Optimal Control for Heterogeneous Node-Based Information Epidemics over Social Networks

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***Abstract*—In this paper, we investigate the optimal control problems of heterogeneous node-based information epidemics. A node-based Susceptible-Infected-Recovered-Susceptible (SIRS) model is introduced to describe the information diffusion processes taking into account heterogeneities in both network structures and individual characters. Aiming at guiding information dissemination processes towards the desired performance, we propose an optimal control framework with respect to two typical scenarios, i.e., impeding the spread of rumors and enhancing the spread of marketing or campaigning information. We prove the existence of the solutions and solve the optimal control problems by Pontryagin Maximum Principle and forward-backward sweep method. Moreover, numerical experiments validate the using of the node-based SIRS model by comparing with the exact 3^N-state Markov chain model. The effectiveness of the proposed control rules are demonstrated on both models. Further discussion on the influence of the parameters provides insights into the strategies of guiding information diffusion processes.**

***Index Terms*—Information epidemics, optimal control, social networks.**

I. Introduction

Information spreading via diverse media, e.g., face-to-face conversations, television and Internet, is indispensable in our daily lives. It is inevitably influential to our decision making, opinions, and activities. Thus the studies on information diffusion processes become attractive and have drawn wide interests in the fields of sociology, psychology, computer science, and control in recent years [1]–[3].

As a fundamental issue, mathematical modeling of information diffusion has been an interesting topic and multitudes of models have been reported [4], [5]. Among the models from diverse points of view, epidemic models have received great attention due to the similarity between virus spreading process and information dissemination [6]. The term, *information epidemics* [7], is deployed to describe the disease-like spread of information. In particular, viral marketing [8], which creates and passes the informative and entertaining messages to the consumers in a virtual environment, paves the way for utilizing information epidemics. It potentially reaches massive audiences at a fast speed by little investment. To this end, various companies adopt this strategy to achieve rapid growth. Besides, other information diffusion processes, e.g., the propagation of rumors and campaigning, have also been reported to have the characteristics of epidemics spreading [9], [10]. These examples establish the significant role played by epidemic models in information diffusion processes.

Armed with information epidemic models, a practical and natural question arises: how can we guide the information spreading in our desired manner? This motivates us to address the optimal control problem for information epidemics. In this article, a node-based susceptible-infected-recovered-susceptible (SIRS) model is utilized and we take into consideration two practical scenarios: 1) impeding rumor propagation and 2) enhancing the spread of marketing or campaign information.

A. Justification for Using SIRS Model

In light of the similarity between epidemic processes and information diffusion, we believe that the SIRS model is suitable for information epidemics over social networks.

In the context of marketing, *susceptible* individuals are potential consumers who may accept the information from certain company. They form the *target market*. An *infected* individual passes the message of the companies through her social contacts, which may result in the infection of her social neighbors. Once the infected forgets to share a message or is tired of passing the message, she belongs to the *recovered* compartment [11]. This susceptible-infected-recovered (SIR) model described above has been widely accepted to study the viral marketing strategy [12], [13]. However, the SIR model assumes that the virus can be life-long immunized, which cannot be guaranteed for many diseases [14]. Inspired by the phenomenon of short-term immunity [15] in the virus dissemination, we argue that there exists *resusceptible* process [16] in the information epidemics. Specifically, a recovered individual can turn to be susceptible again spontaneously since the company will naturally regard her as a member of the target market. Apart from viral marketing, SIRS model has been applied to various information diffusion processes [17]. For example, among all the epidemic models, SIRS model provides the best interpretation of Internet meme (a phenomenon of contents or concepts that spread rapidly online) activity data [18]. Additionally, SIRS model is also applicable for the rumor propagation [19] and information diffusion in blogspace and web forums [20], [21].

It is worth noting that the framework proposed in this article is not restricted to SIRS model but also applicable for other epidemic models, e.g., susceptible-infected (SI), susceptible-infected-susceptible (SIS), and SIR, by simple degeneration.

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B. Related Work and Contribution

Recent years have witnessed the trend of using the mean-field approximation (MFA) to study epidemic models. The pioneering work [22] proposed an N-intertwined SIS model. Further studies concentrate on the equilibrium analysis of the SIS model and its extensions [23], [24]. Although the node-based SIRS model is a generalization of the SIS model, it has not been thoroughly investigated [25].

From epidemiological perspective, previous works on the control of epidemics models mainly focus on the immunization problems. Most of the reported strategies are static, e.g., the random immunization and the target immunization [26]. Recently, the optimal immunization problem has become more attractive in the field of system and control. By utilizing the epidemic threshold [27], the optimal resource allocation problem has been addressed on both single-layer and multi-layer networks [28], [29]. Note that none of them considers time-varying control rules, so their methods belong to the static optimization in essence.

Apart from the immunization problem, some literature consider maximizing the information dissemination under different models [30]–[32]. However, the majority of them consider the macroscopic epidemic models and no network structure is specified. It is equivalent to assume that there exists a well-mixed network, i.e., the interaction and influence between each pair of individuals are identical. This assumption, nonetheless, can hardly hold in real social networks. Although there exist some related work, the optimal control design for node-based information epidemics is still an open problem according to the recent survey [33].

The previous literature [34] studies the node-based SIS model and designs a time-varying control law. However, in their approach, the model needs to be linearized and only the disease free case is set to be the target. More recent works [35], [36] study the optimal control problem for node-based SIR and SI models, respectively. Whereas, they both consider only homogeneous transition rates. To the best of our knowledge, there exists no literature taking into consideration the optimal control framework on the heterogeneous node-based SIRS model and dealing with both the immunization and maximizing dissemination problems.

The main contribution of this article is the design and realization of an optimal control framework for the heterogeneous node-based information epidemics over social networks. As a general extension of previous works in homogeneous cases, we consider the situation where the transition rates between different compartments are generally different, in light of the individual diversities in genders, personalities, preferences, etc. Furthermore, two optimal control problems are proposed taking into consideration the practical scenarios on impeding rumor spreading and enhancing marketing propagation. Therefore, this article provides an insight in optimally guiding information diffusion process from a control theoretical point of view, which is of great importance for marketing and campaigning activities. By directly utilizing the nonlinear model, we prove the existence of the solution and solve the optimal control problems by Pontryagin Maximum Principle and numerical algorithms. Moreover, we compare the node-based SIRS model and the exact 3^N-state Markov chain model by using the Monte Carlo simulations. By implementing the optimal control law on the Markov chain model, we show the effectiveness of the proposed control rules in practice. Further discussion of the influence of the parameters provides valuable hints for achieving desired information spreading performance.

Note that in our previous conference article [37], an optimal control strategy was designed for the SIS model over an undirected graph to enhance information diffusion. In this article, as a general extension, the SIRS model is adopted inspired by the resusceptible process. Besides, we consider information epidemics over directed graphs due to the fact that information flow in social networks is usually directed, which is technically more difficult since the symmetry owing to the undirected graphs is negated. Although the optimal control problem on enhancing the spreading has been studied in [37], we propose a more practical constraint based on the limited budget. Furthermore, an extra instance on impeding rumor spreading is also studied, which is a complement for the optimal control framework.

The rest of the article is organized as follows. The detailed instruction on related fields is given in Section II as preliminaries based on which we introduce the node-based SIRS model and formulate the optimal control problems. The main results on optimal control design (the existence of the solution and the algorithm to solve the problem) are presented in Section III. Numerical studies are given in Section IV.

Notations: The set of real numbers is denoted as $\mathbb{R}$. $\mathbf{1}$ and $\mathcal{I}$ represent a vector whose elements are all 1 and the identity matrix with proper dimension, respectively. $\mathsf{Prob}[\cdot]$ denotes the probability of certain event. For two vectors ξ and ζ, (ξ, ζ) means $[\xi^\mathsf{T}, \zeta^\mathsf{T}]^\mathsf{T}$.

II. Preliminaries and Model Description

In this section, the knowledge of graph theory and the node-based SIRS model is introduced.

