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Master integrals of a planar double-box family for top-quark pair production

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ABSTRACT

We calculate analytically the master integrals of a planar double-box family for top-quark pair production using the method of differential equations. With a proper choice of the bases, the differential equations can be transformed to the d-log form. The square roots of the kinematic variables in the differential equations can be rationalized by defining two dimensionless variables. We find that all the boundary conditions can be fully fixed either by simple integrals or regularity conditions at some special kinematic points. The analytic results for thirty-three master integrals at general kinematics are all expressed in terms of multiple polylogarithms up to transcendental weight four.

1. Introduction

The top-quark pair production is one of the most important processes at a hadron collider, such as the LHC. It has a very large production rate and the decay of the top quarks gives rise to several jets or leptons, which can be considered as an important background in the search of new physics. Moreover, this process can also be used to determine the top-quark mass, the strong coupling constant α_s and the gluon parton distribution functions. As such, it is important to have a precise understanding of this process. So far, the LHC experiment has accumulated a large number of data at 13 TeV. The cross section of the top-quark pair production has been measured with a precision comparable to the most precise theoretical predictions [1,2].

The total cross sections and differential distributions of the top-quark pair production have been calculated up to next-to-next-to-leading order (NNLO) [3–7]. As a part of the calculation, the two-loop virtual corrections have been evaluated numerically [8,9]. However, the analytic results of the two-loop diagrams are still valuable in order to provide a fast and stable evaluation of the virtual corrections and to understand the structure of massive loop integrals, which are usually much more complicated than the massless ones. Some analytic results have already been obtained in refs. [10–13].

In the calculation of the two-loop Feynman diagrams, all the integrals can be reduced to a set of master integrals, e.g. using integration-by-parts (IBP) identities. The master integrals for the top-quark pair production have been widely studied. In ref. [14], the authors calculated the master integrals for the light fermionic two-loop QCD corrections to top-quark pair production in the gluon fusion channel. In refs. [15,16], the master integrals for the NNLO QED corrections to μe scattering have been obtained. Some of these master integrals are also applicable to top-quark pair production. Recently, the planar double-box integrals for top-quark pair production with a closed top-quark loop have been calculated [17,18], of which the results contain elliptic integrals. The method of differential equations [19,20] has played an important role in the above computations.

We have examined all the Feynman integrals relevant to the NNLO corrections to top-quark pair production, and found six planar and seven non-planar double-box integral families remaining to be calculated analytically. We have shown the corresponding planar and non-planar double-box integrals in Fig. 1 and Fig. 2, respectively.

In this paper, we calculate one of the analytically unknown planar double-box integrals, i.e., $P1$ in Fig. 1, for top-quark pair production. We show its topology individually in Fig. 3. It turns out that the differential equations for all the master integrals in this family can be transformed to the d-log form after choosing a proper basis. In addition, the square roots in the logarithms can be rationalized by defining new dimensionless variables. As a con-

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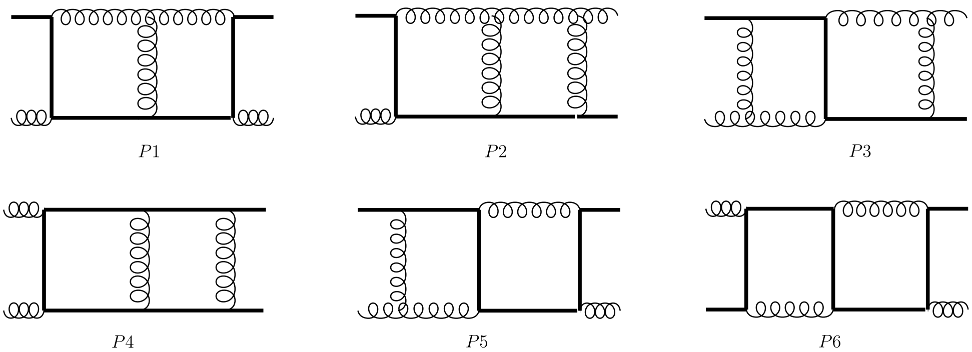

Fig. 1. Analytically unknown planar double-box integrals for top-quark pair production.

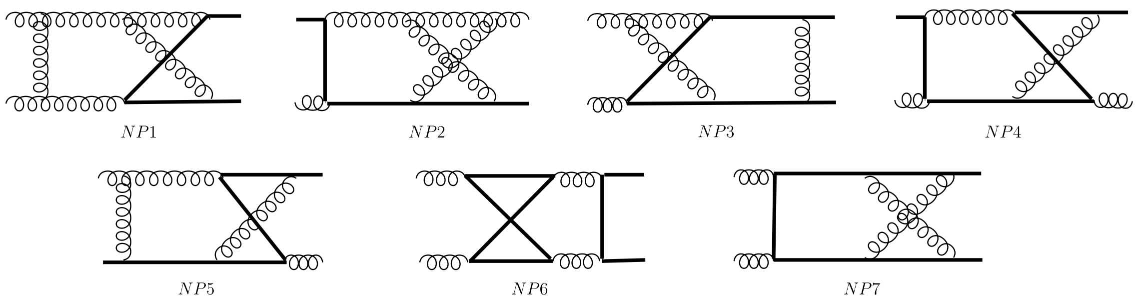

Fig. 2. Analytically unknown non-planar double-box integrals for top-quark pair production.

