DUAL QCD INSIGHT INTO BSM HADRONIC MATRIX ELEMENTS FOR K^0–$\bar{K}^0$ MIXING FROM LATTICE QCD*

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We calculate BSM hadronic matrix elements for K^0–$\bar{K}^0$ mixing in the Dual QCD approach (DQCD). The ETM, SWME and RBC-UKQCD lattice collaborations find the matrix elements of the BSM density–density operators $\mathcal{O}_i$ with $i = 2$–5 to be rather different from their vacuum insertion values (VIA) with $B_2 \approx 0.5$, $B_3 \approx B_5 \approx 0.7$ and $B_4 \approx 0.9$ at $\mu = 3$ GeV to be compared with $B_i = 1$ in the VIA. We demonstrate that this pattern can be reconstructed within the DQCD through the non-perturbative meson evolution from very low scales, where factorization of matrix elements is valid, to scales $\mathcal{O}(1~\text{GeV})$ with subsequent perturbative quark–gluon evolution to $\mu = 3$ GeV. This turns out to be possible in spite of a very different pattern displayed at low scales with $B_2 = 1.2$, $B_3 = 3.0$, $B_4 = 1.0$ and $B_5 \approx 0.2$ in the large-N limit, N being the number of colours. Our results imply that the inclusion of meson evolution in the phenomenology of any non-leptonic transition like K^0–$\bar{K}^0$ mixing and $K \to \pi\pi$ decays is mandatory. While meson evolution, as demonstrated in our paper, is hidden in lattice QCD results, to our knowledge, DQCD is the only analytic approach for non-leptonic transitions and decays which takes this important strong dynamics into account.

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1. Introduction

The K^0–$\bar{K}^0$ mixing induced CP violation in $K \to \pi\pi$ decays described by the parameter ε_K and the $K_{\rm L} - K_{\rm S}$ mass difference ΔM_K have played very important role in the construction of the Standard Model (SM) and, more recently, in tests of its possible extensions. For recent reviews, see [1, 2]. In spite of modest tensions in ε_K [3], in particular in correlation with mass differences $\Delta M_{d,s}$, the SM describes the data on ε_K reasonably well. The situation with ΔM_K is unclear at present because its value within the SM is subject to large long-distance (LD) [4–7] and short-distance (SD) [8] uncertainties. Consequently, there is still considerable room for new physics (NP) contributions in ε_K and ΔM_K which are generally governed by new local operators $\mathcal{O}_i$ with $i = 2$–5 beyond the single SM operator $\mathcal{O}_1$. These BSM density–density operators are known to have enhanced hadronic matrix elements implying very strong constraints on various NP models.

While the Wilson coefficients of the full set of $\Delta S = 2$ operators $\mathcal{O}_i$ were calculated at the NLO level long time ago [9–11], only in this decade, have their hadronic matrix elements been calculated in QCD with respectable precision. In the case of the SM operator $\mathcal{O}_1$, represented by the parameter $\hat{B}_K$, several lattice QCD collaborations confirmed with high precision its value in the framework of Dual QCD approach (DQCD) [7, 12] finding, in agreement with the latter, that its value is rather different from its vacuum insertion estimate (VIA) and very close to its large-N limit $\hat{B}_K = 3/4$. While lattice QCD was not able to explain this result, such an explanation is provided by DQCD as demonstrated in [7] where, relative to the first paper [12], vector meson contributions have been included and the matching to short-distance contributions thereby improved.

Recently, very useful results on hadronic matrix elements of the BSM operators $\mathcal{O}_i$ with $i = 2$–5 have been obtained by the ETM, SWME and RBC-UKQCD lattice collaborations [13–17]. For some of these operators, these results turn out to differ again significantly from the earlier results obtained by vacuum insertion approximation (VIA) [18, 19], with the pattern of deviations summarized already in the abstract and discussed in more detail below. To our knowledge, no attempt has been made so far in the literature to understand the strong dynamics behind this peculiar pattern. The question thus arises whether DQCD approach could again help in getting an insight in the lattice QCD results. The main goal of the present paper is the demonstration that this is indeed the case. This is a significant result as it underlines the importance of meson evolution in a non-leptonic transition in which the controversial role of final-state interactions is absent.

Our paper is organized as follows. In Section 2, we recall very briefly the elements of DQCD relevant for our paper. In Section 3, we give the expressions for the operators $\mathcal{O}_i$, recall the commonly used parametrization of their K^0–$\bar{K}^0$ matrix elements in terms of scale-dependent parameters $B_i(\mu)$, and summarize their values as obtained by lattice QCD at $\mu = 3\,\mathrm{GeV}$ [13–17]. In Section 4, we calculate these matrix elements in DQCD. We begin with the large-N limit that corresponds to an appropriate scale momentarily denoted by μ_0. Next, we demonstrate that starting with the lattice QCD results at $\mu = 3\,\mathrm{GeV}$, performing first renormalization group QCD evolution (quark–gluon evolution) down to scales $\mathcal{O}(1\,\mathrm{GeV})$ and, subsequently, meson evolution in the framework of DQCD down to μ_0 reproduces rather well the pattern of the values of B_i parameters found in the large-N limit. A brief summary is given in Section 5. As we mainly want to understand the pattern of the values of B_i in an analytic approach, we do not aim for precision and perform the $1/N$ meson evolution with the light pseudoscalars only which is sufficient for our purposes.