A. Networks and Graph Theory

We consider a social network described by a weighted directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, W)$ with N ($N \geq 2$) nodes, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the sets of nodes and edges, respectively. The adjacency matrix $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ is nonnegative and with zero diagonal entries. For two distinct nodes, $w_{ij} > 0$ if and only if there exists a link from node j to i. In this case, we say node j listens to node i or node i can influence node j. For the convenience of further presentation, the in-neighborhood of node i is also introduced as

$$\mathcal{N}_i^{\text{in}} = \{j : w_{ij} > 0, j \in \mathcal{V}\}.$$

In this article, we confine ourselves that the graph $\mathcal{G}$ is strongly connected, i.e., W is irreducible.

B. The Node-based SIRS model

In the SIRS model, every node of the graph $\mathcal{G}$ are in one of the three compartments: susceptible (S), infected (I),

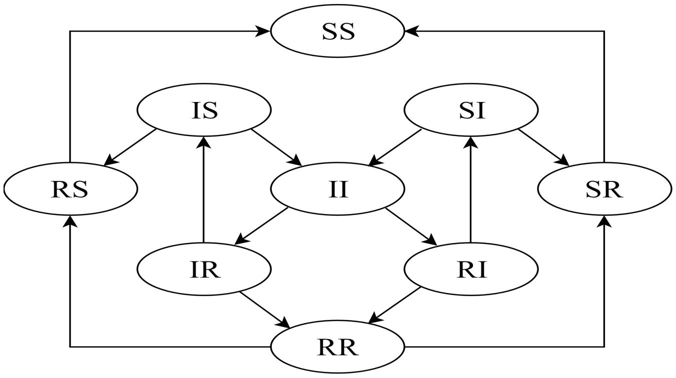

Fig. 1: Compartment transition map of the SIRS model on a 2-node undirected graph. The 9 ellipses represent all the possible compartmental membership of these two nodes. The transitions include the infection, recovering, and resusceptible processes.

or recovered (R), at each time instance. We assume that an individual is infectious once she is infected. Furthermore, it is supposed that individuals have no chance of getting infected unless at least one of her neighbors is infectious. As an illustrative example, the compartment transitions are presented in Figure 1, where a two-node connected graph is considered.

From a microscopic perspective, the information epidemics in the SIRS manner can be modeled as a Markov chain. By denoting $X_i(t)$ as the state of node i at time instance t, the state transitions of node i can be described as the following three Poisson processes:

i) The infection process (S to I). The infection process is considered as a proactive action, i.e., each infected individual j infects her susceptible social neighbors with rate $\alpha_j > 0$ [38]. To this end, we obtain the infection probability in a sufficiently short time Δt as

$$\mathsf{Prob}[X_i(t+\Delta t)=\mathrm{I}|X_i(t)=\mathrm{S}]=\sum_{j\in\mathcal{N}^{\mathrm{in}}}\alpha_j w_{ij}\delta_{X_j(t),\mathrm{I}}\Delta t,$$

where $\delta_{m,n}$ is the Kronecker delta function defined as

$$\delta_{m,n}=\begin{cases}1, & \text{if } m=n\\ 0, & \text{if } m\neq n\end{cases}$$

ii) The recovering process (I to R). The recovering process of node i is passive with transition rate $\beta_i > 0$. Thus we have the recovering probability as

$$\mathsf{Prob}[X_i(t+\Delta t)=\mathrm{R}|X_i(t)=\mathrm{I}]=\beta_i\Delta t.$$

iii) The resusceptible process (R to S). The resusceptible process of node i is passive with transition rate $\gamma_i > 0$. It yields that the resusceptible probability is calculated as

$$\mathsf{Prob}[X_i(t+\Delta t)=\mathrm{S}|X_i(t)=\mathrm{R}]=\gamma_i\Delta t.$$

Note that $\mathsf{Prob}[X_i=\mathrm{I}]=\mathsf{E}[\delta_{X_i,\mathrm{I}}]$ and $\mathsf{Prob}[X_i=\mathrm{R}]=\mathsf{E}[\delta_{X_i,\mathrm{R}}]$. It follows that the time derivative of the expectations of $\delta_{X_i,\mathrm{I}}$ and $\delta_{X_i,\mathrm{R}}$ (marginal probabilities) can be respectively attained as

$$\begin{aligned}\frac{\mathsf{dE}[\delta_{X_i,\mathrm{I}}]}{\mathsf{d}t}=&\mathsf{E}[(1-\delta_{X_i,\mathrm{I}}-\delta_{X_i,\mathrm{R}})\sum_{j=1}^{N}\alpha_j w_{ij}\delta_{X_j,\mathrm{I}}-\beta_i\delta_{X_i,\mathrm{I}}]\\ =&\sum_{j=1}^{N}\alpha_j w_{ij}\mathsf{E}[\delta_{X_j,\mathrm{I}}]-\beta_i\mathsf{E}[\delta_{X_i,\mathrm{I}}]\\ &-\sum_{j=1}^{N}\alpha_j w_{ij}(\mathsf{E}[\delta_{X_i,\mathrm{I}}\delta_{X_j,\mathrm{I}}]+\mathsf{E}[\delta_{X_i,\mathrm{R}}\delta_{X_j,\mathrm{I}}])\\ \frac{\mathsf{dE}[\delta_{X_i,\mathrm{R}}]}{\mathsf{d}t}=&\beta_i\mathsf{E}[\delta_{X_i,\mathrm{I}}]-\gamma_i\mathsf{E}[\delta_{X_i,\mathrm{R}}],\end{aligned}$$

where $\delta_{X_i,\mathrm{I}}\delta_{X_j,\mathrm{I}}$ and $\delta_{X_i,\mathrm{R}}\delta_{X_j,\mathrm{I}}$ are joint events. In the MFA, the components of these joint events are assumed to be mutually independent. Bearing in mind this assumption, the heterogeneous node-based SIRS model for node i in a directed network $\mathcal{G}=(\mathcal{V},\mathcal{E},W)$ can be described as

$$\begin{aligned}\dot{p}_i^{\mathrm{I}}(t)&=(1-p_i^{\mathrm{I}}(t)-p_i^{\mathrm{R}}(t))\sum_{j=1}^{N}\alpha_j w_{ij}p_j^{\mathrm{I}}(t)-\beta_i p_i^{\mathrm{I}}(t),\\ \dot{p}_i^{\mathrm{R}}(t)&=\beta_i p_i^{\mathrm{I}}(t)-\gamma_i p_i^{\mathrm{R}}(t),\end{aligned}\tag{1}$$

where $p_i^{\mathrm{I}}(t)$ and $p_i^{\mathrm{R}}(t)$ are approximations of $\mathsf{Prob}[X_i(t)=\mathrm{I}]$ and $\mathsf{Prob}[X_i(t)=\mathrm{R}]$, respectively.

Remark 1. Since there exist 3^N possible states of the overall network, the Markov chain model can hardly implemented for large N. By using the node-based SIRS model (1), the dimension of the system is significantly reduced to $2N$. Clearly, far less computational consumption is required in large scale networks. In addition, the approximation is accurate if the independence assumption is satisfied, in light of the central limit theorem. In conjugation with its comprehensive physical meaning, we believe it is practical and reasonable to use the model (1) to describe the information diffusion processes. Nevertheless, as a trade-off, there exists a tolerable approximation error if the independence condition does not hold [39]. Numerical experiments to compare the performance of the Markov chain model and the node-based SIRS model in digraphs are conducted in Section IV-A in detail.

Henceforth we denote $p_i=[p_i^{\mathrm{I}},p_i^{\mathrm{R}}]^{\mathsf{T}}$ which takes value in the following domain

$$\Delta\triangleq\{p_i|p_i^{\mathrm{I}}\in[0,1],p_i^{\mathrm{R}}\in[0,1],p_i^{\mathrm{I}}+p_i^{\mathrm{R}}\in[0,1]\}.$$

Now we are on the way to design optimal control rule to achieve the desired information diffusion performance.

III. Optimal Control Design

In this section, an optimal control framework is formulated based on the node-based SIRS model (1) with respect to two different scenarios.