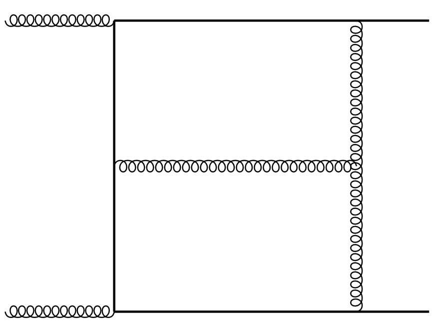

Fig. 3. A planar double-box Feynman diagram for top-quark pair production in gluon-gluon fusion (corresponding to $g(k_1)g(k_2) \to t(k_3)\bar{t}(k_4)$ or $g(k_1)g(k_2) \to t(k_4)\bar{t}(k_3)$).

sequence, all the master integrals in this family can be written in terms of multiple polylogarithms. Note that the integral family we calculate does not contribute to the leading color results in [12]. The other analytically unknown planar double-box integrals involve elliptic functions and will be discussed elsewhere.

The rest of this paper is organized as follows. In section 2 we present the canonical basis of the integral family and their corresponding differential equations in the d-log form. We discuss the determination of boundary conditions in section 3. Conclusions are given in section 4. The analytic results as well as the rational matrices are provided in ancillary files along with this paper.

2. Canonical basis and differential equations

As shown in Fig. 3, the planar double-box Feynman integrals in the family we consider can be formulated as

$$I_{n_1,n_2,\ldots,n_9} = \int \mathcal{D}^d q_1\, \mathcal{D}^d q_2 \frac{D_8^{-n_8}\, D_9^{-n_9}}{D_1^{n_1}\, D_2^{n_2}\, D_3^{n_3}\, D_4^{n_4}\, D_5^{n_5}\, D_6^{n_6}\, D_7^{n_7}} \tag{1}$$

with the propagators given by

$$\begin{aligned}
&D_1 = q_1^2, D_2 = q_2^2, D_3 = (q_1+q_2)^2,\\
&D_4 = (q_1-k_3)^2 - m^2, D_5 = (q_1+k_1-k_3)^2 - m^2,\\
&D_6 = (q_2-k_1-k_2+k_3)^2 - m^2, D_7 = (q_2-k_1+k_3)^2 - m^2,\\
&D_8 = (q_1-k_1-k_2+k_3)^2 - m^2, D_9 = (q_2-k_3)^2 - m^2.
\end{aligned}$$

The measure of the integral is defined as

$$\mathcal{D}^d q_i = \frac{m^{2\epsilon}}{\pi^{D/2}\Gamma(1+\epsilon)} d^d q_i\,, \quad d = 4-2\epsilon. \tag{2}$$

The external gluons and top-quarks are on-shell, i.e., $k_1^2 = 0, k_2^2 = 0, k_3^2 = m^2$ and $k_4^2 = (k_1+k_2-k_3)^2 = m^2$. The Mandelstam variables can be written as

$$s = (k_1+k_2)^2\,, \qquad t = (k_1-k_3)^2\,, \qquad u = (k_2-k_3)^2 \tag{3}$$

with $s+t+u = 2m^2$. Notice that in the definition of the integral family in eq. (1), the denominator $D_i, i = 1, \cdots, 7$ are specified by the propagators in Fig. 3. The denominators D_8 and D_9 are chosen in such a way that the integrals are symmetric under the replacement $(k_1 \leftrightarrow k_2, k_3 \leftrightarrow k_4, q_1 \leftrightarrow q_2)$. And more importantly, they are vanishing in the infrared limit $q_i \to 0$ so that the results have better infrared behavior.

We have adopted the `FIRE` package [21] to construct the IBP identities. All the integrals in this family can be reduced to 66 master integrals. After considering the symmetries of the integrals, e.g. $(k_1 \leftrightarrow k_2, k_3 \leftrightarrow k_4, q_1 \leftrightarrow q_2)$, there are only 33 master integrals that have been shown in Fig. 4.

As proposed in ref. [22], a proper choice of the basis can lead to a rather simple form of differential equations for the master integrals. In this case the differential equations can be transformed to the d-log form, and as a consequence we can present the results of master integrals in terms of multiple polylogarithms. To this aim, we choose the canonical basis as below.