2. Basics of dual QCD approach

The explicit calculation of the contributions of pseudoscalars to hadronic matrix elements of local operators is based on a truncated chiral Lagrangian describing the low-energy interactions of the lightest mesons [7, 20–22]

$$L_{\mathrm{tr}} = \frac{F^2}{8}\left[\mathrm{Tr}\left(D^\mu U D_\mu U^\dagger\right) + r\mathrm{Tr}\left(mU^\dagger + \mathrm{h.c.}\right) - \frac{r}{\Lambda_\chi^2}\mathrm{Tr}\left(mD^2U^\dagger + \mathrm{h.c.}\right)\right], \tag{1}$$

where

$$U = \exp\left(i\sqrt{2}\frac{\Pi}{F}\right), \qquad \Pi = \sum_{\alpha=1}^{8}\lambda_\alpha\pi^\alpha \tag{2}$$

is the unitary chiral matrix describing the octet of lowest-lying pseudoscalars. The parameter F is related to the weak decay constants $F_\pi \approx 130\,\mathrm{MeV}$ and $F_K \approx 156\,\mathrm{MeV}$ through

$$F_\pi = F\left(1 + \frac{m_\pi^2}{\Lambda_\chi^2}\right), \qquad F_K = F\left(1 + \frac{m_K^2}{\Lambda_\chi^2}\right), \tag{3}$$

so that $\Lambda_\chi \approx 1.1\,\mathrm{GeV}$, as expected from higher resonances. The diagonal mass matrix m involving m_u, m_d and m_s is such that

$$r(\mu) = \frac{2m_K^2}{m_s(\mu) + m_d(\mu)}, \tag{4}$$

with $r(1\,\mathrm{GeV}) \approx 3.75\,\mathrm{GeV}$ for $(m_s + m_d)(1\,\mathrm{GeV}) \approx 132\,\mathrm{MeV}$. Compared to large-$N$ Chiral Perturbation Theory, there is one-to-one correspondence with the low-energy parameters introduced in [23, 24]

$$\Lambda_\chi^2 = \frac{F^2}{8L_5}\,, \qquad r = 2B_0\,. \tag{5}$$

In particular, L_5 encodes the scalar state effects in the single-resonance approximation,

$$L_5 = \frac{F^2}{8M_S^2}\,, \tag{6}$$

as shown in [25, 26].

The flavour-singlet η_0 meson decouples due to large mass m_0 generated by the non-perturbative axial anomaly. Consequently, the matrix Π in (2) reads

$$\Pi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix}. \tag{7}$$

In order to calculate the matrix elements of the local operators in question, we need meson representations of colour-singlet quark currents and densities. They are directly obtained from the effective Lagrangian in (1) and are given respectively as follows:

$$\bar{q}_{\mathrm{L}}^b \gamma_\mu q_{\mathrm{L}}^a = i\frac{F^2}{8}\left\{(\partial_\mu U)U^\dagger - U\left(\partial_\mu U^\dagger\right) + \frac{r}{\Lambda_\chi^2}\left[(\partial_\mu U)m^\dagger - m\left(\partial_\mu U^\dagger\right)\right]\right\}^{ab}, \tag{8}$$

$$\bar{q}_{\mathrm{R}}^b q_{\mathrm{L}}^a = -\frac{F^2}{8}r\left[U - \frac{1}{\Lambda_\chi^2}\partial^2 U\right]^{ab}, \tag{9}$$

with U turned into $U^\dagger$ under parity.

At the tree level, corresponding to leading order in $1/N$, one uses these representations to simply express the operators in terms of the meson fields and expands the matrix U in powers of $1/F$. For K^0–$\bar{K}^0$ mixing, the relevant contribution to hadronic matrix elements is read off from terms involving only K^0 and $\bar{K}^0$.

3. BSM matrix elements in lattice QCD

There are two equivalent bases for new operators contributing to K^0–$\bar{K}^0$ mixing. The so-called SUSY basis [18, 19] is given by

$$\begin{aligned}
\mathcal{O}_1 &= \bar{s}^{\,\alpha}\gamma_\mu P_{\rm L} d^\alpha \; \bar{s}^\beta \gamma_\mu P_{\rm L} d^\beta \,, \\
\mathcal{O}_2 &= \bar{s}^{\,\alpha} P_{\rm L} d^\alpha \; \bar{s}^\beta P_{\rm L} d^\beta \,, \\
\mathcal{O}_3 &= \bar{s}^{\,\alpha} P_{\rm L} d^\beta \; \bar{s}^\beta P_{\rm L} d^\alpha \,, \\
\mathcal{O}_4 &= \bar{s}^{\,\alpha} P_{\rm L} d^\alpha \; \bar{s}^\beta P_{\rm R} d^\beta \,, \\
\mathcal{O}_5 &= \bar{s}^{\,\alpha} P_{\rm L} d^\beta \; \bar{s}^\beta P_{\rm R} d^\alpha \,,
\end{aligned} \tag{10}$$

and the BMU one [11] by

$$\begin{aligned}
Q_1^{\rm VLL} &= (\bar{s}\gamma_\mu P_{\rm L} d)\,(\bar{s}\gamma^\mu P_{\rm L} d) = \mathcal{O}_1 \,, \\
Q_1^{\rm LR} &= (\bar{s}\gamma_\mu P_{\rm L} d)\,(\bar{s}\gamma^\mu P_{\rm R} d) = -2\mathcal{O}_5 \,, \\
Q_2^{\rm LR} &= (\bar{s} P_{\rm L} d)\,(\bar{s} P_{\rm R} d) = \mathcal{O}_4 \,, \\
Q_1^{\rm SLL} &= (\bar{s} P_{\rm L} d)\,(\bar{s} P_{\rm L} d) = \mathcal{O}_2 \,, \\
Q_2^{\rm SLL} &= (\bar{s}\sigma_{\mu\nu} P_{\rm L} d)\,(\bar{s}\sigma^{\mu\nu} P_{\rm L} d) = 4\mathcal{O}_2 + 8\mathcal{O}_3 \,,
\end{aligned} \tag{11}$$

where

$$P_{\rm R,L} = \tfrac{1}{2}(1 \pm \gamma_5)\,, \qquad \sigma_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]\,. \tag{12}$$

In (11), we omitted colour indices as they are summed up in each parenthesis.

In what follows, we will use the SUSY basis (10) as most lattice collaborations use it. The K^0–$\bar{K}^0$ matrix elements in this basis are parametrized as follows:

$$\langle \bar{K}^0 |\, \mathcal{O}_1(\mu)\, | K^0 \rangle = \tfrac{2}{3} m_K^2 F_K^2 B_1(\mu)\,, \tag{13}$$
$$\langle \bar{K}^0 |\, \mathcal{O}_2(\mu)\, | K^0 \rangle = -\tfrac{5}{12} R(\mu)\, m_K^2 F_K^2 B_2(\mu)\,, \tag{14}$$
$$\langle \bar{K}^0 |\, \mathcal{O}_3(\mu)\, | K^0 \rangle = \tfrac{1}{12} R(\mu)\, m_K^2 F_K^2 B_3(\mu)\,, \tag{15}$$
$$\langle \bar{K}^0 |\, \mathcal{O}_4(\mu)\, | K^0 \rangle = \tfrac{1}{2} R(\mu)\, m_K^2 F_K^2 B_4(\mu)\,, \tag{16}$$
$$\langle \bar{K}^0 |\, \mathcal{O}_5(\mu)\, | K^0 \rangle = \tfrac{1}{6} R(\mu)\, m_K^2 F_K^2 B_5(\mu)\,, \tag{17}$$

where

$$R(\mu) = \left(\frac{m_K}{m_s(\mu) + m_d(\mu)}\right)^2 = \frac{r^2(\mu)}{4m_K^2} \tag{18}$$

refers to the generic factorized evolution of any density–density operator with μ, a renormalization scale taken at 3 GeV by ETM, SWME and RBC-UKQCD lattice collaborations.