A. Problem Formulation

To guide the information diffusion process, we introduce the word-of-mouth communication which is a common way to influence social neighbors, as is shown in Figure 2. For

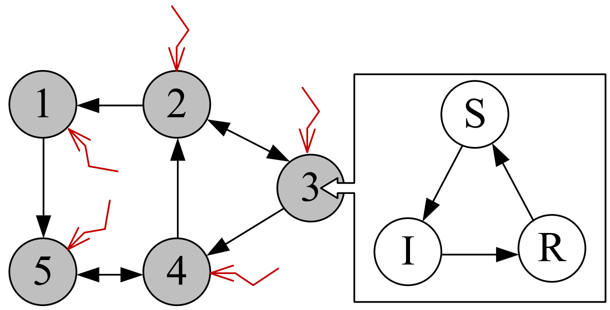

Fig. 2: Control framework for node-based SIRS model.

example, in order to enhance (or impede) the information spreading to node 1, we can influence her in-neighbor (e.g., node 2) by increasing (or decreasing) the infection rate. As a consequence, the infection probability of node 1 will be changed accordingly. This word-of-mouth way is based on the fact that the opinion and decision-making of individuals are influenced by their social neighbors. Inspired by this phenomenon, a control signal is introduced to interact with the infection rate. Note that although this kind of strategy is widely used in the control design for epidemics and information diffusion processes [28], [30], seldom works have implemented it for the node-based models. Specifically, for the node-based SIRS model (1), the controlled system can be written as

$$\dot{p}_i^{\mathrm{I}} = (1 - p_i^{\mathrm{I}} - p_i^{\mathrm{R}}) \sum_{j=1}^{N} (\alpha_j + u_j) w_{ij} p_j^{\mathrm{I}} - \beta_i p_i^{\mathrm{I}},$$
$$\dot{p}_i^{\mathrm{R}} = \beta_i p_i^{\mathrm{I}} - \gamma_i p_i^{\mathrm{R}}.$$

The stacked control input is denoted as $u(t) = [u_1(t), \ldots, u_N(t)]^\mathsf{T}$. Let $U(t) = \mathsf{diag}(u(t))$, the compact form of the controlled information epidemics reads

$$\begin{aligned} \dot{p}^{\mathrm{I}} &= (\mathcal{I} - P^{\mathrm{I}} - P^{\mathrm{R}}) W (A + U) p^{\mathrm{I}} - B p^{\mathrm{I}}, \\ \dot{p}^{\mathrm{R}} &= B p^{\mathrm{I}} - \Gamma p^{\mathrm{R}}, \end{aligned} \tag{2}$$

where $A = \mathsf{diag}(\alpha_1, \ldots, \alpha_N)$, $B = \mathsf{diag}(\beta_1, \ldots, \beta_N)$, $\Gamma = \mathsf{diag}(\gamma_1, \ldots, \gamma_N)$, $p^{\mathrm{I}} = (p_1^{\mathrm{I}}, \ldots, p_N^{\mathrm{I}})$, and $P^{\mathrm{I}} = \mathsf{diag}(p^{\mathrm{I}})$. Similarly, $p^{\mathrm{R}}(t)$ and $P^{\mathrm{R}}(t)$ are denoted for the recovered compartment. Note that we have the following admissible set for the control inputs.

$$\mathcal{U} \triangleq \{u \in \mathbb{R}^N : u_i \text{ is Lebesgue integrable}, \\ u_{\min} \le u_i \le u_{\max}, i = 1, 2, \ldots, N\},$$

where $u_{\min}$ and $u_{\max}$ are scalars. We assume that the product of $u_{\max}$ and $u_{\min}$ is nonnegative, i.e., $u_{\max} u_{\min} \ge 0$. For instance, in order to enhance the diffusion, an intuitive approach is to increase the infection rates, i.e., the bounds are both nonnegative. In this case, $u_{\min}$ is usually chosen as the worst acceptable increment of the infection rates while $u_{\max}$ is roughly estimated according to the budget. It is required that $u_{\min}(t) + \alpha_i \ge 0$ such that the underneath mechanism of the SIRS model is satisfied. Note that $u_i(t) + \alpha_i = 0$ indicates that there is no infection procedure activated by node i. Based on the system (2) and the admissible set $\mathcal{U}$, we propose two optimal control problems for information epidemics.

Situation 1: Aiming at impeding the spread of rumors, we introduce the following optimal control problem.

$$\begin{aligned} &\min_{u(t) \in \mathcal{U}} J_1 = \int_0^{t_f} r \mathbf{1}^\top p^{\mathrm{I}}(t) + u^\top(t) Q u(t) \mathsf{d}t, \\ &\text{s.t. } (2), p^{\mathrm{I}}(0) = p_0^{\mathrm{I}}, p^{\mathrm{R}}(0) = p_0^{\mathrm{R}}, p_i(t) \in \Delta, \forall i \in \mathcal{V}, \end{aligned} \tag{3}$$

where r is a positive scalar and Q is a constant positive definite diagonal matrix. p_0^{I} and p_0^{R} are the given initial conditions. The term $\mathbf{1}^\top p^{\mathrm{I}}(t)$ describes the (approximated) mathematical expectation of the number of infected people. The first item of the cost function in (3) represents the penalty corresponding to the number of individuals who believe the rumor. Note that since rumor-free is the desired performance, we only focus on the compartment I. Besides, the consumption of the control is also considered as a penalty. The consumption is regarded as the incentive for each individual so that they can act in a desired manner.

Remark 2. In immunization problems, the consumption is usually modeled as a quadratic function [30], [40]. On one hand, no consumption of the treatment can be described by a linear function in essence [41]. On the other hand, the quadratic form characterizes the severity of the side effects of the drugs [42]. Inspired by the aforementioned works, the quadratic form is adopted here, since it mimics the nonlinear increment of the required incentive to better impede the rumor propagation. The non-quadratic forms of consumption can be referred to in [33], which is covered by Situation 2.

Situation 2: Aiming at enhancing the information diffusion of campaign and marketing, we consider the optimal control problem as follows.

$$\begin{aligned} &\min_{u(t) \in \mathcal{U}} J_2 = -s_{\mathrm{I}} \mathbf{1}^\top p^{\mathrm{I}}(t_f) - s_{\mathrm{R}} \mathbf{1}^\top p^{\mathrm{R}}(t_f), \\ &\text{s.t. } (2), p^{\mathrm{I}}(0) = p_0^{\mathrm{I}}, p^{\mathrm{R}}(0) = p_0^{\mathrm{R}}, \\ &\int_0^{t_f} \sum_{i=1}^{N} b_i(u_i(t)) \mathsf{d}t \le \mathscr{B}, p_i(t) \in \Delta, \forall i \in \mathcal{V}, \end{aligned} \tag{4}$$

where s_{I} and s_{R} are positive scalars, $b_i(u_i(t))$ is the budget function, and $\mathscr{B} > 0$ is the fixed budget. The cost function in (4) only considers the terminal performance because for the political campaign nothing counts but the final number of supporters on the voting day. Furthermore, the infected and the recovered are not of the same importance for the campaigner or product manager. Thus generally there holds $s_{\mathrm{I}} > s_{\mathrm{R}} > 0$. Apart from the cost function, it is rationally assumed that in the constraint, $b_i(\cdot)$ is continuous, positive, and increasing in u_i. This is built on the fact that the more increment of the infection rate, the more budget is needed as the incentives. Since companies or campaign teams usually have limited budget, the constant $\mathscr{B}$ is introduced as the upper bound for the overtime cost. It is worth noting that one can calculate the maximum resource needed by substituting $u_{\max}$ into the budget function. In this article we only consider the case when $\mathscr{B} < \sum b_i(u_{\max})$. Since the value of $\mathscr{B}$ plays a significant role in the performance of the controlled information epidemics, its further discussion is detailed in Section IV-C.

B. Existence of the Solutions

From a practical point of view, the existence issue should be examined to ensure that an optimal control problem is solvable before attempting to calculate the solution. For Situation 1, we propose the following theorem.

Theorem 1. *Given optimal control problem* (3)*, there exist control signals in* $\mathcal{U}$ *such that the cost function is minimized.*

Theorem 1 confirms the existence of the solution to the optimal control problem in Situation 1. The proof of Theorem 1 is detailed in Appendix A by using Cesari's Theorem [43]. For Situation 2, we conclude the existence result in the following theorem.

Theorem 2. *Given optimal control problem* (4)*, there exist control signals in* $\mathcal{U}$ *such that the cost function is minimized.*

Taking into account that the optimum is unlikely to be achieved without sufficient use of the budget, we rewrite the limited budget constraint in (4) as

$$\dot{h}(t) = \sum_{i=1}^{N} b_i(u_i(t)),\ \ h(0) = 0,\ \ h(t_f) = \mathscr{B}. \tag{5}$$

However, since the convexity of $b_i(u_i(t))$ in u_i is not guaranteed, the proof of Theorem 1 is not applicable. Therefore an alternative approach based on extreme value theorem is provided in Appendix B to prove Theorem 2.

