Cosmology and Particle Physics in Heterotic Orbifolds

Andreas N. Mütter

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Physik–Department T75
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Andreas Nikolaus Mütter

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Cosmology and Particle Physics in Heterotic Orbifolds

Kosmologie und Teilchenphysik in heterotischen Orbifaltigkeiten

Andreas N. Mütter

Abstract

Orbifold compactifications of the heterotic superstring can yield realistic models of particle physics. This thesis aims to study the properties of the resulting four-dimensional field theories. In particular, we show that the cosmological constant in heterotic orbifold theories motivates $\mathcal{N} = 1$ supersymmetric theories. We also provide a rigorous way to study discrete flavor symmetries as remnants of higher-dimensional gauge symmetries. Finally, we show that massive string states provide a viable candidate for dark matter.

Zusammenfassung

Heterotische Superstrings, die auf Orbifaltigkeiten kompaktifiziert sind, können realistische Modelle für die Teilchenphysik liefern. Die vorliegende Arbeit untersucht die daraus resultierenden, vierdimensionalen Feldtheorien. Wir zeigen, dass die kosmologische Konstante in heterotischen Orbifaltigkeitstheorien $\mathcal{N} = 1$ Supersymmetrie motiviert. Darüber hinaus geben wir eine rigorose Methode an, mit der diskrete (Flavor-)Symmetrien als Überreste einer höherdimensionalen Eichtheorie verstanden werden können. Schließlich zeigen wir, dass schwere Stringzustände einen realistischen Kandidaten für Dunkelmaterie darstellen können.

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Introduction

Today, the standard model of particle physics (SM) is by far one of the most thoroughly tested theories known to mankind. It successfully describes the interactions of the known elementary particles in the language of quantum field theory (QFT), where the fundamental entities of our world are point particles. The predictions of the SM have been tested (and confirmed) experimentally to an astonishing precision, with the capstone being the discovery of a scalar boson at the Large Hadron Collider (LHC) in 2012, whose properties match those of the Higgs boson predicted in the 1960s [1, 2].

Despite its tremendous success in explaining observed phenomena in the domain of high-energy particle physics, the standard model is known to have various shortcomings, both of conceptual nature and by being unable to explain some observed phenomena (like, e.g., the origin of dark matter). A commonly agreed viewpoint is that the SM might only be an effective theory valid at low energies, and it is expected that new physics will enter at some energy scale that lies well above the electroweak scale. Throughout the literature, the conceptual shortcomings and their possible solutions have been used as guiding principles on the lookout for signatures of new physics. Among the various conceptual questions being left unanswered by the SM are the following two issues, which we will discuss now in more detail as they will play a major role in the remainder of this thesis. The first issue arises from the following question [3]: due to the fact that the Planck and the electroweak scale are separated by 16 orders of magnitude, the values for the Higgs mass have to be tuned up to an "unnatural" precision in order to obtain the observed value also after including quantum corrections, if new physics effects are expected to enter at a high energy scale. This is commonly referred to as the hierarchy problem. The reason for this fact lies in the corrections to the Higgs mass by its self-interaction, which goes roughly as Λ^2, where Λ is the high energy new physics scale. One of the arguably most elegant solutions to this problem is supersymmetry (SUSY). There, it is postulated that all particles (e.g. in the standard model) have a so-called superpartner that has the exact same quantum numbers except that it obeys the opposite spin-statistics. For each bosonic degree of freedom, there must exist a fermionic one in the same representation of the gauge algebra, and vice versa. It has been shown by both direct calculations and by general theorems that this symmetry removes the quadratic dependence of the Higgs boson mass on the cutoff scale, leading to a situation where much less finetuning is needed in order to match observations. Not only does supersymmetry solve the hierarchy problem, it also makes specific predictions of new physics: as the superpartners of the SM particles are charged under the gauge symmetry, it is expected that they will eventually show up in collider experiments like the LHC. As of today, and unlike the standard model matter, no superparticle has been found by experiment, putting the entire framework of SUSY to question. The second issue is of rather different nature: when examining the renormalization group (RG) evolution of gauge couplings in (supersymmetric extensions of) the standard model, they seem to meet at an energy scale of around 10^{15} GeV. This fact has been interpreted to indicate that the gauge symmetries of the standard model might descend from a more fundamental theory, which is then called a grand unified theory (GUT). Since the conception of GUTs in the 1970s, they have received a lot of attention, as their unifying power proposes a particularly elegant way to describe the intricate dynamics of the standard model. However, there does not exist "the" GUT theory, as there exist various ways to combine the SM matter and gauge bosons to a unified theory. The most straightforward possibility is to place the SM matter in representations of SU(5) [4], however also SO(10) [5] and E_6 [6] conventionally appear in the GUT literature. Much like supersymmetry, also the concept of GUTs makes predictions for new physics, such as that the proton might not be stable, a possibility that is actively being looked for e.g. in the planned HyperKamiokande experiment [7]. Although no observation has been made so far, signatures of nucleon decay remain a "smoking gun" for the search of new physics, allowing one to efficiently constrain the parameter space of new physics models. In particular, combining supersymmetry and grand unification leads to very specific predictions concerning proton stability.

All shortcomings of the SM discussed so far have typically been problems that arise within the field of particle physics. However, one of the conceptually biggest issue lies elsewhere: The connection of particle physics described by quantum field theories and gravity described by general relativity (GR) remains an unsolved question to this day. There is no a priori reason why one should not be able to describe gravity on the same footing as the known gauge forces. On the other hand, such a description is definitely desirable for a more complete understanding of our world. Likewise, the known description of gravity in terms of GR does not allow for a study of quantum effects in gravity. Unfortunately, so far all approaches trying to describe gravity using the framework of conventional quantum field theory suffer from ultraviolet divergences and therefore cannot be the final answer. It has been argued that an ansatz going beyond standard field theory has to be made. One proposed solution in this context is string theory, which replaces the conventional point particles known in QFT by one-dimensional objects. When propagating in spacetime, these one-dimensional objects sweep out a two-dimensional worldsheet, rather than a one-dimensional worldline in the case of a point particle in conventional QFT. On this worldsheet, one now defines a conformal field theory (CFT) in 1+1 dimensions. The fields of the CFT then carry indices of the spacetime the string is embedded in. Hence, studying string constructions amounts to first choosing an appropriate particle content of the CFT, and then an embedding into spacetime. Originally conceived as a theory describing interactions of hadrons, it was soon realized that string theory gives rise to a spin 2 particle whose dynamics is determined by the Einstein equations of GR, the graviton. Over the last decades, string theory has been celebrated as a perturbative theory of quantum gravity, as it has been shown explicitly that the divergences appearing in field theories of gravity are mitigated. On the particle physics side, it has been shown that the known particle spectrum of the standard model can be accommodated in various string constructions, therefore string theory may serve as a framework for successful model building. Likewise, concepts like supersymmetry and grand unification arise from string theory constructions in a very natural manner, especially for the heterotic superstring, as will be demonstrated in the course of this thesis. Due to consistency conditions of the underlying CFT, string theory requires itself to live in more than the four spacetime dimensions we observe. In case of the five classical superstring theories, the critical dimension is ten, which means that the worldsheet degrees of freedom need to be embedded into a ten-dimensional spacetime. In order to make contact with the real world, six dimensions have to be hidden from observation by compactification, that is, by rolling up six dimensions on radii so small that the energy needed to probe them is sufficiently high. From a worldsheet point of view, that means that some of the fields see non-trivial boundary conditions. On the other hand, it is expected that compactification gives rise to myriads of four-dimensional vacua, the so-called landscape [8]. The length scale of the compact dimensions sets the scale where new physics can be expected to appear, namely the energy scale for heavy string modes that would ultimately be needed in order to prove or disprove string theory. As this so-called string scale is expected to lie at values up to the GUT or the Planck scale, direct detection of heavy string modes is notoriously hard, if not impossible, to be seen by experiment. This fact has led to the criticism that string theory by construction lacks predictive power.

However, in recent years it has become clear that not every self-consistent quantum field theory can be compatible with string theory, or any UV-complete theory of quantum gravity. In a context that differs from the scope of this thesis, this is referred to as the swampland program and has attracted much attention in the field over the past decade (see e.g. [9] for a recent review). Turning this logic around, string theory has the potential to give valid input for new ideas in particle physics and cosmology. To be specific, the predictivity of bottom-up models often suffers from the plethora of possible extensions of the standard model, in absence of a guiding principle. In this thesis, it shall be argued that string theory can indeed provide such a guiding principle. This becomes clear as soon as one notices that *string-derived* models (i.e. effective field theories of string models) cannot necessarily accommodate all properties one might think of from a bottom-up perspective. Often the situation in string model building is that requiring a certain feature to be present implies that some other feature also appears (be it desired or undesired). It has been found that some phenomenologically appealing properties generically arise from compactified string theories, the so-called “stringy surprises” [10]. In other cases, it turns out that realizing some property may not be possible at all in string-derived models, which is where the swampland program lies its focus on. Along these lines, the study of *string-inspired* theories, which are field theory models that by design match the properties of string-derived ones (e.g. in the form of orbifold GUTs), has proven itself to be a fruitful playground for approaches to new physics [11], both by constraining the possibilities, and by implicitly ensuring the connection to a perturbative theory of gravity.

As the setup on the string theory side, we choose the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ superstring, compactified on toroidal orbifolds. This choice is made for a number of reasons: Firstly, because unlike in smooth Calabi–Yau (CY) compactifications, orbifolds allow one to trace the effects due to heavy string modes all the way down to four dimensions, as they possess a solvable CFT. That is, one is able to write down a four-dimensional *string* theory, before taking an appropriate field theory limit. This is in contrast to smooth CY compactifications, where one is forced to make a supergravity approximation in ten dimensions and to work then with a dimensionally reduced *field* theory. Specifically, contributions of heavy string modes on scattering amplitudes are particularly well understood in orbifold models, a situation that is still to be achieved in smooth CY compactifications. The second reason that makes heterotic string theory a good playground is that it has been shown that it is possible to obtain the exact (MS)SM spectrum from compactifications of the heterotic string with relative ease, especially in its $\mathrm{E}_8 \times \mathrm{E}_8$ version. Thus, technical problems that arise from the string construction itself are largely avoided, allowing us to focus on the phenomenological aspects. The final reason to pick the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic superstring is that it has many phenomenologically interesting features already built in, among them supersymmetry and grand unified theories, as most known GUT groups are subgroups of E_8. Since these frameworks are both theoretically compelling and yet to be confirmed or falsified by experiment, they will be in the focus of our studies.

Outline. The present thesis aims to show the potential of the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ superstring to act as a guiding principle for new physics using examples from both elementary particle physics and cosmology. The plan of this thesis is as follows:

First, we introduce the heterotic string in *Chapter 2*, where its formulation in ten dimensions and its compactification on toroidal orbifolds is presented along with some technical results that will be needed in the remainder of this thesis.

In *Chapter 3*, the relation of supersymmetry in heterotic compactifications and the cosmological constant problem is studied. The idea put forward in this chapter is the following: In ten dimensions, the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ theory has $\mathcal{N}_{10} = 1$ supersymmetry. If dimensionally reduced to four dimensions, this would correspond to $\mathcal{N}_4 = 4$ SUSY, a non-chiral theory which cannot describe the world we see. Hence, the compactification has not only to incorporate the reduction of dimensionality, but also for some amount of SUSY breaking. While usually the orbifold geometry is chosen such as to lead to $\mathcal{N}_4 = 1$ supersymmetry in four dimensions [12], it is well possible to break all SUSY by orbifolding and obtain non-supersymmetric four-dimensional string models. However, with the breakdown of SUSY, also the cosmological constant problem re-arises: while for supersymmetric string theories the cosmological constant is exactly zero (and is expected to get field theory contributions after SUSY breaking, in order to obtain its small but finite observed value), we show that in non-supersymmetric orbifolds all computational control of the cosmological constant using the known methods is lost. In particular, we are able to trace the unconstrained contributions back to the group theory of the orbifold geometry, allowing us to formulate the (non-)vanishing of the cosmological constant in the language of the representation theory of finite groups.

In *Chapter 4*, we turn our attention to the study of discrete symmetries as they appear in field theory model building. These symmetries play an important role in many model building scenarios, where they are needed in order to suppress dangerous couplings, appear as flavor symmetries, or realize CP violation. Due to general quantum gravity arguments [13], these symmetries, which appear as global symmetries in the low-energy theory, should ultimately be gauged, i.e. be discrete remnants of a gauge symmetry that is broken. While it has been well studied how these symmetries can in fact descend from (four-dimensional) gauge theories that are broken by Higgsing, the situation is less clear for gauge symmetry breaking by orbifold boundary conditions. However, understanding remnant symmetries in orbifold models is desirable, as in these models problems like e.g. the doublet-triplet splitting problem have a simple solution. We begin this chapter with model-independent considerations, where we show how discrete remnants of a gauge symmetry in higher dimensions can survive the orbifolding, and how to enumerate residual symmetry generators systematically. In particular, we are able to create a connection of remnant discrete symmetries and the Weyl group of the unbroken Lie algebra. We conclude the chapter with a discussion of possible applications of this formalism for flavor model building.

Chapter 5 discusses one possibility of realizing dark matter in string theory. We build upon the observation that the spectrum of string theory compactifications generically contains (massive) standard model singlets. Furthermore, some of these singlets can be stabilized by a $\mathbb{Z}_2$ symmetry that ultimately arises from string selection rules. To be specific, we discuss the realizations of Planckian interacting dark matter (PIDM) [14] in a string-derived scenario. In the original, field-theoretic PIDM scenario, it was assumed that the dark matter particle interacts with the standard model (and hence the thermal bath) only via gravitational interactions. However, we show that in string models, there are always heavy string modes that can also act as mediators and therefore lead to a sizable contribution to the dark matter production rate, competing with gravitational interactions. This effect is only possible by taking stringy effects into account, and links observational constraints on the dark matter relic abundance to free parameters (moduli) of the compact space.

Chapter 6 is devoted to a final discussion of the findings of this thesis in the context of (string) model building and particle cosmology.

Bibliography

This thesis is largely based on the following publications by the author [15, 16, 17, 18]

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2

Heterotic string theory on orbifolds

In this chapter, we review textbook knowledge [21, 22, 23] to the extent that is needed to follow the remainder of this thesis. To be specific, we will concentrate on the construction of heterotic strings in ten dimensions in bosonized formulation. Then, toroidal orbifolds will be introduced, and the compactification of closed strings on orbifolds will be discussed. We close our discussion by constructing the heterotic partition function on orbifold spaces, which will be used in the subsequent chapters.

2.1 Overview

Before we dive into more detail, we give an overview of the construction of four-dimensional heterotic string models here. There are five distinct superstring theories: Type I, type IIA/B and the two heterotic theories (with gauge groups $E_8 \times E_8$ and SO(32)). While the type I and II theories may contain open strings, the two heterotic strings have only closed strings in their spectra. As we will see later, it is only possible to formulate a consistent superstring theory in ten dimensions. Therefore, any string model building necessarily involves the specification of a six-dimensional compactification space. Some of the superstring theories are dual in ten dimensions (i.e. they yield equivalent physics), while others are shown to become equivalent upon compactification (see e.g. [24] for a pedagogical review). In fact it is believed that all five theories descend from a common eleven-dimensional theory called M-theory (cf. [25] for an introduction).

In this work, we will concentrate on heterotic strings only, and in particular lay our focus on the heterotic $E_8 \times E_8$ theory. Hence, our starting point is a theory with a gauge group $E_8 \times E_8$ in ten dimensions. Now, there are two ways to proceed. The first one is to perform the so-called supergravity limit (i.e. integrate out all heavy string states) to obtain a ten-dimensional field theory which can then be compactified on a suitable six-dimensional manifold, e.g. a Calabi–Yau (CY) threefold. The other possibility is to compactify the ten-dimensional string theory on a compact space on which the conformal field theory (CFT) on the worldsheet is solvable, to obtain a four-dimensional string theory. A popular example for such compact spaces are orbifolds, which—unlike CY manifolds—have singular points and are no manifolds in the common sense. The resulting four-dimensional string theory has its own field theory limit then. This construction allows one to trace stringy contributions to couplings and scattering amplitudes more directly than the CY constructions.

In orbifold constructions, a particular field theory limit (its gauge group and matter content) depends on the choice of the orbifold geometry as well as the so-called *gauge embedding* of the geometric orbifold action. Hence, the unique ten-dimensional heterotic superstring can result in an entire *landscape* of string-derived field theories with different properties. It should be noted that taking the field theory limit of the four-dimensional theory is not yet the end of the story: In order to find a true *string theory vacuum*, one would need to stabilize the geometric moduli and give vacuum expectation values (VEVs) to various standard model singlets in the theory.

2.2 Heterotic strings in ten dimensions

We start our discussion with the uncompactified heterotic string. As mentioned in the introduction, the starting point of any (super-)string theory is to discuss its underlying worldsheet CFT. As heterotic string theory has closed strings only, our discussion will omit open strings.

2.2.1 The worldsheet action

The same way as a point particle lives on its world*line*, the one-dimensional string sweeps out a two-dimensional world*sheet* that has (for closed strings) the topology of a cylinder. It has become clear that in order to describe the dynamics of the string, one has to define and quantize a conformal field theory on this worldsheet. This construction is not limited to a bosonic theory, one can define a SUSY transformation on the worldsheet and have worldsheet bosons and fermions alongside each other. The most general worldsheet action (after gauge fixing) reads

$$S \;=\; \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \partial_a X^i \partial_b X^j \eta^{ab}\eta_{ij} + \mathrm{i}\overline{\psi}^i\gamma^a\partial_a\psi^j\eta_{ij}\ . \tag{2.1}$$

Here, the fields $X^i(z,\overline{z})$ are worldsheet bosons, and their indices i,j are just labels from the perspective of the worldsheet theory. The ψ^i are worldsheet fermions. The integration goes over the worldsheet time direction σ_1 and the direction along the string σ_2.

Let us for the moment focus on the bosonic side. There, one can define closed string boundary conditions as

$$X^i(\sigma_1,\sigma_2+2\pi) \;=\; X^i(\sigma_1,\sigma_2)\ . \tag{2.2}$$

The equations of motion of the theory read

$$\left(\partial_1^2-\partial_2^2\right)X^i(\sigma_1,\sigma_2) \;=\; 0\ . \tag{2.3}$$

Together with the boundary conditions, one observes that solutions to these equations of motion can be split into functions of $\sigma_1+\sigma_2$ and $\sigma_1-\sigma_2$, called left- and rightmovers, respectively, such that one can write (using obvious notation)

$$X^i(\sigma_1,\sigma_2) \;=\; X^i_{\mathrm{L}}(\sigma_1+\sigma_2)+X^i_{\mathrm{R}}(\sigma_1-\sigma_2)\ . \tag{2.4}$$

Then, the solutions take the general form of Fourier expansions

$$X^i_{\mathrm{L}}(\sigma_1+\sigma_2) \;=x^i_{\mathrm{L}}+\frac{p^i_{\mathrm{L}}}{2}(\sigma_1+\sigma_2)+\mathrm{i}\sum_{n\neq 0}\frac{\alpha^i_n}{n}\mathrm{e}^{-\mathrm{i}(\sigma_1+\sigma_2)n}\ , \tag{2.5}$$

$$X^i_{\mathrm{R}}(\sigma_1-\sigma_2) \;=x^i_{\mathrm{R}}+\frac{p^i_{\mathrm{R}}}{2}(\sigma_1-\sigma_2)+\mathrm{i}\sum_{n\neq 0}\frac{\overline{\alpha}^i_n}{n}\mathrm{e}^{-\mathrm{i}(\sigma_1-\sigma_2)n}\ . \tag{2.6}$$

The first terms in each expansion are the contributions to the center of mass coordinate $x^i=x^i_{\mathrm{L}}+x^i_{\mathrm{R}}$ and the zero modes (momenta) $p^i=p^i_{\mathrm{L}}+p^i_{\mathrm{R}}$. In a quantized theory, the oscillators α^i_n and $\overline{\alpha}^i_n$ are eigenvalues of operators $\boldsymbol{\alpha}^i_n$ and $\overline{\boldsymbol{\alpha}}^i_n$. Canonical quantization amounts to requiring the usual commutation relations for position and momentum operators

$$[\boldsymbol{x}^i,\boldsymbol{p}^j]=\mathrm{i}\,\eta^{ij} \tag{2.7}$$

and for the Fourier modes, which in our case take the form

$$[\boldsymbol{\alpha}^i_n,\boldsymbol{\alpha}^j_m] \;=\; n\,\eta^{ij}\delta_{n+m,0}\ , \tag{2.8a}$$

$$[\overline{\boldsymbol{\alpha}}^i_n,\overline{\boldsymbol{\alpha}}^j_m] \;=\; n\,\eta^{ij}\delta_{n+m,0}\ . \tag{2.8b}$$

One can now express the Hamiltonian of the quantized string action in terms of these oscillators

$$\boldsymbol{H} \;=\; \frac{2}{\alpha'}\left(\boldsymbol{L}_0+\overline{\boldsymbol{L}}_0+\,2a\right)\ , \tag{2.9}$$

where we introduced the Virasoro generators $\boldsymbol{L}_m$ and $\overline{\boldsymbol{L}}_m$ for a bosonic theory

$$\boldsymbol{L}_m \;=\; \frac{1}{2}\sum_{n\in\mathbb{Z}}\ :\boldsymbol{\alpha}_{m-n}\cdot\boldsymbol{\alpha}_n:\,, \tag{2.10a}$$

$$\overline{\boldsymbol{L}}_m \;=\; \frac{1}{2}\sum_{n\in\mathbb{Z}}\ :\overline{\boldsymbol{\alpha}}_{m-n}\cdot\overline{\boldsymbol{\alpha}}_n:\,, \tag{2.10b}$$

and the colons stand for normal ordering. For $\boldsymbol{L}_0$, the normal ordering constant a has to be incorporated.

Similar observations can be made for the worldsheet fermions. In particular, it is as well possible to separate the fermionic modes into left- and rightmovers ψ^i_{L} and ψ^i_{R}. However, due to the double cover, the boundary conditions have a richer structure by allowing for extra signs

$$\psi^i_{\mathrm{L}}(\sigma_1,\sigma_2+2\pi) \;=\; \pm\;\psi^i_{\mathrm{L}}(\sigma_1,\sigma_2)\ , \tag{2.11}$$

$$\psi^i_{\mathrm{R}}(\sigma_1,\sigma_2+2\pi) \;=\; \pm\;\psi^i_{\mathrm{R}}(\sigma_1,\sigma_2)\ . \tag{2.12}$$

The choices for the signs are known as Ramond (for sign $+1$) and Neveu–Schwarz (for sign -1). Depending on the sign choice, the modes have different Fourier expansions

$$\psi^i_{\mathrm{L}}(\sigma_1,\sigma_2) \;=\; \sqrt{\alpha'}\sum_{r\in\mathbb{Z}}\boldsymbol{b}^i_r\,\mathrm{e}^{-\mathrm{i}(\sigma_1+\sigma_2)r} \qquad \text{(R)}\ , \tag{2.13}$$

$$\psi^i_{\mathrm{L}}(\sigma_1,\sigma_2) \;=\; \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\boldsymbol{d}^i_r\,\mathrm{e}^{-\mathrm{i}(\sigma_1+\sigma_2)r} \qquad \text{(NS)}\ , \tag{2.14}$$

with similar expressions for the rightmovers. The Fourier modes now fulfill anticommutation relations

$$\left\{\boldsymbol{b}_r^i, \boldsymbol{b}_s^j\right\} = \eta^{ij}\delta_{r+s,0} , \tag{2.15a}$$

$$\left\{\boldsymbol{d}_r^i, \boldsymbol{d}_s^j\right\} = \eta^{ij}\delta_{r+s,0} . \tag{2.15b}$$

While both sectors are fermions on the worldsheet, only Ramond sector states are target-space fermions, while Neveu-Schwarz sector states are target-space bosons.

We now turn our attention to one-loop partition functions (see [26] for an introduction), which will serve two purposes. First, they are an important tool to understand the construction of heterotic strings, both in ten dimensions and for their compactifications. Second, the partition function will be the main topic of interest in our chapter 3. In general, the one-loop partition function can be viewed as the vacuum-to-vacuum amplitude of a (string) theory. For closed strings, the worldsheet has, as mentioned before, the topology of a cylinder. This cylinder can be closed to form a torus which corresponds to the aforementioned one-loop amplitude, which can also be seen as choosing a boundary condition not only for the coordinate σ_2 (as in (2.2)), but also for the worldsheet time coordinate σ_1. Like any torus, this torus is characterized by a complex structure τ. Then, the partition function for a theory with left- and rightmover Hamiltonians $\boldsymbol{H}_{\mathrm{L,R}}$ is defined as

$$\mathcal{Z}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{\boldsymbol{H}_{\mathrm{L}}}\, \bar{q}^{\boldsymbol{H}_{\mathrm{R}}} , \tag{2.16}$$

where $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\mathcal{H} = \mathcal{H}_{\mathrm{L}} \oplus \mathcal{H}_{\mathrm{R}}$ is the Hilbert space of the theory. Note that not only physical (i.e. level-matched) states contribute to the partition function. Note now that on a geometric level, the worldsheet torus is invariant under modular transformations of τ. More specifically, two complex structures τ and τ' span the same torus if they are related by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} , \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) . \tag{2.17}$$

The projective special linear group $\mathrm{PSL}(2,\mathbb{Z})$ is understood as the quotient

$$\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\left\{\mathbb{1}, -\mathbb{1}\right\} . \tag{2.18}$$

A convenient pair of generators for $\mathrm{SL}(2,\mathbb{Z})$ is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{2.19}$$

which are referred to as the modular T- and S-transformations, respectively. Hence, as the geometric torus stays invariant, the partition function must be invariant as well, which is known as modular invariance. We will see how modular invariance can constrain the theory when constructing and compactifying the heterotic superstring, but first we introduce the particular contributions of worldsheet bosons and fermions to the string partition function.

Let us start with the discussion of the partition function of a free bosonic mode. In the absence of zero modes, a leftmoving boson is labeled by its oscillator excitations $n_k, k = 1, ...$ and contributes with

$$\mathcal{Z}(\tau)_{\text{bos}} \;=\; \text{Tr}_{\mathcal{H}}\, q^{\boldsymbol{L}_0 - \frac{1}{24}} \;=\; q^{-\frac{1}{24}}\, \text{Tr}_{\mathcal{H}}\, q^{\sum_{k=1}^{\infty} k n_k} \;=\; q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \frac{1}{1-q^k} \;=\; \frac{1}{\eta(\tau)}\;, \tag{2.20}$$

where the Dedekind eta-function is introduced (see appendix A for details). The above expression contains only the contributions from stringy oscillators. One can also incorporate zero modes. Assume we consider d bosons, which means that we also have to sum over d-dimensional zero modes P that lie in a certain d-dimensional lattice Λ. Then, the partition function reads

$$\mathcal{Z}(\tau)_{\text{bos}} \;=\; \frac{1}{\eta(\tau)^d} \sum_{P\in\Lambda} q^{\frac{1}{2}P^2}\;. \tag{2.21}$$

Here, the requirement of modular invariance permits us to make two important constraints on the lattice Λ. Namely, invariance under the modular T transformation requires the lattice to be even, i.e. $P^2 \in 2\mathbb{Z}$, and invariance under S implies that Λ must be self-dual.

For fermionic states, the partition function is a little more complicated, as one can have periodic and anti-periodic boundary conditions in order to close the cylinder to a torus, just like for the boundary condition in the σ_2 direction. It turns out that periodic boundary conditions can be achieved by inserting a factor $(-1)^F$, where F is the target-space fermion number in the trace. Let us indicate periodic and anti-periodic boundary conditions by a superscript $+$ or $-$, respectively. For a left-moving fermion in the Neveu–Schwarz sector, the partition function for anti-periodic boundary conditions reads

$$\mathcal{Z}(\tau)^{-}_{\text{NS}} \;=\; \text{Tr}_{\mathcal{H}}\, q^{\boldsymbol{L}_0 - \frac{1}{48}} \;=\; \sqrt{\frac{\vartheta_3(\tau)}{\eta}}\;. \tag{2.22}$$

Here, we introduced the Jacobi theta-function (cf. appendix A)

$$\vartheta_3(\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\;. \tag{2.23}$$

In the same way, a fermionic leftmover with periodic boundary conditions in σ_1 direction and NS boundary conditions in the σ_2 direction contributes with

$$\mathcal{Z}(\tau)^{+}_{\text{NS}} \;=\; \text{Tr}_{\mathcal{H}}\, q^{\boldsymbol{L}_0 - \frac{1}{48}}(-1)^F \;=\; \sqrt{\frac{\vartheta_4(\tau)}{\eta}}\;, \tag{2.24}$$

where

$$\vartheta_4(\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \mathrm{e}^{\mathrm{i}\pi n}\;. \tag{2.25}$$

By the same token, the contribution of a left-moving fermion in the Ramond sector yields

$$\mathcal{Z}(\tau)^{-}_{\text{R}} \;=\; \text{Tr}_{\mathcal{H}}\, q^{\boldsymbol{L}_0 + \frac{1}{24}} \;=\; \sqrt{\frac{\vartheta_2(\tau)}{\eta}}\;, \tag{2.26}$$

where

$$\vartheta_2(\tau) \;=\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \;. \tag{2.27}$$

Moreover, we find that $\mathcal{Z}(\tau)^+_{\mathrm{R}} = 0$. One can now examine the transformation behavior of these expressions under the modular group. One finds that (up to phases)

$$\mathcal{Z}(\tau)^-_{\mathrm{R}} \;\overset{S}{\longleftrightarrow}\; \mathcal{Z}(\tau)^-_{\mathrm{NS}} \;\overset{T}{\longleftrightarrow}\; \mathcal{Z}(\tau)^+_{\mathrm{NS}} \;, \tag{2.28}$$

hence, in order to build a modular invariant theory, all three sectors must be present.

2.2.2 Heterotic strings

As mentioned above, heterotic strings [27, 28, 29] are always closed. Therefore, like in any theory of oriented closed strings, the equations of motion of the worldsheet split into a left- and a rightmover part. However, this does not imply that the left- and rightmoving sector have to resemble each other. Instead, it is possible to define the heterotic superstring to consist of a 26-dimensional bosonic leftmover and a ten-dimensional rightmover. In what follows, we will elaborate on this structure more thoroughly.

Leftmover

The heterotic leftmover is a bosonic string with critical dimension 26. Hence, in order to match the ten dimensions of its rightmoving counterpart, 16 dimensions have to be compactified, so that in these 16 dimensions, zero modes have to be taken into account. These zero modes, as discussed in the previous section, have to lie in a 16-dimensional even and self-dual lattice. There are only two 16-dimensional lattices fulfilling that requirement: the root lattices of SO(32) and the one of $\mathrm{E}_8\times\mathrm{E}_8$. Let us focus on the latter case, especially on the partition function of eight bosons on an E_8 lattice. In the bosonic language, the partition function for one E_8 is given by

$$\mathcal{Z}_{\mathrm{E}_8}(\tau) \;=\; \frac{1}{\eta^8}\sum_{P\in\Lambda_{\mathrm{E}_8}} q^{\frac{1}{2}P^2} \;, \tag{2.29}$$

where Λ_{E_8} denotes the E_8 lattice. This is basically the partition function of eight leftmoving bosons compactified on an E_8 lattice. However, it is useful for our purposes later, especially in chapter 3, to bring this partition function to a different form. Namely, $\mathcal{Z}_{\mathrm{E}_8}(\tau)$ can be rewritten as a sum over all $\mathbf{n}\in\mathbb{Z}^8$ instead of the rather cryptic, implicit summation over Λ_{E_8} vectors. However, not all vectors in $\mathbb{Z}^8$ are also in the Λ_{E_8} lattice, and the other way around there are Λ_{E_8} vectors that are not captured by simply summing over $\mathbb{Z}^8$. We will demonstrate here how this problem can be solved systematically. Recall that there are two classes of vectors in Λ_{E_8}. These classes are parametrized by $n_i\in\mathbb{Z}$ and

$$(n_1,\ldots,n_8) \quad \text{with} \quad \sum_i n_i = \text{even} \;, \tag{2.30}$$

$$\left(n_1+\frac{1}{2},\ldots,n_8+\frac{1}{2}\right) \quad \text{with} \quad \sum_i \left(n_i+\frac{1}{2}\right) = \text{even} \;. \tag{2.31}$$

Hence, the partition function (2.29) can be rewritten by summing over all $\mathbf{n} \in \mathbb{Z}^8$, possibly shifting, and then projecting onto vectors that lie either in $\mathbf{n} \in \Lambda_{\mathrm{E}_8}$ or $\mathbf{n} + \mathbf{s}_8 \in \Lambda_{\mathrm{E}_8}$. Inserting appropriate projectors yields

$$\begin{aligned} \mathcal{Z}_{\mathrm{E}_8}(\tau) \;=\; \frac{1}{\eta^8} \Bigg(& \sum_{\mathbf{n}\in\mathbb{Z}^8} q^{\frac{1}{2}|\mathbf{n}|^2} \, \frac{1}{2} \left(1 + \mathrm{e}^{2\pi \mathrm{i} \mathbf{n}\cdot \mathbf{s}_8}\right) \\ & + \sum_{\mathbf{n}\in\mathbb{Z}^8} q^{\frac{1}{2}|\mathbf{n}+\mathbf{s}_8|^2} \, \frac{1}{2} \left(1 + \mathrm{e}^{2\pi \mathrm{i} (\mathbf{n}+\mathbf{s}_8)\cdot \mathbf{s}_8}\right) \Bigg) \;, \end{aligned} \tag{2.32}$$

where we defined the simultaneous half-integer shift in N dimensions

$$\mathbf{s}_N = \underbrace{\left(\frac{1}{2}, \dots, \frac{1}{2}\right)}_{N \text{ entries}} \;. \tag{2.33}$$

Now, this expression can be further simplified by noticing that none of the eight dimensions is special. Hence, rewriting it in terms of the various entries n_i of $\mathbf{n}$ we get

$$\begin{aligned} \mathcal{Z}_{\mathrm{E}_8}(\tau) \;&=\; \frac{1}{2} \frac{1}{\eta^8} \Bigg(\prod_{i=1}^{8} \sum_{n_i\in\mathbb{Z}} q^{\frac{1}{2}n_i^2} \;+\; \prod_{i=1}^{8} \sum_{n_i\in\mathbb{Z}} q^{\frac{1}{2}n_i^2} \, \mathrm{e}^{2\pi \mathrm{i} n_i \frac{1}{2}} \\ & \qquad + \prod_{i=1}^{8} \sum_{n_i\in\mathbb{Z}} q^{\frac{1}{2}\left(n_i+\frac{1}{2}\right)^2} \;+\; \prod_{i=1}^{8} \sum_{n_i\in\mathbb{Z}} q^{\frac{1}{2}\left(n_i+\frac{1}{2}\right)^2} \, \mathrm{e}^{2\pi \mathrm{i} (n_i+\frac{1}{2})\frac{1}{2}} \Bigg) \\ &=\; \frac{1}{2}\frac{1}{\eta^8} \left(\vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}^8 + \vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}^8 + \vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}^8 + \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}^8 \right) \;=\; \frac{1}{2}\frac{1}{\eta^8} \sum_{r,s=0}^{1} \vartheta\begin{bmatrix} \frac{r}{2} \\ \frac{s}{2} \end{bmatrix}^8 \;, \end{aligned} \tag{2.34}$$

where we used the definitions of the Jacobi theta-functions given in appendix A. As one observes, this is the partition function of 16 leftmoving fermions. Putting the two E_8s and the ten-dimensional leftmoving bosons together, we arrive at the following partition function for the uncompactified heterotic leftmover

$$\mathcal{Z}_{\mathrm{L}}(\tau) \;=\; \frac{1}{4}\frac{1}{\eta^{24}} \left(\sum_{r,s=0}^{1} \vartheta\begin{bmatrix} \frac{r}{2} \\ \frac{s}{2} \end{bmatrix}^8 \right)^2 \;. \tag{2.35}$$

The CFT vertex operator of states counting towards this sector reads

$$\left(\partial X^i\right)^{N_i} \left(\partial X^{\bar{\imath}}\right)^{\widetilde{N}_{\bar{\imath}}} \mathrm{e}^{2\pi \mathrm{i} P\cdot X} \;, \tag{2.36}$$

where N_i and $\widetilde{N}_{\bar{\imath}}$ are oscillator numbers and P is an $\mathrm{E}_8 \times \mathrm{E}_8$ vector. In principle, there can be contributions from zero-modes, however those will be neglected here as the resulting states are generically massive.