Still, the BMU basis in (11) will turn out to be very useful at intermediate steps because all its operators are made of colour singlet bilinears that are most suitable for DQCD calculations.

Now, in the vacuum insertion approximation (VIA), the parameters B_i are, by definition, given by

$$B_1 = B_2 = B_3 = B_4 = B_5 = 1\,, \qquad \text{(VIA)} \tag{19}$$

and thus μ-independent. Already this property of VIA, which is based on the factorization of matrix elements of four-quark operators into products of quark currents or quark densities, is problematic.

In the DQCD approach on the other hand, the factorization of matrix elements in question can be proven to be a property of QCD in the large-N limit [12, 21, 22, 27] because in this limit, QCD at very low momenta becomes a free theory of mesons [28–31]. With non-interacting mesons, the factorization of matrix elements of four-quark operators into matrix elements of quark currents and quark densities is automatic. However, even then, several B_i parameters are not equal to unity as VIA includes the so-called Fierz-terms that are $1/N$ suppressed and thus absent in the large-N limit. The classic example is the parameter B_1 which is equal to unity in the VIA and to $3/4$ in the large-N limit [32].

Another yet advantage of DQCD over VIA is that it tells us that factorization in question does not take place at values of μ used by lattice QCD collaborations but rather at zero momentum transfer between colour-singlet currents or densities. Therefore, it is not surprising that lattice QCD collaborations find $B_1(\mu) \approx 0.53$ at $\mu = 3\,\mathrm{GeV}$, a value significantly below the large-N one, *i.e.*, $3/4$. While within numerical approach like lattice QCD this difference cannot be understood, this decrease of $B_1(\mu)$ with increasing μ can be shown to be the consequence of meson evolution from scale $\mu_0 = 0$ to scale $\mathcal{O}(1\,\mathrm{GeV})$ followed by the usual renormalization group quark–gluon evolution up to $\mu = 3\,\mathrm{GeV}$ [12]. In this particular case, one usually multiplies the result by the corresponding SD renormalization group factor to find the scale- and renormalization scheme-independent $\hat{B}_K = 0.73 \pm 0.02$ [7] in very good agreement with the world average of lattice QCD calculations $\hat{B}_K = 0.766 \pm 0.010$ [33].

In the case of the BSM operators $\mathcal{O}_i$ with $i = 2$–5, the construction of scale-independent $\hat{B}_i$ parameters, although possible, is not particularly useful because $\mathcal{O}_2$ mixes under renormalization with $\mathcal{O}_3$ and $\mathcal{O}_4$ with $\mathcal{O}_5$. This mixing is known at the NLO level [10, 11] and useful NLO expressions for μ dependence of hadronic matrix elements and their Wilson coefficients can be found in [34]. For our purposes, it will be sufficient to work in LO approximation and use LO formulae also given in [34].

The lattice QCD results from three collaborations are displayed in Table I.

TABLE I

Results for the parameters B_i with the first error being statistical and the second systematic obtained by ETM [13], SWME [14] and RBC-UKQCD [15–17] collaborations at $\mu = 3\,\mathrm{GeV}$. Useful comparison can be found in [17].

Collaboration	n_f	B_2	B_3	B_4	B_5
ETM15	$2+1+1$	0.46(3)(1)	0.79(5)(1)	0.78(4)(3)	0.49(4)(1)
SWME15	$2+1$	0.525(1)(23)	0.773(6)(35)	0.981(3)(62)	0.751(7)(68)
RBC-UKQCD	$2+1$	0.488(7)(17)	0.743(14)(65)	0.920(12)(16)	0.707(8)(44)

We observe that:

— there is good agreement between three collaborations as far as the values of B_2 and B_3 are concerned;

— the values of B_4 obtained by SWME15 and RBC-UKQCD are close to unity, while the ETM Collaboration obtains a significantly lower value;

— the values of B_5 obtained by SWME15 and RBC-UKQCD are in the ballpark of 0.7, while the value from ETM Collaboration is in the ballpark of 0.5;

— most values differ significantly from unity, prohibiting the use of VIA.

To our knowledge, no lattice group made an attempt to understand this peculiar pattern of values. On the other hand, as we will demonstrate now, it can be understood within DQCD because in this approach, an insight in the strong dynamics at very low scales up to $1\,\mathrm{GeV}$ can be obtained through *the meson evolution* followed by the usual RG QCD evolution, termed in [7, 12, 21, 22, 27] *the quark–gluon evolution,* from $1\,\mathrm{GeV}$ to $3\,\mathrm{GeV}$.

4. Calculating BSM B_i in DQCD

4.1. Large-N limit

Using the meson representation of quark densities in (9), the K^0–$\bar{K}^0$ matrix elements of the local operators $\mathcal{O}_2$ and $\mathcal{O}_4$ can be easily calculated in the large-N limit because these operators are built out of colour singlet quark densities. We find

$$\left\langle \bar{K}^0 \right| \mathcal{O}_4(\mu_0) \left| K^0 \right\rangle_\infty = -\left\langle \bar{K}^0 \right| \mathcal{O}_2(\mu_0) \left| K^0 \right\rangle_\infty = \tfrac{1}{2} R(\mu_0)\, m_K^2 F_K^2\,, \qquad (20)$$

where we used relations (3). Comparing with (14) and (16) and interpreting the factorization to be valid at $\mu_0 = 0$, we extract

$$B_2(0) = 1.20\,, \qquad B_4(0) = 1.00 \qquad (\text{large-}N \text{ limit})\,. \tag{21}$$

We note that $B_2 \neq 1$ because a $1/N$ Fierz term included in VIA is absent now.