Remark 3. Theorem 1 and 2 are fundamental for the proposed control framework in this article. Firstly, they guarantee the existence of the solutions to the respective optimal control problems. Secondly, from an operational perspective, they confirm the feasibility of designing dynamic resource allocation strategies to guide information epidemics towards the desired performance. Based on these two theorems, it only remains to develop solution techniques.

C. Solutions to the Optimal Control Problems

Since the existence of the solutions is guaranteed, we are now focusing on solving the optimal control problems in (3) and (4).

Solution to (3)*:* Pontryagin's Maximum Principle is utilized here. Denote $p = (p^{\mathrm{I}}, p^{\mathrm{R}})$ and rewrite the system (2) as $\dot{p} = F(p, u)$. the Hamiltonian of the optimal control problem reads:

$$H_1(p, u, \lambda) = -r\mathbf{1}^\top p^{\mathrm{I}} - u^\top Q u + \lambda^\top F(p, u),$$

where $\lambda(t) \in \mathbb{R}^{2N}$ denotes the costate vector and the integrand in (3) is multiplied by -1 to form a maximization problem. Let $\lambda = (\lambda^{\mathrm{I}}, \lambda^{\mathrm{R}})$, where $\lambda^{\mathrm{I}}, \lambda^{\mathrm{R}} \in \mathbb{R}^N$ are the Lagrange multipliers. Then we can compute the costates equations as follows

$$\begin{aligned}
\dot{\lambda}^{\mathrm{I}}(t) &= -\frac{\partial H_1(p, u, \lambda)}{\partial p^{\mathrm{I}}(t)} \\
&= r\mathbf{1} - [(A+U)W^\top(\mathcal{I} - P^{\mathrm{R}}) - B]\lambda^{\mathrm{I}} - B\lambda^{\mathrm{R}} \\
&\quad + \Lambda^{\mathrm{I}} W(A+U)p^{\mathrm{I}} + [\Lambda^{\mathrm{I}} W(A+U)]^\top p^{\mathrm{I}}, \\
\dot{\lambda}^{\mathrm{R}}(t) &= -\frac{\partial H_1(p, u, \lambda)}{\partial p^{\mathrm{R}}(t)} = \Lambda^{\mathrm{I}} W(A+U)p^{\mathrm{I}} + \Gamma\lambda^{\mathrm{R}},
\end{aligned} \tag{6}$$

where $\Lambda^{\mathrm{I}} = \mathsf{diag}(\lambda^{\mathrm{I}})$ and $\Lambda^{\mathrm{R}} = \mathsf{diag}(\lambda^{\mathrm{R}})$. By solving $\frac{\partial H}{\partial u} = 0$ at $p = p^*, u = u^*$, and $\lambda = \lambda^*$, the optimal control rule can be expressed as

$$\begin{aligned}
u^*(t) = \frac{1}{2}Q^{-1}P^{\mathrm{I}*}(t)W^\top(I - P^{\mathrm{I}*}(t) \\
- P^{\mathrm{R}*}(t))\lambda^{\mathrm{I}*}(t), u^*(t) \in \mathcal{U},
\end{aligned} \tag{7}$$

or more specifically, for each node we have

$$\begin{aligned}
u_i^*(t) = \min\Bigg\{ \max\Bigg\{ \frac{p_i^{\mathrm{I}*}(t)}{2q_i} \sum_{j=1}^{N} w_{ji}(1 - p_j^{I*}(t) \\
- p_j^{\mathrm{R}*}(t))\lambda_j^{\mathrm{I}*}(t), u_{\min} \Bigg\}, u_{\max} \Bigg\}.
\end{aligned} \tag{8}$$

The uniqueness of the solution (8) is based on the criteria given in Appendix A and H_1 is strictly concave in u. The precise illustrations are presented in Appendix C.

According to the terminal term of the cost function, the transversality conditions read

$$\lambda^{\mathrm{I}}(t_f) = 0, \quad \lambda^{\mathrm{R}}(t_f) = 0. \tag{9}$$

Although the optimal control inputs can be analytically presented as (8), it cannot be directly calculated because $p^*(t)$ and $\lambda^*(t)$ are unknown beforehand. To tackle this issue, the shooting method is used in [30]. However, in that case, the arbitrary initial condition of a scalar costate is hard to choose, let alone the situation in equation (6) with $2N$-dimension costate vector. Thus we introduce a modified forward-backward sweep method (FBSM) as follows such that it can be used in this N-dimension optimal control problem.

Algorithm 1 Forward-backward sweep method

1: Input: p_0^{I}, p_0^{R}, initial guess $\mathbf{u} = [u(0), \ldots, u(end)]$, tolerance ϵ.
2: **for** $k = 0 : 1 : end$ **do**
3: $\quad p^{\mathrm{I}}(k+1) \leftarrow p^{\mathrm{I}}(k) + \Delta T \dot{p}^{\mathrm{I}}(k)$
4: $\quad p^{\mathrm{R}}(k+1) \leftarrow p^{\mathrm{R}}(k) + \Delta T \dot{p}^{\mathrm{R}}(k)$
5: **end for**
6: **for** $k = end : -1 : 2$ **do**
7: $\quad \lambda^{\mathrm{I}}(k-1) \leftarrow \lambda^{\mathrm{I}}(k) - \Delta T \dot{\lambda}^{\mathrm{I}}(k-1)$
8: $\quad \lambda^{\mathrm{R}}(k-1) \leftarrow \lambda^{\mathrm{R}}(k) - \Delta T \dot{\lambda}^{\mathrm{R}}(k-1)$.
9: **end for**
10: Compute $\hat{\mathbf{u}}$ according to (8).
11: **if** $||\mathbf{u} - \hat{\mathbf{u}}||_2 > \epsilon$ **then**
12: $\quad \mathbf{u} \leftarrow \hat{\mathbf{u}}$
13: $\quad$ **goto** line 2
14: **else**
15: $\quad$ output $\mathbf{u}^* = \hat{\mathbf{u}}$.
16: **end if**

In the FBSM above, Euler method is used such that the continuous model is discretized with sampling period ΔT. The convergence and further properties of FBSM can be referred to in [44]. In the node-based SIRS model on an N-node graph, the information diffusion process is characterized by $2N$ dimensional differential equations. In Algorithm 1, we have to numerically solve $4N$ differential equations in each iteration to calculate $\hat{\mathbf{u}}$. In conjugation of the convergence of the algorithm, our algorithm terminates in finite iterations. Therefore, the complexity is $\mathcal{O}(N)$.

Solution to (4)*:* The Hamiltonian here is written as

$$H_2(p,u,\sigma) = (\sigma^{\mathrm{I}})^\top[(\mathcal{I} - P^{\mathrm{I}} - P^{\mathrm{R}})W(A+U)p^{\mathrm{I}} - Bp^{\mathrm{I}}] + (\sigma^{\mathrm{R}})^\top(Bp^{\mathrm{I}} - \Gamma p^{\mathrm{R}}) + \sigma_h \sum_{i=1}^{N} b_i(u_i),$$

where $\sigma^{\mathrm{I}} \in \mathbb{R}^N$, $\sigma^{\mathrm{R}} \in \mathbb{R}^N$, and $\sigma_h \in \mathbb{R}$ are the Lagrange multipliers and $\sigma := (\sigma^{\mathrm{I}}, \sigma^{\mathrm{R}}, \sigma_h)$. The costates dynamics of σ^{I} and σ^{R} are similar to those in (6) while

$$\dot{\sigma}_h = -\frac{\partial H_2}{\partial h} = 0,$$

which infers that σ_h is a constant scalar but unknown. If the budget function is chosen as quadratic form as that in (3), by using similar techniques to obtain (7), we attain the control law as

$$u^*(t) = \frac{1}{2\sigma_h}Q^{-1}P^{\mathrm{I}*}(t)W^\top(I - P^{\mathrm{I}*}(t) - P^{\mathrm{R}*}(t))\sigma^{\mathrm{I}*}(t), u^*(t) \in \mathcal{U}.$$

Thus to obtain the value of σ_h becomes a natural idea to solve the problem in (4). An approach combining the FBSM and the secant method has been reported to be implemented to solve a similar problem with low dimension in [45]. However, to propose an initial guess which leads to a convergent solution is technically hard, let alone on networks of far larger scale. One alternative method is proposed in [40] where the value of σ_h is obtained by trial-and-error. To deal with this problem, the Matlab function fmincon is utilized in this article. The detailed configurations and further discussions are presented in Section IV. Note that in this case the solution to optimal control problem (4) may not be unique since the solution is highly related to the property of $b_i(u_i)$.

