Fully Distributed Consensus Control for Linear Multi-Agent Systems: A Reduced-Order Adaptive Feedback Approach

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***Abstract*—This paper is concerned with fully distributed consensus control of linear multi-agent systems with undirected graphs. Two kinds of reduced-order adaptive output-feedback protocols are proposed. For the edge-based protocol, each edge is adapted by a scalar that is determined by the output information of the associated two agents; for the node-based protocol, each agent multiplies the connecting weights by a scalar that is determined by the relative output information of all neighbouring agents. Sufficient conditions in terms of the solvability of some matrix equations are derived for the existence of the two protocols. Furthermore, a tractable algorithm is constructed for designing the protocol gains. Compared with the existing related results, the proposed protocols have the following three merits simultaneously: *of lower dimension*, *using relative output information about neighbouring agents* and *in the fully distributed fashion*. A simulation example on formation flying of spacecrafts is presented to illustrate the efficacy of the proposed method.**

***Index Terms*—Consensus, multi-agent systems, output feedback, reduced order, adaptive control.**

I. Introduction

MULTI-AGENT systems (MASs) have drawn considerable attention in the past ten years, and related applications can be found in various areas, e.g., distributed optimization, robot/vehicle formation and social networks [1]–[9]. A fundamental control problem of MASs is consensus, that is, to find a control protocol such that the states of all agents converge to some common trajectory. A basic requirement about a control protocol for MASs is the fact that it can be implemented in a distributed way. This is because agents in practice usually have limited communication capabilities such that each agent can only communicate with its neighbours. Up to now, there have been many results that can manage this challenge. To mention a few, consensus seeking of single-integrator systems was investigated in [10], where it was shown that a directed spanning tree is necessary for consensus of MASs; consensus of general linear MASs was studied in [11], and a kind of dynamic output-feedback protocols with controller interaction were proposed; in [12], [13], a novel dynamic output-feedback protocol was proposed for general linear MASs, and moreover, a unified robust control point of view was provided for the existence of output-feedback protocols without exchanging controller information.

A drawback of the aforementioned results is the fact that the smallest nonzero eigenvalue of the Laplacian matrix is needed for protocol design. This requires that the weights of the communication graph must be exactly known. Moreover, even if the graph is known, it is still difficult to exactly compute the eigenvalues of the Laplacian matrix when the network size is large. As a result, although the protocols in the aforementioned results can be implemented distributively, the corresponding design process is *not* of the distributed nature. Inspired by this limitation, many efforts have been made to explore fully distributed control protocols for consensus of linear MASs, especially those adaptively adjusting scalar gains about the graph by using local information only [14]–[18]. Fully distributed state-feedback protocols were constructed for general linear MASs in [14], [16] with undirected and directed graphs, respectively. Adaptive consensus of linear MASs with external disturbances are further studied under a kind of dynamic state-feedback protocols in [18]. Inspired by the output-feedback protocols in [11], adaptive dynamic output-feedback protocols were proposed in [15] for general linear MASs with undirected graphs, where leader-follower consensus with the leader having an nonzero input was also discussed. To deal with directed graphs, the authors further proposed a kind of sequential observer based adaptive output-feedback protocols in [17] for linear MASs.

Although there have been some results about fully distributed consensus control of MASs as mentioned above, how to reduce the complexity of the constructed protocols is still challenging. Particularly, the order of each local controller of the protocols in [15] is identical to that of the agents. In view of this fact, we term a protocol like those in [15] as a *full-order* protocol. As such, it is seen that the dynamic output-feedback protocols without the adaptive mechanism in [11], [12], [19] also fall into the full-order type. Based on this observation, a natural idea for reducing the protocol complexity is to design *reduced-order* protocols, that is, the order of each local controller is smaller than that of the

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agents; see [20], [21]. However, note that no fully distributed reduced-order protocol has been developed for MASs. The reduced-order protocols in [20], [21] are based on the design theory of traditional reduced-order observer-based controllers. In addition to solving some matrix equations, they also need to know the smallest nonzero eigenvalue of the Laplacian matrix. By using local output information only, it is unknown yet whether the protocols in [20], [21] can be extended to the case that the graph related gains are adaptively adjusted. Moreover, note that the protocols therein have their own shortcomings. Specifically, the one in [20] requires absolute output information of individual agents, which is impractical in some applications; while the one in [21] requires relative input information between neighbouring agents, such that the underlying graph over which the controller information flows is actually different from the original communication graph. It is thus open and challenging to design fully distributed reduced-order output-feedback protocols for consensus of MASs.

Motivated by the above observations, this paper investigates the fully distributed consensus control problem of linear MASs via reduced-order adaptive output-feedback control protocols. The communication graph is assumed to be undirected. Two kind of reduced-order adaptive dynamic output-feedback protocols are constructed: the first one updates the scalar gains from the output difference of the two agents on each edge, termed an edge-based protocol, while the second one from the output difference sum of all neighbouring agents for each node, termed a node-based protocol. By appropriately parameterizing the protocol gains, existence conditions are derived for the two protocols, which require the solvability of some matrix equations related to agent dynamics. Moreover, a design algorithm is presented, which shows that the proposed adaptive output-feedback protocols with (n_x-n_u)th-order local controllers are always feasible, provided that the agents are stabilizable and detectable and the communication graph is connected. Thus, like the full-order results in [15], the proposed results confirm that some reduced-order fully distributed consensus protocols must exist and can be easily found. Moreover, compared with the reduced-order protocols in [20], [21], the proposed protocols make use of relative output information more straightforwardly. The efficacy of the proposed theoretical results are finally demonstrated by a simulation example about formation flying of spacecrafts.

Notation: Represent the set of all $m \times n$ real matrices by $\mathbb{R}^{m\times n}$ and an $n \times n$ identity matrix by $\mathbf{I}_n$, where the subscript is omitted if no confusion is caused. A square, positive definite (semi-definite) matrix is denoted by $P > 0$ (≥ 0). Kronecker product and Hadamard product for two matrices A and B are represented by $A \otimes B$ and $A \circ B$, respectively. For (block) diagonal matrix with $A_1, \dots, A_n$ on the diagonal, we write it as $\mathrm{diag}\{A_1, \dots, A_n\}$. $\mathrm{Image}(\cdot)$ indicates the image of a matrix $(\cdot)$. $\|(\cdot)\|$ denotes the 2-norm of a vector $(\cdot)$.

An undirected graph is denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \dots, N\}$ the set of N nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the edge set. The associated adjacency matrix is denoted by $\mathcal{A} = [a_{ij}]_{N\times N}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For an undirected graph, we mean $a_{ij} = a_{ji}$ for $i, j = 1, \dots, N$. If $(j, i) \in \mathcal{E}$ or $a_{ij} > 0$, node j is said to be a neighbouring node of node i and the set of all the neighbouring node of node i is denoted by $\mathcal{N}_i$. The associated Laplacian matrix $\mathcal{L} = [l_{ij}]_{N\times N}$ is defined as $l_{ii} = \sum_{k\in\mathcal{N}_i} a_{ik}$ and $l_{ij} = -a_{ij}$ for $i, j = 1, 2, \dots, N$ and $i \neq j$. A path of the graph is a sequence of edges connecting two nodes. We say an undirected graph is connected if every node can be reached from every other node over any path.

Lemma 1 ([22, Lemma 1]): Consider an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and suppose it is connected. Then zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$, and $\lambda_2(\mathcal{L}) = \min_{\mathbf{1}^{\mathrm{T}}x=0,\, x\neq 0} \frac{x^{\mathrm{T}}\mathcal{L}x}{x^{\mathrm{T}}x} > 0$ is the smallest nonzero eigenvalue.

II. Main Results

We will present two reduced-order adaptive output-feedback protocols for consensus control of general linear MASs. The consensus problem will be first formulated. Then consensus analysis conditions for the concerned protocols will be provided, and finally a design algorithm will be presented.