In order to find the remaining two matrix elements, we have to bring them to the colour singlet form with the help of relations in (11). In the case of $\mathcal{O}_3$, we use the fact that $\langle Q_2^{\mathrm{SLL}}(\mu_0)\rangle = 0$ to obtain

$$\left\langle \bar{K}^0 \right| \mathcal{O}_3(\mu_0) \left| K^0 \right\rangle_\infty = -\tfrac{1}{2} \left\langle \bar{K}^0 \right| \mathcal{O}_2(\mu_0) \left| K^0 \right\rangle_\infty = \tfrac{1}{4} R(\mu_0)\, m_K^2 F_K^2\,, \tag{22}$$

namely, from (15),

$$B_3(0) = 3.0 \qquad (\text{large-}N \text{ limit})\,. \tag{23}$$

Similarly, for the matrix element of $\mathcal{O}_5$ expressed in terms of the currents in (8), we find using (11)

$$\left\langle \bar{K}^0 \right| \mathcal{O}_5(\mu_0) \left| K^0 \right\rangle_\infty = -\tfrac{1}{2} \left\langle \bar{K}^0 \right| Q_1^{\mathrm{LR}}(\mu_0) \left| K^0 \right\rangle_\infty = \tfrac{1}{4}\, m_K^2 F_K^2\,, \tag{24}$$

and extract, using (17),

$$R(\mu_0) B_5(\mu_0) = \tfrac{3}{2}\,. \tag{25}$$

The result for B_5 is rather peculiar, which is related to the fact that in the BMU basis, $\mathcal{O}_5$ is not a density–density operator but a current–current one like $\mathcal{O}_1$. This implies that the usual parametrization of $\langle \mathcal{O}_5 \rangle$ as given in (17) in terms of B_5 is not useful at low scales, where $R(\mu)$ is not accessible. However, in order to compare it later with lattice QCD, we estimate its value by evaluating $R \approx 6.5$ at the scale around 0.7 GeV, where the meson evolution matches the quark–gluon one for the $\Delta S = 1$ operators [35], to find

$$B_5(0) \approx 0.23 \qquad (\text{large-}N \text{ limit})\,. \tag{26}$$

The main message from this estimate is that at low scales, B_5 is expected to be smaller than the remaining B_i parameters.

In any case, we observe that, except for B_4, the results for parameters B_i obtained in the large-N limit differ significantly from the lattice ones in Table I as well as from VIA in (19).

As the lattice results and large-N results correspond to quite different scales, in order to understand lattice results with the help of DQCD, we will proceed in two steps as follows:

— we will start with the lattice QCD values for B_i at $\mu = 3\,\mathrm{GeV}$ in Table I and evolve these parameters through quark–gluon evolution down to the scale $\mu = 1\,\mathrm{GeV}$;

— in order to see whether the values $B_i(1\,\mathrm{GeV})$ from lattice QCD can be explained in DQCD, we will start with large-N values in (21), (23) and (26), and perform the meson evolution up to the cut-off $\Lambda = (0.65 \pm 0.05)\,\mathrm{GeV}$ defined in Appendix B.

Ideally, one would like to perform meson evolution up to $\mu = 1\,\mathrm{GeV}$ but, as only pseudoscalar contributions are taken into account, one has to stop the evolution around $0.7\,\mathrm{GeV}$ so that the comparison will not be perfect but sufficient to reach firm conclusions by extrapolation.

In this context, we should recall that the parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ relevant for $K \to \pi\pi$ decays equal unity in the large-N limit to be compared with the values in the ballpark of 0.6 and 0.8, respectively, obtained by RBC-UKQCD Collaboration at $\mu = 1.5\,\mathrm{GeV}$. This pattern can be understood at least semi-quantitatively in DQCD [36, 37] and the question arises, whether the pattern

$$B_2 < B_5 \leq B_3 < B_4 \qquad (\text{at } \mu = 3\,\mathrm{GeV}) \tag{27}$$

can be understood within this framework as well, although in DQCD one finds

$$B_5 \ll B_4 < B_2 < B_3 \qquad (\text{at } \mu \ll 1\,\mathrm{GeV})\,. \tag{28}$$

4.2. Quark–gluon evolution

Let us first check whether already the perturbative evolution from $\mu = 3\,\mathrm{GeV}$ to $\mu = 1\,\mathrm{GeV}$ allows us to understand partially the difference between patterns (27) and (28). In order to disentangle the non-factorizable $B_i(\mu)$ from the factorizable $R(\mu)$, we work in the one-loop approximation. The anomalous dimension matrices are then given in units of $\alpha_s/4\pi$ as follows:

$$\hat{\gamma}^{(0)}(\mathcal{O}_2, \mathcal{O}_3) = \begin{pmatrix} -6N + 8 + \frac{2}{N} & 4 - \frac{8}{N} \\ 4N - 4 - \frac{8}{N} & 2N + 4 + \frac{2}{N} \end{pmatrix}, \tag{29}$$

$$\hat{\gamma}^{(0)}(\mathcal{O}_4, \mathcal{O}_5) = \begin{pmatrix} -6N + \frac{6}{N} & 0 \\ -6 & \frac{6}{N} \end{pmatrix}. \tag{30}$$

The two operator systems $(\mathcal{O}_2, \mathcal{O}_3)$ and $(\mathcal{O}_4, \mathcal{O}_5)$ do not mix under renormalization with each other.