IV. Numerical Experiments

In this section, several numerical simulations are conducted to show 1) the approximation performance of the deterministic node-based SIRS model in (1) corresponding to the 3^N-state Markov chain model, 2) the effectiveness of the optimal controllers designed in the previous section, and 3) the influence of the parameters towards the controlled information diffusion process.

A. Comparison between the node-based SIRS model and the 3^N-state Markov chain model

In this subsection we focus on the detailed comparisons between the node-based SIRS model and the 3^N-state Markov chain model over strongly connected directed networks.

The simulations are conducted on scale-free networks with homogeneous transition rates. We generate two strongly connected digraphs with 298 nodes. The first graph, denoted as $\mathcal{G}_1$, has 150 edges on average for each node. Whereas, the average number of edges is 50 for the second graph $\mathcal{G}_2$. The values of the transition rates α, β, and γ are all limited in the set $\{0.1, 1\}$ such that the results can cover major range of the coefficients. Specifically, we choose two sets of configurations: 1) $\alpha = 1$, $\beta = 0.1$, and $\gamma = 0.1$, and 2) $\alpha = 0.1$, $\beta = 1$, and $\gamma = 0.1$. Note that by choosing these two kinds of parameters, the SIRS model reaches the endemic (non-trivial) equilibrium and the disease-free (trivial) equilibrium, respectively. Thus the node-based SIRS model and the 3^N-state Markov chain model are compared in these representative scenarios. The sampling period and the terminal time are set as $\Delta T = 0.01$ and $t_f = 30$, respectively.

Apart from the node-based SIRS model, another challenge is to simulate the Markov chain with large N. Since direct simulations cannot be expected due to the massive number of states, we resort to Monte Carlo simulations (MCS) [46]. By using the foregoing parameters and initial conditions, we conduct MCS 50000 times to show the performance of the 3^N-state Markov chain model. In order to show the difference between the node-based SIRS model and the 3^N-state Markov chain model clearly, we utilize the average of $p^{\mathrm{I}}(t)$, $p^{\mathrm{R}}(t)$, and the infection and recovering probability obtained by the MCS. These indices are respectively denoted as $\bar{p}^{\mathrm{I}}(t)$, $\bar{p}^{\mathrm{R}}(t)$, $\bar{p}^{\mathrm{I}}_{MCS}(t)$, and $\bar{p}^{\mathrm{R}}_{MCS}(t)$. The initial compartments of the nodes are randomly chosen such that the probability of being susceptible, infected, and recovered are $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{2}$, respectively.

The comparison results are presented in Figure 3 and Figure 4 with respect to the aforementioned two categories of configurations. In Figure 3, all the trajectories approach the endemic equilibrium. It is clear that the approximation is accurate on graph $\mathcal{G}_1$. However, there exist evident discrepancies between the results of MCS and the node-based SIRS model on graph $\mathcal{G}_2$. Specifically, at the terminal time, the approximation errors are $|\bar{p}^{\mathrm{I}}(t_f) - \bar{p}^{\mathrm{I}}_{MCS}(t_f)| = 0.0041$ and $|\bar{p}^{\mathrm{R}}(t_f) - \bar{p}^{\mathrm{R}}_{MCS}(t_f)| = 0.0038$ on graph $\mathcal{G}_1$. While on graph $\mathcal{G}_2$, the errors are 0.0358 and 0.0314, respectively. In Figure 4, all the trajectories approach the disease-free equilibrium. Apparently, the approximation in this case is accurate on both graphs. Any of the approximation errors is less than 10^{-4}. Thus we observe that 1) The approximation is more accurate in densely connected communities; 2) the approximation behaves better when the node-based SIRS model converges to the disease-free equilibrium; and 3) the accuracy of the approximation is acceptable. These statements are consistent with the previous work [8], [47] regarding the performance of the MFA applying to epidemic models.

B. Performance of the Optimal Control Law

The performance of the node-based SIRS model under optimal control (3) and (4) are examined to show the effectiveness of the designed control strategy.

We implement the optimal control law (8) on $\mathcal{G}_1$. The transition rates α_i, β_i, and γ_i are randomly chosen in the intervals $(0.55, 0.65)$, $(0.15, 0.25)$, and $(0.3, 0.4)$, respectively. This guarantees the heterogeneity of the SIRS model. The initial conditions are set to be identical to the ones in Section IV-A. Since the steady states of the diffusion process in fixed graphs are highly dependent on the transition rates, optimization rather than optimal control is a more direct way

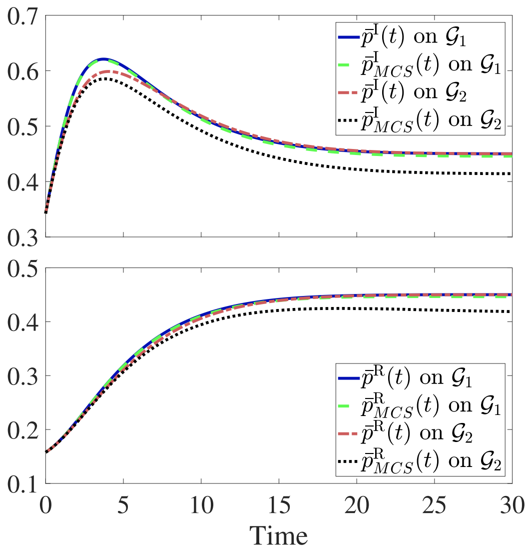

Fig. 3: Comparison between the node-based SIRS model and the 3^N-state Markov chain model on $\mathcal{G}_1$ and $\mathcal{G}_2$ with the coefficients $\alpha = 1$, $\beta = 0.1$, and $\gamma = 0.1$.

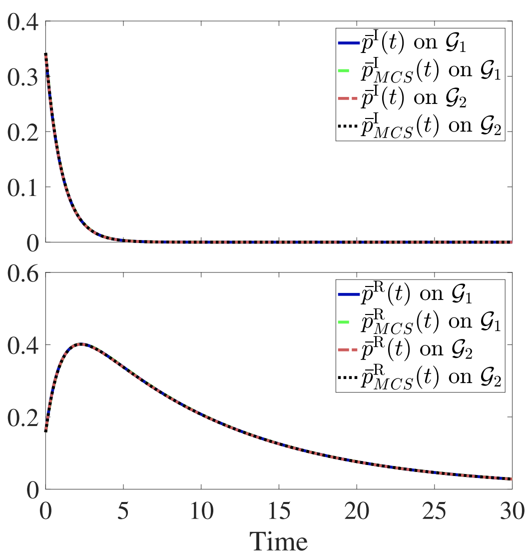

Fig. 4: Comparison between the node-based SIRS model and the 3^N-state Markov chain model on $\mathcal{G}_1$ and $\mathcal{G}_2$ with the coefficients $\alpha = 0.1$, $\beta = 1$, and $\gamma = 0.1$.

to reach the desired performance. In that regard, we mainly focus on the transient states but not the steady states of the information epidemics. Thus in this subsection we set the terminal time as $t_f = 6$ and the sampling period as $\Delta T = 0.1$.