A. Problem Statement

Consider N $(N \geq 2)$ homogeneous dynamic agents with each one represented by a linear time-invariant system:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_u u_i(t), \\ y_i(t) &= C_y x_i(t),\ i = 1, \dots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_x}$, $u_i \in \mathbb{R}^{n_u}$ and $y_i \in \mathbb{R}^{n_y}$ are the state, control input and local output of agent i, respectively, and A, B_u and C_y are appropriately-dimensioned real system matrices. Without loss of generality, we assume that the matrix triple (A, B_u, C_y) is stabilizable and detectable.

The consensus problem is to find a control protocol that drives the states of all the agents to track common trajectories. Denote by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ the communication topology of a distributed protocol. We are interested in distributed control protocols that make only use of relative information between neighbouring agents. Symbols with accent "$\sim$" are used to denote the relative information between neighbouring agents. For instance,

$$\begin{aligned} \tilde{x}_i(t) &\triangleq \sum_{j\in\mathcal{N}_i} a_{ij}(x_i(t) - x_j(t)) = \sum_{j=1}^{N} l_{ij}x_j(t), \\ \tilde{y}_i(t) &\triangleq \sum_{j\in\mathcal{N}_i} a_{ij}(y_i(t) - y_j(t)) = \sum_{j=1}^{N} l_{ij}y_j(t), \end{aligned} \tag{2}$$

where $\mathcal{A} = [a_{ij}]_{N\times N}$ and $\mathcal{L} = [l_{ij}]_{N\times N}$ are the adjacency matrix and Laplacian matrix of $\mathcal{G}(\mathcal{V}, \mathcal{E})$, respectively, and $\mathcal{N}_i$ is the set of neighbouring agents to agent i. Signals $\tilde{x}_i$ and $\tilde{y}_i$ are the relative state and relative output of agent i, respectively. Protocols making use of $\tilde{x}_i$ for consensus control are called state-feedback protocols, while those making use of $\tilde{y}_i$ are output-feedback protocols. Since full (relative) state information is not always available, output-feedback protocols are of more relevance in applications.

To make full use of relative output $\tilde{y}_i$, we propose two distributed adaptive output-feedback protocols for consensus control. The *edge-based* output-feedback protocol is given by

$$\begin{aligned}\dot{r}_i(t) &= Hr_i(t) + F_r \sum_{j\in\mathcal{N}_i} c_{ij}(t)a_{ij}\left(\tilde{y}_{ij}(t) + C_yB_r\tilde{r}_{ij}(t)\right),\\ u_i(t) &= Gr_i(t) + F_u \sum_{j\in\mathcal{N}_i} c_{ij}(t)a_{ij}\left(\tilde{y}_{ij}(t) + C_yB_r\tilde{r}_{ij}(t)\right),\end{aligned} \tag{3}$$

with the law of adaption

$$\begin{aligned}&\dot{c}_{ij}(t) = \alpha_{ij}\left(\tilde{y}_{ij}(t) + C_yB_r\tilde{r}_{ij}(t)\right)^{\mathrm{T}}\left(\tilde{y}_{ij}(t) + C_yB_r\tilde{r}_{ij}(t)\right),\\ &c_{ij}(0) = c_{ji}(0) > 0,\ \alpha_{ij} = \alpha_{ji},\ i = 1,\ldots,N;\ j\in\mathcal{N}_i,\end{aligned} \tag{4}$$

while the *node-based* output-feedback protocol is given by

$$\begin{aligned}\dot{r}_i(t) &= Hr_i(t) + d_i(t)F_r\left(\tilde{y}_i(t) + C_yB_r\tilde{r}_i(t)\right),\\ u_i(t) &= Gr_i(t) + d_i(t)F_u\left(\tilde{y}_i(t) + C_yB_r\tilde{r}_i(t)\right),\end{aligned} \tag{5}$$

with the law of adaption

$$\begin{aligned}&\dot{d}_i(t) = \beta_i\left(\tilde{y}_i(t) + C_yB_r\tilde{r}_i(t)\right)^{\mathrm{T}}\left(\tilde{y}_i(t) + C_yB_r\tilde{r}_i(t)\right),\\ &d_i(0) > 0,\ i = 1,\ldots,N.\end{aligned} \tag{6}$$

Here, α_{ij} and β_i are any positive scalar constants and $r_i \in \mathbb{R}^{n_r}$ is the local controller state for agent i and

$$\begin{aligned}&\tilde{y}_{ij}(t) \triangleq y_i(t) - y_j(t),\ \tilde{r}_{ij}(t) \triangleq r_i(t) - r_j(t),\\ &\tilde{r}_i(t) \triangleq \sum_{j\in\mathcal{N}_i} a_{ij}(r_i(t) - r_j(t)) = \sum_{j=1}^{N} l_{ij}r_j(t).\end{aligned}$$

Matrix gains H, G, B_r, F_r and F_u, which have proper dimensions, are protocol parameters to be designed. Note that the order of the local controller, n_r, is not required to be such that $n_r = n_x$, that is, local controllers are not required to be of full order as that of agents. A kind of adaptive full-order dynamic output-feedback protocols have been addressed in [15]. However, it will be clear that this paper can deal with both full-order and reduced-order cases. Following [15], the two notions, edge-based protocols and node-based protocols are due to the information used for computing the adaptive gains. Adaptive gains are associated with each edge for the former case, and with each node for the latter case.

What is worth pointing out is that our goal is to provide a *tractable* characterization of reduced-order adaptive output-feedback protocols. On one hand, although the above protocols seem to have a special form, it will be seen in Section II-D that the proposed design method provides a *more general* formulation of some existing results. On the other hand, it is known that designing a reduced-order output-feedback controller is not tractable in general. Without a proper parameterization of the gain matrices, even if one can formulate a general linear dynamic output-feedback protocol (e.g., a node-based protocol like $\dot{r}_i = A_rr_i + B_r\tilde{r}_i + B_y\tilde{y}_i$ and $u_i = C_rr_i + D_r\tilde{r}_i + D_y\tilde{y}_i$), finding feasible gain matrices is still *not tractable*.

The consensus control problem to be addressed in this paper is stated as follows: For the MAS (1), find an output-feedback protocol (3) (resp., (5)) such that the states of the resulting closed-loop system satisfy $\lim_{t\to\infty}\|x_i(t) \to x_j(t)\| = 0$ and $\lim_{t\to\infty}\|r_i(t) \to r_j(t)\| = 0$ for all $i, j = 1,\ldots,N$.

Remark 1: Although the protocols (3) and (5) are motivated by combining adaptive designs and reduced-order designs, a main challenge of designing reduced-order adaptive protocols is how to establish a proper law of adaption on some "error" while keeping the tractability of design conditions. For instance, the law in [15, (2)] is given by $\dot{c}_{ij} = \varepsilon_{ij}a_{ij}\left(\tilde{y}_{ij} - C_y\tilde{r}_{ij}\right)^{\mathrm{T}}\left(\tilde{y}_{ij} - C_y\tilde{r}_{ij}\right)$, where $\tilde{y}_{ij} - C_y\tilde{r}_{ij}$ might be intuitively understood as the "error". However, it is unclear what such an "error" should be for the existing reduced-order protocols in [20], [21]. For the protocols (3) and (5), since r_i are some intermediate variables for control but of no physical meaning, it is also difficult to tell what such an "error" should be without carefully designing the laws (4) and (6), let alone, as explained before, the fact that different parameterizations are adopted in (3)/(5) and [20], [21].

Remark 2: On one hand, the first difference between the two protocols is obviously the number of adaptive gains. Note that the smallest number of edges for a connected undirected graph is $N-1$. In such an extreme case, the number of adaptive gains for (3) is $2(N-1)$. Thus, for any connected undirected graph with $N \geq 3$, the number of adaptive gains for (3) is always larger than that for (5). In other words, the edge-based protocol (3) in general involves more adaptive gains than the node-based one (5), which implies that (3) is more complex. On the other hand, if we see $c_{ij}a_{ij}$ and d_ia_{ij} as the time-varying weights of edge (i, j) of the new communication graph, then the Laplacian of the new graph for the former case keeps symmetric while that for the latter does not. This symmetry might benefit system analysis. For instance, if the adaptive gains are fixed after converging, the communication graph of the closed-loop system is undirected with a symmetric Laplacian, to which many existing results for MASs on fixed undirected graphs can be applied.

