omega(u_1, \dots, u_s)$ denote the environment defined by

$$(\omega(u_1, \dots, u_s))(r, i) = \begin{cases} \omega(r,i) & \text{if } (r,i) \notin \{1, \dots, s\} \times J_1 \\ \omega(r, K_{u_r}) & \text{else.} \end{cases}$$

That is, the value of $\omega(u_1, \dots, u_s)$ at (r, J_1) is either $\omega(r, K_0)$ or $\omega(r, K_1)$, depending on u_r. See also Figure 1 for an illustration. For $s = 0$ we set $\omega(\varnothing) := \omega$. Moreover, let $\overline{\omega}(u_1, \dots, u_s)$ be defined as $\omega(u_1, \dots, u_s)$ for all sites except $\{s+1\} \times J_1$, where we set

$$\overline{\omega}(u_1, \dots, u_s)(s+1, J_1) \equiv 1.$$

That is, in $\overline{\omega}(u_1, \dots, u_s)$ we have censored the environment at $\{s+1\} \times J_1$. Let

$$\alpha(\omega, u_1, \dots, u_s) := \frac{E^p\big[F_t^{\beta,A}(\overline{\omega}(u_1, \dots, u_s), X)\mathbb{1}\{X_{s+1} \in K_0\}\big]}{E^p\big[F_t^{\beta,A}(\overline{\omega}(u_1, \dots, u_s), X)\mathbb{1}\{X_{s+1} \in K_0 \cup K_1\}\big]},$$

where we define $\alpha(\omega, u_1, \dots, u_s) := \frac{1}{2}$ if the denominator is zero. Since $\alpha(\omega, u_1, \dots, u_s) \in [0,1]$, we can choose $U_1, \dots, U_t \in \{0,1\}^t$ successively using those probabilities. More precisely, we choose the sequence with $\mathbb{P}(U_1 = 0 | \sigma(\omega)) := \alpha(\omega, \varnothing)$ and, for $s \geq 1$,

$$\mathbb{P}\big(U_{s+1} = 0 \big| \sigma(\omega, U_1, \dots, U_s)\big) := \alpha(\omega, U_1, \dots, U_s).$$

We claim that, for every $s = 0, \dots, t-1$,

$$\mathbb{E}\big[Z_t^{\beta,p,A}(\omega(U_1, \dots, U_s, U_{s+1}))\big|\sigma(\omega, U_1, \dots, U_s)\big] = Z_t^{\beta,p,A}(\omega(U_1, \dots, U_s)). \tag{4.10}$$

Indeed, observe that for $i \in \{0,1\}$,

$$\begin{aligned} Z_t^{\beta,p,A}(\omega(U_1, \dots, U_s, i)) =& E^p[F_t^{\beta,A}(\omega(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \notin J_1\}] \\ &+ \omega(s+1, K_i)E^p[F_t^{\beta,A}(\overline{\omega}(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \in J_1\}]. \end{aligned}$$

Note that (4.10) obviously holds if the second expectation on the r.h.s. is zero, so let us assume that this is not the case. The first term does not depend on i, and therefore

$$\begin{aligned} &\mathbb{E}[Z_t^{\beta,p,A}(\omega(U_1, \dots, U_s, U_{s+1}))|\sigma(\omega, U_1, \dots, U_s)] \\ =& E^p[F_t^{\beta,A}(\omega(U_1, \dots, U_s))\mathbb{1}\{X_{s+1} \notin J_1\}] \\ &+ \alpha(\omega, U_1, \dots, U_s)E^p\big[F_t^{\beta,A}(\omega(U_1, \dots, U_s, 0), X)\mathbb{1}\{X_{s+1} \in J_1\}\big] \\ &+ \big(1 - \alpha(\omega, U_1, \dots, U_s)\big)E^p\big[F_t^{\beta,A}(\omega(U_1, \dots, U_s, 1), X)\mathbb{1}\{X_{s+1} \in J_1\}\big] \\ =& E^p[F_t^{\beta,A}(\omega(U_1, \dots, U_s))\mathbb{1}\{X_{s+1} \notin J_1\}] \\ &+ \omega(s+1, K_0)E^p\big[F_t^{\beta,A}(\overline{\omega}(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \in K_0\}\big] \\ &+ \omega(s+1, K_1)E^p\big[F_t^{\beta,A}(\overline{\omega}(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \in K_1\}\big] \\ =& E^p[F_t^{\beta,A}(\omega(U_1, \dots, U_s))\mathbb{1}\{X_{s+1} \notin J_1\}] \\ &+ E^p\big[F_t^{\beta,A}(\omega(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \in K_0\}\big] \\ &+ E^p\big[F_t^{\beta,A}(\omega(U_1, \dots, U_s), X)\mathbb{1}\{X_{s+1} \in K_1\}\big] \\ =& E^p[F_t^{\beta,A}(\omega(U_1, \dots, U_s))]. \end{aligned} \tag{4.11}$$

We then set $\widehat{\omega} := \omega(U_1, \ldots, U_t)$, and claim that $(\omega, \widehat{\omega})$ is the desired coupling.

First, note that the coupling property (4.9) follows by repeatedly applying (4.10) and the tower-property. To see that $\widehat{\omega}$ has law $\mathbb{P}_{\mathcal{J}}$, note that since $\alpha(\omega, u_1, \ldots, u_s)$ uses the censored environment $\overline{\omega}(u_1, \ldots, u_s)$, $\alpha(\omega, u_1, \ldots, u_s)$ does not depend on the value of ω on $\{s+1\} \times J_1$. Since $\omega(s+1, K_0)$ and $\omega(s+1, K_1)$ are independent and have the same law as $\omega(0,0)$, we indeed get

$$\omega(s+1, U_{s+1}) \overset{\mathrm{d}}{=} \omega(0,0).$$

$\square$

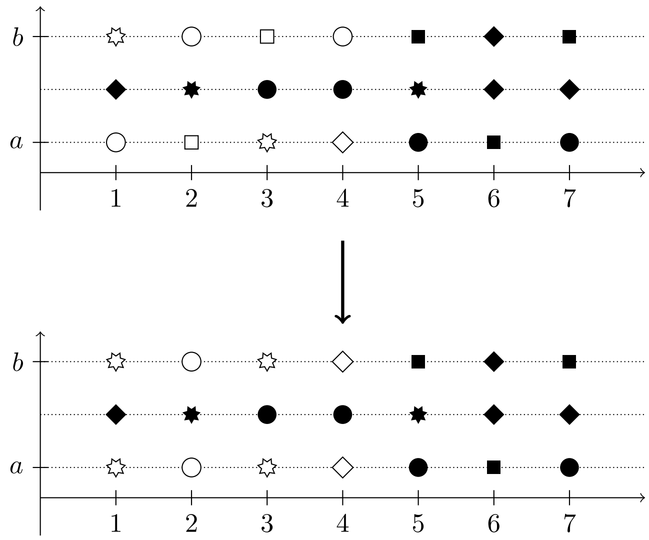

Figure 1: Illustration of the construction in the case where $K_0 = \{a\}$ and $K_1 = \{b\}$ are two singleton sets. The environment takes four values, illustrated by different shapes. We show the original environment ω (above) and the modification $\omega(1,1,0,0)$ (below).

Part II.
Random walk among space-time disasters

5. Introduction

5.1. Motivation

We consider an environment ω consisting of random space-time points $(t, i) \in \mathbb{R}_+ \times \mathbb{Z}^d$, which we interpret as a disaster occurring at site i at time t. Independently of ω, we let $X = \{X(t) \colon t \geq 0\}$ denote a continuous-time simple random walk on $\mathbb{Z}^d$. We say that X survives until time t if it avoids all disasters up to time t, i.e., if its graph does not intersect the environment ω. We study the exponential decay rate of the survival probability, conditioned on the environment. See Figure 2 for a realization of environment and random walk, and Figure 3 for the corresponding quenched survival probability.

5.2. Definition of the model

We introduce the model in slightly more generality: Let I denote a countable state space and Ω the set of locally finite point measures on $\mathbb{R}_+ \times I$. We identify $\omega \in \Omega$ with its support, i.e. we regard it as a random set of space-time points $(t, i) \in \mathbb{R}_+ \times I$. Let $(\omega, \mathbb{P})$ denote the Poisson point process with unit intensity on $\mathbb{R}_+ \times I$.

Moreover, let (X, P) denote a continuous-time Markov process on I with bounded generator A. Given an environment ω, the extinction time $\tau(\omega)$ is the hitting time of ω:

$$\tau(\omega) := \inf\big\{t \geq 0 \colon (t, X(t)) \in \omega\big\}.$$

Note that while the notation does not reflect this, $\tau(\omega)$ is a random variable with respect to both X and ω. If I is equal to the integer lattice $\mathbb{Z}^d$, we write P^κ for the law of simple random walk with jump rate $\kappa > 0$. More precisely, let Δ denote the discrete Laplacian on $\mathbb{Z}^d$, acting on functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ by

$$(\Delta f)(i) = \frac{1}{2d} \sum_{\|j-i\|_1 = 1} (f(j) - f(i)). \tag{5.1}$$

Then P^κ is the Markov process with generator $\kappa\Delta$. It is known that the quenched survival probability $P^\kappa(\tau(\omega) \geq t)$ decays exponentially with a deterministic rate:

Theorem K ([55])**.** *There exists $\mathfrak{p}(\kappa) \in (-\infty, -1]$ such that, almost surely and in L^1,*

$$\mathfrak{p}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\log P^\kappa(\tau(\omega) \geq t)] = \lim_{t\to\infty} \frac{1}{t} \log P^\kappa(\tau(\omega) \geq t).$$

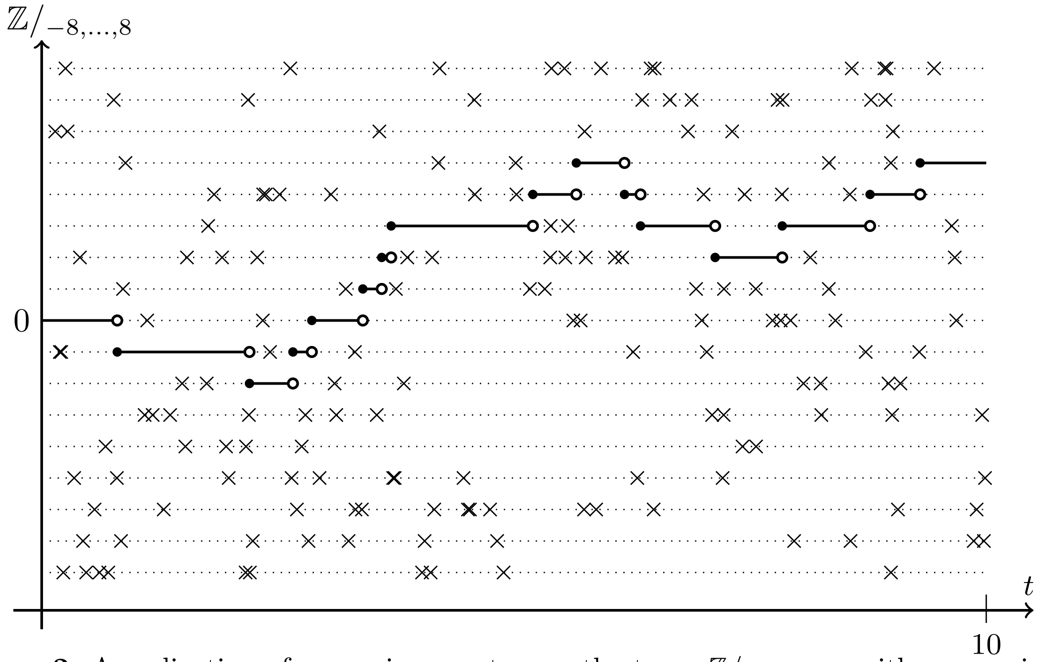

Figure 2: A realization of an environment ω on the torus $\mathbb{Z}/_{\{-8,\ldots,8\}}$, with crosses indicating the space-time disasters. In addition, we have drawn a nearest-neighbor path that does not touch any disasters in $[0, 10]$, and thus survives until time 10.

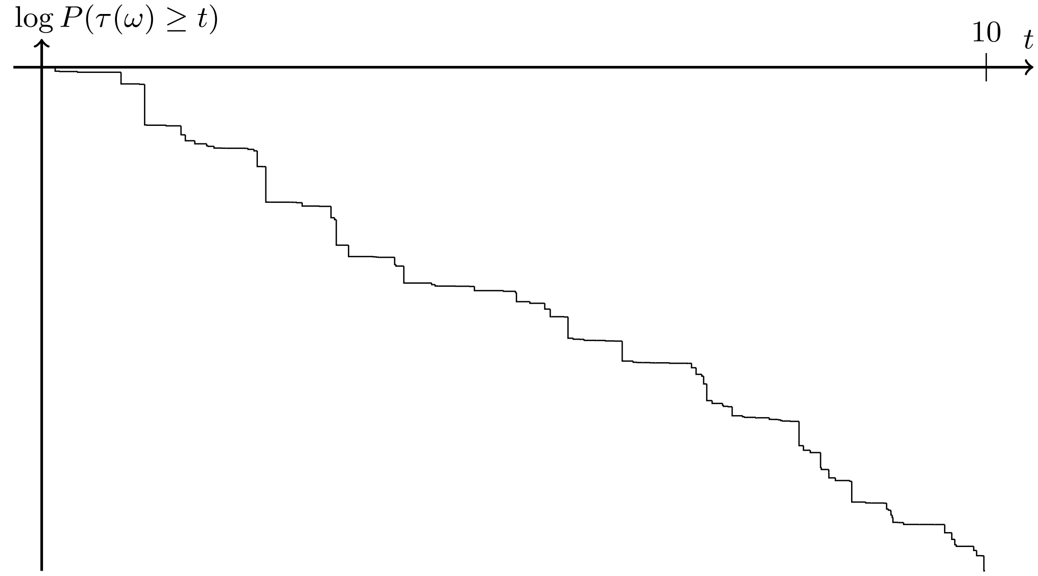

Figure 3: We show the logarithmic survival probability corresponding to the environment from Figure 2. Every jump in the plot corresponds to a disaster of ω. The decay rate $\mathfrak{p}(\kappa)$ is the asymptotic slope of this graph, which is deterministic.

Remark 5.1. In fact, [55] only shows the L^1-convergence in the previous theorem, although the almost-sure statement is used in the literature (for example in [49]). We note that it can be proved along the same lines as in Proposition 4.1, using Proposition 5.3.

Note that $\mathfrak{p}(\kappa)$ also depends on the dimension d, but we do not include this in the notation. The model has originally been introduced in [55] as a special case of the parabolic Anderson model with Lévy noise. We are not going to use this perspective in our discussion, but let us briefly outline this connection for completeness: Let ω and A be as before, and consider the following system of PDEs with stochastic noise:

$$\begin{aligned} &u_\omega(0,\cdot) \equiv 1, \\ &u_\omega(\mathrm{d}t,i) = (Au_\omega(t,\cdot))(i)\,\mathrm{d}t - u_\omega(t^-,i)\omega(\mathrm{d}t,i) \quad \text{for } t \in \mathbb{R} \geq 0, i \in I, \end{aligned} \tag{5.2}$$

This PDE has a unique strong solution [29, Theorem 1.2] satisfying $u_\omega(t,0) \overset{\mathrm{d}}{=} P(\tau(\omega) \geq t)$. More precisely, for fixed $t > 0$, let $\overleftarrow{\omega} \in \Omega$ denote the time-reversal of ω, i.e. for $s \in [0,t]$

$$(s,i) \in \overleftarrow{\omega} \quad \Longleftrightarrow \quad (t-s,i) \in \omega.$$

Then by [55, Lemma 2.1] the solution u_ω to (5.2) has the Feynman-Kac representation

$$u_\omega(t,0) = P(\tau(\overleftarrow{\omega}) \geq t). \tag{5.3}$$

5.3. Related literature

More general space-time environments are discussed in Section 16.2. The model can also be studied with $\mathbb{Z}^d$ replaced by $\mathbb{R}^d$ and X replaced by Brownian motion, which we discuss in Part IV (for disastrous environments) and Section 16.3 (for general Poissonian environments). Replacing instead continuous time $\mathbb{R}_+$ by discrete time $\mathbb{N}$, we arrive at the (directed) random polymer model, which we have discussed in Section 4.1.

The term "parabolic Anderson model" usually refers to the "static" version of (5.2), i.e., to the case where ω is constant in time. The "static" model has quite different behavior from the "dynamic" version, see also the discussion in Section 17.1.

5.4. The main results

Our results are based on the following uniform moment bound, which is interesting in its own right:

Proposition 5.2. *For every $\delta \in (0,1)$ and $\kappa > 0$, there exists $C(\kappa,\delta) \in (0,\infty)$ such that*

$$\sup_{i \in \mathbb{Z}^d} \mathbb{E}\Big[\big(P^\kappa(\tau(\omega) \geq 1 | X(t) = i)\big)^{-\delta}\Big] \leq C(\kappa,\delta). \tag{5.4}$$

Moreover, $\sup_{\kappa \in [\kappa_0,\kappa_1]} C(\kappa,\delta) < \infty$ *for every* $0 < \kappa_0 < \kappa_1 < \infty$.

As a first consequence, we obtain a concentration inequality:

Proposition 5.3. *For all $t \in \mathbb{N}$, $i \in \mathbb{Z}^d$ and $u > 0$*

$$\mathbb{P}\Big(\big|\log P^\kappa(\tau(\omega) \geq t, X(t) = i) - \mathbb{E}[\log P^\kappa(\tau(\omega) \geq t, X(t) = i)]\big| > ut\Big) \\ \leq \begin{cases} 2\exp\Big(-\frac{tu^2}{4C(\kappa,1/2)(1+\sqrt{2})^2}\Big) & \textit{if } u \leq 2C(\kappa, 1/2) \\ 2\exp\Big(-\frac{tu}{2(1+\sqrt{2})^2}\Big) & \textit{if } u > 2C(\kappa, 1/2) \end{cases} \tag{5.5}$$

The same statement holds with $P^\kappa(\tau(\omega) \geq t, X(t) = i)$ replaced by $P^\kappa(\tau(\omega) \geq t)$.

Remark 5.4. For simplicity, we only state the result for $t \in \mathbb{N}$, but it is not hard to obtain a concentration inequality for all $t > 0$. See Remark 7.1 for a sketch of the necessary changes. Moreover, the final part of the proof of Proposition 11.4 in Section 13.2 contains a detailed argument for a related model.

Observe that, somewhat surprisingly, the bound in (5.5) does not depend on $i \in \mathbb{Z}^d$. This is important in the proof of the following result:

Proposition 5.5. *(i) For every $0 < \kappa_0 < \kappa_1 < \infty$ and $\varepsilon \in (0, 1/2)$, there exist C, t_0 such that for all $\kappa \in [\kappa_0, \kappa_1]$ and $t \geq t_0$*

$$\mathfrak{p}(\kappa) - Ct^{-1/2+\varepsilon} \leq \frac{1}{t}\mathbb{E}\big[\log P^\kappa(\tau(\omega) \geq t)\big] \leq \mathfrak{p}(\kappa). \tag{5.6}$$

(ii) $\kappa \mapsto \mathfrak{p}(\kappa)$ is continuous.

Note that the continuity of $\kappa \mapsto \mathfrak{p}(\kappa)$ is already known [29, Corollary 4.1] for the soft-obstacle version of our model, where the random walk gets killed with probability $p \in (0, 1)$ whenever it hits a disaster. Next, we show the existence of the point-to-point free energy:

Proposition 5.6. *(i) For every $i \in \mathbb{R}^d$ and $\kappa > 0$, there exists $\mathfrak{p}(\kappa, i) \in (-\infty, -1]$ such that almost surely*

$$\begin{aligned} \mathfrak{p}(\kappa, i) &= \lim_{t\to\infty, t\in\mathbb{N}} \frac{1}{t} \log P^\kappa(\tau(\omega) \geq t, X(t) = \lfloor ti \rfloor) \\ &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log P^\kappa(\tau(\omega) \geq t, X(t) = \lfloor ti \rfloor)\big]. \end{aligned} \tag{5.7}$$

On the other hand, almost surely

$$\liminf_{t\to\infty} \frac{1}{t} \log P^\kappa(\tau(\omega) \geq t, X(t) = \lfloor ti \rfloor) = -\infty.$$

(ii) For every $\kappa > 0$, the function $i \mapsto \mathfrak{p}(\kappa, i)$ is concave and therefore continuous.

(iii) $\mathfrak{p}(\kappa, 0) = \mathfrak{p}(\kappa)$.

Let us quickly comment on part (iii), which will be crucial in our discussion of a related branching random walk in Part III. Note that if $\{\tau(\omega) \geq t\}$ and $\{X(t) = 0\}$ were independent under P^κ, then it would be clear that

$$P^\kappa(\tau(\omega) \geq t, X(t) = 0)$$

has the same decay rate as $P^\kappa(\tau(\omega) \geq t)$ – it follows from the fact that the return probability of simple random walk decays polynomially. However, note that under $P^\kappa(\cdot|\tau(\omega) \geq t)$ the random walk tries to move to parts of the environment with few disasters, which will drive it further away from the origin. Indeed, in dimension $d = 1$, using KPZ scaling exponents that are conjectured for a wide class of related models, we can expect that under $P^\kappa(\cdot|\tau(\omega) \geq t)$,

$$\max_{s\in[0,t]} |X(s)| \approx t^{2/3}.$$

This is much larger than the traversal fluctuations of P^κ, which are of order $t^{1/2}$. We thus expect that the return probability $P^\kappa(X(t) = 0|\tau(\omega) \geq t)$ decays polynomially, but faster than $P^\kappa(X(t) = 0)$.

5.5. Outline

Section 6 contains preparations for the uniform moment bound (Proposition 5.2):

- In Section 6.1, we prove majorization (recall Section 3.1.3) for the parity of multinomial random variables – no random walks or random environments appear in this section. At first glance, it is not clear how this result relates to Proposition 5.2, but it is a crucial ingredient: We will use it to compare the survival probability of two random walks, under the assumption that one has (stochastically) more jumps than the other.
- This comparison is valid for random walk in state space $I = \mathbb{Z}/_2$. In Section 6.2, we show that the survival probability in $\mathbb{Z}/_2$ is smaller than in $\mathbb{Z}^d$, in the sense of the concave stochastic order.
- Finally, in Section 6.3 we give a lower bound for the survival probability in $\mathbb{Z}/_2$, where explicit computations are possible.

The results in Sections 6.2 and 6.3 are already proved in [55] using analytic methods. For our purpose, we need slightly more general statements, and we give different proofs using probabilistic techniques.

The main results are then proved in their respective subsections of Section 7.

6. Preparation

6.1. Majorization for the parity of multinomial vectors

This section does not contain random environments or random walks, and instead we discuss majorization for the multinomial distribution. Discrete intervals are denoted by

$$[\![N]\!] := \{0, \dots, N\}.$$

Fix $N \in \mathbb{N}$ and weights $p_0, \dots, p_N$, with $p_0 + \cdots + p_N = 1$ and such that

$$0 \leq p_0 \leq p_1 \leq \cdots \leq p_N \leq 1. \tag{6.1}$$

Let P_k denote the multinomial distribution with k trials, each of which has result r with probability p_r. More precisely, let $B_1, B_2, \dots$ be i.i.d. random variables with values in $[\![N]\!]$ and $P(B_k = r) = p_r$, and set

$$M_k(r) \coloneqq \textstyle\sum_{l=1}^{k} \mathbb{1}\{B_l = r\}. \tag{6.2}$$

Let P_k denote the law of the $(N+1)$-dimensional vector $M_k = (M_k(0), \dots, M_k(N))$. The parities of M_k are denoted by

$$\widehat{M}_k \coloneqq \big(\mathbb{1}\{M_k(0) \text{ is odd}\}, \dots, \mathbb{1}\{M_k(N) \text{ is odd}\}\big) \in \{0,1\}^{[\![N]\!]}.$$

Let $\widehat{P}_k$ denote the law of $\widehat{M}_k$, and $\mathcal{E}_{even}$ (resp. $\mathcal{E}_{odd}$) the set of configurations $\pi \in \{0,1\}^{[\![N]\!]}$ with an even (resp. odd) number of ones. A first observation is

$$\widehat{P}_k(\mathcal{E}_{even}) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \tag{6.3}$$

To motivate the main result of this section, we prove that for k, l of the same parity,

$$k \le l \quad \Longrightarrow \quad \widehat{P}_l \preceq_M \widehat{P}_k. \tag{6.4}$$

Since $\preceq_M$ is a transitive relation, it is enough to check this for $l = k+2$. Indeed, in this case we consider the matrix $\Pi \in \mathbb{R}^{[\![N]\!] \times [\![N]\!]}$ defined, for $\pi, \sigma \in \{0,1\}^{[\![N]\!]}$, by

$$\Pi(\pi, \sigma) \coloneqq \begin{cases} \sum_{r=0}^{N} p_r^2 & \text{if } \pi = \sigma \\ 2 p_r p_s & \text{if } \{t \in [\![N]\!] \colon \pi_t \ne \sigma_t\} = \{r, s\}, \text{ with } r \ne s \\ 0 & \text{else.} \end{cases}$$

Clearly Π is symmetric, and thus doubly stochastic. Moreover, we now have $\widehat{P}_l = \widehat{P}_k \cdot \Pi$, and (6.4) therefore follows from the implication $(ii) \implies (i)$ in Theorem D. Note that, using the construction (6.2), we can write

$$\Pi(\pi, \sigma) = P\big(\widehat{M}_{k+2} = \sigma \big| \widehat{M}_k = \pi\big).$$

The aim of this section is to extend (6.4) to mixtures of $\widehat{P}_k$: If K is an integer-valued random variable, we write $\widehat{P}_K$ for the corresponding mixture of $\widehat{P}_1, \widehat{P}_2, \dots$, defined by

$$\widehat{P}_K(\pi) \coloneqq \sum_{k=0}^{\infty} P(K = k) \widehat{P}_k(\pi). \tag{6.5}$$

We prove the following generalization of (6.4):

Theorem 6.1. *Assume that K and L are integer-valued random variables with*

$$\mathbb{P}(K \text{ is even}) = \mathbb{P}(L \text{ is even}) \in \{0,1\}.$$

Furthermore, assume that $K \preceq_{st} L$. Then $\widehat{P}_L \preceq_M \widehat{P}_K$.

Let us first mention that this does not follow from (6.4), because $\preceq_M$ is not stable under taking mixtures.

To see that some assumptions are necessary, it is enough to consider $N = 0$. Define probability measures $\mu_1 := \delta_{\{0\}}$ and $\mu_2 := \delta_{\{1\}}$ on $\{0,1\}$, and random variables K, L by $P(K=1) = P(K=2) = 1/2$ and $P(L=2) = 1$. Then $\mu_2 \preceq_M \mu_1$ and $K \preceq_{st} L$, but

$$\mu_L = \delta_{\{1\}} \not\preceq_M \tfrac{1}{2}(\delta_{\{0\}} + \delta_{\{1\}}) = \mu_K.$$

We will prove Theorem 6.1 with the help of Theorem F, so we start by introducing a partial order $\preceq$ on $\{0,1\}^{[\![N]\!]}$. We set

$$(\pi_0, \ldots, \pi_N) \preceq (\sigma_0, \ldots, \sigma_N) \quad \iff \quad \textstyle\sum_{l=0}^k \pi_l \le \sum_{l=0}^k \sigma_l \quad \text{for all } k = 0, \ldots, N$$

We verify that $\widehat{P}_k$ satisfies the assumptions of Theorem F, for every k:

Lemma 6.2. *Fix $k \in \mathbb{N}$.*

(i) $\widehat{P}_k$ is $\preceq$-decreasing, i.e. $\pi \preceq \sigma$ implies

$$\widehat{P}_k(\pi) \ge \widehat{P}_k(\sigma) \tag{6.6}$$

(ii) Let $A \subseteq \{0,1\}^{[\![N]\!]}$ be $\preceq$-decreasing, i.e. $\sigma \in A$ and $\pi \preceq \sigma$ imply $\pi \in A$. Then

$$\widehat{P}_{k+2}(A) \le \widehat{P}_k(A). \tag{6.7}$$

Both (6.6) and (6.7) are stable under mixtures, so the conclusion follows directly:

Proof of Theorem 6.1. We check that $\widehat{P}_K$ and $\widehat{P}_L$ satisfy the assumptions of Theorem F. First, the definition (6.5) of $\widehat{P}_K$ and (6.6) directly imply that $\widehat{P}_K$ and $\widehat{P}_L$ are $\preceq$-decreasing. Moreover, if A is $\preceq$-decreasing, then $k \mapsto \widehat{P}_k(A)$ is a decreasing function. Therefore $K \preceq_{st} L$ implies

$$\widehat{P}_L(A) \le \widehat{P}_K(A)$$

by the implication $(i) \implies (ii)$ in Theorem A. □

It remains to show the lemma:

Proof of Lemma 6.2. In this proof we will interpret a configuration $\pi \in \{0,1\}^{[\![N]\!]}$ as a subset of $[\![N]\!]$. For $S \subseteq [\![N]\!]$ we write $M_k(S) := \sum_{i \in S} M_k(i)$. We recall the following fact about a binomial random variable $\mathrm{Bin}_{n,p}$ with n trials and success probability p:

$$P(\mathrm{Bin}_{n,p} \text{ is even}) = \frac{1}{2}\big(1 + (1-2p)^n\big). \tag{6.8}$$

Part (i): Let $\pi \preceq \sigma \in \{0,1\}^{[\![N]\!]}$. For $S, T \subseteq [\![N]\!]$ disjoint, we consider the function

$$f_T^S(r) := P\big(M_k(i) \text{ is even } \forall i \in S, M_k(j) \text{ is odd } \forall j \in T \big| M_k(S \cup T) = r\big).$$

Whenever S or T is the empty set, we drop it from the notation and just write f_T or f^S. We first consider two special cases:

Case 1: Assume that $\pi \subseteq \sigma$, and write $\sigma \setminus \pi =: \{a_1, \dots, a_{2m}\}$. Let

$$A := \{\tau \colon \pi \subseteq \tau \subseteq \sigma\},$$

and set $S_j := \{a_{2j-1}, a_{2j}\}$. Then

$$\begin{aligned}\widehat{P}_k(\pi) &= \widehat{P}_k(A)\widehat{E}_k\big[\textstyle\prod_{j=1}^m f^{S_j}(M_k(S_j))\big|A\big]\\ \widehat{P}_k(\sigma) &= \widehat{P}_k(A)\widehat{E}_k\big[\textstyle\prod_{i=1}^m f_{S_j}(M_k(S_j))\big|A\big].\end{aligned}$$

Clearly $f^{S_j}(m)$ is only positive if m is even, and in this case (6.8) implies

$$f^{S_j}(m) = P(\mathrm{Bin}_{m,p}\,\text{is even}) \geq P(\mathrm{Bin}_{m,p}\,\text{is odd}) = f_{S_j}(m)$$

Here $p = p_{a_{2j-1}}/(p_{a_{2j-1}} + p_{a_{2j}})$. This implies $\widehat{P}_k(\pi) \geq \widehat{P}_k(\sigma)$.

Case 2: Assume $|\pi| = |\sigma|$, and that π and σ only differ in two coordinates. That is, $\pi = \alpha \cup \{a\}$ and $\sigma = \alpha \cup \{b\}$ for some $b < a$ and $\alpha \subseteq [\![N]\!]$. Let

$$B := \big\{\tau \subseteq [\![N]\!] \colon \alpha \subseteq \tau \subseteq \alpha \cup \{a, b\}\big\}.$$

and observe

$$\begin{aligned}\widehat{P}_k(\pi) &= \widehat{P}_k(B)\widehat{E}_k\big[f_a^{\,b}\big(M_k(\{a,b\})\big)\big|B\big]\\ \widehat{P}_k(\sigma) &= \widehat{P}_k(B)\widehat{E}_k\big[f_b^{\,a}\big(M_k(\{a,b\})\big)\big|B\big]\end{aligned} \tag{6.9}$$

Observe that $f_a^{\,b}(m) = f_b^{\,a}(m) = 0$ if m is even, and if m is odd

$$f_a^b(m) = P(\mathrm{Bin}_{m,p}\,\text{is even}) \geq P(\mathrm{Bin}_{m,p}\,\text{is odd}) = f_b^a(m)$$

Here $p = p_b/(p_a + p_b)$, and we use that $a > b$ and (6.1) imply $p \leq \frac{1}{2}$. Together with (6.9) we thus also get $\widehat{P}_k(\pi) \geq \widehat{P}_k(\sigma)$.

General case: Observe that for any $\pi \preceq \sigma$, we can find $\pi_0 \preceq \cdots \preceq \pi_r$ such that $\pi_0 = \pi$ and $\pi_r \subseteq \sigma$, and with the property that π_{i+1} and π_i only differ in two coordinates, as defined above. See Figure 4 for an illustration.

Part (ii): We construct a coupling (π_k, π_{k+2}) with marginals $\widehat{P}_k$ and $\widehat{P}_{k+2}$, and such that almost surely

$$\pi_k \preceq \pi_{k+2}. \tag{6.10}$$

Let (B, A) be a random variable with

$$P\big((B, A) = (b, a)\big) = 2p_a p_b \mathbb{1}\{b < a\} + p_a^2 \mathbb{1}\{a = b\}$$

and let M be an independent multinomial random variables with law P_k. On $\{A = B\}$, we define

$$\pi_k = \pi_{k+2} = \big(\mathbb{1}\{M(0) \text{ is odd}\}, \dots, \mathbb{1}\{M(N) \text{ is odd}\}\big). \tag{6.11}$$

On the other hand, on $\{B < A\}$ we define all coordinates of π_k and π_{k+2} except A and B as in (6.11). Set $R \coloneqq M(A) + M(B)$ and $p \coloneqq \frac{p_B}{p_A+p_B}$, and let U be an independent uniform random variable. We define

$$\pi_k(B) \coloneqq \mathbb{1}\{U \le P(\mathrm{Bin}_{R,p} \text{ is odd})\}$$
$$\pi_{k+2}(B) \coloneqq \mathbb{1}\{U \le P(\mathrm{Bin}_{R,p} \text{ is even})\}$$

and, for $l \in \{k, k+2\}$,

$$\pi_l(A) \coloneqq R - \pi_l(B) \mod (2). \tag{6.12}$$

We first check that (π_k, π_{k+2}) has the correct marginals. Note that we can sample a realization of the multinomial distribution M_{k+2} with $k+2$ trials by sampling M together with two additional balls A and B as described above. If the extra balls end up in the same bin, then the parity of all coordinates of M and M_{k+2} will agree, and we can take $\pi_k = \pi_{k+2}$. If they do not fall in the same bin, then adding balls A and B will flip the parity of both $M(A)$ and $M(B)$. So conditioned on $\{M(A) + M(B) = R\}$, $\pi_k(B)$ and $\pi_{k+2}(B)$ indeed have the correct laws, which then forces us to choose $\pi_k(A)$ and $\pi_{k+2}(A)$ as in (6.12).

Next, we check that the coupling property (6.10) holds. Note that (6.1) and $B < A$ imply $p = \frac{p_B}{p_A+p_B} \le \frac{1}{2}$. Together with (6.8), we have

$$P(\mathrm{Bin}_{R,p} \text{ is odd}) = \frac{1}{2} - \frac{1}{2}(1-2p)^R \le \frac{1}{2} \le P(\mathrm{Bin}_{R,p} \text{ is even}).$$

This gives $\pi_k(B) \le \pi_{k+2}(B)$, which implies $\pi_k \preceq \pi_{k+2}$. More precisely, if R is even

$$(\pi_k(B), \pi_k(A)) = (1,1) \implies (\pi_{k+2}(B), \pi_{k+2}(A)) = (1,1)$$

so that $\pi_k \preceq \pi_{k+2}$. If R is odd,

$$(\pi_k(B), \pi_k(A)) = (1,0) \quad \implies \quad (\pi_{k+2}(B), \pi_{k+2}(A)) = (1,0).$$

On the other hand, on $\{(\pi_k(B), \pi_k(A)) = (0,1)\}$ we can have either $\pi_{k+2} = \pi_k$ or $(\pi_{k+2}(B), \pi_{k+2}(A)) = (1,0)$. In both cases $\pi_k \preceq \pi_{k+2}$ holds. $\square$

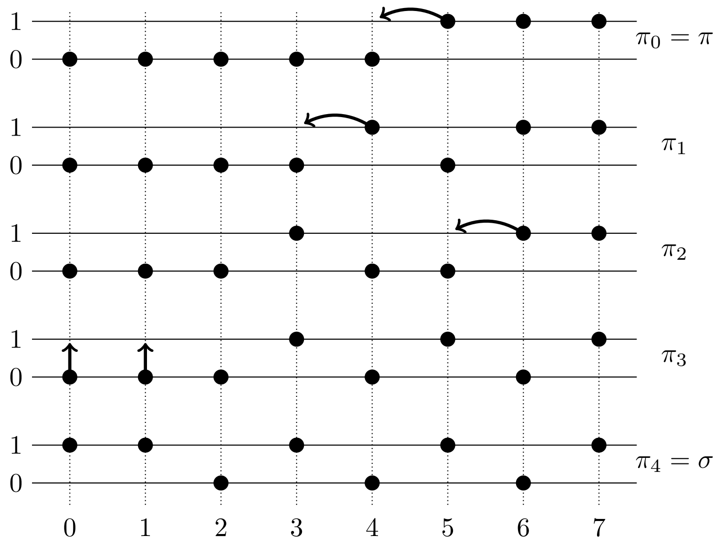

Figure 4: Consider configurations $\pi \coloneqq (0,0,0,0,0,1,1,1)$ and $\sigma \coloneqq (1,1,0,1,0,1,0,1)$, which satisfy $\pi \preceq \sigma$. We find intermediate configurations $\pi = \pi_0 \preceq \pi_1 \preceq \pi_2 \preceq \pi_3 \preceq \pi_4 = \sigma$ such that each intermediate pair falls into case 1 ($\pi_0 \to \pi_1 \to \pi_2 \to \pi_3$) or case 2 ($\pi_3 \to \pi_4$).

6.2. Comparison to smaller state spaces

We show the survival probability is smaller (in the sense of the concave stochastic order) if the environment is more degenerate (i.e., has less randomness). This result is the continuous-time analog to Theorem 4.4, and therefore we only sketch the proof. A different proof using analytic techniques can be found in [55, Lemma 2.2] (under slightly different assumptions).

Assume that $\mathcal{J} = \{J_1, J_2, \dots\}$ is a partition of I. As in Section 4.5, we write $\mathbb{P}_{\mathcal{J}}$ for the law of an environment with the same marginals as $\mathbb{P}$, and such that disasters happen simultaneously at all sites in the same equivalence class. The environment in different equivalence classes is independent.

Lemma 6.3. *Let $\mathcal{J}$ and $\mathcal{K}$ be two partitions of I, and assume that $\mathcal{K}$ is finer than $\mathcal{J}$. That is, for every $K \in \mathcal{K}$ there exists $J \in \mathcal{J}$ such that $K \subseteq J$. Moreover, let $f\colon (0,\infty) \to \mathbb{R}$ be convex and $A \subseteq \mathbb{Z}^d$. Then*

$$\mathbb{E}_{\mathcal{K}}\big[f\big(P(\tau(\omega) \geq t, X(t) \in A)\big)\big] \leq \mathbb{E}_{\mathcal{J}}\big[f\big(P(\tau(\omega) \geq t, X(t) \in A)\big)\big]. \tag{6.13}$$

Proof. Write $\mathcal{J} = \{J_1, J_2, \dots\}$ and $\mathcal{K} = \{K_0, K_1, K_2, \dots\}$. By considering a sequence of intermediate partitions, we can assume $J_1 = K_0 \cup K_1$ and $J_i = K_i$ for all $i \geq 2$. We write

$$Z^{\kappa}_{t,A} \coloneqq P^{\kappa}(\tau(\omega) \geq t, X(t) \in A).$$

We construct a coupling $(\omega, \widehat{\omega})$ with marginals $\mathbb{P}_{\mathcal{K}}$ and $\mathbb{P}_{\mathcal{J}}$, and such that almost surely

$$\mathbb{E}\big[Z^{\kappa}_{t,A}(\widehat{\omega})\big|\sigma(Z^{\kappa}_{t,A}(\omega))\big] = Z^{\kappa}_{t,A}(\omega). \tag{6.14}$$

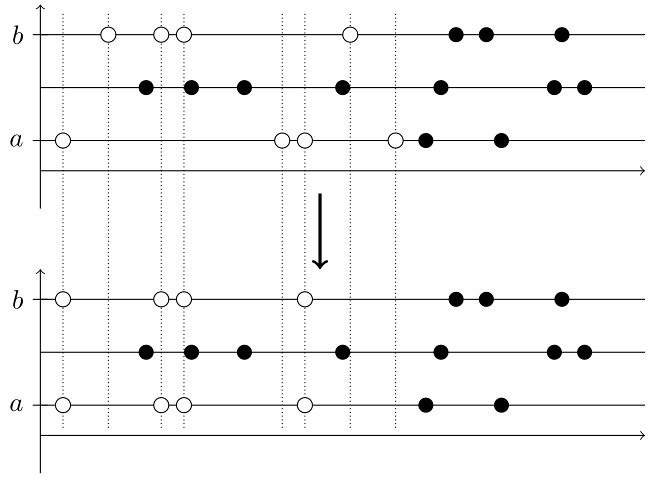

Figure 5: Illustration for the construction in the case $K_0 = \{a\}$, $K_1 = \{b\}$ and $J_1 = \{a, b\}$, with disasters depicted as (black or white) circles. In the environment ω (upper picture) disasters occur at either K_0 or K_1, while in $\omega(0, 0, 1, 1, 1, 0, 0, 1)$ (lower picture) the first 8 disasters are either duplicated or removed.

Let ω have law $\mathbb{P}_{\mathcal{K}}$. Let $T_1 < T_2 < \cdots < T_N$ denote the disaster times at $K_0 \cup K_1$ and $D_1, D_2, \ldots$ the location of the disasters. That is, we set D_j equal to 0 if the disaster at time T_j occurs at K_0, while $D_j = 1$ otherwise.

For $(u_1, \ldots, u_k) \in \{0, 1\}^k$, we write $\omega(u_1, \ldots, u_k)$ for the environment where the disasters corresponding to $T_1, \ldots, T_k$ are either duplicated or removed, according to $u_1, \ldots, u_k$. More precisely, $\omega(u_1, \ldots, u_k)$ contains a disaster at time T_j at $K_0 \cup K_1$ if $D_j = u_j$, while $D_j \neq u_j$ implies that there is no disaster at time T_j. See Figure 5 for an illustration. Furthermore, we write $\overline{\omega}(u_1, \ldots, u_k)$ for the environment where, in addition, the disaster at time T_{k+1} is removed.

We will define $\widehat{\omega} \coloneqq \omega(U_1, U_2, \ldots, U_N)$ for some random choice of $U_1, \ldots, U_N$. More precisely, given $U_1, \ldots, U_k$, we set $U_{k+1} \coloneqq 0$ with probability

$$\alpha(\omega, U_1, \ldots, U_k) \coloneqq \frac{E[Z^{\kappa}_{t,A}(\overline{\omega}(U_1, \ldots, U_k))\mathbb{1}\{X(T_{k+1}) \in K_0\}]}{E[Z^{\kappa}_{t,A}(\overline{\omega}(U_1, \ldots, U_k))\mathbb{1}\{X(T_{k+1}) \in K_0 \cup K_1\}]}$$

and $U_{k+1} \coloneqq 1$ otherwise. Note that in contrast to the discrete-time situation, the denominator is almost surely positive. The same calculation as in (4.11) shows that

$$\mathbb{E}[Z^{\kappa}_{t,A}(\omega(U_1, \ldots, U_k, U_{k+1}))|\sigma(\omega, U_1, \ldots, U_k)] = Z^{\kappa}_{t,A}(\omega(U_1, \ldots, U_k)).$$

Now (6.14) follows the tower-property, and the conclusion follows from the implication $(ii) \implies (i)$ in Theorem C. $\square$

6.3. Survival probability on $\mathbb{Z}/_2$

We estimate the survival probability on $\mathbb{Z}/_2 = \{0, 1\}$, using a variation of the argument from [29, Lemma 2.4].

Lemma 6.4. *Let $I = \{0, 1\}$, and $P^{\bar\kappa}$ the law of a simple random walk X on $\{0, 1\}$ with jump rate $\bar\kappa > 0$, started in 0.*

(i) For every $\delta \in (0, 1)$, there exists $c(\bar\kappa, \delta)$ such that, for $i \in \{0, 1\}$,

$$\mathbb{E}\Big[\big(P^{\bar\kappa}(\tau(\omega) \geq 1, X(1) = i)\big)^{-\delta}\Big] \leq c(\bar\kappa, \delta). \tag{6.15}$$

Moreover, $\sup_{\bar\kappa\in[\bar\kappa_0,\bar\kappa_1]} c(\bar\kappa, \delta) < \infty$ for all $0 < \bar\kappa_0 < \bar\kappa_1 < \infty$.

(ii) $\mathbb{E}\big[\big(P^{\bar\kappa}(\tau(\omega) \geq t)\big)^{-1}\big] = \infty$ for every $t > 0$.

(iii) For every $\delta \in (0, 1)$,

$$\sup_{t\in[0,1]} \mathbb{E}\Big[\big(P^{\bar\kappa}(\tau(\omega) \geq t, X(t) = 0)\big)^{-\delta}\Big] < \infty. \tag{6.16}$$

Proof. **Part (i)**: For $\omega \in \Omega$, let $N := |\omega \cap ([0, 1] \times I)|$ denote the number of disasters until time 1, and observe that N has Poisson distribution with parameter 2. Let $T_1 < \cdots < T_N$ denote the disaster times, and for convenience introduce $T_0 := 0$ and $T_{N+1} := 1$. The interarrival times are denoted by $\Delta_0, \ldots, \Delta_N$, where $\Delta_k := T_{k+1} - T_k$. Finally, let $D_1, \ldots, D_N$ denote the sites of the disasters, and set $D_0 := 0$ and $D_{N+1} := i$. The Markov property for $P^{\bar\kappa}$ implies

$$\begin{aligned} P^{\bar\kappa}(\tau(\omega) \geq 1, X(1) = i) &= \textstyle\prod_{k=0}^{N} P^{\bar\kappa}\big(X(\Delta_k) = D_{k+1} - D_k \mod 2\big) \\ &\geq \textstyle\prod_{k=0}^{N} P^{\bar\kappa}\big(X(\Delta_k) = 1\big) \\ &\geq e^{-\bar\kappa}\bar\kappa^{N+1} \textstyle\prod_{k=0}^{N} \Delta_k \end{aligned}$$

We have used that for simple random walk on $I = \{0, 1\}$, for any $t \geq 0$,

$$P^{\bar\kappa}(X(t) = 0) \geq \tfrac{1}{2} \geq P^{\bar\kappa}(X(t) = 1) \geq e^{-\bar\kappa t}\bar\kappa t.$$

Note that, conditioned on $\{N = n\}$, $(T_1, \ldots, T_n)$ has the same law as the order statistics of n independent random variables, uniformly distributed on $[0, 1]$. Let

$$\begin{aligned} \beta_n(\delta) :=& \frac{1}{n!}\mathbb{E}\Big[\prod_{k=0}^{N} \Delta_k^{-\delta}\Big| N = n\Big] \\ =& \int_0^1 s_1^{-\delta} \int_{s_1}^1 s_2^{-\delta} \cdots \int_{s_n}^1 s_n^{-\delta}(1 - s_n)^{-\delta} \mathrm{d}s_n \ldots \mathrm{d}s_2 \, \mathrm{d}s_1 \\ =& \frac{\Gamma(1 - \delta)^{n+1}}{\Gamma((n + 1)(1 - \delta))}. \end{aligned}$$

For the last equality, observe that $\beta_n(\delta)$ is the normalizing constant in a Dirichlet distribution with $n+1$ categories and parameters $(1-\delta,\dots,1-\delta)$, see [41, Definition 24.25]. Note in particular that $\beta_n(\delta)$ decays superexponentially. Now we can estimate

$$
\begin{aligned}
&\mathbb{E}\Big[\big(P^{\bar\kappa}(\tau(\omega)\ge 1, X(1)=i)\big)^{-\delta}\Big] \\
&\quad\le e^{\bar\kappa\delta}\Big(\mathbb{P}(N=0)\bar\kappa^{-\delta}+\sum_{n\ge 1}\mathbb{P}(N=n)\mathbb{E}\Big[\prod_{k=0}^{N}\Delta_k^{-\delta}\Big|N=n\Big]\bar\kappa^{-(n+1)\delta}\Big) \\
&\quad\le e^{\delta\bar\kappa-2}\Big(\bar\kappa^{-\delta}+\sum_{n\ge 1}2^{-n}\bar\kappa^{-\delta(n+1)}\beta_n(\delta)\Big)=: c(\bar\kappa,\delta).
\end{aligned}
$$

The first factor is increasing in $\bar\kappa$ while the second factor is decreasing, which shows that $c(\bar\kappa,\delta)$ is bounded over compact sets.