Rightmover

Let us now turn to the heterotic rightmover. From a worldsheet perspective, heterotic string theory has eight bosons and eight fermions in the rightmoving sector, leading to a partition function

$$\mathcal{Z}_{\mathrm{R}}(\overline{\tau}) \;=\; \frac{1}{2}\frac{1}{\overline{\eta}^{12}}\left[\overline{\vartheta}_2^4 - \overline{\vartheta}_3^4 - \overline{\vartheta}_4^4\right] \;. \tag{2.37}$$

Let us briefly discuss how this result, especially with the relative minus signs, comes about. We may as well go to a bosonized language in which the eight (worldsheet) fermions can be described by four (worldsheet) bosons, each associated to one complex plane. Their zero-modes are bound to lie in the weight lattice of SO(8). Hence, in that case the partition function (including a factor $\overline{\eta}^8$ for the rightmoving bosons) is given by

$$\mathcal{Z}_{\mathrm{R}}(\overline{\tau}) \;=\; \frac{1}{\overline{\eta}^{12}} \sum_{Q\in\Lambda_{\mathrm{SO}(8)}} \overline{q}^{\frac{1}{2}Q^2} \cdot (-1)^F \;, \tag{2.38}$$

where the space-time fermion operator $(-1)^F$ has to be inserted because the zero-modes Q run over bosonic and fermionic contributions. The SO(8) weight lattice $\Lambda_{\mathrm{SO}(8)}$ is the direct sum of the bosonic ("vector") lattice $\mathbf{8}_v$ and the "spinorial" $\mathbf{8}_s$ lattice

$$\Lambda_{\mathrm{SO}(8)} \;=\; \Lambda_{\mathbf{8}_v} \oplus \Lambda_{\mathbf{8}_s} \;. \tag{2.39}$$

These lattices are in turn given by

$$\Lambda_{\mathbf{8}_v} \;=\; \left\{ \mathbf{n} \in \mathbb{Z}^4 \,|\, \sum_i n_i = \mathrm{odd} \right\} \;, \tag{2.40}$$

$$\Lambda_{\mathbf{8}_s} \;=\; \left\{ \mathbf{n} + \mathbf{s}_4 \,|\, \mathbf{n} \in \mathbb{Z}^4,\, \sum_i n_i = \mathrm{even} \right\} \;. \tag{2.41}$$

All states in the $\Lambda_{\mathbf{8}_v}$ lattice are space-time bosons and have $F = \mathrm{even}$, whereas all states in the $\Lambda_{\mathbf{8}_s}$ lattice have $F = \mathrm{odd}$ as they carry half-integer spin. Thus, one gets a relative minus sign between the NS and R sector of the theory. From here on we can follow the same steps as for the $\mathrm{E}_8 \times \mathrm{E}_8$ lattice and find that

$$\mathcal{Z}_{\mathrm{R}}(\overline{\tau}) \;=\; \frac{1}{2}\frac{1}{\overline{\eta}^{12}} \sum_{r,s=0}^{1} (-1)^{r+s+rs}\; \overline{\vartheta}\begin{bmatrix} \frac{r}{2} \\ \frac{s}{2} \end{bmatrix}^4 \;, \tag{2.42}$$

which coincides with (2.37) and where we can identify $(-1)^{r+s+rs}$ with the GSO projector needed for modular invariance. It is interesting to notice that the fermionic partition function (2.37) vanishes identically (by virtue of Jacobi's "abstruse" identity, see appendix A), which is a manifestation of intact target-space supersymmetry. The corresponding vertex operator for a worldsheet fermion in bosonized formulation reads

$$\mathrm{e}^{2\mathrm{i}q\cdot H} \;, \tag{2.43}$$

where H is a set of four bosonized coordinates whereas the so-called H-momentum q is an SO(8) weight vector from either $\Lambda_{\mathbf{8}_v}$ or $\Lambda_{\mathbf{8}_s}$.[1]

[1]This naming is by convention. Of course, the H-momentum must not be confused with the exponentiation of the worldsheet modulus $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$.

Spectrum

Let us here discuss the physical states $|\Phi\rangle$ in ten-dimensional heterotic string theory, especially the massless ones. One first studies the possible left- and rightmover states $|\Phi\rangle_{\mathrm{L}}$ and $|\Phi\rangle_{\mathrm{R}}$ separately and then considers their tensor products

$$|\Phi\rangle \;=\; |\Phi\rangle_{\mathrm{L}} \otimes |\Phi\rangle_{\mathrm{R}} \;, \tag{2.44}$$

where we have to insist that the masses of the left- and rightmovers have to be equal due to the level-matching condition. We start with the leftmover where a generic state $|\Phi\rangle_{\mathrm{L}}$ is labeled by

$$|\Phi\rangle_{\mathrm{L}} \;=\; \alpha^{I_1}_{-m_1} \ldots \alpha^{I_k}_{-m_k} |P\rangle_{\mathrm{L}} \;, \tag{2.45}$$

hence by the action of a set of oscillators $\alpha^{I_i}_{-m_i}$ on a ground state with $\mathrm{E}_8 \times \mathrm{E}_8$ momentum P. Let us now have a look at the leftmover mass equation with the total leftmoving oscillator number N_{L}

$$\frac{1}{4} M_{\mathrm{L}} \;=\; N_{\mathrm{L}} + \frac{P^2}{2} - 1 \;, \tag{2.46}$$

where we set $\alpha' = 1/2$ and from which we see that a massless state can either have one oscillator $\alpha^{I_i}_{-1}$ and $P = 0$ or no oscillator and $P^2 = 2$. For the rightmover in bosonized formulation, a general state is given by bosonic oscillators acting on a ground state labelled by a certain H-momentum q from $\Lambda_{\mathrm{SO}(8)}$, hence with $q^2 \geq 1$

$$|\Phi\rangle_{\mathrm{R}} \;=\; \overline{\alpha}^{I_1}_{-m_1} \ldots \overline{\alpha}^{I_k}_{-m_k} |q\rangle_{\mathrm{R}} \;. \tag{2.47}$$

As the mass equation for the rightmover reads

$$\frac{1}{4} M_{\mathrm{R}} \;=\; \overline{N}_{\mathrm{R}} + \frac{q^2}{2} - \frac{1}{2} \;, \tag{2.48}$$

we observe that a massless string cannot have rightmoving oscillators and therefore a massless rightmover is labeled by its H-momentum q only. Now, physical states are obtained by tensoring any of the left- and rightmovers we just described with the same mass together. In ten dimensions, if we fix the rightmover to be $|\Phi\rangle_{\mathrm{R}} = |q^2 = 1\rangle$, the most prominent states arising from the various possible choices for the leftmover are:

1. Choosing $|\Phi\rangle_{\mathrm{L}} = \alpha^{\mu}_{-1} |P = 0\rangle$ with a space-time index μ yields the gravity multiplet, hence a symmetric traceless tensor for the graviton, an antisymmetric tensor for the Kalb–Ramond B-field, and the scalar dilaton.

2. Choosing $|\Phi\rangle_{\mathrm{L}} = \alpha^{I}_{-1} |P = 0\rangle$ with an index I in the $\mathrm{E}_8 \times \mathrm{E}_8$ coordinates yields the Cartan generators of $\mathrm{E}_8 \times \mathrm{E}_8$, whereas

3. Choosing $|\Phi\rangle_{\mathrm{L}} = |P^2 = 2\rangle$ gives rise to the ladder operators of the $\mathrm{E}_8 \times \mathrm{E}_8$ Lie-Algebra.

Together with their superpartners, these states are the only massless string states in ten dimensions. From a four-dimensional perspective, the resulting theory looks like an $\mathcal{N} = 4$ supergravity with an $\mathrm{E}_8 \times \mathrm{E}_8$ gauge symmetry. This theory is unattractive because $\mathcal{N} = 4$ is non-chiral. Moreover, one needs to reduce the number of space-time dimensions from ten to four observed dimensions.

2.3 Orbifolds

The purpose of this section is twofold: first, we will introduce toroidal orbifolds as compact spaces, and afterwards we discuss the boundary conditions of closed strings compactified on them. At the end of the chapter, we discuss how the non-chiral ten-dimensional string spectrum gets altered by compactification. In particular, we show how it can yield a chiral four-dimensional spectrum that has ultimately the chance to resemble the (supersymmetric) standard model.

2.3.1 Geometric construction

Formally, orbifolds (initially called V-manifolds) are defined as topological Hausdorff spaces with certain properties [30]. For us, a more practical definition in terms of tori will be useful. In particular, a d-dimensional orbifold can be defined in two steps

1. Define a d-dimensional lattice Λ, and divide out $\mathbb{R}^d/\Lambda$. This compactifies $\mathbb{R}^d$ to a d-dimensional torus $\mathbb{T}^d_\Lambda$ with underlying lattice Λ. In particular, one chooses a vielbein e, i.e. a $d \times d$ matrix containing a set of basis vectors of Λ as columns. Then, the lattice can be parametrized as

$$\Lambda \;=\; \left\{ e\, m \mid m \in \mathbb{Z}^d \right\} \;. \tag{2.49}$$

2. The lattice Λ may possess some isometries, that can be divided out as well. These isometries can be represented as discrete rotations that can (in some cases) be accompanied by translations by fractional lattice vectors (roto-translations). The rotations form a finite discrete group, the point group P which can be Abelian (e.g. $\mathbb{Z}_N, \mathbb{Z}_N \times \mathbb{Z}_M$), but also non-Abelian (e.g. S_3 or A_4). Dividing out these isometries yields the orbifold $\mathcal{O} = \mathbb{T}^d_\Lambda/P$.

This procedure can be reduced to a single step by dividing out the so-called space group $\mathcal{S}$. Elements of the space group can be used to define an equivalence relation for points in $\mathbb{R}^d$

$$X \;\sim\; X' \;\Leftrightarrow\; \exists g \in \mathcal{S} :\; X \;=\; g\, X' \;. \tag{2.50}$$

Furthermore, elements of the space group can be written to have a rotational (coming from the rotations in the point group P) and a translational part (coming from the lattice Λ and possibly the roto-translations). Therefore, one can denote a general space group element g by

$$g \;=\; (\theta \,|\, n_\alpha\, e_\alpha + t_\theta) \;. \tag{2.51}$$

Here, θ is taken to be an element of the finite discrete group P, while t_θ is a possible roto-translation. Usually, t_θ is a fractional lattice vector. The $e_\alpha, \alpha = 1, \ldots, d$ are the basis vectors of Λ, and $n_\alpha \in \mathbb{Z}$. For reasons of readability, we ignore the possibility of roto-translations for the moment, and just keep in mind that they transform as vectors under the rotations. Then, the multiplication law for two space group elements reads

$$(\theta_1 \,|\, n_\alpha\, e_\alpha) \cdot (\theta_2 \,|\, m_\alpha\, e_\alpha) \;=\; (\theta_1\theta_2 \,|\, n_\alpha\, e_\alpha + D_{\mathbf{v}}(\theta_1)\, m_\alpha\, e_\alpha) \;, \tag{2.52}$$

where we introduce the vector representation $D_\mathbf{v}$ of the point group P, i.e. its geometric action on the lattice vectors in the vielbein e (more on this in chapter 3)

$$e \stackrel{\theta}{\longmapsto} D_\mathbf{v}(\theta)\, e\,. \tag{2.53}$$

Because of the group property, there exists a unique identity element $(\mathbb{1}\,|\,0)$, and hence for each element $g = \left(\theta^k \,\middle|\, n_\alpha\, e_\alpha\right)$ there is also its unique inverse

$$g^{-1} \;=\; \left(\theta^{-1} \,\middle|\, -D_\mathbf{v}(\theta^{-1})\, n_\alpha\, e_\alpha\right)\,, \tag{2.54}$$

such that $g^{-1}g = (\mathbb{1}\,|\,0)$. Here, a few comments are in order:

- Even if the point group P is chosen to be Abelian, the space group is in general non-Abelian.

- While the number of elements in $\mathcal{S}$ is infinite, elements with non-trivial rotational part can be arranged in a finite number of conjugacy classes.

For our later purposes we will need orbifolds with an even number of dimensions. Therefore, it is useful to switch from d real coordinates to $d/2$ complex ones, and work with the $d/2$ resulting complex planes rather than pairs of real coordinates. In that setup, it is also useful to diagonalize the $d \times d$ rotation matrices to read

$$D_\mathbf{v}(\theta) \;=\; \mathrm{diag}\left(\mathrm{e}^{2\pi\mathrm{i}v_\theta^{(1)}}, \mathrm{e}^{2\pi\mathrm{i}v_\theta^{(2)}}, \ldots, \mathrm{e}^{2\pi\mathrm{i}v_\theta^{(d/2)}}\right)\,, \tag{2.55}$$

for some element of the point group θ and where v_θ is then (in the string literature) referred to as the twist vector.[2] In our case, where we consider ten-dimensional superstrings, we need a six-dimensional orbifold with three complex planes.

At this point, it is natural to ask which lattices and point groups exist in six dimensions. In fact, a comprehensive enumeration and classification of all possible orbifolds has been performed in the CARAT-classification [31, 32]. Moreover, there exists a complete classification of the subset of those orbifolds that allow for $\mathcal{N} = 1$ target-space supersymmetry in [33]. In general, the classification process is performed in the following manner: First, one picks an (abstract) point group P and makes a choice how this group acts geometrically (i.e. one specifies the vector representation $D_\mathbf{v}$ of P), which gives the so-called $\mathbb{Q}$-class. Note that in general, there can be more than one inequivalent geometric action for an abstract point group (for example with $P = \mathbb{Z}_6$, there are the well studied $\mathbb{Z}_{6-I}$ and $\mathbb{Z}_{6-II}$ orbifolds). Then, one specifies a lattice (in terms of a vielbein e), which yields the so-called $\mathbb{Z}$-class. Finally, a given point group acting in the specified way on a lattice may allow for a number roto-translations and different actions of the point group on the lattice. Fixing this (i.e. making a choice for the roto-translations) yields the affine class.

[2] By abuse of notation, one often denotes the twist vector of a *space* group element g by v_g, where in fact, one means the twist vector of the rotational part of g.

2.3.2 Closed strings on orbifolds

If one considers a closed string theory on a given manifold, one has to take *any* closed string boundary condition possible on that manifold into account, for reasons of self-consistency (see below for details). On orbifolds, the equivalence relation of identified points (2.50) allows one to find more general closed string boundary conditions than (2.2), namely

$$X(\sigma_1, \sigma_2 + 2\pi) \;=\; g\, X(\sigma_1, \sigma_2)\;. \tag{2.56}$$

In other words, the string closes only up to a translation or rotation (or a combination thereof) from the space group $\mathcal{S}$. One can now make a useful distinction between different classes of strings on orbifolds:

1. *Bulk strings.* These strings are closed already in ten dimensions and have hence the trivial constructing element $g = (\mathbb{1}\,|\,0)$. However, generically not the entire spectrum is orbifold invariant, so that some of the ten-dimensional states are projected out.

2. *Winding strings.* In this case, the string closes up to a lattice vector (i.e. already on the torus), hence the boundary condition reads

   $$X(\sigma_1, \sigma_2 + 2\pi) \;=\; X(\sigma_1, \sigma_2) + n_\alpha e_\alpha\;. \tag{2.57}$$

   As with the bulk strings, some states only exist on the torus but not on the final orbifold. Because their non-trivial winding enters the mass equation, these strings are massive at generic points of the moduli space. However, at special values of the orbifold radii, some winding states can become massless.

3. *Twisted strings.* Finally, if the boundary condition involves a rotation, one finds strings that only close on the orbifold. These states are always associated with a non-trivial monodromy around a curvature singularity, in most cases these are fixed points.

2.3.3 Geometric eigenstates

For any string fulfilling (2.56), this boundary condition can be rewritten using another space group element h to read

$$hX(\sigma_1, \sigma_2 + 2\pi) \;=\; hgh^{-1}h\, X(\sigma_1, \sigma_2)\;. \tag{2.58}$$

Then, by using the fact that X and hX are identical by definition of the orbifold, this yields

$$X(\sigma_1, \sigma_2 + 2\pi) \;=\; hgh^{-1}\, X(\sigma_1, \sigma_2)\;, \tag{2.59}$$

which is the same as the original boundary condition (2.56) but with g replaced by hgh^{-1}. Hence, physical states are not in one-to-one correspondence to particular space group elements, but rather to *conjugacy classes* of space group elements, and a general g-twisted state can be written as

$$|[g]\rangle \;=\; \sum_h \alpha(g,h)\, |hgh^{-1}\rangle\;, \tag{2.60}$$

with phases $\alpha(g,h)$ that need to be fixed. Conventionally, these phases are parametrized as $\alpha(g,h) = \mathrm{e}^{-\mathrm{i}\pi\gamma(g,h)}$, where the γ-phase is introduced. The next thing one can read off is that if g and h commute, any physical state must be invariant under the action of h for reasons of consistency. This will become important for the orbifold projection. Note that $\gamma(g,h) = 0$ if g and h commute, hence this phase will play no role for the orbifold projection. To each conjugacy class representing a physical state with boundary condition g, there is the corresponding CFT vertex operator σ_g [34].

2.3.4 Orbifold partition function

Considering the partition function, the same types of orbifold boundary conditions can be chosen for closing the worldsheet cylinder to a torus. With that regard, the discussion closely resembles that of the fermion partition function. However, if g is the constructing element, any element h that is used to close the cylinder to the torus must commute with g, or in the language of group theory, lie in the centralizer of g in the space group. Then, the orbifold partition function can be decomposed into sectors (g,h),

$$\mathcal{Z}_{\mathrm{orb}}(\tau,\overline{\tau}) \;=\; \sum_{g,h} \mathcal{Z}\begin{bmatrix} g \\ h \end{bmatrix}(\tau,\overline{\tau})\;, \tag{2.61}$$

where the sum over g runs over the entire space group and the sum over h is restricted to the centralizer C_g. In a (g,h) sector, h is usually called the projecting element. This is because, as we just discussed, g-twisted states must be invariant under the action of any commuting element h. In the partition function, this can be made visible if one takes the double sum in equation (2.61) apart

$$\mathcal{Z}_{\mathrm{orb}}(\tau,\overline{\tau}) \;=\; \sum_{g} \left(\mathcal{Z}\begin{bmatrix} g \\ \mathbb{1} \end{bmatrix}(\tau,\overline{\tau}) + \mathcal{Z}\begin{bmatrix} g \\ h_1 \end{bmatrix}(\tau,\overline{\tau}) + \mathcal{Z}\begin{bmatrix} g \\ h_2 \end{bmatrix}(\tau,\overline{\tau}) + \ldots \right)\,. \tag{2.62}$$

As one can see, for each element g (which gives the boundary condition (2.56)), one sums over the unprojected part $(g,\mathbb{1})$ and then all elements from the centralizer so that ultimately only the orbifold-invariant states in a sector with constructing element g survive. However, this arrangement in terms of constructing elements becomes meaningless once one studies the transformation behavior of particular g-twisted sectors under modular transformations on τ as they translate into permuting the (g,h) sectors. Specifically, under an $\mathrm{SL}(2,\mathbb{Z})$ element γ, the partition function transforms as

$$\mathcal{Z}\begin{bmatrix} g \\ h \end{bmatrix}(\gamma\tau) \;=\; \mathcal{Z}\begin{bmatrix} g^a h^c \\ g^b h^d \end{bmatrix}(\tau) \quad \text{for } \gamma \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{2.63}$$

The important point to notice here is that modular transformations exchange twisted and untwisted sectors: for example, the S transformation maps a $(g,\mathbb{1})$-twisted sector to a $(\mathbb{1},g)$-twisted one. Therefore, any modular invariant theory must contain twisted and untwisted sectors alongside each other.

2.4 Heterotic strings on orbifolds

Having seen heterotic superstrings in ten dimensions, we are now prepared to see their compactification on orbifolds [35, 36], for other expositions of the formalism see [37, 38]. The key points will be the following: The first step is to study the *geometric* action of translations and rotations on the six coordinates that lie in the orbifold. Second, one may then *embed* this geometric action in the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge sector, thereby breaking the gauge group to some subgroup. Finally, one may put these parts together and—by requiring invariance—study the resulting allowed states. One observes that not all states that exist in ten dimensions survive the orbifolding, hence this step is referred to as the orbifold projection.

2.4.1 Gauge embedding

In what follows, we assume that we are dealing with Abelian $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds. The construction of gauge embeddings in non-Abelian orbifolds is significantly more involved, but will play no role in the remainder of the thesis and is hence omitted.

Recall that the orbifold space group consists of geometric translations $n_\alpha e_\alpha$ and rotations parametrized by some element of the point group θ. These geometric transformations can be embedded into the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge sector by making simultaneous transformations. The translations are embedded as

$$\left.\begin{matrix} X^i \\ X^I \end{matrix}\right\} \quad \longmapsto \quad \begin{cases} X^i + 2\pi n_\alpha e_\alpha^i \\ X^I + \pi n_\alpha W_\alpha^I \end{cases} , \tag{2.64}$$

where we introduced the Wilson lines W_α [39]. Likewise, one can define the so-called shift V_θ associated to a rotation

$$\left.\begin{matrix} X^i \\ X^I \end{matrix}\right\} \quad \longmapsto \quad \begin{cases} D_{\mathrm{v}}(\theta)^{ij} X^j \\ X^I + \pi V_\theta^I \end{cases} . \tag{2.65}$$

In general, one can associate the local shift $V_g = V_\theta + n_\alpha W_\alpha$ to each space group element $g = (\theta \,|\, n_\alpha e_\alpha)$. This way to embed the space group into the gauge degrees of freedom has the nice property that elements from the same conjugacy class automatically have the same gauge embedding $V_g = V_{hgh^{-1}}$ because the gauge embedding only acts by translations in the $\mathrm{E}_8 \times \mathrm{E}_8$ coordinates.

Both the shifts and the Wilson lines are fractional lattice vectors, i.e. an integer multiple of them lies in the $\mathrm{E}_8 \times \mathrm{E}_8$ lattice again, so that

$$N_\alpha W_\alpha \;\in\; \Lambda_{\mathrm{E}_8 \times \mathrm{E}_8} , \tag{2.66}$$

where N_α is then called the order of the Wilson line. The order of a shift is defined in an analogous manner. As we are going to see later on in this chapter, the shifts and Wilson lines have to fulfill a set of constraints from modular invariance.

2.4.2 Twisted states

For strings with constructing element g, the action on the vertex operators is the following: First, the oscillator numbers N_i and $\widetilde{N_{\bar{\imath}}}$ introduced in equation (2.36) can take fractional values for twisted states. The H-momenta of a twisted state get shifted according to

$$q_{\mathrm{sh}} \;=\; q + v_g \;. \tag{2.67}$$

Moreover, the $\mathrm{E}_8 \times \mathrm{E}_8$ momenta P get shifted by the gauge embedding (the local shift)

$$P_{\mathrm{sh}} \;=\; P + V_g \;. \tag{2.68}$$

In general, a non-trivial P_{sh} indicates that the corresponding state is charged under some of the unbroken gauge symmetries.

2.4.3 The heterotic orbifold projection

Now, we turn to the action of projecting elements. Let us start with the geometric action. One needs to consider the oscillator part in equation (2.36) and the bosonized fermions in equation (2.43). Under the action of a space group element h, the bosonic oscillator excitations transform according to

$$\left(\partial X^i\right)^{N_i} \left(\partial X^{\bar{\imath}}\right)^{\widetilde{N}_{\bar{\imath}}} \;\stackrel{h}{\longmapsto}\; \left(\partial X^i\right)^{N_i} \left(\partial X^{\bar{\imath}}\right)^{\widetilde{N}_{\bar{\imath}}} \mathrm{e}^{2\pi \mathrm{i} v_h^i (N_i - \widetilde{N}_{\bar{\imath}})} \;. \tag{2.69}$$

Likewise, the three components of the H-coordinates in equation (2.43) that lie inside the orbifold transform as

$$H^i \;\stackrel{h}{\longmapsto}\; H^i - \pi v_h^i \;. \tag{2.70}$$

In the same fashion, the gauge embedding causes the $\mathrm{E}_8 \times \mathrm{E}_8$ coordinates to shift

$$X^I \;\stackrel{h}{\longmapsto}\; X^I + \pi V_h^I \;. \tag{2.71}$$

In principle, if the constructing element does not commute with the "projecting" h, the geometric eigenstate picks up a phase

$$|[g]\rangle \;\stackrel{h}{\longmapsto}\; \mathrm{e}^{2\pi \mathrm{i}\gamma(g,h)}\, |[g]\rangle \;. \tag{2.72}$$

However, as we mentioned earlier, only commuting elements need to be projected on. In total, projecting on a commuting element h, which means requiring invariance under the orbifold, amounts to the condition

$$P_{\mathrm{sh}} \cdot V_h - \left(q_{\mathrm{sh}} + N - \widetilde{N}\right) \cdot v_h + \Phi_{\mathrm{vac}}(g,h) \;=\; 0 \bmod 1 \;. \tag{2.73}$$

Here, we define the so-called vacuum phase $\Phi_{\mathrm{vac}}(g,h) \;=\; 1/2\,(V_g \cdot V_h - v_g \cdot v_h)$ whose introduction becomes clear in the next section.

2.4.4 The heterotic partition function on orbifolds

In general, the partition function of heterotic strings on orbifolds can be factorized as

$$\mathcal{Z} \;=\; \mathcal{Z}_{\text{Mink}}(\tau,\overline{\tau})\,\mathcal{Z}_{\text{int}}(\tau,\overline{\tau})\;, \tag{2.74}$$

where the non-compact part $\mathcal{Z}_{\text{Mink}}$ corresponds to the uncompactified bosonic coordinates in the Minkowski space-time, and in light-cone gauge reads

$$\mathcal{Z}_{\text{Mink}}(\tau,\overline{\tau}) \;=\; \frac{1}{\tau_2}\left|\frac{1}{\eta^2(\tau)}\right|^2\;. \tag{2.75}$$

The internal part $\mathcal{Z}_{\text{int}}$ can be organized in (g,h)-twisted sectors as in equation (2.61),[3] where the particular blocks read

$$\mathcal{Z}_{\text{int}}\begin{bmatrix}g\\h\end{bmatrix} \;=\; \mathcal{Z}_{\psi}\begin{bmatrix}g\\h\end{bmatrix}(\overline{\tau})\,\mathcal{Z}_{X}\begin{bmatrix}g\\h\end{bmatrix}(\tau,\overline{\tau})\,\mathcal{Z}_{\text{E}_8\times\text{E}_8}\begin{bmatrix}g\\h\end{bmatrix}(\tau)\,\text{e}^{2\pi\text{i}\Phi_{\text{vac}}(g,h)}\;, \tag{2.76}$$

where $\mathcal{Z}_{\psi}(\overline{\tau})$ is the contribution of the rightmoving fermions (including those in the non-compact directions), $\mathcal{Z}_{X}(\tau,\overline{\tau})$ is the partition function of the six internal bosonic coordinates, and finally $\mathcal{Z}_{\text{E}_8\times\text{E}_8}(\tau)$ is the contribution of the leftmoving gauge coordinates.

Modular invariance of the orbifold partition function then requires the following conditions on the gauge embedding [36, 40, 41]:

$$\gcd(N_i,N_j)\,(V_i\cdot V_j - v_i\cdot v_j) \;= 0 \bmod 2\;, \tag{2.77}$$
$$\gcd(N_i,N_\alpha)V_i\cdot W_\alpha \;= 0 \bmod 2\;, \tag{2.78}$$
$$\gcd(N_\alpha,N_\beta)W_\alpha\cdot W_\beta \;= 0 \bmod 2\;. \tag{2.79}$$

Note how the first condition (2.77) mixes properties of the rightmover partition function $\mathcal{Z}_{\psi}(\overline{\tau})$ and the gauge coordinates $\mathcal{Z}_{\text{E}_8\times\text{E}_8}(\tau)$. In effect, neither $\mathcal{Z}_{\psi}$ nor $\mathcal{Z}_{\text{E}_8\times\text{E}_8}$ need to be modular invariant on their own, but only their combination. In reality, it turns out that orbifold geometries where $\mathcal{Z}_{\psi}$ and $\mathcal{Z}_{\text{E}_8\times\text{E}_8}$ can be constructed to be separately modular invariant are the exception rather than the rule. Therefore, in all other cases one is forced to embed the geometric twist v_i into the gauge degrees of freedom with a non-trivial V_i. The inclusion of the vacuum phase becomes necessary to ensure consistency under T^N.

[3] In fact, it was shown in [40] that for a number of orbifold geometries one can introduce discrete torsion phases $\epsilon(g,h)\neq 1$, so that (for a given set of shifts and Wilson lines) the internal partition function reads

$$\mathcal{Z}_{\text{int}} \;=\; \sum_{g,h}\epsilon(g,h)\mathcal{Z}\begin{bmatrix}g\\h\end{bmatrix}\;.$$

However, these phases can be compensated by making an appropriate choice of shifts and Wilson lines [41].

2.5 Heterotic model building

Finding a heterotic four-dimensional vacuum starts with a two-step procedure: First, one chooses one of the admissible orbifold geometries, and second one picks a gauge embedding consisting of shifts and Wilson lines. Apart from reducing the dimensionality to four dimensions, and possibly breaking supersymmetry (see the next chapter for details), the effect of the orbifolding on the ten-dimensional heterotic string is then also twofold.

First, not all ten-dimensional states survive the orbifold projection (2.73). Most notably, a non-trivial gauge embedding leads to projection conditions on the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge bosons

$$P \cdot V_h \stackrel{!}{=} 0 \bmod 1 \,. \tag{2.80}$$

Because the gauge bosons have trivial boundary conditions $g = (\mathbb{1}\,|\,0)$, this condition needs to be fulfilled for all $h \in \mathcal{S}$. One observes that the Cartan generators (with $P_{\mathrm{sh}} = 0$ and an oscillator) trivially survive this projection, and only certain ladder operators (corresponding to simple roots of $\mathrm{E}_8 \times \mathrm{E}_8$) can be projected out. Consequently, the gauge group can be broken to subgroups of $\mathrm{E}_8 \times \mathrm{E}_8$ with rank 16.

On the other hand, we have seen that any self-consistent orbifold theory has to have twisted states in its spectrum. Generically, these localized states transform under non-trivial representations under the unbroken gauge symmetry (namely when their $P_{\mathrm{sh}} \neq 0$), and also they have the chance to be chiral. Moreover, this chiral matter is also charged under the unbroken subgroup of $\mathrm{E}_8 \times \mathrm{E}_8$.

The interplay of the breaking of the ten-dimensional $\mathrm{E}_8 \times \mathrm{E}_8$ to subgroups and the emergence of charged matter open up the road for heterotic model building: If one makes the right choices for the gauge embedding acting on the leftmoving degrees of freedom, one can end up with e.g. the chiral spectrum of the MSSM (provided that the remaining target-space supersymmetry is $\mathcal{N} = 1$) as it was successfully demonstrated in [42, 43, 44]. The emergent picture is that all admissible (i.e. modular invariant) solutions for the gauge embeddings create an entire landscape of models. Those models that give rise to MSSMs are then embedded in this landscape. As studied first in [42, 44], some subsets of the landscape are more likely to host an MSSM than others, giving rise to so-called “fertile islands”, for a depiction see figure 2.1. The statistics of MSSMs versus non-MSSMs were studied in [45], whereas novel approaches to the identification of fertile patches in the heterotic string landscape building on methods from machine learning include [20, 46, 47], see ref. [48] for a comprehensive review of machine learning approaches in string theory. The connection between heterotic orbifold models and heterotic CY model building has been studied e.g. in [49].