Remark 3: The adaptive gains c_{ij} and d_i are non-decreasing when consensus has not been precisely reached. Thus, if the system is subject to external disturbances such that the consensus error is not convergent to zero, then c_{ij} and d_i could continuous increasing to infinity. To circumvent this drawback, one may use the so-called σ-modification technique to damp the adaptive gains. Nevertheless, a trade-off that in general has to be made is no convergence guarantee, but only boundedness guarantee, for the consensus error, if the external disturbances are assumed to be bounded (see [15], [23]).

B. Consensus Analysis Under the Edge-Based Protocol

In this subsection, we present some sufficient conditions under which the closed-loop system resulting from the edge-based protocol (3) reaches consensus. The case for the node-based one (5) will be discussed in the next subsection.

1) Error Dynamics: Let $c_{ii} = \frac{\sum_{j\in\mathcal{N}_i} c_{ij}a_{ij}}{l_{ii}}$ for $i = 1,\ldots,N$ and $c_{ij} = 0$ for $i = 1,\ldots,N$ and $j \notin \mathcal{N}_i$. Thus,

$$\begin{aligned}&\sum_{j\in\mathcal{N}_i} c_{ij}(t)a_{ij}\left(\tilde{y}_{ij}(t) + C_yB_r\tilde{r}_{ij}(t)\right)\\ &= \sum_{j=1}^{N} c_{ij}(t)l_{ij}\left(y_j(t) + C_yB_rr_j(t)\right).\end{aligned}$$

Define the state vector $s \triangleq \mathrm{col}\{s_1,\ldots,s_N\}$ with $s_i \triangleq \mathrm{col}\{x_i, r_i\}$. By combining the agents with the protocols (3), the closed-loop system is given by

$$\dot{s}(t) = \left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C}(t) \circ \mathcal{L}) \otimes \tilde{B}\right] s(t), \tag{7}$$

where $\mathcal{C} \triangleq [c_{ij}]_{N\times N}$ and

$$\tilde{A} \triangleq \begin{bmatrix} A & B_u G \\ 0 & H \end{bmatrix}, \ \tilde{B} \triangleq \begin{bmatrix} B_u F_u C_y & B_u F_u C_y B_r \\ F_r C_y & F_r C_y B_r \end{bmatrix}.$$

Furthermore, define the state transformation $s_e \triangleq (\mathcal{L}_e \otimes \mathbf{I})\, s$, where $\mathcal{L}_e \triangleq \mathbf{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. Denote $s_e = \mathrm{col}\{s_{e1}, \ldots, s_{eN}\}$ and thus $s_{ei} = s_i - \frac{1}{N}\sum_{j=1}^{N} s_j$. Since $c_{ij}(0) = c_{ji}(0)$ and $\alpha_{ij} = \alpha_{ji}$ imply $c_{ij}(t) = c_{ji}(t)$ for all $t \geq 0$ and $\mathcal{L}$ is symmetric, we have $\mathcal{L}_e\,(\mathcal{C}(t) \circ \mathcal{L}) = \mathcal{C}(t) \circ \mathcal{L} = (\mathcal{C}(t) \circ \mathcal{L})\mathcal{L}_e$. Thus, the closed-loop system (7) can be transformed into the following form:

$$\begin{aligned} \dot{s}_e(t) &= (\mathcal{L}_e \otimes \mathbf{I})\left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C}(t) \circ \mathcal{L}) \otimes \tilde{B}\right] s(t) \\ &= \left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C}(t) \circ \mathcal{L}) \otimes \tilde{B}\right] s_e(t), \end{aligned} \tag{8}$$

The following lemma can be obtained, which bridges the consensus property of the original MAS and the convergence property of the system (8) under the protocol (3).

Lemma 2: For the MAS (1) with the protocol (3), consensus is reached if and only if $s_e(t) \to 0$ as $t \to \infty$.

Proof: Note that $\mathcal{L}_e q = 0$ for some vector q if and only if $q \in \mathrm{Image}(\mathbf{1})$. Thus, $s_e(t) = (\mathcal{L}_e \otimes \mathbf{I})\, s(t) \to 0$ as $t \to \infty$ if and only if $s(t) \to \mathrm{Image}(\mathbf{1} \otimes \mathbf{I})$ as $t \to \infty$, which is obviously equivalent to the fact that consensus is reached. ■

2) Existence Condition: Now we present the following result on the existence of a distributed adaptive output-feedback protocol (3) such that consensus is reached.

Theorem 1: Consider the MAS (1) and the protocol (3), and suppose that the communication graph $\mathcal{G}$ is undirected and connected. Then consensus is reached, and $c_{ij}(t)$, $i, j = 1, \ldots, N$, converge to some finite positive constants as $t \to \infty$, if the following statements hold:

1) Matrices H, G and B_r are such that H is Hurwitz and

$$AB_r - B_r H = B_u G. \tag{9}$$

2) Matrices F_u and F_r are such that

$$B_u F_u + B_r F_r = -PC_y^{\mathrm{T}}, \tag{10}$$

where P is a positive definite matrix solving the following ARE for any positive definite matrix Q:

$$PA^{\mathrm{T}} + AP - PC_y^{\mathrm{T}} C_y P + Q = 0. \tag{11}$$

Proof: Introduce the following matrix T and its inverse T^{-1}, which are obviously well-defined:

$$T = \begin{bmatrix} \mathbf{I}_{n_x} & B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix}, T^{-1} = \begin{bmatrix} \mathbf{I}_{n_x} & -B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix}.$$

By substituting the equations (9) and (10), we obtain

$$\begin{aligned} \bar{A} &\triangleq T\tilde{A}T^{-1} = \begin{bmatrix} \mathbf{I}_{n_x} & B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix}\begin{bmatrix} A & B_u G \\ 0 & H \end{bmatrix}\begin{bmatrix} \mathbf{I}_{n_x} & -B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix} \\ &= \begin{bmatrix} A & -AB_r + B_u G + B_r H \\ 0 & H \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & H \end{bmatrix}, \\ \bar{B} &\triangleq T\tilde{B}T^{-1} \\ &= \begin{bmatrix} \mathbf{I}_{n_x} & B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix}\begin{bmatrix} B_u F_u C_y & B_u F_u C_y B_r \\ F_r C_y & F_r C_y B_r \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{I}_{n_x} & -B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix} \\ &= \begin{bmatrix} B_u F_u C_y + B_r F_r C_y & 0 \\ F_r C_y & 0 \end{bmatrix} = \begin{bmatrix} -PC_y^{\mathrm{T}} C_y & 0 \\ F_r C_y & 0 \end{bmatrix}. \end{aligned}$$

Perform a state transformation $\bar{s} = \mathrm{col}\{\bar{s}_1, \ldots, \bar{s}_N\} \triangleq (\mathbf{I} \otimes T)\, s_e$. Then it follows from (8) that

$$\begin{aligned} \dot{\bar{s}}(t) &= (\mathbf{I} \otimes T)\left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C}(t) \circ \mathcal{L}) \otimes \tilde{B}\right] s_e(t) \\ &= \left[\mathbf{I} \otimes \bar{A} + (\mathcal{C}(t) \circ \mathcal{L}) \otimes \bar{B}\right] \bar{s}(t). \end{aligned} \tag{12}$$