Part (iii): This follows by a modification of the previous argument. Let

$$
N_t := |\omega\cap([0,t]\times I)|
$$

and write $\Delta_0^{(t)},\dots,\Delta_{N_t}^{(t)}$ for the interarrival times in $[0,t]$. A substitution shows

$$
\mathbb{E}\Big[\textstyle\prod_{k=0}^{N_t}\big(\Delta_k^{(t)}\big)^{-\delta}\Big|N_t=n\Big]=t^{(1-\delta)n-\delta}\mathbb{E}\Big[\textstyle\prod_{k=0}^{N}\big(\Delta_k^{(1)}\big)^{-\delta}\Big|N_1=n\Big].
$$

Moreover, recalling that $P^{\bar\kappa}(\tau(\omega)\ge t, X(t)=0)\ge 1/2$ on $\{N_t=0\}$, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\big(P^{\bar\kappa}(\tau(\omega)\ge t, X(t)=0)\big)^{-\delta}\Big] \\
&\quad\le \mathbb{P}(N_t=0)2^{\delta}+e^{\bar\kappa\delta}\sum_{n\ge 1}\mathbb{P}(N_t=n)\mathbb{E}\Big[\textstyle\prod_{k=0}^{N_t}\big(\Delta_k^{(t)}\big)^{-\delta}\Big|N_t=n\Big]\bar\kappa^{-(n+1)\delta} \\
&\quad\le e^{-2t}\Big(2^{\delta}+\sum_{n\ge 1}2^n t^{n+(1-\delta)n-\delta}\bar\kappa^{-\delta(n+1)}\beta_n(\delta)\Big).
\end{aligned}
$$

This expression is finite since $\beta_n(\delta)$ decays superexponentially. Moreover, the exponent of t is non-negative for $n\ge 1$, which shows that the last line is bounded for $t\in[0,1]$.

Part (ii): On $\{T_1<t, D_1=0\}$,

$$
P(\tau(\omega)\ge t)\le P(X(T_1)=1)=e^{-\bar\kappa T_1}\sum_{k=0}^{\infty}\frac{(\bar\kappa T_1)^{2k+1}}{(2k+1)!}=e^{-\bar\kappa T_1}\sinh(\bar\kappa T_1).
$$

Note that $\sinh(t)\le 2t$ for $t\le t_0$. Since T_1 is exponentially distributed, we get

$$
\mathbb{E}\big[P(\tau(\omega)\ge t)^{-1}\big]\ge\frac{1}{2}\mathbb{E}\big[T_0^{-1}\mathbb{1}\{T_1<t\wedge t_0, D_1=0\}\big]=\infty.
$$

□

Remark 6.5. The same proof shows that the non-integrability of $P^{\kappa}(\tau(\omega)\ge t)^{-1}$ also holds with state space $I=\mathbb{Z}^d$. This should be compared with Proposition 11.2 for the continuous-space version of this model.

6.4. The uniform moment bound

We define an equivalence relation $\equiv$ on $\mathbb{Z}^d$ by

$$(i_1, \dots, i_d) \equiv (j_1, \dots, j_d) \quad \Longleftrightarrow \quad i_1 = j_1 \mod 2.$$

Let J_0 and J_1 denote the two equivalence classes of $\equiv$, with $0 \in J_0$ and $e_1 \in J_1$. We write $\mathbb{P}_\equiv$ for the law of the degenerate environment corresponding to partition $\mathcal{J} = \{J_0, J_1\}$, as introduced in the beginning of Section 6.2.

Proof of Proposition 5.2. Observe that if ω has law $\mathbb{P}_\equiv$, then $P^{\kappa/2}(\tau(\omega) \geq t)$ has the same law as the survival probability of a simple random walk on $\mathbb{Z}/_2$ with jump rate $\kappa/(2d)$. Moreover, $P^{\kappa/2}(\tau(\omega) \geq 1, X(1) \in J_i)$ has the same law as the quantity in (6.15), for $i \in \{0, 1\}$, where $\bar{\kappa} = \kappa/(2d)$. Now for $i \in \mathbb{Z}^d$,

$$\begin{aligned} &\mathbb{E}\Big[\big(P^\kappa(\tau(\omega) \geq 1 | X(1) = i)\big)^{-\delta}\Big] \\ \leq\ &\mathbb{E}_\equiv\Big[\big(P^\kappa(\tau(\omega) \geq 1 | X(1) = i)\big)^{-\delta}\Big] &(6.17) \\ \leq\ &\mathbb{E}_\equiv\Big[\big(P^{\kappa/2}(\tau(\omega) \geq 1 | X(1) \equiv i)\big)^{-\delta}\Big] &(6.18) \\ \leq\ &c\big(\kappa/(2d), \delta\big). &(6.19) \end{aligned}$$

Indeed, after multiplying both sides by $P^\kappa(X(t) = i)^{-\delta}$, we see that (6.17) is (6.13) with $f(x) = x^{-\delta}$ while (6.19) is (6.15). Finally, (6.18) will be proved in Lemma 6.6 below. The boundedness of $C(\kappa, \delta) := c(\kappa/(2d), \delta)$ over compact sets was shown in Lemma 6.4. $\square$

Lemma 6.6. *For all $i \in \mathbb{Z}^d$ and $f\colon (0, 1] \to \mathbb{R}$ convex*

$$\mathbb{E}_\equiv\big[f\big(P^\kappa(\tau(\omega) \geq 1 | X(1) = i)\big)\big] \leq \mathbb{E}_\equiv\big[f\big(P^{\kappa/2}(\tau(\omega) \geq 1 | X(1) \equiv i)\big)\big].$$

Proof. We treat the case $i \equiv 0$, since $i \equiv e_1$ is identical. We write

$$\omega \cap ([0, 1] \times \{0, e_1\}) = \big\{(T_1, D_1), \dots, (T_N, D_N)\big\}$$

where $T_1 < \dots < T_N$. Let $\mathcal{F} := \sigma(N, T_1, \dots, T_N)$, and write $\mathbb{P}_\equiv^{T_1,\dots,T_N}$ resp. $\mathbb{E}_\equiv^{T_1,\dots,T_N}$ for the conditional law resp. expectation with respect to $\mathcal{F}$. Note that $(D_1, \dots, D_N)$ is uniformly distributed on $\{0, e_1\}^N$ under $\mathbb{P}_\equiv^{T_1,\dots,T_N}$. We define two configurations in $\{0, 1\}^{[\![N]\!]}$:

$$\begin{aligned} \Pi(\omega) &:= (\mathbb{1}_{D_1 \equiv 0}, \mathbb{1}_{D_2 \not\equiv D_1}, \dots, \mathbb{1}_{D_N \not\equiv D_{N-1}}, \mathbb{1}_{D_N \equiv i}) \\ \mathcal{R}(X) &:= (\mathbb{1}_{X(T_1) \not\equiv 0}, \mathbb{1}_{X(T_2) \not\equiv X(T_1)}, \dots, \mathbb{1}_{X(T_N) \not\equiv X(T_{N-1})}, \mathbb{1}_{X(1) \not\equiv X(T_N)}). \end{aligned}$$

Note that $\mathcal{R}(X)(i) = 1$ if X switches sites in $[T_i, T_{i+1})$, for $i \in [\![N]\!]$. Moreover, $\Pi(\omega)(i)$ is equal to one if and only if $\{\tau(\omega) \geq 1, X(1) \equiv i\}$ requires X to switch sites in $[T_i, T_{i+1})$ (using $T_0 := 0$ and $T_{N+1} := 1$ for convenience). That is, if ω has law $\mathbb{P}_\equiv$, then

$$\{\tau(\omega) \geq 1, X(1) \equiv i\} = \{\mathcal{R}(X) = \Pi(\omega)\}.$$

Moreover, recalling the definition of $\mathcal{E}_{even}$ from (6.3), we observe

$$P^{\kappa}(\mathcal{R}(X) \in \mathcal{E}_{even}|X(1) = i) = P^{\kappa/2}(\mathcal{R}(X) \in \mathcal{E}_{even}|X(1) \equiv i) = \mathbb{P}(\Pi(\omega) \in \mathcal{E}_{even}) = 1.$$

For the first two probabilities, this follows from $i \equiv 0$, and for $\Pi(\omega)$ this is a direct consequence of the definition. Now, since $\Pi(\omega)$ is uniformly distributed on $\mathcal{E}_{even}$ under $\mathbb{P}_{\equiv}^{T_1,\dots,T_N}$, we can write, for any $f := (0,1] \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}_{\equiv}^{T_1,\dots,T_N}\left[f(P^{\kappa}(\tau(\omega) \geq 1|X(1) = i))\right] &= 2^{-N} \sum_{\pi \in \mathcal{E}_{even}} f(p(\pi)) \\ \mathbb{E}_{\equiv}^{T_1,\dots,T_N}\left[f(P^{\kappa/2}(\tau(\omega) \geq 1|X(1) \equiv i))\right] &= 2^{-N} \sum_{\pi \in \mathcal{E}_{even}} f(q(\pi)) \end{aligned} \tag{6.20}$$

where p and q are probability measures on $\mathcal{E}_{even}$ satisfying, for $\pi \in \{0,1\}^{[\![N]\!]}$,

$$\begin{aligned} p(\pi_0, \dots, \pi_N) &:= P^{\kappa}\big(\mathcal{R}(X) = (\pi_0, \dots, \pi_N)\big|X(1) = i\big) \\ q(\pi_0, \dots, \pi_N) &:= P^{\kappa/2}\big(\mathcal{R}(X) = (\pi_0, \dots, \pi_N)\big|X(1) \equiv i\big). \end{aligned}$$

Let $\#X$ denote the number of jumps of X in $[0,1]$, i.e.

$$\#X := \big|\{t \in [0,1] \colon X(t) \not\equiv X(t^-)\}\big|,$$

and define

$$\widehat{P}_k(\pi_0, \dots, \pi_N) := P\big(\mathcal{R}(X) = (\pi_0, \dots, \pi_N)|\#X = k\big).$$

We observe that this definition agrees with the definition from Section 6.1: Indeed, conditioned on $\#X = 2k$, each jump occurs in $[T_l, T_{l+1})$ with probability $T_{l+1} - T_l$, independently of the other jumps, and the process switches sites in $[T_l, T_{l+1})$ if and only if there is an odd number of jumps in $[T_l, T_{l+1})$.

Moreover, observe that p and q are mixtures of $\widehat{P}_k$, as defined in (6.5). More precisely, let K and L be integer-valued random variables with

$$\begin{aligned} P(K = 2k) &:= P^{\kappa/2}(\#X = 2k|X(1) \equiv i) \\ P(L = 2k) &:= P^{\kappa}(\#X = 2k|X(1) = i), \end{aligned} \tag{6.21}$$

and note that $p = \widehat{P}_K$ and $q = \widehat{P}_L$. In Lemma 6.7 below we will prove $K \preceq_{st} L$, and thus the conclusion follows from (6.20) and Theorem 6.1. □

Lemma 6.7. *Let K and L be as in (6.21). Then $K \preceq_{st} L$*

Proof. It is easier to show $K \preceq_{lr} L$, which by Theorem B implies $K \preceq_{st} L$. We have to check that for k, l even with $|i_1| \leq k \leq l$,

$$\begin{aligned} &P^{\kappa}(\#X = k|X(1) = i)P^{\kappa/2}(\#X = l|X(1) \equiv i) \\ &\quad \leq P^{\kappa}(\#X = l|X(1) = i)P^{\kappa/2}(\#X = k|X(1) \equiv i) \end{aligned}$$

We apply the definition of conditional probability and cancel the terms that appear on both sides, noting also that $P^{\kappa/2}(X(1) \equiv i | \#X = l) = 1$. Then we can rewrite the equation as

$$\frac{P(Z_k = i_1)}{P(Z_l = i_1)} \leq \frac{P\big(\operatorname{Poi}(\frac{\kappa}{d}) = l\big) P\big(\operatorname{Poi}(\frac{\kappa}{2d}) = k\big)}{P\big(\operatorname{Poi}(\frac{\kappa}{d}) = k\big) P\big(\operatorname{Poi}(\frac{\kappa}{2d}) = l\big)} = 2^{l-k}.$$

Here $(Z_t)_{t \in \mathbb{N}}$ is a discrete time simple random walk on $\mathbb{Z}$, and we write $\operatorname{Poi}(\lambda)$ for a Poissonian random variable with parameter λ. But this inequality holds, since by the Markov property

$$P(Z_l = i_1) \geq P(Z_k = i_1) P(Z_{l-k} = 0) \geq P(Z_k = i_1) 2^{-(l-k)}.$$

$\square$

7. Proof of the main results

The ideas up to this point have been specific to the random walk among disasters discussed in this part. We had to developed careful estimates to take care of the strong degeneracy caused by considering a disastrous environment, while at other times properties of the model have simplified the analysis – for example, we have often used without comment that $t \mapsto \log P^\kappa(\tau(\omega) \geq t)$ is non-positive and decreasing.

We will now apply these estimates to prove the main results from Section 5.4. For the most part, the techniques in this section are variations of well-known ideas from the literature on random polymers and related models. To emphasize the general nature of the arguments, and to make the notation more concise, we will often write

$$\begin{aligned} Z_t^\kappa(\omega) &:= P^\kappa(\tau(\omega) \geq t) \\ Z_{t,i}^\kappa(\omega) &:= P^\kappa(\tau(\omega) \geq t, X(t) = i). \end{aligned} \tag{7.1}$$

7.1. The concentration inequality

We follow the argument from [18, Proposition 3.2.1].

Proof of Proposition 5.3. We will only prove the first statement, since the proof of the concentration inequality for $\log Z_t^\kappa$ is almost identical. Let $\mathcal{F}_t = \sigma(\omega \cap [0,t] \times \mathbb{Z}^d)$, and note that

$$\log Z_{t,i}^\kappa(\omega) - \mathbb{E}[\log Z_{t,i}^\kappa(\omega)] = \textstyle\sum_{s=1}^t V_s,$$

where $V_s := \mathbb{E}[\log Z_{t,i}^\kappa(\omega) | \mathcal{F}_s] - \mathbb{E}[\log Z_{t,i}^\kappa(\omega) | \mathcal{F}_{s-1}]$. For $T \subseteq \mathbb{R}_+$, we define a truncated environment ω_T by

$$\omega_T := \omega \cap (T \times \mathbb{Z}^d).$$

That is, $\omega_{[s-1,s)^c}$ agrees with the untruncated environment ω, except for $[s-1,s)\times\mathbb{Z}^d$, where $\omega_{[s-1,s)^c}$ does not have any disasters. Observe that

$$\mathbb{E}[\log Z^\kappa_{t,i}(\omega_{[s-1,s)^c})|\mathcal{F}_{s-1}] = \mathbb{E}[\log Z^\kappa_{t,i}(\omega_{[s-1,s)^c})|\mathcal{F}_s]$$

and therefore

$$V_s = \mathbb{E}\Big[\log\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big|\mathcal{F}_s\Big] - \mathbb{E}\Big[\log\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big|\mathcal{F}_{s-1}\Big].$$

Note that both expectations are non-positive. We compute, using Jensen's inequality,

$$\mathbb{E}\big[e^{\frac{1}{2}V_s}\big|\mathcal{F}_{s-1}\big] \le \mathbb{E}\Big[\mathbb{E}\Big[\Big(\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big)^{-1/2}\Big|\mathcal{F}_{s-1}\Big]\mathcal{F}_{s-1}\Big] = \mathbb{E}\Big[\Big(\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big)^{-1/2}\Big|\mathcal{F}_{s-1}\Big],$$

and similarly,

$$\mathbb{E}\big[e^{-\frac{1}{2}V_s}\big|\mathcal{F}_{s-1}\big] \le \mathbb{E}\Big[\mathbb{E}\Big[\Big(\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega|_{[s-1,s)^c})}\Big)^{-1/2}\Big|\mathcal{F}_{s}\Big]\mathcal{F}_{s-1}\Big] = \mathbb{E}\Big[\Big(\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big)^{-1/2}\Big|\mathcal{F}_{s-1}\Big].$$

Now observe that we can write

$$\begin{aligned}\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})} &= P^\kappa(\tau(\omega)\ge t\big|\tau(\omega_{[s-1,s)^c})\ge t, X(t)=i\big)\\ &= \sum_{j,k\in\mathbb{Z}^d}\nu_{j,k}(\omega)\alpha_{j,k}(\omega)\end{aligned}$$

where

$$\begin{aligned}\nu_{j,k}(\omega) &:= P^\kappa\big(X(s-1)=j, X(s)=k\big|\tau(\omega_{[s-1,s)^c})\ge t, X(t)=i\big)\\ \alpha_{j,k}(\omega) &:= P^\kappa\big(\tau(\omega)\ge t\big|\tau(\omega_{[s-1,s)^c})\ge t, X(s-1)=j, X(s)=k\big).\end{aligned}$$

Clearly

$$\alpha_{j,k}(\omega) \overset{\mathrm{d}}{=} P^\kappa(\tau(\omega)\ge 1|X(1)=k-j),$$

so that we can apply the uniform moment bound from Proposition 5.2. More precisely, let $\mathcal{F}^* := \sigma(\omega_{[s-1,s)^c}) \supseteq \mathcal{F}_{s-1}$, and observe that $\nu_{j,k}(\omega)$ is $\mathcal{F}^*$-measurable while $\alpha_{j,k}(\omega)$ is independent of $\mathcal{F}^*$, for any j,k. Then Proposition 5.2 shows that, almost surely,

$$\begin{aligned}\mathbb{E}\Big[\Big(\frac{Z^\kappa_{t,i}(\omega)}{Z^\kappa_{t,i}(\omega_{[s-1,s)^c})}\Big)^{-1/2}\Big|\mathcal{F}^*\Big] &\le \sum_{j,k}\nu_{j,k}(\omega)\mathbb{E}\big[\alpha_{j,k}(\omega)^{-1/2}\big]\\ &\le \sup_{j,k}\mathbb{E}\big[\alpha_{j,k}(\omega)^{-1/2}\big]\\ &\le C(\kappa,1/2).\end{aligned}$$

The first inequality follows from Jensen's inequality. The tower property then shows that

$$\mathbb{E}\big[e^{\frac{1}{2}|V_t|}\big|\mathcal{F}_{k-1}\big] \le C(\kappa,1/2)$$

almost surely, and the conclusion follows from Theorem I. □

Remark 7.1. Let us quickly sketch the necessary changes to get rid of the assumption $t \in \mathbb{N}$. Writing

$$A_t(\omega) := \left(Z^\kappa_{\lfloor t \rfloor, i}(\omega)\right)^{-1} Z^\kappa_{t,i}(\omega)$$

we observe

$$\begin{aligned} &\left\{|\log Z^\kappa_{t,i}(\omega) - \mathbb{E}[\log Z^\kappa_{t,i}]| > 2ut\right\} \\ &\quad \subseteq \left\{|\log Z^\kappa_{\lfloor t \rfloor,i}(\omega) - \mathbb{E}[\log Z^\kappa_{\lfloor t \rfloor,i}]| > ut\right\} \cup \left\{|\log A_t - \mathbb{E}[\log A_t]| > ut\right\}. \end{aligned} \tag{7.2}$$

Moreover, note that

$$A_t \geq P^\kappa\big(\tau(\omega) \geq t, X(t) = i \big| \tau(\omega) \geq \lfloor t \rfloor, X(\lfloor t \rfloor) = i\big) \overset{\mathrm{d}}{=} Z^\kappa_{t-\lfloor t \rfloor, 0}$$

Thus, to bound the probability of the second event in (7.2), it is enough to observe that there exist $C > 0$ such that, for all $t > 0$,

$$\mathbb{E}[\log A_t] \geq -2 \log \mathbb{E}\big[A_t^{-1/2}\big] \geq -2 \log \mathbb{E}\big[(Z^\kappa_{t-\lfloor t \rfloor, 0})^{-1/2}\big] > -C.$$

We have used Jensen's inequality and (6.16).

7.2. Continuity of the free energy

The continuity of $\kappa \mapsto \mathfrak{p}(\kappa)$ follows by adapting an argument of [58] for first-passage percolation. We start with the following Lemma:

Lemma 7.2. *There exist t_0 and C_0 such that, for all $\kappa \in [\kappa_0, \kappa_1]$ and $t \geq t_0$,*

$$\mathbb{E}[\log Z^\kappa_{2t}] \leq 2\mathbb{E}[\log Z^\kappa_t] + C_0 t^{1/2+\varepsilon}. \tag{7.3}$$

Proof. First note that for all $t, \kappa > 0$

$$\begin{aligned} \mathbb{E}[\log Z^\kappa_t(\omega)] &\geq \mathbb{E}[\log P^\kappa(\tau(\omega) \geq \lceil t \rceil)] \\ &\geq \mathbb{E}\Big[\log \textstyle\prod_{k=1}^{\lceil t \rceil} P^\kappa\big(\tau(\theta^{k-1,0}\omega) \geq 1, X(1) = 0\big)\Big] \\ &\geq -2\lceil t \rceil \log \mathbb{E}\big[\big(P^\kappa(\tau(\omega) \geq 1, X(1) = 0)\big)^{-1/2}\big] \\ &\geq -2\lceil t \rceil \log C(\kappa, 1/2). \end{aligned} \tag{7.4}$$

Here $\theta^{t,j}\omega$ is the shifted environment, with $(s,k) \in \theta^{t,j}\omega$ if and only if $(s+t, j+k) \in \omega$. In the second line we used Jensen's inequality and the third inequality follows from the uniform moment bound 5.4. Thus, using standard large deviation estimates, we can choose γ large enough that

$$\sup_{\kappa \in [\kappa_0, \kappa_1]} \limsup_{t \to \infty} \frac{1}{t} \log P^\kappa(\|X(t)\| \geq \gamma t) \leq -2 \sup_{\kappa \in [\kappa_0, \kappa_1]} \log C(\kappa, 1/2) - 1. \tag{7.5}$$

We have also used that $C(\kappa, 1/2)$ is bounded over compact intervals. Now we have

$$Z_{2t}^{\kappa}(\omega) \leq \sum_{\|i\| \leq \gamma t} Z_{t,i}^{\kappa}(\omega) Z_t^{\kappa}(\theta^{t,i}\omega) + P^{\kappa}(\|X(t)\| \geq \gamma t).$$

We consider events

$$\begin{aligned}
\mathcal{B}_0 &:= \Big\{ Z_t^{\kappa}(\theta^{t,0}\omega) = \max\big\{ Z_t^{\kappa}(\theta^{t,i}\omega) \colon \|i\| \leq t\gamma \big\} \Big\}, \\
\mathcal{B}_1 &:= \big\{ |\log Z_t^{\kappa}(\omega) - \mathbb{E}[\log Z_t^{\kappa}]| \leq t^{1/2+\varepsilon} \big\}, \\
\mathcal{B}_2 &:= \big\{ |\log Z_t^{\kappa}(\theta^{t,0}\omega) - \mathbb{E}[\log Z_t^{\kappa}]| \leq t^{1/2+\varepsilon} \big\}, \\
\mathcal{B}_3 &:= \big\{ |\log Z_{2t}^{\kappa}(\omega) - \mathbb{E}[\log Z_{2t}^{\kappa}]| \leq (2t)^{1/2+\varepsilon} \big\}.
\end{aligned}$$

Since $\{Z_t^{\kappa}(\theta^{t,i}\omega) \colon i \in \mathbb{Z}^d\}$ is stationary,

$$\mathbb{P}(\mathcal{B}_0) \geq (1 + 2\gamma t)^{-d}.$$

On the other hand, by Proposition 5.3, for all

$$t \geq t_0' := \Big(2 \sup_{\kappa \in [\kappa_0, \kappa_1]} C(\kappa, 1/2) \Big)^{-\frac{1}{1/2-\varepsilon}}$$

we get

$$\mathbb{P}(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \geq 1 - 6 \exp\Big(- \frac{t^{2\varepsilon}}{4(1+\sqrt{2})^2 \sup_{\kappa \in [\kappa_0, \kappa_1]} C(\kappa, 1/2)} \Big).$$

We thus find $t_0'' \geq t_0'$ such that $\mathcal{B} := \mathcal{B}_0 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_4$ has positive probability, for all $t \geq t_0''$ and all $\kappa \in [\kappa_0, \kappa_1]$. In particular, $\mathcal{B} \neq \varnothing$ and we can choose $\omega \in \mathcal{B}$. From $\omega \in \mathcal{B}_0$,

$$\begin{aligned}
Z_{2t}^{\kappa}(\omega) &\leq \sum_{\|i\| \leq t\gamma} Z_{t,i}^{\kappa}(\omega) Z_t^{\kappa}(\theta^{t,i}\omega) + P^{\kappa}(\|X(t)\| \geq \gamma t) \\
&\leq Z_t^{\kappa}(\omega) Z_t^{\kappa}(\theta^{0,t}\omega) + P^{\kappa}(\|X(t)\| \geq \gamma t).
\end{aligned}$$

Moreover, since $\omega \in \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$, we get can replace the logarithm of the probabilities by their expectation with the error terms, which gives

$$\mathbb{E}[\log Z_{2t}^{\kappa}] - (2t)^{1/2+\varepsilon} \leq 2\mathbb{E}[\log Z_t^{\kappa}] + 2t^{1/2+\varepsilon} + \log\Big(1 + \frac{P^{\kappa}(\|X(t)\| \geq \gamma t)}{e^{2\mathbb{E}[\log Z_t^{\kappa}] - 2t^{1/2+\varepsilon}}} \Big).$$

By (7.5) and (7.4), the second term in the logarithm converges to zero exponentially fast, uniformly in κ. ☐

Proof of Proposition 5.5. **Part (i)**: First note that the upper bound follows directly from the definition

$$\mathfrak{p}(\kappa) := \sup_{t>0} \frac{1}{t} \mathbb{E}[\log Z_t^{\kappa}]$$

in the superadditive lemma. For the lower bound, we apply (7.3) repeatedly, which shows that for any $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[\log Z_t^\kappa] &\geq \frac{1}{2}\,\mathbb{E}\big[\log Z_{2t}^\kappa\big] - C_0 t^{1/2+\varepsilon} \\ &\geq \frac{1}{4}\,\mathbb{E}\big[\log Z_{4t}^\kappa\big] - C_0 t^{1/2+\varepsilon} 2^{-2(1/2-\varepsilon)} - C_0 t^{1/2+\varepsilon} \\ &\geq \cdots \\ &\geq \frac{1}{2^k}\,\mathbb{E}\big[\log Z_{2^k t}^\kappa\big] - C_0 t^{1/2+\varepsilon} \textstyle\sum_{i=0}^{k-1} 2^{-i(1/2-\varepsilon)}\end{aligned}$$

Since $\varepsilon \in (0, 1/2)$, the sum in the last line converges for $k \to \infty$, and we get

$$\frac{\mathbb{E}[\log Z_t^\kappa]}{t} + Ct^{-1/2+\varepsilon} \geq \lim_{k\to\infty} \frac{\mathbb{E}[\log Z_{2^k t}^\kappa]}{2^k t} = \mathfrak{p}(\kappa).$$

Part (ii): Take a sequence $(\kappa_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \kappa_n = \kappa_\infty$. Using part (i), we find $C > 0$ such that for $n \in \mathbb{N} \cup \{\infty\}$ and $t \geq t_0$,

$$\mathfrak{p}(\kappa_n) - Ct^{-1/4} \leq \frac{1}{t}\mathbb{E}[\log Z_t^{\kappa_n}] \leq \mathfrak{p}(\kappa_n).$$

Now fix $\delta > 0$, and choose $t \geq t_0$ large enough for $Ct^{-1/4} < \delta$. Then, for all $n \in \mathbb{N}$,

$$|\mathfrak{p}(\kappa_n) - \mathfrak{p}(\kappa_\infty)| \leq 2\delta + \frac{1}{t}\big|\mathbb{E}[\log Z_t^{\kappa_n}] - \mathbb{E}[\log Z_t^{\kappa_\infty}]\big|.$$

The second term now converges to zero for $n \to \infty$. To see this, note that almost surely

$$\lim_{n\to\infty} P^{\kappa_n}(\tau(\omega) \geq t) = P^{\kappa_\infty}(\tau(\omega) \geq t)$$

Moreover, Lemmas 6.4 and 6.3 imply that $(\log Z_t^{\kappa_n})_{n\in\mathbb{N}\cup\infty}$ is bounded in L^2. In particular, the family is uniformly integrable and we can interchange expectation and limit. □

7.3. Existence of the point-to-point free energy

To show the existence of $\mathfrak{p}(\kappa, i)$, we will show that

$$t \mapsto \mathbb{E}[\log Z_{t,\lfloor ti\rfloor}^\kappa(\omega)]$$

is almost-superadditive. The following Lemma is needed to take care of a small discretization error that appears because in general $\lfloor (s+t)i\rfloor \neq \lfloor si\rfloor + \lfloor ti\rfloor$.

Lemma 7.3. *For every $i \in \mathbb{R}^d$, there exists $C, t_0 > 0$ such that for all $t \geq t_0$ and all $j \in \mathbb{Z}^d$ with $\|j\|_\infty \leq 1$,*

$$-C(1 + \log^+ t) \leq \mathbb{E}[\log Z_{t,\lfloor ti\rfloor}^\kappa] - \mathbb{E}[\log Z_{t,\lfloor ti\rfloor+j}^\kappa] \leq C(1 + \log^+ t). \tag{7.6}$$

Proof. It is enough to prove the claim for $j = e_1$. Observe that (see [39, p. 359]) there exists $C > 0$ such that for all $\|k\| \le t$,

$$\frac{P^\kappa(X(1) = k)}{P^\kappa(X(1) = k + e_1)} \in \Big[\frac{1}{Ct}, Ct\Big].$$

This implies that, for all $\|k\| \le \gamma t$,

$$\frac{P^\kappa\big(X(t) = \lfloor ti \rfloor \big| X(t-1) = k\big)}{P^\kappa\big(X(t) = \lfloor ti \rfloor + e_1 \big| X(t-1) = k\big)} \in \Big[\frac{1}{Ct\gamma}, Ct\gamma\Big].$$

Moreover, a similar calculation as in (7.4) shows that there exists $c > 0$ such that, for $k \in \{0, e_1\}$,

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log Z^\kappa_{t,\lfloor ti \rfloor + k}\big] \ge -c.$$

Let γ be large enough that

$$\limsup_{t\to\infty} P^\kappa(\|X(t)\| \ge \gamma t) \le -c - 1. \tag{7.7}$$

Finally, writing

$$A_t(\omega) := (Z^\kappa_{t,\lfloor ti \rfloor})^{-1} P^\kappa(\tau(\omega) \ge t-1, X(t) = \lfloor ti \rfloor),$$

we have

$$\begin{aligned}
Z^\kappa_{t,\lfloor ti \rfloor} A_t &\ge P^\kappa\big(\tau(\omega) \ge t-1, \|X(t-1)\| \le \gamma(t-1), X(t) = \lfloor ti \rfloor\big) \\
&= \sum_{\|k\| \le \gamma(t-1)} Z^\kappa_{t-1,k} P^\kappa\big(X(t) = \lfloor ti \rfloor \big| X(t-1) = k\big) \\
&\ge (Ct\gamma)^{-1} \sum_{\|k\| \le \gamma(t-1)} Z^\kappa_{t-1,k} P^\kappa\big(X(t) = \lfloor ti \rfloor + e_1 \big| X(t-1) = k\big) \\
&= (Ct\gamma)^{-1} P^\kappa\big(\tau(\omega) \ge t-1, \|X(t-1)\| \le \gamma(t-1), X(t) = \lfloor ti \rfloor + e_1\big) \\
&\ge (Ct\gamma)^{-1}\Big(Z^\kappa_{t,\lfloor ti \rfloor + e_1} - P^\kappa\big(\|X(t-1)\| > \gamma(t-1)\big)\Big).
\end{aligned}$$

Taking logarithm and expectation gives

$$\begin{aligned}
\mathbb{E}[\log Z^\kappa_{t,\lfloor ti \rfloor}] \ge \mathbb{E}[\log Z^\kappa_{t,\lfloor ti \rfloor + e_1}] - \mathbb{E}[\log A_t] - C(1 + \log^+ t) \\
+ \log\Big(1 - \frac{P^\kappa(\|X(t-1)\| > \gamma(t-1))}{\mathbb{E}[\log Z^\kappa_{t,\lfloor ti \rfloor + e_1}]}\Big)
\end{aligned}$$

Our choice of γ in (7.7) ensures that the second term in the logarithm converges to zero exponentially fast. Moreover, the same calculation as in the proof of Proposition 5.3 shows that, uniformly in t,

$$\mathbb{E}[\log A_t] = -\mathbb{E}\big[\log P^\kappa\big(\tau(\omega) \ge t \big| \tau(\omega) \ge t-1, X(t) = \lfloor ti \rfloor\big)\big] \le C.$$

This shows the lower bound in (7.6). The argument for the upper bound is identical.

□

Proof of Proposition 5.6. **Part (i)**: Fix $i \in \mathbb{R}^d$, and recall that $\theta^{t,k}\omega$ denotes the environment shifted in space-time by (t,k). From the Markov property of P^κ,

$$Z^\kappa_{s+t,\lfloor (s+t)i\rfloor}(\omega) \geq Z^\kappa_{s,\lfloor si\rfloor}(\omega) Z^\kappa_{t,\lfloor (s+t)i\rfloor - \lfloor si\rfloor}(\theta^{s,\lfloor si\rfloor}\omega)$$

Noting that $\big\| \lfloor si + ti\rfloor - \lfloor si\rfloor - \lfloor ti\rfloor \big\|_\infty \leq 1$, we can thus apply the perturbation result from Lemma 7.3. We obtain that for all $t \geq t_0$,

$$\mathbb{E}[\log Z^\kappa_{s+t,\lfloor (s+t)i\rfloor}(\omega)] \geq \mathbb{E}[\log Z^\kappa_{s,\lfloor si\rfloor}(\omega)] + \mathbb{E}[\log Z^\kappa_{t,\lfloor ti\rfloor}(\omega)] - b_{s+t},$$

where $b_t \coloneqq C(1+\log^+ t)$. Therefore $t \mapsto \mathbb{E}[\log Z^\kappa_{t,\lfloor ti\rfloor}]$ is almost-superadditive in the sense of (3.6). The claim follows by Theorem H.

To show that $(\frac{1}{t}\log Z^\kappa_{t,\lfloor ti\rfloor})_{t\in\mathbb{N}}$ converges to $\mathfrak{p}(\kappa,i)$ almost surely, note that, by Proposition 11.4 and the Borel–Cantelli lemma, we find $t_0(\omega) \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ with $t \geq t_0$,

$$\big|\log Z^\kappa_{t,\lfloor ti\rfloor} - \mathbb{E}[\log Z^\kappa_{t,\lfloor ti\rfloor}]\big| \leq t^{3/4}.$$

Finally, we show that the almost sure convergence cannot hold without the restriction to the integers. For simplicity, we only discuss the case $i=0$. Let $T_1 < T_2 < \dots$ denote the disaster times at the origin, and define $t_n \coloneqq T_n + e^{-n^2}$. Observe that

$$\{\tau(\omega) \geq t_n, X(t_n) = 0\} \subseteq \{X(T_n) \neq X(t_n)\},$$

and since $P^\kappa(X(t)\neq 0) \sim \kappa t$ for $t \downarrow 0$, we have

$$Z^\kappa_{t_n,0} \leq P^\kappa(X(t_n) \neq X(T_n)) \leq C\kappa e^{-n^2}.$$

This shows $\liminf_{t\to\infty} \log Z^\kappa_{t,0} \leq \liminf_{n\to\infty} -\frac{n^2}{t_n} = -\infty$ almost surely. For the last equality, we use that $\frac{t_n}{n} \to 1$ almost surely by the law of large numbers.

Part (ii): Fix $i,j \in \mathbb{R}^d$. Similarly to before, the Markov property for P^κ gives

$$Z^\kappa_{2t,\lfloor ti+tj\rfloor}(\omega) \geq Z_{t,\lfloor ti\rfloor}(\omega) Z^\kappa_{t,\lfloor ti+tj\rfloor - \lfloor ti\rfloor}(\theta^{t,\lfloor ti\rfloor}\omega).$$

Applying again the perturbation result from Lemma 7.3, we thus get

$$\mathbb{E}[\log Z_{2t,\lfloor ti+tj\rfloor}] \geq \mathbb{E}[\log Z_{t,\lfloor ti\rfloor}] + \mathbb{E}[\log Z_{t,\lfloor tj\rfloor}] - C(1+\log^+ t).$$

Dividing by $2t$ and taking the limit $t\to\infty$ then shows

$$\mathfrak{p}(\kappa, i+j) \geq \frac{1}{2}\mathfrak{p}(\kappa,i) + \frac{1}{2}\mathfrak{p}(\kappa,j).$$

Part (iii): From the definition it is clear that $\mathfrak{p}(\kappa,0) \leq \mathfrak{p}(\kappa)$. For the proof of

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}[\log Z^\kappa_{t,0}] \geq \mathfrak{p}(\kappa) \tag{7.8}$$

we follow the argument of [8, Proposition 2.4]. Note that, for $i \in \mathbb{Z}^d$ and $t \in \mathbb{N}$,

$$Z^\kappa_{2t,0}(\omega) \geq Z^\kappa_{t,i}(\omega) Z^\kappa_{t,-i}(\theta^{t,i}\omega)$$

and therefore

$$\mathbb{E}[\log Z^\kappa_{2t,0}] \geq 2\mathbb{E}[\log Z^\kappa_{t,i}]. \tag{7.9}$$

Now fix $\gamma > 0$ large enough that

$$\mathfrak{p}(\kappa) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log P_\omega(\tau \geq t, \|X(t)\| \leq \gamma t)\big]. \tag{7.10}$$

Take now $\varepsilon := t^{-3/4}$ and apply the fractional moment method:

$$\begin{aligned}
\mathbb{E}\big[\log P(\tau(\omega) \geq t, \|X(t)\| \leq \gamma t)\big] &= \frac{1}{\varepsilon}\mathbb{E}\big[\log\big(P(\tau(\omega) \geq t, \|X(t)\| \leq \gamma t)^\varepsilon\big)\big] \\
&\leq \frac{1}{\varepsilon}\log\mathbb{E}\big[P(\tau(\omega) \geq t, \|X(t)\| \leq \gamma t)^\varepsilon\big] \\
&= \frac{1}{\varepsilon}\log\mathbb{E}\Big[\Big(\sum_{\|i\|\leq\gamma t} Z^\kappa_{t,i}(\omega)\Big)^\varepsilon\Big] \\
&\leq \frac{1}{\varepsilon}\log\mathbb{E}\Big[\sum_{\|i\|\leq\gamma t}\big(Z^\kappa_{t,i}(\omega)\big)^\varepsilon\Big] \\
&= \frac{1}{\varepsilon}\log\sum_{\|i\|\leq\gamma t}\mathbb{E}\Big[e^{\varepsilon(\log Z^\kappa_{t,i}(\omega) - \mathbb{E}[\log Z^\kappa_{t,i}])}\Big]e^{\varepsilon\mathbb{E}[\log Z^\kappa_{t,i}]},
\end{aligned} \tag{7.11}$$

where the first inequality is Jensen's inequality, and the second inequality comes from the general estimate $\big(\sum_{j=1}^N a_j\big)^\varepsilon \leq \sum_{j=1}^N a_j^\varepsilon$ for non-negative $a_1, \ldots, a_N$ and $0 < \varepsilon < 1$. Moreover, using the concentration inequality (Proposition 5.3), we find $c, t_0 > 0$ such that for all $t \geq t_0$

$$\mathbb{E}\Big[e^{\varepsilon(\log Z^\kappa_{t,i}(\omega) - \mathbb{E}[\log Z^\kappa_{t,i}])}\Big] \leq 1 + \int_1^\infty \mathbb{P}\Big(\big|\log Z^\kappa_{t,i}(\omega) - \mathbb{E}[\log Z^\kappa_{t,i}]\big| \geq t^{3/4}\log r\Big)\mathrm{d}r \leq c.$$

Both c and t_0 depend only on the constants in (5.5), and are therefore uniform in $i \in \mathbb{Z}^d$. Combining this with (7.11) and (7.9) gives

$$\begin{aligned}
\mathbb{E}[\log P_\omega(\tau \geq t, \|X(t)\| \leq \gamma t)] &\leq \frac{1}{\varepsilon}\log c + \frac{1}{\varepsilon}\log\sum_{\|i\|\leq\gamma t} e^{\varepsilon\mathbb{E}[\log Z^\kappa_{t,i}]} \\
&\leq \frac{1}{\varepsilon}\log c + \frac{1}{\varepsilon}\log(1+2\gamma t)^d + \frac{1}{2}\mathbb{E}[\log Z^\kappa_{2t,0}].
\end{aligned}$$

Dividing by t and taking limits, taking into account (7.10), gives (7.8).

□

Part III. Branching random walk among space-time disasters

8. Introduction

8.1. Motivation

We consider a branching random walk in $\mathbb{Z}^d$ in an environment consisting of random space-time disasters. That is, fix a realization $\omega \subseteq \mathbb{R}_+ \times \mathbb{Z}^d$ of the environment from Part II. Given ω, we consider a process $\mathcal{Z} = (\mathcal{Z}(t))_{t\geq 0}$ that evolves as follows:

- Initially one particle occupies the origin.
- Each particle independently moves as a simple random walk with jump rate $\kappa > 0$.
- Each particle independently branches at rate $\lambda > 0$, i.e., at rate $\lambda > 0$ the particle dies and is replaced by a random number of descendants at the same site. The number of descendants is sampled according to some fixed offspring distribution $q \in \mathcal{M}_1(\mathbb{N})$.
- A disaster $(t, i) \in \omega$ kills all particles that occupy site i at time t.

The model thus has parameters κ (jump rate), λ (branching rate) and q (offspring distribution). An easy calculation (see Lemma 9.1) shows that the model is related to the survival probability from Part II by the following relation:

$$E_\omega[|\mathcal{Z}(t)|] = e^{\lambda t(m-1)} P^\kappa(\tau(\omega) \geq t). \tag{8.1}$$

Here $|\mathcal{Z}(t)|$ denotes the total number of particles and $m := \sum_{k\in\mathbb{N}} kq(k)$ is the expected number of descendants. From the previous section, we already know that $P^\kappa(\tau(\omega) \geq t)$ has (deterministic) exponential decay rate $\mathfrak{p}(\kappa)$. In the following we will show that this is enough to characterize the regime of global survival as a function of the parameters. More precisely, we will show that

$$\begin{aligned} \mathcal{Z} \text{ survives with pos. prob.} \quad &\iff \quad E_\omega[|\mathcal{Z}(t)|] \to \infty \text{ exponentially fast.} \\ &\iff \quad \lambda(m-1) + \mathfrak{p}(\kappa) > 0. \end{aligned} \tag{8.2}$$

The first equivalence is known for a wide class of branching processes, see for example [1, Theorem A.2] for the corresponding result for Galton-Watson processes. Note that the previous statement does not specify with respect to which measure the survival probability is positive. There are two natural candidates:

- Write $\mathbb{P}^{\kappa,\lambda,q}$ for the joint law of environment ω and branching process $\mathcal{Z}$. We have **annealed** survival if $\mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0$.
- Write $P_\omega^{\kappa,\lambda,\kappa}$ for the law of $\mathcal{Z}$ in a fixed realization of the environment. We have **quenched** survival if $P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0$ for almost all ω.

Clearly quenched survival implies annealed survival. It turns out that the two concepts are equivalent and, in particular, we find a zero-one law for the event $\{P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0\}$.

8.2. Definition of the model

We will prove the main result under the assumptions

$$m := \sum_{k=0}^{\infty} kq(\{k\}) < \infty \quad \text{and} \quad q(1) < 1. \tag{8.3}$$

If fact, our result also applies in the case $m = \infty$, see Remark 8.5.

A particle v is a finite sequence of integers, i.e. an element of $\mathbb{N}^* := \bigcup_{k=0}^{\infty} \mathbb{N}^k$. We write $\varnothing$ for the element of length zero, called the root particle. Proceeding recursively, we interpret $(v_1, \ldots, v_k)$ as the v_k-th child of $(v_1, \ldots, v_{k-1})$. The height $|v|$ of a particle $v = (v_1, \ldots, v_k)$ is the length of this sequence, $|v| := k$.

We associate to every particle $v \in \mathbb{N}^*$ an exponential clock of rate λ, and whenever a clock rings the particle is removed and replaced by its children, where the number of children is distributed according to q. The clocks and the numbers of descendants are independent. We will write $\mathcal{V}(t) \subseteq \mathbb{N}^*$ for the set of particles that are alive at time t, starting with $\mathcal{V}(0) = \{\varnothing\}$.

Next, we extend this by associating to each particle $v \in \mathcal{V}(t)$ a position $X(t, v)$ in $\mathbb{Z}^d$. Independently of everything else, each particle performs simple random walk in continuous time with jump rate κ, starting at its birth time until the time it is replaced by its children. The root initially starts at the origin, and all other particles start at the position of their parent at the time of birth.

For $v \in \mathcal{V}(t)$, it will be convenient to extend $X(t, v)$ to a function $X(\cdot, v)\colon [0, t] \to \mathbb{Z}^d$, where for $s \in [0, t]$ we set $X(s, v)$ equal to the position occupied at time s by the unique ancestor of v in $\mathcal{V}(s)$. Given $\omega \in \Omega$ and a realization $\mathcal{V} = (\mathcal{V}(t))_{t \geq 0}$ of the branching random walk, we define

$$\mathcal{Z}(t) := \big\{v \in \mathcal{V}(t)\colon (s, X(s, v)) \notin \omega \text{ for all } s \in [0, t)\big\} \subseteq \mathcal{V}(t).$$

That is, $\mathcal{Z}(t)$ contains all particles v such that no disaster occurred along the trajectory of v before time t. Note that, since we did not assume $q(0) = 0$, it is possible that a particle has zero children, and the process may die out even without the influence of the environment.

We write $\mathbb{P}$ for the law of the environment, and $P^{\kappa,\lambda,q}$ for the law of the branching random walk $(\mathcal{V}, X)$. We write $P_\omega^{\kappa,\lambda,q}$ for the **quenched** law of $\mathcal{Z}$ given ω, and the **annealed** law is obtained by integrating over the environment

$$\mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \in \cdot\,) := \int_\Omega P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \in \cdot\,)\mathbb{P}(\mathrm{d}\omega).$$

8.3. Related literature

After our paper [30] had been published, we became aware that a related model has also been studied in [35, 36]. We refer to Section 8.6 for an in-depth discussion on those results.

Branching random walks in time-dependent environments have been studied extensively in the context of the parabolic Anderson model, see [33, 27]. However, most papers consider the solution to an SDE with random potential which describes the behavior of the **expectation** of the number of particles in a branching random walk in random environment, and not the actual particle system (a notable exception where the two models are compared, is [53]).

In addition, most papers have non-degeneracy conditions on the killing rates which are violated by our environment. In particular, we point out that our model differs from the branching random walks considered in [18] not only because time is continuous instead of discrete, but also because disasters in the environment were excluded, see [18, Formula (1.7)]. The possibility of killing many particles at the same site at once makes our model interesting, but also creates some technical difficulties.

For a survey on the parabolic Anderson model and random walks in random potential, we refer to [42].

8.4. Further notation

Here we collect notation that will be useful in the proof of Theorem 8.1 below.

We first extend the definition of $\mathcal{Z}$ to allow for more than one initial particle. For $\eta = (\eta(i))_{i\in\mathbb{Z}^d}$, let $\mathcal{Z}^\eta$ denote the process as defined before, started with $\eta(i)$ particles at site i, all of which evolve independently but in the same environment. If $A \subseteq \mathbb{R}^d$ and $R \geq 0$ is an integer, we record the special configuration (A, R) where each site $i \in A \cap \mathbb{Z}^d$ is occupied by R particles, that is for $i \in \mathbb{Z}^d$

$$(A, R)(i) \coloneqq R\,\mathbb{1}_{A\cap\mathbb{Z}^d}(i). \tag{8.4}$$

We use $\mathcal{Z}^A$ instead of $\mathcal{Z}^{(A,1)}$ for the process started from exactly one particle on every site in $A \cap \mathbb{Z}^d$. For $t > 0$ and a configuration η, we use

$$\mathcal{Z}^{\{t\}\times\eta} = \left(\mathcal{Z}^{\{t\}\times\eta}(s)\right)_{s\geq t} \tag{8.5}$$

to denote the process started at time t with $\eta(i)$ particles occupying each site i, and we use $\mathcal{Z}^{\{t\}\times A}$ if η is equal to $(A, 1)$. If $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$, we write

$$\{\eta \leq \mathcal{Z}(t)\} \coloneqq \left\{\eta(i) \leq \left|\mathcal{Z}(t) \cap \{i\}\right| \text{ for all } i \in \mathbb{Z}^d\right\}. \tag{8.6}$$

for the event that every site i is occupied by at least $\eta(i)$ particles at time t. If $\eta = \mathbb{1}_C$ for some $C \subseteq \mathbb{Z}^d$, this is simply written as

$$\{C \subseteq \mathcal{Z}(t)\} \coloneqq \{(C, 1) \leq \mathcal{Z}(t)\} = \left\{\text{for all } i \in C \text{ there is } v \in \mathcal{Z}(t) \text{ with } X(t, v) = i\right\}.$$

Moreover, for $B \subseteq \mathbb{R}^d$, we use $(\mathcal{Z}_B(t))_{t\geq 0}$ for the truncated process consisting of all particles that have never left B, i.e.

$$\mathcal{Z}_B(t) := \{v \in \mathcal{Z}(t) \colon X(s,v) \in B \text{ for all } s \in [0,t]\}. \tag{8.7}$$

Finally, for $t \in \mathbb{R}_+$ and $A \subseteq \mathbb{R}^d$, we use

$$\mathcal{Z}(t) \cap A := \{v \in \mathcal{Z}(t) \colon X(t,v) \in A\}. \tag{8.8}$$

for the set of particles of $\mathcal{Z}$ occupying a site $i \in A$ at time t.