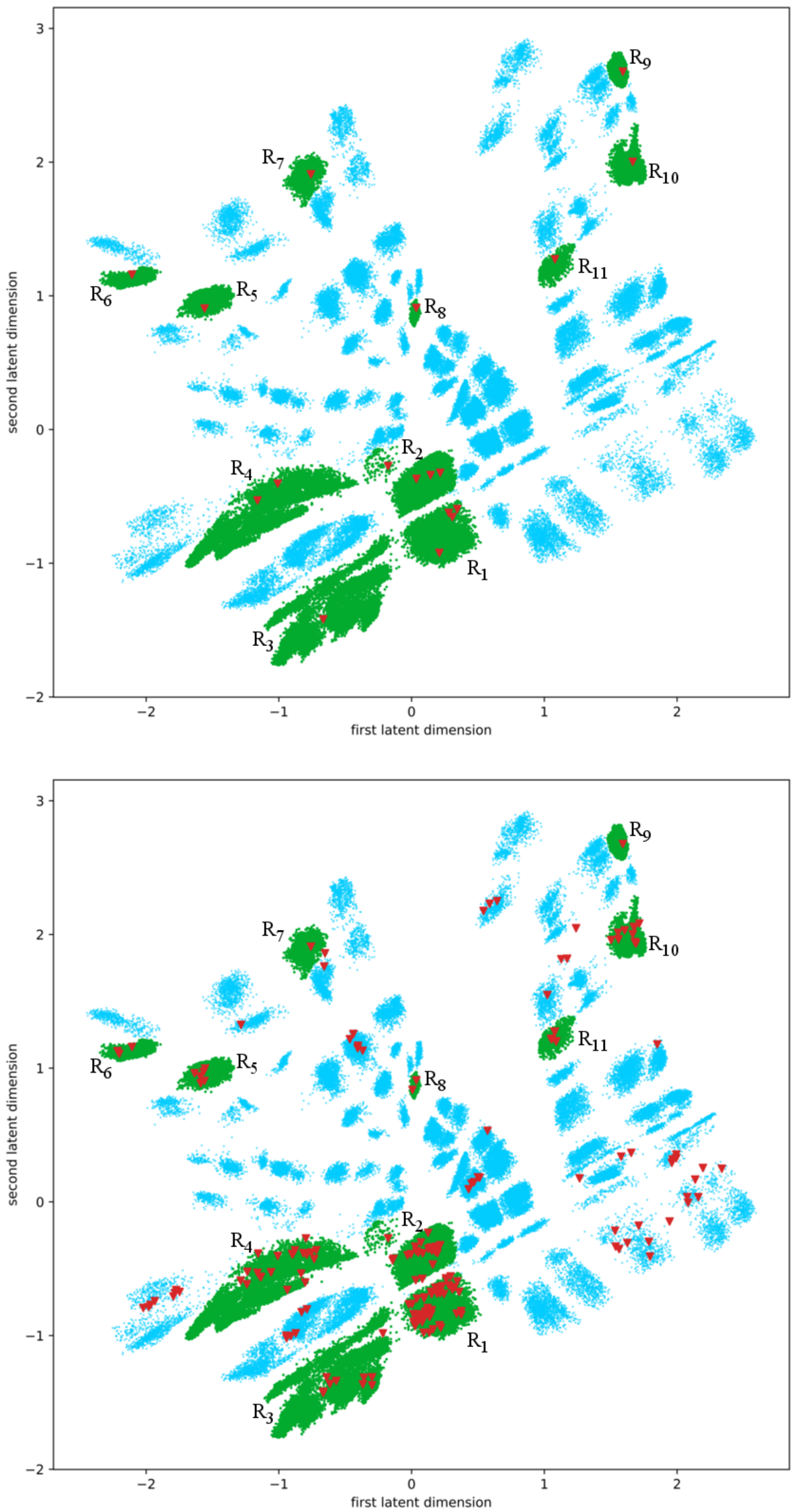


Figure 2.1: *A depiction of the heterotic landscape as it is seen by a neural network. In both frames, blue and green dots represent non-MSSM models, whereas red triangles denote three-generation MSSMs. Upper panel: A small subset of MSSMs is used to identify fertile islands (marked in green). Lower panel: A large portion of all MSSMs indeed lives on these fertile islands, apart from some outliers. Figure adapted from [20].*

3

The cosmological constant in non-supersymmetric compactifications

In our current view of cosmology, a tiny but non-zero cosmological constant drives the accelerated expansion of the universe today [50]. So far, no definite answer has been found why the cosmological constant is many orders of magnitude smaller than the other known scales of fundamental physics, like the Planck, GUT or electroweak scale. As all attempts to compute the cosmological constant in quantum field theory lead to unrealistic results, it is assumed that a UV complete quantum theory of gravity is necessary to determine its value from first principles. In many instances, string theory has been proposed as such a theory. In the case of the heterotic string, the one-loop values of the cosmological constant and the dilaton tadpole are related. As anything but a small dilaton tadpole would either thwart vacuum stability, or, if it is cancelled [51, 52] drastically modify the structure of the resulting model, any physically meaningful heterotic model must have a tiny, if not vanishing, cosmological constant as well. In situations like these, a symmetry is often called to the rescue. Ideally, this symmetry would require an exactly vanishing cosmological constant as long as it is intact. Once it is broken (e.g. spontaneously), small values for the cosmological constant would then be generated.

In fact, all these problems would be solved by a symmetry, namely target-space supersymmetry: not only does the cosmological constant (along with the dilaton tadpole) vanish identically at one loop in SUSY theories, one also has good reasons to assume that this property persists to all orders in perturbation theory. Then, an eventually broken SUSY would reintroduce a small but finite value for the cosmological constant. In recent years, this scenario has lost some of its attractiveness due to the non-observation of superpartners in experiments like e.g. the LHC. As the SUSY breaking scale is driven to higher and higher values in order to match experimental observations, more and more finetuning is necessary for the cosmological constant. In the light of this fact, non-supersymmetric string models, after their conception in [53, 54], have received renewed interest in recent years, especially on the heterotic side. It has been demonstrated in various instances that it is well possible to construct non-supersymmetric string models with almost exactly the spectrum of the standard model, both by considering the manifestly non-supersymmetric $O(16)\times O(16)$ string or in compactifications that break supersymmetry [53, 55]. However, while there have been constructions with exponentially small cosmological constants [56, 57], heterotic models with a vanishing one have yet to be constructed.

The purpose of this chapter is to study why it is so hard to achieve a vanishing cosmological constant in non-supersymmetric heterotic orbifolds. To this end, we start with a discussion of how the model-independent vanishing of the cosmological constant is linked to certain properties of the one-loop partition function, namely that the partition function vanishes in each twisted sector separately. Then, we discuss how this property depends on whether or not a model has target-space supersymmetry. In fact, it turns out that the partition function automatically vanishes sector by sector once there is at least $\mathcal{N} = 1$ SUSY, and that it can never do so once SUSY is broken, which follows from generic arguments. However, as we also show explicitly, it is well possible to construct models where the global amount of supersymmetry is $\mathcal{N} = 1$, but where each twisted sector admits $\mathcal{N} = 2$ or higher. One is tempted to ask why it should be impossible to have $\mathcal{N} = 0$ globally but at least $\mathcal{N} = 1$ in each twisted sector, which would cause the partition function to vanish (and with it the cosmological constant and the dilaton tadpole). This idea has been successfully applied to asymmetric compactifications of non-supersymmetric type II strings [58, 59]. The reason why this idea does not work in the case of heterotic orbifolds is shown to lie in the way target-space rotations act on world-sheet spinors. Hence, we are able to formulate the problem of a non-vanishing partition function in terms of the representation theory of the orbifold point group and its possible spinor embedding(s). Therefore, the rest of the chapter is devoted to a discussion of the group-theoretical properties of finite discrete groups. We show that the sector-per-sector vanishing of the partition function would require the existence of four-dimensional representations of the point group (corresponding to the action of the point group on space-time spinors) that have to fulfill a set of necessary conditions. However, it turns out that, as expected, none of the spinor embeddings has the required properties. Moreover, it is impossible to construct any representation with precisely these properties for any of the possible point groups in six-dimensional orbifolds. Moreover, we show that the non-existence of these representations seems to be independent of string compactifications and is hence formulated as a general group-theoretical conjecture. However, we also observe that there are often sectors in the non-supersymmetric partition function that, although there is no overall SUSY, vanish by themselves, thereby providing valuable input for more involved scenarios. This chapter is in parts based on ref. [15].

3.1 The cosmological constant in heterotic orbifolds

Much like in quantum field theory, the cosmological constant in heterotic orbifold models is defined as the zero-point or vacuum energy. At one-loop level, it is proportional the integral over the partition function

$$\Lambda \sim \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \mathcal{Z}(\tau,\bar\tau) = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \mathcal{Z}_{\text{Mink}}(\tau,\bar\tau) \sum_{g,h} \mathcal{Z}_{\text{int}}\begin{bmatrix} g \\ h \end{bmatrix}(\tau,\bar\tau) \,. \tag{3.1}$$

Here, $\mathcal{F} = \{\tau | -\frac{1}{2} < \tau_1 < \frac{1}{2}, \tau_2 > 0, |\tau| > 1\}$ denotes the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ and $\tau_2^{-2}\mathrm{d}^2\tau$ is the modular invariant Poincaré measure. As discussed in the introduction, observations indicate that the value of Λ takes a finite but tiny value. Moreover a non-zero value of the integral (3.1) would also imply non-vanishing dilaton tadpoles. Therefore, it seems desirable that the cosmological constant given by the integral (3.1) should vanish at the string level, and receive non-zero contributions only by low-energy physics. Therefore, one's primary interest lies in constructions where the integral in (3.1) vanishes. Here, different levels of abstraction come to mind:

1. **The integrand of (3.1) vanishes, because each (g, h)-sector in the internal partition function vanishes identically by itself.**

   This option is the most straightforward one and is realized for supersymmetric compactifications. In the SUSY case, the partition function vanishes due to generalizations of Jacobi's abstruse identity (referred to as Riemann identities). We will show that for generic reasons only the fermionic partition function can vanish at all, both in the supersymmetric theories and in the non-SUSY case. In any case the result does not depend on the gauge embedding and is therefore rather model-independent.

2. **The full partition function vanishes, but some (g, h)-sectors in the internal partition function are non-zero.**

   Because $\mathcal{Z}_{\text{Mink}}$ is non-zero, the full partition function can only vanish if the internal partition function does so. Hence, as we assume that some of the (g, h)-sectors yield non-zero contributions, there need to be non-trivial cancellations between them. This option will therefore depend not only on the geometric action of the orbifold but also on the gauge embedding.

3. **The integrand of (3.1) is non-vanishing, however it integrates to zero over the fundamental domain $\mathcal{F}$.**

   This possibility has been considered in the context of (generalized) Atkin–Lehner symmetry [60, 61], and in general requires (like option 2.) special properties of the entire partition function. Moreover, there exist no-go theorems [62] that indicate that it might be either impossible or at least very hard to construct non-vanishing partition functions that integrate to zero over $\mathcal{F}$ using the known Atkin–Lehner mechanism in models with broken SUSY.

We will see that already due to generic arguments, the partition function cannot vanish once SUSY is broken. Options 1. and 2. are therefore ruled out, and the last hopes for an exactly vanishing cosmological constant in non-supersymmetric heterotic strings lie in a version of the third option, which is, as we mentioned, already challenged by the fact that the known constructions do not work. However, finding generalizations or alternatives to the Atkin–Lehner construction that avoid the no-go theorems is a daunting task that certainly requires a detailed understanding of the non-supersymmetric orbifold partition function. Therefore, the remainder of this chapter is devoted to a study where and why exactly option 1. fails.

The remainder of this chapter is structured as follows. First, we examine the relation of supersymmetry to the representation theory of the orbifold point group. We find that the amount of surviving supersymmetry is related to the number of Killing spinors (i.e. invariant spinors), and see that it is well possible that locally (i.e. in a particular

(g, h)-twisted sector), the number of surviving Killing spinors is higher than globally. Then, we proceed with a discussion of the properties of the partition function and find that the vanishing of a particular (g, h)-twisted sector depends on the fermionic partition function of the sector, as all other contributions are generically non-zero. Finally, we observe that the fermionic partition function in a (g, h)-twisted sector vanishes if and only if g and h admit at least one compatible Killing spinor, which allows us to link the vanishing of the partition function to the representation theory of the point group P.

3.2 Compactification and supersymmetry

The purpose of this section is to clarify how the action of rotations in the orbifold dimensions on target-space vectors and spinors are related. With this information at hand, we will show how the amount of surviving supersymmetry depends on the embedding of the geometric twist living in SO(6) into Spin(6).

3.2.1 Representation theory of the geometric point groups

At this point, it is useful to quickly recall the definition of the point group P in section 2.3.1. The action of a point group element θ is defined as its geometric action on the lattice vielbein e via a six-dimensional representation $D_{\mathbf{v}}$

$$\theta : e \;\mapsto\; D_{\mathbf{v}}(\theta)\, e \,, \tag{3.2}$$

where (for proper rotations) $D_{\mathbf{v}}(\theta)$ fits in a six-dimensional representation **6** of SO(6)[1] and the subscript "**v**" indicates that we are talking about the action of the twist on target-space vectors, which is its geometric action. Each rotation can be block-diagonalized by a change of the lattice basis vectors in the vielbein e to read

$$D_{\mathbf{v}}(\theta) \;=\; \begin{pmatrix} e^{2\pi i v_\theta^{(1)} \mathsf{J}_{12}} & & \\ & e^{2\pi i v_\theta^{(2)} \mathsf{J}_{34}} & \\ & & e^{2\pi i v_\theta^{(3)} \mathsf{J}_{56}} \end{pmatrix} , \tag{3.3}$$

with the generators of rotations in the (X^{2a-1}, X^{2a})-planes $\mathsf{J}_{2a-1,2a}$. Once we switch to complex coordinates $Z^a = X^{2a-1} + iX^{2a}$, the action of the rotation can be written as a simple multiplication by phases

$$Z^a \;\mapsto\; e^{2\pi i v_\theta^a}\, Z^a \,. \tag{3.4}$$

Here, one can observe that the entries of the twist vector are—from a geometric point of view—only defined up to the addition of integers, as v_θ^a and $v_\theta^a + 1$ generate the same rotation. As we will observe shortly, this is no longer true for the twist action on spinors.

[1]There are, in fact, quite many instances of *geometric* orbifolds where the point group is in O(6) rather than SO(6). However, these are out of the picture for phenomenology, as one cannot even speak about spinors on these (non-orientable) manifolds, and hence will be neglected in what follows.

3.2.2 Action of twists on target-space spinors

Since the worldsheet fermions transform trivially under translations on the orbifold, the rotations generated by the point group P already encode the entire relevant information for the orbifold projection in this sector. Each geometric rotation can be embedded into the target-space spinors by the action of

$$D_{\mathbf{s}}(\theta) \;=\; \mathrm{e}^{2\pi \mathrm{i} v_\theta^{(1)}\,\frac{1}{2}\sigma_1} \otimes \mathrm{e}^{2\pi \mathrm{i} v_\theta^{(2)}\,\frac{1}{2}\sigma_2} \otimes \mathrm{e}^{2\pi \mathrm{i} v_\theta^{(3)}\,\frac{1}{2}\sigma_3}\;. \tag{3.5}$$

The first thing to notice here is that the spinor embedding is no longer degenerate under the addition of integers to v_θ^a, but is only invariant under the addition of *even* integers. This is precisely the double cover property of Spin(6) compared to SO(6). From the viewpoint of representation theory, the spinor representation $D_{\mathbf{s}}(\theta)$ can be decomposed into the direct product of four dimensional representations

$$D_{\mathbf{s}}(\theta) \;=\; D_{\mathbf{4}}(\theta) \oplus D_{\overline{\mathbf{4}}}(\theta)\;, \tag{3.6}$$

which fit in a **4** of SU(4) (as SU(4) and Spin(6) are isomorphic). The spinor representation is then related to the vector representation $D_{\mathbf{v}}(\theta)$ by

$$\mathbf{6} \;=\; [\mathbf{4}]_2\;, \tag{3.7}$$

where $[\,\cdot\,]_2$ denotes the two-times antisymmetrized tensor product. The two four-dimensional representations $D_{\mathbf{4}}(\theta)$ and $D_{\overline{\mathbf{4}}}(\theta)$ are not independent and can be obtained from $D_{\mathbf{s}}(\theta)$ by making a chiral projection that yields either of the two, depending on whether one projects on chiral or antichiral spinors. Again, one observes that $+D_{\mathbf{4}}(\theta)$ and $-D_{\mathbf{4}}(\theta)$ yield the same geometric **6**, reflecting the double cover property once more.

Another useful way to understand this relation, which naturally makes the connection to the heterotic partition function is via the SO(8) lattices $\mathbf{8}_v$ and $\mathbf{8}_s$ introduced in the previous chapter, by studying the lattice vectors of length 1. Notably, vectors in the weight lattice with unit length correspond to eight-dimensional representations of SO(8). Beginning with the bosonic vector lattice, we note that the lattice vectors with length 1 can be written as

$$\Lambda_{\mathbf{8}_v} \;\supset\; \left\{(\pm 1,0,0,0)\;,\left(0,\underline{\pm 1,0,0}\right)\right\}\;, \tag{3.8}$$

where the underline indicates that all permutations are to be included. We have split the vectors because in our setting with a six-dimensional orbifold, the first entry gives the four-dimensional spin, while the other three entries correspond to one of the three complex planes of the orbifold. The eight ten-dimensional vectors at this level in $\mathbf{8}_v$ decompose into two four-dimensional vectors (with opposite helicity) and six scalars. In terms of SO(8) representations, the branching into SO(6) precisely predicts this behavior (see e.g. [63])

$$\mathbf{8}_v \;\to\; \mathbf{1}_1 \oplus \mathbf{1}_{-1} \oplus \mathbf{6}_0\;, \tag{3.9}$$

where the subscripts are the space-time spins. It becomes clear that the two space-time vectors are singlets under SO(6) because all their entries corresponding to orbifold dimensions are zero, so there is no non-trivial transformation behavior. One can proceed for the fermionic lattice in a very similar way. Here, the splitting is chosen as

$$\begin{aligned}\Lambda_{\mathbf{8}_s} \supset & \left\{\left(+\frac{1}{2},\underline{+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right),\left(+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right)\right\} \\ & \cup\left\{\left(-\frac{1}{2},\underline{-\frac{1}{2},+\frac{1}{2},+\frac{1}{2}}\right),\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\right\} .\end{aligned} \tag{3.10}$$

This results in four space-time spinors and their partners with opposite helicity. In the words of representation theory, the resulting branching into SU(4) representations reads

$$\mathbf{8}_s \;\rightarrow\; \mathbf{4}_{+\frac{1}{2}} \oplus \mathbf{4}_{-\frac{1}{2}} . \tag{3.11}$$

The SO(6) representation $\mathbf{6}_0$ and the SU(4) representation(s) $\mathbf{4}_{\pm\frac{1}{2}}$ are precisely what we identified with $D_{\mathbf{v}}$ and $D_{\mathbf{4}}$ (or $D_{\overline{\mathbf{4}}}$). It is also clear that, as they transform non-trivially under the orbifold twists, some of the degrees of freedom may be projected out by the orbifold projection, depending on what the precise form of the various $D_{\mathbf{v}}(\theta)$ and $D_{\mathbf{4}}(\theta)$ matrices is for all $\theta \in P$.

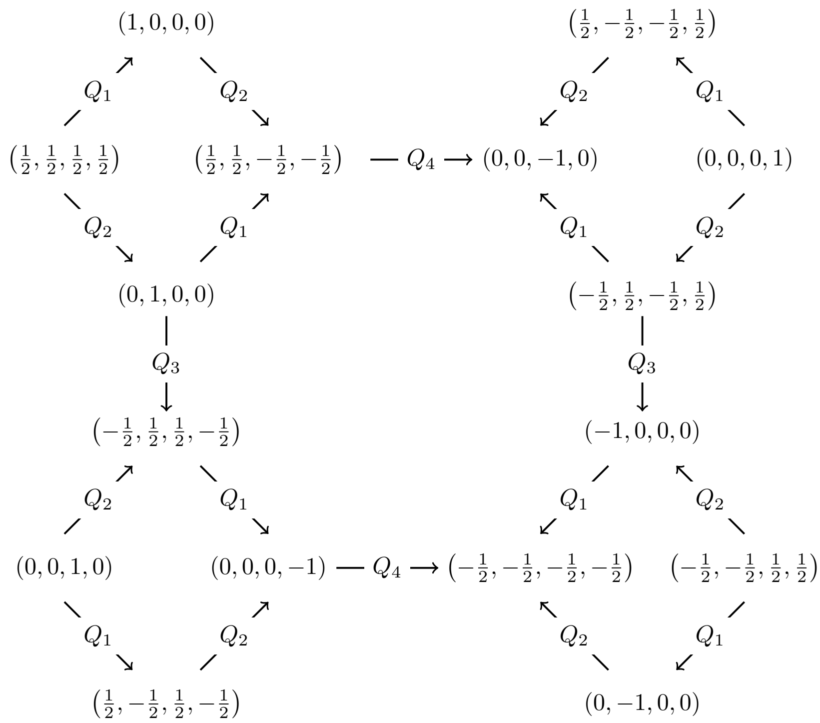
Let us note here that because P is a discrete subgroup of SO(6) (and SU(4)), the vector representation **6** and the spinor representation **4** in turn decompose into irreducible representations of P. The precise form of the vector representation is fixed by the way the rotations act geometrically on the orbifold lattice. Generically, there are then two or more choices for the spinor embedding, which has to match the action of the vector representation via equation (3.7).

3.2.3 Killing spinors and supersymmetry breaking

Put simply, supersymmetry relates space-time fermions and bosons via a transformation that is fermionic itself. In our setup, SUSY therefore has to mediate between the $\mathbf{8}_v$ and $\mathbf{8}_s$ lattices. In ten dimensions, this mapping is facilitated by a set of four supercharges

$$\mathbf{Q} \;=\; \underline{\left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)} . \tag{3.12}$$

By labelling the entries of $\mathbf{Q}$ by Q_i (with the plus sign at the ith entry), one can schematically draw the following diagram for the unbroken ten-dimensional case



(3.13)

where an arrow labelled by Q_i indicates that the respective supercharge is added to the weight at the start of the arrow. In particular, it is straightforward to classify pairs of weights as four-dimensional $\mathcal{N}=1$ chiral or vector multiplets according to their first entry. It also becomes clear why ten-dimensional $\mathcal{N}=1$ SUSY is equivalent to four-dimensional $\mathcal{N}=4$. Just like the components of $\mathbf{8}_v$ and $\mathbf{8}_s$, also the Q_i potentially transform non-trivially under P. Indeed, the set of the Q_i together with the $-Q_i$ transforms as an $\mathbf{8}_c$ under SO(8), which has a branching into non-trivial SU(4) representations as well.

It is now important to make the connection between surviving supercharges and the spinor representation $D_\mathbf{s}$. In particular, we have to make clear what it means when we say that a certain Killing spinor survives at a fixed point. To this end, one needs to distinguish between locally and globally preserved Killing spinors, which are defined as follows: To define the number of globally and locally preserved Killing spinors and to understand their distinction better, we first discuss how to determine the number of Killing spinors preserved by some subgroup $G \subset P$. For example, this may be a $\mathbb{Z}_{N_\theta}$ subgroup $G = \langle\theta\rangle \subset P$ generated by any $\theta \in P$ or the whole point group, $G = P$. Using a four-dimensional Weyl representation $D_\mathbf{4}$ obtained via a chiral projection from $D_\mathbf{s}$, each G-invariant Weyl spinor $\Psi_{\text{inv.}}$ satisfies the condition

$$D_\mathbf{4}(\theta)\,\Psi_{\text{inv.}} \;=\; \Psi_{\text{inv.}}\;, \tag{3.14}$$

for all $\theta \in G$. Consequently, the G-invariant spinor eigenspace can be found using the projection operator

$$\mathcal{P}^G \;=\; \frac{1}{|G|}\sum_{\theta'\,\in G} D_{\mathbf{4}}(\theta')\ , \tag{3.15}$$

which is defined such that $D_{\mathbf{4}}(\theta)\,\mathcal{P}^G = \mathcal{P}^G$ for all $\theta \in G$. Then, the number of G-invariant Killing spinors, namely the dimension of the G-invariant subspace is counted using the trace of the projection operator

$$\mathcal{N}^G \;=\; \mathrm{Tr}\Big(\mathcal{P}^G\Big) \;=\; \frac{1}{|G|}\sum_{\theta'\,\in G} \mathrm{Tr}\left(D_{\mathbf{4}}(\theta')\right)\ . \tag{3.16}$$

In particular, $\mathcal{N}^{\langle\theta\rangle}$ determines the number of *local* Killing spinors compatible with the point group element $\theta \in P$, while $\mathcal{N} = \mathcal{N}^P$ gives the number of *global* Killing spinors and hence the amount of target-space supersymmetry.

The situation is a little more involved for the number of Killing spinors in a (g,h)-twisted sector. There, one has to filter for Killing spinors that are allowed by *both* the constructing element g and the projecting element h. One has to consider the $\mathbb{Z}_N$ groups generated by both elements: let us call $G_g = \langle\theta_g\rangle$ and $G_h = \langle\theta_h\rangle$, where $\theta_{g,h}$ is the rotational part of the respective element. Then, a necessary but not sufficient condition is that both $\mathcal{N}^{G_g} \geq 1$ and $\mathcal{N}^{G_h} \geq 1$. However, in this case one still has to show that in both invariant subspaces there are two Killing spinors that are compatible. Luckily, the partition function consists only of (g,h)-sectors where g and h commute and hence can be block-diagonalized simultaneously. Then, each of the spin embeddings is readily described by the entries of the twist vector v. Indeed, the possible eigenvalues of $D_{\mathbf{s}}(\theta_g)$ are $\exp(\pm 2\pi\mathrm{i}\,\widetilde{v}_g^{(a)})$, $a = 1,2,3$, where

$$\widetilde{v}_g \;=\; \frac{1}{2}\begin{pmatrix} v_g^{(1)} + v_g^{(2)} + v_g^{(3)} \\ -v_g^{(1)} + v_g^{(2)} + v_g^{(3)} \\ v_g^{(1)} - v_g^{(2)} + v_g^{(3)} \\ v_g^{(1)} + v_g^{(2)} - v_g^{(3)} \end{pmatrix}\ , \tag{3.17}$$

and likewise for h. Preserving compatible Killing spinors can be boiled down to the condition that $\widetilde{v}_g$ and $\widetilde{v}_h$ must vanish (modulo integers) at the same entry.

Remark on Scherk–Schwarz breaking. The way to break SUSY via rotations we describe here is not the only way to realize non-supersymmetric, consistent (heterotic) string theories. Another way is via the so-called Scherk–Schwarz mechanism, which has been (in the heterotic context) studied extensively in [56, 57]. The core idea is to have a translation on the orbifold that is equipped with a special Wilson line. Unlike the Wilson lines we have introduced so far for gauge symmetry breaking, this special Wilson line leads to different transformation phases for space-time fermions and bosons, thereby breaking supersymmetry (very much the same way as we break the gauge symmetry). These constructions have the interesting feature that the mass splitting between fermions and bosons is proportional to the radius of the cycle along the translation. In this setup, models with an exponentially suppressed cosmological constant at one (or even two loops) have been constructed, however none with an exactly vanishing one. It is yet to be shown that these results truly persist in perturbation theory.

3.3 Vanishing of the partition function

In this section, we examine under which conditions a (g, h)-twisted sector in the partition function vanishes. In particular, we show that in order for the (g, h)-twisted sector to vanish, g and h must admit at least one common Killing spinor. To this end, we must first determine what the various components of (g, h)-twisted sectors look like (cf. (2.76)), once g and h are chosen to be non-trivial.

3.3.1 Heterotic partition function with twists and gauge embedding

The purpose of this section is to identify the parts of the partition function that can vanish at all, and under which conditions that is possible.

Leftmover. We start with the observation that the leftmover partition function cannot vanish under any circumstance, but rather contains a $1/q$ term in its Fourier expansion that cannot be canceled [56, 64]. To this end, we first have a closer look at the leftmoving heterotic partition function, that is the contribution of leftmoving bosons both in spacetime and in the gauge coordinates. Of particular interest are sectors where $g = \mathbb{1}$ (we will explain towards the end of the section why), where one projects the untwisted sector on some projecting element h. In what follows, it makes a difference if the rotational part of h leaves some of the three complex planes fixed. We count the non-integer entries in v_h with $r \in \{0, 1, 2, 3\}$, and reshuffle v_h such that all non-trivial entries come first. Furthermore, the gauge embedding of h is split as $V_h = \left(V_h^{(1)}, V_h^{(2)}\right)$ according to the first and second E_8. Schematically, the left-mover partition function for a $(\mathbb{1}, h)$-twisted sector then takes the form

$$\mathcal{Z}\begin{bmatrix}\mathbb{1}\\ h\end{bmatrix}_{\mathrm{L}}(\tau) \;=\; \left(\frac{1}{\eta^2}\right)^{4-r} \cdot \frac{\eta^r}{\prod_{i=1}^r \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2} - v_h^{(i)}\end{bmatrix}} \cdot \left(\frac{1}{\eta^8}\sum_P q^{\frac{1}{2}P^2}\,\mathrm{e}^{2\pi\mathrm{i}\,P\cdot V_h^{(1)}}\right) \cdot \left(\frac{1}{\eta^8}\sum_P q^{\frac{1}{2}P^2}\,\mathrm{e}^{2\pi\mathrm{i}\,P\cdot V_h^{(2)}}\right) . \tag{3.18}$$

In this expression, the first line is the combined contribution of the non-compact dimensions and the coordinates in the orbifold, whereas the second lines are the appropriate versions of the E_8 partition function (2.29) after an appropriate projector $\mathrm{e}^{2\pi\mathrm{i}\,P\cdot V_h^{(i)}}$ has been included.

Let us first concentrate on the two E_8 factors in eq. (3.18). There, the terms that contribute to the partition function at lowest order in q have $P = 0$. Hence, each E_8 factor gives a contribution

$$\frac{1}{\eta^8}\sum_P q^{\frac{1}{2}P^2}\,\mathrm{e}^{2\pi\mathrm{i}\,P\cdot V_i^{(a)}} \;=\; \frac{1}{\eta^8}(1+\ldots) \;=\; q^{-\frac{1}{3}} + \ldots \tag{3.19}$$

where the dots indicate the omission of (model dependent) higher weight terms that are irrelevant for our discussion. Moreover, the existence of the $q^{-\frac{1}{3}}$ factor is independent of the chosen gauge embedding $V_i^{(a)}$ for $a = 1, 2$, as long as the term inside the parentheses in the first identity in equation (3.19) does not vanish. We will show later, when we have developed appropriate tools to study the vanishing of such terms, that sensible heterotic theories always have non-vanishing contributions from each E_8. For now, as we have two E_8 factors, these two contributions combine to a $q^{-\frac{2}{3}}$ contribution.

We make use of the product identity for the Jacobi theta-functions with arbitrary spin-structures α and α' in order to analyze the first line in (3.18)

$$\frac{\vartheta\begin{bmatrix}\alpha\\ \alpha'\end{bmatrix}}{\eta} = \mathrm{e}^{2\pi\mathrm{i}\alpha\alpha'}\, q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n+\alpha-\frac{1}{2}}\mathrm{e}^{2\pi\mathrm{i}\alpha'}\right)\left(1+q^{n-\alpha-\frac{1}{2}}\mathrm{e}^{-2\pi\mathrm{i}\alpha'}\right) . \tag{3.20}$$

Using the inverse of this relation and applying it to our case, we can expand the term giving the contribution of the orbifolded coordinates according to

$$\begin{aligned}\frac{\eta^r}{\prod_{i=1}^r \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}-v_h^{(i)}\end{bmatrix}} &= q^{-\frac{r}{12}} \prod_{i=1}^{r} \mathrm{e}^{-\pi\mathrm{i}\left(\frac{1}{2}-v_h^{(i)}\right)} \prod_{n=1}^{\infty} \frac{1}{1+\mathrm{e}^{2\pi\mathrm{i}\left(\frac{1}{2}-v_h^{(i)}\right)}q^n}\,\frac{1}{1+\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{1}{2}-v_h^{(i)}\right)}q^{n-1}}\\ &= q^{-\frac{r}{12}} \prod_{i=1}^{r} \mathrm{e}^{-\pi\mathrm{i}\left(\frac{1}{2}-v_h^{(i)}\right)} + \ldots\ ,\end{aligned} \tag{3.21}$$

because the products at the end of this expression may be written in terms of a geometric series, and where the dots again indicate irrelevant terms that are skipped. Furthermore, the contribution of the uncompactified leftmovers in eq. (3.18) can be expanded by using the definition of the Dedekind eta-function (2.20) as

$$\left(\frac{1}{\eta^2}\right)^{4-r} = q^{-\frac{4-r}{12}} + \ldots\ . \tag{3.22}$$

When putting both contributions (3.21) and (3.22) together, note that the last model-dependence appearing the exponents of q, namely the number of rotated planes r drops out at lowest order as the $q^{-\frac{r}{12}}$ factor in (3.21) cancels with an appropriate term in (3.22). Combining these two expansions with the contribution of the E_8s (twice (3.19)), one finds

$$\mathcal{Z}\begin{bmatrix}\mathbb{1}\\ h\end{bmatrix}_{\mathrm{L}}(\tau) \sim \frac{c_h}{q} + \ldots\ , \tag{3.23}$$

where c_h is a non-zero constant. In particular, $c_h = 1$ for $h = \mathbb{1}$. This result has important implications. The first observation is that sectors of the type studied here (with one element the identity and the other potentially chosen non-trivial) are always present, as every element of the space group commutes with the identity. If one insists that *every* (g, h)-sector vanishes on its own, in particular these $(\mathbb{1}, h)$-sectors must do so. As we have seen, these sectors generically have a non-vanishing $1/q$ term from their leftmover. Therefore, because the leftmover does not vanish, we can conclude that the rightmover partition function in each sector must vanish if the (g, h)-sector vanishes.

Rightmover. For the same reasons as for the bosonic leftmover, we can restrict ourselves to $(\mathbb{1}, h)$-sectors. We start the discussion of the rightmovers by noticing that the contribution of the rightmoving bosons both in the non-compact dimensions as well as in the orbifold essentially looks the same as its leftmoving counterpart (up to complex conjugation). This part is expected to show the same behavior as the first line in eq. (3.18), just with q replaced by $\overline{q}$. Therefore, we can make the same reasoning as for the leftmovers and, using (3.20), arrive at the same result as eq. (3.21), namely that these degrees of freedom generically give a contribution proportional to $\overline{q}^{-\frac{1}{3}}$ and hence do not vanish. By result, if the $(\mathbb{1}, h)$-sector vanishes, it can only do so if the contribution from the worldsheet fermions vanishes. Hence, the discussion in the remainder of this chapter will be centered around this contribution.