From Lemma 2, it is known that consensus is reached if and only if $s_e(t) \to 0$ as $t \to \infty$. Since the state transformation from s_e to $\bar{s}$ is invertible, requiring $s_e(t) \to 0$ as $t \to \infty$ is equivalent to requiring $\bar{s}(t) \to 0$ as $t \to \infty$. Alternatively, the state equations of $\bar{s}(t)$ as above can be represented by

$$\begin{aligned} \dot{\bar{x}}(t) &= \left[\mathbf{I} \otimes A - (\mathcal{C}(t) \circ \mathcal{L}) \otimes PC_y^{\mathrm{T}} C_y\right] \bar{x}(t), \\ \dot{\bar{r}}(t) &= (\mathbf{I} \otimes H)\, \bar{r}(t) + \left[(\mathcal{C}(t) \circ \mathcal{L}) \otimes F_r C_y\right] \bar{x}(t), \end{aligned} \tag{13}$$

where

$$\begin{bmatrix} \bar{x} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \mathrm{col}\{\bar{x}_1, \ldots, \bar{x}_N\} \\ \mathrm{col}\{\bar{r}_1, \ldots, \bar{r}_N\} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \otimes [\mathbf{I}_{n_x}, 0_{n_x \times n_r}] \\ \mathbf{I} \otimes [0_{n_r \times n_x}, \mathbf{I}_{n_r}] \end{bmatrix} \bar{s}.$$

Obviously, consensus is reached if and only if $\bar{x}(t) \to 0$ and $\bar{r}(t) \to 0$ as $t \to \infty$. Note that $\bar{x}$ is not affected by $\bar{r}$. Moreover, $\bar{r}$ is governed by a linear time-invariant system with $\bar{x}$ as the input and this system is asymptotically stable since H is assumed to be Hurwitz. Thus, hereafter we only need to prove that $\mathcal{C}(t)$ is bounded and $\bar{x}(t) \to 0$ as $t \to \infty$.

For the system (13), construct a candidate Lyapunov function as

$$V(t) = \bar{x}^{\mathrm{T}}(t)\left(\mathbf{I} \otimes P^{-1}\right)\bar{x}(t) + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{a_{ij}}{2\alpha_{ij}}\left(c_{ij}(t) - \bar{c}\right)^2,$$

where $\bar{c}$ is a positive constant to be determined and P is the positive definite matrix given in Statement 2 of the theorem. Taking the derivative of $V(t)$ along the solution of $\bar{x}(t)$ in (13) and $c_{ij}(t)$ in (4), we have

$$\begin{aligned} \dot{V} &= 2\bar{x}^{\mathrm{T}}\left(\mathbf{I} \otimes P^{-1}\right)\dot{\bar{x}} + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{a_{ij}}{\alpha_{ij}}\left(c_{ij} - \bar{c}\right)\dot{c}_{ij} \\ &= 2\bar{x}^{\mathrm{T}}\left(\mathbf{I} \otimes P^{-1}\right)\left[\mathbf{I} \otimes A - (\mathcal{C} \circ \mathcal{L}) \otimes PC_y^{\mathrm{T}} C_y\right]\bar{x} \\ &\quad + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \left(a_{ij}c_{ij} - a_{ij}\bar{c}\right)\left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right)^{\mathrm{T}} \\ &\quad \times \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right). \end{aligned}$$

Since

$$\begin{aligned} \tilde{y}_{ij} + C_y B_r \tilde{r}_{ij} &= C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}(s_i - s_j) \\ &= C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}(s_{ei} - s_{ej}) \\ &= C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} T^{-1}(\bar{s}_i - \bar{s}_j) \\ &= C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}\begin{bmatrix} \mathbf{I}_{n_x} & -B_r \\ 0 & \mathbf{I}_{n_r} \end{bmatrix}(\bar{s}_i - \bar{s}_j) \end{aligned}$$

$$= C_y \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} (\bar{s}_i - \bar{s}_j) = C_y (\bar{x}_i - \bar{x}_j) \tag{14}$$

and $c_{ij}(t) = c_{ji}(t)$ for all $t \geq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (a_{ij}c_{ij} - a_{ij}\bar{c}) (\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij})^{\mathrm{T}} (\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}) \\
&= \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (a_{ij}c_{ij} - a_{ij}\bar{c}) (\bar{x}_i - \bar{x}_j)^{\mathrm{T}} C_y^{\mathrm{T}} C_y (\bar{x}_i - \bar{x}_j) \\
&= 2\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} (a_{ij}c_{ij} - a_{ij}\bar{c}) \bar{x}_i^{\mathrm{T}} C_y^{\mathrm{T}} C_y (\bar{x}_i - \bar{x}_j) \\
&= 2\bar{x}^{\mathrm{T}} \left[(\mathcal{C} \circ \mathcal{L} - \bar{c}\mathcal{L}) \otimes C_y^{\mathrm{T}} C_y \right] \bar{x}.
\end{aligned}$$

Substituting this equation into $\dot{V}(t)$ leads to

$$\dot{V} = \bar{x}^{\mathrm{T}} \left[\mathbf{I} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) \right] \bar{x} - 2\bar{c}\bar{x}^{\mathrm{T}} \left(\mathcal{L} \otimes C_y^{\mathrm{T}} C_y \right) \bar{x}.$$

Since $\left(\mathbf{1}^{\mathrm{T}} \otimes \mathbf{I}\right) (\mathbf{I} \otimes C_y) \bar{x} = \left(\mathbf{1}^{\mathrm{T}}\mathcal{L}_e \otimes C_y[\mathbf{I}, 0]T\right) s = 0$, it follows from Lemma 1 that

$$\bar{x}^{\mathrm{T}} \left(\mathcal{L} \otimes C_y^{\mathrm{T}} C_y \right) \bar{x} \geq \lambda_2(\mathcal{L}) \bar{x}^{\mathrm{T}} \left(\mathbf{I} \otimes C_y^{\mathrm{T}} C_y \right) \bar{x}.$$

Let $\bar{c}$ be any constant such that $\bar{c} \geq \frac{1}{2}\lambda_2^{-1}(\mathcal{L})$. Using these relations results in

$$\begin{aligned}
\dot{V} &\leq \bar{x}^{\mathrm{T}} \left[\mathbf{I} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) \right] \bar{x} - 2\bar{c}\lambda_2(\mathcal{L}) \bar{x}^{\mathrm{T}} \left(\mathbf{I} \otimes C_y^{\mathrm{T}} C_y \right) \bar{x} \\
&\leq \bar{x}^{\mathrm{T}} \left[\mathbf{I} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) \right] \bar{x} - \bar{x}^{\mathrm{T}} \left(\mathbf{I} \otimes C_y^{\mathrm{T}} C_y \right) \bar{x} \\
&= \bar{x}^{\mathrm{T}} \left(\mathbf{I} \otimes P^{-1}QP^{-1} \right) \bar{x} \leq 0.
\end{aligned}$$

Since $\dot{V}(t) \leq 0$ and $V(t) \geq 0$ for all $t \geq 0$, it is seen that $V(t)$ is bounded, which implies that $c_{ij}(t)$ and thus $\mathcal{C}(t)$ are bounded. Moreover, since $\dot{c}_{ij}(t) \geq 0$ and $c_{ij}(t) > 0$, it is proved that $c_{ij}(t)$ converge to some finite positive constants. In addition, because $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ implies $\bar{x}(t) = 0$, it is known from Lasalle's theorem (see [24, Theorem 4.4]) that $\bar{x}(t) \to 0$ as $t \to \infty$. Consequently, consensus is reached under the protocol (3). The proof is completed. ■

C. Consensus Analysis Under the Node-Based Protocol

In this subsection, the consensus property of the closed-loop system under the node-based protocol (5) will be analysed. Define $\mathcal{D} \triangleq \mathrm{diag}\{d_1, \ldots, d_N\}$. Then the corresponding closed-loop system is given by

$$\dot{s}(t) = \left(\mathbf{I} \otimes \tilde{A} + \mathcal{D}(t)\mathcal{L} \otimes \tilde{B} \right) s(t).$$

Note that $\mathcal{D}(t)\mathcal{L}\mathcal{L}_e = \mathcal{D}(t)\mathcal{L}$. Thus $s_e(t)$ satisfies

$$\dot{s}_e(t) = \left(\mathbf{I} \otimes \tilde{A} + \mathcal{L}_e\mathcal{D}(t)\mathcal{L} \otimes \tilde{B} \right) s_e(t). \tag{15}$$

Similar to Lemma 2, we can establish the equivalence of the consensus of the original system and the convergence of $s_e(t)$. As a result, we can obtain the following consensus condition through proving the convergence of $s_e(t)$ in (15).