Now, keeping only first leading logarithms, one has for $\mu_2 > \mu_1$

$$\langle\mathcal{O}_i(\mu_2)\rangle = \langle\mathcal{O}_i(\mu_1)\rangle\left(1-\frac{\alpha_s}{4\pi}\hat{\gamma}_{ii}^{(0)}\ln\left(\frac{\mu_2}{\mu_1}\right)\right) - \langle\mathcal{O}_j(\mu_1)\rangle\frac{\alpha_s}{4\pi}\hat{\gamma}_{ij}^{(0)}\ln\left(\frac{\mu_2}{\mu_1}\right). \tag{31}$$

Taking into account that the factorized $R(\mu)$ scales like

$$\frac{R(\mu_2)}{R(\mu_1)} = \left[1+2\frac{\alpha_s}{4\pi}\gamma_m^{(0)}\ln\left(\frac{\mu_2}{\mu_1}\right)\right], \qquad \gamma_m^{(0)} = 6\left(\frac{N^2-1}{2N}\right) \tag{32}$$

and inserting the expressions for $\langle\mathcal{O}_i(\mu)\rangle$ in (14)–(17) into (31), we isolate in this manner the μ dependence of $B_i(\mu)$

$$B_2(\mu_2) = B_2(\mu_1)\left[1-\frac{(8N-4)}{N}\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right)\right]+B_3(\mu_1)\frac{(4N-8)}{5N}\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right), \tag{33}$$

$$B_3(\mu_2) = B_3(\mu_1)\left[1-P_{33}(N)\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right)\right]+B_2(\mu_1)P_{32}(N)\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right), \tag{34}$$

with

$$P_{33}(N) = \frac{(8N^2+4N-4)}{N}, \qquad P_{32}(N) = \frac{(20N^2-20N-40)}{N} \tag{35}$$

and

$$B_4(\mu_2) = B_4(\mu_1), \tag{36}$$

$$B_5(\mu_2) = B_5(\mu_1)\left[1-6N\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right)\right]+B_4(\mu_1)18\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu_2}{\mu_1}\right). \tag{37}$$

We observe:

— a strong suppression of B_2 with increasing μ due to the first term in (33);

— an even stronger suppression of B_3 with $B_3 > B_2$ due to the first term in (34);

— no μ dependence of B_4 in (36);

— a positive shift of B_5 with $B_5 \ll B_4$, due to the second term in (37).

However, in view of large coefficients in front of the logarithms for $N = 3$, one should sum them to all orders. We do this in Appendix A. This allows us to find the values of B_i for $\mu_1 = 1\,\mathrm{GeV}$ from those obtained by RBC-UKQCD at $\mu_2 = 3\,\mathrm{GeV}$ and given in Table I. Inserting the central lattice values in the formulae of Appendix A, we find at $\mu = 1\,\mathrm{GeV}$

$$B_2 = 0.608\,, \qquad B_3 = 1.06\,, \qquad B_4 = 0.920\,, \qquad B_5 = 0.519\,. \tag{38}$$

The corresponding values obtained from central values of ETM and SWME collaborations are collected in Table II.

As already observed from (33)–(37), B_2, B_3 and B_5, all moved towards their large-N values in (21), (23) and (26), while B_4 did not change in LO approximation. These results are already very encouraging. We will next estimate whether the meson evolution can do the rest of the job within DQCD.

4.3. Meson evolution

We first calculate the $1/N$ meson evolution in question in the chiral limit using the technology developed in [38], namely expanding

$$\tilde{U} \equiv \exp\left(i\sqrt{2}\frac{\mathit{\Xi}}{F}\right) U\,, \qquad \mathit{\Xi} = \sum_{\alpha=1}^{8} \lambda_\alpha \xi^\alpha \tag{39}$$

around the classical field U.

It should be emphasized that we do not aim here at achieving the precision of lattice QCD. We would like mainly to demonstrate that DQCD allows us to understand lattice results at the semi-quantitative level by including analytically strong dynamics at very low scales. It is this dynamics which is responsible for the pattern in Table I.

For the non-factorizable evolution of density–density operators like $\mathcal{O}_{2,3,4}$, we find

$$(U)^{ab}(U)^{cd}(\mathit{\Lambda}) = (U)^{ab}(U)^{cd}(0) - 4\,\frac{\mathit{\Lambda}^2}{(4\pi F)^2}\left[(U)^{ad}(U)^{cb} - \frac{1}{3}(U)^{ab}(U)^{cd}\right](0) \tag{40}$$

and

$$\begin{aligned}(U)^{ab}(U^{+})^{cd}(\mathit{\Lambda}) &= (U)^{ab}\left(U^\dagger\right)^{cd}(0) \\ &\quad +4\frac{\mathit{\Lambda}^2}{(4\pi F)^2}\left[\delta^{ad}\left(U^\dagger U\right)^{cb} - \frac{1}{3}(U)^{ab}\left(U^\dagger\right)^{cd}\right](0)\,. \end{aligned}\tag{41}$$

The last terms proportional to $1/3$ arise from the Fierz relation

$$(\lambda_\alpha)^{ij}(\lambda^\alpha)^{kl} = 2\left(\delta^{il}\delta^{kj} - \tfrac{1}{3}\delta^{ij}\delta^{kl}\right) \tag{42}$$

for the Gell-Mann matrices introduced in (2) and (39). Had we worked in the nonet approximation, these terms would have been absent. In particular, the Λ^2 term in (41) would have vanished in remarkable accordance with the absence of short-distance evolution of B_4 in the leading log approximation as seen in (36). Consequently, in our DQCD approach, the purely non-perturbative contribution from the strong anomaly is responsible for $B_4 < 1$. It would be interesting if lattice QCD simulations confirmed this feature one day.

The evolution in (40) can be directly applied to B_2. However, the case of B_3 is more subtle. Working in the BMU basis (11) to extract its value in the large-N limit, we have to stay in the same basis to study the meson evolution of $\mathcal{O}_3$ since otherwise the running of $R(\mu)$ and $B_i(\mu)$ would be interchanged under Fierzing. Consequently, the meson evolution of $\mathcal{O}_3$ is governed by a well-defined linear combination of Q_1^{SLL} and Q_2^{SLL}, namely,

$$\mathcal{O}_3 = -\tfrac{1}{2}\left[Q_1^{\mathrm{SLL}} - \tfrac{1}{4}Q_2^{\mathrm{SLL}}\right] . \tag{43}$$

On the one hand, the Q_1^{SLL} evolution down to the factorization scale μ_0 is fully given by (40) since $\langle Q_2^{\mathrm{SLL}}(\mu_0)\rangle = 0$. On the other, to infer the non-trivial meson evolution of Q_2^{SLL} into Q_1^{SLL} above μ_0, we have to rely on the mixing pattern of the quark–gluon anomalous dimension matrix given in units of $\alpha_{\mathrm{s}}/4\pi$ in the BMU basis at one-loop as follows:

$$\hat{\gamma}^{(0)}\left(Q_1^{\mathrm{SLL}}, Q_2^{\mathrm{SLL}}\right) = \begin{pmatrix} -6N + \frac{6}{N} + 6 & \frac{1}{2} - \frac{1}{N} \\ -24 - \frac{48}{N} & 2N - \frac{2}{N} + 6 \end{pmatrix} . \tag{44}$$