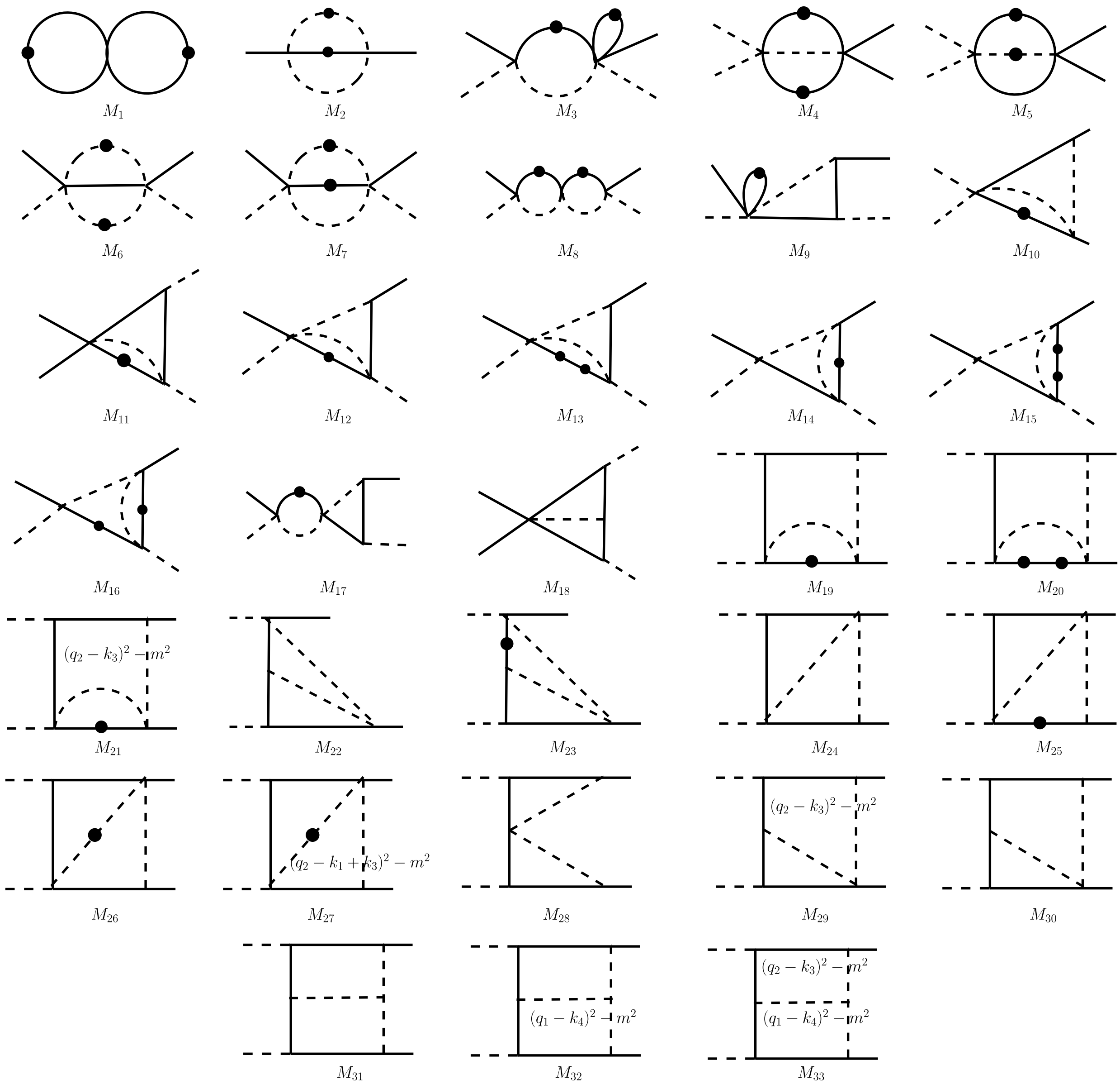

Fig. 4. The master integrals for top-quark pair production shown in Fig. 3. The solid lines represent massive propagators, while the dashed lines indicate massless ones. Each black dot indicates an additional power of the corresponding propagator. For some integrals, we have inserted one or two numerators indicated explicitly on top of the diagram.