We are now going to prove an identity that will allow us to relate the vanishing of the fermionic rightmover partition function in a (g, h)-twisted sector to the existence of Killing spinors compatible with both g and h. To this end, we carefully construct the orbifold analog to the ten-dimensional fermionic partition function presented around equation (2.37). As we have to deal with non-supersymmetric twist vectors, we have to pay special attention to phases in the partition function that would vanish in the SUSY case. In what follows, let us consider twist vectors of the form

$$v_g \;=\; \left(0, v_g^{(1)}, v_g^{(2)}, v_g^{(3)}\right)\;, \tag{3.24}$$

with an analog convention for v_h. Then, the partition function for rightmoving worldsheet fermions reads

$$\mathcal{Z}_\psi\begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) \;=\; \frac{1}{\overline{\eta}^4}\sum_Q \overline{q}^{\frac{1}{2}(Q+v_g)^2}\,\mathrm{e}^{2\pi\mathrm{i}(Q+v_g)\cdot v_h}(-1)^F\;, \tag{3.25}$$

where we include a projector term due to a non-trivial h and where Q is either in the vector lattice $\Lambda_{\mathbf{8}_v}$ (for $F = 0$) or the spinor lattice $\Lambda_{\mathbf{8}_s}$ ($F = 1$) of SO(8). In the previous chapter, we showed how such a sum over lattice vectors can be rewritten in terms of Jacobi theta-functions, which is also possible here. By shifting the spin-structures of the theta-functions appropriately, we can argue that this will become

$$\mathcal{Z}_\psi\begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) \;=\; \frac{1}{2\overline{\eta}^4}\sum_{s,s'=0}^{1}(-1)^{s+s'}\,\Phi\begin{bmatrix} g, s \\ h, s' \end{bmatrix}\prod_{i=0}^{3}\overline{\vartheta}\begin{bmatrix} \frac{s}{2}+v_g^{(i)} \\ \frac{s'}{2}+v_h^{(i)} \end{bmatrix}\;, \tag{3.26}$$

where we yet have to determine the phases Φ in order to match with the initial definition (3.25). To this end, we rewrite (3.25) as a constrained sum over vectors in $\mathbb{Z}^4$

$$\begin{aligned}\mathcal{Z}_\psi\begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) \;=\; &\frac{1}{2\overline{\eta}^4}\prod_{i=0}^{3}\sum_{n_i\in\mathbb{Z}}\overline{q}^{\frac{1}{2}\left(n_i+v_g^{(i)}\right)^2}\mathrm{e}^{2\pi\mathrm{i}\left(n_i+v_g^{(i)}\right)v_h^{(i)}}\\ &-\frac{1}{2\overline{\eta}^4}\prod_{i=0}^{3}\sum_{n_i\in\mathbb{Z}}\overline{q}^{\frac{1}{2}\left(n_i+v_g^{(i)}\right)^2}\mathrm{e}^{2\pi\mathrm{i}\left(n_i+v_g^{(i)}\right)v_h^{(i)}}\mathrm{e}^{2\pi\mathrm{i}n_i\cdot\frac{1}{2}}\\ &-\frac{1}{2\overline{\eta}^4}\prod_{i=0}^{3}\sum_{n_i\in\mathbb{Z}}\overline{q}^{\frac{1}{2}\left(n_i+\frac{1}{2}+v_g^{(i)}\right)^2}\mathrm{e}^{2\pi\mathrm{i}\left(n_i+\frac{1}{2}+v_g^{(i)}\right)v_h^{(i)}}\;.\end{aligned} \tag{3.27}$$

We observe that this expression can be indeed rewritten in terms of theta-functions with the right spin-structures. However, in the second line, we have to add a phase in order to achieve this. The result takes exactly the form of (3.26) with the phases fixed to

$$\Phi \begin{bmatrix} g, s \\ h, s' \end{bmatrix} = \mathrm{e}^{-2\pi \mathrm{i} s' \sum_i v_g^{(i)} \cdot \frac{1}{2}} \,. \tag{3.28}$$

Here, we have reached an interesting point, because for SUSY twists one can always go to a basis where $\sum_i v_g^{(i)} = 0 \bmod 2$, which means that precisely this phase can be safely neglected. For non-SUSY twists, this is no longer guaranteed and the fermionic rightmover partition function then reads

$$\mathcal{Z}_\psi \begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) = \frac{1}{2\overline{\eta}^4} \sum_{s,s'=0}^{1} (-1)^{s+s'} \, \mathrm{e}^{-2\pi \mathrm{i} s' \sum_i v_g^{(i)} \cdot \frac{1}{2}} \prod_{i=0}^{3} \overline{\vartheta} \begin{bmatrix} \frac{s}{2} + v_g^{(i)} \\ \frac{s'}{2} + v_h^{(i)} \end{bmatrix} \,. \tag{3.29}$$

We will see later on that this phase plays a key role in the (non-)vanishing of the non-supersymmetric partition function. For convenience, the product over the four theta-functions can be rewritten in terms of vector-valued theta-functions

$$\mathcal{Z}_\psi \begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) = \frac{1}{2\overline{\eta}^4} \sum_{s,s'=0}^{1} (-1)^{s+s'} \, \mathrm{e}^{-2\pi \mathrm{i} g \, e_4^T \cdot v \, \frac{s'}{2}} \, \overline{\vartheta} \begin{bmatrix} \frac{s}{2} e_4 + v_g \\ \frac{s'}{2} e_4 + v_h \end{bmatrix}_{(4)} \,, \tag{3.30}$$

where we define $e_4^T = (1, 1, 1, 1)$ and the vector-valued theta-functions with arbitrary spin structures according to

$$\overline{\vartheta} \begin{bmatrix} \alpha \\ \alpha' \end{bmatrix}_{(d)} = \prod_{i=0}^{d-1} \overline{\vartheta} \begin{bmatrix} \alpha_i \\ \alpha'_i \end{bmatrix} \,. \tag{3.31}$$

This concludes the construction of the fermionic rightmover partition function. With this result at hand, one now would like to know under which conditions on v_g and v_h this partition function vanishes, similar to the vanishing of the ten-dimensional partition function (2.42).

3.3.2 A Riemann identity for vanishing rightmover partition functions

We can now examine how the expression (3.30) for the fermionic rightmover partition function can be simplified. Using the definition of the Jacobi theta-functions we observe that the vector-valued theta-functions read

$$\overline{\vartheta} \begin{bmatrix} \alpha \\ \alpha' \end{bmatrix}_{(4)} = \sum_{n \in \mathbb{Z}^4} \overline{q}^{\frac{1}{2}|n+\alpha|^2} \mathrm{e}^{2\pi \mathrm{i} (n+\alpha)^T \alpha'} \,. \tag{3.32}$$

It is central to notice that this expression only depends on inner products. In order to rewrite it we can now introduce the orthogonal symmetric matrix S whose choice will be come clear eventually

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{with} \quad S = S^T = S^{-1} . \tag{3.33}$$

The idea now is to transform the right hand side of (3.32) using S in such a way that it matches the fermionic partition function (3.30), such that the sum over theta-functions in (3.30) can be expressed by a single theta-function. Inserting $\mathbb{1} = S^T S$ in all the inner products of (3.32) one obtains

$$\overline{\vartheta}\begin{bmatrix} \alpha \\ \alpha' \end{bmatrix}_{(4)} = \sum_{\widetilde{n}} \overline{q}^{\frac{1}{2}\left|\widetilde{n}+\widetilde{\alpha}\right|^2} \mathrm{e}^{2\pi \mathrm{i}\left(\widetilde{n}+\widetilde{\alpha}\right)^T \widetilde{\alpha'}} , \tag{3.34}$$

where all twiddled quantities are defined by $\widetilde{u} = S\, u$. We observe that the twist vectors undergo the same transformation as in (3.17), which means that we are on a good way of linking our discussion here to the existence of compatible Killing spinors. Moreover, in order to proceed, it is important to notice on a technical level that by the S-transformation, $\widetilde{n}$ is no longer in $\mathbb{Z}^4$. However, the summation over $\widetilde{n}$ may still be carried out after noticing that one can parametrize it as follows

$$\widetilde{n} = m + \frac{s}{2} e_4 , \quad \text{where} \quad m \in \mathbb{Z}^4 , \; s = 0, 1 . \tag{3.35}$$

Then, the summation over all $\widetilde{n}$ can be performed by summing over m and s if one enforces the additional condition that $e_4^T\, m \overset{!}{=} 0 \bmod 2$. As usual, guaranteeing this property is achieved by the introduction of a projector. Inserting the projector via another sum over s', we arrive at a result that can be written in terms of a sum over Jacobi theta-functions (as desired) and find

$$\overline{\vartheta}\begin{bmatrix} \alpha \\ \alpha' \end{bmatrix}_{(4)} = \frac{1}{2} \sum_{s,s'=0}^{1} \sum_{m \in \mathbb{Z}^4} \overline{q}^{\frac{1}{2}\left|m+\frac{s}{2}e_4+\widetilde{\alpha}\right|^2} \mathrm{e}^{2\pi \mathrm{i}\left(m+\frac{s}{2}e_4+\widetilde{\alpha}\right)^T \widetilde{\alpha'}} \mathrm{e}^{2\pi \mathrm{i}\frac{s'}{2}\left(m+\frac{s}{2}e_4\right)^T e_4} \tag{3.36}$$

$$= \frac{1}{2} \sum_{s,s'=0}^{1} \mathrm{e}^{-2\pi \mathrm{i}\frac{s'}{2}\widetilde{\alpha}^T e_4}\, \overline{\vartheta}\begin{bmatrix} \frac{s}{2} e_4 + \widetilde{\alpha} \\ \frac{s'}{2} e_4 + \widetilde{\alpha'} \end{bmatrix}_{(4)} . \tag{3.37}$$

Now, all we have to do is to match the spin-structures α and α' appearing in this expression to those in the partition function (3.30) and take care that all phases are correct. Because S is its own inverse, one may interchange twiddled and non-twiddled vectors. Furthermore, one notices that shifting the zero-component of a non-twiddled spin-structure by an integer, $\alpha_0 \mapsto \alpha_0 + 1$, amounts to a simultaneous half-integer shift in the twiddled one, $\widetilde{\alpha} \mapsto \widetilde{\alpha} + \frac{e_4}{2}$. By making this replacement for both α and α', and using that for physically sensible situations α_0 is an integer (we actually chose it to be zero at the beginning of this discussion), we can generate the $(-1)^{s+s'}$ GSO phase appearing in (3.30). Finally, one arrives at the identity

$$\overline{\vartheta}\begin{bmatrix} \frac{e_4}{2} + \widetilde{\alpha} \\ \frac{e_4}{2} + \widetilde{\alpha'} \end{bmatrix}_{(4)} = \frac{1}{2} \sum_{s,s'=0}^{1} (-1)^{s+s'} \mathrm{e}^{-2\pi \mathrm{i}\frac{s'}{2}\alpha^T e_4}\, \overline{\vartheta}\begin{bmatrix} \frac{s}{2} e_4 + \alpha \\ \frac{s'}{2} e_4 + \alpha' \end{bmatrix}_{(4)} , \tag{3.38}$$

which is a generalization to Jacobi's "abstruse" identity. Comparing the Riemann identity (3.38) to the rightmover partition function (3.30), we first notice that the phase in front of the theta-functions already strongly resembles the non-SUSY phase (3.28). All we have to do is to plug in the spin-structures from (3.30) in place of α and α' and obtain the following simplified expression for the fermionic partition function

$$\mathcal{Z}_\psi \begin{bmatrix} g \\ h \end{bmatrix}(\overline{\tau}) \;=\; \frac{1}{\overline{\eta}^4}\overline{\vartheta}\begin{bmatrix} \frac{e_4}{2} + \widetilde{v}_g \\ \frac{e_4}{2} + \widetilde{v}_h \end{bmatrix}_{(4)} . \tag{3.39}$$

Let us now examine the properties of this simplified expression. One observes that whenever $\widetilde{v}_g$ and $\widetilde{v}_h$ have a vanishing entry (modulo integers) at the *same* position, the partition function is proportional to $\overline{\vartheta}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 0$ and therefore vanishes, whereas it is guaranteed to be a non-zero function otherwise. As we have noted earlier, it is precisely the existence of a Killing spinor that causes an entry of $\widetilde{v}$ to vanish (modulo integers). Hence, we can conclude that the partition function vanishes if and only if both $\widetilde{v}_g$ and $\widetilde{v}_h$ admit the same Killing spinor.

With this statement, we can make the desired connection between the vanishing of the partition function on one side, and the supersymmetry properties of the spin embedding. We can now formulate the (non-)vanishing of the various (g, h)-twisted sectors in the partition function entirely in terms of the representation theory of the point group. To this end, we now identify the minimal necessary conditions that need to be met if we insist that all (g, h)-twisted sectors of a given model vanish. However, let us first make a remark on possible applications of generalized Riemann identities on the leftmover partition function, especially the $\mathrm{E}_8 \times \mathrm{E}_8$ part there.

Remark on Riemann identities for leftmovers. So far, we have derived a Riemann identity (3.38) that can be applied to a set of four fermions with a $\mathbb{Z}_2$ spin structure. As we show in appendix B, our Riemann identity (3.38) can be generalized to sets of d fermions with $\mathbb{Z}_{d/2}$ spin structures. Given that the leftmoving gauge sector of the heterotic string can be understood as an ensemble of 16 fermions (in the fermionic formulation), the possibility that the leftmover partition function vanishes by the virtue of a generalized Riemann identity deserves to be studied.

The first point to notice here is that the generalized Riemann identities cannot be applied to the "standard" $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $\mathrm{E}_8 \times \mathrm{E}_8$ theories, as these theories involve either a set of 16 fermions with a $\mathbb{Z}_2$ spin structure or two sets of each eight fermions equipped again with $\mathbb{Z}_2$ spin structures. The fact that generalized Riemann identities cannot be invoked to achieve a vanishing leftmover partition function does not come as a surprise, in fact we have derived the existence of a $q^{-2/3}$ term in the Fourier expansion of the gauge sector partition function in the foregoing section. In order to put the generalized Riemann identities to work in the leftmoving gauge sector, one would have to consider more exotic settings that have to be invoked:

1. One might take four sets of four fermions, each of which is equipped with a $\mathbb{Z}_2$ spin structure, and make use of (3.38).

2. Moreover, going to more exotic spin structures, one can take two sets of eight fermions with a $\mathbb{Z}_4$ spin structure, or
3. Consider all 16 fermions with a $\mathbb{Z}_8$ spin structure.
4. Finally, also a combination of 1. and 2. is conceivable, i.e. two sets of four fermions, each with a $\mathbb{Z}_2$ spin structure plus one set of eight fermions with a $\mathbb{Z}_4$ spin structure.

While the generalized Riemann identities corresponding to these cases can yield vanishing partition functions, all these theories necessarily suffer from an even worse problem: The exotic spin structures necessarily imply that the ground state (i.e. the leftmoving vacuum) is projected out. However, the massless graviton is then projected out, too, which means that a vanishing cosmological constant is no longer an issue.

3.3.3 Types of orbits and a minimal condition

We notice that there are two types of (g, h)-twisted sectors. Recall that a (g, h)-sector alone is not invariant under modular transformations, but rather gets mapped to a (g', h')-sector, cf. (2.63), where g' and h' depend on g, h and the modular transformation. Then we notice that there are two types of (g, h)-twisted sectors: there are (g, h)-twisted sectors that are connected to a sector of the form $(\mathbb{1}, h')$ by a sequence of modular transformations, for an appropriately chosen h', and some that are not.

The first observation one can make is that as soon as the point group P has only one generator (as for Abelian $\mathbb{Z}_N$ orbifolds), all (g, h)-sectors are of the first kind. However, in $\mathbb{Z}_N \times \mathbb{Z}_M$ or non-Abelian orbifolds, sectors of the second kind can generically appear. As we have argued already in the previous section, the existence of sectors of the second kind is tied to the condition that g and h commute not only on the point group level, but also on the space group level. With that, we mean that while the rotation matrices θ_g and θ_h corresponding to g and h can commute, one can still achieve that g and h do *not* commute by turning on an appropriate roto-translation, and therefore remove this particular (g, h)-twisted sector from the partition function. Note that this trick does precisely not work for sectors of the first kind, because the space group identity commutes with any space group element, with or without roto-translation. Let us demonstrate the power of this trick by an example, showing how the amount of local SUSY can be enhanced by its application.

An $\mathcal{N} = 1 \rightarrow \mathcal{N} = 2$ example. Our example is set in globally $\mathcal{N} = 1$ supersymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. There are two generators of the point group θ and ω which are taken to have twist vectors

$$v_\theta = \left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) , \tag{3.40}$$

$$v_\omega = \left(0, \frac{1}{2}, -\frac{1}{2}, 0\right) . \tag{3.41}$$

Note that each of these twists has a two-dimensional invariant spinor space, so one might be tempted to think at first that the model is globally $\mathcal{N} = 2$ supersymmetric. However, only one spinor is invariant under both twists, and hence there is only a global $\mathcal{N} = 1$ supersymmetry. Does that mean that there are (g, h)-sectors with only $\mathcal{N} = 1$ supersymmetry locally? The answer to this question depends on if there are roto-translations associated with the corresponding space group elements g_θ and g_ω or not. If there are none, the answer to the above question would be yes, namely precisely sectors of the type (g_θ, g_ω). However, if we choose g_θ and g_ω to have roto-translations

$$g_\theta = \left(\theta \,\middle|\, \frac{1}{2}e_2\right) , \tag{3.42}$$

$$g_\omega = (\omega \,|\, 0) , \tag{3.43}$$

g_θ and g_ω no longer commute and we have indeed realized a model with $\mathcal{N} \geq 2$ supersymmetry in each (g, h)-twisted sector but only $\mathcal{N} = 1$ in the intersection.

It is very tempting to hope that a similar construction as in this example might as well work in non-supersymmetric cases, namely that one could be able to enhance a global $\mathcal{N} = 0$ to $\mathcal{N} \geq 1$ supersymmetry in each (g, h)-twisted sector, which, as we just showed, would imply the existence of at least one common Killing spinor and hence cause the cosmological constant to vanish due to generalized Riemann identities. As we just argued, (g, h)-sectors of the second kind are a potential obstacle in this construction, but they can possibly be avoided by the introduction of roto-translations. A minimal condition for the idea of local but not global SUSY is the following

Conditions for local but not global Killing spinors

(i) Every (g, h)-twisted sector admits at least one Killing spinor *locally*. In effect, the partition function vanishes sector per sector provided there are no (dangerous) type 2 sectors.

(ii) Not all (g, h)-twisted sectors preserve the same Killing spinor(s). That is, each element h in the space group must admit at least one local Killing spinor, that is incompatible with the Killing spinor in at least one other (g, h)-sector of the first kind. By result, it is impossible to define any *globally* invariant spinor. Therefore target-space supersymmetry is broken, and our model is $\mathcal{N} = 0$.

The purpose of the next section is to translate this statement to the language of group theory, specifically to the representation theory of the point group P.

3.4 Group theoretical non-existence proof

In the previous section we saw that a necessary condition to ensure the existence of a large class of non-supersymmetric heterotic orbifold theories with vanishing cosmological constant is the following property: For each point group element separately some amount of supersymmetry is preserved but globally, i.e. for the full point group, no Killing spinor exists. In this section we will show that there are no such toroidal orbifolds. As our discussion will be centered around the properties of the point group P, we start out with a study which discrete groups are admissible as point groups at all.

3.4.1 Survey of point groups for toroidal orbifolds

Let us have a look at all possibly available point groups P. Here, we can follow the CARAT-classification of (heterotic) orbifold point groups [31, 32, 33]. As mentioned in chapter 2, the classification of orbifolds goes all the way down to affine classes, hence specific lattices and roto-translations. However, the previous section shows that the rightmover partition function (and with it our minimal condition) only depends on the rotational part of the space group. Hence, out of the 28,927,915 affine and 85,308 $\mathbb{Z}$-classes, we only consider the 7,103 $\mathbb{Q}$-classes that appear. Let us note here again that $\mathbb{Q}$-classes and (abstract) point groups are not in one-to-one correspondence: one abstract point group may possess several inequivalent realizations as $\mathbb{Q}$-classes, which becomes clear once one notices that the 7,103 $\mathbb{Q}$-classes are generated by only 1,594 abstract discrete groups. Hence, one should strictly think of $\mathbb{Q}$-classes as representations rather than groups.

We find that quite many point groups are already ruled out, because they live in O(6) rather than SO(6), so spinors cannot be defined. Out of the remaining ones, again the majority cannot be considered as solutions, because there is no $\mathbb{Q}$-class that implements them in such a way that any element of the point group rotates at least two orbifold planes at once. An element that only rotates one plane however necessarily projects out all four candidates for Killing spinors. This is because it always gives rise to a $-\mathbb{1}_6$ element in the geometric point group, and therefore automatically violates condition (i) presented in the foregoing section. This reduces the number of admissible $\mathbb{Q}$-classes to 106. See table 3.1 for details of this reasoning.

Now, we can use equation (3.7) to construct the various possible spinor representations $D_\mathbf{4}$ of each element separately and then apply equation (3.16) to find the number of allowed local SUSY, see appendix C.1. Note that this allows for element-specific basis changes, and is hence not suitable to identify global properties of the spin embedding, which in turn would require a consistent choice for all elements *simultaneously*. All in all, one finds by explicit constructions that only 63 of the remaining 106 $\mathbb{Q}$-classes allow for a Killing spinor in every (g, h)-sector. However, 60 of these 63 $\mathbb{Q}$-classes have already been identified to allow for global Killing spinors, i.e. $\mathcal{N} = 1$ supersymmetry. Hence, there are two ways to tackle the issue of local but not global SUSY

1. Look for globally non-supersymmetric realizations in the 60 SUSY orbifolds that retain the property of local SUSY in each sector.

2. In the remaining three $\mathbb{Q}$-classes, check if there exists a spinor embedding such that there indeed exists a Killing spinor in each (g, h)-sector.

This is a very tedious task, and moreover one cannot always be sure that all solutions to the defining condition (3.7) have been checked. In the next section we will follow another route, namely via the representation theory of the point group P.

# $\mathbb{Q}$-classes	Restriction
7,103	All inequivalent geometrical point groups $P \subset \mathrm{O}(6)$
1,616	Orientable geometrical point groups $P \subset \mathrm{SO}(6)$
106	No element of P rotates in a two-dimensional plane only
63	Each element $\theta \in P$ admits a choice with $\mathcal{N}^{\langle\theta\rangle} \geq 1$ local Killing spinors
60	Geometrical point group compatible with $\mathcal{N} \geq 1$ global Killing spinors

Table 3.1: *Statistics of $\mathbb{Q}$-classes fulfilling various restrictions on six-dimensional toroidal orbifolds.*

3.4.2 Group-theoretical conditions

As we will work with abstract groups now, it easily happens that one forgets to incorporate the necessary condition that the vector representation $D_{\mathbf{v}}$ and the spinor representation $D_{\mathbf{s}}$ have to generate groups that are isomorphic in order to admit any (local and global) Killing spinors. We demonstrate the failure of any attempt to achieve a vanishing partition function with an example in appendix C, and from now on make the implicit assumption that the groups generated by $D_{\mathbf{v}}$ and $D_{\mathbf{s}}$ are isomorphic whenever we speak of the point group P. Then, we can be sure that the number of Killing spinors allowed by the spinor representation $D_{\mathbf{s}}$ is in fact the number of Killing spinors preserved by the entire model. Let us now reformulate the conditions (i) and (ii) from the previous section in terms of representations of P. In particular, recall that the number of Killing spinors compatible with the action of a group G is counted by the dimension of a G-invariant subspace, obtained by an appropriately chosen projector, see the discussion around equation (3.16). Now, the trace appearing in (3.16) can be rewritten in terms of the character of the representation, $\mathrm{Tr}\, D_{\mathbf{4}}(\theta) = \chi_{\mathbf{4}}(\theta)$ [65]. Then, (3.16) can be recast to read

$$\mathcal{N}^G \;=\; \frac{1}{|G|} \sum_{\theta' \in G} \chi_{\mathbf{4}}(\theta') \;=\; \langle \chi_{\mathbf{4}}, \chi_{\mathbf{1}} \rangle_G \;=\; n^G_{\mathbf{1}}\,. \tag{3.44}$$

Here, we have made use of various group-theoretical concepts: First, we use the fact that the character of the trivial one-dimensional representation $\mathbf{1}$ is a vector of ones only $\chi_{\mathbf{1}} = (1, \ldots, 1)$. Moreover, we make use of the fact that the characters of irreducible representations are pairwise orthogonal with respect to the inner product $\langle\,\cdot\,,\cdot\,\rangle_G$. Hence, the number of Killing spinors allowed by a group G can be determined by counting the number of trivial singlets of G appearing in the branching of the four-dimensional (generally reducible) representation $\mathbf{4}$ into irreducible representations of G.

Now, this observation can be used for our purposes of determining whether a model preserves local and global supersymmetries by making different choices for the group G in (3.44).

Local Killing spinors. If we take G to be the $\mathbb{Z}_N$ subgroup of the point group P that is generated by some point group element θ, we see that the existence of local Killing spinors is precisely given by the number of trivial $\mathbb{Z}_N$ singlets contained in the four-dimensional spinor representation **4**

$$\mathcal{N}^{\langle\theta\rangle} \;=\; n_{\mathbf{1}}^{\langle\theta\rangle} \;. \tag{3.45}$$

Our condition (i), that there is at least one Killing spinor in each sector in order for the partition function to vanish, can be mapped to the condition that the spinor representation **4** branches into at least one trivial singlet of each $\mathbb{Z}_N$ subgroup generated by the elements of P. In other words,

$$n_{\mathbf{1}}^{\langle\theta\rangle} \;\geq\; 1 \;,\; \forall\, \theta \in P \;, \tag{3.46}$$

is a minimal condition that needs to be met.

Global Killing spinors. When making statements about global properties with respect to supersymmetry, we set $G = P$ and consider the full point group at once in (3.44). Following the discussion so far, it is evident that the number of allowed SUSY is given by

$$\mathcal{N} \;=\; n_{\mathbf{1}}^{P} \;. \tag{3.47}$$

It is obvious that $n_{\mathbf{1}}^{P} \geq 1$ also implies $n_{\mathbf{1}}^{H} \geq 1$ for all subgroups $H \subset P$, in particular those $\mathbb{Z}_N$ subgroups that are generated by all elements $\theta \in P$, simply because the trivial singlet of P branches into the trivial singlet of all subgroups. This is in line with the intuition that globally preserved Killing spinors are also preserved locally.

In summary, we can now reformulate the conditions of the previous section as

Conditions for local but not global Killing spinors
(group theory version)

The spinor representation $D_{\mathbf{s}}$ of P must have the following properties

(i) The four-dimensional chiral representation **4** associated with $D_{\mathbf{s}}$ must branch into at least one trivial singlet of every $\mathbb{Z}_N$ subgroup generated by the elements of P.

(ii) The four-dimensional representation does not decompose into a trivial singlet of the entire point group P.

(iii) The group generated by $D_{\mathbf{s}}$ is isomorphic to the group generated by $D_{\mathbf{v}}$.

Obviously, these conditions are straightforward to check for a given group P and a given spinor representation.

3.4.3 Non-existence proof by enumeration

At the beginning of this section, we have seen that the main result of the CARAT-classification of crystallographic orbifolds is that there is only a finite number (namely

7,103) inequivalent six-dimensional representations acting as orbifold twists. As we discussed in the previous paragraph, it would be sufficient to iterate over this list of $\mathbb{Q}$-classes, construct all possible spinor representations for every $\mathbb{Q}$-class, and then check if any of these spinor embeddings fulfills the desired properties. However, if we make use of the group-theoretical considerations we just have presented, we can simplify this process considerably. Namely, the 7,103 $\mathbb{Q}$-classes are representations of only 1,594 distinct discrete groups. If we assert that the geometric and the spinorial action are isomorphic, any four-dimensional representation fulfilling the conditions for local but not global SUSY must be a four-dimensional (possibly reducible) representation of one of these 1,594 discrete groups. Therefore, our strategy also works by iterating over the list of distinct discrete groups, generating all four-dimensional representations of the group, and checking whether the necessary properties are fulfilled for each representation. The upshot of this approach is that one rarely ever needs to work with the explicit 6×6 matrices of the $\mathbb{Q}$-classes, but instead with the abstract properties of the discrete groups and their representations, that are all encoded in the character table. Moreover, all relevant discrete groups and their properties are classified and accessible e.g. in the SmallGroups GAP package [66]. In order to prove that none of the possible four-dimensional representations admitted by any of the possible abstract point groups fulfills the required conditions, we performed these steps in an automated fashion, using the GAP programming language.

Let us now discuss the procedure of the non-existence proof. From an algorithmic point of view, it pays off to adopt the following order of steps for each discrete group G in the list:

1. Generate all four-dimensional representations of the group G.

2. For each representation $\mathbf{4}$:

   2.1 Check if there is a trivial singlet of G in $\mathbf{4}$. If yes, then it can be discarded.

   2.2 Check if $\det \mathbf{4} = 1$ to make sure it is a spinor representation after all. If no, the representation cannot fulfill all conditions and is discarded.

   2.3 Check if the representation $\mathbf{4}$ branches into a trivial singlet of each $\mathbb{Z}_N$ subgroup generated by the elements of G. Again, if the answer is no, the representation can be discarded.

   2.4 Finally, if all foregoing steps were evaluated positive, check if the two-times antisymmetrization $[\mathbf{4}]_2$ corresponds to a geometric action $\mathbf{6}$. If yes, one has found an example for a non-supersymmetric orbifold that fulfills the minimal condition for a vanishing cosmological constant and dilaton tadpole.

We observe that none of the admissible groups possesses four-dimensional representations that pass step 2.1, 2.2 and 2.3, such that step 2.4 is never invoked. Therefore, already the minimal condition for a vanishing partition function, namely that at least all $(\mathbb{1}, g)$-twisted sectors must allow at least one Killing spinor is always violated. Hence, we conclude that there are *no* symmetric toroidal heterotic orbifolds with $\mathcal{N} = 0$ target-space supersymmetry and vanishing cosmological constant and dilaton tadpole.

3.5 Loopholes beyond toroidal symmetric orbifolds

So far, we have elaborated on a no-go result that applies to heterotic strings on symmetric toroidal orbifolds only. As a first idea for more general compactifications beyond symmetric toroidal orbifolds, one could think about compact spaces where the underlying geometry is not a torus but e.g. a smooth Calabi–Yau n-fold, which in turn has to allow for some automorphisms of finite order. Apart from that, one should keep in mind that the type II counterpart of our idea only works in asymmetric orbifolds. It is therefore natural to ask whether an asymmetric action point group action may possibly be the missing ingredient. We will comment on these two possibilities and the conditions necessary for them to work here.

Calabi–Yau manifolds with finite automorphisms. Think about a Calabi–Yau threefold (e.g. an $\mathcal{N} = 2$ supersymmetric K3 $\times$ $\mathbb{T}^2$) with special features, namely that it has an automorphism of finite order, which we assume is a rotation (any translational components will not have any action on target-space spinors whatsoever, so we can ignore this). We also assume that this symmetry exists at least at some point in the Calabi–Yau moduli space. The idea would be now that the finite rotations preserve a set of (common!) Killing spinors that are, however, disjoint from the set of Killing spinors on the Calabi–Yau. That way, one circumvents the negative result of the previous sections, namely that discrete rotations seem to be unable to preserve local Killing spinors without permitting also global ones. On the other hand, one cannot keep too much hope for this construction, for the following reason: It is known that many Calabi–Yau manifolds (especially the phenomenologically favored complete-intersection CYs (CICYs)) can be deformed in one or more ways to look like a (toroidal) orbifold by going to an appropriate limit in moduli space referred to as orbifold point. In our example, the K3 two-fold possesses orbifold limits to e.g. $\mathbb{T}^4/\mathbb{Z}_2$ or $\mathbb{T}^4/\mathbb{Z}_4$. There is no reason why this deformation should no longer be allowed if the discrete automorphism is modded out. Hence, any orbifold limit of a Calabi–Yau with the discrete automorphism modded out will yield just one of the symmetric toroidal orbifolds for which we have already shown that the desired construction (with local but not global Killing spinors) does not exist, and therefore this construction does not seem to provide a promising way out.

Asymmetric (toroidal) orbifolds. Let us now discuss asymmetric orbifolds. Unlike in the type II construction, where this precisely did the trick, the action of the asymmetric twist on the leftmover has practically no influence on the result, because, as we have seen, the only thing to look at is the rightmover partition function. However, allowing for asymmetric constructions still brings us beyond the excluded cases, because the number of available point groups is now by far bigger than in the symmetric case, cf. [67]. For example, in two dimensions, the maximally allowed order of discrete rotations in symmetric orbifolds is six, while it is twelve in the asymmetric case [68, 69]. One can assume that there is a fairly large number of discrete groups we have not looked at so far, that might possibly possess precisely the properties we are looking for to ensure a vanishing one-loop partition function. However, as we will argue in the next section, we cannot be optimistic that any discrete group possesses the right four-dimensional representation at all.

3.6 Conjecture for general discrete groups

Recall one of the observations from the computer-aided non-existence proof in section 3.4.3, namely that we never got so far to check if any of the generated four dimensional representations matches a six-dimensional geometric rotation via $[\mathbf{4}]_2 = \mathbf{6}$. This might be just by coincidence, however, given the sheer number of tested point groups in the CARAT-classification, we still find it intriguing. In particular, the fact that the requirement of an associated geometric action does not matter seems to indicate that this result is in fact independent of orbifold compactifications. However, we do not have a rigorous proof for this statement. Hence, we formulate this claim in the form of the following

Conjecture. *For any given finite group* $\mathbf{H}$*, any representation* D *of* $\mathbf{H}$ *can fulfill no more than three out of the following four properties.*

1. *The dimension of the representation is four.*
2. D *has trivial determinant, that is* $\forall\theta \in \mathbf{H}$*, it holds that* $\det D(\theta) = 1$.
3. D *does not contain a trivial singlet of* $\mathbf{H}$.
4. D *branches into at least one trivial singlet of any* $\mathbb{Z}_N$ *subgroup of* $\mathbf{H}$.

Note that any requirement of isomorphy to a geometric six-dimensional action has disappeared completely. The only connections to heterotic orbifolds are the dimension of the representation (stemming from $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$) and the condition on the determinant (in order to define spinors). As we will show shortly, both these conditions are crucial for the conjecture to hold, as otherwise counter-examples are easily constructed.

We have extended our computer-aided search for four-dimensional representations fulfilling all conditions on the conjecture beyond the 1,594 discrete groups appearing in the CARAT-classification to $\mathcal{O}(100,000)$ other discrete groups from the SmallGroups library of GAP with order less than 500. None of the checked groups possesses a four-dimensional representation that would invalidate our conjecture. Given the fact that the groups in the CARAT-classification have orders of up to 103,680, we regard the non-observation of any such four-dimensional representation as a strong piece of evidence for our conjecture.

Although we have formulated and tested the conjecture without making any references to physics, it would still have physical consequences if it were ultimately true for all finite groups, as then also the last loophole, namely asymmetric orbifolds, would be ruled out, because there are simply no discrete groups that act on the rightmover in the desired manner.

Let us now examine why stronger versions of the conjecture, i.e. where some of the requirements on the properties of the representation are lifted, cannot be formulated. In particular, we have seen in the previous sections that once a representation violates condition 3., namely contains a trivial singlet of the group $\mathbf{H}$, condition 4. is automatically satisfied, no matter what the dimension or the determinant of the representation is ("local Killing spinors are also global ones"). Hence, we will focus on the requirements on the dimensionality and the determinant, which are less intuitive.

Lifting the constraint on the dimensionality. At the first sight, there seems to be no reason why the dimension of the representation matters for its branching behavior. However, as we shall show here, it is possible to construct a five-dimensional representation (violating condition 1.) that still fulfills the conditions 2., 3. and 4. in the conjecture. To this end, we consider the quaternion group Q_8 (see e.g. [70]) and take the five-dimensional representation to decompose into irreducible representations of Q_8 according to

$$\mathbf{5} \;=\; \mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{--} \oplus \mathbf{2}\;. \tag{3.48}$$

Specifically, in this way, the two generators θ_1 and θ_2 of Q_8 take the explicit form

$$D_{\mathbf{5}}(\theta_1) \;=\; \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & & -\mathrm{i} \\ & & & -\mathrm{i} & \end{pmatrix}\;, \quad \text{and} \quad D_{\mathbf{5}}(\theta_2) \;=\; \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & & -1 \\ & & & 1 & \end{pmatrix}\;. \tag{3.49}$$

Note that the representation has unit determinant, and that it does not contain the trivial Q_8-singlet $\mathbf{1}_{++}$. Moreover, in each branching into the various $\mathbb{Z}_N$ subgroups generated by the elements of Q_8, there is at least one trivial element of that subgroup [70]. Therefore, all conditions of the conjecture are fulfilled, except the dimensionality. Hence, in order for the no-go statement of the conjecture to hold, it is imperative to insist on four-dimensional representations.

Lifting the constraint on the determinant. As with the dimensionality, this constraint could be taken as an artifact coming from the spinor interpretation, and it is not clear how this requirement should constrain the branching behavior as predicted by the conjecture. To prove this claim, we again construct an example that fulfills all but one of the conditions in the conjecture, namely in this case conditions 1., 3. and 4., while 2. is violated.