For that purpose, we omit again the $(-6N+6/N)$ term in the $\hat{\gamma}_{11}^{(0)}$ entry since it corresponds to the running of the factorized $R(\mu)$ given in (32). Extending then this mixing pattern below 1 GeV, we expect the left-over relative factor (-4) between $\hat{\gamma}_{21}^{(0)}$ and $\hat{\gamma}_{11}^{(0)}$ to survive hadronization in the large-N and chiral limits. If such is the case, following DQCD, the non-factorizable meson evolution of $\mathcal{O}_3(\Lambda)$ down into $\mathcal{O}_2(0)$ is, in fact, a factor of two faster than the one for $\mathcal{O}_2$ so that the factor -4 in (40) is replaced by -8.

For the evolution of local current–current operators like $\mathcal{O}_5$ in the BMU basis, we obtain

$$\begin{aligned}(J_{\mathrm{L}})^{ab}(J_{\mathrm{R}})^{cd}(\Lambda) &= (J_{\mathrm{L}})^{ab}(J_{\mathrm{R}})^{cd}(0) \\ &\quad + \frac{\Lambda^4}{(4\pi F)^2}\left[\frac{F^4}{8}\right]\left[(U)^{ad}\left(U^\dagger\right)^{cb} - \frac{1}{3}\delta^{ab}\left(U^\dagger U\right)^{cd}\right](0) .\end{aligned} \tag{45}$$

Requiring the measure to be chiral invariant, such a quartic cut-off dependence can be properly cancelled [38]. Computing the quadratic one in the specific case of a Fierz-conjugate $\Delta S = 2$ transition, we find then

$$(J_\mathrm{L})^{ds}(J_\mathrm{R})^{ds}(\Lambda) = (J_\mathrm{L})^{ds}(J_\mathrm{R})^{ds}(0) - \frac{\Lambda^2}{(4\pi F)^2}\left[\frac{F^4 m_K^2}{4}\right](U)^{ds}\left(U^\dagger\right)^{ds}(0)\,. \tag{46}$$

Starting with the large-N values in (21), (23) and (26), and letting them evolve on the basis of (40), (41) and (46), we find at the order of $1/N$

$$B_2(\Lambda) = 1.2\left[1 - \frac{8}{3}\frac{\Lambda^2}{(4\pi F_K)^2}\right], \tag{47}$$

$$B_3(\Lambda) = 3.0\left[1 - \frac{16}{3}\frac{\Lambda^2}{(4\pi F_K)^2}\right], \tag{48}$$

$$B_4(\Lambda) = 1.0\left[1 - \frac{4}{3}\frac{\Lambda^2}{(4\pi F_K)^2}\right], \tag{49}$$

$$B_5(\Lambda) = 0.23\left[1 + 4\frac{\Lambda^2}{(4\pi F_K)^2}\right], \tag{50}$$

where Λ is the cut-off of DQCD which allows us to separate the non-factorizable meson evolution from the quark–gluon one.

The general trend already observed in the quark–gluon evolution is nicely outlined in the meson evolution with

— a strong suppression of B_2;

— an even stronger suppression of B_3;

— a smooth evolution of B_4;

— a strong enhancement of B_5.

In the chiral limit we work and including only pseudoscalar contributions in the loops, a reasonable range for Λ is

$$m_8 \leq \Lambda < m_0 \tag{51}$$

with

$$m_8^2 = \frac{4m_K^2 - m_\pi^2}{3} \approx (0.57\,\mathrm{GeV})^2\,, \tag{52}$$

the Gell-Mann–Okubo mass relation in the octet approximation, and

$$m_0^2 = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 \approx (0.85\,\mathrm{GeV})^2\,, \tag{53}$$

the axial anomaly mass relation in the large-N limit [39]. In the nonet approximation (*i.e.*, $m_0 = 0$), the factors $(-8/3, -16/3, -4/3, +4)$ in (47)–(50) would be replaced by $(-4, -8, 0, +4)$, respectively.

Including vector meson contributions would allow us to raise the cut-off Λ and to approach the critical scale of 1 GeV. However, we do not think such a complication is necessary in order to explain the pattern found by lattice QCD.

Indeed, setting $\Lambda = 0.6\,(0.7)$ GeV in (47)–(50), we obtain

$$\begin{aligned} B_2(\Lambda) &\approx 0.9\,(0.8)\,, \qquad B_3(\Lambda) \approx 1.5\,(1.0)\,, \\ B_4(\Lambda) &\approx 0.9\,(0.8)\,, \qquad B_5(\Lambda) \approx 0.29\,(0.35)\,. \end{aligned} \tag{54}$$

This should be compared with lattice QCD values in (38). Again, the correct pattern is reproduced and it is justified to conclude that these results around 0.65 GeV are already satisfactory.

The $\mathcal{O}_1$ and $\mathcal{O}_5$ current–current operators do not receive one-loop corrections from the flavour-singlet η_0 meson. So, we can safely consider the full chiral corrections given in Appendix B for them to obtain the final results for $\Lambda = (0.65 \pm 0.05)$ GeV given in Table II.

Let us recall that in the case of the parameter $\hat{B}_K$ or equvalently B_1 and in the case of $K \to \pi\pi$ decays within the SM, we have included not only pseudoscalar meson contributions, as done here, but also those of lightest vector mesons. As seen in Tables 4 and 5 in [7], this improvement has an impact on final numerical values and improves the matching of both evolutions but, importantly for us, does not modify the signs of evolutions. We expect this to be also the case here.

Our results, collected in Table II, demonstrate very clearly that the pattern of B_i values obtained by lattice QCD collaborations can be understood in DQCD by performing quark–gluon evolution of lattice values down to $\mu = 1$ GeV and the meson evolution from the factorization scale up to scales $\mathcal{O}(1\,\text{GeV})$.