For Situation 1, the weight r is set to 5. Note that the weighting matrix Q is chosen to be diagonal and its entries read

$$q_i = q|\mathcal{N}_i^{\mathrm{in}}|, \forall i \in \mathcal{V},$$

where q is a scalar set as 0.1 in this case and $|\mathcal{N}_i^{\mathrm{in}}|$ is the cardinality of the in-neighborhood set of node i. The weighting matrix Q mimics the fact that the more individuals

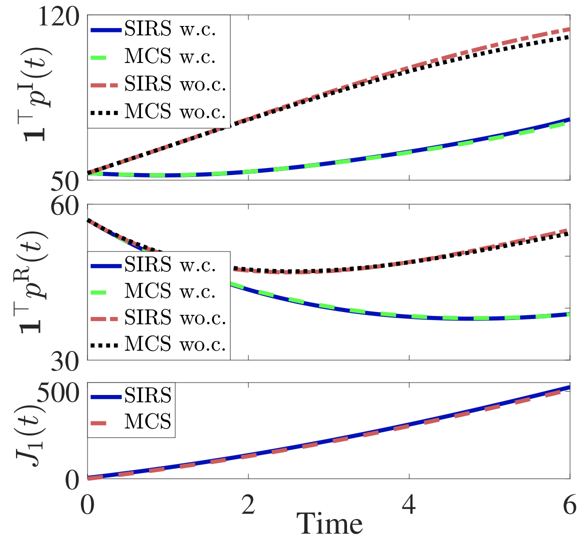

Fig. 5: The performance of the node-based SIRS model and the Markov chain model with optimal control (3) and without control. The subfigures shows the value of $\mathbf{1}^\top p^{\mathrm{I}}(t)$, $\mathbf{1}^\top p^{\mathrm{R}}(t)$, and $J_1(t)$, respectively. The abbreviations w.c. and wo.c. stand for with control and without control, respectively.

one can influence, the more resources she deserves to obtain. By doing this, rumors are less possible to spread via her social connections. Thanks to the strong connectivity of the network, one has $q_i > 0$, i.e., Q is positive definite. The bounds of inputs are set as $[-0.35, 0]$.

Figure 5 shows the performance of optimal control in (3) by comparing with the cases without control. In addition, the optimal control rule is implemented to the Markov chain model by using MCS (50000 times). It is clear that the rumor propagation is impeded since both the trajectories of $\mathbf{1}^\mathsf{T} p^{\mathrm{I}}$ and $\mathbf{1}^\mathsf{T} p^{\mathrm{I}}_{MCS}$ drop significantly within the terminal time. Note that no big discrepancies exist between the controlled node-based SIRS model and Markov chain model. It yields that the proposed control rule is applicable in the real information epidemics. To show the control implemented to the specific individuals, two typical nodes labeled 7 and 292 are selected. These two nodes possess (one of) the most and least number of in-neighbors, respectively. As is presented in Figure 6, the initial condition $p^{\mathrm{I}}_7(0) = 0$ and $p^{\mathrm{I}}_{292}(0) = 1$. Under the optimal control, p^{I}_{292} decreased vastly and the increment of p^{I}_7 is retarded. It is notable to point out that during most of the controlled period, u_{292} stays on the boundary while u_7 is only in a low level. This phenomenon is due to the fact that the consumption is linear in the number of in-neighbors. Since node 7 possesses more in-neighbors, more consumption is needed for implementing the control law; and vice versa for node 292. The terminal value of u_i is zero based on the control law in (8) and the transversality condition of λ^{I} in (9).

For Situation 2, the performance of optimal control to enhance the information spreading with a limited budget is presented in Figure 7. In addition, the optimal control rule is

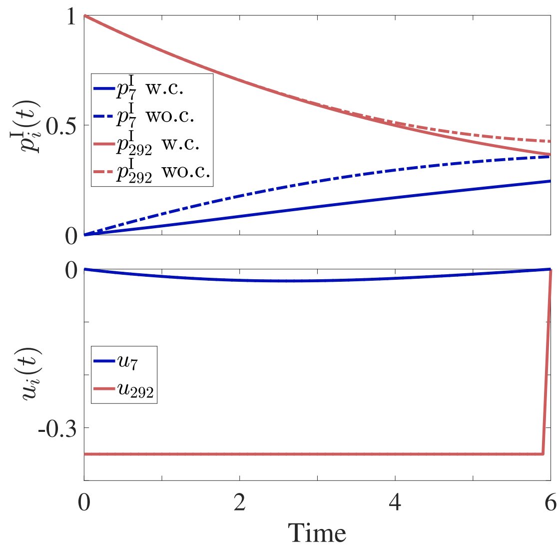

Fig. 6: Selected p_i^{I} of nodes labeled 7 and 292 as well as their control inputs. The abbreviations w.c. and wo.c. stand for with control and without control, respectively.

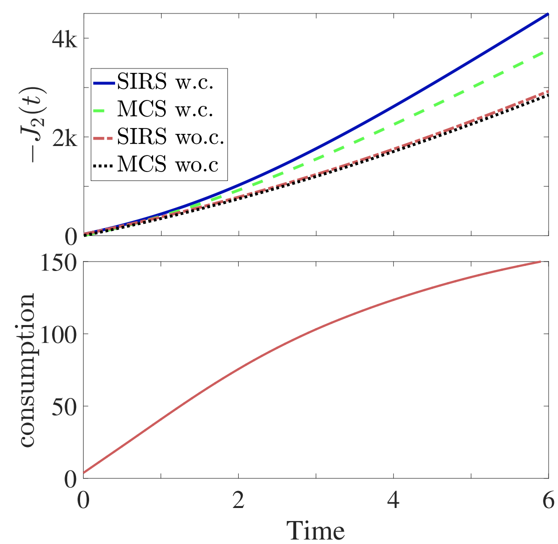

Fig. 7: The performance of the node-based SIRS model and the Markov chain model with optimal control (4) and without control. The subfigures show the value of $-J_2(t)$ and consumption, respectively. The abbreviations w.c. and wo.c. stand for with control and without control, respectively.

implemented to the Markov chain model by conducting MCS 50000 times. The control inputs are now limited in $[0, 1]$, i.e., $u_{\min} = 0$ and $u_{\max} = 1$. The initial conditions are the same with the ones in Section IV-A. The budget $\mathscr{B}$ is chosen as 150. For simplicity, the budget function is still in quadratic form, i.e., $b_i(u(t)) = q_i u_i^2(t)$. Besides, the weights s_{I} and s_{R} are set as 5 and 1, respectively. The solution to (4) is obtained by fmincon, where we choose the initial guess of all the input as $u_{\max}$ and the sqp algorithm is used. As is manifested in Figure 7, the optimal control can increase the diffusion extraordinarily and the budget is adequately utilized during the time interval. Notice that there exists small discrepancy in the value of $-J_2(t)$ between the controlled SIRS model and the Markov chain model. The reason may be the inputs for some nodes are quite small. This tiny increment of the infection rate in the node-based SIRS model is more effective than in the MCS. But it is evident that the proposed control rule does enhance the information diffusion. Detailed information of the two nodes with the most and the least in-neighbors are presented in Figure 8. The contributions to $-J_2$ of these two nodes are remarkably increased at t_f. The control input u_{292} stays at the upper bound most of the time while u_7 stays at a low level. According to the input in Figure 8, it is apparent that the budget is not sufficient enough to support the maximal resources allocated to each individual. It is also worth noting that solving (4) on $\mathcal{G}_1$ by fmincon is slow. Future work should focus on developing fast algorithms for information epidemics on large scale networks.

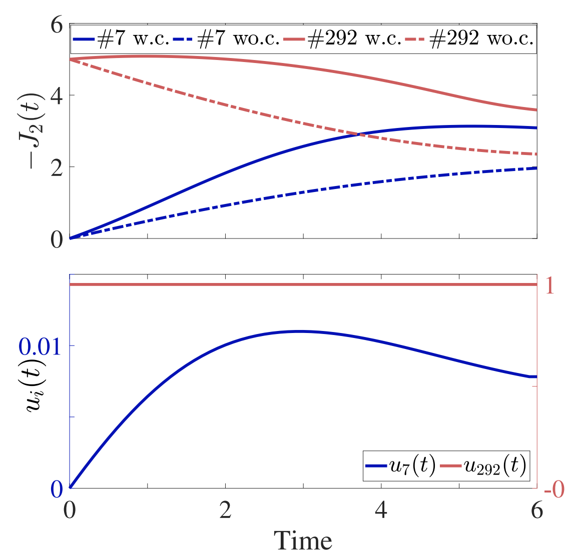

Fig. 8: Selected $-J_2(t)$ of nodes labeled 7 (#7) and 292 (#292) as well as their control inputs. The abbreviations w.c. and wo.c. stand for with control and without control, respectively.