Theorem 2: Consider the MAS (1) and the protocol (5), and suppose that the communication graph $\mathcal{G}$ is undirected and connected. Then consensus is reached, and $d_i(t)$, $i = 1, \ldots, N$, converge to some finite positive constants as $t \to \infty$, if Statements 1) and 2) in Theorem 1 hold.

Proof: Similar to (12), the new state $\bar{s}(t) = (\mathbf{I} \otimes T) s_e$ satisfies

$$\dot{\bar{s}}(t) = \left(\mathbf{I} \otimes \bar{A} + \mathcal{L}_e\mathcal{D}(t)\mathcal{L} \otimes \bar{B} \right) \bar{s}(t).$$

Moreover, similar to (13), the above system can be written as

$$\begin{aligned}
\dot{\bar{x}}(t) &= \left(\mathbf{I} \otimes A - \mathcal{L}_e\mathcal{D}(t)\mathcal{L} \otimes PC_y^{\mathrm{T}}C_y \right) \bar{x}(t), \\
\dot{\bar{r}}(t) &= (\mathbf{I} \otimes H) \bar{r}(t) + (\mathcal{L}_e\mathcal{D}(t)\mathcal{L} \otimes F_r C_y) \bar{x}(t).
\end{aligned} \tag{16}$$

Since the second subsystem is stable, to prove the convergence of $\bar{s}_e(t)$ through $\bar{s}(t)$, we only need to prove the convergence of $\bar{x}(t)$. Construct a candidate Lyapunov function as

$$W(t) = \bar{x}^{\mathrm{T}}(t) \left(\mathcal{L} \otimes P^{-1} \right) \bar{x}(t) + \sum_{i=1}^{N} \frac{1}{\beta_i} \left(d_i(t) - \bar{d} \right)^2,$$

where $\bar{d}$ is a positive constant to be determined and P is the positive definite matrix satisfying (11). Since the communication graph is assumed to be undirected and connected and there holds $\left(\mathbf{1}^{\mathrm{T}} \otimes \mathbf{I}\right) \bar{x} = \left(\mathbf{1}^{\mathrm{T}}\mathcal{L}_e \otimes [\mathbf{I}, 0]T\right) s = 0$, it follows from Lemma 1 that $\bar{x}^{\mathrm{T}} \left(\mathcal{L} \otimes P^{-1} \right) \bar{x} \geq \lambda_2(\mathcal{L}) \bar{x}^{\mathrm{T}} \left(\mathbf{I} \otimes P^{-1} \right) \bar{x} \geq 0$. Thus, $W \geq 0$ and the equality holds only if $\bar{x} = 0$. Taking the derivative of W along the solution of $\bar{x}$ in (16) and d_i in (6), we have

$$\begin{aligned}
\dot{W} &= 2\bar{x}^{\mathrm{T}} \left(\mathcal{L} \otimes P^{-1} \right) \dot{\bar{x}} + \sum_{i=1}^{N} \frac{2}{\beta_i} \left(d_i - \bar{d} \right) \dot{d}_i \\
&= 2\bar{x}^{\mathrm{T}} \left(\mathcal{L} \otimes P^{-1} \right) \left(\mathbf{I} \otimes A - \mathcal{L}_e\mathcal{D}\mathcal{L} \otimes PC_y^{\mathrm{T}}C_y \right) \bar{x} \\
&\quad + 2\sum_{i=1}^{N} \left(d_i - \bar{d} \right) (\tilde{y}_i + C_y B_r \tilde{r}_i)^{\mathrm{T}} (\tilde{y}_i + C_y B_r \tilde{r}_i) \\
&= \bar{x}^{\mathrm{T}} \left[\mathcal{L} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) - 2\mathcal{L}\mathcal{D}\mathcal{L} \otimes C_y^{\mathrm{T}}C_y \right] \bar{x} \\
&\quad + 2s^{\mathrm{T}} (\mathcal{L} \otimes \mathbf{I}) \left[\left(\mathcal{D} - \bar{d}\mathbf{I} \right) \otimes \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} C_y^{\mathrm{T}}C_y \right. \\
&\quad \left. \times \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} \right] (\mathcal{L} \otimes \mathbf{I}) s.
\end{aligned}$$

Similar to the equations in (14), there holds

$$\begin{aligned}
&s^{\mathrm{T}} \left[\mathcal{L} \left(\mathcal{D} - \bar{d}\mathbf{I} \right) \mathcal{L} \otimes \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}^{\mathrm{T}} C_y^{\mathrm{T}}C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} \right] s \\
&= s_e^{\mathrm{T}} \left[\mathcal{L} \left(\mathcal{D} - \bar{d}\mathbf{I} \right) \mathcal{L} \otimes \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}^{\mathrm{T}} C_y^{\mathrm{T}}C_y \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} \right] s_e \\
&= \bar{s}^{\mathrm{T}} \left[\mathcal{L} \left(\mathcal{D} - \bar{d}\mathbf{I} \right) \mathcal{L} \otimes \begin{bmatrix} \mathbf{I} & -B_r \\ 0 & \mathbf{I} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix}^{\mathrm{T}} C_y^{\mathrm{T}}C_y \right. \\
&\quad \left. \times \begin{bmatrix} \mathbf{I} & B_r \end{bmatrix} \begin{bmatrix} \mathbf{I} & -B_r \\ 0 & \mathbf{I} \end{bmatrix} \right] \bar{s} \\
&= \bar{s}^{\mathrm{T}} \left[\mathcal{L} \left(\mathcal{D} - \bar{d}\mathbf{I} \right) \mathcal{L} \otimes \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}^{\mathrm{T}} C_y^{\mathrm{T}}C_y \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \right] \bar{s} \\
&= \bar{x}^{\mathrm{T}} \left[\mathcal{L} \left(\mathcal{D} - \bar{d}\mathbf{I} \right) \mathcal{L} \otimes C_y^{\mathrm{T}}C_y \right] \bar{x}.
\end{aligned}$$

From this equation, we have

$$\dot{W} = \bar{x}^{\mathrm{T}} \left[\mathcal{L} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) - 2\bar{d}\mathcal{L}^2 \otimes C_y^{\mathrm{T}}C_y \right] \bar{x}.$$

Since the matrix $\mathcal{L}$ is symmetric, we can decompose it as $\mathcal{L} = [N^{-1/2}\mathbf{1}, \mathcal{U}]\Lambda[N^{-1/2}\mathbf{1}, \mathcal{U}]^{\mathrm{T}}$, where $[N^{-1/2}\mathbf{1}, \mathcal{U}]$ is a unitary matrix and Λ is a diagonal matrix with the eigenvalues of $\mathcal{L}$ on the diagonal. Particularly, the first entry of Λ is zero. Let $\hat{x} = \mathrm{col}\{x_1, \ldots, x_N\} \triangleq [N^{-1/2}\mathbf{1}, \mathcal{U}]^{\mathrm{T}}\bar{x}$. Thus,

$$\dot{W} = \bar{x}^{\mathrm{T}} \left[[N^{-1/2}\mathbf{1}, \mathcal{U}]\Lambda[N^{-1/2}\mathbf{1}, \mathcal{U}]^{\mathrm{T}} \otimes \left(P^{-1}A + A^{\mathrm{T}}P^{-1} \right) \right.$$

$$
\begin{aligned}
&-2\bar{d}[N^{-1/2}\mathbf{1},\mathcal{U}]\Lambda^2[N^{-1/2}\mathbf{1},\mathcal{U}]^{\mathrm{T}}\otimes C_y^{\mathrm{T}}C_y\Big]\,\bar{x}\\
&=\hat{x}^{\mathrm{T}}\left[\Lambda\otimes\left(P^{-1}A+A^{\mathrm{T}}P^{-1}\right)-2\bar{d}\Lambda^2\otimes C_y^{\mathrm{T}}C_y\right]\hat{x}\\
&=\sum_{i=2}^{N}\lambda_i(\mathcal{L})\hat{x}_i^{\mathrm{T}}\left(P^{-1}A+A^{\mathrm{T}}P^{-1}-2\bar{d}\lambda_i(\mathcal{L})C_y^{\mathrm{T}}C_y\right)\hat{x}_i.
\end{aligned}
$$