This time, we use a representation of the group T_7 of order 21, for details on this group see e.g. [70]. The irreducible representations of T_7 are the following: There are two three-dimensional representations $\mathbf{3}$ and $\overline{\mathbf{3}}$ that are related by complex conjugation. Moreover, aside from the trivial singlet $\mathbf{1}_0$ there are two non-trivial ones, $\mathbf{1}_1$ and $\mathbf{1}_2$. The elements of T_7 generate two different $\mathbb{Z}_N$ subgroups: a $\mathbb{Z}_7$ and a $\mathbb{Z}_3$. As for the branching, a singlet $\mathbf{1}_q$, $q \in \{0,1,2\}$, always branches into a $\mathbf{1}_q$ of $\mathbb{Z}_3$ and in the trivial singlet of $\mathbb{Z}_7$. On the other hand, the branching of the triplets into $\mathbb{Z}_7$ representations never contains a $\mathbb{Z}_7$-singlet, whereas the branching of the triplets into $\mathbb{Z}_3$ representations always contains a $\mathbf{1}_0$ of $\mathbb{Z}_3$. Using this information, we choose the following four-dimensional representation as our counter-example

$$\mathbf{4} \;=\; \mathbf{3} \oplus \mathbf{1}_1\;, \tag{3.50}$$

which does not contain a trivial singlet but does not have determinant one. As for the branching, we find that

$$\begin{array}{rccccl} \mathbf{4} \;= & \mathbf{3} & \oplus & \mathbf{1}_1 & \text{of } T_7 \\ & \downarrow & & \downarrow & \\ & \mathbf{1}_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_2 & \oplus & \mathbf{1}_1 & \text{of } \mathbb{Z}_3 \\ \hline & \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{1}_4 & \oplus & \mathbf{1}_0 & \text{of } \mathbb{Z}_7 \end{array} \tag{3.51}$$

and hence that the representation **4** branches in one $\mathbb{Z}_N$-singlet for each $\mathbb{Z}_N$ subgroup of T_7. Therefore, all conditions except the lifted condition on the determinant are fulfilled, which again tells us that condition 2. in the conjecture is indeed needed in order to make the no-go statement.

3.7 Concluding remarks

In this chapter, we study under which conditions the one-loop cosmological constant (and with it the dilaton tadpole) of heterotic string models on symmetric toroidal orbifolds vanishes. In this setup, the cosmological constant is given as the modular integral over the one-loop partition function. As it is known that the partition function (and then of course also the integral over it) vanishes identically for supersymmetric compactifications, we put a special focus on non-supersymmetric orbifolds. We first identify the various possibilities for the vanishing of the cosmological constant that are tied to properties of the one-loop partition function, which is an expansion in terms of (g, h)-twisted sectors, where g, h can be any pair of commuting space group elements. In general, the heterotic partition function of a (g, h)-twisted sector has a complicated dependence on the specific gauge embedding consisting of shifts and Wilson lines acting on its leftmoving degrees of freedom. We find that if we insist that the cosmological constant vanishes in a model-independent manner (i.e. to some degree independent of the gauge embedding), the (fermionic) rightmover partition function in each (g, h)-twisted sector needs to vanish on its own.

We then proceed to study under which conditions the rightmover partition function vanishes sector per sector. We find that the (non-)vanishing of a (g, h)-sector depends on the embedding of the geometric orbifold twist in spinor space. Specifically, by studying Riemann identities for Jacobi theta-functions appearing in the rightmover partition function, we identify the condition that the spin embedding of the geometric twist must be such that each commuting pair g, h can preserve at least one common Killing spinor. This condition is automatically satisfied in supersymmetric compactifications, whereas the situation is not so clear if SUSY (corresponding to a Killing spinor compatible with all space group elements g) is absent. However, if we were able to find an orbifold where the geometric twist acts on spinors such that each (g, h)-twisted sector preserves at least one "local" Killing spinor but none globally, we would have indeed found an example of a non-supersymmetric heterotic string compactification with a nevertheless vanishing cosmological constant.

Our motivation to look for spinor embeddings with the required properties comes from (asymmetric) compactifications of type II superstrings. It is known that a duality between heterotic and type II strings arises as a side-effect of compactification. It is believed that this duality continues to exist also in non-supersymmetric compactifications [71]. Moreover, there do exist models with a vanishing one-loop cosmological constant in type II constructions [58, 72, 73], although it is debated whether or not this result can truly persist in perturbation theory [74], and it is also not clear if cosmological constant of the heterotic dual receives non-perturbative contributions [75]. At any rate, the type II constructions work because precisely every twisted sector vanishes on its own, while still not preserving a global supercharge, by choosing an asymmetric orbifold action that causes either the rightmover or the leftmover in each (g, h)-twisted sector to vanish. We show in this chapter that the same construction cannot work in the heterotic case, simply because choosing a gauge embedding that causes the leftmover to vanish would also project out the massless graviton and hence render the model phenomenologically unattractive. Still, the positive type II result may cause one to believe that a similar construction has a chance to work on the heterotic side as well, specifically with an appropriate spin embedding acting on the rightmover.

In order to determine whether or not there exists a non-supersymmetric heterotic orbifold with vanishing partition function in each sector, we find that we have to check if the spin embedding of any of the admissible orbifold geometries (tabulated by the CARAT-classification) has the desired properties concerning locally preserved Killing spinors. In principle, this could be done by explicitly constructing the set of possible spinor embeddings for each geometry and then checking each (g, h)-sector for locally preserved Killing spinors as well as the entire space group to ensure the non-existence of globally preserved Killing spinors. However, instead of constructing the explicit representation matrices, we make use of the fact that both the geometric orbifold twist and its embedding into spinor space are representations of a discrete group. In particular, we are able to map the conditions on the spinor embedding to constraints on the representations of the respective finite group, which greatly simplifies the procedure and removes any ambiguity coming from the double cover properties of the spin embedding.

Unlike what one might expect from the type II case, the result of our analysis is that none of the available orbifold point groups admits a spinor embedding that both breaks SUSY globally but still lets the partition function vanish sector per sector. In particular, even the weakest necessary condition, namely that each element g of the space group has to preserve some Killing spinor (which corresponds to the requirement to that all $(\mathbb{1}, g)$-twisted sectors and their images under modular transformations vanish) appears to be violated and hence the rightmover partition function does precisely not vanish sector per sector as soon as one considers a non-supersymmetric compactification. We also comment on how this no-go result may carry over to more involved constructions, such as quotients of Calabi–Yau manifolds by some finite group where the Killing spinors preserved by the CY are incompatible with those of the finite group, or asymmetric orbifolds.

While iterating over the finite set of admissible discrete point groups, we make the observation that our failure to find a four-dimensional representation with the desired properties does not depend at any point on the fact that the geometric counterpart of the spinor action has to act crystallographically on a lattice. This fact prompts us to formulate the observed non-existence of such representations as a mathematical conjecture that is conceived to hold for any finite group. Specifically, we conjecture that no finite group possesses a representation that is (a) four-dimensional, (b) has determinant one, (c) does not contain a trivial singlet of the group but (d) branches into at least one trivial singlet of each cyclic subgroup. While we do not have a rigorous proof, we check a large number of discrete groups and find no counterexample. We do, however, easily find counterexamples as soon as one of the mentioned conditions (a-d) is removed. Hence, we believe that our conjecture is formulated in the strongest version possible (i.e. with the minimal set of assumptions).

Outlook. Let us now start by discussing the impact of our results on constructions that aim to have a vanishing or small cosmological constant in a model-independent way, i.e. mostly independent of the gauge embedding. Given that the no-go result of this chapter seems to extend to any non-supersymmetric heterotic string construction, it motivates constructions with an exponentially suppressed cosmological constant. As mentioned earlier, heterotic models with a small but finite cosmological constant have already been constructed [56], even so with a positive one [76]. In particular, there are constructions where one is able to interpolate between SUSY and non-SUSY models by varying a modulus on the orbifold [64, 77, 78, 79], and some that solve the decompactification problem (see for example [80]).

On the other hand, our general group theory conjecture may be applied in Calabi–Yau constructions without toroidal limits that can be described as gauged linear sigma models (GLSMs) [81, 82] or non-supersymmetric Gepner models [83, 84]. It could also prove useful in the study of more exotic orbifold-like constructions [85]. Finally, studying the representations of the point group (and especially their branching into Abelian subgroups) may find its application in (supersymmetric) heterotic model building on non-Abelian orbifolds [86, 87].

4

Discrete gauge symmetries from orbifolds

4.1 Introduction

Discrete symmetries play an important role in all particle physics model building—especially in bottom-up models. Often, discrete symmetries are invoked in order to forbid dangerous couplings that would otherwise render a theory unrealistic, or to address the flavor problem. Importantly, there exist good reasons to believe that ultimately all global symmetries—discrete or continuous—should be gauged [13].

On the other hand, discrete symmetries frequently arise from gauge symmetries when a gauge symmetry is broken. In four-dimensional physics, breaking a gauge symmetry is often achieved by employing a Higgs mechanism that spontaneously breaks the gauge group $\mathcal{G}$ to a (continuous) subgroup $\mathcal{H}$. Along with the unbroken gauge group $\mathcal{H}$, also discrete symmetries can survive, which then act as outer automorphisms of $\mathcal{H}$. Within the context of heterotic model building (or in models with more than four spacetime dimensions in general), especially when constructing models that contain grand unified theories (GUTs) as intermediate steps, one would of course like to look for similar realizations. However, spontaneous symmetry breaking is notoriously hard to achieve in (heterotic) string vacua. The reason is that the necessary large representations of the gauge group are often not part of the spectrum. For example, in order to break an SO(10) GUT to the Pati–Salam group, one needs a Higgs in a **54** of SO(10), but there is no known stringy SO(10) model that possesses a Higgs in a **54** in its spectrum, making it impossible to achieve the corresponding gauge symmetry breaking by a vacuum expectation value for the Higgs.

However, it is known that in models with extra dimensions, gauge symmetries cannot only be broken spontaneously, but alternatively by orbifolding [35, 36, 88], namely by building on the fact that some of the higher-dimensional gauge bosons may not survive the orbifold projection as soon as non-trivial shifts and Wilson lines are turned on. In these settings, one starts with an "upstairs" gauge theory that lives in extra dimensions, and then compactifies the theory on an orbifold. If the orbifold is chosen such that it acts non-trivially on the gauge degrees of freedom, this allows one to break the upstairs gauge group to some subgroup. Roughly, the picture is then the same as for spontaneous symmetry breaking, namely that the low-energy theory has a smaller symmetry, whereas the full (UV) symmetry eventually gets restored once the relevant energy scales surpass a certain threshold. It also turns out that—as in the case of spontaneous breaking—there can be discrete symmetries that survive the breaking along with the continuous ones. These discrete symmetries (that are global from a low-energy perspective) have the appealing feature that they ultimately descend from a gauge theory, which is why they are often called *discrete gauge symmetries* [89, 90]. With the existence of these discrete gauge symmetries, the idea of gauge symmetry breaking through orbifold boundary conditions was popularized by the observation that many problems in GUT model building, such as the doublet-triplet splitting, can be solved in a simple fashion [91, 92, 93, 94, 95, 96, 97, 98]. While the unbroken low-energy continuous gauge symmetries are studied thoroughly [36, 97, 99], the same cannot be said about the *gauge* origin of discrete remnant symmetries that survive the orbifold projection along with the continuous ones.

Another way to study discrete symmetries in orbifold theories is directly via string theory. In a full string model, the possible couplings between orbifold-invariant states are restricted by the so-called string selection rules. The two most prominent examples for this class of restrictions are the point group and space group selection rules. These selection rules allow or forbid couplings between strings based on their localization on the orbifold, by considering whether or not a given set of strings can split and join on a geometrical level. Therefore, these symmetries arise from the constructing space group elements of states in the theory. Specifically, the string selection rules operate on the level of conjugacy classes of space group elements. From a four-dimensional perspective, these conjugacy classes correspond to orbifold-invariant states in the low-energy effective (field) theory. The low-energy states carry their localization in the orbifold dimensions (bulk versus brane fields) as additional quantum numbers. On these quantum numbers, the stringy point group and space group selection rules generically act as finite groups. It has been demonstrated in various instances how phenomenologically interesting discrete groups can arise from this construction. However, it is unclear if this approach captures all surviving discrete symmetries or whether some are overlooked. Moreover, in this setting, it is not obvious how the origin of the discrete symmetries can lie in a conventional gauge symmetry. While there have been attempts to make this link within string theory (in the context of symmetry enhancement) [100], the results are hard to carry over to a generic field theory model. The reason for this is that by far not every conceivable gauge symmetry can be realized by symmetry enhancement (only simply-laced groups are possible), and that the intuition is that one should be able to study the survival of a discrete remnant symmetry from a gauge symmetry no matter how the gauge symmetry is realized.

Therefore, the purpose of this chapter is to study the emergence of discrete symmetries that survive an orbifolding procedure from gauge symmetries. We start with a purely field-theoretic discussion that is largely independent of the localized matter and focuses on the origin of discrete remnant symmetries in higher-dimensional gauge theories. To this end, we study the compactification of six-dimensional Yang–Mills first on a torus and then on an orbifold in section 4.2. In section 4.3, we revisit the known conditions for unbroken continuous gauge symmetries, and explain how these conditions have to be modified to capture the existence of all discrete symmetries surviving the orbifold projection that were overlooked previously. After the theoretical foundations are in place, we then study the applications to orbifold GUT model building. In particular, we demonstrate how a SO(10) GUT that is broken to its Pati–Salam subgroup by orbifolding exhibits a discrete left-right parity (referred to as D-parity) at low energies. Moreover, we present two examples with promising properties for flavor model building in orbifold GUT models. This chapter is in parts based on ref. [16, 18].

4.2 Gauge theories in extra dimensions

In this section, we will review the relevant setup for the study of remnant symmetries after orbifolding. To this end, we will study a six-dimensional Yang–Mills theory with gauge group $\mathcal{G}$ which we will refer to as the upstairs theory. Then, we will discuss how this theory can be compactified to four dimensions on a toroidal $\mathbb{T}^2/\mathbb{Z}_N$ orbifold.

In terms of the field content, the upstairs theory contains gauge bosons $V^M(x,y)$, $M = 0,\ldots,5$ associated with the upstairs gauge group $\mathcal{G}$, and possibly their fermionic superpartners. For our purposes, it is sufficient to study the bosonic degrees of freedom only, whose Lagrangian is the standard one and reads

$$\mathcal{L} \;=\; -\frac{1}{2}\,\mathrm{tr}\left(F_{MN}F^{MN}\right)\;. \tag{4.1}$$

Here, F_{MN} denotes the field strength tensor. As we are dealing with a non-Abelian gauge theory, the gauge bosons can be expanded in terms of the generators T_a of the Lie algebra associated with $\mathcal{G}$. A particularly useful basis for the Lie algebra generators is the Cartan–Weyl basis $\mathsf{T}_a^{(\mathrm{CW})}$. It consists of the Cartan generators H_I, $I = 1,\ldots,\mathrm{rank}(\mathcal{G})$, that span the maximal commuting subalgebra, that is $[H_I, H_J] = 0$. Moreover, there are the ladder operators E_w, where w is an element of the set W that consists of the non-trivial roots of $\mathcal{G}$. By expanding $V^M(x,y)$ in terms of these operators

$$V^M(x,y) \;=\; \sum_I V_I^M(x,y)\,H_I + \sum_{w\in W} V_w^M(x,y)\,E_w \;=\; \sum_a V_a^M(x,y)\,\mathsf{T}_a^{(\mathrm{CW})}\;, \tag{4.2}$$

we obtain the component fields $V_I^M(x,y)$ and $V_w^M(x,y)$ associated with the corresponding Lie algebra generators. We choose this particular spitting because, as we will see later on, the fields associated with Cartan generators will transform differently than those associated with ladder operators.

As we will eventually consider compactifications of the six-dimensional theory to four dimensions, we can also split the vector field $V^M(x,y)$ depending on the value of the vector index M. For notational simplicity, we have already split the spacetime coordinates into coordinates x in the Minkowski spacetime and coordinates $y = (y_1, y_2)^T$ that will eventually lie in the extra dimensions. In particular, we take the basis vectors of y to coincide with the lattice basis of a two-torus $\mathbb{T}^2$ that will later be orbifolded to a $\mathbb{T}^2/\mathbb{Z}_N$ orbifold, as described in section 4.2. To this end, we also have to split the gauge fields $V^M(x,y)$ into components with index $M = \mu$ in the four-dimensional Minkowski spacetime and with index $M = 4, 5$ in the internal compact dimensions on the torus/orbifold. In order to perform this splitting, let us recall that if the setting would descend from a heterotic string model, the rightmoving H-momenta of the six-dimensional vector fields $V^M(x,y)$ would be

$$q \;\in\; \left\{\left(\underline{\pm 1, 0}, 0, 0\right)\right\}\;. \tag{4.3}$$

For the two cases $M = 0, \ldots, 3$ and $M = 4, 5$, the vector fields $V^M(x,y)$ get split into four-dimensional gauge bosons

$$V^\mu(x,y) \quad \text{with} \quad q \;\in\; \{(\pm 1, 0, 0, 0)\} \;, \tag{4.4}$$

and a pair of complex four-dimensional scalars

$$\chi^\pm(x,y) \;=\; \frac{1}{\sqrt{2}}\left(V^4 \pm \mathrm{i}\, V^5\right) \quad \text{with} \quad q \;\in\; \{(0, \pm 1, 0, 0)\} \;, \tag{4.5}$$

all of which transform in the adjoint representation of the upstairs gauge group $\mathcal{G}$. Let us note, however, that because the scalar fields χ originate from a higher dimensional vector, they still carry a vector index in the extra dimensions, and hence are going to transform non-trivially under rotations in these dimensions.[1] Let us now discuss the compactification of this theory on a $\mathbb{T}^2/\mathbb{Z}_N$ orbifold. We will follow the usual procedure and consider first the compactification on a torus.

Torus compactification. As discussed in chapter 2, compactifying on a torus amounts to imposing periodicities for the coordinates y

$$y \;\sim y + e_1 \;, \tag{4.6}$$
$$y \;\sim y + e_2 \;, \tag{4.7}$$

where "$\sim$" indicates that two points are now identified and where we have introduced the two linearly independent basis vectors e_i that span the lattice of the torus $\mathbb{T}^2$. Depending on the orbifold, we will choose different torus metrics $G_{ij} = e_i \cdot e_j$. In order to span the entire lattice of the $\mathbb{T}^2$ one can now take a general, integral linear combination $n_i e_i$ for $n_i \in \mathbb{Z}$, where summation over $i = 1, 2$ is implied. Because the space has a periodicity, we also have to impose boundary conditions on the fields $V^\mu_a(x,y)$ and $\chi_a(x,y)$. Specifically, torus periodicity implies that for all $n_i \in \mathbb{Z}$

$$V^\mu_a(x, y + n_i e_i) \;=\; A(n_i)\, V^\mu_a(x,y)\, A(n_i)^{-1} \;, \tag{4.8}$$
$$\chi_a(x, y + n_i e_i) \;=\; A(n_i)\, \chi_a(x,y)\, A(n_i)^{-1} \;. \tag{4.9}$$

Here, the matrices $A(n_i)$ are discrete gauge transformations and can be parametrized as

$$A(n_i) \;=\; A_1^{n_1} A_2^{n_2} \;, \tag{4.10}$$

where we used that $[A_i, A_j] = 0$ for consistency. In a UV-complete model descending from the heterotic string, A_1 and A_2 would be in one-to-one correspondence with discrete Wilson lines. However, in our case we will choose $A_1 = A_2$, since this choice of boundary conditions corresponds to the case of a torus with trivial gauge background fields, i.e. with trivial Wilson lines. Hence, the boundary conditions (4.8) and (4.9) then imply that the fields $V^\mu_a(x,y)$ and $\chi_a(x,y)$ are periodic on the torus. Therefore, the usual Kaluza–Klein

[1] It is nice to notice here that the vector fields V^M, together with their fermionic superpartners, fit into a *six-dimensional* $\mathcal{N} = 1$ representation. However, upon dimensional reduction, the resulting *four-dimensional* vectors and scalars (again together with their superpartners) fit into a *four-dimensional* $\mathcal{N} = 2$ vector multiplet. The extra supersymmetry is later on broken by the orbifold twist.

expansion yields zero-modes $V_a^\mu(x)$ and $\chi_a(x)$, for both $V_a^\mu(x,y)$ and $\chi_a(x,y)$, which are constant functions in the extra dimensions and massless from the four-dimensional point of view. Because there are zero-modes for V_a^μ, the upstairs gauge symmetry $\mathcal{G}$ remains unbroken after torus compactification, i.e. the vector and scalar fields in the fundamental representation see transformations of the full gauge group $\mathcal{G}$

$$V^\mu \overset{\mathcal{G}}{\longmapsto} U\,V^\mu\,U^{-1} - \frac{\mathrm{i}}{g}\,(\partial^\mu U)\,U^{-1}\,, \tag{4.11}$$

$$\chi \overset{\mathcal{G}}{\longmapsto} U\,\chi\,U^{-1}\,, \tag{4.12}$$

where $U = U(x)$ is in the fundamental representation of $\mathcal{G}$ and g denotes the associated gauge coupling. Let us now proceed to the orbifolding, i.e. dividing out a discrete rotational symmetry, in the next step.

Orbifold GUT breaking. As we have seen in the string theory construction, a geometric orbifold twist can in general be accompanied by a non-trivial action on the gauge degrees of freedom. In the same spirit, we will now consider a purely geometric rotation and then study the inclusion of a simultaneous gauge transformation. In order to divide out the discrete geometric rotation, we have to choose the torus-lattice such that it exhibits a $\mathbb{Z}_N$ rotational symmetry ϑ with $\vartheta^N = \mathbb{1}$. In two dimensions, the allowed orders of the wallpaper groups for ϑ are $N = 2,3,4,6$. In the case $N = 2$ the basis vectors e_1 and e_2 simply have to be linear independent. For all other orders we set $\vartheta\,e_1 = e_2$. In order to orbifold the two-torus $\mathbb{T}^2$ to a $\mathbb{T}^2/\mathbb{Z}_N$ orbifold, we mod out this $\mathbb{Z}_N$ symmetry, by identifying points y on $\mathbb{T}^2$ which are related by a $(360/N)^\circ$ rotation generated by ϑ,

$$y \overset{\mathbb{Z}_N^{\text{geom.}}}{\longmapsto} \vartheta\,y \;\sim\; y\,. \tag{4.13}$$

Note that this rotation is an element of the higher-dimensional Lorentz group. Hence, under this geometrical action our six-dimensional fields transform according to

$$V^\mu(x,y) \overset{\mathbb{Z}_N^{\text{geom.}}}{\longmapsto} V^\mu(x,\vartheta^{-1}\,y)\,, \quad \text{and} \quad \chi(x,y) \overset{\mathbb{Z}_N^{\text{geom.}}}{\longmapsto} \exp\left(\frac{2\pi\mathrm{i}}{N}\right)\,\chi(x,\vartheta^{-1}\,y)\,, \tag{4.14}$$

where, as discussed above, the χ fields obtain an extra phase because they transform as the internal components of the six-dimensional vector $V^M(x,y)$ of the six-dimensional Lorentz symmetry. Now, let us discuss how the $\mathbb{Z}_N$ orbifold can be extended from its pure geometric action (4.13) to include a discrete $\mathbb{Z}_N$ transformation from the gauge symmetry $\mathcal{G}$. In addition to the geometric action (4.13), we take the generators of the Lie algebra of $\mathcal{G}$ to transform non-trivially under the gauge embedding

$$\mathsf{T}_a^{(\mathrm{CW})} \overset{\mathbb{Z}_N^{\text{gauge}}}{\longmapsto} P\,\mathsf{T}_a^{(\mathrm{CW})}\,P^{-1} \quad \text{with} \quad P^N \;=\; \mathbb{1}\,, \tag{4.15}$$

where $P \in \mathcal{G}$ acts as a discrete gauge transformation. That is, we set $U(x) = U = P =$ constant in (4.11). Because we take P to be an element of $\mathcal{G}$, we project on an *inner* automorphism. We hereby exclude the possibility to choose the gauge action P as an element of the outer automorphism group of $\mathcal{G}$, which is studied in [97]. Moreover, the order of P can in general differ from the order of ϑ. Like any Lie group element, P can be written as the exponentiation of a linear combination of Lie algebra generators

$$P \;=\; \exp\left(2\pi\mathrm{i}\, V_a \mathsf{T}_a^{(\mathrm{CW})}\right)\;. \tag{4.16}$$

Since we restrict ourselves to Abelian point groups, P must always be an element of the maximal torus of $\mathcal{G}$, i.e. an element of the maximal Abelian subgroup which is isomorphic to a $\mathrm{U}(1)^{\mathrm{rank}(\mathcal{G})}$. Therefore, we can always choose the Cartan–Weyl basis such that P can be expanded exclusively in the Cartan generators

$$P \;=\; \exp\left(2\pi\mathrm{i}\, V\cdot H\right)\;, \tag{4.17}$$

where the vector V is "quantized" such that $P^N = \mathbb{1}$. In this way, one can achieve that—as soon as explicit matrices are considered—the explicit matrix form of P is always a diagonal matrix. As we will see below, making this choice of basis in order for P being diagonal has the effect that the orbifold boundary conditions for the V^μ and χ fields are automatically diagonal. Again, in a model derived from the heterotic string, the vector V would be associated with a shift vector.

All in all, the orbifold action O, generated by $\mathbb{Z}_N^{\mathrm{orb.}}$ is then given by the simultaneous action of the geometric $\mathbb{Z}_N^{\mathrm{geom.}}$ on the extra-dimensional coordinate y and the discrete gauge transformation $\mathbb{Z}_N^{\mathrm{gauge}}$ generated by P. In particular,

$$O:\left(\begin{array}{c} V^\mu(x,y)\\ \chi(x,y)\end{array}\right)\;\longmapsto\;\left(\begin{array}{c} P\,V^\mu(x,\vartheta^{-1}y)\,P^{-1}\\ \exp\left(\frac{2\pi\mathrm{i}}{N}\right)\,P\,\chi(x,\vartheta^{-1}y)\,P^{-1}\end{array}\right)\;. \tag{4.18}$$

Then, in order to retain a four-dimensional zero-mode, fields must have zero-modes on the torus (4.8) and (4.9), and be invariant under the action of O.

Orbifold conditions. Let us now discuss the boundary conditions for the six-dimensional fields V^μ and χ that arise from the combined action of (4.13) and (4.15). One observes that in addition to the torus boundary conditions (4.8) and (4.9), the V^μ and χ fields have to fulfill the following orbifold boundary conditions

$$V^\mu(x,\vartheta\, y) \;=\; P\,V^\mu(x,y)\,P^{-1}\;, \tag{4.19}$$

$$\chi(x,\vartheta\, y) \;=\; \exp\left(\frac{2\pi\mathrm{i}}{N}\right)\,P\,\chi(x,y)\,P^{-1}\;, \tag{4.20}$$

to be invariant under O. If the gauge action P is chosen to be non-trivial, not all zero-modes of V^μ that exist on the torus can survive the orbifolding, which means that the gauge group gets broken to some subgroup. For any zero-mode of the scalar χ fields to survive, a non-trivial P is necessary to cancel the $\exp\left(\frac{2\pi\mathrm{i}}{N}\right)$ phase that arises from the vector index of the χ fields in the orbifold plane. It is apparent that the boundary conditions for the component fields V_a^μ and χ_a depend on the commutator of the respective $\mathsf{T}_a^{(\mathrm{CW})}$ with P. Now, in order to write down closed expressions for the boundary conditions, one makes use of the commutation relations of the Lie algebra in Cartan–Weyl basis

$$[H_I, H_J] \;=0\;, \tag{4.21}$$

$$[H_I, E_w] \;= w_I\, E_w\;, \tag{4.22}$$

$$[E_w, E_{w'}] \;=\; \begin{cases} E_{w+w'} & \text{if } w+w'\in W\\ 0 & \text{else}\end{cases}\;, \tag{4.23}$$

where W denotes the set of roots. Because P can be expanded in terms of the Cartan generators (see eq. (4.17)), one finds that transformation of the component fields corresponding to Cartan generators induced by P is trivial

$$P\, H_I\, P^{-1} \;=\; H_I\;. \tag{4.24}$$

This is in accordance with the situation in string theory, where the breaking of $E_8 \times E_8$ to subgroups by shifts and (discrete) Wilson lines has no effect on the Cartan generators. On the other hand, the ladder operators acquire a phase

$$P\, E_w\, P^{-1} \;=\; \exp\left(2\pi\mathrm{i}\, V\cdot w\right)\, E_w\;, \tag{4.25}$$

where w denotes the root vector of E_w. By rewriting the boundary conditions (4.19) and (4.20) in terms of the component fields V_a^μ and χ_a using this information, we obtain the following diagonal boundary conditions for the four-dimensional vector fields

$$\begin{aligned} V_I^\mu(x,\vartheta\, y) \;&=\; V_I^\mu(x,y)\;, &(4.26)\\ V_w^\mu(x,\vartheta\, y) \;&=\; \exp\left(2\pi\mathrm{i}\, V\cdot w\right)\, V_w^\mu(x,y)\;, &(4.27)\end{aligned}$$

and for the scalars χ_a

$$\begin{aligned} \chi_I(x,\vartheta\, y) \;&=\; \exp\left(\frac{2\pi\mathrm{i}}{N}\right)\, \chi_I(x,y)\;, &(4.28)\\ \chi_w(x,\vartheta\, y) \;&=\; \exp\left(2\pi\mathrm{i}\,\left(V\cdot w + \frac{1}{N}\right)\right)\, \chi_w(x,y)\;. &(4.29)\end{aligned}$$

We observe that because the $V_I^\mu(x,y)$ have zero-modes on the torus, and moreover have only trivial orbifold boundary conditions, they always have four-dimensional zero-modes even after orbifolding. Hence, the unbroken gauge symmetry after orbifolding is at least a $U(1)^{\mathrm{rank}(\mathcal{G})}$, that is, the rank of the unbroken gauge group is always equal to the rank of $\mathcal{G}$. If some of the fields $V_w^\mu(x,y)$ corresponding to ladder operators have trivial orbifold boundary conditions as well, namely when

$$V\cdot w \;=\; 0 \bmod 1\;, \tag{4.30}$$

the unbroken gauge symmetry can also be non-Abelian. In the next section, we will study a systematic approach to determine the unbroken gauge group, that in addition reveals any discrete remnant that survives the orbifold action O along the continuous gauge symmetry. On the other hand, whenever

$$V\cdot w \;=\; -\frac{1}{N} \bmod 1\;, \tag{4.31}$$

we know that there are massless scalars in the four-dimensional spectrum, that are coming from the corresponding $\chi_w(x,y)$ fields. However, in what follows we will focus on the unbroken gauge symmetry and hence we will be more interested in surviving modes of V^μ rather than the χ fields.

4.3 Residual gauge symmetries

Let us now study the surviving modes of the six-dimensional gauge symmetry $\mathcal{G}$ in more detail. We have already seen that Cartan generators always survive provided the ad hoc condition (4.30) in order to obtain a non-Abelian unbroken continuous gauge symmetry after orbifolding is fulfilled. We now consider also the possibility to retain unbroken discrete symmetries from $\mathcal{G}$ in addition to the continuous ones. To this end, we study under which circumstances a global transformation $U \in \mathcal{G}$ is compatible with the orbifolding procedure. In particular, a surviving symmetry transformation from $\mathcal{G}$ has to commute with the orbifold boundary conditions (4.19) and (4.20) in order to remain unbroken, i.e.

$$(U \circ O)(f) \stackrel{!}{=} (O \circ U)(f)\ , \tag{4.32}$$

for every field f in the theory. Note that this condition amounts to the requirement that U must be a homomorphism of the orbifolded theory. Therefore, this condition ensures that orbifold invariant states come in well-defined representations of any symmetry generated by the various surviving elements U. In terms of the four-dimensional vector fields $V_a^\mu(x,y)$ this condition becomes

$$\begin{array}{ccc}
V_a^\mu(x,y)\,\mathsf{T}_a & \stackrel{O}{\longmapsto} & V_a^\mu(x,\vartheta^{-1}y)\,P\,\mathsf{T}_a\,P^{-1} \\
\Big\downarrow \mathcal{G} & & \Big\downarrow \mathcal{G} \\
 & & V_a^\mu(x,\vartheta^{-1}y)\,U\,P\,\mathsf{T}_a\,P^{-1}\,U^{-1} \\
 & & \| ! \\
V_a^\mu(x,y)\,U\,\mathsf{T}_a\,U^{-1} & \stackrel{O}{\longmapsto} & V_a^\mu(x,\vartheta^{-1}y)\,P\,U\,\mathsf{T}_a\,U^{-1}\,P^{-1}\ ,
\end{array} \tag{4.33}$$

where the global transformation $U \in \mathcal{G}$ is defined according to (4.11), with $U(x) \equiv U$ and hence $(\partial^\mu U)U^{-1} = 0$. Because the transformation of the $\chi_a(x,y)$ fields is the same up the the $\exp\left(\frac{2\pi\mathrm{i}}{N}\right)$ phase, the corresponding condition is automatically fulfilled once (4.33) is satisfied. We now observe that the component fields $V_a^\mu(x,\vartheta^{-1}\,y)$ in (4.33) can be factored out, so that the resulting condition can be formulated in terms of P, U and the Lie algebra generators $\mathsf{T}_a^{(\mathrm{CW})}$. It then reads

$$\mathsf{T}_a^{(\mathrm{CW})}\left(P^{-1}\,U^{-1}\,P\,U\right) \;=\; \left(P^{-1}\,U^{-1}\,P\,U\right)\mathsf{T}_a^{(\mathrm{CW})}\ . \tag{4.34}$$

Hence, the combination $P^{-1}\,U^{-1}\,P\,U$ has to commute with all Lie algebra generators. Now, Schur's lemma states that any matrix that commutes with all generators of a Lie algebra must be proportional to the identity, hence it follows that

$$P^{-1}\,U^{-1}\,P\,U \;=:\; [P,U] \;\propto\; \mathbb{1}\ . \tag{4.35}$$

Here, we used the definition of the group-theoretical commutator [101]

$$[A,B] = A^{-1}B^{-1}A\,B \tag{4.36}$$

for two *group* elements $A, B \in \mathcal{G}$ (as opposed to the usual commutator for Lie *algebra* elements). To fix the proportionality in (4.35), we use that P is of order N (i.e. $P^N = \mathbb{1}$).

Since $[\mathbb{1}, U] = \mathbb{1}$ for any U, we see that the proportionality factor in (4.35) can be at most an N-th root of unity. This yields our main condition for any unbroken symmetries after orbifolding

$$[P, U] \;=\; \omega^k \,\mathbb{1} \qquad \text{for} \quad k \;\in\; \{0, 1, \ldots, N-1\}\;, \tag{4.37}$$

where $\omega = \exp \frac{2\pi \mathrm{i}}{N}$. In order for a transformation U to survive the orbifolding, it does not have to completely commute with the gauge action P (which would correspond to $[P, U] = \mathbb{1}$), it only has to do so up to an element that is proportional to the identity (corresponding to the solutions where $k \neq 0$). In the existing literature, this fact has not been appreciated yet.