8.5. The main result

We consider two types of survival:

$$\begin{aligned}\{\mathcal{Z} \text{ survives}\} &:= \{|\mathcal{Z}(t)| \geq 1 \text{ for infinitely large } t\}\\ \{\mathcal{Z} \text{ survives locally}\} &:= \{|\mathcal{Z}(t) \cap \{0\}| \geq 1 \text{ for infinitely large } t\}\end{aligned}$$

The following is the main result, which confirms the intuition from (8.2).

Theorem 8.1. *If $\lambda(m-1) + \mathfrak{p}(\kappa) > 0$, then almost surely $P_\omega^{\kappa,\lambda,q}(\mathcal{Z}$ survives locally$) > 0$. On the other hand, if $\lambda(m-1) + \mathfrak{p}(\kappa) \leq 0$, then $\mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z}$ survives$) = 0$.*

In particular, we have the following consequences:

Corollary 8.2. *(i) Annealed and quenched survival are equivalent, i.e.*

$$\mathbb{P}\big(P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0\big) = 1 \quad \Longleftrightarrow \quad \mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0.$$

In particular, $\mathbb{P}(P_\omega^{\kappa,\lambda,q}(\mathcal{Z}$ survives$) > 0) \in \{0,1\}$. Moreover, the same statements hold for local survival.

(ii) Global and local survival are equivalent, i.e.

$$\mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0 \iff \mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives locally}) > 0.$$

In analogy to classical branching processes, we define three regimes:

Definition 8.3. We say that $\mathcal{Z}$ is

$$\begin{array}{ccc} \text{subcritical} & & \lambda(m-1) + \mathfrak{p}(\kappa) < 0,\\ \text{critical} & \text{if} & \lambda(m-1) + \mathfrak{p}(\kappa) = 0,\\ \text{supercritical} & & \lambda(m-1) + \mathfrak{p}(\kappa) > 0.\end{array}$$

We point out that our proof shows that the number of particles grows exponentially on the event of survival:

Corollary 8.4. *Assume $\lambda(m-1) + \mathfrak{p}(\kappa) > 0$. Then for every $c \in (0, \lambda(m-1) + \mathfrak{p}(\kappa))$, almost surely*

$$\{\mathcal{Z} \text{ survives}\} = \Big\{\liminf_{t\to\infty} |\mathcal{Z}(t)| e^{-ct} = \infty\Big\}$$

The proof can be found at the end of Section 9.

Remark 8.5. Note that, by an obvious truncation argument, the assumption $m < \infty$ can be dropped: if $m = \infty$, then our result implies that $P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0$ almost surely.

In Section 9 we prove relation (8.1) and the non-critical cases of Theorem 8.1. The sub-critical case follows immediately from the first moment method. For the supercritical case, we compare $\mathcal{Z}$ to an embedded Galton-Watson process with i.i.d. offspring distributions. This section is relatively short, but builds heavily on the preparation from Part II, in particular Proposition 5.6. The proof in the critical case in Section 10 will take up the remainder of our discussion. We will explain its strategy in Section 10.1.

8.6. The critical regime with infinitely many particles

In this section, we discuss the relation between Theorem 8.1 and the results from [35, 36]. The model discussed in those works is related to the annealed version of our process.

We start by presenting their notation: Consider transition probabilities $(p(i,j))_{i,j\in\mathbb{Z}^n}$ and $(q(i,j))_{i,j\in\mathbb{Z}^n}$, rates b, d and m (corresponding to the rate of birth, death and movement) as well as a parameter $p \in (0,1)$ measuring cooperation between individuals. All results are in dimensions $n \geq 3$ and under the assumption $b = d$. Let $(\eta_t)_{t\geq 0}$ denote a Markov process in $\mathbb{N}^{\mathbb{Z}^n}$ with the following dynamics:

- At rate m a particle at site i jumps according to $p(i,\cdot)$.
- At rate b a particle at site i gives birth to one descendant, whose position has law $q(i,\cdot)$. The parent particle is not removed from the system.
- Each particle dies at rate $(1-p)d$.
- At rate pd all particles occupying the same site are killed. This corresponds to the disasters from our model.

All of these transitions happen independently. Up to some restrictions, we can translate this notation to our model:

- We have assumed that disasters occur at rate 1, so we compare our model to the process $(\widetilde{\eta}_t)_{t\geq 0}$ obtained from η by re-scaling time, i.e. $\widetilde{\eta}_t := \eta_{t/dp}$.
- The jump rate m/dp in the re-scaled model corresponds to κ. We restrict to simple random walk, i.e. we assume that $p(i,\cdot)$ is the uniform distribution over the neighbors of i.
- We only allow descendants at the same site as their parents, so we assume that $q(i,\cdot)$ is the Dirac-distribution in i.
- Each individual dies at rate $(1-p)/p$, and has one descendant at rate b/dp (in addition to itself). In our model, this corresponds to $\lambda = ((1-p)d + b)/dp$ and offspring distribution q with $q(\{0\}) = 1 - q(\{2\}) = (1-p)d/((1-p)d + b)$.

Inserting those parameters in the expression from Theorem 8.1 yields

$$\mathfrak{p}\Big(\frac{m}{dp}\Big) + 1 + \frac{b-d}{dp}.$$

Since $\mathfrak{p}(\kappa) \leq -1$ and $p = q$ by assumption, the model is either critical or subcritical (in the sense of Definition 8.3). In particular, Theorem 8.1 shows that every finite initial configuration dies out eventually. In [35, 36], the initial configuration η_0 is sampled with distribution $\mu \in \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^n})$, which is assumed to be ergodic and translation-invariant. That is, in contrast to our model there are always infinitely many occupied sites, and the question is whether the system dies out locally:

Theorem L. *There exists $0 < p^* < 1$ such that*

(i) For $p \in (p^*, 1)$ there is local extinction, i.e. $\mathbb{P}^p(\eta_t \in \cdot) \xrightarrow{d} \delta_0$, the Dirac-measure on $0 \in \mathbb{N}^{\mathbb{Z}^n}$.

(ii) For $p \in (0, p^*)$ the system is persistent, i.e. there exists a non-degenerate invariant measure $\nu \in \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^n})$ such that $\mathbb{P}^p(\eta_t \in \cdot) \xrightarrow{d} \nu$.

From the monotonicity of $p \mapsto \mathfrak{p}(m/dp)$ (see Corollary 16.6), we find that there exists $\widehat{p}$ such that the model is critical for $p \leq \widehat{p}$, and subcritical for $p > \widehat{p}$. The fact that criticality holds for $p = \widehat{p}$ follows from the continuity of $p \mapsto \mathfrak{p}(m/dp)$ (see Proposition 5.5).

It seems reasonable to expect $\widehat{p} = p^*$, but this question remains open. Note, however, that local extinction is also conjectured at $p = p^*$, see [37]. This would imply that in the critical regime $0 < p \leq \widehat{p}$ both persistence (for $0 < p < \widehat{p}$) and local extinction (for $p = \widehat{p}$) can occur.

9. The non-critical cases

The following lemma explains the connection between the branching random walk and the decay rate $\mathfrak{p}(\kappa)$ from the previous section.

Lemma 9.1 (Many-to-one Lemma)**.** *For all $\omega \in \Omega$ and $t \geq 0$,*

$$E_\omega^{\kappa,\lambda,q}[|\mathcal{Z}(t)|] = e^{\lambda t(m-1)} P^\kappa(\tau(\omega) \geq t)$$
$$E_\omega^{\kappa,\lambda,q}[|\mathcal{Z}(t) \cap \{0\}|] = e^{\lambda t(m-1)} P^\kappa(\tau(\omega) \geq t, X(t) = 0).$$

Proof. Let us explicitly write

$$\tau(\omega, x) := \inf\{t \geq 0 \colon (t, x(t)) \in \omega\}$$

for the extinction time of a trajectory $x \colon \mathbb{R}_+ \to \mathbb{Z}^d$. Note that, for every particle $v \in \mathbb{N}^*$,

$$\{v \in \mathcal{Z}(t)\} = \{v \in \mathcal{V}(t)\} \cap \{\tau(\omega, X(\cdot, v)) \geq t\}.$$

Since the movement of particles is independent of the branching, the probability of the second event is $P^\kappa(\tau(\omega) \geq t)$, independently of $v \in \mathcal{V}(t)$. Therefore

$$\begin{aligned} E_\omega^{\kappa,\lambda,q}[\mathcal{Z}(t)] &= \sum_{v \in \mathbb{N}^*} E_\omega\big[\mathbb{1}\{v \in \mathcal{V}(t)\}\mathbb{1}\{\tau(\omega, X(\cdot, v)) \geq t\}\big] \\ &= P^\kappa(\tau(\omega) \geq t) E[|\mathcal{V}(t)|] \\ &= P^\kappa(\tau(\omega) \geq t) e^{\lambda(m-1)}. \end{aligned}$$

The expected number $E[|\mathcal{V}(t)|]$ of particles without disasters can be computed using standard arguments. The second claim follows by an analogous calculation. $\square$

Proof of Theorem 8.1 (subcritical case). Assume $-\varepsilon := \lambda(m-1) + \mathfrak{p}(\kappa) < 0$. For almost all ω, we find $T = T(\omega)$ such that for all $t \geq T$

$$e^{\lambda t(m-1)} P^\kappa(\tau(\omega) \geq t) \leq e^{-\frac{\varepsilon}{2}t}.$$

Then, by the Markov inequality and Lemma 9.1,

$$\sum_{t\in\mathbb{N}} P_\omega^{\kappa,\lambda,q}(|\mathcal{Z}(t)| \geq 1) \leq T(\omega) + \sum_{t\geq T(\omega)} E_\omega^{\kappa,\lambda,q}[|\mathcal{Z}(t)|] \leq T(\omega) + \sum_{t\geq T(\omega)} e^{-\frac{\varepsilon}{2}t} < \infty.$$

From Borel–Cantelli, we get $P_\omega^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) \leq P_\omega^{\kappa,\lambda,q}(|\mathcal{Z}(t)| \geq 1 \text{ i.o.}) = 0$. $\square$

Proof of Theorem 8.1 (supercritical case). Assume

$$\lambda(m-1) + \mathfrak{p}(\kappa) > 0\,. \tag{9.1}$$

We will find a branching process with i.i.d. offspring distributions embedded in $\mathcal{Z}$ and show that it has positive survival probability. More precisely, for large T we introduce a process $(\mathcal{A}(k))_{k\in\mathbb{N}}$ such that $\mathcal{A}(k) \subseteq \mathcal{Z}(kT)$ for all $k \in \mathbb{N}$, and such that almost surely

$$P_\omega^{\kappa,\lambda,q}(\mathcal{A} \text{ survives}) > 0.$$

Recalling Proposition 5.6(iii), we find $T \in \mathbb{N}$ such that

$$\mathbb{E}\big[\log P^\kappa(\tau(\omega) \geq T, X(T) = 0)\big] + \lambda T(m-1) > 0. \tag{9.2}$$

We define $\mathcal{A}(0) := \mathcal{Z}(0)$ and

$$\mathcal{A}(k) := \big\{v \in \mathcal{Z}(kT)\colon X(iT, v) = 0 \text{ for all } i = 0, \dots, k\big\}$$

That is, for $\mathcal{A}$ we only consider particles which at times $T, 2T, 3T, \dots$ return to the origin. Recall that we use $(\mathcal{Z}^{(k-1)T,\{0\}}(t))_{t\geq(k-1)T}$ to denote a branching random walk started at time $(k-1)T$ with one particle at the origin, and moreover that $\mathcal{Z}(t) \cap \{0\}$ denotes the set of particles occupying the origin at time t.

Claim. *Fix $\omega \in \Omega$. Then $\{|\mathcal{A}(k)|\colon k \in \mathbb{N}\}$ is an inhomogeneous Galton-Watson process with offspring distributions $(q_\omega^{(k)})_{k\in\mathbb{N}}$, where $q_\omega^{(k)} \in \mathcal{M}_1(\mathbb{N})$ is defined by*

$$q_\omega^{(k)}(\{j\}) = P_\omega^{\kappa,\lambda,q}\big(\big|\mathcal{Z}^{(k-1)T,\{0\}}(kT) \cap \{0\}\big| = j\big), \quad j \in \mathbb{N}.$$

Proof of the claim. Observe that every particle in generation k of $\mathcal{A}$ is the descendant of some particle (possibly itself) in generation $k-1$. For $v \in \mathcal{A}(k-1)$, we let $N_{k-1}(v)$ denote the number of descendants of v in $\mathcal{A}(k)$. We point out that $q_\omega^{(k)}$ is the law of $N_{k-1}(v)$. From the definition of the model, $N_{k-1}(v)$ and $N_{k-1}(w)$ are independent for $v \neq w \in \mathcal{A}(k-1)$, conditioned on ω. Moreover, since every $v \in \mathcal{A}(k-1)$ occupies the origin at time $(k-1)T$, the law of $N_{k-1}(v)$ does not depend on $v \in \mathcal{A}(k)$. $\square$

We note that $q_\omega^{(k)}$ only depends on the environment in $[(k-1)T, kT)$, so that $(q_\omega^{(k)})_k$ is an i.i.d. sequence in $\mathcal{M}_1(\mathbb{N})$. We write $m_\omega^{(k)}$ for the expectation of $q^{(k)}$, and note that by Lemma 9.1

$$m_\omega^{(k)} = e^{\lambda T(m-1)} P^\kappa(\tau(\theta^{(k-1)T,0}\omega) \geq T, X(T) = 0).$$

Thus (9.2) gives $\mathbb{E}[\log m_\omega^{(k)}] > 0$. By a well-known result on branching processes with i.i.d. offspring distributions, see [56, 57], the survival probability of $(|\mathcal{A}(k)|)_{k\in\mathbb{N}}$ is positive for almost all environments, provided the following non-degeneracy condition holds:

$$\mathbb{E}[\log(1 - q_\omega^{(1)}(0))] > -\infty \tag{9.3}$$

To see this, note that $|\mathcal{Z}(T) \cap \{0\}| = 1$ if the root-particle does not branch in $[0, T]$, avoids all disasters and returns to the origin. That is, we have the following lower bound:

$$\begin{aligned} 1 - q_\omega^{(1)}(0) &= P_\omega(\mathcal{Z}(T) \cap \{0\} \neq \varnothing) \\ &\geq e^{-\lambda T} P(\tau(\omega) \geq T, X(T) = 0) \\ &\geq e^{-\lambda T} \textstyle\prod_{i=1}^T P(\tau(\theta^{i-1}\omega) \geq 1, X(1) = 0) \end{aligned}$$

Proposition 5.2 and Jensen's inequality show that the logarithm of the last line is integrable. □

Proof of Corollary 8.4. This follows from [57, Theorem 5.5(iii)], applied to $(\mathcal{A}(k))_{k\geq 0}$. □

10. The critical case

10.1. Outline

We adapt the technique used in [5] to show that the critical contact process dies out. The same approach has been used in [32] for a discrete time, non-degenerate version of our model. In this section, we provide the main part of the proof, while two essential propositions will be proved in the remainder of this section.

Proof of Theorem 8.1 (critical case). Fix κ, λ and q such that

$$\lambda(m-1) + \mathfrak{p}(\kappa) = 0 \tag{10.1}$$

and assume for contradiction that

$$\mathbb{P}^{\kappa,\lambda,q}(\mathcal{Z} \text{ survives}) > 0. \tag{10.2}$$

We first observe that, using the subcritical part of Theorem 8.1, for every $\mu < \lambda$

$$\mathbb{P}^{\kappa,\mu,q}(\mathcal{Z} \text{ survives}) = 0. \tag{10.3}$$

In the following, we will discuss subsets $A \subseteq \mathbb{R}^d$ of sites, with the understanding that an intersection with $\mathbb{Z}^d$ has to be taken wherever appropriate. For example, in the next display

$[-n,n]^d$ should be replaced by $\{-n,\dots,n\}^d$ and $[L,3L]\times[-L,L]^{d-1}$ by $\{L,\dots,3L\}\times\{-L,\dots,L\}^{d-1}$. The aim is to improve the readability of formulas.

To arrive at the contradiction, we compare $\mathscr{Z}$ to an embedded oriented percolation, which we now introduce. Recall the notation from Section 8.4. For $s,L,T\in\mathbb{R}^+$, $j\in\mathbb{Z}^d$ and $n,S\in\mathbb{N}$, let

$$A^{s,j}(L,T,n,S) := \left\{ \begin{array}{c} \exists i\in[L,3L]\times[-L,L]^{d-1}, t\in[5T,6T] \\ \text{such that } (i+[-n,n]^d,S^2)\le \mathscr{Z}^{\{s\}\times(j+[-n,n]^d,S^2)}_{[-5L,5L]\times[-3L,3L]^{d-1}}(t). \end{array} \right\} \tag{10.4}$$

In words, $A^{0,0}(L,T,n,S)$ is the event that starting from configuration $([-n,n]^d,S^2)$ at time 0, the initial particles will propagate such that at some time $t\in[5T,6T]$ we find a copy $i+[-n,n]^d$ of $[-n,n]^d$ where every site is occupied by at least S^2 particles. The truncation ensures that we only consider particles which do not leave the space-time box $[0,T]\times[-5L,5L]\times[-3L,3L]^{d-1}$. The additional parameters s,j correspond to a space-time shift of the initial configuration. See Figure 6 for an illustration.

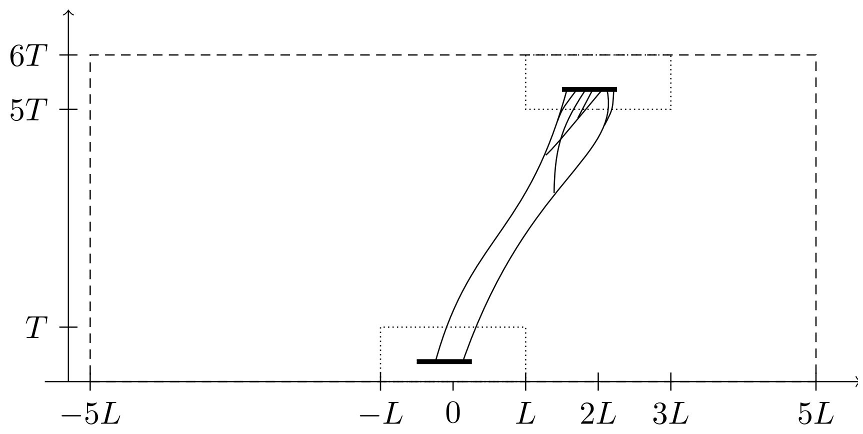

Figure 6: Illustration of the event $A^{s,j}(L,T,n,S)$ in $d=1$, with $S=1$. The black bars represent intervals $[-n,n]$ where every site is occupied by at least one particle. The dotted boxes represent the origin and target areas $[0,T]\times[-L,L]$ and $[5,T]\times[L,3L]$, while the dashed box is the truncation $[0,6T]\times[-5T,5T]$. Only particles which contribute to $A^{s,t}(L,T,n,S)$ are shown.

The main task in the remainder of this section is to show that this event has high probability, uniformly over some set of initial configurations:

Proposition 10.1. *Assume (10.2). For every $\varepsilon>0$, there exist $L,T>0$ and $n,S\in\mathbb{N}$ such that*

$$\inf_{s\in[0,T],j\in[-L,L]^d} \mathbb{P}^{\kappa,\lambda,q}\left(A^{s,j}(L,T,n,S)\right) > 1-\varepsilon. \tag{10.5}$$

Using $A^{s,j}$, we can introduce an embedded oriented percolation $\eta = (\eta(k,l))_{k,l\in\mathbb{N}^2}$. We follow the arguments from [45, Chapter I.2]. Consider shifted copies of the box $[0,T]\times[-L,L]^d$ that appeared in (10.4): For $k,l\in\mathbb{N}^2$, let

$$\mathbb{B}(k,l) := \big(5kT, (-2k+4l)L, 0, \dots, 0\big) + [0,T]\times[-L,L]^d \tag{10.6}$$

See Figure 7 below. We call $(k,l)\in\mathbb{N}^2$ **occupied**, if there exist $(t,i)\in\mathbb{B}(k,l)$ such that

$$(i+[-n,n]^d, S^2) \le \mathcal{Z}^{([-n,n]^d,S^2)}(t).$$

We call a path $(k_0,l_0),\dots,(k_m,l_m)$ in $\mathbb{N}^2$ an **oriented path** from (k_0,l_0) to (k_m,l_m), if $k_{h+1}=k_h+1$ and either $l_{h+1}=l_h$ or $l_{h+1}=l_h+1$, for all $h=0,\dots,m-1$. We write

$$\eta(k,l) = \begin{cases} 1 & \text{if there is an oriented path } (0,0)\to(k,l) \text{ using only occupied sites} \\ 0 & \text{else.} \end{cases}$$

This defines a random variable $\eta\in\{0,1\}^{\mathbb{N}^2}$ from every realization of $\mathcal{Z}^{([-n,n]^d,S^2)}$. Moreover, $\mathcal{Z}$ dominates η in the sense that

$$\{\eta(k,l)=1 \text{ infinitely often}\} \subseteq \{\mathcal{Z} \text{ survives}\}$$

Now let $\xi\in\{0,1\}^{\mathbb{N}^2}$ denote independent site percolation. That is, every site $(k,l)\neq(0,0)$ is occupied independently with probability $p\in(0,1)$, and we set $\xi(k,l):=1$ if and only if (k,l) is reachable from $(0,0)$ along an oriented path of occupied pairs. Let $\mathbb{P}^p$ denote the law of ξ. We show that ξ is dominated by η, for some choice of p:

Proposition 10.2. *Assume* (10.5). *There exists $p=p(\varepsilon)$ such that $\xi\preceq_{st}\eta$, where ξ is independent oriented site percolation with parameter p. Moreover, $p(\varepsilon)\uparrow 1$ for $\varepsilon\downarrow 0$.*

Here $\xi\preceq_{st}\eta$ means that there exists a coupling $(\widehat{\xi},\widehat{\eta})$ such that $\eta\overset{\mathrm{d}}{=}\widehat{\eta}$ and $\xi\overset{\mathrm{d}}{=}\widehat{\xi}$, and almost surely $\widehat{\xi}(k,l)\le\widehat{\eta}(k,l)$ for all $(k,l)\in\mathbb{N}^2$.

We can now derive a contradiction: Fix $\varepsilon>0$ such that $p(2\varepsilon)$ is larger than the critical parameter of oriented site percolation, i.e. such that

$$\{\xi(k,l)=1 \text{ infinitely often}\}$$

has positive probability. Next, use Proposition 10.1 to choose L, T, n and S such that (10.5) holds for this value of ε. Since $A^{s,j}(L,T,n,S)$ is a local event, its probability depends continuously on the parameters, and thus we can find $\mu<\lambda$ such that also

$$\inf_{s\in[0,T],j\in[-L,L]^d} \mathbb{P}^{\kappa,\mu,q}\left(A^{s,j}(L,T,n,S)\right) > 1-2\varepsilon.$$

Proposition 10.2 therefore shows

$$\mathbb{P}^{\kappa,\mu,q}\big(\mathcal{Z}^{([-n,n]^d,S^2)} \text{ survives}\big) \ge \mathbb{P}^{\kappa,\mu,q}(\eta(k,l)=1 \text{ i.o.}) \ge \mathbb{P}^{p(2\varepsilon)}(\xi(k,l)=1 \text{ i.o.}) > 0.$$

Clearly, this also implies $\mathbb{P}^{\kappa,\mu,q}(\mathcal{Z}^{\{0\}} \text{ survives})>0$ and thus contradicts (10.3). □

In the remainder of this section we will prove the missing results:

- The comparison to oriented percolation in Proposition 10.2 follows from standard arguments for interacting particle systems, see Section 10.2.

It remains to show Proposition 10.1, which is the technical core of the proof. We proceed along the following steps:

- In Section 10.3 we collect some technical results which are necessary to choose L, T, n and S in Proposition 10.9.
- In Section 10.4 we discuss the numbers N and M of particles leaving a space-time box through the top and the faces.
  - This notation is defined rigorously in Section 10.4.1.
  - In Section 10.4.2 we use well-known techniques for branching processes to show that on the event of survival, $N + M$ has to be large.
  - We need to show that **both** N and M are large on the event of survival (and not just one of them). This will following with the help of an FKG-inequality, which we derive in Section 10.4.3.
- In Section 10.5 we state an auxiliary result, and use it to prove Proposition 10.1.
- Finally, Section 10.6 contains the proof of the auxiliary result. Depending on ε, we have to treat two cases, one of which is fairly easy (Section 10.6.2). The other case makes use of all preparation outlined here (Section 10.6.1).

Note that throughout this section the parameters κ, λ and q are fixed, and we assume (10.1) and (10.2). We will sometimes drop them from the notation, and write $\mathbb{P}$ instead of $\mathbb{P}^{\kappa,\lambda,q}$.

10.2. Comparison to oriented percolation

Proof of Proposition 10.2. We consider a truncation corresponding to $\mathbb{B}(k,l)$:

$$\overline{\mathbb{B}}(k,l) := \big(5kT, (-2k+4l)L, 0, \ldots, 0\big) + [0, 6T] \times [-5L, 5L] \times [-3L, 3L]^{d-1}$$

We now recursively construction an auxiliary process $(\widetilde{\eta}(k,l))_{(k,l)\in\mathbb{N}^2}$ that is dominated by η (that is, $\widetilde{\eta}(k,l) \le \eta(k,l)$ for every k,l). Assume that the first k levels $\{\widetilde{\eta}(k',l') \colon k' \le k, l' \le k\}$ have been constructed.

If $\widetilde{\eta}(k,l) = \widetilde{\eta}(k,l-1) = 0$, then we set $\widetilde{\eta}(k+1,l) := 0$. Assume that $\widetilde{\eta}(k,l) = 1$. This implies $\eta(k,l) = 1$, so by the definition of η we find $(t_{k,l}, i_{k,l}) \in \mathbb{B}(k,l)$ such that at time $t_{k,l}$ every site of $i_{k,l} + [-n,n]^d$ is occupied by at least S^2 particles (in the underlying branching random walk). If there is more than one such pair, we choose the one that has the smallest time-coordinate. We set $\widetilde{\eta}(k+1,l) := 1$ if there exists $(t_{k+1,l}, i_{k+1,l}) \in \mathbb{B}(k+1,l)$ such that

$$([-n,n]^d + i_{k+1,l}, S^2) \le \mathcal{Z}_{\overline{\mathbb{B}}(k,l)}^{\{t_{k,l}\}\times(i_{k,l}+[-n,n]^d, S^2)}(t_{k+1,l}).$$

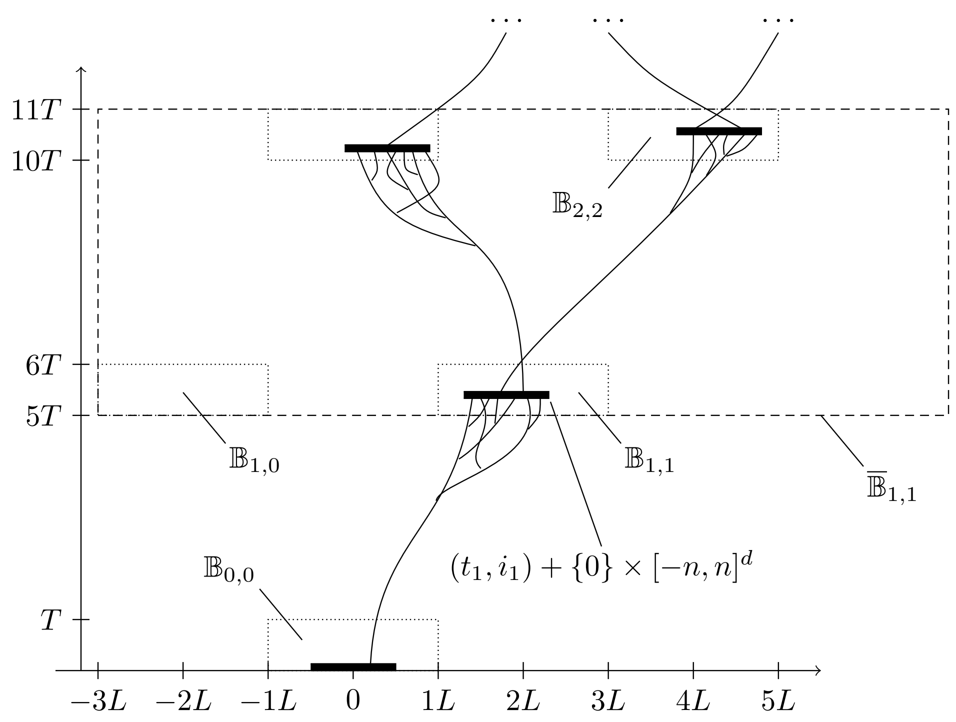

Figure 7: Illustration for the comparison to oriented percolation. The black bars represent shifted copies of $\{0\} \times [-n, n]$ where each site is occupied by at least one particle. The dashed box represents the truncation from (10.5), and we observe that two truncation boxes are disjoint whenever they are well-separated. As before, only particles which contribute to the construction are drawn.

See Figure 7 for an illustration. Observe that, up to space-time shifts, this is the same event as in (10.5), so that

$$\mathbb{P}^{\kappa,\lambda,q}(\widetilde{\eta}(k+1,l) = 1 | \widetilde{\eta}(k,l) = 1) > 1 - \varepsilon.$$

We have to deal with the problem that $\widetilde{\eta}(k+1,l)$ and $\widetilde{\eta}(k+1,l')$ are not independent for $l \neq l'$, given $\sigma(\eta(k',l) \colon k' \leq k, l \leq k)$, since the truncation boxes $\overline{\mathbb{B}}(k+1,l)$ and $\overline{\mathbb{B}}(k+1,l')$ are in general not disjoint. Observe however that $|l - l'| > 2$ implies

$$\overline{\mathbb{B}}(k+1,l) \cap \overline{\mathbb{B}}(k+1,l') = \varnothing,$$

and therefore, conditioned on $\sigma(\eta(k',l) \colon k' \leq k, l \leq k)$, the families $(\widetilde{\eta}(k+1,l))_{l \in S_1}$ and $(\widetilde{\eta}(k+1,l))_{l \in S_2}$ are independent whenever

$$\min\{|l_1 - l_2| \colon l_1 \in S_1, l_2 \in S_2\} > 2.$$

Now [45, Theorem B26] shows that we can couple $(\widetilde{\eta}(k+1,l))_{l=0,\dots,k+1}$ with an independent family of Bernoulli random variables $(\xi(k+1,l))_{l \leq k+1}$ such that $\widetilde{\eta}$ dominates ξ, and such that if either $\xi(k,l-1)=1$ or $\xi(k,l)=1$, then $\xi(k+1,l)=1$ with probability at least

$$p(\varepsilon) \coloneqq (1-\sqrt[5]{\varepsilon}).$$

Since η in turn dominates $\widetilde{\eta}$, this gives the claim. $\square$

10.3. Elementary preparations

Recall that we have fixed λ, κ and q such that (10.2) holds. In Lemma 10.3 below we show some simple consequences of this assumption:

- Part (i): By enlarging the initial configuration, the survival probability gets arbitrarily close to 1.
- Part (ii): Consider particles that survive locally (using only two sites) until time 1. With high probability, there are many such particles, provided the number of initial particles at the origin is large enough.
- Part (iii): With high probability, starting from many particles occupying the origin, we end up with a configuration where every site of $[-n,n]^d$ is occupied by many particles at time 1.

Lemma 10.3. *(i) For every $\varepsilon>0$ there is $n\in\mathbb{N}$ with*

$$\mathbb{P}\big(\mathcal{Z}^{[-n,n]^d} \text{ survives}\big) > 1-\varepsilon.$$

(ii) Recall (8.8). *For every $\varepsilon>0$ and $M\in\mathbb{N}$, there is $N\in\mathbb{N}$ such that*

$$\mathbb{P}\Big(\big|\mathcal{Z}^{(\{0\},N)}_{\{0,e_1\}}(1)\cap\{0\}\big|\geq M\Big) > 1-\varepsilon.$$

(iii) Recall (8.6). *For every $\varepsilon>0$ and $n,S\in\mathbb{N}$, there is an $N\in\mathbb{N}$ such that*

$$\min_{v\in\{0,ne_1\}}\Big\{\mathbb{P}\big((v+[-n,n]^d,S)\leq \mathcal{Z}^{(\{0\},N)}_{v+[-n,n]^d}(1)\big)\Big\} > 1-\varepsilon.$$

Proof. **Part (i)**: Define a collection $(Y_i)_{i\in\mathbb{Z}^d}$ with $Y_i \coloneqq \mathbb{1}\{\mathcal{Z}^{\{i\}} \text{ survives}\}$. We have

$$\mathbb{P}\big(|\mathcal{Z}^{[-n,n]^d}(t)|>0\ \forall t\big) = \mathbb{P}\Big(\sum_{i\in[-n,n]^d} Y_i>0\Big) = \mathbb{E}\Big[P_\omega\Big(\sum_{i\in[-n,n]^d} Y_i>0\Big)\Big]$$

Writing $S_n \coloneqq \sum_{i\in[-n,n]^d} Y_i$, we have

$$P_\omega\big(S_n=0\big) \leq P_\omega\left(|S_n-E_\omega[S_n]|\geq E_\omega[S_n]\right) \leq \frac{\operatorname{Var}_\omega(S_n)}{\big(E_\omega[S_n]\big)^2} \tag{10.7}$$

Now, due to the spatial ergodic theorem (see [44, Theorem 4.9]), almost surely

$$\frac{1}{|[-n,n]^d\cap\mathbb{Z}^d|}E_\omega[S_n] \to \mathbb{E}[Y_0]>0.$$

On the other hand, almost surely

$$\frac{1}{|[-n,n]^d \cap \mathbb{Z}^d|}\mathrm{Var}_\omega\big(S_n\big) = \frac{1}{|[-n,n]^d \cap \mathbb{Z}^d|}\sum_{i\in[-n,n]^d} \mathrm{Var}_\omega(Y_i) \to \mathbb{E}\big[\mathrm{Var}_\omega(Y_0)\big]$$

where we used the fact that $\{Y_i, i \in \mathbb{Z}^d\}$ are independent with respect to P_ω. We conclude from (10.7) that $P_\omega(S_n = 0) \to 0$ almost surely and therefore $\mathbb{P}\left(S_n = 0\right) \to 0$ as well.

Part (ii): For a particle v, let $B(v)$ denote the event that v

- does not branch before time 1
- satisfies $X([0,1], v) \subseteq \{0, e_1\}$ and $X(1, v) = 0$
- and is not killed by the environment until time 1.

For $\alpha \in (0,1]$, let $A(\alpha) := \{P_\omega(B(\varnothing)) \geq \alpha\}$. Note that the events $A(\alpha)$ are increasing as $\alpha \downarrow 0$ and that their union over all $\alpha \in (0,1] \cap \mathbb{Q}$ has probability 1. Fix η such that $(1-\eta)^2 \geq 1-\varepsilon$, and $\alpha > 0$ small enough that $\mathbb{P}(A(\alpha)) \geq 1-\eta$.

Now take $\omega \in A(\alpha)$. Then, starting with N initial particles at the origin, the number of particles v such that $B(v)$ occurs dominates the number of successes of a binomial random variable with N trials and success probability α. We can thus find N such that, with probability at least $1-\eta$, this number is larger than M. Then

$$\begin{aligned}\mathbb{P}\big(\big|\mathcal{Z}^{(\{0\},N)}_{\{0,e_1\}}(1) \cap \{0\}\big| \geq M\big) &\geq \mathbb{P}(A(\alpha)) \inf_{\omega\in A(\alpha)} P_\omega(\#\{v \in \mathcal{Z}^{(\{0\},N)}(0) : B(v) \text{ occurs}\} \geq M) \\ &\geq (1-\eta)^2 \geq 1-\varepsilon.\end{aligned}$$

Part (iii): Fix ε, n, S. The arguments for both probabilities in the minimum are identical, so we only show that, for N large enough,

$$\mathbb{P}\big(([-n,n]^d, S) \leq \mathcal{Z}^{(\{0\},N)}_{[-n,n]^d}(1)\big) > 1-\varepsilon$$

Consider

$$R(i) := P_\omega\big(|\mathcal{Z}^{\{0\}}_{[-n,n]^d}(t)| = 1 \text{ for all } t \in [0,1], |\mathcal{Z}^{\{0\}}_{[-n,n]^d}(1) \cap \{i\}| = 1\big).$$

That is, $R(i)$ is the (quenched) probability that one initial particle started in the origin survives, does not branch or leave $[-n,n]^d$ until time 1, and occupies site i at time 1. Let

$$A(\alpha) := \Big\{\min_{i\in[-n,n]^d\cap\mathbb{Z}^d} R(i) \geq \alpha\Big\}$$

Fix $\eta > 0$ such that $(1-\eta)^2 > 1-\varepsilon$, and note that by the same argument as above, we find $\alpha > 0$ such that $\mathbb{P}(A(\alpha)) \geq 1-\eta$.

For $i \in [-n,n] \cap \mathbb{Z}^d$, let $W_N(i)$ count the number of particles in $\mathcal{Z}^{(\{0\},N)}(0)$ that satisfies the event in the definition of $R(i)$. Observe that, for $\omega \in A(\alpha)$, the collection $\{W_N(i) \colon i \in [-n,n]^d \cap \mathbb{Z}^d\}$ dominates a multinomial random variable with N trials and $|[-n,n]^d \cap \mathbb{Z}^d|$

categories, each of which occurs with probability at least α. Thus we find N such that, for $\omega \in A(\alpha)$,

$$P_\omega\big(\min_{i\in[-n,n]\cap\mathbb{Z}^d} W_N(i) \geq S\big) \geq 1-\eta.$$

Together with $\mathbb{P}(A(\alpha)) \geq 1-\eta$, this finishes the proof.

□

We prove an auxiliary result that is needed for the FKG-inequality, see Remark 10.7

Lemma 10.4. *Let $m, S \geq 1$ and consider random vector $(X_0,\dots,X_m)$ taking values in $\{0,1\}^{m+1}$ (not necessarily independent). Then*

$$\prod_{i=0}^m P\big(X_i = 0\big)^S \leq P(X_i = 0 \textit{ for all } i) + \Big(\frac{m}{m+1}\Big)^{(m+1)S}.$$

Proof. For $I \subseteq [\![m]\!] = \{0,\dots,m\}$, let us define

$$p_I \coloneqq P(\{i\colon X_i = 0\} = I).$$

We need to show that

$$\prod_{i=0}^m \Big(p_{[\![m]\!]} + \sum_{\{i\}\subseteq I \subsetneq [\![m]\!]} p_I\Big)^S \leq p_{[\![m]\!]} + \Big(\frac{m}{m+1}\Big)^{(m+1)S}. \tag{10.8}$$

Define q by $q_{[\![m]\!]} \coloneqq p_{[\![m]\!]}$, $q_I \coloneqq 0$ for $|I| < m$ and

$$q_{[\![m]\!]\setminus\{i\}} \coloneqq \frac{1-p_{[\![m]\!]}}{m+1} \quad \text{for } i = 0,\dots,m.$$

Observe that the l.h.s. of (10.8) takes a maximum over all admissible values of $\{p_I\colon I \neq [\![m]\!]\}$ at q, and it is enough to check that

$$\Big(\frac{m+p_{[\![m]\!]}}{m+1}\Big)^{(m+1)S} \leq p_{[\![m]\!]} + \Big(\frac{m}{m+1}\Big)^{(m+1)S}.$$

Since the function on the l.h.s. is convex in $p_{[\![m]\!]}$ while the r.h.s. is linear, the conclusion follows by checking that the inequality indeed holds for $p_{[\![m]\!]}$ equal to 0 and to 1. □

10.4. Space-time boxes

10.4.1. Notation

We can think of $(\mathcal{Z}^\eta(t))_{0\leq t\leq T}$ as a process in space-time, which we emphasize by writing

$$[0,T\,] \times \mathcal{Z}^\eta \coloneqq \big\{(t,v)\colon 0 \leq t \leq T, v \in \mathcal{Z}^\eta(t)\big\} \subseteq [0,T] \times \mathbb{N}^*.$$

For convenience, we define the sign of zero to be 1, that is

$$\operatorname{sign}(i) \coloneqq \mathbb{1}_{i\geq 0} - \mathbb{1}_{i<0} \quad \text{for } i \in \mathbb{Z}. \tag{10.9}$$

For $L \in \mathbb{N}$ and $T > 0$, we now consider a **space-time box** $\mathbb{B} \subseteq \mathbb{R}_+ \times \mathbb{Z}^d$ of the form

$$\mathbb{B}(L,T) \coloneqq [0,T] \times [-L,L]^d.$$

We denote the **top** of this box by

$$\mathbb{T}(L,T) \coloneqq \{T\} \times [-L,L]^d$$

and, for $u \in \{\pm e_i \colon i = 1,\dots,d\} \eqqcolon \mathcal{U}$, the **face** in direction u by

$$\mathbb{F}(L,T,u) \coloneqq [0,T] \times \Big([-L,L]^{i-1} \times \{0\} \times [-L,L]^{d-i} + Lu\Big).$$

The **boundary** $\partial\mathbb{B}$ of $\mathbb{B}$ by consists of the top and the sides, that is

$$\partial\mathbb{B}(L,T) \coloneqq \mathbb{T}(L,T) \cup \textstyle\bigcup_{u\in\mathcal{U}} \mathbb{F}(L,T,u).$$

Note that the bottom $\{0\} \times [-L,L]^d$ of the box is not part of the boundary. We further subdivide $\mathbb{B}(L,T)$ into 2^d smaller boxes of equal sizes. That is, for $\theta \in \Theta \coloneqq \{\pm 1\}^d$, let

$$\mathbb{B}(L,T,\theta) \coloneqq \{(t,i_1,\dots,i_d) \colon \operatorname{sign}(i_1) = \theta_1, \dots, \operatorname{sign}(i_d) = \theta_d\}.$$

The **orthants** are obtained by intersecting these boxes with the top resp. the faces of $\mathbb{B}$. That is, for $u \in \mathcal{U}$ and $\theta \in \Theta$,

$$\begin{aligned} \mathbb{T}(L,T,\theta) &\coloneqq \mathbb{T}(L,B) \cap \mathbb{B}(L,T,\theta) \\ \mathbb{F}(L,T,u,\theta) &\coloneqq \mathbb{F}(L,B,u) \cap \mathbb{B}(L,T,\theta). \end{aligned}$$

See Figure 8 for an illustration in $d = 2$. We will omit the dependence on L and T if it is clear from the context.

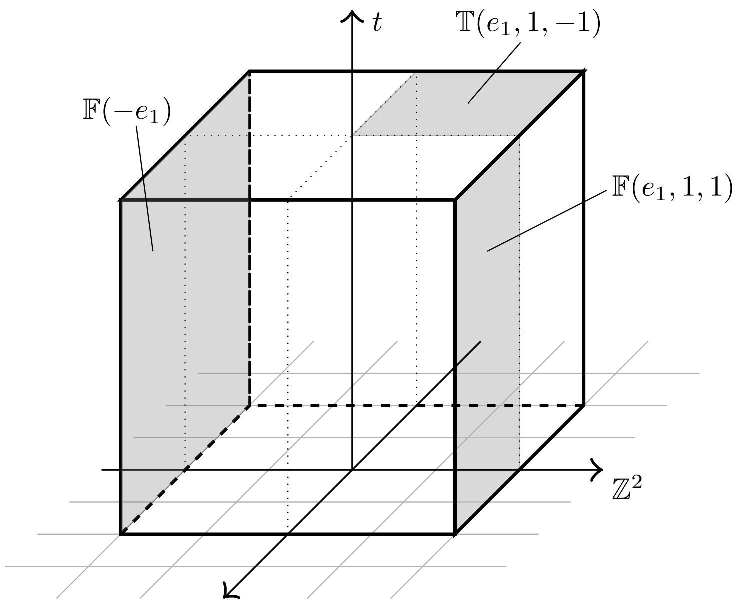

Figure 8: The space-time box $\mathbb{B}$ in dimension 2, with one face and two orthants marked.

Let now η be an initial configuration. For $u \in \mathcal{U}$ and $\theta \in \Theta$, let

$$N^\eta(L,T,u,\theta) \coloneqq \big|\{(t,v) \in [0,T] \times \mathcal{Z}^\eta \colon X(t,v) \in \mathbb{F}(L,T,u,\theta), X(s,v) \notin \partial\mathbb{B} \text{ for } s < t\}\big| \tag{10.10}$$

count the number of particles leaving $\mathbb{B}$ through $\mathbb{F}(L,T,u,\theta)$. That is, $N^\eta(L,T,u,\theta)$ is the number of times such that a particle of $\mathcal{Z}^\eta$ hits $\partial\mathbb{B}$ for the first time at some $(t,i) \in \mathbb{F}(L,T,u,\theta)$. Furthermore, for $\theta \in \Theta$, let $M^\eta(L,T,\theta)$ count the particles exiting $\mathbb{B}$ through $\mathbb{T}(L,T,\theta)$, that is

$$M^\eta(L,T,\theta) \coloneqq \big|\{v \in \mathcal{Z}^\eta(T) \colon X(T,v) \in \mathbb{T}(L,T,\theta), X(s,v) \notin \partial\mathbb{B} \text{ for } s < T\}\big|. \tag{10.11}$$

We interpret $M^\eta(L,T)$ (respectively $N^\eta(L,T)$) as 2^d-dimensional (resp. $d2^d$-dimensional) vectors, and $\sum M^\eta$ (resp. $\sum N^\eta$) always refers to summation over the coordinates:

$$\sum M^\eta(L,T) \coloneqq \sum_{\theta\in\Theta} M^\eta(L,T,u,\theta)$$
$$\sum N^\eta(L,T) \coloneqq \sum_{u\in\mathcal{U},\theta\in\Theta} N^\eta(L,T,u,\theta).$$

10.4.2. The number of particles on the boundary

In part (i) of the next lemma, we show that $\mathcal{Z}$ survives if and only if the number of particles is unbounded, which is a common feature of branching processes. Interpreting $\mathcal{Z}$ as a random process embedded in space-time, this means that many particles occupy the top of a space-time box.