Let $\bar{d}$ be any constant such that $\bar{d}\geq\frac{1}{2}\lambda_2^{-1}(\mathcal{L})$. Then

$$
\begin{aligned}
\dot{W}&\leq\sum_{i=2}^{N}\lambda_i(\mathcal{L})\hat{x}_i^{\mathrm{T}}\left(P^{-1}A+A^{\mathrm{T}}P^{-1}-C_y^{\mathrm{T}}C_y\right)\hat{x}_i\\
&=\sum_{i=2}^{N}\lambda_i(\mathcal{L})\hat{x}_i^{\mathrm{T}}P^{-1}QP^{-1}\hat{x}_i\leq 0.
\end{aligned}
$$

Since $\dot{W}(t)\leq 0$ and $W(t)\geq 0$, $W(t)$ is bounded. Thus, $d_i(t)$ and $\mathcal{D}(t)$ are bounded. From the facts that $\dot{d}_i(t)\geq 0$ and $d_i(t)>0$, it follows that $d_i(t)$ converge to some finite positive constants. Moreover, because $\dot{W}(t)\leq 0$ and $\dot{W}(t)=0$ implies $\bar{x}(t)=0$, it follows from Lasalle's theorem that $\bar{x}(t)\to 0$ as $t\to\infty$. Consequently, consensus is reached under the protocol (5). The proof is completed. ■

The consensus conditions, Theorems 1 and 2, show that the consensus problem can be solved by the proposed adaptive output-feedback protocols (3) and (5), as long as the protocol gains are properly parameterized. Especially, designing the protocol gains only needs to know the system matrices of agents but does not need to compute the eigenvalues of the Laplacian matrix. The next natural question to ask is, whether the protocol gains that satisfy the specifications in the theorems are feasible under some common assumptions about the agents. This will be answered in the next subsection.

Remark 4: Theorems 1 and 2 can only deal with consensus on undirected graphs. Note that, even for state-feedback consensus control, adaptive protocols on directed graphs are quite different from those on undirected graphs (please refer to [15], [16] for related results). The main difficulty in extending Theorems 1 and 2 to directed graphs is due to the loss of symmetry of the Laplacian, which will make the previous derivations invalid. Designing reduced-order adaptive protocols on directed graphs deserves investigation in the future.

Remark 5: The consensus problem studied in this paper is leaderless. However, it is not difficult to extend Theorems 1 and 2 to the leader-follower case if the followers interact through some connected undirected graphs. To brief the edge-based case, let us denote the leader as $\dot{x}_0=Ax_0; y_0=C_yx_0$. Then the tracking error of agent i is $e_i=y_i-y_0$. An edge-based reduced-order adaptive protocol can be constructed as

$$
\begin{aligned}
\dot{r}_i&=Hr_i+F_r\sum_{j=1}^{N}c_{ij}a_{ij}\left(\tilde{y}_{ij}+C_yB_r\tilde{r}_{ij}\right)\\
&\quad+c_{i0}b_iF_r\left(e_i+C_yB_rr_i\right),\\
u_i&=Gr_i+F_u\sum_{j=1}^{N}c_{ij}a_{ij}\left(\tilde{y}_{ij}+C_yB_r\tilde{r}_{ij}\right)\\
&\quad+c_{i0}b_iF_u\left(e_i+C_yB_rr_i\right),
\end{aligned}
$$

where c_{ij}, $i,j=1,\ldots,N$, are given by (4), c_{i0} are given by

$$
\dot{c}_{i0}=\alpha_{i0}\left(e_i+C_yB_rr_i\right)^{\mathrm{T}}\left(e_i+C_yB_rr_i\right),
$$

$$
c_{i0}(0)>0,\ \alpha_{i0}>0,\ i=1,\ldots,N, \tag{17}
$$

and b_i is a positive constant if agent i is connected to the leader, and $b_i=0$ otherwise. It can be verified that the lumped state $\varepsilon=\mathrm{col}\{\varepsilon_1,\ldots,\varepsilon_N\}$ with $\varepsilon_i=\mathrm{col}\{e_i,r_i\}$ satisfies

$$
\dot{\varepsilon}=\left[\mathbf{I}\otimes\tilde{A}+(\mathcal{C}\circ\mathcal{L}+\mathcal{C}_0\mathcal{B})\otimes\tilde{B}\right]\varepsilon, \tag{18}
$$

where $\mathcal{C}_0=\mathrm{diag}\{c_{10},\ldots,c_{N0}\}$ and $\mathcal{B}=\mathrm{diag}\{b_1,\ldots,b_N\}$. By following the proof for the convergence of s_e in (8), it can be verified that Statements 1) and 2) in Theorem 1 ensure the convergence of ε, implying that consensus is reached. A node-based counterpart can also be constructed. Related details are omitted here for brevity.

D. Design of Protocol Gains

In this subsection, we discuss the feasibility of the protocol gains that are specified in Theorems 1 and 2. For simplicity, suppose that B_u has full column rank; if this is not satisfied, one can extract the linearly independent columns of B_u as the new input matrix and then repeat the design procedures. Motivated by [25], the following algorithm is proposed for designing the protocol gains of (3) and (5).

1) Select a matrix $J\in\mathbb{R}^{n_x\times(n_x-n_u)}$ such that $\begin{bmatrix} B_u & J\end{bmatrix}$ is nonsingular. Partition the inverse of $\begin{bmatrix} B_u & J\end{bmatrix}$ as
$$
\begin{bmatrix} B_u & J\end{bmatrix}^{-1}=\begin{bmatrix} S_u\\ S_r\end{bmatrix}\begin{array}{l}\}\ n_u \text{ rows}\\ \}\ n_x-n_u \text{ rows}\end{array}.
$$
2) Compute the matrices B_r, H and G from the known matrices J, S_u and S_r that are obtained as above and another matrix $K\in\mathbb{R}^{n_u\times(n_x-n_u)}$ such that
$$
\begin{aligned}
&H=S_rA(J-B_uK)\text{ and }H\text{ is Hurwitz},\\
&B_r=J-B_uK,\ G=\left(S_u+KS_r\right)AB_r.
\end{aligned}\tag{19}
$$
3) Compute the protocol gains F_u and F_r as
$$
F_u=-\left(S_u+KS_r\right)PC_y^{\mathrm{T}},\ F_r=-S_rPC_y^{\mathrm{T}}, \tag{20}
$$
where P is given as in Theorem 1.

Following the proof of [25, Theorem 2], it can be verified that the matrices obtained as above satisfy the specifications stated in the proposed consensus analysis conditions.

Remark 6: Some comments about the proposed protocols and the above algorithm are provided as follows:

1) The matrix H is controlled by the variable K (S_r and J are determined in the first step). It is known from [25, Remark 3] that the matrix pair (S_rAJ, S_rAB_u) are stabilizable, provided that S_r and J are given as above and the pair (A,B_u) is stabilizable. Thus, by some standard design methods in linear control theory, it is easy to find a matrix K such that H is Hurwitz. Moreover, it is well known that the Riccati equation (11) is always feasible for some positive definite matrix P, since the pair (A,C_y) is detectable. Consequently, it is seen that all the steps in the algorithm are feasible under the usual assumption about agents that (A,B_u,C_y) are stabilizable and detectable.
2) Without accounting the laws of adaption, each local controller of the protocols designed by the algorithm is of

$(n_x - n_u)$th order (in view of $H \in \mathbb{R}^{(n_x-n_u)\times(n_x-n_u)}$). In this sense, the dynamic output-feedback protocols (3) and (5) resulting from this algorithm are some reduced-order protocols, which are computationally *less demanding* than the full-order adaptive dynamic output-feedback ones in [15].