However, we can immediately derive a simple necessary condition for when (4.37) can have such non-trivial solutions. Because P, U are both elements of $\mathcal{G}$, also their commutator $[P, U]$ must be from $\mathcal{G}$. On the other hand, Schur's lemma requires that $[P, U] \propto \mathbb{1}$, which means that $[P, U]$ must be an element of the center $Z(\mathcal{G})$ of $\mathcal{G}$. That is, the condition (4.37) can only have non-trivial solutions if

$$\omega^k \,\mathbb{1} \;\in\; Z(\mathcal{G}) \qquad \text{for some} \quad k \;\in\; \{0, 1, \ldots, N-1\}\;. \tag{4.38}$$

Depending on the properties of the group $\mathcal{G}$, this condition constrains the allowed values of k. Clearly, as every group contains the identity, the value $k = 0$ is always allowed. However, it may be that no other solutions with $k \neq 0$ exist. Let us illustrate the restrictiveness of this conditions for the two classes of gauge groups that appear most frequently in GUT-related model building, namely $\mathrm{SU}(M)$ and $\mathrm{SO}(2M)$.

We start by considering an example where $\mathcal{G} = \mathrm{SU}(M)$ and we use a $\mathbb{Z}_N$ orbifold for compactification. As it is well known, the center of $\mathrm{SU}(M)$ is $\mathbb{Z}_M$. In order for the commutator $\omega^k \mathbb{1}$ to be in the center of $\mathrm{SU}(M)$ for $k \neq 0$, we have to require the dimension of the group M and the order of the orbifold twist N to be not coprime.

For the choice $\mathcal{G} = \mathrm{SO}(2M)$ the conditions are even more restrictive. As the center of an $\mathrm{SO}(2M)$ is always a $\mathbb{Z}_2$, we immediately observe that the order of the orbifold twist has to be even in order for solutions to (4.37) with non-zero k to exist.

Note that these conditions are only necessary ones, which do not automatically guarantee that any discrete remnants survive. To this end, we examine the possible solutions to the condition (4.37) in more detail now. Let us start with the continuous symmetries from $\mathcal{G}$ that survive the orbifolding.

4.3.1 Unbroken continuous gauge symmetries

We start with the continuous gauge symmetries as they are the main link to the usual Higgs mechanism employed in four-dimensional field theory. Moreover, in practical model building, one often has specific choices in mind for both the upstairs group $\mathcal{G}$ and the continuous subgroup it is broken to (one would for example want to break an SO(10) to the Pati–Salam group, see example 4.4.1). There exist two equivalent methods to identify the unbroken continuous gauge symmetries that survive the orbifolding.

The first, well known, way is by noticing that the unbroken continuous gauge interactions are mediated by the zero-modes of the higher dimensional gauge bosons. In our $\mathbb{T}^2/\mathbb{Z}_N$ setup, these are the modes that fulfill trivial boundary conditions on both the torus (4.8) and the orbifold (4.26) and (4.27). Thus, as discussed already earlier, the gauge bosons $V_I^\mu(x,y)$ which are associated with the Cartan generators H_I together with those gauge bosons $V_w^\mu(x,y)$ that are associated with roots w for which $V \cdot w \in \mathbb{Z}$, have trivial boundary conditions and therefore generate the unbroken continuous gauge group in four dimensions as they possess massless modes.

Our main condition (4.37) provides a second way to identify the unbroken continuous symmetries as already noted in [97]. In particular, the unbroken continuous symmetries are continuously connected to the identity $U = \mathbb{1}$. Hence, we observe that in order to find the continuous gauge symmetries that survive the orbifold projection, we have to set $k = 0$ in (4.37). Then, we expand a general transformation U to linear order around the identity

$$U = \exp\left(\mathrm{i}\,\alpha_a\,\mathsf{T}_a^{(\mathrm{CW})}\right) \approx \mathbb{1} + \mathrm{i}\,\alpha_a\,\mathsf{T}_a^{(\mathrm{CW})}\,. \tag{4.39}$$

Here, we assume for the moment that the parameters α_a are constant, i.e. they do not depend on the spacetime coordinate x. This is necessary in order to make contact with the commuting diagram in equation (4.33), where the transformation U was taken to be global (and hence no $(\partial^\mu U)U^{-1}$ term appears there). However, as we take U to be continuously connected to the identity, there exists no obstruction to promote the solutions we are about to find to local symmetries later on. Equation (4.37) yields the condition that needs to be fulfilled by the various generators of the unbroken gauge symmetry

$$P^{-1}\left(\alpha_a\,\mathsf{T}_a^{(\mathrm{CW})}\right)\,P \;=\; \alpha_a\,\mathsf{T}_a^{(\mathrm{CW})}\,. \tag{4.40}$$

It is now straightforward to verify that this condition is equivalent to the requirement of trivial boundary conditions. Since the gauge action associated with the orbifold twist P is expanded in terms of the Cartan generators H_I, as defined in (4.17), one can make use of the definition of the $\mathbb{Z}_N^{\mathrm{gauge}}$ transformation (4.24) to confirm that both the Cartan generators H_I and the generators E_w with $V \cdot w \in \mathbb{Z}$ satisfy (4.40), and remain unbroken after orbifolding.

One may now be tempted to believe that all solutions to (4.37) with $k = 0$ are automatically elements of the unbroken continuous gauge group, and only the ones with $k \neq 0$ are discrete gauge symmetries. However, as we shall see now, this is not true.

4.3.2 Unbroken discrete gauge symmetries

As we have seen in the previous section, the condition (4.37) can be used to identify generators that survive the orbifold projection and hence have trivial boundary conditions. In particular, one needs to set $k = 0$ to obtain such zero-modes, which then are propagating degrees of freedom and give rise to continuous gauge symmetries. Now, in order to determine which discrete symmetries survive along with the continuous ones, one examines the condition (4.37) for additional solutions, for both $k = 0$ and $k \neq 0$.

Let us start with the case $k \neq 0$ because it is more straightforward. Solutions to (4.37) for some $k \neq 0$ do not correspond to zero-modes of the higher dimensional gauge field. Hence, these symmetries look like global symmetries from a four-dimensional perspective, as they do not correspond to dynamical gauge fields. Moreover, as they are not connected to the identity, they give rise to discrete symmetries.

In addition, there is also the possibility to obtain discrete symmetries from solutions to (4.37) with $k = 0$. In order to see this, assume that the continuous surviving subgroup of upstairs gauge group $\mathcal{G}$ after orbifolding is $\mathcal{H}$. Now, $\mathcal{H}$ might not a simple, but rather a semi-simple Lie group that can be written as

$$\mathcal{H} \; = \; \mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \;, \tag{4.41}$$

where each $\mathcal{H}_i$ is a simple Lie group. Then, it can happen that some solution U that fulfills the condition (4.37) with $k = 0$ is not an element of any of the gauge factors $\mathcal{H}_i$. In particular, the elements R of $\mathcal{H}$ are always block-diagonal

$$R \; = \; \begin{pmatrix} R_1 & & \\ & R_2 & \\ & & \ddots \end{pmatrix} \;, \tag{4.42}$$

where the individual submatrices R_i are elements of the factor groups $\mathcal{H}_i$. However, it is well possible that some solution U has a similar block-diagonal form, $U = \mathrm{diag}(U_1, U_2, \ldots)$, but the various U_i are not elements of the corresponding $\mathcal{H}_i$ factor groups of $\mathcal{H}$. Then, the transformation U does not correspond to a propagating gauge boson of $\mathcal{H}$. Hence, this symmetry again gives rise to a discrete symmetry that looks like a global symmetry from the four-dimensional perspective, and whose local nature only becomes clear by its origin in the gauge group $\mathcal{G}$.

We will now study a set of examples that illustrate the appearance of discrete remnants as solutions to the condition (4.37) from a higher-dimensional gauge group $\mathcal{G}$. To this end, we consider discrete remnants from the case $k = 0$ in 4.4.1, where we show how the known left-right parity in Pati–Salam GUT models can have a gauge origin. Moreover, we also present how solutions from cases with $k \neq 0$ can provide valuable input for e.g. flavor model building in section 4.4.2.

4.4 Examples and applications

In this section, we show how solutions to the main condition (4.37) can be obtained in various orbifold GUT scenarios. We start our discussion with an example for discrete symmetries from a solution with $k = 0$ in (4.37) that yields an Abelian symmetry. Afterwards, we show how non-Abelian discrete symmetries can be generated by solutions with $k \neq 0$.

4.4.1 Gauge origin of D-parity and left-right parity

In GUT model building, the Pati–Salam model [102] has the continuous gauge group

$$G_{\mathrm{PS}} \; = \; \mathrm{SU}(4) \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \;. \tag{4.43}$$

The idea here is that the SU(4) can be broken to the standard model $\mathrm{SU}(3)_{\mathrm{C}}$ group. In addition, matter charged under G_{PS} then falls into complete standard model generations of chiral matter. Moreover, G_{PS} can be understood as an intermediate step when breaking an SO(10) GUT to the standard model, as it fits into the sequence

$$\text{SO}(10) \longrightarrow G_{\text{PS}} \longrightarrow G_{\text{LR}} \longrightarrow \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} . \tag{4.44}$$

Here we also introduced the left-right symmetric subgroup G_{LR} of G_{PS}

$$G_{\text{LR}} = \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}} \times \text{U}(1)_{B\text{–}L} . \tag{4.45}$$

Apart from these continuous symmetries, the Pati–Salam model also has a $\mathbb{Z}_2$ symmetry that exchanges $\text{SU}(2)_{\text{L}}$ with $\text{SU}(2)_{\text{R}}$ and acts as complex conjugation on SU(4) representations. In the literature, this symmetry is referred to as D-parity on the level of the Pati–Salam group. There are field-theoretic models [103, 104] in which the D-parity originates as part of the SO(10) supergroup that contains G_{PS}. In particular, there are four-dimensional GUT models where this symmetry can be preserved by breaking the SO(10) GUT by giving a VEV to an appropriate Higgs field that transforms as a **54**-plet of SO(10). Moreover, the D-parity in general also survives the breaking of G_{PS} to its left-right symmetric subgroup G_{LR}. At the level of the left-right symmetric subgroup this $\mathbb{Z}_2$ is the well-known left-right parity [105]. Hence, the full symmetry of these settings, including both continuous and discrete ones, is

$$[\text{SU}(4) \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}] \rtimes \mathbb{Z}_2 \text{ or } [\text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}} \times \text{U}(1)_{B\text{–}L}] \rtimes \mathbb{Z}_2 . \tag{4.46}$$

We will now demonstrate how this $\mathbb{Z}_2$ symmetry can be obtained as part of an SO(10) GUT that is broken to G_{PS} by orbifold boundary conditions rather than by Higgsing, which to our knowledge has been overlooked in the existing literature on orbifold GUTs.

To this end, we will consider a theory with an upstairs gauge symmetry $\mathcal{G} = \text{SO}(10)$ in higher dimensions that is then compactified on a $\mathbb{Z}_2$ orbifold. We will implicitly assume to work with a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, but for this case we could have as well picked a five-dimensional setting that is then compactified on a one-dimensional $\mathbb{S}^1/\mathbb{Z}_2$ orbifold. We choose the gauge action associated with the $\mathbb{Z}_2$ twist that breaks the GUT by boundary conditions to be

$$P_{\text{PS}} = \text{diag}(-\mathbb{1}_6; \mathbb{1}_4) . \tag{4.47}$$

This setting has been studied earlier in [98], where it was shown that the continuous gauge symmetry after orbifolding indeed is the Pati–Salam group G_{PS}. However, we now show that there is an additional $\mathbb{Z}_2$ symmetry.

To this end, we will carefully rederive the result of [98] by using the methods described earlier, and see how besides the continuous gauge group also a discrete $\mathbb{Z}_2$ survives. In this setting, our main condition (4.37) yields

$$[P_{\text{PS}}, U_{(k)}] = (-1)^k \, \mathbb{1} \qquad \text{for} \quad k \in \{0,1\} , \tag{4.48}$$

and we search for unbroken elements $U_{(k)}$ that are elements of SO(10). Let us first show that setting $k = 1$ in our main condition (4.48) does not yield further unbroken symmetries. To show this, let us parameterize $U_{(1)}$ to read

$$U_{(1)} = \begin{pmatrix} O_6 & A \\ B & O_4 \end{pmatrix} . \tag{4.49}$$

Then, the condition (4.48) with $k = 1$ amounts to

$$O_6 = O_4 = 0\ , \tag{4.50}$$

and requiring $U_{(1)}$ to be an orthogonal matrix yields, among other conditions, that

$$A^T A \ = \ \mathbb{1}_6\ , \tag{4.51}$$

which does not have a solution. Hence we have to focus on symmetries that commute with the orbifold boundary condition. If we set $k = 0$, our condition (4.48) reads

$$P_{\mathrm{PS}}\, U_{(0)} \ = \ U_{(0)}\, P_{\mathrm{PS}}\ . \tag{4.52}$$

The most general SO(10) matrix satisfying this condition is block-diagonal, with a six-by-six and a four-by-four block

$$U_{(0)} \ = \ \begin{pmatrix} O_6 & 0 \\ 0 & O_4 \end{pmatrix} \ \in\ \mathrm{SO}(10)\ . \tag{4.53}$$

As $U_{(0)}$ must be an element of SO(10), the six-by-six and the four-by-four block are subject to conditions on their determinants. In particular, we find the conditions

$$O_6^T\, O_6 \ = \ \mathbb{1}_6 \quad \text{and} \quad O_4^T\, O_4 \ = \ \mathbb{1}_4 \quad \text{and} \quad \det O_6 \ = \ \det O_4 \ = \ \pm 1\ . \tag{4.54}$$

The individual blocks are in general elements of O(6) and O(4), respectively. However, the continuous subgroup of SO(10) that survives the orbifolding generates only an SO(6) $\times$ SO(4), that is where both blocks have determinant one. Taking now $U_{(0)}$ with $\det O_6 = \det O_4 = +1$ yields

$$O_6 \ \in\ \mathrm{SO}(6) \ \simeq\ \mathrm{SU}(4) \quad \text{and} \quad O_4 \ \in\ \mathrm{SO}(4) \ \simeq\ \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}\ . \tag{4.55}$$

Now, we precisely find ourselves in the case described in the previous section, namely where the unbroken continuous gauge group consists of more than one simple Lie group. Moreover, we observe that the solutions with negative determinants in both blocks of $U_{(0)}$ (i.e. with $\det O_6 = \det O_4 = -1$) can be generated from the same SO(6) and SO(4) we just found, namely by multiplying each of these solutions with an appropriate matrix

$$O_6 \ = \ \mathrm{diag}(1,1,1,1,1,-1)\, O_6' \quad \text{and} \quad O_4 \ = \ \mathrm{diag}(1,1,1,-1)\, O_4'\ , \tag{4.56}$$

where again $O_6' \in \mathrm{SO}(6) \simeq \mathrm{SU}(4)$ and $O_4' \in \mathrm{SO}(4) \simeq \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$, and where it should be noted that the "$\simeq$" means "up to $\mathbb{Z}_2$ factors". These $\mathbb{Z}_2$'s should not be confused with the D-parity we are going to discuss next. That is, there is a matrix D'

$$D' \ = \ \mathrm{diag}(1,1,1,1,1,-1;1,1,1,-1) \tag{4.57}$$

that facilitates the transition from $\det O_6 = \det O_4 = -1$ to $\det O_6 = \det O_4 = 1$. Note that this matrix is not an element of SO(6) $\times$ SO(4). Hence, we have found a discrete symmetry that survives the orbifold projection and is not an element of the low-energy continuous gauge group, but rather acts as an outer automorphism of SO(6) × SO(4). As a result, the $\mathbb{Z}_2$ orbifold boundary condition P_{PS} breaks the SO(10) GUT to

$$G_{\text{PS}} \;=\; [\text{SU}(4) \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}] \rtimes \mathbb{Z}_2 \;. \tag{4.58}$$

Here the additional $\mathbb{Z}_2$ remnant symmetry does not commute with the gauge transformations, hence there is a semidirect product instead of a direct one. Now, as we will shortly look at how the extra $\mathbb{Z}_2$ acts on matter fields sitting in fundamental representations of the gauge group, let us note that we can choose another, more suggestive generator D for the $\mathbb{Z}_2$ instead of D'

$$D \;=\; \text{diag}(-1,1,1,1,1,1;1,-1,-1,-1) \;. \tag{4.59}$$

This suggestive form of D is chosen because it makes the action of the D-parity on the fundamental representations of the Pati–Salam group G_{PS} more obvious. In terms of group theory, we could have chosen any diagonal ten-by-ten matrix with entries ±1, with the only condition that the number of -1's in the first six-by-six block and the second four-by-four block must be odd.

Let us now discuss the action of the D-parity on fundamental representations of G_{PS}. We start by considering the $\text{SO}(4) \simeq \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$ subblock. Note that we have chosen the corresponding subblock of D in (4.59) such that it resembles the parity transformation in 4D Euclidean spacetime. In that spirit, it can be understood as the action of parity on 4D spinor representations $(1/2, 0) \oplus (0, 1/2)$ of $\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$. As is well known, the parity interchanges these SU(2) representations. Applied to the Pati–Salam model, the D-parity acts on representations $(\mathbf{r}_{\text{L}}, \mathbf{r}_{\text{R}})$ of $\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$ according to

$$(\mathbf{r}_{\text{L}}, \mathbf{r}_{\text{R}}) \;\xmapsto{D}\; (\mathbf{r}_{\text{R}}, \mathbf{r}_{\text{L}}) \;, \tag{4.60}$$

see also appendix D for an explicit discussion how the D-parity acts on representations of $\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$. In a similar fashion, we can study how the D-parity acts on the $\text{SO}(6) \simeq \text{SU}(4)$ subgroup. In our definition of D in (4.59), we have chosen the action on this subgroup such that it acts in analogy to an Euclidean version of time reversal by choosing the first entry to be -1. Hence, for any SU(4) representation $\mathbf{r}_4$ we find that D acts as charge conjugation

$$\mathbf{r}_4 \;\xmapsto{D}\; \overline{\mathbf{r}}_4 \tag{4.61}$$

on the fundamental representation of SU(4). Altogether a representation $(\mathbf{r}_4, \mathbf{r}_{\text{L}}, \mathbf{r}_{\text{R}})$ of $\text{SU}(4) \times \text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$ transforms under D as

$$(\mathbf{r}_4, \mathbf{r}_{\text{L}}, \mathbf{r}_{\text{R}}) \;\xmapsto{D}\; (\overline{\mathbf{r}}_4, \mathbf{r}_{\text{R}}, \mathbf{r}_{\text{L}}) \;. \tag{4.62}$$

Among the rest, the interesting representations of the Pati–Salam group are $(\mathbf{4}, \mathbf{2}, \mathbf{1})$s and $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$s, which (upon further breaking) give rise to the usual standard model representations for matter. On the SO(10) side, one $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and one $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ make up a **16** of SO(10). From the action of D-parity on representations (4.62) one deduces that it exchanges the $(\mathbf{4}, \mathbf{2}, \mathbf{1})$ and $(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ representations. In terms of standard model matter, that means that the left- and right-handed matter fields get exchanged. In effect, this field-theoretic orbifold GUT implements the well-known left-right parity [105]. As we show, its origin lies in a full SO(10) GUT theory and is hence a discrete remnant of the upstairs gauge symmetry. The way it acts motivates the terminology of a "parity" [105], although it must not be mixed up with the ordinary spacetime parity: Despite its name, the left-right symmetric model is chiral and therefore does not preserve the conventional spacetime parity even in its unbroken phase.

Let us summarize the findings of this section. We have shown that if an SO(10) GUT gets broken to the Pati–Salam group by orbifold boundary conditions, there is an additional $\mathbb{Z}_2$ symmetry that survives the breaking

$$\mathrm{SO}(10) \xrightarrow{\mathbb{Z}_2 \text{ orbifold}} [\mathrm{SU}(4) \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}] \rtimes \mathbb{Z}_2 \,. \tag{4.63}$$

This $\mathbb{Z}_2$ corresponds to the left-right parity and is in particular a non-trivial outer automorphism of G_PS. A thorough discussion of the phenomenological implications of the $\mathbb{Z}_2$ is given e.g. in [106] and references therein.

4.4.2 Non-Abelian residual symmetries

We now come to two examples of discrete remnant symmetries that commute with the orbifold boundary condition only up to a non-trivial element from the center, i.e. they fulfill the condition (4.37) with $k \neq 0$. In effect, the higher-dimensional gauge group gets broken to a semidirect product of the unbroken remnant continuous gauge symmetry with some discrete $\mathbb{Z}_N$ symmetry. As in the Pati–Salam example presented above, this $\mathbb{Z}_N$ may have implications on the phenomenology of the resulting four-dimensional model. Although we do not consider matter fields, one can observe (from a group-theoretical point of view), that the additional $\mathbb{Z}_N$ in our examples acts on matter representations of the GUT group as a flavor symmetry.

$\mathbb{T}^2/\mathbb{Z}_4$ Orbifold GUT

For the torus in our first example, we choose two base vectors e_1 and e_2 with the same length, $|e_1| = |e_2|$, that form an orthogonal lattice, that is $e_1 \cdot e_2 = 0$. By doing so, we ensure that the lattice possesses a rotational $\mathbb{Z}_4$ symmetry generated by a rotation ϑ

$$\vartheta : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \longmapsto \begin{pmatrix} e_2 \\ -e_1 \end{pmatrix} \tag{4.64}$$

that we mod out in order to construct a $\mathbb{T}^2/\mathbb{Z}_4$ orbifold. On this orbifold, we compactify a six-dimensional theory with gauge symmetry $\mathcal{G} = \mathrm{SU}(2)$. To this end, we choose the gauge action P associated with the geometric rotation ϑ to be given by

$$P = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \in \mathrm{SU}(2) \quad \text{where} \quad P^4 = \mathbb{1} \,. \tag{4.65}$$

Following the discussion leading to the general condition (4.37), we find that the unbroken symmetries correspond to those matrices $U_{(k)}$ of SU(2) that satisfy the condition

$$[P, U_{(k)}] = \exp\left(\frac{2\pi \mathrm{i}\, k}{4}\right) \mathbb{1} \quad \text{where} \quad k \in \{0, 1, 2, 3\} \,. \tag{4.66}$$

For which non-zero values of k can we now expect to find solutions? Because both P and $U_{(k)}$ are elements of SU(2), so must be the right-hand side of (4.66). However, we have also seen earlier that the right-hand side of (4.66) must be an element of the center $Z(\mathcal{G})$ of the upstairs gauge group. Because the center $Z(\mathrm{SU}(2))$ of SU(2) is only a $\mathbb{Z}_2$ (rather than a $\mathbb{Z}_4$), the possible values of k are now restricted, and therefore (4.66) can at most allow for solutions with $k \in \{0, 2\}$. Of course, as we have seen in the earlier example, the existence of solutions (especially with non-zero values of k) is not guaranteed by this restriction.

In order to find all solutions of (4.66) for both choices of k, we now explicitly construct a general ansatz for $U_{(k)}$ and then require that the condition (4.66) must be fulfilled. To this end, we use the parameterization of a general SU(2) matrix $U_{(k)}$

$$U_{(k)} \;=\; \begin{pmatrix} p_k & q_k \\ -\bar{q}_k & \bar{p}_k \end{pmatrix} \;\in\; \mathrm{SU}(2) \quad \text{with} \quad \det(U_{(k)}) = |p_k|^2 + |q_k|^2 = 1\;, \tag{4.67}$$

where the two parameters $p_k, q_k \in \mathbb{C}$ have to be fixed. In this explicit form, condition (4.66) becomes

$$[P, U_{(k)}] \;=\; \begin{pmatrix} |p_k|^2 - |q_k|^2 & 2\bar{p}_k q_k \\ -2p_k\bar{q}_k & |p_k|^2 - |q_k|^2 \end{pmatrix} \;\stackrel{!}{=}\; \exp\left(\frac{2\pi\mathrm{i}\,k}{4}\right) \mathbb{1}\;. \tag{4.68}$$

Solving this equation, we find the following constraints on the parameters p_k and q_k

$$|p_k|^2 - |q_k|^2 \;\stackrel{!}{=}\; \exp\left(\frac{2\pi\mathrm{i}\,k}{4}\right) \tag{4.69}$$

for the diagonal entries, and, from the off-diagonal elements

$$\bar{p}_k q_k \;\stackrel{!}{=}\; 0\;. \tag{4.70}$$

Now, because the diagonal entries $|p_k|^2 - |q_k|^2 \in \mathbb{R}$, we immediately observe that (4.68) cannot possess any solutions for $k \in \{1, 3\}$, as we predicted earlier, where we argued that any solution with $k \in \{1, 3\}$ would not lie in the center of SU(2) and would therefore not be admissible.

By setting $k = 0$ in (4.69), we find the unbroken gauge symmetry. The solution for $k = 0$ requires $|p_0|^2 = 1$, and consequently $q_0 = 0$. The most general solution $U_{(0)}$ is therefore a function of a single free parameter α

$$U_{(0)} \;\equiv\; U_{(0)}(\alpha) \;=\; \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \;\in\; \mathrm{SU}(2)\;, \tag{4.71}$$

where $\alpha \in [0, 2\pi)$. Because there are no further constraints, this yields an unbroken U(1) gauge symmetry. This result was to be expected: It is easy to identify the generator of this U(1) symmetry with the Cartan generator of SU(2).

On the other hand, as opposed to the Pati–Salam example we have seen earlier, there are now solutions also for non-zero k. In particular, setting $k = 2$ in the main condition (4.69) yields $p_2 = 0$ and $|q_2|^2 = 1$. Hence, we set $q_2 = \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}$, and obtain

$$U_{(2)} \;=\; \begin{pmatrix} 0 & \mathrm{i}\mathrm{e}^{\mathrm{i}\alpha} \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha} & 0 \end{pmatrix}\;, \tag{4.72}$$

where again $\alpha \in [0, 2\pi)$. However, this solution is not fully independent of the one we found for $k = 0$ as it only differs up to the multiplication with a constant matrix

$$U_{(2)}(\alpha) \;=\; \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \;=\; U_{(0)}(\alpha) \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} . \tag{4.73}$$

Consequently, the orbits in gauge space parameterized by $U_{(0)}(\alpha)$ and $U_{(2)}(\alpha)$ cannot be disentangled completely and the full unbroken symmetry arising from the original SU(2) is generated by a U(1) generator and a discrete $\mathbb{Z}_4$, which are given by

$$U_{(0)}(\alpha) \;=\; \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix} \quad \text{and} \quad U_{(2)} \;=\; \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} . \tag{4.74}$$

The orbits of these generators are interlinked, as one can observe that

$$(U_{(2)})^2 = -\mathbb{1} = U_{(0)}(\pi) \in \mathrm{U}(1) . \tag{4.75}$$

Although the order of the matrix $U_{(2)}$ is four, its square is connected with the identity up to a U(1) gauge transformation. This fact needs to be taken into account when determining the full residual symmetry, and becomes even more apparent in the action of the $\mathbb{Z}_4$ transformation $U_{(2)}$ on the gauge bosons. Namely, $U_{(2)}$ acts only as a $\mathbb{Z}_2$ on the gauge bosons, i.e.

$$V_a^\mu(x,y)\,\mathsf{T}_a^{(\mathrm{CW})} \;\mapsto\; V_a^\mu(x,y)\,U_{(2)}\,\mathsf{T}_a^{(\mathrm{CW})}\,U_{(2)}^{-1} , \tag{4.76}$$

see diagram (4.33).

In summary, the six-dimensional SU(2) gauge theory is broken by the $\mathbb{Z}_4$ orbifold boundary conditions to

$$\mathrm{SU}(2) \;\xrightarrow{\mathbb{Z}_4^{\mathrm{orb.}}}\; (\mathrm{U}(1) \rtimes \mathbb{Z}_4)\,/\mathbb{Z}_2 , \tag{4.77}$$

where “mod $\mathbb{Z}_2$” addresses the fact that the additional $\mathbb{Z}_4$ acts only as a $\mathbb{Z}_2$ on the gauge boson level. Note that we have not incorporated matter fields in this example. Adding both localized and bulk matter can be done in an ad hoc fashion in field theory (where one at most has to worry about anomalies arising from a certain charge assignment), or in a more systematic way in string theory (where the matter content of the theory is fixed by the original ten-dimensional heterotic string spectrum). In the string theory case, one also has no freedom to choose the charges of the respective matter fields.

One may now break this symmetry further, e.g. by breaking the U(1) factor to some $\mathbb{Z}_N$, so that in the end one arrives at

$$\mathrm{SU}(2) \;\xrightarrow{\mathbb{Z}_4^{\mathrm{orb.}}+\mathrm{VEV}}\; (\mathbb{Z}_N \rtimes \mathbb{Z}_4)\,/\mathbb{Z}_2 . \tag{4.78}$$

In field theory, this additional breaking can be performed by Higgsing. On the other hand, as we will see later, in string theory the U(1) factor can be broken to some subgroup by giving a certain vacuum expectation value to geometric moduli. Then, this setup can give rise to a variety of discrete groups. For example, it is possible to obtain the binary dihedral groups Q_{2N} with $N =$ even as subgroups [70], including the quaternion group for $N = 4$.

$\mathbb{T}^2/\mathbb{Z}_3$ Orbifold GUT

Next, we consider another setup where solutions with $k \neq 0$ exist, namely again a two-dimensional torus with a $\mathbb{Z}_3$ orbifold action. This setting has been studied in the context of string theory before, e.g. in [100]. There, identifying the residual symmetry is not very straightforward, because there is no analogue to our field theoretical condition (4.37), and one has to rely in studying potentially accidental symmetries of the matter spectrum (e.g. via stringy selection rules for couplings) instead, which makes the true gauge nature of the resulting discrete symmetry a little unclear. Moreover, our discussion so far gives rise to the intuition that the residual (discrete and continuous) symmetry should be largely independent of any matter fields present in the upstairs theory. It is interesting to see how the findings in the string theory setup are supported by the more rigorous methods available in field theory orbifolds.

For this example, we choose our six-dimensional gauge symmetry of the upstairs theory to be $\mathcal{G} = \mathrm{SU}(3)$. For the torus, we make again the choice that the two lattice vectors have equal length $|e_1| = |e_2|$ but enclose an angle of 120°, that is $e_1 \cdot e_2 = -{|e_1|^2}/{2}$. This lattice has a $\mathbb{Z}_3$ rotational symmetry ϑ that acts according to

$$\vartheta : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \longmapsto \begin{pmatrix} e_2 \\ -e_1 - e_2 \end{pmatrix} , \tag{4.79}$$

hence a 120° counter-clockwise rotation. This rotation is modded out in order to construct a $\mathbb{T}^2/\mathbb{Z}_3$ orbifold.

Note that this setting can be achieved in a string theory setup in a very particular way, by starting with a $\mathrm{U}(1) \times \mathrm{U}(1)$ theory on this special torus. Then, one sets the overall size of the torus to a special value (the so-called self-dual point), which upon fixing the Kalb–Ramond background as well, amounts to fixing the Kähler T-modulus at a particular value. Because of this special choice for the radii, a set of stringy states becomes massless, which enhances the $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry to an SU(3) on the torus. This SU(3) is then broken by the $\mathbb{Z}_3$ boundary conditions. In the string theory setup, the transformation behavior of the various gauge bosons of SU(3) is fixed by their geometric (i.e. localization) properties, as opposed to a generic field theory setup, where the choice for the gauge action P associated with the geometric rotation is not restricted by first principles. In order to compare the outcomes of the string and field theory constructions, we choose our gauge action P in this example in such a way that it produces the same breaking patterns as the string theory setup.

If one were to match the transformation behavior of particular states in the string theory setup, one would have to choose an off-diagonal P, because in string theory, the boundary conditions are non-diagonal. However, a diagonal P can be chosen by making a basis change in the Lie algebra of the string theory construction, which also amounts to diagonalizing the string theory boundary conditions. As the results are basis-independent, the resulting breaking patterns are the same. To be specific, the diagonal gauge embedding P associated with ϑ is chosen as

$$P = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SU}(3) \quad \text{with} \quad P^3 = \mathbb{1} , \tag{4.80}$$

where $\omega = \exp{}^{2\pi\mathrm{i}}/_3$. It is instructive for later uses to study the adjoint action of this gauge embedding on the Lie algebra generators in the Cartan–Weyl basis. To this end, we fix the basis by arranging the Cartan generators and ladder operators of SU(3) in an eight-dimensional vector of matrices defined as

$$\mathsf{T}^{(\mathrm{CW})} = \Big(H_1,\, H_2,\, E_{(1,0)},\, E_{(-1,-1)},\, E_{(0,1)},\, E_{(-1,0)},\, E_{(1,1)},\, E_{(0,-1)}\Big)^T . \tag{4.81}$$

On this basis, the gauge embedding P then acts by an eight-by-eight matrix according to

$$\mathsf{T}_a^{(\mathrm{CW})} \xmapsto{\mathbb{Z}_3^{\mathrm{orb.}}} \left(R^{\mathrm{orb.}}\right)_{ab} \mathsf{T}_b^{(\mathrm{CW})} \qquad \text{where} \qquad R^{\mathrm{orb.}} = \begin{pmatrix} \mathbb{1}_2 & 0 & 0 \\ 0 & \omega\, \mathbb{1}_3 & 0 \\ 0 & 0 & \omega^2\, \mathbb{1}_3 \end{pmatrix} , \tag{4.82}$$

where we have arranged the ladder operators in such a way that they are grouped according to their transformation phases. Following the same steps as in the foregoing examples, the unbroken symmetry is given by the set of matrices $U_{(k)}$ from SU(3) that satisfy the condition

$$[P, U_{(k)}] = \exp\left(\frac{2\pi\mathrm{i}\, k}{3}\right) \mathbb{1} \quad \text{where} \quad k \in \{0, 1, 2\} . \tag{4.83}$$

As before, since $P, U_{(k)} \in \mathrm{SU}(3)$, also their commutator (i.e. the right-hand side of (4.83)) has to be an element of SU(3). And, because $[P, U_{(k)}] \propto \mathbb{1}$, the right hand side of (4.83) has to be from the center $Z(\mathrm{SU}(3))$ of SU(3), which is a $\mathbb{Z}_3$. Because the order of the center and the order of the orbifold twist coincide, the values of k are not restricted and (4.83) may allow for solutions for all possible cases $k \in \{0, 1, 2\}$.