Lemma 10.5. *(i) For every configuration η, almost surely*

$$\{\mathcal{Z}^\eta \text{ survives}\} = \big\{\lim_{t\to\infty} |\mathcal{Z}^\eta(t)| = \infty\big\}.$$

(ii) Let $(T_j)_j$ and $(L_j)_j$ be two sequences increasing to infinity. Then for any configuration η, almost surely

$$\big\{\mathcal{Z}^\eta \text{ survives}\big\} \subseteq \Big\{\lim_{j\to\infty} \sum N^\eta(L_j, T_j) + \sum M^\eta(L_j, T_j) = \infty\Big\}.$$

Proof. We define

$$\begin{aligned}\alpha &:= \mathbb{P}\big((\omega \cap [0,1] \times \{0\}) \neq \varnothing\big)\\ \beta &:= P\big(|\mathcal{V}^{\{0\}}(t)| = |\mathcal{V}^{\{0\}}(t) \cap \{0\}| = 1 \text{ for all } t \in [0,1]\big)\end{aligned}$$

Part (i): Note that "$\supseteq$" is elementary. For the other inclusion, note that $\mathcal{Z}^\eta$ dies out if every particle $v \in \mathcal{Z}^\eta(t)$ does not move or branch in $[t, t+1]$, and if there is a disaster in $[t, t+1]$ at $X(t, v)$. For $v \in \mathcal{Z}^\eta(t)$, these events have probabilities α and β, and we get

$$\mathbb{P}\big(\mathcal{Z}^A \text{ dies out}|\sigma(\mathcal{Z}^A_s \colon s \le t)\big) \ge (\alpha\beta)^{|\mathcal{Z}^A(t)|}.$$

For $t \to \infty$, the l.h.s. converges to $\mathbb{1}\{\mathcal{Z}^\eta \text{ dies out}\} \in \{0,1\}$. But, since $\alpha\beta > 0$, the r.h.s. only converges to zero on $\{|\mathcal{Z}^\eta_t| \to \infty\}$. This shows that, almost surely,

$$\{\mathcal{Z}^\eta \text{ survives}\} \subseteq \big\{\lim_{t\to\infty} |\mathcal{Z}^\eta(t)| = \infty\big\}.$$

Part (ii): We consider the space-time box $\mathbb{B}_j := [0, T_j] \times [-L_j + 1, L_j - 1]^d$. We denote by $\mathcal{F}_{L_j,T_j}$ the sigma algebra generated by the environment in $\mathbb{B}_j$ as well as the branching times and positions of particles inside $\mathbb{B}_j$. We consider the particles that exit $\mathbb{B}_j$,

$$\begin{aligned}E_j := &\Big\{(s,v) \in [0,T] \times \mathcal{Z}^\eta \colon \|X(s,v)\|_\infty = L_j, \|X(r,v)\|_\infty < L_j \text{ for all } r < s\Big\}\\ &\cup\ \Big\{(T,v) \in \{T\} \times \mathcal{Z}^\eta \colon \|X(r,v)\| < L_j \text{ for all } r \le T\Big\}.\end{aligned}$$

Here $\|\cdot\|_\infty$ denotes the maximum norm. Note that $(s,v) \in E_j$ implies that the particle v has just left $\mathbb{B}_j$ (for the first time) at time s, either through one of the sides or through the top. Clearly E_j is $\mathcal{F}_{L_j,T_j}$ measurable and we have $|E_j| = \sum N^\eta(L_j, T_j) + \sum M^\eta(L_j, T_j)$. By the same argument as before,

$$\mathbb{P}(\mathcal{Z}^\eta \text{ dies out}|\mathcal{F}_{L_j,T_j}) \ge (\alpha\beta)^{|E_j|}. \tag{10.12}$$

Moreover, taking $j \to \infty$, the l.h.s. converges to zero on $\{\mathcal{Z}^\eta \text{ survives}\}$, and therefore almost surely

$$\{\mathcal{Z}^\eta \text{ survives}\} \subseteq \big\{\lim_{j\to\infty} |E_j| = \infty\big\}.$$

□

10.4.3. An FKG-inequality

We prove that two independent trees in the same environment satisfy an FKG-inequality:

Theorem 10.6. *Let η_1 and η_2 be two configurations, and $\mathcal{V}_1^{\eta_1}$ and $\mathcal{V}_2^{\eta_2}$ two independent realizations of the process started from η_1 resp. η_2. We let $\mathcal{Z}_1^{\eta_1}$, $M_1^{\eta_1}$ and $N_1^{\eta_1}$ (resp. $\mathcal{Z}_2^{\eta_2}$, $M_2^{\eta_2}$ and $N_2^{\eta_2}$) be the random variables corresponding to $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). Moreover let $f, g\colon \mathbb{N}^{(2^d)} \times \mathbb{N}^{(d2^d)} \to \mathbb{R}^+$ be increasing. Then*

$$\mathbb{E}\Big[f\big(M_1^{\eta_1}, N_1^{\eta_1}\big)g\big(M_2^{\eta_2}, N_2^{\eta_2}\big)\Big] \geq \mathbb{E}\Big[f\big(M_1^{\eta_1}, N_1^{\eta_1}\big)\Big]\mathbb{E}\Big[g(M_2^{\eta_2}, N_2^{\eta_2}\big)\Big] \tag{10.13}$$

An intuitive explanation is that if many particles of $\mathcal{V}_1^{\eta_1}$ survive, then the environment is probably not too hostile, which improves the chance that many particles of $\mathcal{V}_2^{\eta_2}$ are alive.

Proof of Theorem 10.6. Let $\mathcal{H} := \sigma(\mathcal{V}_1^{\eta_1}, \mathcal{V}_2^{\eta_2})$. We will show that almost surely

$$\mathbb{E}\big[\big(M^{\eta_1}, N^{\eta_1}\big)g\big(M^{\eta_2}, N^{\eta_2}\big)\big|\mathcal{H}\big] \geq \mathbb{E}\big[f\big(M^{\eta_1}, N^{\eta_1}\big)\big|\mathcal{H}\big]\mathbb{E}\big[g\big(M^{\eta_2}, N^{\eta_2}\big)\big|\mathcal{H}\big] \tag{10.14}$$

Note that in the previous line the expectation is only over the environment, while the branching processes are fixed. The claim follows by integrating (10.14) and using the independence of $\mathcal{V}_1^{\eta_1}$ and $\mathcal{V}_2^{\eta_2}$. Now conditioned on $\mathcal{H}$, we can find $K \in \mathbb{N}$ and

$$0 = U_0 < U_1 < \cdots < U_K < U_{K+1} = T$$

such that both trees are constant on $[U_k, U_{k+1})$ for all $k = 0, \ldots, K$. That is, neither $\mathcal{V}_1^{\eta_1}$ nor $\mathcal{V}_2^{\eta_2}$ jumps or branches in $[0, T] \setminus \{U_1, \ldots, U_K\}$. Consider

$$\chi(k, i) := \mathbb{1}\{\omega \cap ([U_k, U_{k+1}) \times \{i\}) = \varnothing\}$$

the indicator function of the event that there is no disaster in $[U_k, U_{k+1})$ at site i. Let $\mathcal{G} := \sigma(\chi(k, i)\colon 0 \leq k \leq K, i \in \mathbb{Z}^d)$, and note that $M^{\eta_1}, N^{\eta_1}, M^{\eta_2}$ and N^{η_2} are $\mathcal{G}$-measurable and increasing in χ. Thus by the assumptions, $f(M^{\eta_1}, N^{\eta_1})$ and $g(M^{\eta_2}, N^{\eta_2})$ are also increasing in χ. Now (10.14) follows from the usual FKG inequality, see [44, Corollary 2.12]. The law of χ satisfies the FKG lattice condition because it is a product measure. □

Remark 10.7. Note that our application below would be much easier if Theorem 10.6 held for the same tree, i.e., with $\mathcal{V}_1$ equal to $\mathcal{V}_2$. The corresponding statement holds for the contact process, for which this technique was originally developed, see display (13) in [5]. The branching random walk, however, does not satisfy an FKG-inequality. To understand why, consider the situation with one initial particle, without disasters or branching. Then there will always only be exactly one particle, which can only exit the space-time box once. Thus for example the events $\{M(\theta_1) \geq 1\}$ and $\{M(\theta_2) \geq 1\}$ are actually negatively associated, for every $\theta_1 \neq \theta_2$. We are grateful to an anonymous referee for pointing out this problem.

We have introduced the parameter S in (10.4) to deal with this problem, with the help of the following Corollary:

Corollary 10.8. *For any $L, K, K' \in \mathbb{N}, T > 0$, any configuration η and any $S \in \mathbb{N}$,*

$$\prod_{\theta\in\Theta} \mathbb{P}\big(M^{S\eta}(L,T,\theta) \le K\big) \le \mathbb{P}\Big(\sum M^{\eta}(L,T) \le 2^d K\Big) + (2^d)^{-2^d S} \tag{10.15}$$

$$\prod_{\theta\in\Theta, u\in\mathcal{U}} \mathbb{P}\big(N^{S\eta}(L,T,u,\theta) \le K\big) \le \mathbb{P}\Big(\sum N^{\eta}(L,T) \le d2^d K\Big) + (d2^d)^{-d2^d S} \tag{10.16}$$

and

$$\begin{aligned} &\mathbb{P}\Big(\sum N^{S\eta}(L,T) \le K\Big)\mathbb{P}\Big(\sum M^{S\eta}(L,T) \le K'\Big) \\ \le &\mathbb{P}\Big(\sum M^{\eta}(L,T) + \sum N^{\eta}(L,T) \le K + K'\Big) + 4^{-S} \end{aligned} \tag{10.17}$$

Proof. We only give the proof of (10.15), since the other claims follow in the same way. We have

$$\begin{aligned} \prod_{\theta\in\Theta} P_\omega(M^{S\eta}(L,T,\theta) \le K) &\le \prod_{\theta\in\Theta} \big(P_\omega(M^{\eta}(L,T,\theta) \le K)\big)^S \\ &\le P_\omega\big(M^{\eta}(L,T,u,\theta) \le K \text{ for all } \theta\in\Theta\big) + (2^d)^{-2^d S} \\ &\le P_\omega\Big(\sum M^{\eta}(L,T) \le 2^d K\Big) + (2^d)^{-2^d S} \end{aligned} \tag{10.18}$$

The second inequality is by Lemma 10.4, and the third inequality is elementary. To see the first inequality, observe $\mathcal{Z}^{S\eta}$ can be interpreted as S independent copies of $\mathcal{Z}^{\eta}$, in the same environment. Therefore

$$\big\{M^{S\eta}(L,T,\theta) \le K\big\} \overset{d}{\subseteq} \big\{\widetilde{M}_i \le K \text{ for } i = 1,\dots,S\big\},$$

where $\widetilde{M}_i$ are independent copies of $M^{\eta}(L,T,\theta)$. The claim follows by taking expectation in (10.18), and applying Theorem 10.6 to the l.h.s.. □

10.5. Proof of the key proposition

We now start proving Proposition 10.1. For $\varepsilon > 0$, we find $\delta > 0$ such that

$$\min\Big\{\Big(1-(3\delta)^{(2^d)^{-1}}\Big)\Big(1-(2\delta)^{(d2^d)^{-1}}\Big)(1-\delta)^3, 1-3\delta\Big\} \ge (1-\varepsilon)^{1/10}. \tag{10.19}$$

Using Lemma 10.3, we find $n \in \mathbb{N}$ such that

$$\mathbb{P}\big(\mathcal{Z}^{[-n,n]^d} \text{ survives}\big) \ge 1-\delta^2. \tag{10.20}$$

Moreover, let S be an integer such that

$$\max\Big\{\Big(1-\frac{1}{d2^d}\Big)^{d2^d S}, \Big(1-\frac{1}{2^d}\Big)^{2^d S}, 4^{-S}\Big\} \le \frac{\delta^2}{2}.$$

We state an auxiliary proposition:

Proposition 10.9. *There exist $L, T > 0$ such that*

$$\mathbb{P}^{\kappa,\lambda,q}\left(\begin{array}{c}\exists i \in [L+n, 2L+n] \times [0,2L]^{d-1}, t \in [T+2, 2T+2] \\ \textit{such that } (i+[-n,n]^d, S^2) \leq \mathcal{Z}^{([-n,n]^d, S^2)}_{[-L,3L]\times[-3L,3L]^{d-1}}(t)\end{array}\right) > (1-\varepsilon)^{1/10}. \quad (10.21)$$

This proposition will be proved in the next section.

Proof of Proposition 10.1. Note that the event (10.21) already looks very similar to the event $A^{s,j}$. It remains to show that the bound is uniform in the initial configuration. Let L' and T' be large enough that (10.21) holds, and set $L := 2L' + n$ and $T := 2T'$. Observe that, by symmetry, we can replace the target area $[L'+n, 2L'+n] \times [0, 2L']^{d-1}$ in (10.21) by

$$\mathbb{W}_{L,n}(w) := [w_1(L'+n), w_1(2L'+n)] \times [0, 2w_2L'] \times \cdots \times [0, 2w_dL'],$$

for any direction $w \in \{\pm 1\}^d$. Here an interval $[b,a]$ is to be interpreted as $[a,b]$ if $a < b$. We now repeatedly apply the result from Proposition 10.9, each time making a suitable choice for the direction w. See Figure 9 for an illustration.

More precisely, we choose space-time sites $(s^k, j^k)_{k\in\mathbb{N}}$ and directions $(w^k)_{k\in\mathbb{N}}$, starting with $(s^0, j^0) = (s, j)$ as in the statement of the proposition. Assuming that we have constructed $(s^0, j^0), \dots, (s^k, j^k)$, we choose $w^k = (w^k_1, \dots, w^k_d)$ in the following way:

- For the first coordinate, if $j^0_1, \dots, j^k < L + L' + n$, then $w^k_1 := 1$. Otherwise we choose alternating signs, i.e. $w^k_1 := -w^{k-1}_1$.
- For all other coordinates $l \neq 1$, choose $w^k_l := -\operatorname{sign}(j^k_l)$.

Then, given (s^k, j^k) and direction w^k, consider the event

$$\left\{\begin{array}{c}\exists i \in j^k + \mathbb{W}_{L',n}(w^k), t \in s^k + [T'+2, 2T'+2] \\ \text{such that } (i+[-n,n]^d, S^2) \leq \mathcal{Z}^{(j^k+[-n,n]^d, S^2)}_{j^k+[-w^k_1L', 3w^k_1L']\times[-3L',3L']^{d-1}}(t)\end{array}\right\}.$$

If this event succeeds, we set (s^{k+1}, j^{k+1}) equal to (i, t). If there is more than one such pair, we choose the one with minimal time-coordinate – this ensures that the steps are independent. Moreover, we note that our choice of direction ensures that

- $|j^k_l| \leq 2L' \leq L$ for $l = 2, \dots, d$, for all $k \geq 0$.
- $j^k_1 \in [L, 3L]$ for all $k \geq 4$: Note that $j^k_1 \geq L + L' + n$ after at most 4 iterations, and the alternating the sign of u ensures that $L \leq j^l_1 \leq 3L$ for all $l \geq k$.
- $s^k \in [5T, 6T]$ for some $4 \leq k \leq 10$: After 4 iterations we have $s^4 \in [4T'+8, 8T'+8] \subseteq [2T, 4T]$. Moreover, the duration of one iteration is $[T'+2, 2T'+2] \subseteq [T/2, T]$, so we reach the target interval after at most 10 iterations.

Each iteration has success probability at least $(1-\varepsilon)^{1/10}$, and since we need at most 10 successes the total success probability is at least $1 - \varepsilon$. $\square$

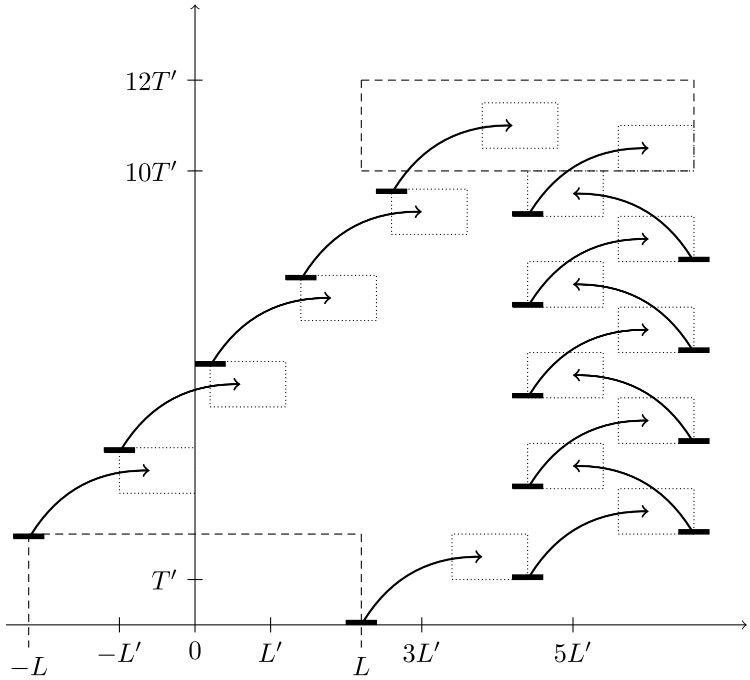

Figure 9: In the auxiliary Proposition 10.9 we have shown that starting from configuration $\{s^k\} \times (j^k + [-n, n])$ (depicted as black bars), there is high probability of finding a shifted copy of $[-n, n]$ centered in the target box $(s^k, j^k) + \mathbb{W}_{L',n}(w^k)$ (depicted as small, dotted boxes at the end of the arrows). We have drawn two "trajectories" to illustrate that 4–10 iterations are enough to reach $[5T, 6T] \times [L, 3L]$, from any initial configuration in $[0, T] \times [-L, L]$.

10.6. Proof of the auxiliary proposition

One of the following two statements is true, and we give separate proofs in each case:

$$\exists L \in \mathbb{N}\colon \mathbb{P}\big(\mathcal{Z}_L^{([-n,n]^d,S)} \text{ survives}\big) \geq 1 - 2\delta \qquad \text{(local case)}$$

$$\forall L \in \mathbb{N}\colon \mathbb{P}\big(\mathcal{Z}_L^{([-n,n]^d,S)} \text{ survives}\big) < 1 - 2\delta. \qquad \text{(non-local case)}$$

Intuitively, in the local case we can restrict the process to a large box without decreasing the survival probability too much, whereas in the non-local case all truncations of $\mathcal{Z}$ have small survival probability.

10.6.1. The non-local case

Proof of Proposition 10.9. We start by choosing L and T. We first identify a suitably large number R to apply Lemma 10.5: Let

$$\alpha \coloneqq \min_{v \in \{0, ne_1\}} \left\{ \mathbb{P}\Big((v + [-n, n]^d, S^2) \leq \mathcal{Z}^{\{0\}}_{v+[-n,n]^d}(1) \Big) \right\}. \tag{10.22}$$

and choose R_1 such that $(1-\alpha)^{R_1} < \delta$. By part (iii) of Lemma 10.3, we find R_2 such that

$$\min_{v \in \{0, ne_1\}} \Big\{ \mathbb{P}\Big((v + [-n,n]^d, S^2) \le \mathcal{Z}^{(\{0\},R_2)}_{v+[-n,n]^d}(1) \Big) \Big\} > 1 - \delta.$$

Finally, due to part (ii) of Lemma 10.3, we can choose R_3 large enough that

$$\mathbb{P}\Big(\big| \mathcal{Z}^{(\{0\},R_3)}_{\{0,e_1\}}(1) \cap \{0\} \big| \ge R_2 \Big) > 1 - \delta.$$

Now set $R \coloneqq ((4n)^d R_1 \vee nR_3)^2$. From Lemma 10.5(i) and the definition of n, we obtain

$$\lim_{T\to\infty} \lim_{L\to\infty} \mathbb{P}\Big(\sum M^{[-n,n]^d}(L,T) > 2^d R \Big) = \lim_{T\to\infty} \mathbb{P}(|\mathcal{Z}^{[-n,n]^d}(T)| > 2^d R) \ge 1 - \delta^2.$$

That is, for all $T \ge T_0$ there exists $L(T)$ such that for $L \ge L(T)$

$$\mathbb{P}\Big(\sum M^{([-n,n]^d,S)}(L,T) > 2^d R \Big) \ge \mathbb{P}\Big(\sum M^{[-n,n]^d}(L,T) > 2^d R \Big) \ge 1 - \delta. \tag{10.23}$$

We have shown that, by choosing L and T large enough, the probability of finding $2^d R$ particles at the top of a box $[0,T] \times [-L,L]^d$ can be made large. We want a similar result for the number of particles leaving through the sides of $[0,T] \times [-L,L]^d$. Using (10.23) and the definition of the non-local case, we can define two increasing sequences $(L_k)_{k\ge 0}$ and $(T_k)_{k\ge 0}$, starting with T_0 as introduced before in (10.23), and $L_0 \coloneqq L(T_0) + 1$. We proceed by

$$\begin{aligned} L_{k+1} &\coloneqq \max\big\{ L_k + 1, L(T_k + 1) \big\} \\ T_{k+1} &\coloneqq \inf\Big\{ T > T_k \colon \mathbb{P}\Big(\sum M^{([-n,n]^d,S)}(L_{k+1},T) > 2^d R \Big) < 1 - 2\delta \Big\}. \end{aligned}$$

Observe that since $T \mapsto \mathbb{P}(\sum M^{[-n,n]^d}(L,T) > 2^d R)$ is continuous, for all $k \in \mathbb{N}$

$$\mathbb{P}\Big(\sum M^{([-n,n]^d,S)}(L_k,T_k) \le 2^d R \Big) = 2\delta. \tag{10.24}$$

Note that $\mathbb{B}(L,T)$ has a total number of $(d+1)2^d$ orthants in its boundary. We apply part (i) of Lemma 10.5 with the sequences $(L_k)_k$ and $(T_k)_k$ defined before, which gives

$$\begin{aligned} \liminf_{k\to\infty} \mathbb{P}\Big(\sum N^{[-n,n]^d}(L_k,T_k) + \sum M^{[-n,n]^d}(L_k,T_k) > (d+1)2^d R \Big) \\ \ge \mathbb{P}(\mathcal{Z}^{[-n,n]^d} \text{ survives}). \end{aligned}$$

Thus, using (10.20), there exists k_0 such that for all $k \ge k_0$

$$\mathbb{P}\Big(\sum N^{[-n,n]^d}(L_k,T_k) + \sum M^{[-n,n]^d}(L_k,T_k) \le (d+1)2^d R \Big) \le \frac{3}{2}\delta^2.$$

We set $L \coloneqq L_{k_0}$ and $T \coloneqq T_{k_0}$, which finishes the constructive part of the proof.

Next, we claim that, for any choice of $\theta \in \Theta$ and $u \in \mathcal{U}$,

$$\mathbb{P}\big(M^{([-n,n]^d,S^2)}(L,T,\theta) \le R\big)^{2^d} \le 3\delta \tag{10.25}$$
$$\mathbb{P}\big(N^{([-n,n]^d,S^2)}(L,T,u,\theta) \le R\big)^{d2^d} \le 2\delta. \tag{10.26}$$

Note that, the (annealed) laws of $N^{([-n,n]^d,S^2)}(L,T,u,\theta)$ and $M^{([-n,n]^d,S^2)}(L,T,u,\theta)$ do not depend on θ or u. Therefore (10.25) follows from (10.15) together with (10.24) and the definition of S. For the second claim, note that

$$\begin{aligned}\frac{3}{2}\delta^2 &\ge \mathbb{P}\Big(\sum N^{[-n,n]^d}(L,T) + \sum M^{[-n,n]^d}(L,T) \le (d+1)2^d R\Big)\\ &\ge \mathbb{P}\Big(\sum N^{([-n,n]^d,S)}(L,T) \le d2^d R\Big)\mathbb{P}\Big(\sum M^{([-n,n]^d,S)}(L,T) \le 2^d R\Big) - \frac{\delta^2}{2}.\end{aligned}$$

For the second inequality we have used (10.17) and the definition of S. Using (10.24), we get

$$\mathbb{P}\Big(\sum N^{([-n,n]^d,S)}(L,T) \le d2^d R\Big) \le \delta. \tag{10.27}$$

Now (10.26) follows from (10.16) and the definition of S.

With these estimates, we can now verify (10.21): We need to bound the probability of finding a copy of $[-n,n]^d$ shifted to the correct space-time location where every site is occupied by at least S^2 particles. From now on, we keep

$$\theta := (1,\dots,1) \in \Theta$$

fixed. We show that each of the following steps occurs has high probability:

1. The tree $\mathcal{Z}^{([-n,n]^d,S^2)}$ has many particles leaving through $\mathbb{F}(e_1,\theta)$.
2. There exist $(t,i) \in \mathbb{F}(e_1,\theta)$ such that the particles occupying i at time t grow into a fully occupied copy $\{t+1\} \times (i + ne_1 + [-n,n]^d, S^2)$ of $([-n,n]^d, S^2)$.
3. Consider now the box $\overline{\mathbb{B}} := \big([0,T] \times [-L,L]^d\big) + (t+1, i+ne_1)$. The tree growing from $\{t+1\} \times (i+ne_1+[-n,n]^d, S^2)$ will have many descendants that leave through the top $\overline{\mathbb{T}}(e_1,\theta)$ of $\overline{\mathbb{B}}$.
4. There exists $(\bar t, \bar i) \in \overline{\mathbb{T}}(e_1,\theta)$ that grows into a new copy of the box $\{\bar t+1\} \times (\bar i + [-n,n]^d, S^2)$.

See also Figure 10 for an illustration. We have already estimated the probability of the **first step** in (10.26). For the **second step**, we have to consider the set $\mathcal{R}$ of space-time-points in $\mathbb{F}(e_1,\theta)$ where a particle leaves $[0,T] \times [-L,L]^d$ for the first time, i.e.

$$\mathcal{R} := \Big\{(t,i) \in \mathbb{F}(e_1,\theta) \colon \exists v \in \mathcal{Z}_L^{([-n,n]^d,S^2)}(t) \text{ s.t. } i = X(t,v), X(s,v) \notin \partial\mathbb{B}\ \forall s < t\Big\}$$

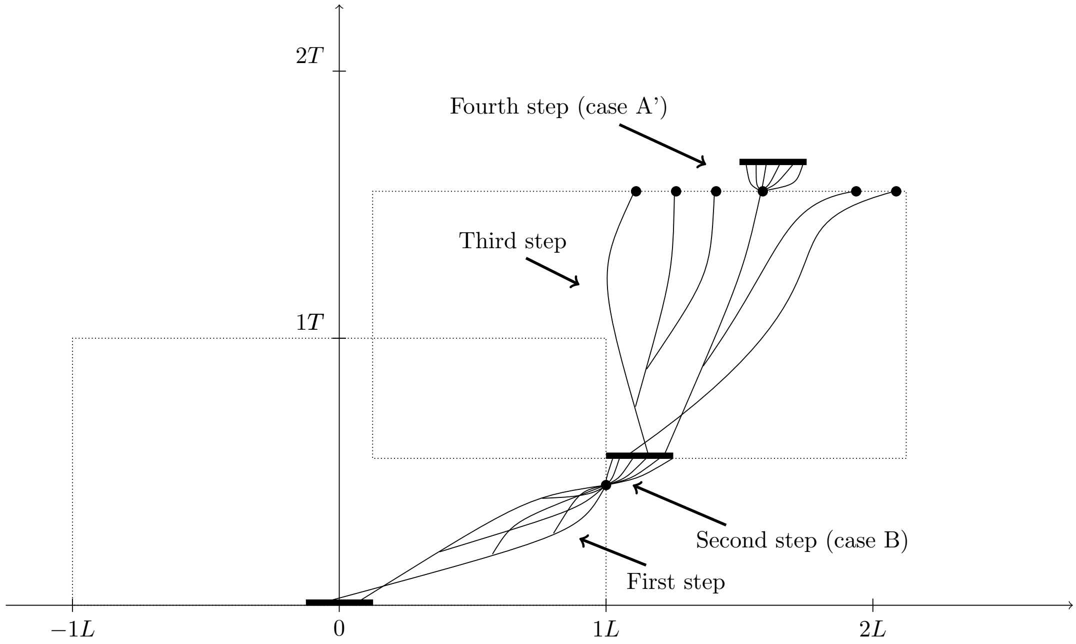

Figure 10: To prove (10.21) we show that each step depicted above has high probability. For the second and fourth step, we use that if a space-time box has many particles on its boundary, then either many well-separated sites are occupied by at least one particle (case A/A') or there is one site that is occupied by many particles (case B/B').

That is, $N^{([-n,n]^d,S^2)}(e_1,\theta) = |\mathcal{R}|$. For $(t,i) \in \mathcal{R}$, let $E_{t,i}$ be the event that the particle exiting $\mathbb{B}$ in (t,i) grows into a shifted copy of $[-n,n]^d$, using only particles that do not move far away:

$$E_{t,i} \coloneqq \big\{\big((t, i+ne_1) + [-n,n]^d, S^2\big) \le \mathcal{Z}^{\{t\}\times\{i\}}_{i+ne_1+[-n,n]^d}(1)\big\}.$$

We show that this happens with high probability:

$$\mathbb{P}\big(\textstyle\bigcup_{(t,i)\in\mathcal{R}} E_{(t,i)} \big| N^{([-n,n]^d,S^2)}(e_1,\theta) > R\big) \ge (1-\delta)^2 \tag{10.28}$$

Assuming (10.28) and recalling (10.26), we then have

$$\mathbb{P}\begin{pmatrix} \exists i \in \{L+n\}\times[0,L]^{d-1}, t\in[0,T+1] \\ \text{s. th.}(i+[-n,n]^d, S^2) \le \mathcal{Z}^{([-n,n]^d,S^2)}_{[-L,L+2n]\times[-L-n,L+n]^{d-1}}(t)\end{pmatrix} \ge \Big(1-(2\delta)^{(d2^d)^{-1}}\Big)(1-\delta)^2 \tag{10.29}$$

Proof of (10.28). We need to deal with the correlations between $E_{t,i}$ and $E_{s,j}$, for $(t,i) \neq (s,j) \in \mathcal{R}$. We will consider two cases: Either $\mathcal{R}$ contains many space-time points that are far apart (then events are independent), or there is one area where many particles exit (then we will use Lemma 10.3(ii)–(iii)).

We introduce a tiling $\mathbb{F}(e_1,\theta) \subseteq \bigcup_{(t,i)\in I} \big((t,i)+H\big)$ of $\mathbb{F}(e_1,\theta)$, with

$$I := \big(\mathbb{N} \times \{L\} \times n\mathbb{Z}^{d-1}\big) \cap \mathbb{F}(e_1,\theta)$$
$$H := [0,1] \times \{0\} \times [0,n)^{d-1}$$

On $\{N^{([-n,n]^d,S^2)}(e_1,\theta) > R\}$, one of the following statements holds:

(case A) There exist at least $\sqrt{R}$ distinct indices $(t,i) \in I$ such that $\mathcal{R} \cap \big((t,i)+H\big) \neq \varnothing$.
(case B) There exists $(t_0,i_0) \in I$ such that $\big|\mathcal{R} \cap \big((t_0,i_0)+H\big)\big| \geq \sqrt{R}$.

First consider case A: Since $\sqrt{R} \geq (4n)^d R_1$, we can find R_1 distinct indices (s_1,j_1), $\ldots,(s_{R_1},j_{R_1}) \in I$ such that $|s_l - s_k| \geq 2$ and $\|j_l - j_k\|_\infty \geq 4n$ for $l \neq k$. Choose (in some deterministic way) $(t_l,i_l) \in \mathcal{R} \cap (s_l,j_l)+H$. Because of the truncation, E_{t_l,i_l} and E_{t_k,i_k} are independent for $k \neq l$. Moreover, recalling (10.22),

$$\mathbb{P}\big(E_{t_k,i_k}\big|N^{([-n,n]^d,S^2)}(e_1,\theta) > R\big) \geq \alpha$$

Now (10.28) follows from the definition of R_1. In case B, there is $j_0 \in \{0\} \times [0,n-1)^{d-1}$ such that

$$\big|\mathcal{R} \cap [t_0,t_0+1] \times \{i_0+j_0\}\big| \geq \frac{\sqrt{R}}{n} \geq R_4$$

Let G be the event that

- at least R_3 of those particles survive until time t_0+1
- while not leaving the set $\{i_0+j_0, i_0+j_0+e_1\}$,
- and occupying i_0+j_0 at time t_0+1.

By our choice of R_4 and part (ii) of Lemma 10.3,

$$\mathbb{P}(G) \geq \mathbb{P}\Big(\Big|\mathcal{Z}^{(\{0\},R_4)}_{\{0,e_1\}}(1) \cap \{0\}\Big| \geq R_3\Big) \geq 1-\delta. \tag{10.30}$$

Moreover, by our choice of R_3 and part (iii) of Lemma 10.3, we find

$$\mathbb{P}\Big((ne_1+[-n,n]^d,S^2) \leq \mathcal{Z}^{\{t_0+1\}\times(\{i_0+j_0\},R_3)}_{ne_1+[-n,n]^d}(t_0+2)\Big) \geq 1-\delta. \tag{10.31}$$

Combining (10.30) and (10.31) finishes the proof. □

Third step: We now write $\overline{\mathbb{P}}$ for $\mathbb{P}$ conditioned on the event in (10.29), and denote the first such pair by (t,i). From now on, we consider the process

$$\big(\overline{\mathcal{Z}}_L(s)\big)_{s\geq t} := \Big(\mathcal{Z}^{\{t\}\times(i+[-n,n]^d,S^2)}_{i+[-L,L]^d}(s)\Big)_{s\geq t}$$

started from $\{t\}\times(i+[-n,n]^d,S^2)$. Observe that under $\overline{\mathbb{P}}$, the process $\overline{\mathcal{Z}}_L$ is independent of the process up to time t. We consider a shifted space-time box

$$\overline{\mathbb{B}} := (t,i)+[0,T]\times[-L,L]^d.$$

and let $\overline{M}$ count the number particles of $\overline{\mathcal{Z}}_L$ that leave $\overline{\mathbb{B}}$ through $\overline{\mathbb{T}}$. By (10.25) we have

$$\overline{\mathbb{P}}(\overline{M}(\theta) \geq R) \geq 1 - (3\delta)^{2^{-d}} \tag{10.32}$$

Fourth step: Let

$$\overline{\mathcal{R}} := \{v \in \overline{\mathcal{Z}}_L(T) \colon (T, \overline{X}(T,v)) \in \overline{\mathbb{T}}(\theta), (s, \overline{X}(s,v) \notin \partial\overline{\mathbb{B}} \text{ for } s \in [0,T)\}$$

denote the set of particles of $\overline{\mathcal{Z}}$ that leave $\overline{\mathbb{B}}$ through the orthant in direction θ of the top $\overline{\mathbb{T}}$. Note that, in contrast to $\mathcal{R}$ before, $\overline{\mathcal{R}}$ is a set of nodes, since two particles can exit through the same site. For $v \in \overline{\mathcal{R}}$, let

$$E_v := \Big\{(\overline{X}(v,T) + [-n,n]^d, S^2) \leq \overline{\mathcal{Z}}^{\{t+T\}\times\{\overline{X}(v,T)\}}_{\overline{X}(v,T)+[-n,n]^d}(t+T+1)\Big\}$$

As before, we will show

$$\overline{\mathbb{P}}\big(\textstyle\bigcup_{v\in\overline{\mathcal{R}}} E_v | \overline{M}(\theta) \geq R\big) \geq 1 - \delta. \tag{10.33}$$

Combining (10.33) with (10.32) then gives

$$\overline{\mathbb{P}}\left(\exists \bar{i} \in i + [0,L]^d \text{ s. th. } (\bar{i} + [-n,n]^d, S^2) \leq \overline{\mathcal{Z}}_L(t+T+1)\right) \geq \Big(1 - (3\delta)^{(d2^d)^{-1}}\Big)(1-\delta). \tag{10.34}$$

Since $i + \bar{i} \in [L+n, 2L+n] \times [0,2L]^{d-1}$ and $t + T + 1 \in [T+2, 2T+2]$, the claim now follows from (10.34), (10.29) and our choice of δ. □

Proof of (10.32). Similar to before, one of these two cases will occur:

(Case A') There are distinct particles $v_1, \dots, v_{\sqrt{R}} \in \overline{\mathcal{R}}$, all of which occupy distinct sites.
(Case B') There are distinct particles $v_1, \dots, v_{\sqrt{R}} \in \overline{\mathcal{R}}$, all of which occupy the same site.

In case A' we have $\sqrt{R} \geq (4n)^d R_1$, so that we find particles $\widehat{v}_1, \dots, \widehat{v}_{R_1}$ in $\overline{\mathcal{R}}$ satisfying, for $i \neq j$,

$$\|\overline{X}(\widehat{v}_i, T) - \overline{X}(\widehat{v}_j, T)\|_\infty \geq 2n + 1.$$

Because of the truncation, $\overline{E}_{\widehat{v}_i}$ and $\overline{E}_{\widehat{v}_j}$ are independent for $i \neq j$, under $\overline{\mathbb{P}}$. Moreover, by definition of α, we have

$$\overline{\mathbb{P}}(E_{\widehat{v}_i} | \overline{M}(e_1, -\theta) \geq R) \geq \alpha$$

for $i = 1, \dots, R_1$, and thus (10.33) follows from the definition of R_1. On the other hand, in case B' our choice of R_3 directly implies (10.33).

□

10.6.2. The local case

Proof of Proposition 10.9 in the local case: Take $L \in \mathbb{N}$ large enough such that the condition of the local case holds. On $\{\mathcal{Z}_L^{([-n,n]^d,S^2)}$ survives$\}$, let $(v_k)_{k\in\mathbb{N}}$ be a sequence of particles with $v_k \in \mathcal{Z}_{L'}^{([-n,n]^d,S^2)}(k)$ (chosen in some deterministic way). For $k \in \mathbb{N}$, let

$$A_k := \left\{((L+n)e_1 + [-n,n]^d, S^2) \leq \mathcal{Z}_{[-L,3L]\times[-3L,3L]^{d-1}}^{\{k\}\times\{X(k,v_k)\}}((k+1))\right\}.$$

That is, A_k is the event that we spontaneously generate the desired particles in time $[k, k+1]$, starting from one particle at time k at site $X(k, v_k)$. Since $X(tk, v_k)$ can take only finitely many values, we have

$$\alpha := \inf_k \operatorname{ess\,inf} \mathbb{P}(A_k|\mathcal{F}_k) > 0 \tag{10.35}$$

where $\mathcal{F}_t$ denotes the sigma-field of environment and branching process up to time t. Let $T \in \mathbb{N}$ be large enough that $(1-\alpha)^T \leq \delta$, and observe that

$$\begin{aligned}
&\left\{\mathcal{Z}_L^{([-n,n]^d,S^2)} \text{ survives}\right\} \cap \bigcup_{k=T+2}^{2T+2} A_k \\
&\qquad \subseteq \left\{\begin{array}{l} \exists i \in [L+n, 2L+n]\times[0,2L]^{d-1}, t \in [T+2, 2T+2] \\ \text{such that } (i+[-n,n]^d, S^2) \leq \mathcal{Z}_{[-L,3L]\times[-3L,3L]^{d-1}}^{([-n,n]^d,S^2)}(t) \end{array}\right\}.
\end{aligned}$$

The definition of α and our choice of T show that the event on the l.h.s. has probability at least $(1-\delta^2)(1-\delta) \geq 1-\varepsilon$. $\square$

Part IV.
Brownian motion among space-time disasters

11. Introduction

11.1. Motivation

In this section, we study the continuous-space analogue of the model from Part II.

The environment ω is a collection of random space-time disasters $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, and we replace the random walk on $\mathbb{Z}^d$ by d-dimensional Brownian motion B. The intersection between the graph of B and ω is almost surely empty, so to define the extinction time of B it is natural to enlarge ω in the space-coordinate, i.e., to define

$$\mathcal{D} := \bigcup_{(s,x)\in\omega} \{s\} \times U(x). \tag{11.1}$$

Here $U(x) \subseteq \mathbb{R}^d$ is the ball of unit volume around x. Observe that the enlargement is only in the space-coordinate, and that each disaster still has "duration" zero. The extinction time of B is the hitting time of the enlarged obstacle set $\mathcal{D}$, i.e. the first time t such that $B(t)$ is close to x, for some disaster $(t,x) \in \omega$, see Figure 11.

As in Part II, we are primarily interested in the (quenched) decay rate of the survival probability, see Figure 12. More precisely, we show that the survival probability (almost surely) decays at a deterministic exponential rate $\mathfrak{p}$.

We also consider a "soft" version of these dynamics, where B gets killed with a certain probability $p \in (0,1)$ whenever it hits a disaster. The "disastrous" model described above then corresponds to the case $p = 1$. As a second result, we show that the decay rate $\mathfrak{p}$ is a continuous function of p, even at the boundary $p \uparrow 1$.

Note that this is somewhat surprising, as one might think that two cases behave quite differently – after all, for $p < 1$ the Brownian motion can handle unfavorable parts of the environment simply by crossing some disasters, even if this carries some cost. For $p = 1$ on the other hand, we have to "counteract" the degeneracy of the environment using only the spatial freedom of Brownian motion.

We have already discussed a similar transition from soft to hard disasters in discrete time in Section 4.4, where the question of continuity in the zero-temperature limit is an open problem. This question was a partial motivation for our work on Brownian polymers, which can be seen as the natural generalization to continuous time and space: It is reasonable to conjecture that continuous time/space help smooth out the strong degeneracy of a disastrous environment, and indeed this is what we prove.

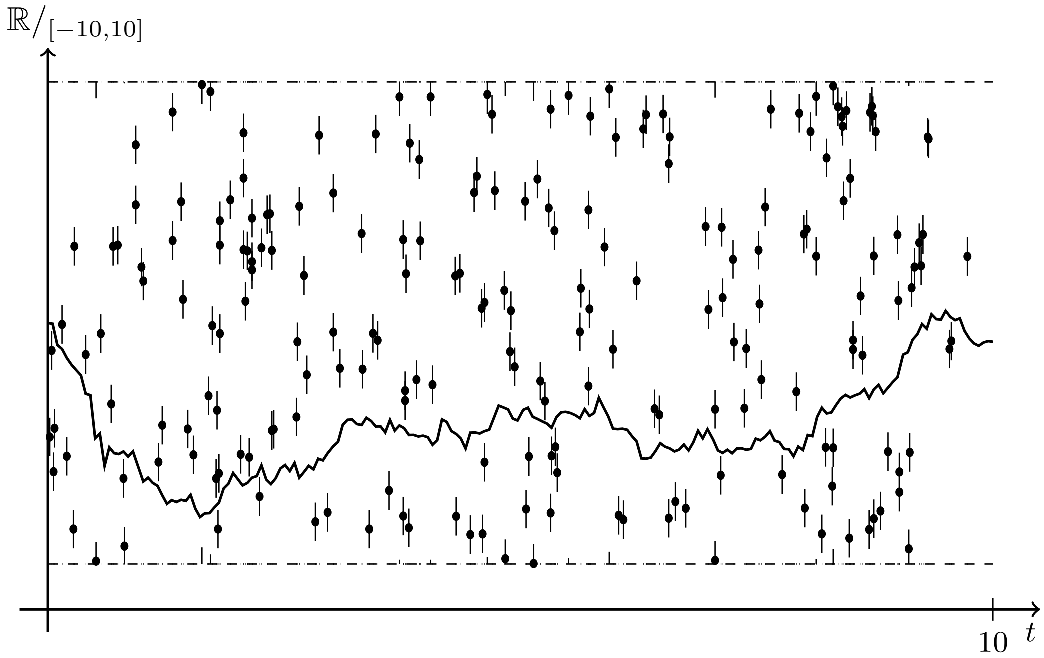

Figure 11: Example for an environment ω on the torus $\mathbb{R}/_{[-10,10]}$, together with a surviving path: The circles represent disasters and the bars the enlarged obstacle set $\mathcal{D}$. The graph of B does not intersect $\mathcal{D}$, and we therefore say that B survives (up to time 10).

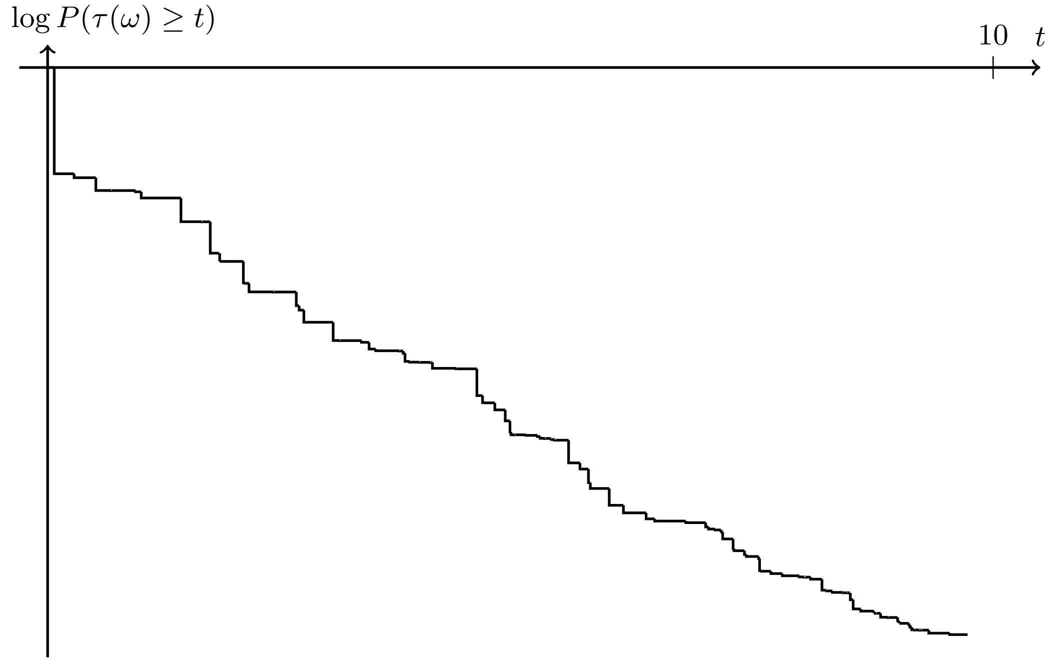

Figure 12: Plot of the quenched survival probability in the environment ω from above. Every jump in the graph corresponds to a disaster of ω.

11.2. Definition and known results

Let Ω denote the set of locally finite point measures on $\mathbb{R}_+ \times \mathbb{R}^d$. As always, we identify $\omega \in \Omega$ with its support, i.e. we will regard $\omega \subseteq \mathbb{R}_+ \times \mathbb{R}^d$. Let $(\omega, \mathbb{P})$ denote the Poisson point process with unit intensity on $\mathbb{R}_+ \times \mathbb{R}^d$, and let (B, P) denote d-dimensional Brownian motion. We will interpret P as a probability measure on $\mathcal{I}$, the set of càdlàg paths $b\colon \mathbb{R}_+ \to \mathbb{R}^d$.

Let $\beta \in [0, \infty]$ be the **inverse temperature** of the model. We will refer to $\beta < \infty$ as **positive temperature**, and to $\beta = \infty$ as the **zero-temperature** case. For $b \in \mathcal{I}$ and $t \geq 0$, we define the tube $V_t(b)$ of volume t around b by

$$V_t(b) \coloneqq \big\{(s, x) \in [0, t) \times \mathbb{R}^d \colon x \in U(b(s))\big\}.$$

Recall that $U(x)$ is the unit volume ball centered around $x \in \mathbb{R}^d$, and note that

$$\big|\{s \in [0, t) : (s, B(s)) \in \mathcal{D}\}\big| = |V_t(B) \cap \omega| = \omega(V_t(B)).$$

For the last expression, recall that ω is a measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Let ξ denote an exponential random variable of parameter 1, independent of B but defined on the same probability space. We introduce the extinction time

$$\tau_\beta(\omega) \coloneqq \begin{cases} \inf\big\{t \geq 0 \colon \beta|\omega \cap V_t(B)| \geq \xi\big\} & \text{if } \beta < \infty \\ \inf\big\{t \geq 0 \colon \omega \cap V_t(B) \neq \varnothing\big\} & \text{if } \beta = \infty. \end{cases} \tag{11.2}$$

Note that, as in Part II, the notation does not reflect that $\tau_\beta(\omega)$ is also a random variable depending on B. An easy calculation shows that in positive temperature

$$P(\tau_\beta(\omega) \geq t) = E\big[e^{-\beta|V_t(B)\cap\omega|}\big] \tag{11.3}$$

while in the zero-temperature case

$$P(\tau_\infty(\omega) \geq t) = P(V_t(B) \cap \omega = \varnothing).$$

Let us also introduce the **polymer measure**, i.e. the random probability measure $\mu^t_{\omega,\beta}$ on $\mathcal{I}$ defined by

$$\mu^t_{\omega,\beta}(B \in \cdot) \coloneqq P(B \in \cdot|\tau_\beta(\omega) \geq t). \tag{11.4}$$

Under this measure, the polymer is repulsed by the environment. In positive temperature, we can try to understand (11.4) by comparing the entropic cost of avoiding the disasters to the cost of intersections with the environment. This is more difficult in zero temperature, since the Brownian motion is not allowed to intersect the environment and the influence of the disasters is therefore larger.

Remark 11.1. In the earlier works [16, 19, 15, 11, 20] on Brownian directed polymers, the partition function in positive temperature is defined by (11.3). However, in those works, there is no negative sign in front of β, and general $\beta \in \mathbb{R}$ are considered. Since we focus on the disastrous case, we have chosen to deviate from their notation.

We study the logarithmic decay rate of the survival probability $P(\tau_\beta(\omega) \geq t)$. Existence and continuity are known in positive temperature:

Theorem M ([16, Theorem 2.2.1])**.** *For every $\beta \in [0, \infty)$ there exists $\mathfrak{p}(\beta) \in (-\infty, e^{-\beta}-1]$ such that almost surely*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log P(\tau_\beta(\omega) \geq t)\big] = \lim_{t\to\infty} \frac{1}{t} \log P(\tau_\beta(\omega) \geq t) = \mathfrak{p}(\beta). \tag{11.5}$$

Moreover, $\beta \mapsto \mathfrak{p}(\beta)$ is continuous.

In the following, we will extend this existence and continuity result to the zero-temperature case. This does not seem to follow from the methods in [16], due to the following issues:

- In positive temperature, $\log P(\tau_\beta(\omega) \geq t)$ is integrable and standard arguments show that $t \mapsto \mathbb{E}[\log P(\tau_\beta(\omega) \geq t)]$ is superadditive. However, in zero temperature $\log P(\tau_\infty(\omega) \geq t)$ is not integrable, see Proposition 11.2 below.
- Finally, an easy application of Hölder's inequality yields that $\beta \mapsto \mathfrak{p}(\beta)$ is convex, and therefore continuous. But convexity tells us nothing about continuity at the boundary $\beta = \infty$.