$$F_1 = M_1\,, \qquad F_2 = m^2\, M_2\,,$$

$$F_3 = t\, M_3\,, \qquad F_4 = \frac{\sqrt{s(s-4m^2)}}{2}\,(2M_4 + M_5)\,,$$

$$F_5 = -s\, M_5\,, \qquad F_6 = (t-m^2)\, M_6 - 2m^2\, M_7\,,$$

$$F_7 = t\, M_7\,, \qquad F_8 = t^2\, M_8\,,$$

$$F_9 = (t-m^2)\, M_9\,, \qquad F_{10} = \sqrt{s(s-4m^2)}\, M_{10}\,,$$

$$F_{11} = -s\, M_{11}\,, \qquad F_{12} = (t-m^2)\, M_{12}\,,$$

$$F_{13} = (t-m^2)m^2\, M_{13}\,, \qquad F_{14} = (t-m^2)\, M_{14}\,,$$

$$F_{15} = (t-m^2)m^2\, M_{15}\,,$$

$$F_{16} = (t-2m^2)m^2\, M_{16} - 2m^4 M_{15} - 3m^2 M_{14}\,,$$

$$F_{17} = t(t-m^2)\, M_{17}\,, \qquad F_{18} = -s\, M_{18}\,,$$

$$F_{19} = \sqrt{(t-m^2)[t(s-m^2)^2 - (s^2 - 6sm^2 + m^4)m^2]}\, M_{19}\,,$$

$$F_{20} = \sqrt{s(s-4m^2)}(t-m^2)\left(M_{19} + m^2 M_{20}\right)\,,$$

$$F_{21} = (t-m^2)\left(M_{21} - m^2 M_{19}\right) - 2(s+t-m^2)\, M_{11}\,,$$

$$F_{22} = (t-m^2)\, M_{22}\,,$$

$$F_{23} = (t-m^2)m^2\, M_{23}\,,$$

$$F_{24} = (m^2-s-t)\, M_{24}\,,$$

$$F_{25} = (t-m^2)m^2\, M_{25}\,,$$

$$F_{26} = \sqrt{s(s-4m^2)}(t-m^2)\, M_{26}\,,$$

$$F_{27} = \frac{1}{4}(m^2-s-t)\,(4M_{27} - 2M_{10} + 2M_4 + M_5)\,,$$

$$F_{28} = (t-m^2)^2\, M_{28}\,,$$

$$F_{29} = (t-m^2)\,(M_{29} + M_{24} - M_{21} + m^2 M_{19}) + 2(m^2-s-t)(M_{18} - M_{11})\,,$$

$$
\begin{aligned}
\mathrm{F}_{30} &= \sqrt{s(s-4m^2)}(t-m^2)\,(\mathrm{M}_{30}-m^2\mathrm{M}_{20}-\mathrm{M}_{19})\,,\\
\mathrm{F}_{31} &= \sqrt{s(s-4m^2)}(t-m^2)^2\,\mathrm{M}_{31}\,,\\
\mathrm{F}_{32} &= (t-m^2)^2\,\mathrm{M}_{32}+(t-m^2)(2m^2-s-2t)\,\mathrm{M}_{30}\,,\\
\mathrm{F}_{33} &= t(\mathrm{M}_{33}-4\mathrm{M}_{29}-2\mathrm{M}_{24})-2(t-m^2)m^2\,\mathrm{M}_{32}\\
&\quad +2(s+2t-2m^2)m^2\,\mathrm{M}_{30}-(t(s+t)-2m^2t+m^4)\mathrm{M}_{28}\\
&\quad +4(s+t)\mathrm{M}_{18}-2m^4\,\mathrm{M}_{25}-2m^4\,\mathrm{M}_{23}-2m^2\,\mathrm{M}_{22}\\
&\quad -2m^2\,\mathrm{M}_{21}+2m^4\,\mathrm{M}_{19}+2m^2\,t\,\mathrm{M}_{17}+4m^4\,\mathrm{M}_{15}+2m^2\mathrm{M}_{14}\\
&\quad -2m^4\mathrm{M}_{13}-2m^2\mathrm{M}_{12}+4m^2\mathrm{M}_{11}-2m^2\mathrm{M}_9-2m^2\mathrm{M}_6\\
&\quad -\frac{8m^4}{t-m^2}\mathrm{M}_2+\frac{4(t+m^2)m^2}{t-m^2}\mathrm{M}_7\\
&\quad +\frac{t}{2(1-2\epsilon)}(\mathrm{M}_3+\mathrm{M}_7-t\,\mathrm{M}_8)
\end{aligned} \tag{4}
$$