C. Influence of the Parameters

The solutions to the optimal control problems (3) and (4) are inevitably influenced by the parameters therein. The impacts of the main parameters are discussed in this subsection. An E-R random graph of 30 nodes with connectivity probability

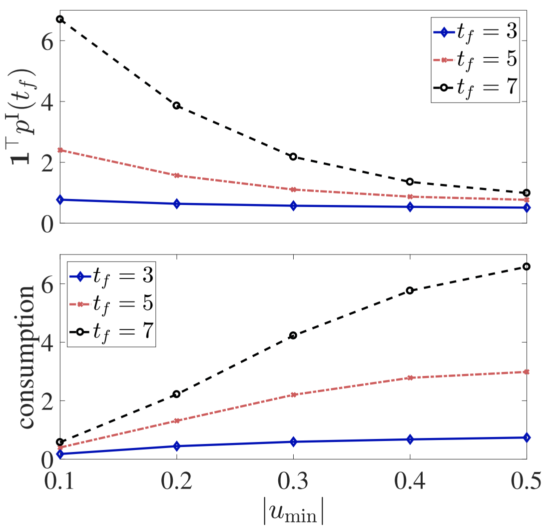

Fig. 9: Influence of $u_{\min}$ and t_f in (3).

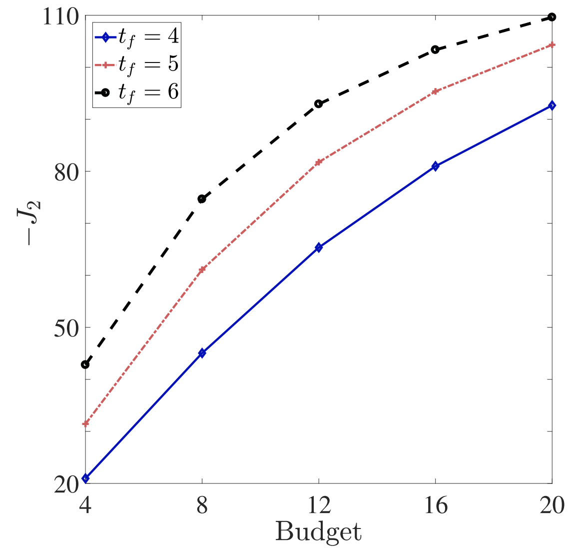

Fig. 10: Influence of $u_{\max}$ and t_f in (4).

0.1 is generalized. The initial conditions are randomly chosen in the interval $(0, 0.01)$. The transition rates in Section IV-B are adopted with slight modification.

In Situation 1, the boundary of the input and the terminal time play significant role to impede the rumor spreading. The comparisons among different configurations of $u_{\min}$ and t_f are studied. In this case, α_i is within the interval $(0.75, 0.85)$ and $u_{\max}$ is set to 0. The simulations are conducted under the same initial conditions and network topology while $u_{\min}$ changes from -0.1 to -0.5 with step 0.1 and t_f increases from 3 to 7 with step 2. As is shown in Figure 9, we can conclude that generally with shorter terminal time and larger input bound, better performance can be achieved to impede rumor spreading. Specifically, with the same terminal time, the lager $|u_{\min}|$ is, the less people tend to believe the rumor. However, the decrement of the infected is retarded. From the view of time limit, with identical $u_{\min}$, the longer time we plan to impede the dissemination, the more widely the rumor spreads. The reason underneath is that the rumor also spreads meanwhile we take the control action. Note that for the case when $u_{\min} = -0.5$, the approximated numbers of infected nodes at the terminal time are very similar for different t_f, which infers that for certain long period of time, the bound of input plays the dominant role in the performance of the system. Thus to impede the spread of rumors, we should wisely make decisions on the resources we can allocate and start as early as possible.

In Situation 2, the role of the budget as well as terminal time are selected for further inspection. In this case, α_i is within the interval $(0.25, 0.35)$ and $u_{\min}$ is set to 0. The simulations are conducted via changing $\mathscr{B}$ from 4 to 20 with step 4 and t_f from 4 to 6 with step 1. In Figure 10, we show the corresponding variation of $-J_2$ under different configurations. It yields that the more budget we have, the better diffusion we can obtain, which also means that more people would buy the product or vote for the desired campaigner. Akin to the result on impeding rumors, for identical budget, more people turn to be infected in longer diffusion time. However, the effect of time expansion is weakened as time interval increases because the limited sources are distributed for longer periods. To conclude, the budget is the dominant factor in marketing or campaigning in a short period of time.

V. Conclusion

Focusing on information diffusion processes on social networks, a heterogeneous node-based SIRS model is introduced in this article. An optimal control framework based on interacting with the infection rate is proposed, following which two scenarios, i.e., to impede rumor spreading and to enhance the diffusion in marketing or campaign, are separately described. The solutions to the optimal control problems are proved to exist and obtained by Pontryagin Maximum Principle. A modified forward-backward sweep algorithm and fmincon are utilized to obtain the solution numerically. Several simulations are conducted to show the performance of the optimal control law, as well as the effectiveness of the SIRS model as an approximation of the Markov chain model. By comparing the performance of the system under different configurations, we conclude that 1) it is effective and critical to start to impede the rumor spreading as early as possible and 2) enough budget is the key factor to enhance the diffusion in a short period of time.

Future works on this topic, which are still open and promising, lie on establishing diffusion models with uncertainties, designing decentralized control strategies, and developing fast algorithms for large scale networks.

Appendix

A. Proof of Theorem 1

We first show the properties of the node-based SIRS model in the following two lemmas, which are necessary for the proof of the existence of the solution to (3).

Lemma 1. *Consider the system in* (1). *If* $p_i(0) \in \Delta$, *then* $p_i(t) \in \Delta$, $\forall t > 0$.

Proof. Assume that for a time instance $\tau \geq 0$ there exists $p_i(\tau) \in \Delta$. We have the following three cases: 1) if $p_i^{\mathrm{I}}(\tau) = 0$, then $\dot{p}_i^{\mathrm{I}}(\tau) \geq 0$, $\forall p_i^{\mathrm{R}}(\tau) \in [0,1]$, 2) if $p_i^{\mathrm{R}}(\tau) = 0$, then $\dot{p}_i^{\mathrm{R}}(\tau) \geq 0$, $\forall p_i^{\mathrm{I}}(\tau) \in [0,1]$, and 3) if $p_i^{\mathrm{I}}(\tau) + p_i^{\mathrm{R}}(\tau) = 1$, then $\dot{p}_i^{\mathrm{I}}(\tau) + \dot{p}_i^{\mathrm{R}}(\tau) \leq 0$. From 1) and 2), we have $p_i^{\mathrm{I}}, p_i^{\mathrm{R}} \geq 0$ and in conjugation with 3), we have $p_i^{\mathrm{I}} + p_i^{\mathrm{R}} \leq 1$, which is equivalent to $p_i(t) \in \Delta$, $\forall t > 0$, if $p_i(0) \in \Delta$. $\square$

Lemma 2. *The node-based SIRS model* (2) *is globally Lipschitz continuous in* $p(t)$, *where* $p(t) = (p^{\mathrm{I}}, p^{\mathrm{R}})$.