Let us observe why the integrability of $\log P(\tau_\infty(\omega) \geq t)$ is violated:

Proposition 11.2. *For any $t > 0$, $\mathbb{E}[\log P(\tau_\infty(\omega) \geq t)] = -\infty$.*

Proof. Let F be the first disaster time close to the origin:

$$F(\omega) := \inf\left\{t \geq 0 \colon \omega \cap \left([0,t] \times \tfrac{1}{2}U(0)\right) \neq \varnothing\right\}.$$

Note that the Brownian motion B gets killed if $B(F(\omega)) \in \frac{1}{2}U(0)$, see Figure 13. Therefore, on $\{F < t\}$,

$$P(\tau_\infty(\omega) \geq t) \leq P\left(B(F(\omega)) \notin \tfrac{1}{2}U(0)\right) \leq \exp\Big(-\frac{C}{F(\omega)}\Big).$$

Since F is exponentially distributed,

$$\mathbb{E}[\log P(\tau_\infty(\omega) \geq t)] \leq -C\mathbb{E}\big[F^{-1}\mathbb{1}\{F < t\}\big] = -\infty.$$

$\square$

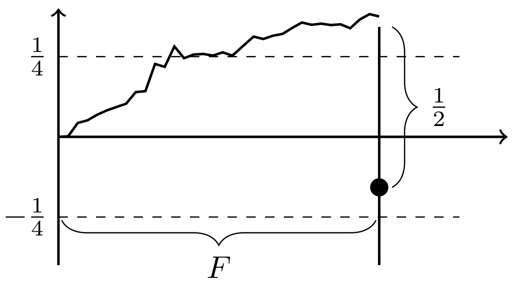

Figure 13: The time of the first disaster (black circle) close to the origin is denoted by $F(\omega)$. Notice that the Brownian motion has to move at least distance 1/4 until time F in order to survive.

The proof of this proposition suggests that the non-integrability is caused by the possibility of having a disaster near the starting point of the Brownian motion. It is reasonable to think that this is the only source of non-integrability, and we will in fact confirm this intuition in the proof. For an illustration, note that in Figure 12 the cost of avoiding the first disaster is much larger than that of any other disaster.

11.3. Related literature

The Brownian polymer model was introduced in [16] and has since been studied by many authors. We refer to [20] for a recent survey of known results.

In this section we discuss other works on the zero-temperature limit. There are not so many results in this direction for the discrete-time random polymer model. This is mainly because we have a simple answer in a large class of settings. To see this, let us consider the case of nearest neighbor simple random walk. Deviating slightly from the notation in Section 4, the environment is given by real-valued random variables $\omega = (\omega(s,i))_{s\in\mathbb{N},i\in\mathbb{Z}^d}$ and the polymer measure is defined by

$$\mu^t_{\omega,\beta}(x) = \frac{1}{Z^\beta_t(\omega)} \exp\Big(-\beta \sum_{s=1}^t \omega(s,x(s))\Big) \mathbb{1}\{x \in \mathcal{N}_t\},$$

where $\mathcal{N}_t$ denotes the set of nearest neighbor paths of length t on $\mathbb{Z}^d$. Now if the **time constant** for the directed first passage percolation

$$\mu = \lim_{t\to\infty} \frac{1}{t} \min_{x\in\mathcal{N}_t} \sum_{s=1}^t \omega(s,x(s))$$

is non-zero, then it is easy to deduce a continuity result

$$\lim_{\beta\to\infty} \lim_{t\to\infty} \frac{1}{\beta t} \log Z^\beta_t(\omega) = -\mu.$$

On the other hand, if $\operatorname{ess\,inf} \omega = 0$ and the set $\{(s,i) \colon \omega(s,i) = 0\}$ percolates, then we have $\mu = 0$. In this case, $Z^\infty_t(\omega) := \lim_{\beta\to\infty} Z^\beta_t(\omega)$ is the number of open paths in directed site percolation, and $\lim_{t\to\infty} \frac{1}{t}\log Z^\infty_t(\omega)$ represents the exponential growth rate. As mentioned in the discussion in Section 4.4, the existence of this limit is proved in [31], but it is not known whether it equals $\lim_{\beta\to\infty}\lim_{t\to\infty}\frac{1}{t}\log Z^\beta_t(\omega)$.

Two recent works [12, 51] study this type of problem in a non-nearest neighbor model on $\mathbb{N}\times\mathbb{Z}^d$ defined by

$$P^t_{\omega,\beta}(x) = \frac{1}{Z^\beta_t(\omega)} \exp\Big(-\sum_{s=1}^t \big[\beta\omega(s,x(s)) + |x(s-1) - x(s)|^\alpha\big]\Big) \tag{11.6}$$

and proved the continuity of the free energy at $\beta = \infty$. In this case, $\log Z^\beta_t(\omega)$ is integrable and hence the existence follows from the superadditivity argument.

Finally, there is a recent work [3] where the zero-temperature limit of the polymer measure is discussed for the model on $\mathbb{N}\times\mathbb{R}$ defined by

$$\mu^t_{\omega,\beta}(\mathrm{d}x) = \frac{1}{Z^\beta_t(\omega)} \exp\Big(-\beta \sum_{s=1}^{t} \big[\omega(s,x(s)) + |x(s-1)-x(s)|^2\,\big] \Big) \prod_{s=1}^{t} \mathrm{d}x(s).$$

In the preceding works [2, 4], the infinite volume polymer measure is constructed for every given asymptotic slope, at zero and positive temperature, respectively. Then in [3], it is shown that as $\beta\to\infty$, not only the free energy but also the infinite volume polymer measure converges. This model is similar to our model, since the polymer measure in (11.4) has a heuristic representation

$$\mu^t_{\omega,\beta}(\mathrm{d}x) = \frac{1}{Z^\beta_t(\omega)} \exp\Big(-\beta\omega(V_t) - \frac{1}{2}\int_0^t |\dot{x}(s)|^2 \mathrm{d}s\Big)\mathrm{d}x.$$

However, we do not multiply the term $\int_0^t |\dot{x}(s)|^2\mathrm{d}s$ by β and thus the two models behave quite differently as $\beta\to\infty$. The zero temperature model in [2] is of last passage percolation type and concentrates on a single path, whereas our result implies that the entropy is non-degenerate at zero temperature.

11.4. The main results

In view of Proposition 11.2, it is natural to consider a modified death time where the disasters up to time 1 are not taken into account. For $I\subset\mathbb{R}_+$, let us write ω_I for the restriction $\omega|_{I\times\mathbb{R}^d}$ as a measure, and define

$$\tau^1_\beta(\omega) := \tau_\beta(\omega_{[0,1]^c}) = \begin{cases} \inf\big\{t\geq 1\colon \beta\,|\omega_{[0,1]^c}\cap V_t(B)|\geq \xi\big\} & \text{for } \beta<\infty, \\ \inf\big\{t\geq 1\colon |\omega_{[0,1]^c}\cap V_t(B)|\geq 1\big\} & \text{for } \beta=\infty. \end{cases} \tag{11.7}$$

It is convenient to restrict the Brownian motion to a domain growing at polynomial speed:

$$\mathcal{A}_t := \Big\{ \sup_{0\leq s\leq t} |B(s)-B(0)| \leq \lceil t\rceil^2 \Big\}. \tag{11.8}$$

The probability of $\mathcal{A}^c_t$ is bounded by $\exp(-ct^3)$ by the reflection principle and hence it should be much smaller than the survival probability. We state the key technical estimate that will be used in the main results:

Proposition 11.3. *For every $p\in\mathbb{N}$, there exists $C>0$ such that for all $\beta\in[0,\infty]$, all $t\geq C$ and $r,s>0$ such that $1\leq r\leq r+s\leq t$ and either $r+s\leq t-1$ or $r+s=t$,*

$$\mathbb{E}\Big[\Big|\log P\big(\tau^1_\beta(\omega)\geq t\big|\tau^1_\beta(\omega_{[r,r+s]^c})\geq t, \mathcal{A}_t\big)\Big|^p\,\Big] \leq C(1+s^p) + C(1+\log^+ t)^C. \tag{11.9}$$

That is, we control the cost of surviving in the time-slice $[r,r+s]\times\mathbb{R}^d$, conditioned on surviving all disasters until time t outside $[r,r+s]\times\mathbb{R}^d$.

With the help of this estimate, we obtain a polynomial concentration bound for the logarithmic survival probability, uniformly in β:

Proposition 11.4. *For every $\varepsilon \in (0,1/2)$ and $r \in \mathbb{N}$, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $\beta \in [0,\infty]$,*

$$\mathbb{P}\left(\left|\log P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t) - \mathbb{E}[\log P(\tau^1_\beta \geq t, \mathcal{A}_t)]\right| \geq t^{1/2+\varepsilon}\right) \leq t^{-r}. \tag{11.10}$$

Remark 11.5. For $\beta \in (0,\infty)$, an exponential concentration bound is obtained in [16, Theorem 2.4.1(b)]. However, it does not cover the case $\beta = \infty$ since it contains a constant that degenerates at $\beta = \infty$.

We show the existence of the decay rate $\mathfrak{p}$ in zero temperature:

Theorem 11.6. *There exists $\mathfrak{p}(\infty) \in (-\infty,-1]$ such that*

(i) $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t)] = \mathfrak{p}(\infty)$.

(ii) Almost surely, $\lim_{t\to\infty} \frac{1}{t} \log P(\tau_\infty(\omega) \geq t) = \mathfrak{p}(\infty)$.

We point out, that both the restriction to $\mathcal{A}_t$ and the truncation of the environment are only present in part (i), while the almost sure limit in part (ii) is for the unrestricted survival probability. To close this gap, we therefore need an argument that the limit is not affected by replacing τ_∞ with the modified extinction time τ^1_∞, i.e. that almost surely

$$\lim_{t\to\infty} \frac{1}{t}\left| \log P(\tau_\infty(\omega) \geq t, \mathcal{A}_t) - \log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t)\right| = 0. \tag{11.11}$$

Notice that we can replace the interval $[0,1]$ in the modification (11.7) by $[0,\varepsilon]$, for any $\varepsilon > 0$, and that we can show, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log P\big(\tau_\infty(\omega_{[0,\varepsilon]^c}) \geq t, \mathcal{A}_t\big)\big] = \mathfrak{p}(\infty).$$

Thus, it seems natural to use that almost surely

$$\log P(\tau_\infty(\omega_{[0,\varepsilon]^c}) \geq t, \mathcal{A}_t) \xrightarrow{\varepsilon\downarrow 0} \log P(\tau_\infty(\omega) \geq t, \mathcal{A}_t).$$

However, the problem is that we first take a limit $t \to \infty$. We would have to justify that

$$P\big(B(\varepsilon) \in [-R,R] \big| \tau_\infty(\omega_{[0,\varepsilon]^c}) \geq t, \mathcal{A}_t\big)$$

decays subexponentially, for R some large constant. Our methods do not give enough control over this measure, and (11.11) will instead follow from the following result:

Proposition 11.7. *Assume $d = 1$. There exists a random variable $A(\omega) \in (0,\infty)$ such that, for all $x \in \mathbb{R}$,*

$$P\big(B(2) \in \mathrm{d}x, \tau_\infty(\omega) \geq 2\big) \geq A(\omega)P\big(B(1) \in \mathrm{d}x\big).$$

That is, we provide a (random) bound on the additional cost of the disasters in $[0,1] \times \mathbb{R}$. We expect that $A(\omega)$ is small, since Proposition 11.2 in particular implies $\mathbb{E}[\log A] = -\infty$, but for our purposes $A(\omega) > 0$ will be enough.

Remark 11.8. Note that except for this point, the proofs of our results are almost identical for $d = 1$ and $d \geq 2$. For this reason, and to keep the notation simple, we carry out the proofs mostly in the one-dimensional setting. We were not able to prove Proposition 11.7 in dimensions other than $d = 1$, and for this reason we provide a different proof of (11.11) in higher dimensions. We postpone the description of the necessary modifications in dimensions $d \geq 2$ to Section 13.6.

Finally, we show that the continuity from Theorem M also extends to the boundary $\beta = \infty$:

Theorem 11.9. *(i) For every $\varepsilon \in (0, 1/2)$, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $\beta \in [0, \infty]$,*

$$-t^{1/2+\varepsilon} \leq \mathbb{E}\big[\log P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t)\big] - t\mathfrak{p}(\beta) \leq t^\varepsilon. \tag{11.12}$$

(ii) $\lim_{\beta\to\infty} \mathfrak{p}(\beta) = \mathfrak{p}(\infty)$.

11.5. Outline

In this section, we explain the high-level structure of the proofs. The necessary preparations for Proposition 11.3 are carried out in Section 12, which is the technical core of the argument. Let us sketch the main steps:

In Section 12.1, we provide a bound for the first moment of the logarithmic survival probability in a space-time tube. To deal with the non-integrability from Proposition 11.2, we condition on the first disaster time F, so that the resulting bound also depends on F. In fact, it is convenient to replace Brownian motion by the Brownian bridge, and consequently the bound also depends on the time L of the last disaster.

In Section 12.2 we strengthen this to higher moments by duplicating the tube strategy. More precisely, we study the logarithmic survival probability in time interval $[r, r + s]$, assuming that the endpoints $(B(r), B(r + s))$ have some arbitrary law ν. We apply the result from the previous section in a large number of parallel tubes (see Figure 15), and use the fact that it is unlikely to have small survival probability in each of them. The resulting bound depends on the cost of spreading out over multiple tubes, i.e., on a quantity $M(\nu)$ measuring the dispersion of ν, see (12.24).

In order to prove Proposition 11.3, we need to consider the situation where the endpoints $(B(r), B(r + s))$ are distributed according to a random probability measure ν_ω depending on the environment outside $[r, r + s] \times \mathbb{R}$. In Section 12.3, we therefore estimate the dispersion $M(\nu_\omega)$ of this measure.

Finally, Section 12.4 contains the proof of Proposition 11.3, making use of all preparations up to this point.

The role of Proposition 11.3 is very similar to the uniform moment bound (Proposition 5.2) in Part II, and we will use it repeatedly in the proofs of the main results in Section 13:

- In Section 13.1 we show that $t \mapsto \mathbb{E}[\log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t)]$ defines an almost-superadditive sequence in the sense of Theorem H.
- In Section 13.2 we prove the concentration inequality (Proposition 11.4).
- In Section 13.3 we prove the bound on the cost of replacing τ_∞ with the modified extinction time τ^1_∞ in dimension $d = 1$ (Proposition 11.7).
- In Section 13.4 we combine those results and prove Theorem 11.6 in $d = 1$.
- In Section 13.5 we prove continuity at $\beta = \infty$ (Theorem 11.9). We follow the same argument as on the lattice (Proposition 5.5), using a technique from [58].
- The extra work for the proof of Theorem 11.6 in dimension $d \geq 2$ is carried out in Section 13.6.

12. Preparation

12.1. Survival probability in a tube

In this section, we provide a lower bound for the survival probability of the Brownian motion, which is conditioned to end at a fixed point and restricted to a tube.

We start by introducing some notation. We write $P^{r,x;s,y}$ for the Brownian bridge measure between (r, x) and (s, y). For technical reasons, we consider a sequence of nested intervals:

- x and y will be chosen from $J^{(5)} := [-\frac{5}{2}, \frac{5}{2}]$,
- the Brownian motion will be restricted to $J^{(6)} := [-3, 3]$,
- then the survival probability depends only on the disasters in $J^{(7)} := [-\frac{7}{2}, \frac{7}{2}]$,
- $J^{(5)}$ is divided into $\mathcal{J} = \{J^{(1)}_{-2}, \dots, J^{(1)}_{2}\}$, where $J^{(1)}_x := x + [-\frac{1}{2}, \frac{1}{2})$.

The role of $J^{(7)}$ is to ensure independence of the survival probabilities in different tubes in our duplication strategy, see Figure 15. For $t > 0$, let F_t denote the first disaster in $[0, t] \times J^{(7)}$, that is,

$$F_t := \inf\big\{r \in [0, t] \colon \exists z \in J^{(7)} \text{ such that } (r, z) \in \omega\big\}$$

with the convention $F_t = t$ if $\omega \cap ([0, t] \times J^{(7)}) = \varnothing$. Similarly we let

$$L_t := \sup\big\{r \in [0, t] \colon \exists z \in J^{(7)} \text{ such that } (r, z) \in \omega\big\}$$

denote the last disaster in $[0, t] \times J^{(7)}$, where we set $L_t = 0$ if there is no such disaster. The following lemma provides a lower bound on the survival probability in $[0, t] \times J^{(6)}$:

Lemma 12.1. *There exists $C > 0$ such that the following hold almost surely:*

(i) For all $x, y \in J^{(5)}$,

$$\begin{aligned} &\mathbb{E}\Big[\log P^{0,x;t,y}\Big(\tau_\infty(\omega) \geq t, B(s) \in J^{(6)} \textit{ for all } 0 \leq s \leq t\Big) \,\Big|\, F_t, L_t\Big] \\ &\quad \geq -C\big(t + \mathbb{1}\{F_t < t\}(F_t^{-1} + (t - L_t)^{-1})\big), \end{aligned} \tag{12.1}$$

(ii) $\mathbb{E}\left[\log P\left(\tau^1_\infty(\omega) \geq t, B(s) \in J^{(6)} \text{ for all } 0 \leq s \leq t\right)\right] \geq -C(t+1)$.

Remark 12.2. In contrast to the situation in Figure 15, the tube is assumed to be "parallel" to the time axis. However, this is not restrictive, as we can change the terminal point of the Brownian bridge by applying a time-space affine transformation which leaves the law of ω invariant, see also Figure 16. We include this generalization to Lemma 12.6.

The term F_t^{-1} (resp. $(t-L_t)^{-1}$) is the cost for the Brownian motion to avoid the first (resp. last) disasters in $[0,t] \times J^{(7)}$. This justifies the intuition discussed after Proposition 11.2, that a disaster appearing immediately after the starting point is the only source of non-integrability. To see the reason why the cost is inverse proportional to F_t, we state simple estimates for Brownian motion without proof, which we will use repeatedly.

Lemma 12.3. *There exists $C > 0$ such that for every $s, t > 0$ and $x, y \in \{-2, \dots, 2\}$, almost surely on $\{B(t) \in J^{(1)}_x\}$,*

$$\begin{aligned} &P\left(B(s+t) \in J^{(1)}_y \text{ and } B(u+t) \in J^{(6)} \text{ for all } u \in [0,s] \,\middle|\, B(t)\right) \\ &\geq \begin{cases} e^{-\frac{C}{s} - Cs} & \text{if } x \neq y, \\ e^{-Cs} & \text{if } x = y. \end{cases} \end{aligned} \tag{12.2}$$

We are going to bound the probability in (12.1) from below by constructing a specific survival strategy for the Brownian motion. We will introduce various terminologies in the course of describing the strategy. Given an environment ω, we can find $T_i \geq 0$ and $D_i \in J^{(7)}$ such that

$$\omega \cap \left(\mathbb{R}_+ \times J^{(7)}\right) = \left\{\left(T_0, D_0\right), \left(T_1, D_1\right), \dots\right\}$$

and such that $T_0 < T_1 < \dots$. We denote the interarrival times by $\Delta_0 \coloneqq T_0$ and

$$\Delta_i \coloneqq T_i - T_{i-1}$$

for $i \geq 1$, which are independent exponential random variables with parameter 7. We say that $J^{(1)}_x \in \mathcal{J}$ is **contaminated** by (T_j, D_j) if

$$J^{(1)}_x \cap U(D_j) \neq \varnothing.$$

It is simple to check that if $J^{(1)}_x \in \mathcal{J}$ is not contaminated by (T_j, D_j) and $B(T_j) \in J^{(1)}_x$, then the Brownian motion is not affected by the disaster at time T_j. Clearly every disaster can contaminate at most two sites, and since $|\mathcal{J}| = 5$, there exists a sequence $(s(0), s(1), \dots) \in \{0, 1, \dots, 4\}^{\mathbb{N}}$ such that $J^{(1)}_{s(j)}$ is not contaminated by (T_j, D_j) or (T_{j+1}, D_{j+1}). See Figure 14. The interval $J^{(1)}_{s(j)}$ is **safe** in the sense that the Brownian motion can survive during $[T_j, T_{j+2})$ simply by staying there.

Note that if there is no disaster in $[0,t] \times J^{(7)}$ (that is, on $\{F_t = t\} = \{F_t = t, L_t = 0\}$), we get (12.1) from Lemma 12.3, since

$$P(\tau_\infty(\omega) \geq t) \geq P\left(B(s) \in J^{(6)} \text{ for all } s \in [0,t]\right) \geq e^{-Ct}.$$

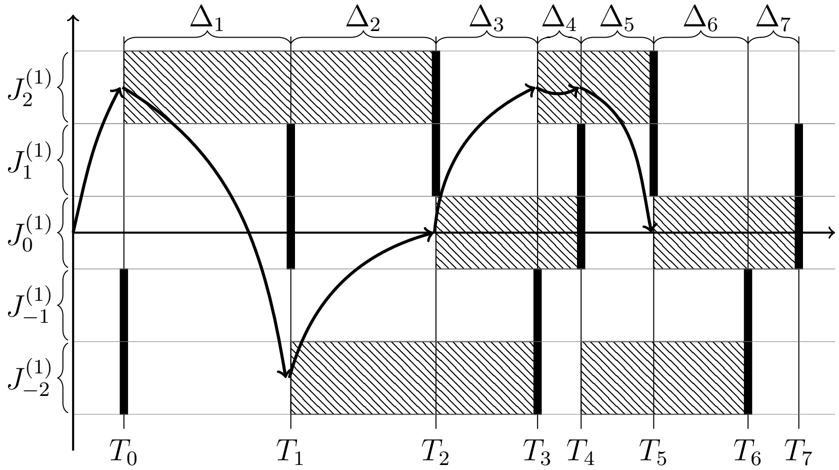

Figure 14: An illustration of the survival strategy until the first regeneration time R_1. In this figure we have $\rho_1 = 5$. At every disaster time, (typically) two intervals are contaminated (marked by the thick lines). The left ends of the striped regions are safe intervals. The arrows indicate to which interval the Brownian motion is supposed to move.

For the remainder of this section we therefore discuss the case $\{F_t < t\} = \{F_t < t, L_t > 0\}$.

The first interval: The survival strategy up to $T_0 = F_t$ is prescribed by the event

$$\mathcal{S}(0) := \left\{ B(T_0) \in J^{(1)}_{s(0)} \text{ and } B(u) \in J^{(6)} \text{ for } u \in [0, T_0] \right\}. \tag{12.3}$$

From the estimates in Lemma 12.3, we get

$$\log P(\mathcal{S}(0)) \geq -C(F_t + F_t^{-1}).$$

Renewal construction: After $T_0 = F_t$, we define the sequence of survival strategies by using a renewal structure. Let $\rho_0 := 0$ and for $i \geq 0$,

$$\rho_{i+1} = \inf \{ j > \rho_i + 1 \colon \Delta_j > \Delta_{j-1} \}.$$

We write the corresponding disaster time by

$$R_i := T_{\rho_i}.$$

We now recursively define events $\mathcal{S}(i)$ $(i \geq 1)$ as follows: $B(u) \in J^{(6)}$ for all $u \in [R_{i-1}, R_i)$, and in addition

$$B(T_j) \in J^{(1)}_{s(j)} \text{ for } j = \rho_{i-1}, \ldots, \rho_i - 2; \tag{S1}$$

$$B(u) \in J^{(1)}_{s(\rho_i - 2)} \text{ for } u \in [T_{\rho_i - 2}, T_{\rho_i - 1}]; \tag{S2}$$

$$B(T_{\rho_i}) \in J^{(1)}_{s(\rho_i)}. \tag{S3}$$

In words, the Brownian motion moves to the next safe interval in each time interval (T_j, T_{j+1}) except for $j = \rho_i - 2$. Note that we may have $\rho_i = \rho_{i-1} + 2$, and then the step (S1) is to be skipped. The second step (S2) is possible in this case since we have $B(T_{\rho_i - 2}) = B(T_{\rho_{i-1}}) \in J^{(1)}_{s(\rho_{i-1})}$ by the definition of $\mathcal{S}(0)$ ($i = 1$) and $\mathcal{S}(i-1)$ ($i \geq 2$). Now on the event $\{\rho_1 = k\}$ ($k \geq 2$), Lemma 12.3 yields

$$\log P(\mathcal{S}(1) \mid \mathcal{S}(0)) \geq -C \sum_{\substack{i=1,\dots,k \\ i \neq k-1}} \Delta_i^{-1} - C \sum_{i=1}^{k} \Delta_i. \tag{12.4}$$

It is important that the term $\Delta_{k-1}^{-1} = \max\{\Delta_1^{-1}, \dots, \Delta_k^{-1}\}$ is omitted from the first sum on the r.h.s., due to the unusual strategy in (S2) above. Indeed, if that sum was taken over $1 \leq i \leq k$, it would be the sum of inverse exponential random variables, which is not $\mathbb{P}$-integrable. On the other hand, the other terms $\{\Delta_1^{-1}, \dots, \Delta_{k-2}^{-1}, \Delta_k^{-1}\}$ gain one extra degree of integrability from the knowledge that they are the $k-1$ smallest members from the collection $\{\Delta_1^{-1}, \dots, \Delta_k^{-1}\}$.

Last interval: It remains to prescribe the behavior after the last renewal time before time t. Let us denote by

$$\begin{aligned} N(s) &\coloneqq \sum_{i=1}^{\infty} \mathbb{1}\{T_i \leq s\} \text{ and} \\ M(s) &\coloneqq \sum_{i=1}^{\infty} \mathbb{1}\{R_i \leq s\} \end{aligned}$$

the numbers of disasters and renewals up to time s, respectively. We further set

$$\begin{aligned} \sigma &\coloneqq N(L_t) - M(L_t) = \text{the number of disasters in } [R_{M(L_t)}, L_t] \times J^{(7)} \\ U &\coloneqq L_t - R_{M(L_t)} = \text{the duration from the last renewal to } L_t. \end{aligned}$$

Then the survival strategy in $[R_{M(L_t)}, t]$ is prescribed by the event $\mathcal{T}$ defined as follows: $B(u) \in J^{(6)}$ for all $u \in [R_{M(L_t)}, t]$, and in addition

$$B(T_j) \in J^{(1)}_{s(j)} \text{ for } j = M(L_t), \dots, N(L_t) - 1, \tag{S4}$$

$$B(u) \in J^{(1)}_{s(N_t - 1)} \text{ for } u \in [T_{N(L_t)-1}, L_t)), \tag{S5}$$

$$B(t) = y. \tag{S6}$$

In the case where the last disaster time L_t is a renewal time, both (S4) and (S5) are to be skipped. In words, the strategy $\mathcal{T}$ for the terminal part is the same as for the previous cases except that we choose to remain in $J^{(1)}_{s(N(L_t)-1)}$ after the last disaster before L_t, regardless of whether a renewal occurs after L_t or not. Then exactly as in (12.4), on the event $\{\sigma = n\}$, we have

$$\begin{aligned} &\log P^{0,x;t,y}(\mathcal{T} \mid \mathcal{S}(0), \dots, \mathcal{S}(M(L_t))) \\ &\geq -C \left(\sum_{i=1}^{n-1} \Delta_i^{-1} + \sum_{i=1}^{n} \Delta_i + (t - L_t) + (t - L_t)^{-1} \right), \end{aligned}$$

where the last term $(t - L_t)^{-1}$ appears since the Brownian motion has to move from $J^{(1)}_{s(N(L_t)-1)}$ to the endpoint y during $[L_t, t]$. Note that since there is no renewal in $[R_{M(L_t)}, L_t]$, the strategy $\mathcal{T}$ makes the Brownian motion survive without moving in the shortest interval among $\{[T_j, T_{j+1}]\}_{j=M(L_t)}^{N(L_t)-1}$. Therefore for the same reason as before, we can expect that the sum $\sum_{i=1}^{n-1} \Delta_i^{-1}$ gains an extra degree of integrability.

Collecting the above strategies, we define

$$\mathcal{S}_t \coloneqq \mathcal{S}(0) \cap \bigcap_{i=1}^{M(L_t)} \mathcal{S}(i) \cap \mathcal{T}.$$

Then the probability that the Brownian motion survives in the tube $[0,t] \times J^{(6)}$ is bounded from below by

$$\begin{aligned} &\log P^{0,x;t,y}(\tau_\infty(\omega) \geq t, B(s) \in J^{(6)} \text{ for all } 0 \leq s \leq t) \\ &\quad \geq \log P(\mathcal{S}_t) \\ &\quad = \log P(\mathcal{S}(0)) + \sum_{i=1}^{M(L_t)} \log P(\mathcal{S}(i) \mid \mathcal{S}(i-1)) + \log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(M(L_t))). \end{aligned} \tag{12.5}$$

Expectation conditioned on $\{R_i\}_{i\geq 1}$: The previous intuition about "extra integrability" will be made precise in Lemma 12.4 below. Using Part (iii), it is then not hard to show that

$$\mathbb{E}\big[\log P(\mathcal{S}(i) \mid \mathcal{S}(i-1))\big] < \infty.$$

Unfortunately, we also have to take into account that the number of summands $M(L_t)$ in (12.5) is random, which makes the argument more complicated. We found that, instead of conditioning directly on $M(L_t)$, it is easier to first estimate

$$\mathbb{E}\left[\log P(\mathcal{S}(i) \mid \mathcal{S}(i-1)) \mid R_i\right], \tag{12.6}$$

which does not depend on the other renewals. Similarly, the last term in (12.5) also depends on $R_{M(L_t)}$ through U, and hence we will consider

$$\mathbb{E}\big[\log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(M(L_t))) \mid U, L_t\big]. \tag{12.7}$$

In Lemma 12.5 we will provide bounds on (12.6) and (12.7) in terms of R_i and U, and later in the proof of Lemma 12.1 we will show that the random sum over those bounds is integrable.

To this end, it is instrumental to understand the inter-dependence among $\{\Delta_i\}_{i\geq 1}$, $\{\rho_i\}_{i\geq 0}$ and $\{R_i\}_{i\geq 1}$.

Lemma 12.4. *The following hold:*

(i) Both

$$\{\rho_j\}_{j\geq 1} \text{ under } \mathbb{P} \text{ and}$$
$$\big\{\big(\Delta_{\rho_j+k}\big)_{k=1,\dots,\rho_{j+1}-\rho_j} : j \geq 1\big\} \text{ under } \mathbb{P}(\cdot|\rho_j : j \geq 1)$$

are independent families.

(ii) The $\rho_{j+1} - \rho_j$ $(j \geq 1)$ has the same law as ρ_1, which is given by

$$\mathbb{P}(\rho_1 = k) = \frac{k-1}{k!} \qquad \textit{for all } k \geq 2.$$

Moreover, conditioned on $\{\rho_1 = k\}$, $R_1 - R_0$ is Gamma distributed with parameter $(k, 7)$. That is, it has probability density

$$\frac{7^k}{(k+1)!} r^{k-1} e^{-7r} \mathbb{1}\{r \geq 0\}.$$

(iii) Let $\{E_i\}_{i\in\mathbb{N}}$ be independent exponential random variables with rate 7. Conditioned on $\{\rho_1 = k, T_k - T_0 = s\}$,

$$\sum_{i=\{1,\dots,k\}\setminus\{k-1\}} \Delta_i^{-1} \stackrel{\mathrm{d}}{=} \frac{1}{s} \sum_{i=2}^{k} \frac{\sum_{j=1}^{k} E_j}{\sum_{j=1}^{i} \frac{1}{k-j} E_j}$$

Proof. The first assertion follows from the fact that $(\rho_j)_{j\geq 1}$ are stopping times for the process $(T_i)_{i\geq 0}$.

To prove the second and third assertions, it is useful to realize the interarrival times in such a way that the dependence structure between ρ_1, $T_k - T_0 = \sum_{i=1}^k \Delta_i$ and Δ_i^{-1} is clear. To this end, let $(\Delta_i^{(k)})_{i=1}^k$ be an increasing order statistics of independent Exp(7) random variables and let π be a uniform random variable on the permutations $\mathfrak{S}_k$ of $\{1, 2, \dots, k\}$, which is independent of $\Delta^{(k)}$. Then we can realize the interarrival times as

$$(\Delta_i)_{1\leq i\leq k} = \left(\Delta_{\pi(i)}^{(k)}\right)_{1\leq i\leq k}. \tag{12.8}$$

Now, since $\{\rho_1 = k\}$ depends only on π, we find

$$\mathbb{P}(\rho_1 = k) = \mathbb{P}(\Delta_1 > \Delta_2 > \cdots > \Delta_{k-1} \text{ and } \Delta_{k-1} < \Delta_k) = \frac{k-1}{k!}$$

by simply counting the number of permutations satisfying the above order. For the same reason, $\{\rho_1 = k\}$ is independent of $\sum_{i=1}^k \Delta_i = \sum_{i=1}^k \Delta_i^{(k)}$, which is Gamma distributed with parameter $(k, 7)$. Thus the second assertion is proved.

Finally, $\sum_{i=1}^k \Delta_i$ is independent of $\{\Delta_j / \sum_{i=1}^k \Delta_i\}_{j=1}^k$, see [24, Theorem IX.4.1]. Therefore, conditioned on $\{\rho_1 = k, \sum_{i=1}^k \Delta_i = s\}$, we have

$$\sum_{i\in\{1,\dots,k\}\setminus\{k-1\}} \Delta_i^{-1} \stackrel{\mathrm{d}}{=} \frac{1}{s} \sum_{i=2}^{k} \left(\frac{\tilde{\Delta}_i^{(k)}}{\sum_{i=1}^k \tilde{\Delta}_i^{(k)}}\right)^{-1}, \tag{12.9}$$

where $\tilde{\Delta}^{(k)}$ is an independent copy of $\Delta^{(k)}$. The third assertion follows from the following distributional identity proved in [54, §1]:

$$\Big(\tilde{\Delta}_1^{(k)}, \tilde{\Delta}_2^{(k)}, \ldots, \tilde{\Delta}_k^{(k)}\Big) \stackrel{\mathrm{d}}{=} \Big(\frac{E_1}{k-1}, \sum_{j=1}^{2} \frac{E_j}{k-j}, \ldots, \sum_{j=1}^{k} \frac{E_j}{k-j}\Big).$$

□

Now we state the bounds on the conditional expectations mentioned before.

Lemma 12.5. *(i) There exists $C > 0$ such that almost surely,*

$$\mathbb{E}\big[\log P(\mathcal{S}(1) \mid \mathcal{S}(0)) \,\big|\, \rho_1, R_1\big] \geq -C\Big(R_1 + \frac{\rho_1^3}{R_1}\Big), \tag{12.10}$$

and

$$\begin{aligned} &\mathbb{E}\big[\log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(M(L_t))) \,\big|\, U, \sigma, L_t\big] \mathbb{1}\{U > 0\} \\ &\geq -C\Big(U + \frac{\sigma^3}{U} + (t - L_t) + (t - L_t)^{-1}\Big). \end{aligned} \tag{12.11}$$

(ii) There exists $C > 0$ such that almost surely,

$$\mathbb{E}\big[\log P(\mathcal{S}(1) \mid \mathcal{S}(0)) \,\big|\, R_1\big] \geq -C\big(R_1 + R_1^{-1}\big), \tag{12.12}$$

and

$$\begin{aligned} &\mathbb{E}\big[\log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(M(L_t))) \,\big|\, U, L_t\big] \mathbb{1}\{U > 0\} \\ &\geq -C\big(U + U^{-1} + (t - L_t) + (t - L_t)^{-1}\big). \end{aligned} \tag{12.13}$$

Proof. **Part (i)**: By (12.4) and Lemma 12.4, we get for $\omega \in \{\rho_1 = n+2, T_{n+2} - T_0 = s\}$,

$$\begin{aligned} \log P(\mathcal{S}_1 \mid \mathcal{S}_0) &\geq -C\Big(s + \sum_{\substack{i=1,\ldots,n+2 \\ i \neq n+1}} \frac{1}{\Delta_i}\Big) \\ &\stackrel{\mathrm{d}}{=} -C\Big(s + \frac{1}{s} \sum_{i=2}^{n+2} \frac{\sum_{j=1}^{n+2} E_j}{\sum_{j=1}^{i} \frac{1}{n+2-j} E_j}\Big). \end{aligned} \tag{12.14}$$

Thus it suffices to show that the expectation over $\{E_1, \ldots, E_{n+2}\}$ in the last line is bounded by $(n+1)^3$. To this end, we first bound the expectation of the sum as follows:

$$\begin{aligned} \mathbb{E}\Big[\sum_{i=2}^{n+2} \frac{\sum_{j=1}^{n+2} E_j}{\sum_{j=1}^{i} \frac{1}{n+2-j} E_j}\Big] &\leq \sum_{i=2}^{n+2} (n+2-i) \mathbb{E}\Big[\frac{\sum_{j=1}^{n+2} E_j}{\sum_{j=1}^{i} E_j}\Big] \\ &= \sum_{i=2}^{n+2} (n+2-i) \Bigg(1 + \mathbb{E}\Big[\sum_{j=i+1}^{n+2} E_j\Big] \mathbb{E}\Big[\Big(\sum_{j=1}^{i} E_j\Big)^{-1}\Big]\Bigg). \end{aligned} \tag{12.15}$$

This is the point where we use the extra integrability brought by omitting $i = 1$, which corresponds to the largest value of $\{\Delta_i^{-1}\}_{i=1}^n$. Indeed, since $\sum_{j=1}^i E_j$ is Gamma distributed with parameters $(i, 1)$, for $i \geq 2$, we can compute

$$\mathbb{E}\Big[\sum_{j=i+1}^{n+2} E_j\Big] = n + 2 - i \quad \text{and} \quad \mathbb{E}\Big[\Big(\sum_{j=1}^{i} E_j\Big)^{-1}\Big] = \frac{1}{i-1}.$$

Substituting into (12.15), we arrive at

$$\mathbb{E}\Big[\sum_{i=2}^{n+2} \frac{\sum_{j=1}^{n+2} E_j}{\sum_{j=1}^{i} \frac{1}{n+2-j} E_j}\Big] \leq n \sum_{i=2}^{n+2} \frac{n+1}{i-1} \leq (n+1)^3.$$

The proof of (12.11) is essentially the same. We assume $U > 0$ and $\sigma = n$. Then recall that by (12.1), we have

$$\log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(M(L_t))) \geq -C\Big(\sum_{i=1}^{n-1} \Delta_i^{-1} + U + (t - L_t) + (t - L_t)^{-1}\Big).$$

Since the interarrival times of disasters in $[R_{M(L_t)}, L_t]$ are decreasing, the largest member of $\{\Delta_i^{-1}\}_{i=1}^n$ is omitted in the sum on the right-hand side. This is the same situation as in Lemma 12.4-(iii), and thus conditioned on U, we have

$$\sum_{i=1,\ldots,n-1} \Delta_i^{-1} \stackrel{\mathrm{d}}{=} U^{-1} \sum_{i=2}^{n} \frac{\sum_{j=1}^{n} E_j}{\sum_{j=1}^{i} \frac{1}{n-j} E_j}.$$

Then the same computation as in the previous case yields the desired bound.

Part (ii): In order to take an expectation over ρ_1 conditioned on R_1, we estimate the conditional probability

$$\begin{aligned}
\mathbb{P}(\rho_1 = n+2 \mid R_1 = r) &= \mathbb{P}(\rho_1 = n+2 \mid T_{\rho_1} - T_0 = r) \\
&= \frac{\mathbb{P}(\rho_1 = n+2, T_{n+2} - T_0 = r)}{\mathbb{P}(T_{\rho_1} - T_0 = r)},
\end{aligned}$$

where here and in what follows, conditions like $T_{n+2} - T_0 = r$ should be understood in the sense of probability density. Since $\{\rho_1 = n+2\}$ and $T_{n+2} - T_0$ are independent, by using Lemma 12.4, we can bound the numerator from above by

$$\mathbb{P}(\rho_1 = n+2, T_{n+2} - T_0 = r) \leq \frac{(n+1)}{(n+2)!} \frac{(7r)^{n+1}}{(n+1)!} e^{-7r}. \tag{12.16}$$

On the other hand, the denominator is bounded from below by

$$\begin{aligned}
&\mathbb{P}(\rho_1 = 2, T_2 - T_0 = r) \\
&\quad = \mathbb{P}(T_1 - T_0 < T_2 - T_1, T_2 - T_0 = r) \\
&\quad = \tfrac{1}{2}\mathbb{P}(T_2 - T_0 = r) \\
&\quad = \tfrac{3}{2} r e^{-7r}.
\end{aligned} \tag{12.17}$$

Combining (12.16) and (12.17), we find the bound

$$\mathbb{P}(\rho_1 = n+2 \mid R_1 = r) \leq \frac{(n+1)}{(n+2)!}\frac{(7r)^{n+1}}{(n+1)!}\frac{2}{3r} \leq 4\frac{(7r)^n}{(n!)^2}.$$

In particular, we get that if $R_1 \leq \frac{1}{7}$ then

$$\mathbb{P}(\rho_1 = n+2|R_1) \leq \frac{4}{(n!)^2}$$

and consequently,

$$\mathbb{E}\left[\rho_1^3 \mid R_1\right] = \sum_{n=0}^{\infty}(n+2)^3\mathbb{P}(\rho_1 = n+2 \mid R_1) \leq \sum_{n=0}^{\infty} 4\frac{(n+2)^3}{(n!)^2} < \infty. \tag{12.18}$$

If $R_1 > \frac{1}{7}$, then we use $n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$ to see that for all $n > \sqrt{28R_1}$, we have

$$\mathbb{P}(\rho_1 = n+2 \mid R_1) \leq \frac{4}{n^3 2^n}$$

and consequently,

$$\mathbb{E}\left[\rho_1^3 \mid R_1\right] \leq 28^2 R_1^2 + 4\sum_{n>\sqrt{28R_1}} \frac{(n+2)^3}{n^3 2^n}. \tag{12.19}$$

Since the sum on the right-hand side converges, we can combine the two estimates (12.18) and (12.19) to find $C > 0$ such that for all $R_1 > 0$,

$$\mathbb{E}\left[\rho^3 \mid R_1\right] \leq C(1 + R_1^2).$$

Plugging this in (12.10), we get (12.12).

Finally, (12.13) follows in a similar way. We consider the probability of $\{\sigma = n\}$ conditioned on $\{U = u, L_t = l\}$, which can be written as

$$\begin{aligned}&\mathbb{P}(\sigma = n \mid U = u, L_t = l)\\ &\quad= \frac{\mathbb{P}\left(\sum_{i=M(l)+1}^{M(l)+n+1} \Delta_i = u, \Delta_{M(l)+1} > \cdots > \Delta_{M(l)+n+1}\right)}{\mathbb{P}\left(\sum_{i=M(l)+1}^{M(l)+\sigma+1} \Delta_i = u, \Delta_{M(l)+1} > \cdots > \Delta_{M(l)+\sigma+1}\right)}.\end{aligned}$$

The two events in the numerator are independent and hence the numerator is bounded (in the sense of density) from above by

$$\frac{1}{(n+1)!}\frac{1}{n!}(7u)^n e^{-7u}. \tag{12.20}$$

On the other hand, the denominator is bounded from below by considering the special case $\sigma = 0$:

$$\begin{aligned}
&\mathbb{P}\Bigg(\sum_{i=M(l)+1}^{M(l)+\sigma+1} \Delta_i = u, \Delta_{M(l)+1} > \cdots > \Delta_{M(l)+\sigma+1}\Bigg) \\
&\geq \mathbb{P}\left(\Delta_{M(l)+1} = u\right) \\
&= 7e^{-7u}.
\end{aligned} \tag{12.21}$$

From (12.20) and (12.21), we find that

$$\mathbb{P}(\sigma = n \mid U = u, L_t = l) \leq \frac{1}{n+1} \frac{(7u)^n}{(n!)^2}.$$

The rest of the argument is the same as for (12.12). □

We are now ready to prove the main result of this section:

Proof of Lemma 12.1. **Part (i)**: Note that on $\{M(t) = m\}$, we have

$$\log P(\mathcal{S}_t) = \log P(\mathcal{S}(0)) + \sum_{i=1}^{m} \log P(\mathcal{S}(i) \mid \mathcal{S}(i-1)) + \log P(\mathcal{T} \mid \mathcal{S}(0), \ldots, \mathcal{S}(m)).$$

By using the bounds (12.12) and (12.13) and denoting $R_i - R_{i-1}$ by $\mathit{\Delta} R_i$, we get on $\{F_t < t\}$

$$\begin{aligned}
\mathbb{E}\left[\log P(\mathcal{S}_t) \mid F_t, L_t\right] \geq -C\Big(&F_t + F_t^{-1} + \mathbb{E}\Big[\sum_{i=1}^{M(t)} \mathit{\Delta} R_i + U\Big] \\
&+ \mathbb{E}\Big[\sum_{i=1}^{M(t)} (\mathit{\Delta} R_i)^{-1} + U^{-1}\Big] + (t - L_t) + (t - L_t)^{-1}\Big).
\end{aligned} \tag{12.22}$$

Since we have $F_t + \sum_{i=1}^{M(t)} \mathit{\Delta} R_i + U + (t - L_t) = t$ by definition, it remains to show that the third expectation in (12.22) is bounded by Ct. We use that $A_i' \preceq_{st} A_i \preceq_{st} \mathit{\Delta} R_i$, where A_i is Gamma distributed with parameter $(2, 7)$ and A_i' is exponentially distributed with parameter 7, respectively. Since

$$(r_1, \ldots, r_i) \mapsto \frac{1}{r_1} \mathbb{P}(r_1 + \cdots r_i \leq t)$$

is decreasing, the above stochastic domination implies

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{M(t)} (\mathit{\Delta} R_i)^{-1}\right] &= \sum_{i=1}^{\infty} \mathbb{E}\big[(\mathit{\Delta} R_1)^{-1} \mathbb{1}\{\mathit{\Delta} R_1 + \cdots + \mathit{\Delta} R_i \leq t\}\big] \\
&\leq \sum_{i=1}^{\infty} \mathbb{E}\big[A_1^{-1} \mathbb{1}\{A_1 + A_2' + \cdots + A_i' \leq t\}\big].
\end{aligned}$$

By using the form of the probability density of A_1, we find

$$\begin{aligned}&\mathbb{E}\big[A_1^{-1}\mathbb{1}\{A_1+A_2'+\cdots+A_i'\le t\}\big]\\&\quad=\int_0^\infty a^{-1}\mathbb{P}(a+A_2'+\cdots+A_i'\le t)49ae^{-7a}\mathrm{d}a\\&\quad=7\mathbb{P}(A_1'+\cdots+A_i'\le t)\end{aligned}$$

and hence

$$\mathbb{E}\left[\sum_{i=1}^{M(t)}(\varDelta R_i)^{-1}\right]=7\sum_{i=1}^{\infty}\mathbb{P}(A_1'+\cdots+A_i'\le t).$$

The sum on the right-hand side is the expectation of a Poisson process with intensity 7 on $[0,t]$, which is equal to $7t$.

Part (ii): We follow the same strategy as in (i), but we skip (S6) in our strategy. Then we obtain the bound

$$\mathbb{E}\left[\log P(\tau_\infty^1(\omega)\ge t, B(s)\in J^{(6)}\text{ for all }0\le s\le t)\;\middle|\;F_t\right]\ge -C(t+\mathbb{1}\{F_t<t\}F_t^{-1}).$$

Since $F_t(\omega_{[0,1]^c})\ge 1$, we are done. □

12.2. Higher moments with general endpoints distribution

In this section we use Lemma 12.1 to get bounds on higher moments for the survival probability with more general initial and terminal distribution for the Brownian bridge. We first introduce some more notation. Given $0\le r<s$ and $\nu\in\mathcal{M}(\mathbb{R}^2)$, we denote by $P^{\nu,r,s}$ the law of the Brownian bridge in the interval $[r,s]$ with initial and terminal points chosen according to ν. More precisely, recall that $P^{r,x;s,y}$ denotes the Brownian bridge between (r,x) and (s,y). Let

$$P^{\nu,r,s}(\cdot):=\int_{\mathbb{R}^2}P^{r,x;s,y}(\cdot)\nu(\mathrm{d}(x,y)).\tag{12.23}$$

As mentioned in Section 11.5, we will derive our moment bound by considering the survival probability in disjoint tubes. For $x\in\mathbb{R}$ and $i\ge 1$, let

$$\begin{aligned}J_x^{(5)}(i)&:=x+7i+[-\tfrac{5}{2},\tfrac{5}{2}]\subseteq\mathbb{R},\\J_x^{(6)}(i)&:=x+7i+[-3,3]\subseteq\mathbb{R},\end{aligned}$$

and for a given probability measure $\nu\in\mathcal{M}(\mathbb{R}^2)$ and $p\ge 1$,

$$M^p(\nu):=\sup_{x,y\in\mathbb{R}}\min_{i=0,\dots,p}\nu(J_x^{(5)}(i)\times J_y^{(5)}(i)).\tag{12.24}$$

This is a measure of (local) dispersion of ν. If $M^p(\nu)$ is large, then under $P^{\nu,r,s}$, there is a good chance to find the initial and terminal points of the Brownian motion in $J_x^{(5)}(i)\times J_y^{(5)}(i)$, for each $i=0,1,\dots,p$. See also Figure 15.