with

$$
\begin{aligned}
&\mathrm{M}_1=\epsilon^2\,I_{0,0,0,2,0,2,0,0,0}\,, && \mathrm{M}_2=\epsilon^2\,I_{0,1,2,2,0,0,0,0,0}\,,\\
&\mathrm{M}_3=\epsilon^2\,I_{1,0,0,0,2,2,0,0,0}\,, && \mathrm{M}_4=\epsilon^2\,I_{0,0,2,2,0,1,0,0,0}\,,\\
&\mathrm{M}_5=\epsilon^2\,I_{0,0,1,2,0,2,0,0,0}\,, && \mathrm{M}_6=\epsilon^2\,I_{0,2,2,0,1,0,0,0,0}\,,\\
&\mathrm{M}_7=\epsilon^2\,I_{0,1,2,0,2,0,0,0,0}\,, && \mathrm{M}_8=\epsilon^2\,I_{1,1,0,0,2,0,2,0,0}\,,\\
&\mathrm{M}_9=\epsilon^3\,I_{0,1,0,2,0,1,1,0,0}\,, && \mathrm{M}_{10}=\epsilon^3\,I_{1,0,1,1,0,2,0,0,0}\,,\\
&\mathrm{M}_{11}=\epsilon^3\,I_{0,0,1,1,1,2,0,0,0}\,, && \mathrm{M}_{12}=\epsilon^3\,I_{0,1,1,0,2,1,0,0,0}\,,\\
&\mathrm{M}_{13}=\epsilon^2\,I_{0,1,1,0,3,1,0,0,0}\,, && \mathrm{M}_{14}=\epsilon^3\,I_{0,1,1,2,0,0,1,0,0}\,,\\
&\mathrm{M}_{15}=\epsilon^2\,I_{0,1,1,3,0,0,1,0,0}\,, && \mathrm{M}_{16}=\epsilon^2\,I_{0,1,1,2,0,0,2,0,0}\,,\\
&\mathrm{M}_{17}=\epsilon^3\,I_{1,1,0,0,2,1,1,0,0}\,, && \mathrm{M}_{18}=\epsilon^4\,I_{0,0,1,1,1,1,1,0,0}\,,\\
&\mathrm{M}_{19}=\epsilon^3\,I_{0,1,1,2,0,1,1,0,0}\,, && \mathrm{M}_{20}=\epsilon^2\,I_{0,1,1,3,0,1,1,0,0}\,,\\
&\mathrm{M}_{21}=\epsilon^3\,I_{0,1,1,2,0,1,1,0,-1}\,, && \mathrm{M}_{22}=\epsilon^4\,I_{0,1,1,1,1,0,1,0,0}\,,\\
&\mathrm{M}_{23}=\epsilon^3\,I_{0,1,1,1,1,0,2,0,0}\,, && \mathrm{M}_{24}=\epsilon^4\,I_{0,1,1,1,1,1,0,0,0}\,,\\
&\mathrm{M}_{25}=\epsilon^3\,I_{0,1,1,1,1,2,0,0,0}\,, && \mathrm{M}_{26}=\epsilon^3\,I_{0,1,2,1,1,1,0,0,0}\,,\\
&\mathrm{M}_{27}=\epsilon^3\,I_{0,1,2,1,1,1,-1,0,0}\,, && \mathrm{M}_{28}=\epsilon^4\,I_{1,1,0,1,1,1,1,0,0}\,,\\
&\mathrm{M}_{29}=\epsilon^4\,I_{0,1,1,1,1,1,1,0,-1}\,, && \mathrm{M}_{30}=\epsilon^4\,I_{0,1,1,1,1,1,1,0,0}\,,\\
&\mathrm{M}_{31}=\epsilon^4\,I_{1,1,1,1,1,1,1,0,0}\,, && \mathrm{M}_{32}=\epsilon^4\,I_{1,1,1,1,1,1,1,-1,0}\,,\\
&\mathrm{M}_{33}=\epsilon^4\,I_{1,1,1,1,1,1,1,-1,-1}\,.
\end{aligned} \tag{5}
$$

The differential equations of the canonical basis contain two different square roots, i.e.,

$$
\sqrt{s(s-4m^2)},\quad \sqrt{(t-m^2)[t(s-m^2)^2-(s^2-6sm^2+m^4)m^2]}. \tag{6}
$$

Notice that only F_{19} contains the second square root explicitly. In order to solve the differential equations in terms of multiple polylogarithms, we have to rationalize these square roots. Therefore, we define two dimensionless variables,

$$
\begin{aligned}
y &= -\frac{\sqrt{s}-\sqrt{s-4m^2}}{\sqrt{s}+\sqrt{s-4m^2}}\,,\\
x &= (y^2-y+1)\frac{\sqrt{t-m^2(s^2-6s\,m^2+m^4)/(s-m^2)^2}}{\sqrt{t-m^2}}
\end{aligned} \tag{7}
$$

corresponding to the following transformation of variables

$$
\begin{aligned}
s &= -\frac{(1-y)^2}{y}m^2\,,\\
t &= \frac{x^2-(y-1)y[y(y+3)-2]-1}{x^2-((y-1)y+1)^2}m^2\,.
\end{aligned} \tag{8}
$$