Proof. The system in (2) is denoted as $\dot{p}(t) = F(p(t), u(t))$ for simplicity. Since $u(t)$ is a function of t, we can directly consider the Lipschitz continuity of $F(p,t)$ in p. Let $\hat{p} := (\hat{p}^{\mathrm{I}}, \hat{p}^{\mathrm{R}})$ satisfy (2). Then we use 1-norm to prove the Lipschitz continuity.

$$
\begin{aligned}
\|F(p,t) - F(\hat{p},t)\|_1 &= \|\dot{p}^{\mathrm{I}} - \dot{\hat{p}}^{\mathrm{I}}\|_1 + \|\dot{p}^{\mathrm{R}} - \dot{\hat{p}}^{\mathrm{R}}\|_1 \\
&\leq N w_{\max}(\alpha_{\max} + u_{\max})\|p^{\mathrm{I}} - \hat{p}^{\mathrm{I}}\|_1 \\
&+ w_{\max}(\alpha_{\max} + u_{\max}) \sum_{i=1}^{N}\sum_{j=1}^{N} |p_i^{\mathrm{I}} p_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}}| \\
&+ w_{\max}(\alpha_{\max} + u_{\max}) \sum_{i=1}^{N}\sum_{j=1}^{N} |p_i^{\mathrm{R}} p_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{R}} \hat{p}_j^{\mathrm{I}}| \\
&+ 2\beta_{\max}\|p^{\mathrm{I}} - \hat{p}^{\mathrm{I}}\|_1 + \gamma_{\max}\|p^{\mathrm{R}} - \hat{p}^{\mathrm{R}}\|_1,
\end{aligned} \tag{10}
$$

where $w_{\max} = \max_{i,j} w_{ij}$, $\alpha_{\max} = \max_i \alpha_i$, $\beta_{\max} = \max_i \beta_i$, and $\gamma_{\max} = \max_i \gamma_i$. By rewriting $p_i^{\mathrm{I}} p_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}}$ as $p_i^{\mathrm{I}} p_j^{\mathrm{I}} - p_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}} + p_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}}$, one can have

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N} |p_i^{\mathrm{I}} p_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{I}} \hat{p}_j^{\mathrm{I}}| \\
&\leq \sum_{i=1}^{N} |p_i^{\mathrm{I}}| \sum_{j=1}^{N} |p_j^{\mathrm{I}} - \hat{p}_j^{\mathrm{I}}| + \sum_{j=1}^{N} |\hat{p}_j^{\mathrm{I}}| \sum_{i=1}^{N} |p_i^{\mathrm{I}} - \hat{p}_i^{\mathrm{I}}| \\
&\leq 2N\|p^{\mathrm{I}} - \hat{p}^{\mathrm{I}}\|_1.
\end{aligned} \tag{11}
$$

and similarly there holds

$$
\sum_{i=1}^{N}\sum_{j=1}^{N} |p_i^{\mathrm{R}} p_j^{\mathrm{I}} - \hat{p}_i^{\mathrm{R}} \hat{p}_j^{\mathrm{I}}| \leq N(\|p^{\mathrm{I}} - \hat{p}^{\mathrm{I}}\|_1 + \|p^{\mathrm{R}} - \hat{p}^{\mathrm{R}}\|_1). \tag{12}
$$

By Substituting (11) and (12) into (10), it follows that

$$
\|F(p,t) - F(\hat{p},t)\|_1 \leq L\|p - \hat{p}\|_1,
$$

where $L = \max\{4N w_{\max}(\alpha_{\max} + u_{\max}) + 2\beta_{\max}, N w_{\max} (\alpha_{\max} + u_{\max}) + \gamma_{\max}\} > 0$ is the Lipschitz constant. $\square$

Based on Lemma 1 and 2, we provide the following proof for the existence of the solution to (3) by examining the conditions given by Cesari's Theorem in [43].

i) The admissible input set $\mathcal{U}$ and the set of solutions to Cauchy problem, i.e. $\dot{p} = F(p,u)$, $p^{\mathrm{I}}(0) = p_0^{\mathrm{I}}$, $p^{\mathrm{R}}(0) = p_0^{\mathrm{R}}$, is apparently non-empty since $F(p,u)$ is Lipschitz in p [48, Theorem 3.2].
ii) We prove that $F(p,u)$ is bounded by $\hat{C}_1(1 + \|p\| + \|u\|)$, where $\hat{C}_1$ is a constant. Since u is bounded, we only need to show there exists a constant C_1 such that $C_1(1 + \|p\|)$ upper bounds $F(p,u)$. Note that

$$
\begin{aligned}
&\|F(p,u)\|_1 \\
&\leq (\alpha_{\max} + u_{\max}) \sum_{i=1}^{N}\sum_{j=1}^{N} (1 - p_i^{\mathrm{I}} - p_i^{\mathrm{R}})|w_{ij} p_j^{\mathrm{I}}| \\
&+ \sum_{i=1}^{N} |\beta_i p_i^{\mathrm{I}}| + \sum_{i=1}^{N} |\beta_i p_i^{\mathrm{I}} - \gamma_i p_i^{\mathrm{R}}| \\
&\leq N(w_{\max}(\alpha_{\max} + u_{\max}) + 2\beta_{\max})\|p^{\mathrm{I}}\|_1 \\
&+ \gamma_{\max}\|p^{\mathrm{R}}\|_1.
\end{aligned}
$$

Generally, since $N \gg \gamma_{\max}$, we choose $C_1 = N(w_{\max}(\alpha_{\max} + u_{\max}) + 2\beta_{\max})$ such that $\|F(p,u)\|_1 \leq C_1(1 + \|p\|_1)$.
iii) $F(p,u)$ is linear in u and the integrand satisfies the inequality $r\mathbf{1}^\top p^{\mathrm{I}} + u^\top Q u \geq C_2\|u\|_2^{C_3} - C_4$ with constants C_2, C_3, and C_4. It is required that $C_2 > 0$, $C_3 > 1$, which can be fulfilled by choosing $C_2 = \lambda_{\min}\{Q\}$, $C_3 = 2$ and $C_4 = 0$.

Thus all the conditions of Cesari's Theorem are satisfied, which infers the existence of the solution to the problem described as (3).

B. Proof of Theorem 2

Based on the extreme value theorem [49, Theorem 4.16], we prove the existence of the solution to (4). First we show that the solution of the node-based SIRS model (1) is continuous. From Lemma 2, the model (2) is Lipschitz continuous. Moreover, $F(p(t), u(t))$ is obviously bounded. To attain the continuous dependence on parameter $u(t)$, the following proposition needs to be validated.

Proposition 1. *Given* $\|u - \hat{u}\| < \delta$, $\delta > 0$, *there exists* $\mu > 0$ *such that* $\|F(p,u) - F(p,\hat{u})\| < \mu$.

Proof. 1-norm is utilized here to prove the proposition. By direct calculation, we have

$$
\begin{aligned}
&\|F(p,u) - F(p,\hat{u})\|_1 \\
&= \sum_{i=1}^{N} \left|(1 - p_i^{\mathrm{I}} - p_i^{\mathrm{R}}) \sum_{j=1}^{N} (u_j - \hat{u}_j) w_{ij} p_j^{\mathrm{I}}\right| \\
&\leq \sum\nolimits_{i=1}^{N} \left|1 - p_i^{\mathrm{I}} - p_i^{\mathrm{R}}\right| w_{\max} \sum_{j=1}^{N} |u_j - \hat{u}_j| \\
&\leq N w_{\max}\|u - \hat{u}\|_1
\end{aligned}
$$

Let $\mu = N w_{\max}\delta$, the result in the proposition can be obtained. $\square$

Now we can come to the result that $F(p,u)$ is continuous in u. Define the following set for the constraint

$$
\mathcal{S} = \{u : \int_0^{t_f} \sum_{i=1}^{N} b_i(u_i(\tau))\mathsf{d}\tau = \mathscr{B}, u_{\min} \leq u_i(t) \leq u_{\max}\},
$$

which is compact. Along with the compact set $\mathcal{U}$, the product $\mathcal{S} \times \mathcal{U}$ is also compact. Since the conditions of [49, Theorem 4.16] are all satisfied, there exists a solution to (4).

C. Proof of the uniqueness of the solution to (3)

This proof follows the idea of the proof in [43, Theorem 6.2]. If the optimal control problem (3) is solved by (8), it is evident that p^* and λ^* are continuous on interval $[t_0, t_f]$. Hence, they are bounded therein. In conjugation with ii) in Appendix A, there exists some constant C_5 such that the following inequality holds

$$|\lambda^\top F(p,u)| \leq \|\lambda\|\hat{C}_1(1 + \|p\| + \|u\|) \leq C_5(1 + \|u\|).$$

Bearing in mind that iii) in Appendix A holds, it yields that

$$-H_1 = r\mathbf{1}^\top p^{\mathrm{I}} + u^\top Qu - \lambda^\top F(p,u) \geq C_2\|u\|^2 - C_5(1 + \|u\|).$$

Consequently, there holds $-\|u\|^{-1}H_1 \to \infty$ as $\|u\| \to \infty$. Besides, since Q is positive definite, it implies H_1 is strictly concave in u. Based on the proof in [43, Theorem 6.2], H_1 reaches its maximum at a unique u^* on $\mathcal{U}$ for every $t \in [t_0, t_f]$.

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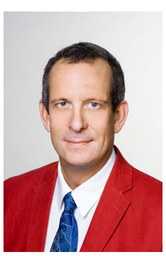

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