In judging the quality of the agreement of DQCD with lattice QCD, we should realize that there is still a gap between the values of the cut-off Λ and $\mu = 1$ GeV at which the values of B_i obtained by lattice QCD can be evaluated. Moreover, except for B_5, our calculations have been performed in the chiral limit. Going beyond this limit and also including the vector meson contributions would allow us to improve the calculation and raise the values of Λ at least to 0.9 GeV.

TABLE II

When mesons (almost) meet quarks and gluons. Upper half: summary of the central values of B_i parameters at $\mu = 3\,\text{GeV}$ obtained by lattice QCD collaborations and their quark–gluon (QG) evolution down to $\mu = 1\,\text{GeV}$. Lower half: summary of the values of B_i parameters obtained in DQCD and their meson evolution up to $\Lambda = (0.60 \to 0.70)\,\text{GeV}$.

μ	B_2	B_3	B_4	B_5
3 GeV (ETM15)	0.46	0.79	0.78	0.49
3 GeV (SWME)	0.52	0.77	0.98	0.75
3 GeV (RBC-UKQCD)	0.49	0.74	0.92	0.71
$\Downarrow$ (QG)				
1 GeV (ETM15)	0.58	1.22	0.78	0.24
1 GeV (SWME)	0.66	1.07	0.98	0.55
1 GeV (RBC-UKQCD)	0.61	1.06	0.92	0.52
$(0.60 \to 0.70)$ GeV	$0.90 \to 0.79$	$1.50 \to 0.96$	$0.87 \to 0.83$	$0.27 \to 0.30$
$\Uparrow$ (DQCD)				
0	1.2	3.0	1.0	0.23
Λ	B_2	B_3	B_4	B_5

Extrapolating Table II with some caution, we can make the following observations:

— The parameter B_2 is still visibly larger than lattice values at 1 GeV but the difference between the lattice values at 3 GeV and its large-N limit value of roughly 0.7 has been decreased below 0.2 for $\Lambda = 0.70\,\text{GeV}$. It would disappear already for $\Lambda = 0.80\,\text{GeV}$.

— The parameter B_3 is already in very good agreement with lattice values, although its value is subject to visible uncertainty. Yet, what is impressive is that the initial difference between DQCD and lattice QCD values of 2.2 has been decreased to 0.2 at $\Lambda = 0.70\,\text{GeV}$.

— The meson evolution decreases B_4 by roughly 15%. At $\Lambda = 0.70\,\text{GeV}$, we find the value of B_4 between those obtained by the ETM Collaboration and RBC-UKQCD. It will be interesting to see how this comparison will look like when the accuracy of lattice results will improve.

– Our result for B_5 is between those obtained by ETM and the two other collaborations.

5. Summary

While not as precise as ultimate lattice QCD calculations, the DQCD approach offered over many years an insight in the lattice results and often, like was the case of the $\Delta I = 1/2$ rule [27] and the parameter $\hat{B}_K$ [12], provided results almost three decades before this was possible with lattice QCD. The agreement between results from DQCD approach and lattice QCD is remarkable, in particular considering the simplicity of the former approach and the very sophisticated and tedious numerical calculations of the latter. The most recent example is the good agreement for hadronic matrix elements of the chromomagnetic penguin operator between DQCD [35] and results from the ETM Collaboration [40].

In the present paper, we have demonstrated that the pattern of hadronic matrix elements of BSM operators affecting K^0–$\bar{K}^0$ mixing obtained by lattice QCD can also be understood in DQCD at the semi-quantitative level. The crucial role in this insight, as seen in Table II, is played by meson evolution, an important ingredient of DQCD, which could be exhibited here clearer than in $K \to \pi\pi$ decays because of the absence of final-state interactions in K^0–$\bar{K}^0$ mixing. In turn, our results imply that the inclusion of meson evolution in the phenomenology of $K \to \pi\pi$ decays is mandatory.

It is truly remarkable that this insight has been obtained without any free parameters beyond the value of the cut-off Λ. The remaining input were the values of the pseudoscalar masses, F_K and of $\alpha_{\rm s}$. In particular, no values of low-energy constants from lattice QCD, used often in chiral perturbation studies, were involved. We are not aware of any analytical approach that could provide such insight in lattice QCD results in question. This makes us confident that the pattern for $B_6^{(1/2)}$ and $B_8^{(3/2)}$ obtained in DQCD [36, 37] will be confirmed by improved lattice calculations making the existing ε'/ε anomaly more pronounced. If such is the case, the same methodology applied to the hadronic matrix elements of all BSM operators affecting $K \to \pi\pi$ decay amplitudes [41, 42] will be useful.

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Appendix A

Summing leading logarithms

In order to sum the leading logarithms, formulae (4.4)–(4.7) and (4.12)–(4.15) of [34] derived in the BMU basis (11) have to be transferred into the SUSY one (10).

We find now

$$\begin{aligned} B_2(\mu_2) &= T(\mu_2,\mu_1)\left[B_2(\mu_1)r_{22} - \tfrac{1}{5}B_3(\mu_1)r_{23}\right], && \text{(A.1)}\\ B_3(\mu_2) &= T(\mu_2,\mu_1)\left[B_3(\mu_1)r_{33} - 5B_2(\mu_1)r_{32}\right], && \text{(A.2)} \end{aligned}$$

and

$$\begin{aligned} B_4(\mu_2) &= T(\mu_2,\mu_1)\left[B_4(\mu_1)r_{44} + \tfrac{1}{3}B_5(\mu_1)r_{45}\right], && \text{(A.3)}\\ B_5(\mu_2) &= T(\mu_2,\mu_1)\left[B_5(\mu_1)r_{55} + 3B_4(\mu_1)r_{54}\right]. && \text{(A.4)} \end{aligned}$$

Here,

$$T(\mu_2,\mu_1) = \frac{R(\mu_1)}{R(\mu_2)}\left[\frac{\alpha_{\mathrm{s}}^{(4)}(\mu_2)}{\alpha_{\mathrm{s}}^{(4)}(m_c)}\right]^{24/25}\left[\frac{\alpha_{\mathrm{s}}^{(3)}(m_c)}{\alpha_{\mathrm{s}}^{(3)}(\mu_1)}\right]^{8/9} \tag{A.5}$$

and the coefficients r_{ij} are just linear combinations of the coefficients ρ_{ij} for which explicit expressions can be found in (4.4)–(4.7) and (4.12)–(4.15) of [34]. The ρ_{ij} are just functions of QCD factors like the ones in (A.5).