After changing to the x and y variables, the differential equations for $\mathbf{F}=\{\mathrm{F}_1,\ldots,\mathrm{F}_{33}\}$ can be written as

$$
d\,\mathbf{F}(x,y;\epsilon)=\epsilon\,(d\,\mathbf{A})\,\mathbf{F}(x,y;\epsilon), \tag{9}
$$

where

$$
d\,\mathbf{A}=\sum_{i=1}^{13}\mathbf{R}_i\,d\log(l_i) \tag{10}
$$

with $\mathbf{R}_i$ rational matrices independent of the kinematics and the space-time dimension. Their explicit forms are provided in ancillary files. These d-log forms contain all the information of the kinematics. The set of the arguments l_i is referred to as the *alphabet* and it consists of the following 13 letters

$$
\begin{aligned}
&l_1=x-(y^2+y-1)\,, && l_2=x+(y^2+y-1)\,,\\
&l_3=x-(y^2-y-1)\,, && l_4=x+(y^2-y-1)\,,\\
&l_5=x-(y^2-y+1)\,, && l_6=x+(y^2-y+1)\,,\\
&l_7=x-(y^2-3y+1)\,, && l_8=x+(y^2-3y+1)\,,\\
&l_9=x^2-[y(y-3)+1]\left(y^2+y+1\right)\,,\\
&l_{10}=x^2-y(y-1)[y(y+3)-2]-1\,,\\
&l_{11}=y\,, && l_{12}=y+1\,,\\
&l_{13}=y-1\,.
\end{aligned} \tag{11}
$$

One can see from Fig. 4 that the master integrals $\{\mathrm{M}_{29},\mathrm{M}_{30},\mathrm{M}_{31},\mathrm{M}_{33}\}$ contain M_{19} as a sub-topology. As a consequence, their differential equations may contain M_{19}. However, we choose the canonical basis $\{\mathrm{F}_{29},\mathrm{F}_{30},\mathrm{F}_{31},\mathrm{F}_{33}\}$ in such a way that their dependence on F_{19} has been removed. At the end, only the differential equations of $\{\mathrm{F}_{19},\mathrm{F}_{20},\mathrm{F}_{21},\mathrm{F}_{32}\}$ involve F_{19}.

The differential equations of the remaining integrals do not depend on F_{19} and thus do not contain the square root $\sqrt{(t-m^2)[t(s-m^2)^2-(s^2-6s\,m^2+m^4)m^2]}$. Since the result of F_{19} starts at transcendental weight three as indicated by Eq. (10), we can use the variables y and $z\equiv t/m^2$, instead of y and x, to express the results of the remaining integrals, namely all the master integrals except $\{\mathrm{F}_{19},\mathrm{F}_{20},\mathrm{F}_{21},\mathrm{F}_{32}\}$, up to transcendental weight four. In this way, we obtain more compact results for the remaining master integrals. Here, for illustration, we show the differential equations for F_{31},

$$
\begin{aligned}
\frac{\partial\,\mathrm{F}_{31}}{\partial\,y} = \epsilon\Bigg[&\frac{1}{y-z}(4\mathrm{F}_2+\mathrm{F}_6-2\mathrm{F}_7+\mathrm{F}_9+3\mathrm{F}_{12}+4\mathrm{F}_{13}-4\mathrm{F}_{14}\\
&-6\mathrm{F}_{15}-2\mathrm{F}_{17}+2\mathrm{F}_{20}+2\mathrm{F}_{22}+2\mathrm{F}_{23}-\mathrm{F}_{26}+2\mathrm{F}_{31}-2\mathrm{F}_{32})\\
&-\frac{1}{y-\frac{1}{z}}(4\mathrm{F}_2+\mathrm{F}_6-2\mathrm{F}_7+\mathrm{F}_9+3\mathrm{F}_{12}+4\mathrm{F}_{13}-4\mathrm{F}_{14}\\
&-6\mathrm{F}_{15}-2\mathrm{F}_{17}-2\mathrm{F}_{20}+2\mathrm{F}_{22}+2\mathrm{F}_{23}+\mathrm{F}_{26}-2\mathrm{F}_{31}-2\mathrm{F}_{32})\\
&+\frac{1}{2y}(5\mathrm{F}_1+36\mathrm{F}_2-4\mathrm{F}_3-2\mathrm{F}_5+4\mathrm{F}_6-12\mathrm{F}_7+8\mathrm{F}_9\\
&-10\mathrm{F}_{11}+8\mathrm{F}_{12}+8\mathrm{F}_{13}-24\mathrm{F}_{14}-24\mathrm{F}_{15}-8\mathrm{F}_{16}-8\mathrm{F}_{17}\\
&-20\mathrm{F}_{18}+8\mathrm{F}_{20}+8\mathrm{F}_{22}+8\mathrm{F}_{23}+16\mathrm{F}_{24}-4\mathrm{F}_{26}\\
&+4\mathrm{F}_{27}-4\mathrm{F}_{28}-8\mathrm{F}_{29}+4\mathrm{F}_{31}+8\mathrm{F}_{32}-8\mathrm{F}_{33})\\
&-\frac{2}{y-1}(2\mathrm{F}_{20}-\mathrm{F}_{26}+\mathrm{F}_{31})-\frac{2}{y+1}(4\mathrm{F}_{20}-2\mathrm{F}_{26}+3\mathrm{F}_{31})\Bigg],
\end{aligned}
$$