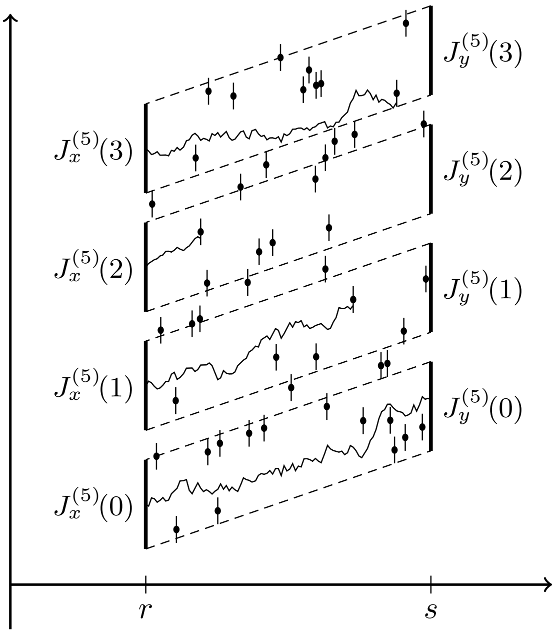

Figure 15: Illustration for the duplication strategy, with black bars indicating the beginning and end of the tubes. The survival probability in each tube is controlled by Lemma 12.1. If the distribution of $(B(r), B(s))$ is sufficiently dispersed, then we can choose the best tube among many to construct a good survival strategy in $[r, s]$.

Note that from our choice of $J^{(7)}$, the tubes connecting $J_x^{(5)}(i)$ and $J_y^{(5)}(i)$ are independent for different i. Since we can apply Lemma 12.1 to get a lower bound on the survival probability for each tube by itself, we should be able to get a better bound on the survival probability in the time interval $[r, s]$. The following lemma confirms this intuition:

Lemma 12.6. *For every $p \geq 1$ there exists $C > 0$ such that for any $0 \leq r < s$, $t \in [0, s-r]$, and $\nu \in \mathcal{M}(\mathbb{R}^2)$,*

$$\mathbb{E}\left[\left|\log P^{\nu,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t)\right|^p\right] \leq C(1+t^p) + |\log M^p(\nu)|^p. \tag{12.25}$$

If in addition ν is supported on $[-A, A]^2 \subseteq \mathbb{R}^2$ for some $A \geq \frac{7(3+p)}{2}$, then

$$\begin{aligned}&\mathbb{E}\left[\left|\log P^{\nu,r,s}\left(\tau_\infty(\omega_{[r,s]}) \geq r+t, |B(u)| \leq A \text{ for all } u \in [r, r+t]\right)\right|^p\right]\\ &\quad \leq C(1+t^p) + |\log M^{p+2}(\nu)|^p.\end{aligned} \tag{12.26}$$

Proof. We assume that the supremum in (12.24) is attained at $x, y \in \mathbb{R}$, and set

$$\nu_i(\mathrm{d}(u,v)) := \frac{\nu|_{J_x^{(5)}(i)\times J_y^{(5)}(i)}(\mathrm{d}(u,v))}{\nu\big(J_x^{(5)}(i)\times J_y^{(5)}(i)\big)}.$$

Then we have

$$\begin{aligned}&P^{\nu,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t)\\ &\quad \geq \max_{i=0,\dots,p} \int_{(u,v)\in J_x^{(5)}(i)\times J_y^{(5)}(i)} P^{r,u;s,v}(\tau_\infty(\omega_{[r,s]}) \geq r+t)\nu(\mathrm{d}(u,v))\\ &\quad = \max_{i=0,\dots,p} \nu\big(J_x^{(5)}(i)\times J_y^{(5)}(i)\big) P^{\nu_i,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t)\\ &\quad \geq \min_{i=0,\dots,p} \nu\big(J_x^{(5)}(i)\times J_y^{(5)}(i)\big) \max_{i=0,\dots,p} P^{\nu_i,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t).\end{aligned}$$

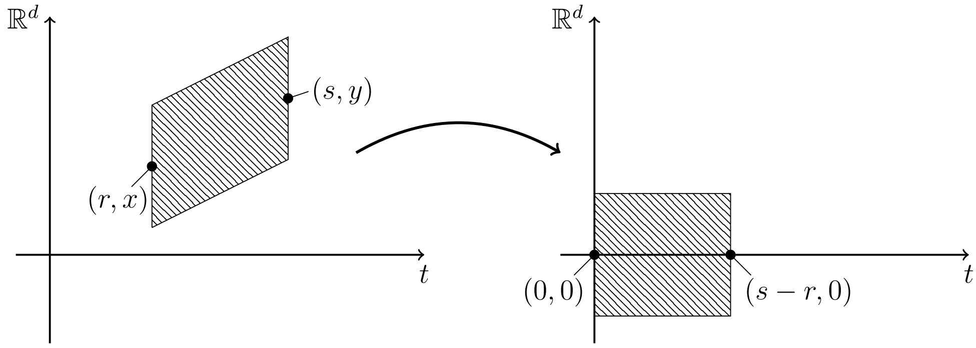

Figure 16: The law of ω is invariant under the affine transformation that maps (r,x) to $(0,0)$ and (s,y) to $(r-s,0)$. Note that the shifted tube connecting $\{r\}\times(x+[-3,3])$ and $\{s\}\times(y+[-3,3])$ is mapped onto $[0,s-r]\times J^{(6)}$, the tube considered in Lemma 12.1.

In order to apply Lemma 12.1 to the probability in the last line, we perform a time-space affine transformation that maps (r,x) to $(0,0)$ and (s,y) to $(s-r,0)$ (see Figure 16), and write $\bar{\omega}$ for the image of ω and $\bar{\nu}_i \in \mathcal{M}(J_0^{(5)}(i)^2)$ for the image measure of ν_i, respectively. Under this transformation, $\bar{\omega}$ has the same law as ω while $P^{\nu_i,r,s}$ is transformed to $P^{\bar{\nu}_i} = P^{\bar{\nu}_i,0,s-r}$. Therefore we have

$$P^{\nu_i,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t) \geq P^{\bar{\nu}_i}(\tau_\infty(\bar{\omega}) \geq t, B(u) \in J_0^{(6)}(i) \text{ for all } 0 \leq u \leq t) \tag{12.27}$$

and for different i's, the probabilities on the right-hand side depend on $\bar{\omega}$ in disjoint sets and hence are independent under $\mathbb{P}$. Let us introduce

$$X_i := \left| \log P^{\bar{\nu}_i}\big(\tau_\infty(\bar{\omega}) \geq t, B(u) \in J_0^{(6)}(i) \text{ for all } 0 \leq u \leq t\big) \right|, \tag{12.28}$$

so that we can write

$$|\log P^{\nu,r,s}(\tau_\infty(\omega_{[r,s]}) \geq r+t)|^p \leq 2^{p-1}\Big(|\log M^p(\nu)|^p + \big(\min_{i=0,\dots,p} X_i\big)^p \Big).$$

It remains to bound the p-th moment of $\min_{i=0,\dots,p} X_i$. Recall that $\bar{\omega}$ has the same law as ω and that $X_i = X_0 \circ \theta_{0,7i}$, where $\theta_{0,7i}$ is the time-space shift operator. To simplify the notation, we write $F_{i,t}$ and $L_{i,t}$ for $F_t \circ \theta_{0,7i}$ and $L_t \circ \theta_{0,7i}$, respectively. Then by Lemma 12.1, we have the following upper bound on the first moment of X_i for $i = 0,1,\dots,p$:

$$\mathbb{E}[X_i \mid F_{i,t}, L_{i,t}] \leq c_1 \Big(t + \mathbb{1}\{F_{i,t} < t\} \Big(\frac{1}{F_{i,t}} + \frac{1}{t - L_{i,t}} \Big) \Big),$$

where $c_1 > 0$ is a constant. In this proof we keep the constants indexed and clarify their dependence on parameters. Using Jensen's inequality, the above bound and that the marginal laws of $F_{i,t}$ and $L_{i,t}$ are the exponential law with rate 7 truncated at t, we obtain that for any $\varepsilon > 0$ there exists $c_2(\varepsilon)$ such that for all $i = 0, \dots, p$,

$$\mathbb{E}\Big[(X_i - c_1 t)_+^{1-\varepsilon}\Big] \le c_1 \mathbb{E}\Big[\mathbb{1}\{F_{i,t} < t\}\Big(\frac{1}{F_{i,t}} + \frac{1}{t - L_{i,t}}\Big)^{1-\varepsilon}\Big] \le c_2(\varepsilon).$$

This bound and the Markov inequality yield

$$\mathbb{P}\big((X_t - c_1 t)_+ \ge u\big) \le c_2(\varepsilon) u^{\varepsilon - 1}$$

for all $i = 0, \dots, p$ and $u > 0$. As a consequence, if we choose ε sufficiently small, we have

$$\mathbb{P}\Big(\min_{i=0,\dots,p}(X_i - c_1 t)_+ \ge u\Big) = \prod_{i=0}^{p} \mathbb{P}\big((X_i - c_1 t)_+ \ge u\big) \le c_2(\varepsilon)^{p+1} u^{-p-1/2}$$

for all $u > 0$. We have used that $X_0, \dots, X_p$ are independent. This tail bound then gives

$$\begin{aligned}
\mathbb{E}\Big[\Big(\min_{i=0,\dots,p} X_i\Big)^p\Big] &\le \mathbb{E}\Big[\Big(\min_{i=0,\dots,p}(X_i - c_1 t)_+ + c_1 t\Big)^p\Big] \\
&\le c_3(p)\Big(t^p + \mathbb{E}\Big[\Big(\min_{i=0,\dots,p}(X_i - c_1 t)_+\Big)^p\Big]\Big) \\
&\le c_3(p)\Big(t^p + c_2(\varepsilon)^{p+1}\int_0^\infty p u^{p-1} u^{-p-1/2} \mathrm{d}u\Big) \\
&\le c_4(\varepsilon, p)(t^p + 1).
\end{aligned} \tag{12.29}$$

This completes the proof of the first assertion. The second assertion is essentially proved in the above argument once we account for some issues with the boundary. Note that the bound (12.26) is trivial unless $M^{p+2}(\nu) > 0$, and in that case we again write (x, y) for the values where the supremum in (12.24) is attained. Since A is large enough, we observe that among

$$\Big\{J_x^{(5)}(i) \times J_y^{(5)}(i) \colon i = 0, \dots, p+2\Big\}$$

there are at least $p + 1$ indices $i_0, \dots, i_p$ such that

$$J_x^{(6)}(i_j) \times J_y^{(6)}(i_j) \subseteq [-A, A]^2 \quad \text{for all } j = 0, \dots, p.$$

For such an index i_j we note that the event considered in (12.28) ensures that the Brownian motion does not leave $[-A, A]$ in $[r, r + t]$. We then obtain (12.26) by the same calculation as in (12.29) where $\min_{i=0,\dots,p} X_i$ has to be replaced by $\min_{j=0,\dots,p} X_{i_j}$. □

12.3. Midpoint distribution of the polymer

In order to prove Proposition 11.3, we will apply Lemma 12.6 to the midpoint distribution under the polymer measure

$$\nu_{\omega,\beta}^{r,s,t}(\mathrm{d}(x,y)) := P\left((B(r), B(s)) \in \mathrm{d}(x,y) \;\middle|\; \tau_\beta^1(\omega_{[r,s]^c}) \ge t, \mathcal{A}_t\right) \in \mathcal{M}(\mathbb{R}^2) \tag{12.30}$$

Thus we need to estimate the dispersion M^p of this measure, which is the goal of this section:

Lemma 12.7. *Let $p \geq 0$ and $q \geq 1$. There exists $C > 0$ such that for all $\beta \in [0, \infty]$ and all $1 \leq r^- \leq r^+ \leq t$ such that either $r^+ \leq t-1$ or $r^+ = t$,*

$$\mathbb{E}\left[\left|\log M^p\left(\nu_{\omega,\beta}^{r^-,r^+,t}\right)\right|^q\right] \leq C(1+\log^+ t)^C. \tag{12.31}$$

Proof. Let us recall the notation

$$\begin{aligned} J_x^{(1)} &= x + \left[\tfrac{1}{2}, \tfrac{1}{2}\right), \\ J_x^{(5)}(i) &= x + 7i + \left[-\tfrac{5}{2}, \tfrac{5}{2}\right], \\ J_x^{(6)}(i) &= x + 7i + [-3, 3], \\ M^p(\nu) &= \sup_{x,y\in\mathbb{R}} \min_{i=0,\dots,p} \nu\left(J_x^{(5)}(i) \times J_y^{(5)}(i)\right). \end{aligned}$$

Observe first that, thanks to the truncation $\mathcal{A}_t$, for every $0 \leq r < s \leq t$ and every ω, there exist $x, y \in \mathbb{R}$ such that

$$\nu_{\omega,\beta}^{r,s,t}\left(J_x^{(1)} \times J_y^{(1)}\right) \geq ct^{-4}. \tag{12.32}$$

The bound (12.31) for $p = 0$ follows by setting $r = r^-$ and $s = r^+$. In order to prove (12.31) for $p \geq 1$, we need to find sets of intervals $\{J_x^{(5)}(i)\}_{i=0}^p$ and $\{J_y^{(5)}(i)\}_{i=0}^p$ for which

$$\min_{i\in\{0,1,\dots,p\}} \nu_{\omega,\beta}^{r^-,r^+,t}\big(J_x^{(5)}(i) \times J_y^{(5)}(i)\big)$$

is not too small. Our strategy is to use (12.32) for some $r < r^-$ and $s > r^+$ first, and then **sprinkle** the mass on the time-intervals $[r, r^-]$ and $[r^+, s]$, see also Figure 17. To this end, we have to find $r < r^- < r^+ < s$ and $x, y \in \mathbb{R}$ such that

- (12.32) is satisfied,
- there are no obstacles inside $[r, r^-]$ and $[r^+, s]$, close to (r, x) or (s, y) (that is, in the gray areas in Figure 17).

The latter condition ensures that the disasters do not prevent sprinkling the mass. For now, let us assume that $r^- \geq 2$ and $r^+ \leq t-1$. We denote $r_0^- := r^- - 1$ and $r_0^+ := r^+ + 1$ and for $i \geq 1$,

$$r_i^- := r_0^- + \frac{6}{\pi^2}\sum_{j=1}^i j^{-2} \quad \text{and} \quad r_i^+ := r_0^+ - \frac{6}{\pi^2}\sum_{j=1}^i j^{-2}. \tag{12.33}$$

Note that $r_i^- < r^-$ and $r_i^+ > r^+$ for all i. From (12.32), we know that there exists (j_i^+, j_i^-) such that

$$\nu_{\omega,\beta}^{r_i^-,r_i^+,t}\left(J_{j_i^-}^{(1)} \times J_{j_i^+}^{(1)}\right) \geq ct^{-4}. \tag{12.34}$$

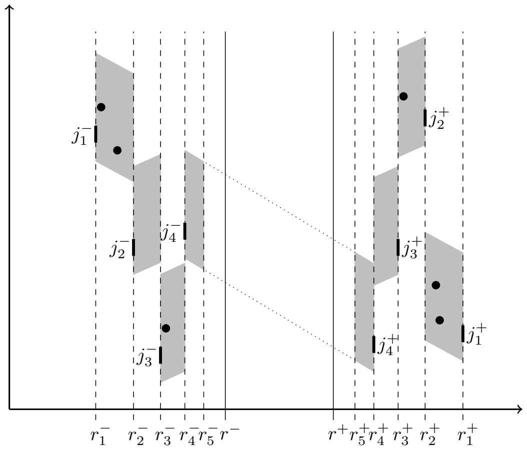

Figure 17: An illustration of the resampling procedure to find $S_G^{\pm}$. The dots represent disasters, and the gray areas corresponds to $S_i^{\pm}$. The short, black intervals $J^{(1)}_{j_i^-} \times J^{(1)}_{j_i^+}$ have not-too-small probability under the polymer measure. In this figure, we have $G = 4$ since S_4^+ and S_4^- are free of disasters.

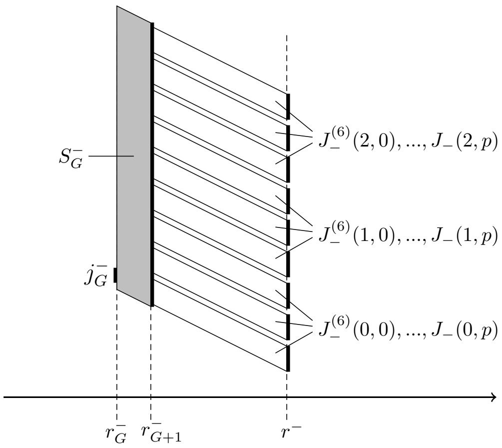

Figure 18: An illustration for $p = q = 2$. Since S_G^- contains no disasters, the mass can be sprinkled from $J^{(1)}_{j_G^-}$ (small black bar) onto the long black bar at time r_{G+1}^-. To see that it can further be sprinkled onto three consecutive, medium-sized black bars at time r^-, we divide the tubes into bundles of three. By the same argument as in Lemma 12.6, it is unlikely that the survival probability is small in all bundles.

For $i \geq 0$, let $\lambda_i\colon [0,\infty) \to \mathbb{R}$ be the affine linear function with $\lambda_i(r_i^-) = j_i^-$ and $\lambda_i(r_i^+) = j_i^+$, and introduce the slanted space-time boxes

$$S_i^{\pm} \coloneqq \left\{(u,x)\colon u \in [r_i^{\pm}, r_{i+1}^{\pm}), \lambda_i(u) - \tfrac{7}{2} \leq x \leq \lambda_i(u) + 7(p+1)(q+1) - \tfrac{7}{2}\right\}. \qquad (12.35)$$

Here we interpret the time-interval $[r_i^+, r_{i+1}^+)$ as $(r_{i+1}^+, r_i^+]$ by a slight abuse of notation. The same convention applies in the rest of this proof. Let us define the event

$$C_i \coloneqq \left\{\omega \cap (S_i^+ \cup S_i^-) = \varnothing\right\}.$$

Observe that since the boxes $S_i^{\pm}$ are disjoint and have decreasing volume, the events are independent and $\mathbb{P}(C_i) \geq \mathbb{P}(C_0) > 0$ for all $i \geq 0$. Therefore $G \coloneqq \inf\{i \geq 0\colon C_i \text{ holds}\}$ has a geometric tail,

$$\mathbb{P}(G \geq i) \leq (1 - \mathbb{P}(C_0))^i. \qquad (12.36)$$

In particular G is almost surely finite and hence j_G^- and j_G^+ are well-defined. Now for

$k \in \{0,\dots,q\}$, $l \in \{0,\dots,p\}$ and $u \geq 0$, let

$$J^{(5)}(k,l,u) := J^{(5)}_{\lambda_G(u)+7(p+1)k}(l),$$
$$J^{(6)}(k,l,u) := J^{(6)}_{\lambda_G(u)+7(p+1)k}(l),$$

and for $\pm \in \{+,-\}$, consider space-time tubes

$$J^{(6)}_{\pm}(k,l) := \left\{(u,x)\colon u \in [r^{\pm}_{G+1}, r^{\pm}], x \in J^{(6)}_{\pm}(k,l,u)\right\},$$
$$J^{(6)}_{\pm}(k) := J^{(6)}_{\pm}(k,0) \cup \dots \cup J^{(6)}_{\pm}(k,p).$$

See also Figure 18. We define the events

$$\mathcal{A}^{\pm}_1(k,l) := \left\{(u,B(u)) \in S_G \text{ for all } u \in [r^{\pm}_G, r^{\pm}_{G+1}], B(r^{\pm}_{G+1}) \in J^{(5)}_{\pm}(k,l,r^{\pm}_{G+1})\right\},$$
$$\mathcal{A}^{\pm}_2(k,l) := \left\{(u,B(u)) \in J^{(6)}_{\pm}(k,l) \setminus \mathcal{D} \text{ for all } u \in [r^{\pm}_{G+1}, r^{\pm}], B(r^{\pm}) \in J^{(5)}_{\pm}(k,l,r^{\pm})\right\},$$

where $\mathcal{D}$ is the set of disasters defined in (11.1). In words, $\mathcal{A}^-_1(k,l)$ is the event that the Brownian motion moves from $J^{(1)}_{j^-_G}$ to the left end of the tube $J^{(6)}_-(k,l)$ in $[r^-_G, r^-_{G+1}]$, without leaving S^-_G. This guarantees survival since by the definition of G, there are no disasters in S^-_G. On the other hand, $\mathcal{A}^-_2(k,l)$ is the event that the Brownian motion survives inside tube $J^{(6)}_-(k,l)$ in $[r^-_{G+1}, r^-]$. We set

$$\mathcal{A}(k,l) := \mathcal{A}^-_1(k,l) \cap \mathcal{A}^-_2(k,l) \cap \mathcal{A}^+_2(k,l) \cap \mathcal{A}^+_1(k,l).$$

By definition, we know that $M^p(\nu^{r^-,r^+,t}_{\omega,\beta})$ is bounded from below by the $\max_{k\in\{0,1,\dots,q\}}$ $\min_{l\in\{0,1,\dots,p\}}$ of the following probability:

$$\begin{aligned}
&\nu^{r^-,r^+,t}_{\omega,\beta}\left(J^{(5)}(k,l,r^-) \times J^{(5)}(k,l,r^+)\right) \\
&\quad \overset{\text{def}}{=} P\left((B(r^-),B(r^+)) \in J^{(5)}(k,l,r^-) \times J^{(5)}(k,l,r^+) \;\middle|\; \tau^1_\beta(\omega_{[r^-,r^+]^c}) \geq t, \mathcal{A}_t\right) \\
&\quad \geq P\left((B(r^-_G),B(r^+_G)) \in J^{(1)}_{j^-_G} \times J^{(1)}_{j^+_G}, \mathcal{A}(k,l) \;\middle|\; \tau^1_\beta(\omega_{[r^-_G,r^+_G]^c}) \geq t, \mathcal{A}_t\right),
\end{aligned}$$

where in the last line, we have used that

$$\mathcal{A}(k,l) \cap \left\{\tau^1_\beta(\omega_{[r^-_G,r^+_G]^c}) \geq t\right\} \subseteq \left\{\tau^1_\beta(\omega_{[r^-,r^+]^c}) \geq t\right\}.$$

Let us introduce the distribution

$$\alpha^{x_1,y_1}(\mathrm{d}x_2,\mathrm{d}y_2) := P^{r^-_{G+1},x_1;r^+_{G+1},y_2}\big((B(r^-),B(r^+)) \in \mathrm{d}(x_2,y_2)\big).$$

and denote

$$\begin{aligned}
p^{x_1,y_1}(\mathrm{d}x_2,\mathrm{d}y_2) := P^{r^-_G,x_1;r^+_G,y_1}\Big(&B(r^-_{G+1}) \in \mathrm{d}x_2, B(r^+_{G+1}) \in \mathrm{d}y_2, (u,B(u)) \in S^-_G \cup S^+_G \\
&\text{for all } u \in [r^-_G, r^-_{G+1}] \cup [r^+_{G+1}, r^+_G]\Big).
\end{aligned}$$

Note that $r^-_{G+1} - r^-_G = r^+_G - r^+_{G+1} = 6/(\pi^2(G+1)^2)$ and therefore

$$\inf_{(x_1,y_1)\in J^{(1)}_{j^-_G}\times J^{(1)}_{j^+_G}} \left\{ \int_{J^{(5)}_-(k,l,r^+_{G+1})\times J^{(5)}_+(k,l,r^-_{G+1})} p^{x_1,y_1}(\mathrm{d}x_2,\mathrm{d}y_2)\right\} \geq e^{-cG^2}.$$

Note also that since S^-_G and S^+_G are slanted parallel to the line connecting (r^-_G, j^-_G) and (r^+_G, j^+_G), we can apply an affine transformation and use invariance of the Brownian bridge to see that this estimate does not depend on the distance between j^-_G and j^+_G. Using the above notation and estimate, we get

$$\begin{aligned}
&P\left((B(r^-_G), B(r^+_G)) \in J^{(1)}_{j^-_G} \times J^{(1)}_{j^+_G}, \mathcal{A}(k,l) \,\middle|\, \tau^1_\beta(\omega_{[r^-_G,r^+_G]^c}) \geq t, \mathcal{A}_t\right)\\
&= \int_{J^{(1)}_{j^-_G}\times J^{(1)}_{j^+_G}} \nu^{r^-_G,r^+_G,t}_{\omega,\beta}(\mathrm{d}(x_1,y_1)) \int_{J^{(5)}(k,l,r^-_{G+1})\times J^{(5)}(k,l,r^+_{G+1})} p^{x_1,y_1}(\mathrm{d}x_2,\mathrm{d}y_2)\\
&\quad \int_{J^{(5)}(k,l,r^-)\times J^{(5)}(k,l,r^+)} \alpha^{x_2,y_2}(\mathrm{d}x_3,\mathrm{d}y_3) P^{r^-_{G+1},x_2;r^-,x_3}\left(\mathcal{A}^-_2(k,l)\right) P^{r^+,y_3;r^+_{G+1},y_2}\left(\mathcal{A}^+_2(k,l)\right)\\
&\geq \nu^{r^-_G,r^+_G,t}_{\omega,\beta}\left(J^{(1)}_{j^-_G}\times J^{(1)}_{j^+_G}\right) e^{-cG^2} \inf_{x_2,x_3,y_2,y_3} P^{r^-_{G+1},x_2;r^-,x_3}\left(\mathcal{A}^-_2(k,l)\right) P^{r^+,y_3;r^+_{G+1},y_2}\left(\mathcal{A}^+_2(k,l)\right),
\end{aligned}$$

where the infimum is over $J^{(5)}(k,l,r^-_{G+1}) \times J^{(5)}(k,l,r^-) \times J^{(5)}(k,l,r^+) \times J^{(5)}(k,l,r^+_{G+1})$. Recalling (12.34) and noting that G has all moments by (12.36), we only need to prove that $\min_{k\in\{0,1,\dots,q\}} \max_{l\in\{0,1,\dots,p\}}$ of

$$Z_{k,l} := \left| \log \inf_{(x_2,x_3,y_2,y_3)} P^{r^-_{G+1},x_2;r^-,x_3}\left(\mathcal{A}^-_2(k,l)\right) P^{r^+,y_3;r^+_{G+1},y_2}\left(\mathcal{A}^+_2(k,l)\right) \right|$$

has all moments. Now letting $F^\pm_k$ and $L^\pm_k$ denote the first and last disasters in $J^{(6)}_\pm(k)$, respectively, we get from (12.1) that there exists $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\max_{l\in\{0,1,\dots,p\}} Z_{k,l} \Big| \omega_{[r^-_{G+1},r^+_{G+1}]^c}, F^+_k, F^-_k, L^+_k, L^-_k\Big]\\
&\leq C\Bigg(1 + \mathbb{1}\{F^-_k < r^- - r^-_{G+1}\}\Big(\frac{1}{F^-_k} + \frac{1}{r^- - L^-_k}\Big)\\
&\qquad + \mathbb{1}\{F^+_k < r^+_{G+1} - r^+\}\Big(\frac{1}{F^+_k} + \frac{1}{r^+_{G+1} - L^+_k}\Big)\Bigg).
\end{aligned}$$

Then we argue in the same way as in the proof of Lemma 12.6 to obtain

$$\mathbb{E}\Big[\Big(\min_{k\in\{0,1,\dots,q\}} \max_{l\in\{0,1,\dots,p\}} Z_{k,l}\Big)^q\Big] \leq C.$$

This finishes the proof for $r^- \geq 2$ and $r^- \leq t-1$.

In the case $r^- < 2$, we use the interval $[0,1]$, which is free of disasters, in place of $[r^-_G, r^-_{G+1}]$, and set $j^-_i = 0$ for all $i \geq 0$. More precisely, define r^+_i as above and let j^+_i be such that

$$\nu^{1,r^+_i,t}_{\omega,\beta}\left(\mathbb{R}\times J^{(1)}_{j^+_i}\right) \geq Ct^{-2}.$$

Let λ_i be the linear function with $\lambda_i(0) = 0$ and $\lambda_i(r_i^+) = j_i^+$ and define S_i^+ as in (12.35). Using an affine transformation similar to before, we see that there exists $C > 0$ (independent of ω, i or j_i^+) such that for all $y \in J^{(1)}_{j_i^+}$ and all $k = 0, \dots, (p+1)(q+1)-1$,

$$\begin{aligned}
&P^{0,0;r_i^+,y}\Big(B(1) \in J^{(5)}_{\lambda_i(1)}(k), B(r_{i+1}^+) \in J^{(5)}_{\lambda_i(r_{i+1}^+)}(k), (u, B(u)) \in S_i^+ \text{ for all } u \in [r_{i+1}^+, r_i^+]\Big) \\
&\quad = P^{0,0;r_i^+,y-j_i^+}\Big(B(1) \in J^{(5)}_0(k), B(r_{i+1}^+) \in J^{(5)}_0(k), (u, B(u)) \in \widetilde{S}_i^+ \text{ for all } u \in [r_{i+1}^+, r_i^+]\Big) \\
&\quad \geq C^{-1} e^{-Ci^2},
\end{aligned} \tag{12.37}$$

where $\widetilde{S}_i := [r_{i+1}^+, r_i^+] \times [-\frac{7}{2}, 7(p+1)(q+1) - \frac{7}{2}]$. Now let $G := \inf\{i \geq 0\colon S_i \cap \omega = \varnothing\}$ and note that G has a geometric tail, so that in particular j_G^+ is well-defined. By the same consideration as before, it follows that

$$\min_{k=0,\dots,(p+1)(q+1)-1} \nu_{\omega,\beta}^{1,r_{G+1}^+,t}\Big(J^{(5)}_{\lambda_G(1)}(k) \times J^{(5)}_{\lambda_G(r_{G+1}^+)}(k)\Big) \geq C^{-1} e^{-CG^2} t^{-2}. \tag{12.38}$$

The rest of the argument is identical to before.

Finally, in the case $r^+ = t$, we simply restrict to $x = y$ in (12.24) to get

$$M^p(\nu_{\omega,\beta}^{r^-,t,t}) \geq C \sup_{x\in\mathbb{R}} \min_{i=0,\dots,p} P\left(B(r^-) \in J^{(5)}_x(i), B(t) \in J^{(5)}_x(i) \,\middle|\, \tau_\beta^1(\omega_{[r^-,t]^c}) \geq t\right).$$

Since we do not need to consider the survival strategy after time t, we can modify the previous argument by setting $r_i^+ = t$ and $j_i^+ = j_i^-$ for all $i \geq 0$ to show that

$$\begin{aligned}
&\mathbb{E}\left[\Big(\log \sup_{x\in\mathbb{R}} \min_{i=0,\dots,p} P\big(B(r^-) \in J^{(5)}_x(i), B(t) \in J^{(5)}_x(i) \big| \tau_\beta^1(\omega_{[r^-,t]^c}) \geq t\big)\Big)^q\right] \\
&\quad \leq C(1 + \log^+ t)^C.
\end{aligned}$$

□

12.4. Proof of the key proposition

The key proposition now follows easily from our preparation in Lemmas 12.6 and 12.7.

Proof of Proposition 11.3. We define a random probability measure $\nu(\omega)$ by

$$\nu(\omega)(\mathrm{d}(x,y)) := P\left(\big(B(r+s), B(s)\big) \in \mathrm{d}(x,y) \,\middle|\, \tau_\beta^1(\omega_{[r,r+s]^c}) \geq t, \mathcal{A}_t\right). \tag{12.39}$$

Then we can write

$$\begin{aligned}
&\mathbb{E}\left[\left|\log P\left(\tau_\beta^1(\omega) \geq t \,\middle|\, \tau_\beta^1(\omega_{[r,r+s]^c}) \geq t, \mathcal{A}_t\right)\right|^p\right] \\
&\quad \leq \mathbb{E}\Big[\Big|\log P^{\nu(\omega),r,r+s}\big(\tau_\infty^1(\omega_{[r,r+s]}) \geq t, \sup_{t'\in[r,r+s]} |B(t')| \leq t^2\big)\Big|^p\Big].
\end{aligned}$$

Since $\nu(\omega)$ depends only on the environment outside of $[r, r+s] \times \mathbb{R}$, we may integrate $\omega_{[r,r+s]}$ conditionally on $\nu(\omega)$. Then, by using (12.26) and part (i) of Lemma 12.7, we can find $C > 0$ such that the above r.h.s. is bounded by

$$C(1+s^p) + \mathbb{E}\big[|\log M^{p+2}(\nu(\omega))|^p\big] \leq C(1+s^p) + C(1+\log^+ t)^C.$$

□

13. Proof of the main results

13.1. Almost-superadditivity of the mean

With the key proposition we can bound the error-term in the almost-superadditive lemma:

Proposition 13.1. *Let $a_\beta(t) := \mathbb{E}[\log P(\tau_\beta^1(\omega) \geq t, \mathcal{A}_t)]$. For every $\varepsilon \in (0, 1/2)$, there exists $t_0 > 0$ such that for all $\beta \in [0, \infty]$ and $s, t \geq t_0$,*

$$a_\beta(s+t) \geq a_\beta(s) + a_\beta(t) - (s+t)^\varepsilon. \tag{13.1}$$

Proof of Proposition 13.1. We introduce a variation of the truncation $\mathcal{A}_t$: For $s, t \geq 0$, let

$$\mathcal{A}_s^t := \big\{ \sup\{|B(r)| \colon 0 \leq r \leq s\} \leq \lceil t \rceil^2 \big\}.$$

We define a random distribution

$$\mu(\omega)(\mathrm{d}x) := P(B(s) \in \mathrm{d}x \mid \tau_\beta^1(\omega) \geq s, \mathcal{A}_s^{s+t}) \in \mathcal{M}(\mathbb{R})$$

and we write P^μ for the law of Brownian motion with initial distribution μ. Then

$$\begin{aligned}
a_\beta(s+t) &= \mathbb{E}\big[\log P(\tau_\beta^1(\omega) \geq s+t, \mathcal{A}_{s+t})\big] \\
&= \mathbb{E}\big[\log P(\tau_\beta^1(\omega) \geq s, \mathcal{A}_s^{s+t})\big] + \mathbb{E}\big[\log P^{\mu(\omega)}(\tau_\beta(\theta_s(\omega)) \geq t, \mathcal{A}_t^{s+t})\big] \\
&=: \mathbb{E}\big[\log P(\tau_\beta^1(\omega) \geq s, \mathcal{A}_s^{s+t})\big] + \mathbb{E}\big[\log P^{\mu(\omega)}(\tau_\beta^1(\theta_s(\omega)) \geq t, \mathcal{A}_t^{s+t})\big] + b(s,t) \\
&\geq a_\beta(s) + \mathbb{E}\big[\log P^{\mu(\omega)}(\tau_\beta(\theta_s(\omega)) \geq t, \mathcal{A}_t)\big] + b(s,t),
\end{aligned}$$

where the remainder term is, by Proposition 11.3,

$$\begin{aligned}
b(s,t) &:= \mathbb{E}\Big[\log P\left(\tau_\beta^1(\omega) \geq s+t \;\middle|\; \tau_\beta^1(\omega_{[s,s+1]^c}) \geq s+t, \mathcal{A}_{s+t}\right)\Big] \\
&\geq -C(1+\log^+(s+t))^C.
\end{aligned}$$

Now (13.1) follows from Jensen's inequality:

$$\mathbb{E}\big[\log P^{\mu(\omega)}(\tau_\beta^1(\theta_s(\omega)) \geq t, \mathcal{A}_t)\big] \geq \mathbb{E}\Big[\int \mathbb{E}[\log P^{\delta_x}\left(\tau_\beta^1(\theta_s(\omega)) \geq s, \mathcal{A}_t\right)]\mu(\omega)(\mathrm{d}x)\Big] = a_\beta(t).$$

□

13.2. The concentration inequality

Proof of Proposition 11.4. First consider the case $t \in \mathbb{N}$. We regard ω as the sum of independent random measures $\omega = \sum_{i \geq 0} \omega_{[i,i+1]}$ and apply Theorem J. Let ω and ω' be two independent realizations of the environment, and for $i = 1, \ldots, t$, let

$$\omega_i := \omega_{[i,i+1]^c} + \omega'_{[i,i+1]}.$$

That is, ω_i is obtained by resampling the disasters of ω in $[i-1, i) \times \mathbb{R}$. We set

$$X := \log P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t),$$
$$X_i := \log P(\tau^1_\beta(\omega_i) \geq t, \mathcal{A}_t).$$

Then, from Theorem J and Jensen's inequality, there exists $C > 0$, depending only on q, such that

$$\begin{aligned}
&\mathbb{E}\left[|X - \mathbb{E}[X]|^{2q}\right] \\
&\quad \leq C\mathbb{E}\left[\left(\sum_{i=0}^{t-1} \mathbb{E}\left[((X - X_i)^+)^2 \Big| \omega\right]\right)^q\right] + C\mathbb{E}\left[\left(\sum_{i=0}^{t-1} \mathbb{E}\left[((X - X_i)^-)^2 \Big| \omega\right]\right)^q\right] \\
&\quad \leq Ct^{q-1} \sum_{i=0}^{t-1} \left(\mathbb{E}\left[((X - X_i)^+)^{2q}\right] + \mathbb{E}\left[((X - X_i)^-)^{2q}\right]\right).
\end{aligned} \tag{13.2}$$

Since $(X - X_i)^+$ and $(X - X_i)^-$ have the same law, we focus on the first one. In our setting, we have

$$\begin{aligned}
(X - X_i)^+ &= \mathbb{1}\{X_i \leq X\} \left(\log P\left(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t\right) - \log P\left(\tau^1_\beta(\omega_i) \geq t, \mathcal{A}_t\right)\right) \\
&\leq \mathbb{1}\{X_i \leq X\} \left(\log P\left(\tau^1_\beta(\omega_{[i,i+1]^c}) \geq t, \mathcal{A}_t\right) - \log P\left(\tau^1_\beta(\omega_i) \geq t, \mathcal{A}_t\right)\right) \\
&\leq \left|\log P\left(\tau^1_\beta(\omega_i) \geq t \mid \tau^1_\beta(\omega_{[i,i+1]^c}) \geq t\right)\right|.
\end{aligned}$$

Noting that the r.h.s. depends only on ω_i that has the same law as ω, we may apply Proposition 11.3 to find a constant $C > 0$ independent of t and β such that

$$\begin{aligned}
\mathbb{E}\left[\left((X - X_i)^+\right)^{2q}\right] &\leq \mathbb{E}\left[\left|\log P\left(\tau^1_\beta(\omega) \geq t \mid \tau^1_\beta(\omega_{[i,i+1]^c}) \geq t, \mathcal{A}_t\right)\right|^{2q}\right] \\
&\leq C(1 + \log^+ t)^{Cq}.
\end{aligned}$$

Substituting this into (13.2), we obtain

$$\mathbb{E}\Big[\left|X - \mathbb{E}[X]\right|^{2q}\Big] \leq Ct^q(1 + \log^+ t)^{Cq}$$

and the desired bound (11.10) for $t \in \mathbb{N}$ follows readily.

It remains to show that it suffices to consider $t \in \mathbb{N}$. By Proposition 11.3, we find $C > 0$ such that for all $t \geq 1$ and all $\beta \in [0, \infty]$,

$$\mathbb{E}\big[\log P(\tau_\beta(\omega) \geq t, \mathcal{A}_t)\big] - \mathbb{E}\big[\log P(\tau_\beta(\omega) \geq \lceil t \rceil, \mathcal{A}_t)\big] \leq C(1 + \log^+ t)^C.$$

Moreover by the same proposition with $p = 2q + 2$, we see that for t sufficiently large,

$$
\begin{aligned}
&\mathbb{P}\Big(\log P(\tau_\beta^1(\omega) \geq t, \mathcal{A}_t) - \log P(\tau_\beta^1(\omega) \geq \lceil t \rceil, \mathcal{A}_t) \geq t^{\frac{1}{2}} \Big) \\
&\quad \leq t^{-(q+1)} \mathbb{E}\Big[\big(\log P(\tau_\beta^1(\omega) \geq t, \mathcal{A}_t) - \log P(\tau_\beta^1(\omega) \geq \lceil t \rceil, \mathcal{A}_t) \big)^{2q+2} \Big] \\
&\quad \leq t^{-(q+\frac{1}{2})}.
\end{aligned} \tag{13.3}
$$

These two bounds allow us to extend (11.10) to $t \in \mathbb{R}_+$. □

13.3. Disasters close to the starting point

Proof of Proposition 11.7. First note that there exists $C > 0$ such that, for all $x \in \mathbb{R}$,

$$P(B(2) \in \mathrm{d}x) \geq C P(B(1) \in \mathrm{d}x) e^{Cx^2}. \tag{13.4}$$

The factor e^{Cx^2} can be regarded as a gain from the 1 extra time. We are going to impose the additional constraint $\{\tau_\infty(\omega) \geq 2\}$ on the l.h.s. and show that the additional cost is much smaller than the gain. More precisely, we show that there exists $K(\omega)$ such that for some (deterministic) $c > 0$ and all $x \geq K(\omega)$,

$$P^{0,0;2,x}(\tau_\infty(\omega) \geq t) \geq c^{-1} \exp\left(-c|x|^{\frac{3}{2}} \right). \tag{13.5}$$

To see this, denote by λ_k the linear function with $\lambda_k(0) = 0$ and $\lambda_k(2) = 5k$, and let $S_k \subseteq \mathbb{R}_+ \times \mathbb{R}$ denote the slanted time-space box

$$S_k := \big\{ (s, x) \colon s \in [0, 2], x \in [\lambda_k(s) - 4, \lambda_k(s) + 4] \big\}. \tag{13.6}$$

We write $R_k := |\omega \cap S_k|$ for the number of disasters in S_k, and $0 < T_1^{(k)} < \cdots < T_{R_k}^{(k)} < 2$ for the corresponding ordered disaster times. It is convenient to further define $T_0^{(k)} := 0$ and $T_{R_k+1}^{(k)} := 2$. As in Section 12.1, we also consider the interarrival times between disasters:

$$\Delta_i^{(k)} := T_{i+1}^{(k)} - T_i^{(k)} \qquad \text{for } i = 0, \ldots, R_k.$$

Note that by our convention $\Delta_0 = T_1^{(k)}$ and $\Delta_{R_k} = 2 - T_{R_k}^{(k)}$. Let us define events

$$
\begin{aligned}
\mathcal{E}_k &:= \{ R_k \leq C \log |k| \}, \\
\mathcal{F}_k &:= \Big\{ \min_{i=0,\ldots,R_k} \Delta_i^{(k)} > k^{-\frac{5}{4}} \Big\}.
\end{aligned}
$$

Since R_k is Poisson distributed with parameter 8, which has an exponentially decaying tail, we find $C > 0$ such that

$$\sum_{k \in \mathbb{Z}} \mathbb{P}(\mathcal{E}_k^c) < \infty.$$

Thus we have $\mathbb{P}(\mathcal{E}) = 1$ for $\mathcal{E} := \{\mathcal{E}_k \text{ for all but finitely many } k\}$ by the Borel–Cantelli lemma. Next, note that $\mathcal{F}_k^c$ is nothing but the event that the Poisson process with rate 8 on $[0,2]$ has a point in the $k^{-5/4}$ neighborhood of the boundary or has two points within distance $k^{-5/4}$. It is easy to see that such probability decays like $\mathbb{P}(\mathcal{F}_k^c) \le ck^{-5/4}$.

Setting $\mathcal{F} := \{\mathcal{F}_k \text{ for all but finitely many } k\}$ and using the Borel–Cantelli lemma again, we find that $\mathbb{P}(\mathcal{E} \cap \mathcal{F}) = 1$. Now for $\omega \in \mathcal{E} \cap \mathcal{F}$, we find $K(\omega) \ge 2$ such that for all $|k| \ge K(\omega)$, we have $R_k \le C \log |k|$ and $\min\{\Delta_0^{(k)}, \dots, \Delta_{R_k}^{(k)}\} > k^{-5/4}$. Observe that every $x \in \mathbb{R}$ is contained in $[5k(x) - 3, 5k(x) + 3]$ for some $k(x) \in \mathbb{Z}$, and in particular $(2, x) \in S_{k(x)}$. Then for all x with $|k(x)| \ge K(\omega)$, we use the estimates from Lemma 12.3 to get

$$\begin{aligned} P^{0,0;2,x}(\tau_\infty(\omega) \ge t) &\ge P^{0,0;2,x}(\tau_\beta(\omega) \ge 2, B(u) \in \lambda_{k(x)}(u) + [-3,3] \text{ for } u \in [0,2]) \\ &\ge \exp\Big(-c - \sum_{i=0}^{R_k} \frac{c}{\Delta_i^{(k)}}\Big) \\ &\ge \exp\Big(-c - c|k|^{\frac{5}{4}} \log|k|\Big). \end{aligned} \tag{13.7}$$

This finishes the proof of (13.5).