We find

$$\begin{aligned} r_{22} &= [\rho_{11}]_{\mathrm{SLL}} + 4\,[\rho_{21}]_{\mathrm{SLL}}, \qquad r_{23} = 8\,[\rho_{21}]_{\mathrm{SLL}}, \\ r_{33} &= [\rho_{22}]_{\mathrm{SLL}} - 4\,[\rho_{21}]_{\mathrm{SLL}}, && \text{(A.6)}\\ r_{32} &= \tfrac{1}{2}\left([\rho_{22}]_{\mathrm{SLL}} - [\rho_{11}]_{\mathrm{SLL}}\right) - 2\,[\rho_{21}]_{\mathrm{SLL}} + \tfrac{1}{8}\,[\rho_{12}]_{\mathrm{SLL}}, && \text{(A.7)} \end{aligned}$$

and

$$\begin{aligned} r_{44} &= [\rho_{22}]_{\mathrm{LR}}, \qquad r_{45} = -2\,[\rho_{12}]_{\mathrm{LR}}, \\ r_{55} &= [\rho_{11}]_{\mathrm{LR}}, \qquad r_{54} = -\tfrac{1}{2}\,[\rho_{21}]_{\mathrm{LR}}. && \text{(A.8)} \end{aligned}$$

We have suppressed the scales $\mu_K = \mu_1$ and $\mu_{\mathrm{L}} = \mu_2$ as well as the superscript (0) which indicates LO approximation. In fact, the formulae above are valid including NLO corrections given in that paper except that formula (A.5) must be then generalized beyond LO. For our purposes, LO is sufficient.

Using then

$$\begin{aligned} \alpha_{\mathrm{s}}^{(4)}(3\,\mathrm{GeV}) &= 0.241\,, \qquad \alpha_{\mathrm{s}}^{(4)}(m_c) = \alpha_{\mathrm{s}}^{(3)}(m_c) = 0.365\,, \\ \alpha_{\mathrm{s}}^{(3)}(1\,\mathrm{GeV}) &= 0.437\,, && \text{(A.9)} \end{aligned}$$

we find

$$\begin{aligned} B_2(\mu_2) &= 0.794\,B_2(\mu_1) + 0.005B_3(\mu_1)\,, && \text{(A.10)}\\ B_3(\mu_2) &= 0.395\,B_3(\mu_1) + 0.532\,B_2(\mu_1)\,, && \text{(A.11)} \end{aligned}$$

and

$$
\begin{aligned}
B_4(\mu_2) &= B_4(\mu_1)\,, && \text{(A.12)}\\
B_5(\mu_2) &= 0.532\,B_5(\mu_1) + 0.468\,B_4(\mu_1)\,. && \text{(A.13)}
\end{aligned}
$$

These results generalise the ones in (33)–(37) to include the summation of leading logarithms to all orders in perturbation theory. Using the relation for three quark theory

$$
\frac{\alpha_{\rm s}(\mu_2)}{\alpha_{\rm s}(\mu_1)} = 1 - 9\frac{\alpha_{\rm s}}{2\pi}\ln\frac{\mu_2}{\mu_1} \tag{A.14}
$$

and keeping only the first leading logarithm, one verifies the results in (33)–(37).

Appendix B

Meson evolution beyond chiral limit

For the $\mathcal{O}_1$ and $\mathcal{O}_5$ current–current operators, chiral corrections imply

$$
B_i(\Lambda) = B_i(0)\left(1 + \frac{\Delta_i(\Lambda)}{2F_K^2}\right) \tag{B.1}
$$

with

$$
\begin{aligned}
\Delta_1(\Lambda) &= -\left[1 + \frac{m_\pi^2}{m_K^2}\right] I\left(m_\pi^2\right) - 3\left[1 + \frac{m_8^2}{m_K^2}\right] I\left(m_8^2\right) - 4m_K^2 I'\left(m_K^2\right)\,,\\
\Delta_5(\Lambda) &= +\left[1 + \frac{m_\pi^2}{m_K^2}\right] I\left(m_\pi^2\right) + 3\left[1 + \frac{m_8^2}{m_K^2}\right] I\left(m_8^2\right) - 4m_K^2 I'\left(m_K^2\right) && \text{(B.2)}
\end{aligned}
$$

and

$$
\begin{aligned}
I\left(m_i^2\right) &= \frac{i}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{q^2 - m_i^2} = \frac{1}{16\pi^2}\left[\Lambda^2 - m_i^2\ln\left(1 + \frac{\Lambda^2}{m_i^2}\right)\right]\,, && \text{(B.3)}\\
I'\left(m_i^2\right) &\equiv \frac{\mathrm{d}I\left(m_i^2\right)}{\mathrm{d}m_i^2} = \frac{1}{16\pi^2}\left[\frac{\Lambda^2}{\Lambda^2 + m_i^2} - \ln\left(1 + \frac{\Lambda^2}{m_i^2}\right)\right]\,. && \text{(B.4)}
\end{aligned}
$$

Using the well-known Gell-Mann–Okubo mass relation

$$
m_\pi^2 + 3m_8^2 = 4m_K^2\,, \tag{B.5}
$$

we obtain

$$
B_1(0.6\,\mathrm{GeV}) = 0.75\,(1 - 0.11)\,, \qquad B_1(0.7\,\mathrm{GeV}) = 0.75\,(1 - 0.18) \tag{B.6}
$$

and

$$B_5(0.6\,\mathrm{GeV}) = 0.23\,(1+0.19)\,, \qquad B_5(0.7\,\mathrm{GeV}) = 0.23\ (1+0.29)\,. \quad \text{(B.7)}$$

For B_1, the resulting DQCD evolution is quite impressive with

$$B_1(0)\!=\!0.75\!\Rightarrow\! B_1(0.7\,\mathrm{GeV}) = 0.62\ ||\ 0.61\!=\!B_1(1\,\mathrm{GeV})\!\Leftarrow\!0.53\!=\!B_1(3\,\mathrm{GeV})\,. \quad \text{(B.8)}$$

The values of B_1 at the end of meson and quark–gluon evolutions are close to each other and are expected to be equal when the gap || between these two evolutions will be filled by vector mesons. For B_5, the resulting DQCD evolution is displayed in Table II.

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