$$\frac{\partial F_{31}}{\partial z} = \epsilon\Bigg[\frac{1}{z-y}(4F_2+F_6-2F_7+F_9+3F_{12}+4F_{13}-4F_{14} - 6F_{15}-2F_{17}+2F_{20}+2F_{22}+2F_{23}-F_{26}+2F_{31}-2F_{32}) - \frac{1}{z-\frac{1}{y}}(4F_2+F_6-2F_7+F_9+3F_{12}+4F_{13}-4F_{14} - 6F_{15}-2F_{17}-2F_{20}+2F_{22}+2F_{23}+F_{26}-2F_{31}-2F_{32}) - \frac{4F_{31}}{z-1}\Bigg]. \tag{12}$$

3. Boundary conditions and analytic results

In order to obtain analytic results from differential equations for the canonical basis shown in the previous section, we need to determinate the boundary conditions first.

The bases $\{F_1, F_2\}$ are just single-scale integrals, corresponding to a vacuum diagram with virtual massive particles or a self-energy diagram of a massive particle, and their results are already known in ref. [23]. Explicitly,

$$F_1 = 1\,,\ F_2 = -\frac{1}{4} - \epsilon^2\frac{\pi^2}{6} - 2\epsilon^3\zeta(3) - \epsilon^4\frac{8\pi^4}{45} + \mathcal{O}(\epsilon^5). \tag{13}$$

The boundary condition for F_6 at $t=0$ $(z=0)$ can be evaluated using the Mellin-Barnes method, implemented in the Mathematica packages MB [24] and AMBRE [25], and we obtain

$$F_6|_{z=0} = 1 + \epsilon^2\frac{\pi^2}{3} - 2\epsilon^3\zeta(3) + \epsilon^4\frac{\pi^4}{10} + \mathcal{O}(\epsilon^5). \tag{14}$$

All the master integrals do not have singularities at $t=0$. Due to the prefactor t for the bases $\{F_3, F_7, F_8, F_{17}\}$, we can deduce that they are vanishing at $t=0$. We derive the boundary of F_{16} at $t=0$ from the regularity condition of the corresponding differential equation at $t=0$. Specifically, the differential equation for F_{16} can be formulated as

$$\frac{dF_{16}}{dt} = -\frac{\epsilon}{2t}(3F_1+12F_2-2F_3-6F_{14}-6F_{16}) + \ldots, \tag{15}$$

where the ellipses stand for less singular terms at $t=0$. The regularity condition at $t=0$ leads to a relation

$$\lim_{t\to 0}(3F_1+12F_2-2F_3-6F_{14}-6F_{16}) = 0. \tag{16}$$

Thus, we obtain the boundary condition of F_{16} at $t=0$ $(z=0)$ from the above equation.

All the master integrals are regular at $s=0$. The canonical bases $\{F_4, F_5, F_{10}, F_{11}, F_{18}, F_{31}\}$ have a prefactor s or $\sqrt{s(s-4m^2)}$ and thus they are vanishing at $s=0$. The boundary condition of F_{33} is determined from the regularity condition of the corresponding differential equation at $s=0$ $(y=1)$.

The same logic leads us to know that the bases $\{F_9, F_{12}, F_{13}, F_{14}, F_{15}, F_{22}, F_{23}, F_{25}, F_{28}, F_{30}\}$ are vanishing at $t=m^2$, and $F_{29} = 2F_{18} - 2F_{11}$ at $t=m^2$ $(z=1)$.

The base F_{19} does not have a singularity at $t = \frac{(s^2-6sm^2+m^4)m^2}{(s-m^2)^2}$ but has a prefactor $\sqrt{t(s-m^2)^2-(s^2-6sm^2+m^4)m^2}$, so it is vanishing at $t = \frac{(s^2-6sm^2+m^4)m^2}{(s-m^2)^2}$ $(x=0)$.

The planar integrals we are considering do not have a u-channel singularity. Due to the prefactor $(m^2-s-t) = (u-m^2)$ for $\{F_{24}, F_{27}\}$, they are vanishing at $t = m^2 - s$.

The boundary conditions of $\{F_{20}, F_{21}, F_{26}, F_{32}, F_{33}\}$ are determined from the regularity conditions of the corresponding differential equations at $t = \frac{m^4}{u}$ $(z=y)$.