3) Theorems 1 and 2 are also applicable for designing full-order protocols. For this case, it is more straightforward to find the protocol gains such that (9) and (10) are satisfied. Specifically, by directly selecting $B_r = -\mathbf{I}$ and $F_u = 0$, the equation (9) reduces to $H = A + B_u G$ and the matrix F_r in (10) is given by $F_r = PC_y^{\mathrm{T}}$. It is easy to see that the protocols (3) and (5) recover the full-order ones in [15, (2) and (3)]. Thus, the results therein can be viewed as special cases of the proposed protocols and consensus analysis conditions in this paper.
4) By resorting to the results in [20], one could construct another kind of reduced-order adaptive protocols, for which a node-based one might be given by

$$\dot{r}_i = Hr_i + Fy_i + TB_u u_i,\ u_i = d_i G K_1 \tilde{y}_i + d_i G K_2 \tilde{r}_i$$

with $H \in \mathbb{R}^{(n_x-n_y)\times(n_x-n_y)}$. In no regard of the feasibility, an obvious drawback of the protocol, however, is the fact that it needs absolute output information about agents, so that it cannot be applied in some scenarios where only relative information between agents is available. On the contrary, the proposed protocols (3) and (5) do not need absolute output information. Another work about reduced-order protocols is [21], based on which one might overcome this issue. However, the protocol therein needs relative input information between agents, which makes each controller require the information about neighbours' neighbours. On the contrary, the information flow in the protocols (3) and (5) is directly based on the communication graph.
5) It is well known that reduced-order output-feedback controllers are difficult to design in general. The reduced order of each local controller in the algorithm is set to $n_x - n_u$, so as to illustrate that both theorems always guarantees the existence of some reduced-order protocols under standard assumptions. However, it should be stressed that they do *not* exclude the possible existence of protocols that satisfy $n_r < n_x - n_u$. For instance, let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B_u = \begin{bmatrix} 7.4 \\ 11.1 \\ \frac{16.9}{3} \end{bmatrix},\ C_y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^{\mathrm{T}}.$$

A feasible solution satisfying (9) and (10) is as follows:

$$B_r = \begin{bmatrix} 3.7 \\ 3.7 \\ \frac{2.9}{3} \end{bmatrix},\ G = 1,\ H = -1,\ F_u = F_r = -1,$$
$$P = \begin{bmatrix} 61.5 & 51.3 & 11.1 \\ 51.3 & 56.9 & 14.8 \\ 11.1 & 14.8 & 6.6 \end{bmatrix},$$
$$Q = \begin{bmatrix} 0.21 & 0.18 & -0.24 \\ 0.18 & 2.64 & 0.08 \\ -0.24 & 0.08 & 0.76 \end{bmatrix}.$$

Unfortunately, under the usual assumption that the agent matrices (A, B_u, C_y) are stabilizable and detectable, there is no general, tractable procedure, like those in the algorithm, for checking the existence of a reduced-order protocol with $n_r < n_r - n_u$ and further re-constructing it. Except the case, $n_r = n_x - n_u$, as in the algorithm, checking and exploring such a reduced-order protocol should be on a case-by-case basis.

III. Simulation Example

In this section, to illustrate the efficacy of the propose protocols, we provide an applied example on formation flying of spacecrafts. As in the [11], the problem setting is stated as follows. A group of spacecrafts are supposed to move in a circular orbit with a virtual spacecraft as the origin, while the linearized equations of relative dynamics of the ith spacecraft are given by

$$\dot{x}_i = \begin{bmatrix} 0 & \mathbf{I}_3 \\ A_1 & A_2 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \mathbf{I}_3 \end{bmatrix} u_i,$$

where $x_i = \mathrm{col}\{p_i, \dot{p}_i\} \in \mathbb{R}^6$, $u_i \in \mathbb{R}^3$ and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 2\omega_0 & 0 \\ 2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the equation, $p_i \in \mathbb{R}^3$ and $\dot{p}_i \in \mathbb{R}^3$ are the relative position and velocity of the ith spacecraft in the x, y, z-axis of 3D space, and ω_0 is the angular rate of the virtual spacecraft. Formation flying of spacecraft means $p_i - h_i \to p_j - h_j$ and $\dot{p}_i \to \dot{p}_j$, where $h_i - h_j \in \mathbb{R}^3$ denotes the desired, fixed relative position between spacecrafts i and j. That is, the spacecrafts have the same final velocity while keeping a fixed formation.

Suppose that each spacecraft can measure the relative positive $p_i - p_j$ and know the desired position h_i and h_j. Motivated by the edge-based protocol (3), the following distributed controllers are proposed for formation flying:

$$\begin{aligned} \dot{r}_i &= Hr_i + F_r \sum_{j\in\mathcal{N}_i} c_{ij} a_{ij} \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right), \\ u_i &= Gr_i + F_u \sum_{j\in\mathcal{N}_i} c_{ij} a_{ij} \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right) - A_1 h_i, \end{aligned} \tag{21}$$

where c_{ij} is in (4), $\tilde{y}_{ij} = (p_i - h_i) - (p_j - h_j)$ and $H \in \mathbb{R}^{3\times3}$. Motivated by the node-based one (5), the following alternative distributed controllers can also be constructed:

$$\begin{aligned} \dot{r}_i &= Hr_i + d_i F_r \sum_{j\in\mathcal{N}_i} \left(\tilde{y}_i + C_y B_r \tilde{r}_i\right), \\ u_i &= Gr_i + d_i F_u \sum_{j\in\mathcal{N}_i} \left(\tilde{y}_i + C_y B_r \tilde{r}_i\right) - A_1 h_i, \end{aligned} \tag{22}$$

where d_i is in (6), $\tilde{y}_i = \sum_{i=1}^{N} l_{ij} (p_j - h_j)$ and $H \in \mathbb{R}^{3\times3}$.

Remark 7: The idea of the reduced-order protocol in [20] is to use the absolute output to re-construct the absolute state of each agent. Thus, each local reduced-order controller therein is actually a reduced-order observer. However, rather than this physical meaning, the controller states in our protocols are just some intermediate variables for the control purpose, similar to most of the existing results like [11], [12], [15], [16].

Example 1: Consider a group of 4 spacecrafts which communicate with each other according to a line graph. Let the edge weights are all 1. Thus, the Laplacian matrix is

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Let $\omega_0 = 0.005$. By the proposed algorithm, a feasible solution of the gains of the protocols (21) and (22) is given by

$$H = -\mathbf{I},\ B_r = \begin{bmatrix} \mathbf{I} \\ -\mathbf{I} \end{bmatrix},\ G = \begin{bmatrix} -1 & -0.01 & 0 \\ 0.01 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$F_u = \begin{bmatrix} -2.7320 & -0.0058 & 0 \\ 0.0058 & -2.7321 & 0 \\ 0 & 0 & -2.7320 \end{bmatrix},$$
$$F_R = \begin{bmatrix} -1.7320 & 0 & 0 \\ 0 & -1.7321 & 0 \\ 0 & 0 & -1.7320 \end{bmatrix},$$

and the matrix P, which is used to obtain the above F_u and F_r but is not presented here, is computed from the equation (11) with $Q = \mathbf{I}$. Note that, different from the results in [15], each local controller of the above protocols is only of 3rd order (in no regard of the laws of adaption). Thus, the proposed control protocols as above are less computationally demanding.