For $x \in \mathbb{R}$ with $|k(x)| \le K(\omega)$, we can still use the second line in (13.7) as a lower bound. Therefore we conclude that

$$\begin{aligned} &P(B(2) \in \mathrm{d}x, \tau_\infty(\omega) \ge 2) \\ &\quad \ge P(B(1) \in \mathrm{d}x)\bigg(C \inf_{x \in \mathbb{Z}^d} e^{Cx^2 - c|x|^{3/2}} \wedge \min_{|k| \le K(\omega)} \exp\Big(-c - \sum_{i=0}^{R_k} \frac{c}{\Delta_i^{(k)}}\Big)\bigg). \end{aligned} \tag{13.8}$$

□

13.4. Existence of the free energy in dimension $d = 1$

Proof of Theorem 11.6. **Part (i)**: In Proposition 13.1 we have shown that

$$t \mapsto \mathbb{E}[\log P(\tau_\infty^1(\omega) \ge t, \mathcal{A}_t)]$$

is almost-superadditive in the sense of (3.6). Thus, the conclusion follows from Theorem H.
Part (ii): The almost sure convergence

$$\lim_{t \to \infty, t \in \mathbb{N}} \frac{1}{t} \log P(\tau_\infty^1(\omega) \ge t, \mathcal{A}_t) = \mathfrak{p}(\infty)$$

along $t \in \mathbb{N}$ follows by choosing $r = 2$ in Proposition 11.4 and the Borel–Cantelli lemma. Let us extend this convergence to $t \in \mathbb{R}_+$. Note that the definition of the truncation in (11.8) implies $\mathcal{A}_t = \mathcal{A}_{\lceil t \rceil}$. Therefore we have

$$P(\tau_\infty^1(\omega) \ge \lceil t \rceil, \mathcal{A}_{\lceil t \rceil}) \le P(\tau_\infty^1(\omega) \ge t, \mathcal{A}_t) \le P(\tau_\infty^1(\omega) \ge \lfloor t \rfloor, \mathcal{A}_{\lceil t \rceil}).$$

On the other hand, one can easily deduce from Proposition 11.3 and the Borel–Cantelli lemma that almost surely,

$$
\begin{aligned}
\log P(\tau^1_\infty(\omega) \geq \lfloor t \rfloor, \mathcal{A}_{\lceil t \rceil}) &= \log P(\tau^1_\infty(\omega_{[\lfloor t \rfloor, \lceil t \rceil]^c}) \geq \lceil t \rceil, \mathcal{A}_{\lceil t \rceil}) \\
&\leq \log P(\tau^1_\infty(\omega) \geq \lceil t \rceil, \mathcal{A}_{\lceil t \rceil}) + t^{1/2}
\end{aligned}
$$

for all sufficiently large t. Combining the above two bounds, we find

$$
\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t) = \mathfrak{p}(\infty).
$$

Next we get rid of $\mathcal{A}_t$. Note that Lemma 12.1 implies $\mathfrak{p}(\infty) > -\infty$ while $P(\mathcal{A}_t^c) \leq e^{-ct^3}$. It follows that almost surely,

$$
\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t) = \lim_{t\to\infty} \frac{1}{t} \log P(\tau^1_\infty(\omega) \geq t) = \mathfrak{p}(\infty).
$$

Finally, we replace τ^1_∞ by τ_∞. By definition, it is clear that $P(\tau_\infty(\omega) \geq t) \leq P(\tau^1_\infty(\omega) \geq t)$, and therefore we only have to show that almost surely

$$
\liminf_{t\to\infty} \frac{1}{t} \log P(\tau_\infty(\omega) \geq t) \geq \mathfrak{p}(\infty). \tag{13.9}
$$

Using Proposition 11.7, we find that

$$
\begin{aligned}
P(\tau_\infty(\omega) \geq t) &= \int_{\mathbb{R}} P^{2,x}(\tau_\infty(\omega) \geq t-2) P(B(2) \in \mathrm{d}x, \tau_\infty(\omega) \geq 2) \\
&\geq A(\omega) \int_{\mathbb{R}} P^{2,x}(\tau_\infty(\omega) \geq t-2) P(B(1) \in \mathrm{d}x) \\
&\geq A(\omega) P(\tau^1_\infty(\theta^{1,0}\omega) \geq t-1).
\end{aligned} \tag{13.10}
$$

Since $\lim_{t\to\infty} t^{-1} \log P(\tau^1_\infty(\theta^{1,0}\omega) \geq t-1) = \mathfrak{p}(\infty)$ almost surely, we are done. □

13.5. Continuity of the free energy

Proof of Theorem 11.9. **Part (i)**: The upper bound in (11.12) follows from (13.1) together with the bound (3.7) in the almost-superadditive lemma (Theorem H), which gives

$$
\frac{a_\beta(t)}{t} \leq \mathfrak{p}(\beta) + 4 \int_{2t}^\infty s^{-(2-\varepsilon)} \mathrm{d}s \leq \mathfrak{p}(\beta) + \tfrac{4}{2-\varepsilon} t^{-(1-\varepsilon)}.
$$

For the lower bound, we first prove that there exists t_0 such that for all $t \geq t_0$,

$$
a_\beta(2t) \leq 2a_\beta(t) + Ct^{1/2+\varepsilon}. \tag{13.11}
$$

To this end, we define for all $x \in [-\lceil t \rceil^2, \lceil t \rceil^2 - 1] \cap \mathbb{Z}$,

$$
\begin{aligned}
p_x &:= P(B(t) \in [x, x+1), \tau^1_\beta(\omega) \geq t, \mathcal{A}_t), \\
\mu_x(\omega)(\mathrm{d}x) &:= P(B(t) \in \mathrm{d}x \mid B(t) \in [x, x+1), \tau^1_\beta(\omega) \geq t, \mathcal{A}_t) \in \mathcal{M}_1([x, x+1)), \\
X_x &:= P^{\mu_x(\omega)}(\tau^1_\beta(\theta^t(\omega)) \geq t, \mathcal{A}_t), \\
Y_x &:= P^{\delta_x}(\tau^1_\beta(\theta^t(\omega)) \geq t, \mathcal{A}_t),
\end{aligned}
$$

where as before P^μ denotes the law of Brownian motion started with $B(0)$ distributed according to μ. Moreover we consider events

$$\begin{aligned}
\mathcal{B}_0 &:= \big\{Y_0 = \max\{Y_x \colon x \in [-t^2, t^2] \cap \mathbb{Z}\}\big\}, \\
\mathcal{B}_1 &:= \Big\{\big|\log P(\tau^1_\beta(\omega) \geq 2t, \mathcal{A}_{2t}) - \mathbb{E}[\log P(\tau^1_\beta(\omega) \geq 2t, \mathcal{A}_{2t})\big| \leq (2t)^{1/2+\varepsilon}\Big\}, \\
\mathcal{B}_2 &:= \Big\{\big|\log P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t) - \mathbb{E}[\log P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t)]\big| \leq t^{1/2+\varepsilon}\Big\}, \\
\mathcal{B}_3 &:= \Big\{|Y_0 - \mathbb{E}[Y_0]| \leq t^{1/2+\varepsilon}\Big\}.
\end{aligned}$$

Since $\{Y_x \colon x \in \mathbb{Z}\}$ is a stationary sequence, we have

$$\mathbb{P}(\mathcal{B}_0) \geq (2t^2+1)^{-1}. \tag{13.12}$$

Note that there exists $C > 0$ such that for all $\mu \in \mathcal{M}([0,1])$ and all $x \in \mathbb{R}$,

$$P^{\delta_0}(B(1) \in \mathrm{d}x) \vee P^{\delta_1}(B(1) \in \mathrm{d}x) \geq C P^\mu(B(1) \in \mathrm{d}x).$$

This implies that almost surely for all $x \in [-\lceil t \rceil^2, \lceil t \rceil^2 - 1] \cap \mathbb{Z}$,

$$Y_x \vee Y_{x+1} \geq C X_x.$$

By Proposition 11.10, there exists $C > 0$ such that for all t and all $\beta \in [0, \infty]$

$$\mathbb{P}(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \geq 1 - Ct^{-3}. \tag{13.13}$$

Combining (13.12) and (13.13), we find that $\mathcal{B} := \mathcal{B}_0 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_4$ has positive probability for all sufficiently large t. In particular, it is non-empty and we can pick an $\omega \in \mathcal{B}$. Then, since $\omega \in \mathcal{B}_0$, we have

$$\begin{aligned}
&P(\tau^1_\beta(\omega) \geq 2t, \mathcal{A}_{2t}) \\
&\quad \leq \sum_{x \in [-t^2, t^2] \cap \mathbb{Z}} p_x X_x + P\left(\sup_{r \in [0,t]} |B(r)| > \lceil t \rceil^2 \text{ or } \sup_{r \in [t, 2t]} |B(r) - B(t)| > \lceil t \rceil^2\right) \\
&\quad \leq C \sum_{x \in [-t^2, t^2] \cap \mathbb{Z}} p_x (Y_x \vee Y_{x+1}) + 2e^{-Ct^3} \\
&\quad \leq C P(\tau^1_\beta(\omega) \geq t, \mathcal{A}_t) Y_0 + 2e^{-Ct^3}.
\end{aligned}$$

Next, by using $\omega \in \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$, we can replace the logarithm of the probabilities by their $\mathbb{P}$-expectation with the error terms, which yields for t sufficiently large,

$$\begin{aligned}
a_\beta(2t) - (2t)^{1/2+\varepsilon} &\leq 2a_\beta(t) + 2t^{1/2+\varepsilon} + \log\left(1 + 2e^{-t^3 + 2a_\beta(t)}\right) + C \\
&\leq 2a_\beta(t) + 2t^{1/2+\varepsilon} + 2e^{-t^3 + 2C(1+t)} + C \\
&\leq 2a_\beta(t) + Ct^{1/2+\varepsilon},
\end{aligned}$$

where we have used Lemma 12.1(ii) in the second inequality. This finishes the proof of (13.11), and by applying it repeatedly we obtain, for any $k \in \mathbb{N}$,

$$\begin{aligned} a_\beta(t) &\geq \tfrac{1}{2} a_\beta(2t) - Ct^{1/2+\varepsilon} \\ &\geq \tfrac{1}{4} a_\beta(4t) - Ct^{1/2+\varepsilon} 2^{-2(1/2-\varepsilon)} - Ct^{1/2+\varepsilon} \\ &\geq \dots \\ &\geq \left(\tfrac{1}{2}\right)^k a_\beta(2^k t) - Ct^{1/2+\varepsilon} \textstyle\sum_{i=0}^{k-1} 2^{-i(1/2-\varepsilon)} \end{aligned}$$

By assumption $\varepsilon < 1/2$, so that the sum in the last line converges for $k \to \infty$ and we get

$$\frac{a_\beta(t)}{t} + Ct^{-(1/2-\varepsilon)} \geq \lim_{k\to\infty} \frac{a_\beta(2^k t)}{2^k t} = \mathfrak{p}(\beta).$$

Part (ii): Fix an arbitrary $\delta > 0$. By part (i), we find $t_0 > 0$ such that for all $\beta \in [0, \infty]$,

$$\left| \frac{a_\beta(t)}{t} - \mathfrak{p}(\beta) \right| \leq \delta.$$

Since $a_\beta(t)$ is the expectation of a random variable depending on the disasters in a finite area, it is clear that $\beta \mapsto a_\beta(t)$ is continuous. So there exists β_0 such that for all $\beta \geq \beta_0$,

$$|\mathfrak{p}(\beta) - \mathfrak{p}(\infty)| \leq 2\delta + \frac{1}{t} |a_\beta(t) - a_\infty(t)| \leq 3\delta.$$

This implies the desired continuity.

□

Remark 13.2. When $d \geq 2$, we have

$$\mathbb{P}(\mathcal{B}_0) \geq (1 + t^{2d})^{-1}$$

instead of (13.12) and we have to replace (13.13) by

$$\mathbb{P}(\mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3) \leq Ct^{-2d-1}.$$

This causes no problem since Proposition 11.4 gives arbitrary fast polynomial decay.

13.6. Existence of the free energy in dimension $d \geq 2$

In this section, we prove the almost sure convergence

$$\lim_{t\to\infty} \frac{1}{t} \log P(\tau_\infty(\omega) \geq t) = \mathfrak{p}(\infty)$$

in dimension $d \geq 2$. As mentioned in Remark 11.8, the only point that requires an extra argument is (13.9).

Note that Proposition 11.7 does not generalize to higher dimensions: For $\mathbf{k} \in \mathbb{Z}^d$, let $L_{\mathbf{k}}$ denote the last disaster in a multi-dimensional version of the time-space box from (13.6).

Then almost surely, there exists a point $\mathbf{k} \in \mathbb{Z}^d$ with $\|\mathbf{k}\| \leq K$ such that $2 - L_{\mathbf{k}} < K^{-d+1/2}$ for all sufficiently large K. If x is behind the last disaster for such $\mathbf{k}$, then in $d \geq 3$ the second line in (13.7) is smaller than $\exp(-cK^{d-1/2}) = o(\exp(-CK^2))$ and hence cannot be compensated by the factor e^{Cx^2} in (13.4). The problem in this argument is that we have too many $\mathbf{k}$'s.

We solve this problem in the following steps:

- First restrict $B(1)$ to an essentially one-dimensional slab $H_{\mathbf{0}}$,
- then show that this restriction does not affect the limit.

For $\mathbf{k} = (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}$, let

$$H_{\mathbf{k}} := \mathbb{R} \times \left[k_2 - \tfrac{1}{2}, k_2 + \tfrac{1}{2}\right) \times \cdots \times \left[k_d - \tfrac{1}{2}, k_d + \tfrac{1}{2}\right)$$

and set

$$b_t(\omega, \mathbf{k}) := P\left(\tau_\infty^1(\omega) \geq t, B(1) \in H_{\mathbf{k}}, \mathcal{A}_t\right).$$

An easy extension of Proposition 11.7 shows that there exists some positive and finite random variable $A'(\omega)$ such that for all $x \in H_{\mathbf{0}}$ and $t \geq 2$,

$$P(\tau_\infty(\omega) \geq 2, B(2) \in \mathrm{d}x, \mathcal{A}_t) \geq A'(\omega) P(B(1) \in \mathrm{d}x, \mathcal{A}_t).$$

Then, by the same argument as in (13.10), we have

$$\begin{aligned} P(\tau_\infty(\omega) \geq t, \mathcal{A}_t) &\geq P\left(\tau_\infty(\omega) \geq t, B(2) \in H_{\mathbf{0}}, \mathcal{A}_t\right) \\ &\geq A'(\omega) b_t(\theta^{1,0}(\omega), \mathbf{0}). \end{aligned}$$

Thus (13.9) follows once we show that $\mathbb{P}$-almost surely,

$$\liminf_{t \to \infty} \frac{1}{t} \log b_t(\omega, \mathbf{0}) \geq \mathfrak{p}(\infty). \tag{13.14}$$

The proof of (13.14) is divided into the following two lemmas, which are analogous to Proposition 11.4 and the lower bound in Theorem 11.9(i):

Lemma 13.3. *There exists $t_0 > 0$ such that for all $t \geq t_0$,*

$$\mathbb{P}\Big(\big|\log b_t(\omega, \mathbf{0}) - \mathbb{E}[\log b_t(\omega, \mathbf{0})]\big| \geq t^{3/4}\Big) \leq t^{-2d-1}. \tag{13.15}$$

Lemma 13.4. *There exists $t_0 > 0$ such that for all $t \geq t_0$,*

$$\mathbb{E}[\log b_t(\omega, \mathbf{0})] \geq \mathbb{E}[\log P(\tau_\infty^1(\omega) \geq t, \mathcal{A}_t)] - t^{3/4}. \tag{13.16}$$

Proof of Lemma 13.3. The proof is almost identical to that of Propositions 11.4. Let us introduce a multidimensional version of the notation used before:

$$\begin{aligned} J_x^{(1)} &:= x + \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)^d, \\ J_x^{(5)}(i) &:= x + 7i\mathbf{e}_1 + \left[-\tfrac{5}{2}, \tfrac{5}{2}\right) \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right)^{d-1}, \\ M^p(\nu) &:= \sup_{x,y \in \mathbb{R}^d} \min_{i=0,\ldots,p} \nu\big(J_x^{(5)}(i) \times J_y^{(5)}(i)\big), \end{aligned}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denotes the canonical basis of $\mathbb{R}^d$. With these definitions, Lemmas 12.1 and 12.6 readily extend to $d \geq 2$. Moreover, Lemma 12.7 holds for

$$\widehat{\nu}^{r^-, r^+, t}_{\omega, \infty}(\mathrm{d}(x,y)) := P\left((B(r^-), B(r^+)) \in \mathrm{d}(x,y) \;\middle|\; \tau^1_\infty(\omega_{[r^-, r^+]^c}) \geq t, \mathcal{A}_t, B(1) \in H_{\mathbf{0}}\right)$$

in place of $\nu^{r^-, r^+, t}_{\omega, \infty}$. Given these ingredients, we can follow the same argument to prove Proposition 11.4.

Let us explain how to verify Lemma 12.7 for $\widehat{\nu}$. Since (12.32) holds with ν replaced by $\widehat{\nu}$, the proof of Lemma 12.7 works without change in the case $r^- \geq 2$. The case $r^- < 2$ requires some care because we need to sprinkle the mass on the time interval $[0,1]$ under the additional constraint $\{B(1) \in H_{\mathbf{0}}\}$. We define r_i^+ as in Section 12.3 and choose j_i^+ such that

$$\widehat{\nu}^{1, r_i^+, t}_{\omega, \infty}\big(H_{\mathbf{0}} \times J^{(1)}_{j_i^+}\big) \geq C t^{-d}. \tag{13.17}$$

Then, define $\lambda_i(u) = (\lambda_i^1(u), \lambda^2(u))$ with $\lambda_i^1 \colon \mathbb{R}_+ \to \mathbb{R}$ and $\lambda_i^2 \colon \mathbb{R}_+ \to \mathbb{R}^{d-1}$ such that

- $\lambda_i(r_i^+) = j_i^+$,
- λ_i^1 is linear ($\lambda_i^1(0) = 0$),
- λ_i^2 is piecewise affine linear with $\lambda_i^2(0) = \lambda_i^2(1) = 0$.

Using this definition, we can replace ν by $\widehat{\nu}$ in (12.38). Observe that, unlike in the one-dimensional case, S_i^+ is not a slanted time-space box in the last $d-1$ coordinates. This is in order to ensure $\{B(1) \in H_{\mathbf{0}}\}$. As a consequence, we have to consider the Brownian bridge conditioned on $\{B(1) \in H_{\mathbf{0}}\}$ in (12.37). However, this does not impose any additional cost since we have the same conditioning in the definition of $\widehat{\nu}$. For the coordinates in time and $\mathbf{e}_1$-direction, we can apply an affine transformation and we get (12.38) for $\widehat{\nu}$. $\square$

Proof of Lemma 13.4. We argue in a similar way to the proof of Theorem 11.9(i). Let us introduce events

$$\begin{aligned}
\mathcal{B}_0(t) &:= \Big\{ b_t(\omega, \mathbf{0}) \geq \max_{\mathbf{k} \in \{-t^2, \ldots, t^2\}^{d-1}} b_t(\omega, \mathbf{k}) - e^{-Ct^3} \Big\}, \\
\mathcal{B}_1(t) &:= \Big\{ \big| \log b_t(\omega, \mathbf{0}) - \mathbb{E}[\log b_t(\omega, \mathbf{0})] \big| \leq t^{3/4} \Big\}, \\
\mathcal{B}_2(t) &:= \Big\{ \big| \log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t) - \mathbb{E}[\log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t)] \big| \leq t^{3/4} \Big\}.
\end{aligned}$$

Note that from here on out t should be replaced by $\lceil t \rceil$, which we omit to ease the notation. Proposition 11.4 and (13.15) yield that for all t large enough

$$\mathbb{P}(\mathcal{B}_1(t)^c \cup \mathcal{B}_2(t)^c) \leq 2t^{-2d-1}. \tag{13.18}$$

Moreover we claim that

$$\mathbb{P}(\mathcal{B}_0(t)) \geq \big(1 + 2t^2\big)^{-(d-1)}. \tag{13.19}$$

Postponing the claim for the moment, note that from (13.18) and (13.19), we get that $\mathcal{B}_0(t) \cap \mathcal{B}_1(t) \cap \mathcal{B}_2(t)$ has a positive probability for all t large enough. In particular the intersection is not empty and we can choose $\omega \in \mathcal{B}_0(t) \cap \mathcal{B}_1(t) \cap \mathcal{B}_2(t)$. Therefore

$$\begin{aligned}\mathbb{E}[\log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t)] &\leq \log P(\tau^1_\infty(\omega) \geq t, \mathcal{A}_t) + t^{3/4} \\ &= \log \Big(\sum_{\mathbf{k}=\{-t^2,\dots,t^2\}^{d-1}} b_t(\omega, \mathbf{k}) \Big) + t^{3/4} \\ &\leq \log \left((2t^2+1)^d \left(e^{\log b_t(\omega,\mathbf{0})} + e^{-Ct^3} \right) \right) + t^{3/4} \\ &\leq \log \left((2t^2+1)^d \left(e^{\mathbb{E}[\log b_t(\omega,\mathbf{0})]+t^{3/4}} + e^{-Ct^3} \right) \right) + t^{3/4},\end{aligned}$$

where the first and the last inequality follow from $\omega \in \mathcal{B}_1(t) \cap \mathcal{B}_2(t)$, and the second inequality follows from $\omega \in \mathcal{B}_0(t)$. From Lemma 12.1(ii), we see that $\mathbb{E}[\log b_t(\omega, \mathbf{0})]$ decays linearly, which finishes the proof of (13.16).

It remains to show (13.19). This is intuitively obvious since $b_t(\omega, \mathbf{0})$ should have the highest chance to be the maximum as it imposes least constraint on $[0, 1]$, and there are $(1 + 2t^2)^{(d-1)}$ candidates. To make this argument rigorous, it is better to drop the truncation $\mathcal{A}_t$ and work with

$$c_t(\omega, \mathbf{k}) := P\left(\tau^1_\infty(\omega) \geq t, B(1) \in H_{\mathbf{k}}\right).$$

For $\mathbf{k} \in \mathbb{Z}^{d-1}$, let $\omega_{\mathbf{k}} := \theta^{0,(0,\mathbf{k})}(\omega)$ be obtained by shifting ω by $(0, \mathbf{k}) \in \mathbb{Z}^d$ in space, and let $\mathbf{K} = \mathbf{K}(\omega)$ be the random index such that

$$c_t(\omega_{\mathbf{K}}, 0) = \max_{\mathbf{k}=\{-t^2,\dots,t^2\}^{d-1}} c_t(\omega_{\mathbf{k}}, 0).$$

Then by this definition, for every $\mathbf{k} \in \{-t^2, \dots, t^2\}^{d-1}$,

$$\begin{aligned}c_t(\omega_{\mathbf{K}}, \mathbf{k}) &= \int_{H_{\mathbf{0}}} P^{1,z+(0,\mathbf{k})}(\tau^1_\infty(\omega_{\mathbf{K}}) \geq t) P(B(1) \in (0, \mathbf{k}) + \mathrm{d}z) \\ &\leq \int_{H_{\mathbf{0}}} P^{1,z+(0,\mathbf{k})}(\tau^1_\infty(\omega_{\mathbf{K}}) \geq t) P(B(1) \in \mathrm{d}z) \\ &= c_t(\omega_{(0,-\mathbf{k})}, 0) \\ &\leq c_t(\omega_{\mathbf{K}}, 0).\end{aligned}$$

Here we use $P^{s,x}$ for the law of Brownian motion started at time s with initial distribution δ_x. Together with $P(\mathcal{A}_t^c) \leq e^{-Ct^3}$, we get

$$\begin{aligned}b_t(\omega_{\mathbf{K}}, 0) \geq c_t(\omega_{\mathbf{K}}, 0) - e^{-Ct^3} &= \max_{\mathbf{k}=\{-t^2,\dots,t^2\}^{d-1}} c_t(\omega_{\mathbf{K}}, \mathbf{k}) - e^{-Ct^3} \\ &\geq \max_{\mathbf{k}=\{-t^2,\dots,t^2\}^{d-1}} b_t(\omega_{\mathbf{K}}, \mathbf{k}) - e^{-Ct^3}.\end{aligned}$$

Now let $\mathbf{L}$ be independent of ω and uniformly distributed on $\{-t^2, \ldots, t^2\}^{d-1}$, and set $\widetilde{\omega} \coloneqq \omega_{\mathbf{L}}$. Since $\widetilde{\omega}$ has the same distribution as ω, we have

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_0(t)) &= \mathbb{P}\Big(b_t(\widetilde{\omega}, 0) \geq \max_{\mathbf{k}=\{-t^2,\ldots,t^2\}^{d-1}} b_t(\widetilde{\omega}, \mathbf{k}) - e^{-Ct^3}\Big) \\
&\geq \mathbb{P}\big(\mathbf{L} = \mathbf{K}(\omega)\big) \\
&= (1+2t^2)^{-(d-1)}
\end{aligned}$$

and we are done. $\square$

Part V. Stochastic comparison in space-time random environments

14. Introduction

14.1. Motivation

We discuss a question that is relevant to all models introduced up to this point. We start by presenting an intuition, using the random walk among disasters from Part II. Recall from (5.3) that the quenched survival probability $P^\kappa(\tau(\omega) \geq t)$ has the same law as $u_\omega(t, 0)$, the solution to

$$\begin{aligned} &u(0, \cdot) \equiv 1, \\ &u(\mathrm{d}t, i) = \kappa\, (\Delta u)(t, i)\mathrm{d}t - u(t^-, i)\omega(\mathrm{d}t, i) \quad \text{ for } t \geq 0, i \in \mathbb{Z}^d. \end{aligned} \tag{14.1}$$

Since the jump rate κ appears as a parameter in this model, it is natural to ask whether we can compare solutions for different values of κ.

Consider the following interpretation for the dynamics: Initially, every site has mass one. Whenever $(t, i) \in \omega$ for some $t \geq 0$, all mass currently at site i is removed from the system (that is, the mass at site i is set to zero). On the other hand, the Laplacian Δ spreads the mass at i to the neighboring sites at rate κ. We think of the environment ω as a random disorder which is smoothed by the Laplacian. The parameter κ adjusts the strength of this smoothing, and we expect that (in some sense) the solution to (14.1) is less random if κ is larger, and indeed we will prove this.

Let us try to make this a bit more precise: Observe that, since $X \equiv 0$ under P^0, we can approximate, for $\kappa \downarrow 0$,

$$P^\kappa(\tau(\omega) \geq t) \approx \mathbb{1}\{\omega \cap ([0, t] \times \{0\}) = \varnothing\}. \tag{14.2}$$

On the other hand, if the jump rate is high, then the random walk can visit a large part of the environment before time t, so from the spatial ergodicity of ω we expect, for $\kappa \uparrow \infty$,

$$P^\kappa(\tau(\omega) \geq t) \approx \mathbb{E}[P^\kappa(\tau(\omega) \geq t)] = e^{-t}. \tag{14.3}$$

At the qualitative level, we note that (14.2) depends on a small part of the environment and therefore has large fluctuations, while (14.3) is (almost) deterministic. In Theorem 15.1 we will see that this observation holds more generally and not only for the extremal cases $\kappa \approx 0$ and $\kappa \approx \infty$. More precisely, we show that $\kappa \mapsto P^\kappa(\tau(\omega) \geq t)$ is increasing in concave stochastic order, i.e. for all $\kappa_1 \leq \kappa_2$ and all real, concave functions f

$$\mathbb{E}\big[f\big(P^{\kappa_1}(\tau(\omega) \geq t)\big)\big] \leq \mathbb{E}\big[f\big(P^{\kappa_2}(\tau(\omega) \geq t)\big)\big]. \tag{14.4}$$

The function $f(x) = \log(x)$ is of special importance since it corresponds to the free energy, see Theorem K.

Note that the concave stochastic order is a measure of the "riskiness" of random variables. To see why $P^{\kappa_1}(\tau(\omega) \geq t)$ has more risk than $P^{\kappa_2}(\tau(\omega) \geq t)$ for $\kappa_1 \leq \kappa_2$, note that the survival probability is small if disasters in the environment form a trap close to the origin. Here the term "trap" refers to a configuration of disasters that forces the random walk to behave atypically to avoid it. A high jump rate is helpful because it allows the random walk more flexibility to go around problematic areas. However, (14.2) shows that a high jump rate is not optimal for every ω, since there are environments with no disasters at the origin, see also Figure 19. In that sense, we can view a low jump rate as an all-or-nothing gamble where the survival probability is large if there is no trap, and small otherwise.

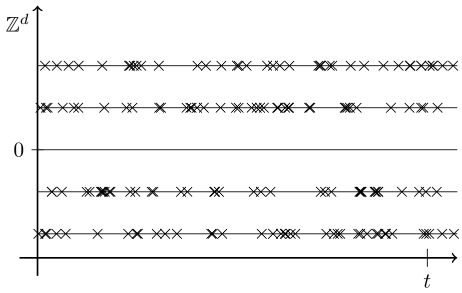

Figure 19: Example of an environment where the survival probability is maximal at $\kappa \approx 0$. Note that this is only true if the random walk starts at the origin – if it starts at any other site, then a high jump rate is better.

An alternative interpretation is that $P^{\kappa}(\tau(\omega) \geq t)$ represents a variational problem where one has to balance out the benefit of moving to a "good" part of the environment with the entropic cost of the random walk behaving "atypical". In that interpretation, a high jump rate is helpful because the random walk can move to good parts of the environment with smaller cost.

Writing $Z(\omega) := P^{\kappa}(\tau(\omega) \geq t)$ for the partition function, we show that the implication

$$\text{“more randomness in } X\text{”} \quad \Longrightarrow \quad \text{“less randomness in } Z\text{”} \tag{14.5}$$

is a universal property of all the models discussed before. We will present the result in an abstract way that is suitable for discrete and continuous time, and where the underlying random walk moves in an abelian group. Our aim is to identify which assumptions are necessary, regardless of the specific features of the model.

14.2. Outline

In Section 14.3 we will introduce notation that is suitable for a general class of models, using the random walk from Part II as an ongoing example. In Section 15 we state and prove the main result (Theorem 15.1). We then discuss implications for various models:

- The discrete-time random polymer model (Section 16.1)
- Continuous-time random walk in Lévy-type environments (Section 16.2)
- Brownian motion in Poissonian environments (Section 16.3)

- Discrete-time branching random walk in space-time environments (Section 16.4).
- Continuous-time branching random walk among disasters (Section 16.5)

Finally, we discuss limitations of our result: Note that (14.1) is more commonly studied with ω replaced by a random field which is static in time or has long-time correlations. Here we do not expect that the implication (14.5) holds, see the discussion in Section 17.1.

Moreover, recall the convolution property of the continuous-time random walk: If X and X' are independent random walks of jump rates κ and κ', then $X + X'$ is a random walk of jump rate $\kappa + \kappa'$. This fact is crucial in the proof of Theorem 15.1, but we expect that one can also obtain a comparison between partition functions under weaker assumptions. In Section 17.2 we make a conjecture for the optimal criterion, and in Section 17.3 we show that our criterion is indeed necessary and sufficient if the integer lattice is replaced by a tree.

14.3. Definition of the model

The notation presented here is suitable for models in discrete and in continuous time, and on the lattice $\mathbb{Z}^d$ as well as in Euclidean space $\mathbb{R}^d$. Let T be equal to $\mathbb{N}$ or $\mathbb{R}_+$ and let I be a commutative group. We use "+" for the group action and 0 for the neutral element. Let $\mathcal{I}$ denote the set of càdlàg paths $x\colon T \to I$. For $x, y \in \mathcal{I}$, we use $x + y$ to denote the path obtained by coordinate-wise addition

$$(x + y)(t) := x(t) + y(t).$$

By a slight abuse of notation, we also use 0 for the trivial path $x \equiv 0$.

Let $\mathcal{G}$ be the smallest sigma field such that all projections $x \mapsto x(t)$ are $\mathcal{G}$-measurable. Moreover, let $(\Omega, \mathcal{F})$ be a probability space and let

$$\{\theta^x\colon x \in \mathcal{I}\}$$

be a family of $\mathcal{F}$-measurable bijections $\theta^x\colon \Omega \to \Omega$. An element $\omega \in \Omega$ is called an **environment** while θ^x is called the **shift** associated to the path x.

Example (Random walk among disasters). *Let $T = \mathbb{R}_+$, $I = \mathbb{Z}^d$ and Ω the set of locally finite subsets of $\mathbb{R}_+ \times \mathbb{Z}^d$. For $x \in \mathcal{I}$ and $\omega \in \Omega$, the shifted environment $\theta^x\omega$ is obtained by moving all disasters in ω according to the displacement of x. That is,*

$$(t, i) \in \omega \qquad \Longleftrightarrow \qquad (t, i - x(t)) \in (\theta^x\omega). \tag{14.6}$$

Definition 14.1. Let $F\colon \Omega \times \mathcal{I} \to \mathbb{R}_+$ be $\mathcal{F} \otimes \mathcal{G}$-measurable. We say that F is **consistent** if $F(\omega, x + y) = F(\theta^y\omega, x)$ for every $\omega \in \Omega$, $x, y \in \mathcal{I}$.

Definition 14.2. Let $F\colon \Omega \times \mathcal{I} \to \mathbb{R}_+$ be consistent. We call

$$Z(\omega) := \int_{\mathcal{I}} F(\omega, x) P(\mathrm{d}x). \tag{14.7}$$

the **partition function** of $P \in \mathcal{M}_1(\mathcal{I})$.

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ and write $\mathbb{E}$ for its expectation.

Example (Continued)**.** *For $\omega \in \Omega$, $x \in \mathcal{I}$, let*

$$F_t(\omega, x) := \mathbb{1}\{(s, x(s)) \notin \omega \text{ for all } s \in [0, t]\} \tag{14.8}$$

the indicator function of the event that x survives until time t. Note that F_t is consistent: For $\omega \in \Omega$, $x, y \in \mathcal{I}$

$$\begin{aligned} F_t(\omega, x+y) = 1 &\iff (s, x(s)+y(s)) \notin \omega \text{ for all } s \in [0,t] \\ &\iff (s, x(s)) \notin (\theta^y \omega) \text{ for all } s \in [0,t] \iff F_t(\theta^y \omega, x) = 1. \end{aligned}$$

Let $\mathbb{P}$ denote the Poisson point process with unit intensity on $\mathbb{R}_+ \times \mathbb{Z}^d$ and P^κ the law of the continuous time simple random walk with jump rate κ. Then the partition function $Z_t^\kappa(\omega)$ of P^κ agrees with the quenched survival probability $P^\kappa(\tau(\omega) \geq t)$ from Part II.

We make the following assumptions:

Assumption A1 (Shift invariance)**.** $\mathbb{P}(A) = \mathbb{P}\big((\theta^x)^{-1}(A)\big)$ for all $A \in \mathcal{F}$, $x \in \mathcal{I}$.

Assumption A2 (Integrability)**.** $\mathbb{E}[F(\cdot, 0)] < \infty$.

The partition function is a well-defined random variable:

Lemma 14.3. *Let Z be the partition function of $P \in \mathcal{M}_1(\mathcal{I})$ and $\mathbb{P}$ such that* (A1) *and* (A2) *hold. Then Z is $\mathcal{F}$-measurable and $\mathbb{E}[Z] < \infty$. In particular, $\mathbb{P}$-almost surely $Z < \infty$. Moreover, if Z' is the partition function of $P' \in \mathcal{M}_1(\mathcal{I})$, then $\mathbb{E}[Z] = \mathbb{E}[Z']$.*

This follows easily from Fubini's theorem and the definition of consistency, so we skip the proof.

We have now finished setting up the model. Next, we need to make the notion "X has more randomness" from (14.5) precise, so we introduce an order relation for probability measures on $\mathcal{I}$. Note that, since $\mathcal{I}$ inherits the group structure from I, we can define a convolution on $\mathcal{I}$:

Definition 14.4. Let $P, Q \in \mathcal{M}_1(\mathcal{I})$. Then $P * Q$ is the law of $X + Y$, where X and Y are independent and have laws P and Q.

This defines an order $\preceq_*$ on $\mathcal{M}_1(\mathcal{I})$:

Definition 14.5. For $P^1, P^2 \in \mathcal{M}_1(\mathcal{I})$ we write $P^2 \preceq_* P^1$ if $P^2 = P^1 * Q$ for some $Q \in \mathcal{M}_1(\mathcal{I})$.

Example (Continued)**.** *Assumption* (A2) *is clear from $F_t \leq 1$ while* (A1) *follows from the spatial invariance of $\mathbb{P}$. The annealed partition function is indeed independent of κ:*

$$\mathbb{E}[Z_t^\kappa] = \mathbb{E}[F_t(\omega, 0)] = \mathbb{P}\big(|\omega \cap ([0,t] \times \{0\})| = 0\big) = e^{-t}$$

Moreover, $P^\kappa * P^{\kappa'} = P^{\kappa+\kappa'}$ and therefore $P^{\kappa_2} \preceq_* P^{\kappa_1}$ whenever $\kappa_1 \leq \kappa_2$.

Remark 14.6. It is somewhat counter-intuitive to use $P^2 \preceq_* P^1$ if P^2 is more random than P^1, but we adopt this notation for consistency with the majorization order $\preceq_M$ (recall Section 3.1.3) that will play a role later.

Remark 14.7. Note that $\preceq_*$ is a reflexive and transitive relation, i.e. it defines a pre-order on $\mathcal{M}_1(\mathcal{I})$. We can extend $\preceq_*$ to a partial order by identifying P^1 and P^2 if there exists a deterministic $z \in \mathcal{I}$ such that P^1 is the law of $P^2(\cdot + z)$. In this case $Z^1 \overset{\mathrm{d}}{=} Z^2$, where Z^1 (resp. Z^2) is the partition function of P^1 (resp. P^2).

15. The main result

The following result confirms the informal implication (14.5):

Theorem 15.1. *Assume* (A1) *and* (A2), *and let* $P^1, P^2 \in \mathcal{M}_1(\mathcal{I})$ *with* Z^1 *resp.* Z^2 *the partition function of* P^1 *resp.* P^2. *Assume that*

$$P^2 \preceq_* P^1.$$

There exists a coupling $(\widehat{Z}^1, \widehat{Z}^2)$ *such that* $\widehat{Z}^i \overset{\mathrm{d}}{=} Z^i$ *for* $i = 1, 2$, *and almost surely*

$$\widehat{Z}^2 = \mathbb{E}\big[\widehat{Z}^1 \big| \widehat{Z}^2\big]. \tag{15.1}$$

Proof. From Definition 14.5 we find $Q \in \mathcal{M}_1(\mathcal{I})$ such that

$$P^2 = P^1 * Q. \tag{15.2}$$

Let ω and Y be independent with laws $\mathbb{P}$ and Q, define $\widehat{\omega} := \theta^Y \omega$ and

$$\begin{aligned}
\widehat{Z}^1 &:= \int_{\mathcal{I}} F(\widehat{\omega}, x) P^1(\mathrm{d}x) \\
\widehat{Z}^2 &:= \int_{\mathcal{I}\times\mathcal{I}} F(\omega, x + y) P^1(\mathrm{d}x) Q(\mathrm{d}y).
\end{aligned}$$

From (15.2) it is clear that $\widehat{Z}^2 \overset{\mathrm{d}}{=} Z^2$. Moreover Y and ω are independent so (A1) yields $\widehat{\omega} \overset{\mathrm{d}}{=} \omega$. We therefore have $\widehat{Z}^1 \overset{\mathrm{d}}{=} Z^1$ as well, and setting $\mathcal{A} := \sigma(\omega)$ we compute

$$\begin{aligned}
\mathbb{E}\big[\widehat{Z}^1 \big| \mathcal{A}\big] &= \int_{\mathcal{I}} \int_{\mathcal{I}} F(\theta^y \omega, x) P^1(\mathrm{d}x) Q(\mathrm{d}y) \\
&= \int_{\mathcal{I}} \int_{\mathcal{I}} F(\omega, x + y) P^1(\mathrm{d}x) Q(\mathrm{d}y) = \widehat{Z}^2.
\end{aligned}$$

We have used the consistency of F in the second equality. Since $\widehat{Z}^2$ is $\mathcal{A}$-measurable, (15.1) follows from the tower property of conditional expectation. □

Corollary 15.2. *The assumptions of Theorem 15.1 imply* $Z^1 \preceq_{cv} Z^2$. *That is, for all* $f\colon \mathbb{R}_+ \to [-\infty, \infty)$ *concave*

$$\mathbb{E}\big[f(Z^1)\big] \le \mathbb{E}\big[f(Z^2)\big]. \tag{15.3}$$

Proof. First note that the expectations on both sides are well-defined in $[-\infty, \infty)$ by the concavity of f and (A2). There is nothing to do if the l.h.s. equals $-\infty$, so in the following we can assume $f(Z^1) \in L^1$. Using the coupling from Theorem 15.1,

$$\begin{aligned}\mathbb{E}\big[f(Z^2)\big] = \mathbb{E}\big[f\big(\widehat{Z}^2\big)\big] &= \mathbb{E}\big[f\big(\mathbb{E}[\widehat{Z}^1|\widehat{Z}^2]\big)\big] \\ &\geq \mathbb{E}\big[\mathbb{E}\big[f\big(\widehat{Z}^1\big)\big|\widehat{Z}^2\big]\big] = \mathbb{E}\big[f\big(\widehat{Z}^1\big)\big] = \mathbb{E}\big[f(Z^1)\big].\end{aligned}$$

The inequality is by Jensen's inequality for conditional expectations. The second-to-last equality follows from the tower-property of conditional expectation, using $f(Z^1) \in L^1$. □

Remark 15.3. Note that this also follows from the implication (ii) $\Longrightarrow$ (i) in Theorem C. We carry out the proof because we want to stress that f may take the value $-\infty$.

16. Applications

In this section we discuss implications of Theorem 15.1 for various models.

16.1. The discrete-time random polymer model

Recall the notation from Section 4.1. We set $T := \mathbb{N}$, $I := \mathbb{Z}^d$ and $\Omega := [0, \infty)^{T \times I}$, and fix an inverse temperature $\beta \geq 0$. Let $\mathbb{P}$ be i.i.d. with

$$\mathbb{E}\big[\omega(0,0)^\beta\big] < \infty. \tag{16.1}$$

For $\omega \in \Omega$, $x \in \mathcal{I}$, we define the shifted environment $\theta^x\omega$ by

$$(\theta^x\omega)(t,i) := \omega(t, i + x(t)).$$

That is, θ^x acts on ω by shifting the environment in each "time-slice" according to the corresponding displacement of x. Assumption (A1) follows because $\mathbb{P}$ is i.i.d. and (A2) follows from (16.1). Let us check that F_t^β, defined in (4.1), is consistent:

$$F_t^\beta(\omega, x+y) = \prod_{s=1}^t \big(\omega(s, x(s) + y(s))\big)^\beta = \prod_{s=1}^t \big((\theta^y\omega)(s, x(s))\big)^\beta = F_t^\beta(\theta^y\omega, x).$$

In discrete time the order $\preceq_*$ does not have a natural interpretation as a jump rate. We can, for example, compare random walks with binomial increment distributions:

Example. *Let X resp. Y be such that $X(t+1) - X(t) \sim \mathrm{Bin}(a_t, p_t)$ and $Y(t+1) - Y(t) \sim \mathrm{Bin}(b_t, p_t)$ for all $t \in \mathbb{N}$, for $a_t, b_t \in \mathbb{N}$ and $p_t \in [0,1]$. Then $Y \preceq_* X$ holds if $a_t \leq b_t$ for all $t \in \mathbb{N}$.*

From Theorem 15.1 we get the following consequence:

Corollary 16.1. *Let $p, q \in \mathcal{M}_1(I)$ be such that $p \preceq_* q$. Then for all $f\colon \mathbb{R} \to [-\infty, \infty)$ concave*

$$\mathbb{E}\big[f(Z_t^{\beta,q})\big] \leq \mathbb{E}\big[f(Z_t^{\beta,p})\big] \tag{16.2}$$

Remark 16.2. We point out that the assumptions can be weakened: It is enough if $\mathbb{P}$ is independent in time and stationary in space, i.e. if $\omega(t,\cdot)$ and $\omega(s,\cdot)$ are independent for $s \neq t$, and for each $t \in \mathbb{N}$, $i \in I$

$$\{\omega(t,j) \colon j \in I\} \overset{\mathrm{d}}{=} \{\omega(t,i+j) \colon j \in I\}.$$

Note that we have not assumed $\mathbb{P}(\omega(0,0) > 0) = 1$, so that Corollary 16.1 also holds for the disastrous case (in the sense of (4.7)). If we exclude this case, then the free energy exists and we obtain another comparison result:

Corollary 16.3. *Assume* (16.1) *and* $\mathbb{E}[\log \omega(0,0)] > -\infty$*, so that the free energy* $\mathfrak{p}(\beta,p)$ *exists for all* $p \in \mathcal{M}_1(\mathbb{Z}^d)$ *by Proposition 4.1. Then* $p \preceq_* q$ *implies*

$$\mathfrak{p}(\beta,q) \leq \mathfrak{p}(\beta,p).$$

We obtain a second consequence of Corollary 16.1 by considering the martingale $(W_t^{\beta,p})_{t\in\mathbb{N}}$ from (4.4). Since $W^{\beta,p}$ is non-negative, the almost sure limit $\lim_{t\to\infty} W_t^{\beta,p} =: W_\infty^{\beta,p}$ exists, and by Kolmogorov's 0-1 law

$$\mathbb{P}\big(W_\infty^{\beta,p} > 0\big) \in \{0,1\}. \tag{16.3}$$

It is known that $W^{\beta,p}$ converges in L^1 if and only if $\mathbb{P}(W_\infty^{\beta,p} > 0) = 1$, see [17, Proposition 3.1]. Recall from the discussion in Section 4.2 that the behavior of the polymer measure has a phase transition, characterized by whether $W_\infty^{\beta,p}$ converges in L^1 or not. We use the dichotomy (16.3) to show that the phase transition is monotone in $\preceq_*$:

Corollary 16.4. *Assume* (16.1)*. If* $W_\infty^{\beta,q} > 0$ *and* $p \preceq_* q$*, then also* $W_\infty^{\beta,p} > 0$*.*

Proof. Consider the fractional moments of $W^{\beta,p}$, i.e. the supermartingales

$$\big\{(W_t^{\beta,p})^{1/2} \colon t \in \mathbb{N}\big\}.$$

Since this processes is L^2-bounded, it is uniformly integrable, and therefore

$$\mathbb{E}\big[(W_\infty^{\beta,q})^{1/2}\big] = \lim_{t\to\infty} \mathbb{E}\big[(W_t^{\beta,q})^{1/2}\big] \leq \lim_{t\to\infty} \mathbb{E}\big[(W_t^{\beta,p})^{1/2}\big] = \mathbb{E}\big[(W_\infty^{\beta,p})^{1/2}\big].$$

The inequality follows from Corollary 16.1. So if $W_\infty^{\beta,q} > 0$, then $\mathbb{E}\big[(W_\infty^{\beta,p})^{1/2}\big] > 0$, which by the zero-one-law implies $W_\infty^{\beta,p} > 0$ almost surely. □

To the best of our knowledge, Corollaries 16.3 and 16.4 are the first results in this direction, possibly because in discrete time there is no natural parameter that corresponds to the jump rate in continuous time. At present, the only work on stochastic orders in the context of the random polymer model seems to be [52]. However, in that work the underlying random walk is kept fixed, and the comparison is between environments at different temperatures.

16.2. Random walk in a Lévy-type random environment

We extend the model from Part II to more general environments than Poissonian disasters.

Let $T = \mathbb{R}^+$, $I = \mathbb{Z}^d$ and Ω the set of $\omega\colon T \times I \to \mathbb{R}$ such that for every $i \in I$ the mapping $t \mapsto \omega(t, i)$ is càdlàg with jump sizes bounded from below by -1, i.e. such that

$$\omega(i,t) \geq \omega(i,t^-) - 1 \quad \text{ for all } t \in T.$$

Let $\mathcal{F}$ be the smallest sigma-field such that all projections $\omega \mapsto \omega(t,i)$ are measurable, and $(\omega, \mathbb{P})$ a collection of independent Lévy processes. That is, let $\sigma^2 \geq 0$ and let $\rho \in \mathcal{M}_{<\infty}([-1,\infty))$ be a finite measure, satisfying

$$R := \int_{[-1,\infty)} (1+r)\rho(\mathrm{d}r) < \infty. \tag{16.4}$$

Let $\{B(\cdot,i)\colon i \in I\}$ be a family of independent Brownian motions and η an independent Poisson point process on $\mathbb{R}_+ \times I \times [-1,\infty)$ with intensity measure

$$\lambda_{[0,\infty)} \otimes \mu_I \otimes \rho.$$

Here $\lambda_{[0,\infty)}$ denotes the Lebesgue measure on $\mathbb{R}_+$ and μ_I the counting measure on I. Let

$$\omega(t,i) := -\frac{\sigma^2}{2}t + \sigma B(t,i) + \int_{[0,t]\times\{i\}\times[-1,\infty)} r\,\eta(\mathrm{d}(s,j,r)) \tag{16.5}$$

As before, we identify η with its support, i.e., we write $(s,i,r) \in \eta$ if $\eta(\{(s,i,r)\}) = 1$. Note that this is equivalent to $\omega(\cdot,i)$ having a jump of size r at time s.

Let $\mathcal{I}$ denote the set of right-continuous paths $x\colon \mathbb{R}_+ \to I$ having finitely many jumps in every compact interval, and $\mathcal{G}$ the sigma field generated by the coordinate processes on $\mathcal{I}$. For $\omega \in \Omega$, $x \in \mathcal{I}$ and $t \in \mathbb{R}_+$ we define

$$F_t\colon \Omega \times \mathcal{I} \to \mathbb{R}_+, (\omega,x) \mapsto e^{H_t(\omega,x)+H_t'(\omega,x)} G_t(\omega,x), \tag{16.6}$$

where

$$\begin{aligned}
H_t(\omega,x) &:= \textstyle\sum_{i\in I} \int_0^t \mathbb{1}\{x(s)=i\} B(\mathrm{d}s,i)\\
H_t'(\omega,x) &:= \textstyle\int_{[0,t]\times I\times(-1,\infty)} \mathbb{1}\{x(s)=i\}\log(1+r)\eta(\mathrm{d}(s,i,r))\\
G_t(\omega,x) &:= \mathbb{1}\big\{ \textstyle\int_{[0,t]\times I\times\{-1\}} \mathbb{1}\{x(s)=i\}\eta(\mathrm{d}(s,i,r)) = 0\big\}.
\end{aligned}$$

Note that, if x has jump times $0 = s_0 < s_1 < \cdots < s_N < s_{N+1} = t$ in $[0,t]$, we can write

$$F_t(x,\omega) = e^{-\frac{\sigma^2}{2}t} \prod_{k=0}^{N} e^{B(s_{k+1},x(s_k)) - B(s_k,x(s_k))} \times \prod_{(s,i,r)\in\eta\cap x} (1+r).$$

Here the second product is over all (s,i,r) in η that lie on the graph of x, i.e. over all $(s,i,r) \in \eta$ with $s \in [0,t]$ and $x(s) = i$. The definition is similar to (4.1), but observe that the factors in the product depend on the increments of ω along the path x. We have chosen this presentation for consistency with the existing literature.

Let us give an interpretation for the factors in $F_t(\omega,x)$:

- To maximize the first product, the path needs to mostly visit regions where the Brownian motion component of ω is increasing.
- If $(s, i, -1) \in \eta$ and $x(s) = i$, then immediately $F_t(\omega, x) = 0$. We can therefore interpret $(s, i, -1) \in \eta$ as a disaster at space-time site (s, i).
- Similarly, we think of $(s, i, r) \in \eta$ for some $r \in (-1, 0)$ as a soft obstacle at (s, i), where the process survives with probability $(1 + r) \in (0, 1)$.
- The event $(s, i, r) \in \eta$ for $r > 0$ can be interpreted as a bonus at (s, i), that will result in a contribution $(1 + r) > 1$. To maximize F_t, the path x should collect such bonuses.

Observe that if we choose intensity measure $\rho \coloneqq \delta_{\{-1\}}$, then the model agrees with the random walk among disasters from Part II.

Let A be a bounded generator on I, P the law of the corresponding Markov chain on $(\mathcal{I}, \mathcal{G})$, and Z_t the partition function of P. By Lemma 14.3, (16.4) and our choice for the deterministic drift-term in (16.13), we can compute the annealed partition function as

$$\mathbb{E}[Z_t] = e^{t(R-1)}. \tag{16.7}$$

We point out that the partition function Z_t satisfies the same Feynman-Kac representation,

$$Z_t(\omega) \stackrel{\mathrm{d}}{=} u_\omega(t, 0),$$

as in the disastrous case, see [29, Theorems 2.1 and 2.2]. Here u_ω is the solution to (14.1) with ω as defined in (16.13).

We finish the definition of the model by specifying shifts $\{\theta^x \colon x \in \mathcal{I}\}$. Similar to the previous section, θ^x acts on ω by shifting each time-slice according to the displacement of the random walk. More precisely, for $t \in T, i \in I$,

$$(\theta^x \omega)(t, i) \coloneqq \sum_{j \in I} \int_0^t \mathbb{1}\{j = i + x(s)\} \omega(\mathrm{d}s, j).$$

We write $\theta^x B$ resp. $\theta^x \eta$ for the Brownian motion resp. the jump part of the shifted environment. We check that H_t is consistent: For $\omega \in \Omega$, $t > 0$ and $x, y \in \mathcal{I}$

$$\begin{aligned}
H_t(\omega, x + y) &= \sum_{j \in I} \int_0^t \mathbb{1}\{x(s) + y(s) = j\} B(\mathrm{d}s, j) \\
&= \sum_{i,j \in I} \int_0^t \mathbb{1}\{x(s) = i\} \mathbb{1}\{j = i + y(s)\} B(\mathrm{d}s, j) \\
&= \sum_{i \in I} \int_0^t \mathbb{1}\{x(s) = i\} (\theta^y B)(\mathrm{d}s, i) \\
&= H_t(\theta^y \omega, x).
\end{aligned}$$

A similar calculation for H'_t and G_t shows that F_t is consistent. Moreover, (A2) follows from (16.7). Finally, since $\omega(\cdot, i)$ has independent increments and $\mathbb{P}$ is spatially homogeneous, it is clear that $(\theta^x\omega)(\cdot, i) \overset{\mathrm{d}}{=} \omega(\cdot, i)$ for all $i \in I$. Note that $(\theta^x\omega)(\cdot, i)$ is a function of the increments of ω in

$$g^x(i) := \{(t, x(t) + i) : t \in T\} \subseteq T \times I$$

the graph of x shifted by i. Clearly $g^x(i) \cap g^x(j) = \varnothing$ for $i \neq j$, and this together with the independent increments of ω implies that $(\theta^x\omega)(\cdot, i)$ and $(\theta^x\omega)(\cdot, j)$ are independent. These considerations show that (A1) is satisfied.