With the discussion above, we finish the determination of all boundary conditions that are necessary to obtain the full analytic results. Then all the master integrals can readily be calculated from the differential equations. The analytic results for $\{F_1, \ldots, F_{33}\}$ up to transcendental weight four are expressed in terms of multiple polylogarithms [26], which are provided in an ancillary file. For illustration, we show the result for F_{33} up to transcendental weight four

$$\begin{aligned} F_{31} = {}& \epsilon^3\Big[2G_{0,0,0}(y) + \frac{\pi^2}{3}G_0(y)\Big] + \epsilon^4\Big[-\frac{5\pi^4}{18} \\ & + \frac{\pi^2}{3}(2G_{1,0}(y) + 2G_{-1,0}(y) - 7G_{0,0}(y)) + 4G_{-1,0,0,0}(y) \\ & - 12G_{0,0,-1,0}(y) - 4G_{0,0,0,0}(y) + 4G_{0,0,1,0}(y) \\ & + 8G_{0,1,0,0}(y) + 4G_{1,0,0,0}(y) - 8G_{0,0,0}(y)G_1(z) \\ & + \frac{2}{3}G_0(y)[\pi^2(3G_0(z_1) - 2G_{\frac{1}{z}}(z_1) - 2G_1(z)) \\ & - 6(G_{\frac{1}{z},0,0}(z_1) - G_{\frac{1}{z},1,0}(z_1) + G_{z,1,0}(z_1) + G_{1,0,1}(z) \\ & + G_{\frac{1}{z},1,0}(1) - G_{z,1,0}(1) + 2G_{0,1,0}(z_1) + 3G_{1,0,0}(z_1) \\ & - G_{0,0,1}(z) - 3G_{0,0,0}(z_1)) - 9\zeta(3)]\Big] + \mathcal{O}(\epsilon^5), \end{aligned} \tag{17}$$

where $z_1 = (z+1+\sqrt{z^2+2z-3})/2$ corresponds to the boundary conditions of F_{24}, F_{27} discussed above. Here $G_{a_1,a_2,\ldots,a_n}(x)$ are multiple polylogarithms [26], defined recursively by

$$G_{a_1,a_2,\ldots,a_n}(x) = \int_0^x \frac{dt}{t-a_1}G_{a_2,\ldots,a_n}(t) \tag{18}$$

with $G(x) = 1$ and

$$G_{\underbrace{0,0,\ldots,0}_{n}}(x) = \frac{1}{n!}\ln^n x\,. \tag{19}$$

The number n is referred to as the transcendental weight of the result. We see that the result of F_{31} begins with $O(\epsilon^3)$. This is anticipated. From the discussion above, we see that the only non-vanishing boundary values are that of F_6 in Eq. (14) and the two single-scale integrals F_1, F_2 in Eq. (13). Though they have $O(\epsilon^0)$ contribution, their combination in Eq. (12) is in such a way that the right-hand side starts at $O(\epsilon^3)$.

The multiple polylogarithms can be numerically evaluated by the GINAC implementation [27,28]. They can also be transformed to the functions like $\mathrm{Li}_n(x)$ and $\mathrm{Li}_{2,2}(x,y)$ up to transcendental weight four with the methods described in [29].

We are interested in the physical region for top-quark pair production, i.e., $(s > 4m^2, t < -s(1-\sqrt{1-4m^2/s})^2/4)$. The proper analytic continuation can be achieved by starting from the Euclidean space where $s < 4m^2$ and $t < 0$. Then we assign s a small positive imaginary part $(s \to s + 0i)$ to produce the correct result when $s > 4m^2$ [30].

The analytic results of all the master integrals have been checked with the numerical package FIESTA [31], and good agreements have been achieved. For example, we show the result of M_{31} at a kinematic point $(s = 3.41, t = -0.91, m = 1)$,

$$\begin{aligned} M_{31}^{\text{analytic}} &= \epsilon^3(0.656683) \\ &\quad + \epsilon^4(4.006860), \\ M_{31}^{\text{FIESTA}} &= \epsilon^3(0.656684 \pm 0.000002) \\ &\quad + \epsilon^4(4.006867 \pm 0.000019). \end{aligned} \tag{20}$$

4. Conclusions

In summary, we calculate analytically a planar double-box Feynman integral family for top-quark pair production. After choosing a canonical basis, the differential equations for the corresponding basis are expressed in canonical form. The boundary conditions are determined either by simple integrals or by regularity conditions at certain kinematic points without physical singularities. Therefore, the analytic results of the basis can be expressed in terms of multiple polylogarithms. These results and the rational matrices in the differential equations are provided in ancillary files. In the future, it will be interesting to calculate the other unknown integral families, especially those with elliptic integrals, to obtain a fully analytic result of the two-loop corrections to the top-quark pair production.

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