Let us now study the solutions to condition (4.83) for all possible values of k. Setting $k = 0$, one observes that the unbroken continuous symmetry is generated by matrices of the form

$$U_{(0)} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\alpha+\beta)} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\alpha-\beta)} & 0 \\ 0 & 0 & \mathrm{e}^{-2\mathrm{i}\alpha} \end{pmatrix} \in \mathrm{SU}(3) , \tag{4.84}$$

which gives rise to a $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry. This $\mathrm{U}(1) \times \mathrm{U}(1)$ is generated by the two Cartan generators of SU(3), and is the minimal continuous gauge symmetry preserved by any diagonal orbifold boundary condition, because the gauge fields corresponding to the Cartans are always guaranteed to have trivial boundary conditions. For the two non-zero values of k, we find the following solutions

$$U_{(1)} = \begin{pmatrix} 0 & 0 & \mathrm{e}^{\mathrm{i}(\alpha+\beta)} \\ \mathrm{e}^{\mathrm{i}(\alpha-\beta)} & 0 & 0 \\ 0 & \mathrm{e}^{-2\mathrm{i}\alpha} & 0 \end{pmatrix} \quad \text{and} \quad U_{(2)} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}(\alpha+\beta)} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}(\alpha-\beta)} \\ \mathrm{e}^{-2\mathrm{i}\alpha} & 0 & 0 \end{pmatrix} . \tag{4.85}$$

As in the $\mathbb{T}^2/\mathbb{Z}_4$ example, these two solutions are not independent of the $\mathrm{U}(1) \times \mathrm{U}(1)$ generated by $U_{(0)}$, but yield two discrete transformations in addition to the continuous remnants

$$U_{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{SU}(3) \quad , \quad U_{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SU}(3) . \tag{4.86}$$

Note that these two transformations are as well not independent

$$U_{(2)} = (U_{(1)})^2 \; . \tag{4.87}$$

As $(U_{(1)})^3 = \mathbb{1}$, we find that there is an additional unbroken $\mathbb{Z}_3$ generated by $U_{(1)}$. Like in the $\mathbb{T}^2/\mathbb{Z}_4$ example, this $\mathbb{Z}_3$ does not commute with the unbroken gauge symmetry but rather acts as an outer automorphism. As a result, the upstairs theory with a SU(3) gauge symmetry in six dimensions is broken by the $\mathbb{Z}_3$ orbifold according to

$$\mathrm{SU}(3) \xrightarrow{\mathbb{Z}_3^{\text{orb.}}} \Big[\mathrm{U}(1) \times \mathrm{U}(1)\Big] \rtimes \mathbb{Z}_3 \; , \tag{4.88}$$

which is the same result found for the string theory equivalents (see cf. [100, 107]). As discussed above, the emergence of the two U(1) factors can be understood by the standard discussion of gauge symmetry breaking via orbifold boundary conditions by taking only the commuting subgroup into account, see for example [97, equation (6)]. We note, however, that a rigorous derivation of the non-commuting $\mathbb{Z}_3$ factor without relying on accidental symmetries of the matter spectrum is not possible without our condition (4.37).

As in the $\mathbb{T}^2/\mathbb{Z}_4$ example, the resulting $\Big[\mathrm{U}(1) \times \mathrm{U}(1)\Big] \rtimes \mathbb{Z}_3$ symmetry may be broken further, by breaking the two U(1)s to discrete $\mathbb{Z}_N$ subgroups. A particularly interesting case is obtained when both U(1)s are broken down to $\mathbb{Z}_3$ symmetries. In this case, one is left with a $[\mathbb{Z}_3 \times \mathbb{Z}_3] \rtimes \mathbb{Z}_3$ symmetry, which is known as $\Delta(27)$ and has found various applications as a flavor symmetry, see e.g. [108, 109], and within the context of CP violation [110].

Larger remnant symmetries with outer automorphisms. As it has been noted already in the string setup [100, 107, 111, 112, 113], it is possible to obtain a larger symmetry than just $\Big[\mathrm{U}(1) \times \mathrm{U}(1)\Big] \rtimes \mathbb{Z}_3$. This possibility opens up if the matter content of the upstairs theory is chosen appropriately, which results in a situation where the remnant symmetry is more model-dependent than in the previous examples. However, in string-derived models the matter content (which is fixed and cannot be chosen at will), seems to allow for these extra symmetries in most known cases. In particular, if the matter charged under the SU(3) gauge symmetry is symmetric under SU(3) charge conjugation, the full upstairs symmetry is not just SU(3) but SU(3) $\rtimes \mathbb{Z}_2$, where the additional $\mathbb{Z}_2$ acts in the Lie algebra generators according to

$$\mathsf{T}_a^{(\mathrm{CW})} \xmapsto{\mathbb{Z}_2^{\text{out.}}} -\left(\mathsf{T}_a^{(\mathrm{CW})}\right)^T \; . \tag{4.89}$$

Hence, it maps the adjoint **8** of SU(3) to itself, while it interchanges the fundamental representation **3** with its conjugate $\overline{\mathbf{3}}$. Therefore, in order for this symmetry to be present in the first place, the number of **3**s and $\overline{\mathbf{3}}$s in the upstairs theory must be equal (the same is of course true for any other representation that transforms non-trivially under the $\mathbb{Z}_2$). On the generators in the Cartan–Weyl basis (where we fixed all basis choices), the corresponding action can be determined to be

$$H_I \xmapsto{\mathbb{Z}_2} s_{IJ} H_I \; , \tag{4.90}$$

$$E_w \xmapsto{\mathbb{Z}_2} -E_{\rho_s^{-1} w} \; , \tag{4.91}$$

where the matrix s is given by

$$s \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.92}$$

and ρ_s is a $\mathrm{GL}(2,\mathbb{Z})$ matrix such that $se = e\rho_s$. In a full string orbifold picture, the $\mathbb{Z}_2$ has a geometric interpretation. However in order to understand the resulting discrete groups after orbifolding, one does not have to rely on such a geometric interpretation. Explicitly, the outer automorphism acts on the Cartan–Weyl generators as an eight-by-eight matrix $R^{\text{out.}}$ which is given in the basis (4.81) by

$$\mathsf{T}_a^{(\mathrm{CW})} \;\xmapsto{\mathbb{Z}_2^{\text{out.}}}\; \left(R^{\text{out.}}\right)_{ab} \mathsf{T}_b^{(\mathrm{CW})}\;. \tag{4.93}$$

The explicit matrix representing this transformation is given by

$$R^{\text{out.}} \;=\; \left(\begin{array}{cc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right)\;, \tag{4.94}$$

where we have indicated the two-by-two block of the Cartan generators in the upper right, followed by the two three-by-three blocks containing the ladder operators with orbifold transformation phases ω and ω^2, respectively, cf. the matrix $R^{\text{orb.}}$ in equation (4.82). Because it is an outer automorphism, the $\mathbb{Z}_2$ action cannot be represented by conjugation with SU(3) matrices. Therefore, whether or not it survives the orbifold boundary conditions is not captured by our condition (4.37). Using the analog of the commuting diagram (4.33) in this case, one arrives at the condition that in order for the $\mathbb{Z}_2$ outer automorphism to survive the orbifolding procedure, the matrices $R^{\text{orb.}}$ and $R^{\text{out.}}$ have to commute, which is indeed the case, given their respective block-structures.

If this non-commuting $\mathbb{Z}_2$ factor is included in the upstairs theory (i.e. if the matter content allows it to be present), the full remnant symmetry after orbifolding may be as large as

$$\mathrm{SU}(3) \rtimes \mathbb{Z}_2 \;\xrightarrow{\mathbb{Z}_3^{\text{orb.}}}\; \left(\Big[\mathrm{U}(1) \times \mathrm{U}(1)\Big] \rtimes \mathbb{Z}_3\right) \rtimes \mathbb{Z}_2\;. \tag{4.95}$$

Now, if the two U(1) factors are broken to $\mathbb{Z}_3$s, the resulting discrete group is a $\Delta(54)$, which is also known to be the basis of many (bottom-up) models of flavor physics [114, 115, 116], as well as top-down approaches to CP violation [117].

4.5 Discrete remnant symmetries from Weyl reflections

Let us put our results so far into perspective. We have seen that, given an upstairs gauge group $\mathcal{G}$ any matrix U from $\mathcal{G}$ survives the orbifold boundary conditions given by the gauge embedding P if the two simultaneous conditions are satisfied

$$[P, U] = \omega^k \mathbb{1} , \tag{4.96}$$

$$\omega^k \mathbb{1} \in Z(\mathcal{G}) , \tag{4.97}$$

for some $k \in \{0, \ldots, N-1\}$, where N is the order of the orbifold twist. Using this condition, it is therefore a straightforward task to decide whether or not a given matrix U is part of the unbroken gauge symmetry after orbifolding. On the other hand, behind every beautiful thing, there is some kind of pain:[2] performing an exhaustive enumeration of all matrices U for a given P can be very hard, because U can be any element of the (infinite) group $\mathcal{G}$.[3] For the determination of the residual *continuous* gauge symmetries, one can still use the standard approach and check only the finite number of Lie algebras for invariant linear combinations that commute with the orbifold boundary conditions, but for the *discrete* ones there is no other way but to solve the condition (4.37). In our examples, it was still fairly straightforward to find all solutions, either because $\mathcal{G}$ was small enough to allow for an explicit parametrization (as in the SU(2) example), or because the boundary condition P was chosen in a particularly simple way (as in the SO(10) example). For a general P and a large group, e.g. E_6, this approach stops being viable because of the multitude of possible transformations U that need to be checked. These problems would be largely mitigated if there were (except for the continuous gauge symmetries from the $k = 0$ sector) only a large but finite number of additional possibilities as candidates for a surviving transformation U that needs to be taken into account.

Luckily for us, our findings support the intuition that the set of possible candidates for additional discrete symmetries is given by the action of Weyl reflections in the Lie algebra root lattice. A Weyl reflection w_α with respect to a root α acts on the root system of a Lie algebra $\mathfrak{g}$ as the reflection along the hyperplane perpendicular to α [118], that is, for some β from the root system

$$w_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha , \tag{4.98}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product. On the level of the Lie group $\mathcal{G}$ associated with $\mathfrak{g}$, the set of all Weyl reflections on $\mathfrak{g}$ gives rise to the Weyl group $W(\mathcal{G})$, which is a finite group. In all our examples, the discrete remnant symmetries are in fact Weyl reflections with respect to roots that are *broken* by the orbifold boundary conditions. Instead of having to check the infinite number of elements of an upstairs gauge group $\mathcal{G}$, one only has to check the *finite* number of transformations that correspond to Weyl

[2] Quote attributed to Bob Dylan.

[3] In that sense, the task of determining the unbroken discrete symmetry strongly resembles problems from the complexity class NP in theoretical computer science: while *verifying* the correctness of a proposed solution is easy, *finding* a solution can be hard.

reflections at broken roots, which yields the desired shortcut. Let us illustrate this procedure in the three examples presented earlier.

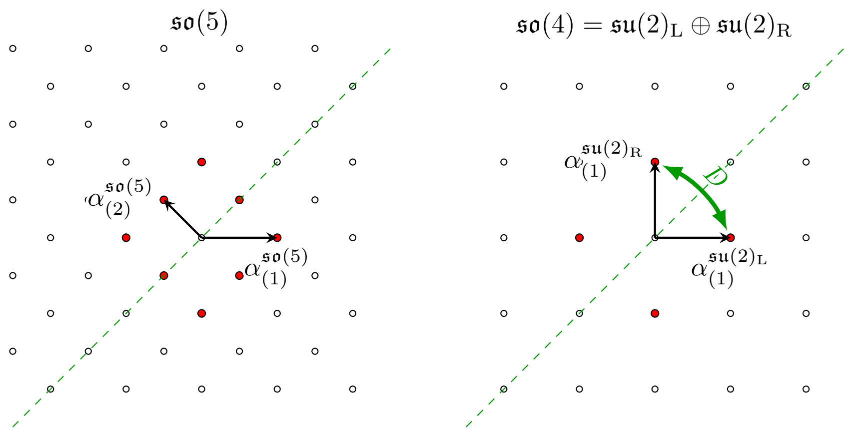


Figure 4.1: *A depiction of the root lattices of the $\mathfrak{so}(5)$ (left) and the $\mathfrak{so}(4) = \mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$ algebra (right). It is evident that a Weyl reflection at the plane perpendicular to $\alpha^{\mathfrak{so}(5)}_{(2)}$ exchanges the $\mathfrak{su}(2)$ algebras. Therefore, the action of the D-parity on $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$ can be understood as a Weyl reflection at the broken root $\alpha^{\mathfrak{so}(5)}_{(2)}$ of $\mathfrak{so}(5)$.*

***D*-parity in the Pati–Salam model.** The interpretation of the D-parity in terms of a geometric reflection in the $\mathfrak{so}(10)$ root lattice is most interesting for the $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$ sublattice, where one would like to observe how the two $\mathfrak{su}(2)$s get interchanged by the action of the D-parity.

It turns out that this particular property can be understood already by considering a lower-dimensional example that is easier to visualize. To this end, we consider a specific SO(5) subgroup of SO(10). The breaking of an SO(5) upstairs theory to an SO(4) can be facilitated by a $\mathbb{Z}_2$ orbifold boundary condition

$$P_5 \;=\; \mathrm{diag}(1,-1,-1,-1,-1) \in \mathrm{SO}(5)\;. \tag{4.99}$$

The root lattice of $\mathfrak{so}(5)$ is two dimensional ($\mathfrak{so}(5)$ corresponds to the non-simply-laced B_2 in the Dynkin classification of Lie algebras), where the two roots $\alpha^{\mathfrak{so}(5)}_{(1)}$ and $\alpha^{\mathfrak{so}(5)}_{(2)}$ can be chosen according to

$$\left|\alpha^{\mathfrak{so}(5)}_{(1)}\right|^2 \;=2\;, \tag{4.100}$$

$$\left|\alpha^{\mathfrak{so}(5)}_{(2)}\right|^2 \;=1\;, \tag{4.101}$$

$$\alpha^{\mathfrak{so}(5)}_{(1)} \cdot \alpha^{\mathfrak{so}(5)}_{(2)} \;=\;-1\;. \tag{4.102}$$

The corresponding root lattice is shown on the left panel of figure 4.1. Of the two simple roots, the shorter one, $\alpha^{\mathfrak{so}(5)}_{(2)}$, is broken by the orbifold boundary condition, leading to the coarser sublattice of $\mathfrak{so}(4)$ shown in the right panel of figure 4.1. This coarser sublattice of the original $\mathfrak{so}(5)$ lattice is spanned by the two simple roots of $\mathfrak{so}(4)$, or equivalently by the roots of $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$. In particular, the two $\mathfrak{su}(2)$ roots are given in terms of the $\mathfrak{so}(5)$ roots as

$$\alpha_{(1)}^{\mathfrak{su}(2)_\mathrm{L}} = \alpha_{(1)}^{\mathfrak{so}(5)} , \tag{4.103}$$

$$\alpha_{(1)}^{\mathfrak{su}(2)_\mathrm{R}} = \alpha_{(1)}^{\mathfrak{so}(5)} + 2\alpha_{(2)}^{\mathfrak{so}(5)} . \tag{4.104}$$

However, the Weyl reflection $w_{\alpha_{(2)}}$ with respect to the hyperplane orthogonal to the "broken" root $\alpha_{(2)}^{\mathfrak{so}(5)}$ acts on the roots of $\mathfrak{so}(5)$ according to

$$\alpha_{(1)}^{\mathfrak{so}(5)} \xmapsto{w_{\alpha_{(2)}}} \alpha_{(1)}^{\mathfrak{so}(5)} + 2\alpha_{(2)}^{\mathfrak{so}(5)} , \tag{4.105}$$

$$\alpha_{(2)}^{\mathfrak{so}(5)} \xmapsto{w_{\alpha_{(2)}}} -\alpha_{(2)}^{\mathfrak{so}(5)} . \tag{4.106}$$

This transformation is a symmetry of the $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$ sublattice, and in particular it exchanges all generators of the $\mathfrak{su}(2)$ algebras, as depicted in figure 4.1.

We can lift this action such that it holds also in the full Pati–Salam example, because the $\mathfrak{so}(5)$ lattice can be seen as a particular two-dimensional projection of the $\mathfrak{so}(10)$ lattice. An explicit depiction of the transformation D as a Weyl reflection is more difficult in this case since the rank of $\mathfrak{so}(10)$ (and hence the required dimensionality of the visualization) is five. Let us now study how also the residual transformations in the examples $k \neq 0$ can be understood as elements of the Weyl group of the upstairs Lie algebra.

$\mathbb{T}^2/\mathbb{Z}_4$ Orbifold GUT. In this example, we make an explicit choice for the Cartan–Weyl basis of the SU(2) upstairs theory as $\mathsf{T}^{(\mathrm{CW})} = (H, E_+, E_-)$ with

$$H = \frac{1}{\sqrt{2}}\sigma_3 , \tag{4.107}$$

$$E_\pm = \frac{1}{2}\left(\sigma_1 \pm \mathrm{i}\sigma_2\right) . \tag{4.108}$$

The $\mathfrak{su}(2)$ Lie algebra possesses a single reflection in its root lattice, namely with respect to the hyperplane perpendicular to E_+ (or, equivalently, to E_-) that gives rise to the Weyl group $W(\mathrm{SU}(2)) \cong \mathbb{Z}_2$. This reflection w acts on the generators according to

$$w : \begin{pmatrix} H \\ E_+ \\ E_- \end{pmatrix} \xmapsto{w} \begin{pmatrix} -H \\ E_- \\ E_+ \end{pmatrix} = R^w \begin{pmatrix} H \\ E_+ \\ E_- \end{pmatrix} , \quad \text{with} \quad R^w = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.109}$$

and survives the orbifold boundary conditions. The action of the boundary condition P on this basis is given by the three-by-three matrix

$$R^{\mathrm{orb.}} = \mathrm{diag}\left(1, -1, -1\right) . \tag{4.110}$$

Note that both E_+ and E_- are broken by the orbifold boundary condition. One observes that the orbifold action $R^{\mathrm{orb.}}$ and the Weyl reflection R^w commute and hence the Weyl transformation is left unbroken. It is straightforward to verify that R^w is precisely the adjoint action of the unbroken element $U_{(2)}$ in (4.76). This remnant symmetry can be understood as the action of the unbroken element w of the Weyl group of $\mathfrak{su}(2)$.

$\mathbb{T}^2/\mathbb{Z}_3$ Orbifold GUT. As in the previous examples, the surviving discrete transformations can be traced back to specific elements of the Weyl group of SU(3). Let us denote the Weyl reflection with respect to the root α by w_α. Then, the Weyl group of SU(3) acts on the roots of $\mathfrak{su}(3)$ as

$$w_{\alpha_{(1)}} \;:\; \begin{pmatrix}\alpha_{(1)}\\ \alpha_{(2)}\end{pmatrix} \;\longmapsto\; \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_{(1)}\\ \alpha_{(2)}\end{pmatrix}\;, \tag{4.111}$$

$$w_{\alpha_{(2)}} \;:\; \begin{pmatrix}\alpha_{(1)}\\ \alpha_{(2)}\end{pmatrix} \;\longmapsto\; \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_{(1)}\\ \alpha_{(2)}\end{pmatrix}\;. \tag{4.112}$$

As both $\alpha_{(1)}$ and $\alpha_{(2)}$ are broken by the orbifold twist, we must in principle check any composition of the Weyl reflections $w_{\alpha_{(1)}}$ and $w_{\alpha_{(2)}}$, hence, the entire Weyl group. To this end, let us first determine the Weyl group of SU(3) in terms of these two Weyl reflections. We first note that, although both $w_{\alpha_{(1)}}$ and $w_{\alpha_{(2)}}$ are order two matrices, they do not commute and give rise to a larger group. If we make the replacement

$$a \;=\; w_{\alpha_{(1)}} w_{\alpha_{(2)}}\;, \tag{4.113}$$

$$b \;=\; w_{\alpha_{(1)}} w_{\alpha_{(2)}} w_{\alpha_{(1)}}\;, \tag{4.114}$$

we observe that $a^3 = b^2 = \mathbb{1}$ and that $bab = a^2$, so that the full group can be obtained by

$$W(\mathrm{SU}(3)) \;=\; \left\{\mathbb{1}, a, a^2, b, ab, a^2 b\right\}\;, \tag{4.115}$$

which is an S_3 that is generated by the order three element a and the order two element b. Not all elements of this group survive the orbifold boundary condition. In particular, the explicit form of the adjoint action of b on the generators in the Cartan–Weyl basis is given by

$$R^b \;=\; \begin{pmatrix} r^b_H & & \\ & & r^b_E \\ & r^b_E & \end{pmatrix}\;, \tag{4.116}$$

where the various blocks are given by

$$r^b_H \;=\; \begin{pmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix} \quad\text{and}\quad r^b_E \;=\; \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}\;. \tag{4.117}$$

It is easy to verify that this matrix does not commute with the orbifold boundary conditions $R^{\text{orb.}}$ and hence does not survive. On the other hand, the action of the $\mathbb{Z}_3$ generator $a = w_{\alpha_{(1)}} w_{\alpha_{(2)}}$ can be shown to be exactly the same as conjugating each generator with $U_{(1)}$. Therefore, the unbroken $\mathbb{Z}_3$ symmetry generated by $U_{(1)}$ can be identified with the Weyl reflection $w_{\alpha_{(1)}} w_{\alpha_{(2)}}$.

4.6 Summary

In this chapter, we are concerned with the emergence of discrete symmetries in a low-energy effective theory from a gauge theory that is broken towards lower energies by orbifold boundary conditions. In order to illustrate this mechanism, we consider field theory settings that are inspired by string theory compactifications. In particular, we consider setups where a gauge theory in higher dimensions (the "upstairs" theory) is compactified to four dimensions on an orbifold. As in string theory, the orbifold twist is chosen to have a non-trivial action on the gauge degrees of freedom. Then, the orbifold boundary conditions break the higher-dimensional gauge group to some subgroup, which yields the "downstairs" theory. As in conventional scenarios (e.g. spontaneous symmetry breaking), discrete remnants can survive along with the unbroken continuous gauge group. In the upstairs theory, these discrete remnants are still part of the continuous gauge group, hence they are inner automorphisms of the upstairs theory. However, in the downstairs theory, they act as outer automorphisms.

In order to identify these discrete remnant symmetries, we study the group-theoretical conditions that need to be fulfilled in order to preserve a remnant symmetry. As our findings show, discrete remnants do not have to commute with the orbifold boundary condition, but only have to fulfill a weaker condition. Let us consider an upstairs gauge group $\mathcal{G}$ and denote the orbifold boundary condition by P, where the order of P is taken to coincide with the order of the orbifold twist, that is $P^N = \mathbb{1}$ for a $\mathbb{Z}_N$ orbifold. A transformation $U \in \mathcal{G}$ survives the orbifolding procedure if it fulfills the condition that it has to commute with the boundary condition only up to a diagonal element, i.e.

$$P^{-1}U^{-1}P\,U \;=\; \omega^k\,\mathbb{1}\;, \tag{4.118}$$

where ω is an N-th root of unity and $k = 0, \ldots, N-1$. Additionally, the diagonal element on the right hand side of (4.118) must be an element of the center of $\mathcal{G}$

$$\omega^k\mathbb{1}\;\in Z(\mathcal{G})\;, \tag{4.119}$$

which restricts the admissible values of k, depending on the properties of the group $\mathcal{G}$. The fact that U does not strictly have to commute with P is the consequence of the representation theory of Lie algebras. The case $k = 0$ amounts to the "standard" case where the transformation U and the boundary condition P commute and includes the unbroken continuous gauge group (among possible additional discrete symmetries). On the other hand, the cases with $k \neq 0$ have been overlooked by the existing literature and always correspond to discrete remnants of the gauge symmetry.

We apply this set of rules to a few examples and find solutions for residual discrete symmetries in both $k = 0$ sectors (the so-called left-right or D-parity of the Pati–Salam model from an SO(10) GUT), and in $k \neq 0$ sectors, where the resulting remnant symmetries are candidate for a flavor symmetries, e.g. $\Delta(27)$ or $\Delta(54)$, and may also address the issue of CP violation. Additionally, it is conceivable that the $\mathbb{Z}_3$ symmetry that permutes the SU(3)s in trinification models (where E_6 is broken to SU(3) × SU(3) × SU(3)) can be explained in a similar fashion.

However, we find that this approach has an important drawback at this level. Namely, while it is easy to check whether conditions (4.118) and (4.119) are fulfilled for a given U, making an exhaustive enumeration of *all* transformations U from $\mathcal{G}$ that are left unbroken is hard, because one would have to check an infinite number of matrices. By studying the existing examples in more detail, we are able to restrict the candidates for unbroken discrete remnants to a finite set, namely to elements of the Weyl group of $\mathcal{G}$. In particular, one only has to check elements of the Weyl group that involve Weyl reflections with respect to roots that are broken by the orbifold boundary conditions.

Our results show that it is possible to identify large classes of phenomenologically interesting discrete symmetries as remnants of a UV gauge symmetry. While we base our derivation on a field-theoretical approach, we believe that the results also apply to string constructions on orbifolds as well, because the crucial steps lie in the group-theoretical properties of the upstairs theory and the orbifold boundary conditions.

In contrast to existing approaches on the string theory side, our results do not rely on the presence and nature of twisted matter. However, as we quickly comment, making assumptions on the representations of twisted and bulk matter (and the multiplicities), may allow one to build even more interesting models. On a more conceptual level, the inclusion of charged matter is necessary, for example should one ever try to attack the ultimate goal of making predictions (or at least deriving implications) on the measured CP phases e.g. in the standard model CKM matrix. Moreover, on a technical level, we make the explicit simplifying assumption that the order of the gauge embedding P is equal to the order of the orbifold twist. It would be interesting to see which patterns emerge if this restriction is relaxed. Finally, the relation of the discrete symmetries constructed in this chapter with modular symmetries appearing in flavor model building [119, 120] deserves to be studied, especially in connection with string theory [121, 122, 123, 124].

5

String scale interacting dark matter

5.1 Introduction

There is a wide array of observational evidence for the existence of a non-luminous but gravitating matter component that couples at most weakly to the standard model, called dark matter (DM) [125]. It has been shown that the most straightforward models of dark matter involve some species of dark matter that is cold at present-day [126] in order to allow for the formation of large-scale structures. The Lambda–Cold Dark Matter (ΛCDM) model has proven itself as successful in explaining cosmological phenomena as the standard model of particle physics. However, the microscopic nature of dark matter has not yet been explained.

It is known that dark matter makes up the majority of the matter content in the universe today [50]. Therefore, a successful explanation for particle dark matter must provide that the dark matter candidate is produced to sufficient amounts in the early universe. However, the dark matter candidate must also be long-lived enough to persist until present times, instead of simply decaying. Otherwise it cannot explain the observed relic density. Among other mechanisms that ensure the longevity of the population of dark matter particles, symmetries (e.g. $\mathbb{Z}_2$s) are often invoked in order to stabilize the dark matter candidate against decay.

There exist various ways to produce dark matter, but the most prominent (and for many years most promising) one is freeze-out production (see e.g. [127] for a pedagogical introduction). There, the dark matter candidate is in thermal equilibrium with the heat bath of the universe right after inflation. As the universe expands and cools down, equilibrium is maintained for some time, for example, by a $2 \to 2$ scattering with other particles in the bath. However, at some point this scattering becomes inefficient and the dark matter species drops out of thermal equilibrium, which causes its relic density to remain essentially constant from that time on. The prime example for a dark matter particle which is produced by freeze-out is the weakly interacting massive particle (WIMP) [128]. While the paradigm of WIMP dark matter has become very popular by the observation that a weak-scale particle naturally yields the correct relic abundance ("WIMP miracle"), some tension has built recently due to the non-observation of WIMPs in experiments [129]. Much as the WIMP is not the only dark matter candidate, freeze-out is not the only mechanism for (thermal) production. The freeze-out mechanism relies on the fact that the dark matter candidate is in thermal equilibrium for at least some time before freezing out eventually. However, if the dark matter particle is heavy enough (or its cross section with the other particles is sufficiently small), it will never attain thermal equilibrium, making freeze-out production infeasible. In this case, it may still be produced *non-thermally* in sizable amounts, e.g. by gravitational production [130, 131], or during reheating [132, 133]. Moreover, it has become clear that sufficient amounts of a dark matter species may also be produced *thermally* even if it never attains thermal equilibrium ("freeze-in") [134], provided that the temperature reached during reheating is high enough. A very peculiar type of dark matter produced in this way comes from Planckian interacting dark matter (PIDM) [14, 135, 136, 137], where the dark matter candidates are extremely heavy (with masses above the GUT scale) and have couplings that are suppressed by $1/m_{\mathrm{Pl}}^2$.

The ideas presented in this chapter mainly build on this observation. In particular, any string model has heavy states with masses above the GUT scale as part of its spectrum. It is generically not a hard task to find such heavy states that are complete standard model singlets, and only couple to the thermal bath with couplings that are suppressed by $1/m_{\mathrm{s}}^2$, where m_{s} is the string scale that lies between the GUT and the Planck scale. These couplings can either be the exchange of gravitons (as in the pure PIDM scenario) or be the result of stringy interactions. Moreover, as we have already seen in the foregoing chapters, discrete symmetries appear frequently in heterotic string models on orbifolds. If the topology of the compact space has the right properties, it is straightforward to obtain a setup where a subset of these extremely heavy string states is stable [138], e.g. as the consequence of an Abelian $\mathbb{Z}_2$ symmetry [139]. Hence, in string models like these, situations that are very similar to those of the PIDM scenarios are very common, and it deserves to be studied how these models can give rise to realizations of dark matter in string theory. It is then interesting to compare these setups to other attempts to explain dark matter in the context of string theory and extra dimensions [140, 141, 142, 143].

Outline. The goal of this chapter is to show how a large class of string models give rise to a dark matter candidate produced by freeze-in in a very generic way. To this end, we first review the freeze-in production of dark matter, in particular for the production of multi-component dark matter with coannihilations. Then, we explain how a topologically stable string state can be constructed. For specificity, we consider a setup in the framework of heterotic string theory with six extra dimensions compactified on a special class of orbifolds, but as we shall comment later, it is conceivable that our scheme is valid more generally. In our setup, the dark matter candidate becomes stable by winding around a certain non-contractible cycle in the extra-dimensional compact space. From the viewpoint of the low-energy effective field theory, this winding gives rise to a charge under an exact discrete symmetry. This symmetry originates from the topological property of the compactification space to be non-simply connected. The resulting discrete symmetry is exact on the classical level and it is believed that any possible anomaly may be cancelled by a (generalized) Green–Schwarz mechanism. Therefore, the discrete stabilizing symmetry can only be broken non-perturbatively due to the discrete anomaly [144, 145]. Due to the winding, the stable string state is generically very heavy (with a mass at the compactification or GUT scale). We proceed to study its interactions with the standard model and other massless states in the thermal plasma. On the one hand, these interactions are mediated by gravity, but we find that also the exchange of other heavy winding modes contributes to the $2 \to 2$ scattering cross section. We find that these extra contributions can come from both Kähler and superpotential terms in the effective Lagrangian. In particular, we show that the contribution coming from Kähler terms is present in any scenario, unlike the superpotential terms that are subject to a special choice of the localization of certain fields. The Kähler terms can be traced back to the exchange of a massive U(1) gauge boson, where the charges of the standard model particles and dark matter are not fixed by first principles and can be varied without affecting the unification of the standard model gauge couplings. We then consider the freeze-in production of our winding dark matter candidate via the operators obtained from this effective field theory. Demanding that the relic density of our dark matter candidate matches the observed value allows one to relate the coupling strength of the dark matter candidate to the thermal bath with the Hubble rate after inflation, which is constrained to lie below a certain bound due to the non-observation of tensor modes in the cosmic microwave background (CMB) by experiments like Planck and Bicep. We observe that for large portions of the parameter space, it is possible to obtain the correct observed relic abundance for dark matter without violating any existing observational bounds from the CMB. However, this situation might change in the near future due to refined bounds from experiment. We conclude our discussion by studying how well our specific model generalizes to a generic string model that contains variable numbers of (vector-like) standard model exotics and hidden sector matter in its spectrum. We are able to show that due to the nature of freeze-in production, the required values of the cosmological parameters change only marginally if such model-dependent details are taken into account, which means that our results can be expected to hold in more or less any string-derived model if the topological properties of the compact space are chosen in the right way. Hence, the model-dependence enters mostly on the cosmology side, i.e. how inflation and reheating are realized. This chapter is in parts based on ref. [17].

5.2 Thermal production of dark matter

Here, we review the thermal production mechanisms for (cold) dark matter in the early universe. We shall closely follow [127] (for the general setup and freeze-out production) and [134] (for freeze-in production).

5.2.1 Boltzmann equation

If the production of a particle species χ is considered, the fundamental object of interest is its phase space distribution $f_\chi(p^\mu, x^\mu)$. Then, the dynamics of this distribution are governed by a Boltzmann equation, which takes the general form

$$\mathbf{L}[f_\chi] \;=\; \mathbf{C}_\chi[f_\chi]\,. \tag{5.1}$$

Here, the (relativistic) Liouville operator $\mathbf{L}$ describes the evolution of the phase space density in an expanding universe in terms of cosmological quantities. On the other hand, the collision term $\mathbf{C}_\chi$ takes care of all particle physics interactions the particle χ is involved in. Therefore, it is not only model-dependent but (by contrast to the Liouville operator) may also vary in its form for different particle species in a given model, hence we include the subscript χ.

In a general relativistic case, the Liouville operator incorporates the geometry of space-time via the affine connections (i.e. the Christoffel symbols) and reads

$$\mathbf{L} \;=\; p^{\mu}\frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta}\frac{\partial}{\partial p^{\mu}}\;. \tag{5.2}$$

In an spatially isotropic homogeneous Friedman–Lemaître–Robertson–Walker universe with scale factor a, the Boltzmann equation can be significantly simplified by noticing first that the phase space density $f_\chi(p^\mu, x^\mu)$ in this case takes a simplified form

$$f_\chi(p^\mu, x^\mu) \;\equiv\; f_\chi(E, t)\;, \tag{5.3}$$

which can be used to rewrite the Liouville operator in a more concise form. By using that most of the partial derivatives in the Liouville operator vanish (all but the ∂x^0 and ∂p^0 derivatives) due to the simplified form, and that the Christoffel symbols in Cartesian coordinates are given by $\Gamma^0_{ij} = H(t)g_{ij}$ in terms of the Hubble rate $H(t) = \dot{a}/a$, we arrive at the following expression for the Liouville operator

$$\mathbf{L} \;= E\frac{\partial}{\partial t} - H(t)|\vec{p}|^2\frac{\partial}{\partial E}\;. \tag{5.4}$$

In order to eventually compare the final result with observed quantities, it is advantageous to switch from the phase space density f_χ to the particle number density n_χ

$$n_\chi \;=\; g\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}f_\chi(E,t)\;, \tag{5.5}$$

where g counts the internal (spin) degrees of freedom of χ. Using this substitution, the Boltzmann equation becomes

$$\dot{n}_\chi + 3H(t)\,n_\chi \;=\; g\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3E}\mathbf{C}_\chi[f_\chi]\;, \tag{5.6}$$

where, due to the momentum-space integration, only reactions that change the number density of χ contribute to the right-hand side of the equation. The left-hand side of this equation now has the desired form in terms of observable quantities. The remainder of this section will be devoted to a discussion of the right hand side of this equation. To this end, we need to specify the overall form of the collision terms. The overall form of the final Boltzmann equation depends on what type of reaction involving the species χ appears in the collision term. As we are ultimately interested in an application of this formalism to a dark matter scenario, we will always assume that there is some stabilizing symmetry that forbids decays of the particle χ and allows only annihilations. To simplify the discussion, we will make no distinction between χ and its antiparticle $\overline{\chi}$. When considering annihilations, it is useful to distinguish two cases:

1. A particularly simple but instructive case arises if the only reactions appearing in the collision term are of the form

$$\chi\chi \;\leftrightarrow\; \psi_a\psi_b\;, \tag{5.7}$$

where ψ_a and ψ_b can be any other particle species in the thermal bath, i.e. standard model particles. As we will see in the later course of this section, this case already captures most of the relevant dynamics of thermal dark matter production in both freeze-out and freeze-in scenarios.