For simulation, we specify the desired positions as $h_1 = [-100; 0; 0]$, $h_2 = [-100; 0; 100]$, $h_3 = [0; -100; 100]$ and $h_4 = [0; -100; 0]$. For the protocol (21), let the scalars $\alpha_{ij} = 0.01$ and the initial conditions $c_{ij}(0) = 10^{-4}$, $i, j = 1, \ldots, 4$. For the protocol (22), let $\beta_1 = 0.01$, $\beta_2 = 0.02$, $\beta_3 = 0.03$ and $\beta_4 = 0.04$ and the initial conditions $d_i(0) = 10^{-4}$, $i = 1, \ldots, 4$. The initial states of the protocols are set to zero. Figure 1 shows the simulation result under the above settings, where the initial conditions of the spacecrafts are not presented for saving space. It can be seen that the spacecrafts maintain the specified flying formation, while the adaptive gains c_{ij} and d_i converge to some positive constants. Thus, the effectiveness of the proposed theoretical results are clearly illustrated.

IV. Conclusion

In this paper, the problem of fully distributed consensus control of linear MASs has been investigated, and novel reduced-order adaptive output-feedback protocols have been constructed and analyzed. The edge-based protocol associates each edge with a scalar gain that is adaptively updated by the output difference of the two agents on each edge, while the node-based one associates each agent with a scalar gain that is updated by the output difference sum of all neighbouring agents. Sufficient existence conditions have been derived and a design algorithm has been presented for the proposed protocols. It is shown that, under the common assumption that the agents are stabilizable and detectable, the proposed protocols with $(n_x - n_u)$th-order local controllers must exist and can be easily found. Compared with the existing reduced-order protocols, the propose ones rely on relative output information of neighbouring agents only and can be designed and implemented in a fully distributed way. A simulation example on formation flying of spacecrafts has been provided for illustrating the efficacy of the proposed method.

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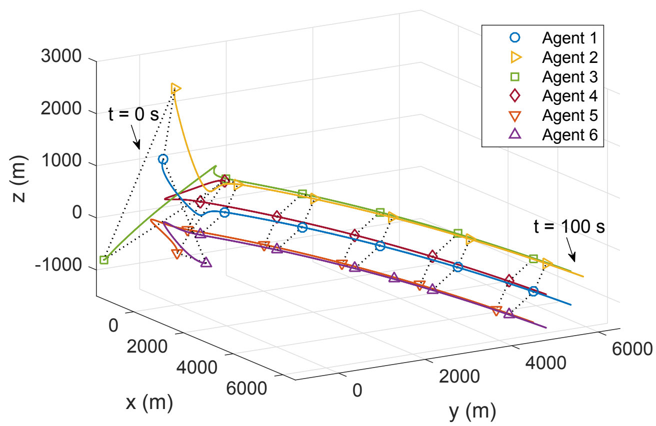
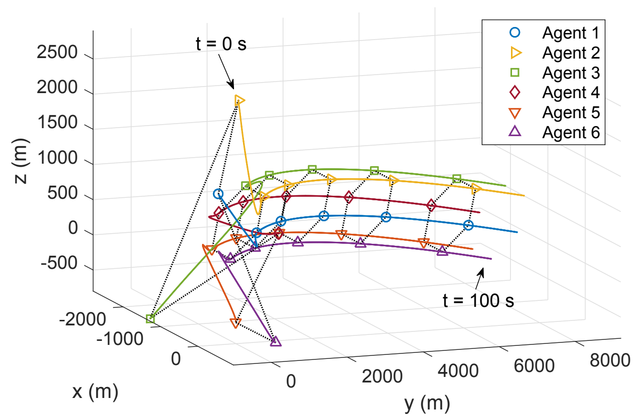
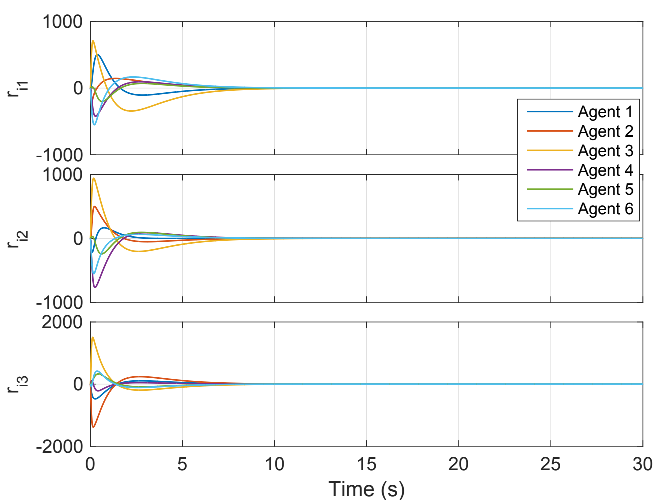


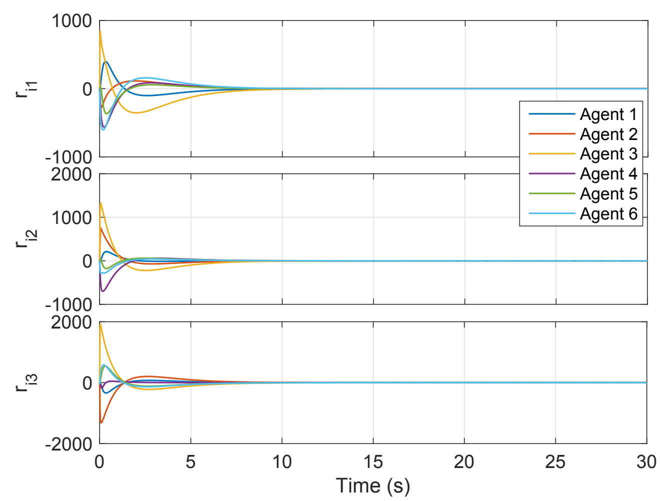
(a) Trajectories of spacecrafts under the edge-based protocol (21)



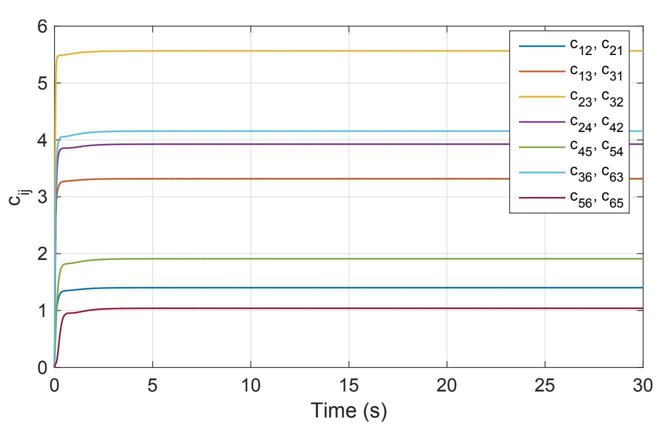
(b) Trajectories of spacecrafts under the node-based protocol (22)



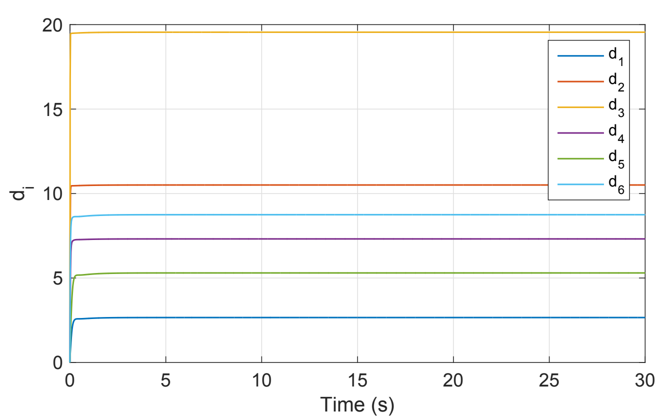
(c) State response of the edge-based protocol (21)



(d) State response of the node-based protocol (22)



(e) c_{ij} of the edge-based protocol (21)



(f) d_i of the node-based protocol (22)

Fig. 1. Simulation results

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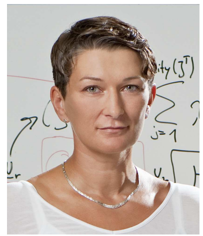

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