Remark 16.5. Note that, in contrast to [29], we have chosen to present the model only in the case where the intensity measure ρ is finite, i.e., $\omega(\cdot, i)$ has finitely many discontinuities in every bounded interval. The reason is to keep the notation simple by avoiding the compensated jump measure, but Corollary 16.6 below also holds in the general case.

As before, we focus on the case of simple random walk, where A is the discrete Laplacian $\kappa\Delta$, see (5.1). Let P^κ denote the law of the Markov process corresponding to $\kappa\Delta$ and write Z_t^κ for the partition function of P^κ. In Section 14.3 we have already observed that P^κ is $\preceq_*$-decreasing, i.e. $\kappa_1 \leq \kappa_2$ implies $P^{\kappa_2} \preceq_* P^{\kappa_1}$. Thus, from Theorem 15.1 we obtain the following consequence:

Corollary 16.6. *For every $t > 0$, $\kappa_1 \leq \kappa_2$ and $f : \mathbb{R} \to [-\infty, \infty)$ concave*

$$\mathbb{E}[f(Z_t^{\kappa_1})] \leq \mathbb{E}[f(Z_t^{\kappa_2})].$$

Remark 16.7. Similar to the situation in discrete time, we can consider slightly more general settings: It is enough to assume that ω has independent increments, not necessarily stationary in time.

We discuss some consequences: Note that the conclusion of Theorem K also holds for the environment discussed above, see [55, 29, 21, 22]. That is, there exists $\mathfrak{p}_0(\kappa) \in \mathbb{R}$ such that almost surely and in L^1

$$\lim_{t\to\infty} \tfrac{1}{t}\mathbb{E}[\log Z_t(\omega)] = \lim_{t\to\infty} \tfrac{1}{t} \log Z_t = \mathfrak{p}_0(\kappa). \tag{16.8}$$

We also consider the r^{th}-annealed Lyapunov exponent,

$$\lim_{t\to\infty} \tfrac{1}{t} \log \mathbb{E}\big[(Z_t^\kappa)^r\big]^{1/r} =: \mathfrak{p}_r(\kappa). \tag{16.9}$$

Existence of this limit follows from the sub-/superadditive lemma. As a direct consequence of Corollary 16.6,

$$\kappa \mapsto \mathfrak{p}_r(\kappa) \text{ is } \begin{cases} \text{increasing} & \text{if } r < 1 \\ \text{constant} & \text{if } r = 1 \\ \text{decreasing} & \text{if } r > 1 \end{cases}. \tag{16.10}$$

Moreover, as in Section 16.1, we can consider the martingales $(W_t^\kappa)_{t\geq 0}$ defined by

$$W_t^\kappa := Z_t^\kappa e^{-t(R-1)}.$$

The same argument as in Section 16.1 shows

Corollary 16.8. *Assume that $(W_t^{\kappa_1})_{t\ge 0}$ converges in L^1. Then $(W_t^{\kappa_2})_{t\ge 0}$ converges in L^1 for all $\kappa_1 \le \kappa_2$.*

16.3. Brownian motion in a Poissonian environment

In this section, we discuss a generalization of the model from Part IV.

Recall that in the previous section, the environment had a Brownian-motion component, i.e., an i.i.d. collection $\{B(\cdot, i)\colon i \in \mathbb{Z}^d\}$ of Brownian motions indexed by the sites of $\mathbb{Z}^d$. The natural generalization to continuous space is to consider environments with a space-time white noise component, which however introduces considerable technical difficulties. In the following, we therefore restrict ourselves to Poissonian environments, and refer to [9] for a discussion of Brownian motion in a space-time white noise environment.

Let $T = \mathbb{R}^+$, $I = \mathbb{R}^d$ and Ω the set of locally finite point measures on $\mathbb{R}_+ \times \mathbb{R}^d \times [-1, \infty)$. Let $(\omega, \mathbb{P})$ denote the Poisson point process on Ω with intensity measure $\lambda_{[0,\infty)} \otimes \lambda_{\mathbb{R}^d} \otimes \rho(\mathrm{d}r)$. Here $\lambda_{[0,\infty)}$ (resp. $\lambda_{\mathbb{R}^d}$) denotes the Lebesgue measure on $[0, \infty)$ (resp. $\mathbb{R}^d$) and ρ is a finite measure on $[-1, \infty)$ such that

$$R := \int_{[-1,\infty)} (1 + r)\rho(\mathrm{d}r) < \infty. \tag{16.11}$$

In particular, for $\beta \in [0, \infty]$, let $\mathbb{P}_\beta$ denote the environment with intensity measure

$$\rho := \begin{cases} \delta_{\{-1+e^{-\beta}\}} & \text{if } \beta < \infty \\ \delta_{\{-1\}} & \text{if } \beta = \infty. \end{cases}$$

We set

$$\begin{aligned} H_t(\omega, x) &:= \textstyle\int_{[0,t]\times\mathbb{R}^d\times(-1,\infty)} \mathbb{1}\{i \in U(x(s))\} \log(1 + r)\omega(\mathrm{d}(s, i, r)) \\ G_t(\omega, x) &:= \mathbb{1}\big\{ \textstyle\int_{[0,t]\times\mathbb{R}^d\times\{-1\}} \mathbb{1}\{i \in U(x(s))\}\omega(\mathrm{d}(s, i, r)) = 0\big\} \\ F_t(\omega, x) &:= e^{H_t(\omega,x)} G_t(\omega, x). \end{aligned}$$

Here $U(i)$ denotes the ball of unit volume around $i \in \mathbb{R}^d$. We define the shifted environment $\theta^x\omega$ by the relation

$$(s, i, r) \in \omega \quad \iff \quad (s, i - x(s), r) \in \theta^x\omega.$$

The same proof as in Section 16.2 shows that F_t is consistent and that (A1) and (A2) are satisfied.

Let now P^{σ^2} be the law of Brownian motion with variance σ^2 and write $Z_t^{\sigma^2}$ for the partition function of P^{σ^2}. It is well-known that $P^{\sigma_1^2} * P^{\sigma_2^2} = P^{\sigma_1^2+\sigma_2^2}$, so that $\sigma^2 \mapsto P^{\sigma^2}$ is decreasing in $\preceq_*$. Applying Theorem 15.1, we therefore get the following consequence:

Corollary 16.9. *For $\sigma_1^2 \le \sigma_2^2$ and $f\colon \mathbb{R} \to [-\infty, \infty)$ concave,*

$$\mathbb{E}\big[f\big(Z_t^{\sigma_1^2}\big)\big] \le \mathbb{E}\big[f\big(Z_t^{\sigma_2^2}\big)\big].$$

Remark 16.10. As before, we point out that the assumptions can be relaxed, since it is enough that the environment has independent increments and is stationary in space. In particular, the conclusion also holds if ω is replaced by $\omega_{[0,1]^c}$, the modified environment considered in Part IV.

Observe that if ω has law $\mathbb{P}_\beta$, then $Z_t^1(\omega)$ has the same law as $P(\tau_\beta(\omega) \geq t)$, the quenched survival probability from Part IV. A re-run of Theorem 11.6 shows that, for any $\sigma^2 > 0$ and $\beta \in [0, \infty]$, there exists $\mathfrak{p}(\beta, \sigma^2) \in (-\infty, e^{-\beta} - 1]$ such that $\mathbb{P}_\beta$-almost surely

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_\beta\big[\log Z_t^{\sigma^2}(\omega_{[0,1]^c})\big] = \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\sigma^2}(\omega) = \mathfrak{p}(\beta, \sigma^2). \tag{16.12}$$

Thus, from Corollary 16.9, we see that $\sigma^2 \mapsto \mathfrak{p}(\beta, \sigma^2)$ is increasing for every $\beta \in [0, \infty]$.

16.4. Branching random walk in discrete time

The random polymer model has a natural connection to branching processes in space-time random environments. In this section we show that Theorem 15.1 can be applied to prove a phase transition for this model. Let $T = \mathbb{N}$, $I = \mathbb{Z}^d$ and Ω the set of $\eta = \{\eta(t,i) \colon t \in T, i \in I\}$, with $\eta(t,i) \in \mathcal{M}_1(\mathbb{N})$ for every $t \in T, i \in I$. For $\eta \in \Omega$, we define $\omega = \omega(\eta) \in [0,\infty)^{T\times I}$ by

$$\omega(t,i) \coloneqq \sum_{k\in\mathbb{N}} k\eta(t,i)(\{k\}). \tag{16.13}$$

In words, an element $\eta \in \Omega$ defines an offspring distribution for every space-time site and $\omega(\eta)$ is the expected number of descendants. We assume

$$\mathbb{E}\big[\omega(0,0) + \omega(0,0)^{-1}\big] < \infty, \tag{16.14}$$

$$\mathbb{P}\big(\eta(0,0)(\{0\}) > 0\big) > 0, \tag{16.15}$$

$$\mathbb{P}\big(\eta(0,0)(\{0,1\}) < 1\big) > 0. \tag{16.16}$$

Let $p \in \mathcal{M}_1(I)$ be an increment distribution and let $(\{X(t) \colon t \in \mathbb{N}\}, P^p)$ denote the corresponding random walk, as defined in Section 4.1.

Given $\eta \in \Omega$, we consider a branching process $\mathcal{Z} = \{\mathcal{Z}(t,i) \colon t \in T, i \in I\}$ with values in $\mathbb{N}^{T\times I}$ that is informally defined in the following way:

- At time $t = 0$, there is one particle at the origin, i.e. $\mathcal{Z}(0,i) = \mathbb{1}\{i = 0\}$.
- Suppose the process has been defined until time time t and consider a particle that in generation t occupies site i:
  - For generation $t+1$, this particle is replaced by a random number of descendants, which are sampled according to the offspring distribution $\eta(t,i)$.
  - Each descendant then independently moves from i to a random site $i+D$, where the displacement D has law p.
- This procedure is applied independently for each particle in generation t, and we let $\mathcal{Z}(t+1,j)$ denote the number of particles that occupy site j in generation $t+1$.

We use P^p_η to denote the law of $\mathcal{Z}$ for a fixed realization of η, and $\mathbb{P}^p$ for the joint law of η and $\mathcal{Z}$. As in Part III, we are interested in the event

$$\{\mathcal{Z} \text{ survives}\} := \big\{\text{For every } t \in \mathbb{N} \text{ there exists } i \in \mathbb{Z}^d \text{ such that } \mathcal{Z}(t,i) > 0\big\}.$$

Clearly, $\mathbb{P}^p(\mathcal{Z} \text{ survives}) = 1$ if (16.15) does not hold. If (16.15) holds but (16.16) fails, then $\mathbb{P}^p(\mathcal{Z} \text{ survives}) = 0$. We may thus exclude those trivial cases. Assuming (16.14)–(16.16), the event $\{\mathcal{Z} \text{ survives}\}$ can be characterized using the free energy, similar to Part III. That is, define $F_t \colon \Omega \times \mathcal{I} \to \mathbb{R}_+$ by

$$F_t(\eta, x) = \prod_{s=1}^{t} (\omega(\eta))(s, x(s))$$

and let Z^p_t denote the partition function of P^p associated with F_t. In [18, Lemma 1.4] they show the following analogue to Lemma 9.1: For $\eta \in \Omega$,

$$E^p_\eta\Big[\sum_{i \in I} \mathcal{Z}(t,i)\Big] = Z^p_t(\eta). \tag{16.17}$$

From (16.14) and Proposition 4.1, there exists $\mathfrak{p}(p) \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log Z^p_t] =: \mathfrak{p}(p).$$

It is thus intuitively clear that we can expect different behavior depending on the sign of $\mathfrak{p}(p)$. As in Theorem 8.1, this is indeed enough to characterize global survival:

Theorem N ([18, Theorem 2.1.1] and [32, Theorem 1])**.** *Assume* (16.14)–(16.16)*. Then*

$$\mathbb{P}^p(\mathcal{Z} \textit{ survives}) > 0 \quad \Longleftrightarrow \quad \mathfrak{p}(p) > 0. \tag{16.18}$$

Combining this with Corollary 16.6, we thus have the following consequence:

Corollary 16.11. *Assume* (16.14)–(16.16)*. Then $p \preceq_* q$ implies*

$$\mathbb{P}^q(\mathcal{Z} \textit{ survives}) > 0 \quad \Longrightarrow \quad \mathbb{P}^p(\mathcal{Z} \textit{ survives}) > 0.$$

Remark 16.12. Note that we cannot expect the stronger conclusion

$$\text{“}\mathbb{P}^q(\mathcal{Z} \text{ survives}) \leq \mathbb{P}^p(\mathcal{Z} \text{ survives})\text{”}.$$

To see this, consider two classical Galton-Watson processes $(Z_n)_{n\in\mathbb{N}}$ and $(Z'_n)_{n\in\mathbb{N}}$, each with constant deterministic offspring distribution. Corollary 16.3 corresponds to a first moment comparison, i.e. we assume $\mathbb{E}[Z_1] \leq \mathbb{E}[Z'_1]$. It is well-known that a Galton-Watson process Z has positive survival probability if and only if $\mathbb{E}[Z_1] > 0$, and therefore we can conclude that $\mathbb{P}(Z \text{ survives}) > 0$ implies $\mathbb{P}(Z' \text{ survives}) > 0$. There are, however, examples where $\mathbb{E}[Z_1] < \mathbb{E}[Z'_1]$ but $\mathbb{P}(Z \text{ survives}) > \mathbb{P}(Z' \text{ survives})$, see below.

Example. *Let $Z^{(k)} = (Z_n^{(k)})_{n\in\mathbb{N}}$ be a classical Galton-Watson process with*

$$\mathbb{P}\big(Z_1^{(k)} = k^2\big) = 1 - \mathbb{P}\big(Z_1^{(k)} = 0\big) = \frac{1}{k}.$$

Then $\mathbb{E}\big[Z_1^{(k)}\big] = k$, and therefore $\mathbb{P}(Z^{(k)}$ survives$) > 0$ for all $k \geq 2$. Moreover, since $\mathbb{P}\big(Z^{(k)}$ survives$\big) \leq \mathbb{P}\big(Z_1^{(k)} \neq 0\big)$, we have

$$\mathbb{P}\big(Z^{(k)} \text{ survives}\big) < \mathbb{P}\big(Z^{(2)} \text{ survives}\big)$$

for k large enough. On the other hand, we clearly have $\mathbb{E}\big[Z_1^{(k)}\big] > \mathbb{E}\big[Z_1^{(2)}\big]$ for all $k \geq 2$.

16.5. Branching random walk among disasters

Recalling Theorem 8.1, the same conclusion holds for the branching random walk among disasters from Part III:

Corollary 16.13. *Under the assumptions of Theorem 8.1, for $\lambda > 0$, $q \in \mathcal{M}_1(\mathbb{N})$ and $\kappa_1 \leq \kappa_2$,*

$$\mathbb{P}^{\lambda,\kappa_1,q}(\mathcal{Z} \text{ survives}) > 0 \quad\implies\quad \mathbb{P}^{\lambda,\kappa_2,q}(\mathcal{Z} \text{ survives}) > 0.$$

17. Outlook

In this section, we discuss two possible ways in which Theorem 15.1 might be generalized:

(a) One can try to weaken the assumption that the environment has independent increments (which is implicit in (A1)). This is particularly interesting for the parabolic Anderson model, which is often studied in the case where the environment has long-time correlations.

(b) One can also try to weaken the relation $\preceq_*$, so that in Theorem 15.1 we obtain $Z^1 \preceq_{cv} Z^2$ for more partition functions. Here it is particularly interesting to consider the discrete-time setting, where $\preceq_*$ is a rather unnatural condition.

We will keep the discussion of (a) short since one cannot expect the conclusion of Theorem 15.1 to hold in this case (Section 17.1). For (b), we argue that the notion of majorization from Section 3.1.3 is a natural candidate to extend $\preceq_*$. We conjecture that, under some additional assumptions, it is necessary and sufficient for the conclusion of Theorem 15.1 (Section 17.2). As evidence for this conjecture, we then prove that $\preceq_M$ is indeed optimal if we consider random walks on trees instead of on the lattice (Section 17.3).

17.1. Environments with long-time correlations

We consider the parabolic Anderson model

$$\begin{aligned} u(0,\cdot) &\equiv 0, \\ u(\mathrm{d}t,i) &= \kappa\,(\Delta u)(t,i)\mathrm{d}t + u(t^-,i)\omega(\mathrm{d}t,i) \qquad \text{for all } t\in\mathbb{R}_+, i\in\mathbb{Z}^d \end{aligned}$$

with environment $\omega = \{\omega(t,i) \colon t\in\mathbb{R}_+, i\in\mathbb{Z}^d\}$, but this time we do not assume that ω has independent increments. Examples include:

1. **Static environment**: Let $\{\xi(i) \colon i \in \mathbb{Z}^d\}$ be i.i.d. real random variables satisfying some integrability conditions, and set $\omega(t, i) := \xi(i)t$ for $t \in \mathbb{R}_+, i \in \mathbb{Z}^d$.
2. **Independent simple random walks**: We consider a field $\{\eta(t, i) \colon t \in \mathbb{R}_+, i \in \mathbb{Z}^d\}$ where $\eta(t, i)$ counts the number of particles occupying site i. Assume that the initial values $\eta(0, \cdot)$ are independent and Poisson distributed, and that afterwards all particles independently move as simple random walks. We set $\omega(\mathrm{d}t, i) := \beta\eta(t, i)\mathrm{d}t$ with β some (positive or negative) parameter.
3. **Simple exclusion process/voter model**: We consider a field $\{\eta(t, i) \colon t \in \mathbb{R}_+, i \in \mathbb{Z}^d\}$ with values in $\{0, 1\}$, where $\eta(0, \cdot)$ is i.i.d. Bernoulli, while the field afterwards evolves according to the simple exclusion process/the voter model. We again define $\omega(\mathrm{d}t, i) := \beta\eta(t, i)\mathrm{d}t$ for β some (positive or negative) parameter.

Note that we have only informally presented these examples, and we refer to [34, 27, 26] as well as the survey [42] for a precise definition and an overview of known results. We point out that in these models it is already an interesting problem to study the annealed partition function $\mathbb{E}[Z_t^\kappa]$, while in our setup the annealed partition function does not depend on κ (recall Lemma 14.3).

Intuitively, for these models we expect that the environment exhibits islands with significantly larger values than in an average environment and which are persistent over a long time. Let us, for example, consider a static environment taking two values $a < b \in \mathbb{R}$:

$$\mathbb{P}(\xi(0) = a) = 1 - \mathbb{P}(\xi(0) = b) = p \in (0, 1)$$

Then (in dimension one) we can expect that among the sites $[-\sqrt{t}, \sqrt{t}] \cap \mathbb{Z}$ there exists some interval I with $\xi \equiv b$, whose length is of order $\log(t)$. Now both

- the probability of moving from the origin to the center of I before time εt and
- the probability of not leaving I in $[\varepsilon t, t]$

are subexponential. We therefore expect that $u(t, 0) \approx e^{bt+o(t)}$ for large t. While this localization strategy gives the optimal contribution in the first order of Z_t, it is not immediately clear that it is the optimal strategy. This is however known in closely related models, see for example [25].

Note that the behavior is qualitatively different from before: the optimal strategy is to immediately move to an area where the environment is good and then never again leave this area. It is intuitively clear that for the second part of this strategy, a low jump rate κ is better. We thus expect Z_t^κ to be (in some sense) decreasing in κ, at least for t large.

We expect the same phenomenon for the other models: For example, consider an environment consisting of independent random walks that have repulsive interaction with the random walk (Example 2 with $\beta < 0$). This model was considered in [26], where it was shown [26, Proposition 2.1] that

$$\mathbb{E}[Z_t^\kappa] \leq \mathbb{E}[Z_t^0] \quad \text{for all } \kappa > 0.$$

In the physics literature, this phenomenon is known under the name "Pascal's principle", see the discussion in [26].

Similarly, in [27, Section 1.4] they discuss the attractive case of Example 2, and conjecture that the free energy $\mathfrak{p}_0(\kappa)$ is first increasing and then decreasing in the jump rate κ.

Heuristically, we believe that the concave order is not the correct criterion: Recall that for the random walk among disasters, the quenched survival probability takes small values if there is a trap in the environment. A high jump rate was thus beneficial because it provides a "hedge" against such a scenario by spreading the mass over a large part of the environment. On the other hand, in a static environment the partition function Z_t^κ is large if the environment has a persistent island of large values. In that situation, having a low jump rate is beneficial, because we can take full advantage of the benefit without being forced to move away. This is an indication that in the static case, the correct notion is to study $\kappa \mapsto \mathbb{E}[f(Z_t^\kappa)]$ for f convex (i.e., risk-seeking) instead of concave (i.e., risk-averse).

It is an interesting question for future research to investigate the transition between static environments and the Lévy-type environments covered by Theorem 15.1. More precisely, for $\kappa_1 \leq \kappa_2$ one might conjecture that if the environment has long-time correlations (as in Examples 1–3) then $Z_t^{\kappa_2} \preceq_{cx} Z_t^{\kappa_1}$, while $Z_t^{\kappa_1} \preceq_{cv} Z_t^{\kappa_2}$ holds if the environment has correlations that decay fast in time.

17.2. Weakening the convolution property

In this section, we only discuss the one-dimensional random polymer model, so let $T = \mathbb{N}$, $I = \mathbb{Z}$, $\Omega := [-1,\infty)^{T\times I}$, F_t and Z_t^p as in Section 4.1, with $\beta = 1$.

Assume for simplicity that $p, q \in \mathcal{M}_1(I)$ are compactly supported increment distributions, and recall that in Corollary 16.1 we have seen that $p \preceq_* q$ implies $Z_t^q \preceq_{cv} Z_t^p$. As explained in Section 14.1, we interpret $p \preceq_* q$ as p having more randomness than q. That is, under P^p the mass is spread out more evenly than under P^q, i.e. it is distributed over a larger set of paths. In this section we ask if this is enough. More precisely:

Question. *Under what conditions on p and q do we have $Z_t^q \preceq_{cv} Z_t^p$ for all $\mathbb{P}$ satisfying* (A1) *and* (A2)*?*

Recall the majorization order from Section 3.1.3. We observe that $p \preceq_M q$ is necessary:

Proposition 17.1. *Assume $Z_t^q \preceq_{cv} Z_t^p$ for all $\mathbb{P}$ satisfying* (A1) *and* (A2)*. Then $p \preceq_M q$.*

Proof. We construct a suitable environment: For $r \in \{-K, \dots, K\}$, define $\omega^{(r)} \in \Omega$ by

$$\omega^{(r)}(t,i) = \begin{cases} 0 & \text{if } t = 1 \text{ and } i \not\equiv_K r \\ 1 & \text{else.} \end{cases}$$

Here $\equiv_K$ denotes equivalence in $\mathbb{Z}/_{\{-K,\dots,K\}}$, the torus of size $2K+1$. In words, at time $t = 1$ there are hard obstacles on $\{-K,\dots,K\} \setminus \{r\}$ and the environment is trivial everywhere else. Note that for all $t \geq 1$,

$$Z_t^p(\omega^{(r)}) = p(r).$$

Let $\mathbb{P}$ be the uniform distribution on $\{\omega^{(-K)}, \dots, \omega^{(K)}\}$, which clearly satisfies (A1) and (A2). By assumption, for every concave function f,

$$\begin{aligned}(2K+1)\mathbb{E}[f(Z_t^p)] &= \sum_{r=-K}^{K} f(Z_t^p(\omega^{(r)})) = \sum_{r=-K}^{K} f(p(r)) \\ &\le \sum_{r=-K}^{K} f(q(r)) = \sum_{r=-K}^{K} f(Z_t^q(\omega^{(r)})) = (2K+1)\mathbb{E}[f(Z_t^q)].\end{aligned}$$

This is equivalent to $p \preceq_M q$ by the implication (iii) $\Longrightarrow$ (i) in Theorem D. $\square$

Let us also point out that the relation $\preceq_*$ from Definition 14.5 is stronger than $\preceq_M$, i.e. $p \preceq_* q$ implies $p \preceq_M q$. This follows from the previous proposition together with Theorem 15.1, or it can be checked directly (see [47, Proposition 12.N.1]).

However, $\preceq_M$ is not a sufficient criterion on its own: Note that if p is obtained from q by permuting the weights in $\{-K, \dots, K\}$, then $p \preceq_M q$ and $q \preceq_M p$. So if $\preceq_M$ was sufficient, we would get $Z_t^p \preceq_{cv} Z_t^q \preceq_{cv} Z_t^p$, hence $Z_t^p \overset{\mathrm{d}}{=} Z_t^q$. But on the lattice this is clearly not true for $t > 1$. Let us therefore place some restriction on the order of the weights:

Assumption A3. Both p and q are symmetric and unimodal, i.e. the functions $i \mapsto p(i)$ and $i \mapsto q(i)$ are symmetric and decreasing for $i \ge 0$.

We conjecture that this is enough:

Conjecture 17.2. *Assume* (A3). *Then $p \preceq_M q$ if and only if $Z_t^q \preceq_{cv} Z_t^p$ for all $\mathbb{P}$ satisfying* (A1) *and* (A2).

We close by mentioning a model where we expect monotonicity, but for which Theorem 15.1 does not apply: In [12, 51] the authors consider an i.i.d. Bernoulli environment of hard obstacles, i.e.

$$\mathbb{P}(\omega(0,0) = 0) = 1 - \mathbb{P}(\omega(0,0) = -1) = p \in (0,1).$$

For $\alpha > 0$, let P^α denote the random walk with increment distribution

$$P^\alpha(X(t+1) = i | X(t) = j) = C(\alpha) e^{-|i-j|^\alpha},$$

where $C(\alpha)$ is the normalizing constant. They show that there exists $\mathfrak{p}(\alpha) \in (-\infty, 0]$ such that almost surely

$$\lim_{t\to\infty} \tfrac{1}{t}\mathbb{E}[\log Z_t^\alpha] = \lim_{t\to\infty} \tfrac{1}{t} \log Z_t^\alpha = \mathfrak{p}(\alpha).$$

Observe that in this case, large values of α mean that the increments concentrate mostly on $\{-1, 0, 1\}$ while for $\alpha \to 0$ the distribution becomes more spread out. One would thus expect that $\mathfrak{p}(\alpha)$ is decreasing in α.

17.3. A comparison result for random walks on trees

In this section, we show that $\preceq_M$ is a necessary and sufficient criterion for polymers on trees. We begin by defining the model: Let V denote the K-ary tree. That is, V is cycle-free and has a distinguished vertex o of degree K, while all other vertices have degree $K+1$. We say that o is the root and write $|v|$ for the graph-distance between v and o. We call $|v|$ the height of v and let V_t denote the set of vertices of height t. For $v \in V$ and $i \in \{1, \dots, K\}$, we let (v, i) denote the i^{th} descendant of v, and we write $D(v)$ for the set of descendants of v.

A path in V is a function $x \colon \mathbb{N} \to V$ such that $|x(t)| = t$ and $x(t+1) \in D(x(t))$ for every $t \in \mathbb{N}$. That is, a path moves away from the root in each step. Let $\mathcal{I}$ denote the set of paths on V. An elementary observation is that if $x, y \in \mathcal{I}$ satisfy $x(s) \neq y(s)$, then $x(t) \neq y(t)$ for all $t \geq s$. Clearly this is a special property of the tree which does not hold on the lattice.

Let $\Omega \coloneqq [-1, \infty)^V$ denote the set of environments and define F_t as in (4.1). Since there is no group structure on $\mathcal{I}$ we have to re-define (A1):

Definition 17.3. A **shift** is a bijection $\theta \colon V \to V$ such that if v is a descendant of w, then $\theta(v)$ is a descendant of $\theta(w)$.

In words, shifts are bijections that respect the tree structure.

Assumption A4. $\mathbb{P}$ is invariant under all shifts θ. That is for all $A \in \mathcal{F}$ and all shifts θ

$$\mathbb{P}(\{\omega(v) \colon v \in V\} \in A) = \mathbb{P}(\{\omega(\theta(v)) \colon v \in V\} \in A).$$

Note that this assumption is satisfied for the canonical example of an i.i.d. environment. We introduce a special class of shifts:

Definition 17.4. A shift θ is called **elementary** if there exist $v \in V$ and a permutation π of $\{1, \dots, K\}$ such that $\theta(w) = w$ if w is not a (strict) descendant of v, and such that $\theta(w) = (v, \pi(a), v')$ if $w = (v, a, v')$ is a descendant of v.

In words, an elementary shift permutes all subtrees attached to the node v according to the permutation π, and leaves everything else unchanged. See Figure 20 for an illustration.

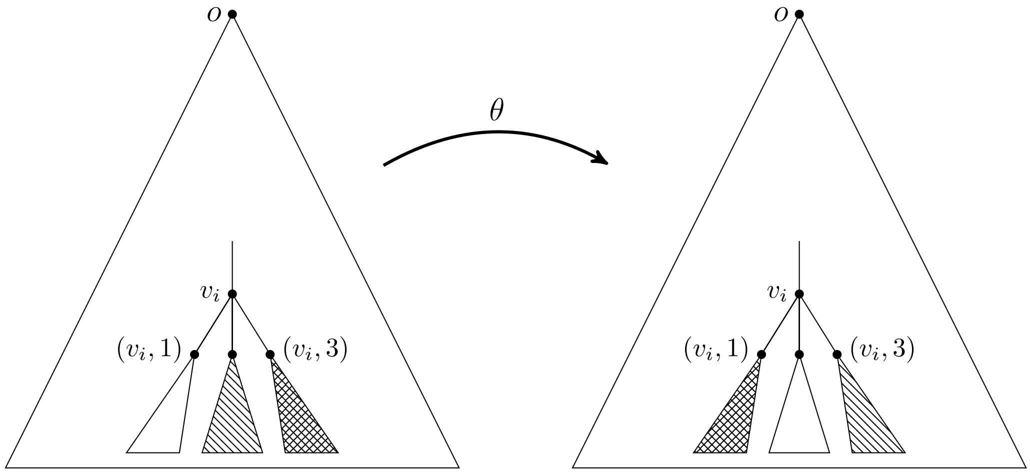

Figure 20: In the case $K = 3$ and $(\pi(1), \pi(2), \pi(3)) = (2, 3, 1)$ the elementary shift θ associated to v and π permutes the subtrees attached to the descendants of v while keeping the rest of the tree untouched.

We finish the model by defining random walks on V: For $p \in \mathcal{M}_1(\{1, \dots, K\})$, let P^p denote the law of the Markov chain with increment distribution p. That is, $X(0) = \mathrm{o}$ and for every $t \in \mathbb{N}, a \in \{1, \dots, K\}$

$$P^p(X(t+1) = (v, a)|X(t) = v) = p(a).$$

In anticipation of the proof of Theorem 17.5, we also introduce inhomogeneous random walks: If $p = \{p^{(v)} : v \in V\}$ is a collection of increment distributions on $\{1, \dots, K\}$, we write P^p for the Markov chain with transition probabilities

$$P^p\big(X(t+1) = (v, a)\big|X(t) = v\big) = p^{(v)}(a).$$

We can now state the main result of this section:

Theorem 17.5. *Let $p, q \in \mathcal{M}_1(\{1, \dots, K\})$ and write Z_t^p resp. Z_t^q for the partition function of P^p resp. P^q. Then $p \preceq_M q$ if and only if $Z_t^q \preceq_{cv} Z_t^p$ for all $\mathbb{P}$ satisfying* (A2) *and* (A4)*.*

Proof of "⇐". This is identical to the proof of Proposition 17.1, since only the first step of the random walk is relevant in that proof. □

Proof of "⇒". We show $Z_{t+1}^q \preceq_{cv} Z_{t+1}^p$. Let $\overline{V}_t$ denote the set of node of height at most t and fix an enumeration $\overline{V}_t = \{v_1, v_2, \dots, v_N\}$ such that $i \mapsto |v_i|$ is decreasing. Note that this ensures that all descendants of v_i in $\overline{V}_t$ are contained in $\{v_1, \dots, v_i\}$, for every $i = 1, \dots, N$. Our aim is to incrementally transform P^q into P^p. We consider a sequence of inhomogeneous random walks $P^{r_0}, \dots, P^{r_N}$ such that from P^{r_i} to $P^{r_{i+1}}$ we only change the increment distribution at one node. More precisely, for $i = 0, \dots, N$ and $v \in \overline{V}_t$ let

$$r_i^{(v_j)} := \begin{cases} p & \text{if } j < i \\ q & \text{else.} \end{cases} \tag{17.1}$$

In particular $P^{r_0} = P^q$ and $P^{r_N} = P^q$. Let W_i denote the partition function of P^{r_i}, and note that it is enough to show for all $i = 0, \dots, N-1$

$$W_i \preceq_{cv} W_{i+1}. \tag{17.2}$$

Let x be a path that visits v_i, and note that if $F_{|v_i|}(\omega, x) = 0$, then $F_t(\omega, y) = 0$ for all $t \geq |v_i|$ and all paths y visiting v_i. So on $\{F_{|v_i|}(\omega, x) = 0\}$, there is nothing to prove since $W_i(\omega) = W_{i+1}(\omega)$. We therefore assume $F_{|v_i|}(\omega, x) > 0$, and in this case we can write

$$\begin{aligned} W_i(\omega) &= A(\omega) + b(\omega) \textstyle\sum_{a=1}^K q(a) \widehat{W}_i(a, \omega) \\ W_{i+1}(\omega) &= A(\omega) + b(\omega) \textstyle\sum_{a=1}^K p(a) \widehat{W}_i(a, \omega), \end{aligned}$$

where

$$\begin{aligned} \widehat{W}_i(a, \omega) &:= E^{\,r_i}\Big[\frac{F_{t+1}(\omega, X)}{F_{|v_i|}(\omega, X)}\Big| X(|v_i| + 1) = (v_i, a)\Big] \\ A(\omega) &:= E^{r_i}\big[F_{t+1}(\omega, X) \mathbb{1}\{X(|v_i|) \neq v_i\}\big] \\ b(\omega) &:= E^{r_i}\big[F_{|v_i|}(\omega, X) \mathbb{1}\{X(|v_i|) = v_i\}\big]. \end{aligned}$$

In words, $A(\omega)$ is the contribution from paths that do not visit v_i. The contribution from paths that do visit v_i can be split into the common part $b(\omega)$ (collected at the nodes $o \to v_i$) and the remaining contribution $\widehat{W}_i(a, \omega)$ depending on which descendant of v_i the path visits at time $|v_i| + 1$. Note that such a decomposition is only possible on trees.

For π a permutation of $\{1, \dots, K\}$, let θ^π denote the elementary shift (recall Definition 17.4) associated to v_i and π. Let $\omega(\pi)$ be the environment obtained by

$$(\omega(\pi))(v) := \omega(\theta^\pi(v)).$$

By (A4), $\omega(\pi)$ has the same law as ω. Moreover, by our choice of enumeration and (17.1), we know that all descendants of v_i have the same increment distribution, so that

$$\widehat{W}_i(a, \omega(\pi)) = \widehat{W}_i(\pi^{-1}(a), \omega). \tag{17.3}$$

Let $\mathcal{S}$ denote the set of permutations of $\{1, \dots, K\}$, and for $\pi \in \mathcal{S}$ define

$$\begin{aligned} c(\pi, \omega) &:= \sum_{a=1}^K p(a) \widehat{W}_i(a, \omega(\pi)) \\ d(\pi, \omega) &:= \sum_{a=1}^K q(a) \widehat{W}_i(a, \omega(\pi)) \end{aligned}$$

In what follows, we regard $c(\cdot, \omega)$ and $d(\cdot, \omega)$ as $K!$-dimensional real vectors.

Claim. *For all ω, it holds that $c(\cdot,\omega) \preceq_M d(\cdot,\omega)$.*

Let us first see how we can apply the claim: Let

$$\begin{aligned}
C(\pi,\omega) &:= W_{i+1}(\omega(\pi)) = A(\omega(\pi)) + b(\omega(\pi))c(\omega,\pi) = A(\omega) + b(\omega)c(\pi,\omega),\\
D(\pi,\omega) &:= W_i(\omega(\pi)) = A(\omega(\pi)) + b(\omega(\pi))d(\omega,\pi) = A(\omega) + b(\omega)d(\pi,\omega).
\end{aligned}$$

We have used that ω and $\omega(\pi)$ disagree only on the descendants of v_i, so that $A(\omega) = A(\omega(\pi))$ and $b(\omega) = b(\omega(\pi))$. From the claim it is clear that

$$C(\cdot,\omega) \preceq_M D(\cdot,\omega).$$

Due to Theorem D, for all f concave,

$$\frac{1}{|\mathcal{S}|}\sum_{\pi\in\mathcal{S}} f(C(\pi,\omega)) \geq \frac{1}{|\mathcal{S}|}\sum_{\pi\in\mathcal{S}} f(D(\pi,\omega)). \tag{17.4}$$

Let ω and Π be independent, ω with distribution $\mathbb{P}$ and Π chosen uniformly from $\mathcal{S}$. By (A4), we see that $\omega(\Pi)$ has law $\mathbb{P}$ as well, and from (17.4),

$$\begin{aligned}
\mathbb{E}[f(W_{i+1}(\omega))] &= \mathbb{E}\Big[\frac{1}{|\mathcal{S}|}\sum_{\pi\in\mathcal{S}} f(W_{i+1}(\omega(\pi)))\Big]\\
&\geq \mathbb{E}\Big[\frac{1}{|\mathcal{S}|}\sum_{\pi\in\mathcal{S}} f(W_i(\omega(\pi)))\Big] = \mathbb{E}[f(W_i(\omega))].
\end{aligned}$$

This proves (17.2). □

Proof of the claim. Consider the matrix $M \in [0,\infty)^{\{1,\ldots,K\}\times\mathcal{S}}$ defined by

$$M(a,\pi) := \widehat{W}_i(a,\omega(\pi)).$$

Interpreting p and q as K-dimensional row vectors, we can write $c = pM$ and $d = qM$. The claim then follows Theorem E, which states that it is enough to check the following: If $m = (m_1,\ldots,m_K)$ is a column of M and $\sigma\in\mathcal{S}$, then

$$\widehat{m} := (m_{\sigma(1)},\ldots,m_{\sigma(K)})$$

is also a column of M. To see this, let m be the column corresponding to $\pi\in\mathcal{S}$, i.e. $m_a = \widehat{W}_i(a,\omega(\pi))$. Then from (17.3),

$$(\widehat{m})_a = m_{\sigma(a)} = \widehat{W}_i(\sigma(a),\omega(\pi)) = \widehat{W}_i(a,\omega(\sigma^{-1}\circ\pi)).$$

Thus, $\widehat{m}$ is the column corresponding to $\sigma^{-1}\circ\pi\in\mathcal{S}$. □

List of Figures

References

[1] Krishna B. Athreya and Peter E. Ney. *Branching processes*. Dover Publications, Inc., Mineola, NY, 2004.

[2] Yuri Bakhtin. Inviscid Burgers equation with random kick forcing in noncompact setting. *Electron. J. Probab.*, 21(37):50, 2016.

[3] Yuri Bakhtin and Liying Li. Zero temperature limit for directed polymers and inviscid limit for stationary solutions of stochastic Burgers equation. *J. Stat. Phys.*, 172(5):1358–1397, 2018.

[4] Yuri Bakhtin and Liying Li. Thermodynamic limit for directed polymers and stationary solutions of the Burgers equation. *Comm. Pure Appl. Math.*, 72(3):536–619, 2019.

[5] Carol Bezuidenhout and Geoffrey Grimmett. The critical contact process dies out. *Ann. Probab.*, 18(4):1462–1482, 1990.

[6] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities*. Oxford University Press, Oxford, 2013.

[7] Richard A. Brualdi and Geir Dahl. Majorization for partially ordered sets. *Discrete Math.*, 313(22):2592–2601, 2013.

[8] Philippe Carmona and Yueyun Hu. Fluctuation exponents and large deviations for directed polymers in a random environment. *Stochastic Process. Appl.*, 112(2):285–308, 2004.

[9] René A. Carmona and Stanislav A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108(518):1–125, 1994.

[10] Francis Comets. *Directed polymers in random environments*, volume 2175 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. Lecture notes from the 46th Probability Summer School held in Saint-Flour, 2016.

[11] Francis Comets and Clément Cosco. Brownian polymers in poissonian environment: A survey. *arXiv preprint arXiv:1805.10899*, 2018.

[12] Francis Comets, Ryoki Fukushima, Shuta Nakajima, and Nobuo Yoshida. Limiting results for the free energy of directed polymers in random environment with unbounded jumps. *J. Stat. Phys.*, 161(3):577–597, 2015.

[13] Francis Comets, Tokuzo Shiga, and Nobuo Yoshida. Directed polymers in a random environment: Path localization and strong disorder. *Bernoulli*, 9(4):705–723, 2003.

[14] Francis Comets, Tokuzo Shiga, and Nobuo Yoshida. Probabilistic analysis of directed polymers in a random environment: A review. In *Stochastic analysis on large scale interacting systems*, volume 39 of *Adv. Stud. Pure Math.*, page 115–142. Math. Soc. Japan, Tokyo, 2004.

[15] Francis Comets and Nobuo Yoshida. Some new results on Brownian directed polymers in random environment. *RIMS Kôkyûroku*, 1386:50–66, 2004.

[16] Francis Comets and Nobuo Yoshida. Brownian directed polymers in random environment. *Comm. Math. Phys.*, 254(2):257–287, 2005.

[17] Francis Comets and Nobuo Yoshida. Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.*, 34(5):1746–1770, 2006.

[18] Francis Comets and Nobuo Yoshida. Branching random walks in space-time random environment: Survival probability, global and local growth rates. *J. Theoret. Probab.*, 24(3):657–687, 2011.

[19] Francis Comets and Nobuo Yoshida. Localization transition for polymers in Poissonian medium. *Comm. Math. Phys.*, 323(1):417–447, 2013.

[20] Clément Cosco. The Intermediate Disorder Regime for Brownian Directed Polymers in Poisson Environment. *arXiv preprint arXiv:1804.09571*, 2018. *preprint*, arXiv:1804.09571.

[21] Michael Cranston and Thomas S. Mountford. Lyapunov exponent for the parabolic Anderson model in $\mathbb{R}^d$. *J. Funct. Anal.*, 236(1):78–119, 2006.

[22] Michael Cranston, Thomas S. Mountford, and Tokuzo Shiga. Lyapunov exponent for the parabolic Anderson model with Lévy noise. *Probab. Theory Related Fields*, 132(3):321–355, 2005.

[23] Frank den Hollander. *Random polymers*, volume 1974 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007.

[24] L. Devroye. *Non-Uniform Random Variate Generation*. Springer New York, 2013.

[25] Jian Ding and Changji Xu. Poly-logarithmic localization for random walks among random obstacles. *Ann. Probab.*, 47(4):2011–2048, 2019.

[26] Alexander Drewitz, Jürgen Gärtner, Alejandro F. Ramírez, and Rongfeng Sun. Survival probability of a random walk among a Poisson system of moving traps. In *Probability in complex physical systems*, volume 11 of *Springer Proc. Math.*, pages 119–158. Springer, Heidelberg, 2012.

[27] Dirk Erhard, Frank den Hollander, and Grégory Maillard. The parabolic Anderson model in a dynamic random environment: basic properties of the quenched Lyapunov exponent. *Ann. Inst. Henri Poincaré Probab. Stat.*, 50(4):1231–1275, 2014.

[28] Ryoki Fukushima and Stefan Junk. Zero temperature limit for the brownian directed polymer among poissonian disasters. *arXiv preprint arXiv:1810.09600*, 2018.

[29] Tasuku Furuoya and Tokuzo Shiga. Sample Lyapunov exponent for a class of linear Markovian systems over $\mathbb{Z}^d$. *Osaka J. Math.*, 35(1):35–72, 1998.

[30] Nina Gantert and Stefan Junk. A branching random walk among disasters. *Electron. J. Probab.*, 22(67):1–34, 2017.

[31] Olivier Garet, Jean-Baptiste Gouéré, and Régine Marchand. The number of open paths in oriented percolation. *Ann. Probab.*, 45(6A):4071–4100, 2017.

[32] Olivier Garet and Régine Marchand. The critical branching random walk in a random environment dies out. *Electron. Commun. Probab.*, 18(9):1–15, 2013.

[33] Jürgen Gärtner and Frank den Hollander. Intermittency in a catalytic random medium. *Ann. Probab.*, 34(6):2219–2287, 2006.

[34] Jürgen Gärtner, Frank den Hollander, and Grégory Maillard. Quenched Lyapunov exponent for the parabolic Anderson model in a dynamic random environment. In *Probability in complex physical systems*, volume 11 of *Springer Proc. Math.*, page 159–193. Springer, Heidelberg, 2012.

[35] Andreas Greven. A phase transition for the coupled branching process. I. The ergodic theory in the range of finite second moments. *Probab. Theory Related Fields*, 87(4):417–458, 1991.

[36] Andreas Greven. On phase-transitions in spatial branching systems with interaction. In *Stochastic models (Ottawa, ON, 1998)*, volume 26 of *CMS Conf. Proc.*, page 173–204. Amer. Math. Soc., Providence, RI, 2000.

[37] Andreas Greven and Frank den Hollander. Phase transitions for the long-time behavior of interacting diffusions. *Ann. Probab.*, 35(4):1250–1306, 2007.

[38] John M. Hammersley. Generalization of the fundamental theorem on sub-additive functions. *Proc. Cambridge Philos. Soc.*, 58:235–238, 1962.

[39] Jakob Horn. Über lineare Differentialgleichungen mit einem veränderlichen Parameter (in German). *Math. Ann.*, 52(2):340–362, June 1899.

[40] Stefan Junk. Comparison of partition functions in a space-time random environment. *arXiv preprint arXiv:1811.03684*, 2018.

[41] Achim Klenke. *Wahrscheinlichkeitstheorie.* Springer-Lehrbuch Masterclass Series. Springer Berlin Heidelberg, 2008.

[42] Wolfgang König. *The parabolic Anderson model.* Pathways in Mathematics. Birkhäuser/Springer, Cham, 2016. Random walk in random potential.

[43] Hubert Lacoin. New bounds for the free energy of directed polymers in dimension $1+1$ and $1+2$. *Comm. in Math. Phys.*, 294(2):471–503, 2010.

[44] Thomas M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.

[45] Thomas M. Liggett. *Stochastic interacting systems: Contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[46] Quansheng Liu and Frédérique Watbled. Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. *Stochastic Process. Appl.*, 119(10):3101–3132, 2009.

[47] Albert W. Marshall, Ingram Olkin, and Barry C. Arnold. *Inequalities: Theory of majorization and its applications*. Springer Series in Statistics. Springer, New York, second edition, 2011.

[48] James B. Martin. Linear growth for greedy lattice animals. *Stochastic Process. Appl.*, 98(1):43–66, 2002.

[49] Thomas S. Mountford. A note on limiting behaviour of disastrous environment exponents. *Electron. J. Probab.*, 6:no. 1, 9, 2001.

[50] Alfred Müller and Dietrich Stoyan. *Comparison Methods for Stochastic Models and Risks*. Wiley Series in Probability and Statistics. Wiley, 2002.

[51] Shuta Nakajima. Concentration results for directed polymer with unbounded jumps. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(1):1–20, 2018.

[52] Vu Lan Nguyen. A note about domination and monotonicity in disordered systems. *arXiv preprint arXiv:1606.01835*, 2016.

[53] Marcel Ortgiese and Matthew I. Roberts. Intermittency for branching random walk in Pareto environment. *Ann. Probab.*, 44(3):2198–2263, 2016.

[54] Alfréd Rényi. On the theory of order statistics. *Acta Math. Hungar.*, 4(3-4):191–231, 1953.

[55] Tokuzo Shiga. Exponential decay rate of survival probability in a disastrous random environment. *Probab. Theory Related Fields*, 108(3):417–439, 1997.

[56] Walter L. Smith. Necessary conditions for almost sure extinction of a branching process with random environment. *Ann. Math. Statist*, 39:2136–2140, 1968.

[57] David Tanny. Limit theorems for branching processes in a random environment. *Ann. Probab.*, 5(1):100–116, 1977.

[58] Yu Zhang. On the concentration and the convergence rate with a moment condition in first passage percolation. *Stochastic Process. Appl.*, 120(7):1317–1341, 2010.