2. In general, a dark sector is not made up by a single particle species. In particular, in supersymmetric models, particles and their superpartners carry either strictly the same charges under the stabilizing symmetry, or the stabilizing symmetry would be an R-symmetry (like e.g. the R-parity in the most naive models of LSP dark matter). In any case, e.g. if a supersymmetric dark sector is stabilized by a non-R-symmetry, but also in other scenarios with multi-component dark matter, there can be interactions of the form

$$\chi_i \chi_j \ \leftrightarrow \ \psi_a \psi_b \ , \tag{5.8}$$

where χ_i and χ_j are fields from the dark sector. These interactions are usually referred to as coannihilations. The complication in this case arises from the fact that the Boltzmann equations followed by the number densities of each of the species χ_i and χ_j are now coupled through common collision terms on the right hand sides. Note that distinguishing between a dark sector particle χ and its antiparticle $\overline{\chi}$ may be seen as a special case of coannihilations.

In what follows, we will first derive the general result for the final form of the Boltzmann equation in the simpler situation of case 1, and then discuss how this result can be generalized to include coannihilations as well. At first, we also consider only a single channel for the interaction $\chi\chi \ \leftrightarrow \ \psi_a\psi_b$ as it is straightforward to include a variable number of channels, i.e. different pairs of thermal bath particles coupling to the dark sector via $2 \to 2$ scattering. Our final assumption throughout this section is that all particles coupling to the dark sector are themselves in thermal equilibrium, i.e. $f_{a,b}(E_{a,b}, t) = f_{a,b}^{\text{eq.}}(E_{a,b}, t)$. This assumption is well justified if the thermal bath particles are sufficiently light and tightly coupled enough in order to maintain thermal equilibrium at least longer than the dark matter particle. This condition is in general fulfilled for standard model particles, as they are (at the energy scales of the early universe) massless and couple sufficiently strongly to one another through the standard model gauge couplings.

Under these assumptions and simplifications, the right hand side of the Boltzmann equation containing the collision terms can be rearranged as a phase-space integral to read

$$\begin{aligned} g \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 E} \mathbf{C}_\chi[f_\chi] \ = \ & - \int \mathrm{d}\Pi_\chi \mathrm{d}\Pi'_\chi \mathrm{d}\Pi_a \mathrm{d}\Pi_b \, (2\pi)^4 \, \delta^{(4)}\Big(\sum p_i\Big) \ |\mathcal{M}_{\psi_a\psi_b\leftrightarrow\chi\chi}|^2 \\ & \cdot \left[f_\chi(E,t) f_\chi(E',t) - f_{\psi_a}(E_a,t) f_{\psi_b}(E_b,t) \right] \ , \end{aligned} \tag{5.9}$$

where Fermi-blocking has been taken care of by standard methods and where we used that the spin-averaged matrix elements for the reaction $\psi_a\psi_b \leftrightarrow \chi\chi$ are the same no matter in which direction the reaction takes place (due to assumed CP invariance)

$$|\mathcal{M}_{\chi\chi\to\psi_a\psi_b}|^2 \ = \ |\mathcal{M}_{\psi_a\psi_b\to\chi\chi}|^2 \ \equiv \ |\mathcal{M}_{\psi_a\psi_b\leftrightarrow\chi\chi}|^2 \ . \tag{5.10}$$

Using now that the particles ψ_a and ψ_b are in thermal equilibrium, and moreover making use of the principle of detailed balance, one obtains

$$f_{\psi_a}(E_a,t)f_{\psi_b}(E_b,t) \;=\; f^{\text{eq.}}_{\psi_a}(E_a,t)f^{\text{eq.}}_{\psi_b}(E_b,t) \;=\; f^{\text{eq.}}_{\chi}(E,t)f^{\text{eq.}}_{\chi}(E',t)\,, \tag{5.11}$$

which already simplifies the collision term (5.9). While the equilibrium densities actually follow either Bose–Einstein or Fermi statistics, it turns out that taking them to follow a Maxwell–Boltzmann statistic is a good approximation. This approximation is nearly perfect for freeze-out production, where the relevant temperature scales are sufficiently far below the dark matter mass. For the case of freeze-in production, the final relic abundance may be suppressed or enhanced by a few percent as noted in [146], which does not noticeably alter our findings. Plugging in the equilibrium densities $f^{\text{eq.}}_{a,b}(E_{a,b},t) = \mathrm{e}^{-E_{a,b}/T}$ and performing the phase space integrals one finally arrives at

$$\dot{n} + 3H(t)\,n \;=\; -\,\langle\sigma v\rangle\left(n^2 - n^2_{\text{eq}}\right)\,. \tag{5.12}$$

Here, $\langle\sigma v\rangle$ is the effective thermally averaged cross section for the $2\to 2$ dark matter production. It is given by an appropriate integral over the Mandelstam variable s

$$\langle\sigma v\rangle \;=\; \frac{T}{n^2_{\text{eq}}}\frac{g^2}{8\pi^4}\int_{4m_\chi^2}^{\infty}\mathrm{d}s\sqrt{s}\,p^2\sigma_{\chi\chi\to\psi_a\psi_b}\,K_1\left(\frac{\sqrt{s}}{T}\right)\,, \tag{5.13}$$

where $\sigma_{\chi\chi\to\psi_a\psi_b}$ is the spin-averaged cross section for the process $\chi\chi\to\psi_a\psi_b$ and K_1 is the modified Bessel function of the second kind of order 1. The equilibrium density n_{eq} is given by

$$n_{\text{eq}} \;=\; \frac{T}{2\pi^2}g\,m_\chi^2 K_2\left(\frac{m_\chi}{T}\right)\,, \tag{5.14}$$

where K_2 is the modified Bessel function of the second kind of order 2. Finally, the relative momentum of the annihilating dark matter particles is given by

$$p \;=\; \sqrt{\frac{s}{4} - m_\chi^2}\,. \tag{5.15}$$

It is now straightforward to generalize this result to the case where the dark matter particle couples to multiple states in the thermal bath, i.e. more than one pair of ψ_a, ψ_b. Then, the spin-averaged cross section $\sigma_{\chi\chi\to\psi_a\psi_b}$ in the thermal average (5.13) gets replaced by

$$\sigma_{\text{tot.}} \;=\; \sum_{a,b}\sigma_{\chi\chi\to\psi_a\psi_b}\,, \tag{5.16}$$

that is by the sum over all possible final states in the thermal bath.

Coannihilations. Let us now consider the more general case where more than one particle species exists in the dark sector. We make the simplifying assumption that all particles in the dark sector have the same mass, which is always the case in the models with intact supersymmetry considered in this chapter. We work with a set χ_i in the dark sector and denote their number densities by n_i. As mentioned earlier, each of the number densities fulfills then its own Boltzmann equation that follows the form of (5.12)

$$\dot{n}_i + 3H(t)\, n_i \;=\; -\sum_j \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_{i,\mathrm{eq}} n_{j,\mathrm{eq}} \right) \,, \tag{5.17}$$

where the index j runs over all fields in the dark sector, and where the cross sections σ_{ij} correspond to the various coannihilation processes, each individually summed over its final states in the thermal bath. Note how the non-coannihilation case (5.12) is naturally included by $i = j$. While already solving the simpler Boltzmann equation (5.12) can be numerically hard, solving the coupled system of Boltzmann equations (5.17) for an arbitrary set of off-diagonal cross sections $\langle \sigma_{ij} v_{ij} \rangle$ is infeasible. However, oftentimes one is not interested in the evolution of the individual number densities. In particular, if one wants to determine the total dark matter relic abundance, one is only interested in the evolution of their sum $n = \sum_i n_i$. By taking the sum over the individual Boltzmann equations (5.17), one finds that the sum of the individual number densities obeys the Boltzmann equation

$$\dot{n} + 3H(t)\, n \;=\; -\langle \sigma_{\mathrm{eff}} v \rangle \left(n^2 - n_{\mathrm{eq}}^2 \right) \,, \tag{5.18}$$

which has the same convenient form as the Boltzmann equation in the case without coannihilations. Here, n is the number density of all states in the dark matter sector, and on the right hand side $\langle \sigma_{\mathrm{eff}} v \rangle$ is the effective thermally averaged cross section for the various $2 \to 2$ dark matter production channels, taking also coannihilations into account [147, 148]. In terms of the coupled cross sections it reads

$$\langle \sigma_{\mathrm{eff}} v \rangle \;=\; \sum_{i,j} \langle \sigma_{ij} v_{ij} \rangle \frac{n_{i,\mathrm{eq}} n_{j,\mathrm{eq}}}{n_{\mathrm{eq}}^2} \tag{5.19}$$

which, using $m_i = m_\chi$ can be recast to become

$$\langle \sigma_{\mathrm{eff}} v \rangle \;=\; \frac{T}{n_{\mathrm{eq}}^2} \frac{1}{8\pi^4} \int_{4m_\chi^2}^{\infty} \mathrm{d}s \sqrt{s}\, p^2 \left(\sum_{i,j} g_i g_j \sigma_{ij}(s) \right) K_1\!\left(\frac{\sqrt{s}}{T} \right) \,. \tag{5.20}$$

Here, g_i counts the internal degrees of freedom of each species χ_i, which in contrast to the mass, can vary in the dark sector. In particular, $g_i = 2$ for a Weyl fermion and $g_i = 1$ for a real scalar, and as before the summation indices i and j is understood as an unrestricted sum over all fields in the dark sector. The total cross sections $\sigma_{ij}(s)$ given by

$$\sigma_{ij} \;=\; \sum_{a,b} \sigma_{ij \to \psi_a \psi_b} \,, \tag{5.21}$$

where the contributions

$$\sigma_{ij \to \psi_a \psi_b} \;=\; \frac{1}{16\pi s \left(s - 4m_\chi^2 \right)} \int_{t_-}^{t_+} \mathrm{d}t \, \left| \mathcal{M}_{ij \to \psi_a \psi_b}(t) \right|^2 \,. \tag{5.22}$$

Here $t_\pm = -\left(\sqrt{s/4} \mp \sqrt{s/4 - m_\chi^2} \right)^2$ and $\mathcal{M}_{ij \to \psi_a \psi_b}(t)$ denotes the (spin-averaged) matrix element for the respective process. The equilibrium density n_{eq} is given by

$$n_{\mathrm{eq}} \;=\; \sum_i \frac{T}{2\pi^2} g_i\, m_i^2 K_2\!\left(\frac{m_i}{T} \right) \;=\; \frac{T}{2\pi^2}\, m_\chi^2 K_2\!\left(\frac{m_\chi}{T} \right) \sum_i g_i \,, \tag{5.23}$$

where we again used that the only thing that may vary over the spectrum of the dark sector is the spin but not the mass of the various particles. In order to obtain an equality between the sum over the coupled Boltzmann equations (5.17) and the simplified expression (5.18), we have to demand that all individual number densities deviate from their particular equilibrium densities by roughly the same factor, or in other words that

$$\frac{n_i}{n_{i,\mathrm{eq}}} \simeq \frac{n_j}{n_{j,\mathrm{eq}}} \; \forall i,j \quad \text{and hence} \quad \frac{n_i}{n_{i,\mathrm{eq}}} \simeq \frac{n}{n_{\mathrm{eq}}} \,. \tag{5.24}$$

This assumption is fulfilled in all practicable cases, especially when (in addition to the masses) all the couplings of dark sector particles are roughly equal, which is again guaranteed in a supersymmetric model.

5.2.2 Freeze-out production

Although we will not make direct use of it, let us introduce the production of dark matter via freeze-out, as it is by far the most thoroughly understood way of thermal production of particle species (not just dark matter) in the early universe. To this end, it is useful to rewrite the Boltzmann equation in terms of the so-called yield $Y = n/s$, where n is the particle number density and s is the entropy density. Moreover, instead of spelling out the time dependence of all quantities, one often switches to the dimensionless variable $x = m_\chi/T$ instead, where the time-dependence is implicitly given by the time-dependence of the temperature T. Then, the Boltzmann equation (5.18) becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\lambda \frac{\langle \sigma_{\mathrm{eff}} v \rangle}{x^2} \left(Y^2 - Y_{\mathrm{eq}}^2 \right) , \tag{5.25}$$

where λ represents a set of constants that arises from the introduction of Y. Note that this non-linear differential equation (called the *Riccati* equation) does not have an exact closed analytic solution and has to be solved numerically in practice. The idea behind freeze-out production is now that the dark matter particle is in thermal equilibrium at early times, or equivalently that $Y \sim Y_{\mathrm{eq}}$. During this time, the yield can be approximated to be

$$Y = Y_{\mathrm{eq}} + \frac{x^2}{2\lambda \langle \sigma_{\mathrm{eff}} v \rangle} , \tag{5.26}$$

hence it is always strictly larger than the equilibrium yield Y_{eq}. Often, it is an excellent approximation to assume that $\langle \sigma_{\mathrm{eff}} v \rangle$ is constant in x. Then, one observes that the gap between Y and Y_{eq} grows with x^2 and that at some point the approximation that $Y \sim Y_{\mathrm{eq}}$ is no longer valid. The picture is then that the dark matter particle is able to maintain thermal equilibrium for some time until it is diluted so much that the $2 \to 2$ scattering becomes too inefficient to keep it in equilibrium with the thermal plasma. Let us assume that the dark matter particle falls out of thermal equilibrium at $x = x_{\mathrm{f}}$. Then, instead of being tied to Y_{eq}, the actual yield is much larger than the equilibrium yield and becomes constant after some time. In fact, one can show that $Y_\infty = c\, Y(x_{\mathrm{f}})$ for some $\mathcal{O}(1)$ constant c. The longer the dark matter particle can keep up thermal equilibrium, the lower is the value of $Y(x_{\mathrm{f}})$, because Y_{eq} has a negative slope throughout. However, as we have just seen, the dark matter can stay *longer* in thermal equilibrium for *larger* cross sections, as the thermally averaged cross section enters (5.26) with negative power. Hence, the main feature of freeze-out production is that *weaker* coupling leads to *larger* relic densities of a particle species.

5.2.3 Freeze-in production

If the dark matter particle is either so heavy or so weakly coupled that it never attains thermal equilibrium, its production via the freeze-out mechanism is not possible. However, as first demonstrated in [134], it can still be produced thermally in large quantities, via freeze-in, a process that largely differs from freeze-out production. The most notable point is that in contrast to freeze-out which takes place mostly during the radiation dominated phase of the expansion of the universe, the amount of dark matter produced via freeze-in depends on the specific dynamics of inflation and especially reheating as it takes place only at the highest temperatures. This dependence is more pronounced if the dark matter candidate is very heavy. Instead of starting with a non-zero value for the number density at early times, one assumes that $n = 0$ directly after inflation. Then, starting from the onset of reheating, the production of dark matter can take place. We will come back to the specific dependence on the reheating dynamics later in more detail. If we start with $n = 0$ after inflation, this means that the actual density in the freeze-in case is always much smaller than the equilibrium one, $n \ll n_{\rm eq}$. Then, the right hand side of the full Boltzmann equation (5.18) can be approximated to sufficient accuracy by neglecting n^2 compared to $n_{\rm eq}^2$, and hence the Boltzmann equation in this case becomes

$$\dot{n} + 3H(t)\, n \;=\; \langle \sigma_{\rm eff} v \rangle \, n_{\rm eq}^2 \,, \tag{5.27}$$

which will turn out to be much simpler to solve as it is a linear equation. Much like the yield helped solving the freeze-out case, one can proceed like in ref. [14] and simplify the discussion by introducing the dimensionless abundance

$$X \;=\; \frac{n a^3}{T_{\rm rh}^3} \,, \tag{5.28}$$

where a is the scale factor and $T_{\rm rh}$ is the reheating temperature. This parameterization proves its usefulness after noticing that

$$\frac{{\rm d}X}{{\rm d}t} \;=\; \frac{a^3}{T_{\rm rh}^3} \left(\dot{n} + 3H\, n\right) \;=\; \frac{a^3}{T_{\rm rh}^3} \langle \sigma_{\rm eff} v \rangle \, n_{\rm eq}^2 \,, \tag{5.29}$$

where we used the approximated Boltzmann equation (5.27). Moreover, if one has to take different phases of expansion into account, the time dependence of the temperature (and hence the cross section and the Hubble rate) gets complicated soon. Here, it proves useful to replace the time derivative by a derivative with respect to the scale factor, so that ${\rm d}X/{\rm d}t$ can be replaced by $\dot{a}({\rm d}X/{\rm d}a)$ and the entire Boltzmann equation becomes

$$\frac{{\rm d}X}{{\rm d}a} \;=\; \frac{a^2}{H(a) T_{\rm rh}^3} \langle \sigma_{\rm eff} v \rangle \, n_{\rm eq}^2 \,. \tag{5.30}$$

This relation can be simply solved by separation of variables and yields after integration

$$X_\infty \;=\; \frac{1}{T_{\mathrm{rh}}^3}\int_1^\infty \mathrm{d}a \frac{a^2}{H(a)}\,\langle\sigma_{\mathrm{eff}}v\rangle\; n_{\mathrm{eq}}^2\;. \tag{5.31}$$

Here, we adopted the convention that the scale factor at the end of inflation can be chosen to be 1, and that the abundance of dark matter immediately after inflation vanishes. Note that by contrast to the freeze-out scenario, the final relic density is now larger for a greater cross section. Moreover, one has to be aware that the abundance X becomes comparable to the equilibrium abundance X_{eq} at some point during the integration (5.31). However, this crossing takes place at late times at which the thermal production is already inefficient and $X(a)$ is practically constant. In order to compare to the observed dark matter relic density $\Omega_X h^2 = 0.12$ [50], one can use (cf. [135])

$$X_\infty^{\mathrm{crit.}} \;=\; 0.29\cdot 10^{-5}\cdot\frac{\mathrm{GeV}}{m_\chi}\cdot\Omega_X h^2\;, \tag{5.32}$$

where we assumed that the massless matter content of the thermal bath is the three-generation MSSM (with three right-handed neutrinos). Hence, for a GUT scale dark matter particle ($m_\chi \sim 10^{16}\,\mathrm{GeV}$), the critical abundance is of order 10^{-23}. It has been shown in various instances how this scenario can indeed yield the correct relic abundance, even for particles with masses up to the Planck scale and highly suppressed couplings.

5.3 Stable dark matter at the string scale

We now turn our attention to massive string states that are stabilized by a discrete symmetry and hence can in principle give rise to a dark matter candidate. To this end, we first introduce the stabilizing symmetry that arises from stringy selection rules and is ultimately the result of the geometry of the underlying orbifold.

5.3.1 Non-contractible cycles and stable particles

In order to construct a stable dark matter candidate, we first determine which topological properties the orbifold needs to fulfill. In field theory discussions, often the so-called Kaluza–Klein parity is cited as a stabilizing symmetry without further specification of the geometry. Typically, it is argued, that a field with an odd Kaluza–Klein number cannot decay into fields with an even Kaluza–Klein number, leading hence to a $\mathbb{Z}_2$ symmetry with charge

$$q_{\mathrm{KK}} \;=\; (-1)^{n_{\mathrm{KK}}}\;, \tag{5.33}$$

where n_{KK} is the KK number of the field. As this symmetry survives the orbifold boundary conditions in most cases, it is then typically assumed that the stabilizing nature of this symmetry persists even when brane fields are introduced. It is instructive to study under which circumstances this is actually the case. Let us start with a very simple non-working example that, in spite of its simplicity, exhibits all interesting features and lays out a route to a possible solution. Consider the $\mathbb{S}^1/\mathbb{Z}_2$ orbifold whose space group $\mathcal{S}$ is generated by the constructing elements

$$\mathcal{S} \;=\; \langle(\mathbb{1}\,|\,v)\,,(\theta\,|\,0)\rangle\ , \tag{5.34}$$

where v is some arbitrary lattice vector and the twist θ acts as $\theta : v \mapsto -v$. This simple orbifold possesses two conjugacy classes of fixed points that correspond to brane fields Φ_1 and Φ_2. As representative space group elements, it is customary to choose

$$\Phi_1 \;\leftrightarrow\; (\theta\,|\,0) \tag{5.35}$$

$$\Phi_2 \;\leftrightarrow\; (\theta\,|\,v)\ . \tag{5.36}$$

In what follows, we are going to argue that an untwisted state χ with odd KK number can decay in these two brane fields via a stringy operator that is not obvious to appear in a pure field theory. In order to study the couplings, it is useful to go to a T-dual picture, where the KK state is instead a winding state with constructing element $(\mathbb{1}\,|\,v)$. Then, the decay of this state in the brane fields is allowed if the space group selection rule [149]

$$[(\mathbb{1}\,|\,v)]\cdot[(\theta\,|\,v)]\cdot[(\theta\,|\,0)] \;\supset\; (\mathbb{1}\,|\,0)\ , \tag{5.37}$$

is satisfied, where $[\,\cdot\,]$ denotes the conjugacy class of the respective space group element. Indeed, we find that this condition is fulfilled if we choose $(\mathbb{1}\,|\,{-v})$, $(\theta\,|\,v)$ and $(\theta\,|\,0)$ as representatives, and hence decay processes of the type

$$\chi \;\rightarrow\; \Phi_1\Phi_2 \tag{5.38}$$

are allowed, causing χ to be unstable despite its odd charge under KK parity. Note that it is in general hard to explain the appearance of such couplings in field theory because Φ_1 and Φ_2 are localized at different points in the orbifold. To see why this type of decay operator happens to be allowed, let us examine the condition (5.37) a little closer. We observe that the coupling of the winding state to the brane fields becomes possible because the cycle around which χ is winding can be generated by a combination of the constructing elements of the brane fields. In other words, we could have generated the space group $\mathcal{S}$ by using brane fields only

$$\mathcal{S} \;=\; \langle(\theta\,|\,0)\,,(\theta\,|\,v)\rangle\ . \tag{5.39}$$

In the language of orbifold topology, the subset of $\mathcal{S}$ generated by brane fields is called the fixed point set $\langle F\rangle$. We observe that the winding/KK states in the $\mathbb{S}^1/\mathbb{Z}_2$ orbifold are not stabilized because every available cycle on the orbifold is already contained in the fixed point set, which allows one to construct decay operators of winding/KK states to brane fields by fulfilling (5.37). More formally, this is because the fundamental group π_1, defined as

$$\pi_1 \;=\; \mathcal{S}/\langle F\rangle \tag{5.40}$$

is the trivial group for the $\mathbb{S}^1/\mathbb{Z}_2$ orbifold as $\mathcal{S} = \langle F\rangle$ in this case. Hence, if we are looking for topologically stable strings, we have to restrict ourselves to orbifolds (or in general, compact spaces) with non-trivial π_1. The non-trivial fundamental group π_1 is in one-to-one correspondence with a so-called freely-acting element (often called τ in the literature). As we have seen by studying the condition (5.37), any string with winding or Kaluza–Klein momentum in this freely-acting direction is guaranteed to be stabilized against decay into brane fields, leading to topologically stable strings as first noted in [138]. In fact, it has been shown in a field-theoretic setting that orbifold with this property exhibit an interesting phenomenology [150], where it is referred to as non-local gauge symmetry breaking. The construction of MSSM-like string models on this type of geometry has been studied in [151, 152, 153, 154, 155, 156]. In the string setups, the appealing phenomenology usually appears if one turns on a Wilson line in the freely-acting direction, which breaks some intermediate orbifold GUT to the standard model. This type of symmetry breaking naturally appears also in smooth Calabi–Yau compactifications [157], which makes the concept of stabilizing particles by winding them around freely-acting cycles even more general.

5.3.2 Stable dark matter in the $\mathbb{Z}_2 \times \mathbb{Z}_2$–5–1 orbifold

For definiteness, we will pick an explicit example geometry. To be specific, we consider a string model compactified on the so-called $\mathbb{Z}_2 \times \mathbb{Z}_2$–5–1 orbifold in the classification of ref. [33]. The $\mathbb{Z}_2 \times \mathbb{Z}_2$–5–1 orbifold can be constructed in three steps. First, one defines a set of basis vectors e_i, $i = 1, \ldots, 6$ that span a six-torus $\mathbb{T}^6$ that factorizes into three $\mathbb{T}^2$ tori. If one would now mod out isometries of this torus via discrete rotations, the resulting orbifold would have a trivial fundamental group π_1 and would hence be no good for our purposes. However, before we mod out any twist, we first define another lattice vector that is going to become the freely-acting cycle on the orbifold. We denote it by τ and choose it to be

$$\tau \;=\; \frac{1}{2}(e_2 + e_4 + e_6)\;, \tag{5.41}$$

which, when it is added as a seventh lattice vector, leads to a non-factorizable orbifold lattice. In principle, one could use the additional lattice vector τ to replace one of the old lattice vectors e_2, e_4 or e_6. However, our discussion is simpler if we keep all seven lattice vectors and restrict n_τ to $n_\tau \in \{0, 1\}$. Now, one can divide out a $\mathbb{Z}_2 \times \mathbb{Z}_2$ twist generated by

$$\theta \;\leftrightarrow\; v_\theta = \left(0, 0, \frac{1}{2}, -\frac{1}{2}\right) \tag{5.42}$$

$$\omega \;\leftrightarrow\; v_\omega = \left(0, \frac{1}{2}, -\frac{1}{2}, 0\right)\;. \tag{5.43}$$

Then, any element of the space group with $n_\tau = 1$ is *not* an element of the fixed point group and the resulting orbifold has fundamental group $\pi_1 = \mathbb{Z}_2$. This can be understood by noting that any such element necessarily has some winding in the fixed torus of a twist, because the τ-element has components in all three orbifold planes.

As discussed in detail in [139], the stringy selection rules give rise to a $\mathbb{Z}_4$ symmetry with charges

$$Q \;=\; n_\tau + 2(n_2 + n_4 + n_6) \mod 4\;, \tag{5.44}$$

such that $Q \in [0,1,2,3]$. Because any element with $n_\tau \neq 0$ is not in the fixed point set, the corresponding states are generically massive due to the winding in the fixed torus. Upon examining the charge assignment of the $\mathbb{Z}_4$ symmetry, one discovers that all massless states (including states from the standard model) have *even* $\mathbb{Z}_4$ charges, while some of the massive ones have odd charge. Hence, the $\mathbb{Z}_4$ possesses a subgroup $\mathbb{Z}_2^{\mathrm{DM}}$ with charges

$$Q_{\mathbb{Z}_2^{\mathrm{DM}}} \;=\; n_\tau \mod 2 \;. \tag{5.45}$$

This $\mathbb{Z}_2$ symmetry stabilizes any string state whose constructing element has $n_\tau = 1 \mod 2$ against decay into massless particles. Hence, the lightest state with an odd $\mathbb{Z}_4$ charge (corresponding to a non-trivial $\mathbb{Z}_2^{\mathrm{DM}}$ charge) is completely stable as long as the discrete symmetry is intact. Therefore, we choose our dark matter candidate to be a winding string along the τ direction,[1] i.e. with constructing element

$$g_{\mathrm{DM}} \;=\; (\mathbb{1}\,|\,\tau) \;. \tag{5.46}$$

In fact, there is an entire class of stable particles that share this constructing element but differ in their choices for the KK momentum. We assume the dark matter candidate to be the lightest representative of this class. Because of the winding, the dark matter candidate is in general charged under some of the $\mathrm{E}_8 \times \mathrm{E}_8$ gauge bosons, depending on the choice for the Wilson line W_τ. It is possible to choose the Wilson lines such that the dark matter particle is a complete standard model singlet. We assume that this is the case in the remainder of this chapter. However, it is also possible to choose the Wilson line configuration such that the dark matter candidate is an SU(2) doublet with zero electric charge.[2] As we discuss in more detail in the appendix, the mass of the dark matter candidate lies between the GUT and the string scale, i.e. around 10^{16} GeV. In contrast to other winding strings, we find that it is hard to make the dark matter candidate much lighter than that without going to extreme (and unrealistic) points in parameter space.

5.4 Interactions between dark matter and the standard model

While we construct a stable dark matter candidate in the previous section, we now consider its couplings to other states, especially the standard model, in the theory. These couplings are crucial to ensure that the dark matter candidate is produced in sufficient quantities in the early universe.

As in many other instances, the dark matter production is expected to be dominated by $2 \to 2$ processes. In principle, one would now have to compute the relevant four-string amplitudes for any $\mathrm{DM}^2 \to \mathrm{SM}^2$ process. However, it is to be expected that the stringy nature of these couplings has only a subleading effect on the overall outcome. Therefore, we instead concentrate on the construction of an effective field theory that contains all

[1] It is certainly possible to construct a dark matter candidate with constructing element $g_{\mathrm{DM}} = \left(\theta^k\omega^\ell\,\middle|\,\tau\right)$. However, because this state would be localized, its couplings to the massless states (which can be localized and bulk states) are highly model-dependent and do not generalize very well to other geometries.

[2] In this case, one might refer to it as a "String-Scale Interacting Heavy Higgsino".

Figure 5.1: *Dark matter production mediated by the exchange of a field M.*

relevant fields and couplings, and captures all relevant properties of the model. As a result of the constraining $\mathbb{Z}_4$ symmetry, the most general coupling of dark matter to thermal bath particles looks like the diagram in the left panel in figure 5.1, where the blob is representative for any allowed operator. At tree level, this reduces to the exchange of mediator fields M, cf. the diagram in the right panel of figure 5.1, which is expected to give the dominant contribution. Notably, the full four-string amplitude allows for a similar (s-channel) factorization. One can now identify which string states are allowed as mediator fields, and then turn on the corresponding couplings in the field theory model.

The analysis of allowed string couplings shows that there are two classes of mediator fields, corresponding to their orbifold boundary conditions. To be specific, these classes correspond to solutions g_M to the conditions

$$[g_{\mathrm{DM}}] \cdot [g_{\mathrm{DM}}^{-1}] \cdot [g_M] \;\supset\; (\mathbb{1}\,|\,0) \tag{5.47}$$

$$[g_{\mathrm{SM}}] \cdot [g_{\mathrm{SM}}^{-1}] \cdot [g_M] \;\supset\; (\mathbb{1}\,|\,0)\ , \tag{5.48}$$

where we denoted the constructing elements of standard model matter with g_{SM}. These two conditions basically state whether or not the $\mathrm{SM}^2 - M$ and $\mathrm{DM}^2 - M$ vertices in the right panel of figure 5.1 are allowed. The solutions to this conditions are on the level of space group elements

1. *Bulk states* with trivial constructing element. One example for this class are gravitational interactions, where the exchanged particle is the graviton. This case has been studied extensively in the PIDM program [14, 135] and, as we will see, will give in our setup a contribution that is in general subdominant. Another notable representative of this class are (at the string level) massless gauge bosons. As we have chosen our dark matter candidate to be a standard model singlet, only gauge bosons from a hidden gauge group come into question here. The standard model couplings of these hidden sector interactions are severely restricted by experiment (i.e. the absence of additional long-range forces), so that these gauge bosons must be given a mass (e.g. through SSB) in any realistic low-energy model. Moreover, the hidden gauge bosons must couple both to the dark matter and the standard model, which makes their role as mediators rather model-dependent.[3] In what follows, we

[3] Given our choice for the dark matter candidate, it is a lot easier to construct a full string model where

hence assume that hidden sector gauge interactions play no role for the production of our superheavy dark matter candidate.

2. *Winding states* with a certain winding along such that the couplings to the dark matter candidate and to the standard model are allowed by stringy selection rules. This possibility opens up once (part of) the standard model is localized, i.e. $g_{\text{SM}} = \left(\theta^k \omega^\ell \,\middle|\, v\right)$ for some lattice vector v. As they are winding, the mediators are very massive at generic points in the orbifold moduli space. However, depending on their choice of KK momenta, they can also be massless at special values for the radii, while (if no special choice is made) the upper bound for their mass lies a little below twice the mass of the dark matter candidate. Unlike the hidden sector gauge bosons, the winding mediators are generically present, and we will base all forthcoming considerations on these states.

The string construction allows the winding mediators to come either in $\mathcal{N} = 1$ chiral or vector multiplets, and we will consider both options in our field theory model. Moreover, there are actually three independent solutions to the stringy selection rules that couple to different subsets of localized standard model matter, which we will (for simplicity) treat as one and the same field. As both the dark matter candidate and the mediator are massive, we know that their CP partners must exist, too, in order to write down a mass term in the superpotential. As the mediator is a complete gauge singlet, and it has $\mathbb{Z}_4$ charge 0, we assume that it is its own mass partner, while we have to introduce a second field for the dark matter candidate. Table 5.1 shows all relevant fields for the model. We now turn our attention to the couplings of these fields to each other. As we work with an $\mathcal{N} = 1$ supersymmetric theory, we have to distinguish between couplings arising from the Kähler and the superpotential.

	superfield	type of strings	$\mathbb{Z}_4$ charge
SM	$\Phi_i = (f_i, \tilde{f}_i)$	localized	0 or 2
DM	$\Phi_{\text{DM}} = (\chi, \varphi)$	τ-winding	1
	$\Phi'_{\text{DM}} = (\chi', \varphi')$	$-\tau$-winding	3
mediator	$\Phi_{\text{M}} = (\chi_{\text{M}}, M)$	winding	0
	$V^{(\text{M})} = (V_\mu, \lambda)$	winding	0

Table 5.1: *Summary of the relevant (on-shell) fields for the effective field theory model, and their origin in string theory.* Φ'_{DM} *denotes the mass partner of the dark matter multiplet* Φ_{DM}*. We omit the auxiliary fields.*

there is no hidden sector gauge boson that couples to both dark matter and the SM, than constructing a model where a hidden sector gauge boson can act as mediator.

5.4.1 Kähler potential terms

In order to determine the possible couplings of the dark matter candidate, we start with the most general Kähler potentials for both the standard model and the dark matter sector. For the standard model, we find

$$K_{\mathrm{SM}} \supset \Phi_i^\dagger \left[\mathrm{e}^{2g_1 V^{(\mathrm{M})}} + \frac{g_1'}{\Lambda}\left(\Phi_{\mathrm{M}} + \Phi_{\mathrm{M}}^\dagger\right) + \frac{\mathrm{i}g_1''}{\Lambda}\left(\Phi_{\mathrm{M}} - \Phi_{\mathrm{M}}^\dagger\right)\right] \Phi_i \,. \tag{5.49}$$

As one can see, all allowed couplings except for that of the vector multiplet $V^{(\mathrm{M})} = (V_\mu, \lambda)$ with coupling g_1 are non-renormalizable, and therefore only the coupling to the vector multiplet will be considered in what follows. The vector superfield corresponds to a massive U(1). As all couplings of the standard model to the chiral superfield are ruled out, we also consider only the vector superfield $V^{(\mathrm{M})}$ for the coupling to the dark matter candidates Φ_{DM} and Φ'_{DM} in the dark sector Kähler potential

$$K_{\mathrm{DM}} \supset \Phi_{\mathrm{DM}}^\dagger \,\mathrm{e}^{2g_2 V^{(\mathrm{M})}} \Phi_{\mathrm{DM}} + \Phi_{\mathrm{DM}}'^\dagger \,\mathrm{e}^{-2g_2 V^{(\mathrm{M})}} \Phi'_{\mathrm{DM}} \,. \tag{5.50}$$

If we parameterize the SM chiral multiplets as in table 5.1 as $\Phi_i = (f_i, \tilde{f}_i)$ and the dark matter multiplets as $\Phi_{\mathrm{DM}} = (\chi, \varphi)$ and $\Phi'_{\mathrm{DM}} = (\chi', \varphi')$, the relevant Lagrangian for the $2 \to 2$ production of dark matter from the D-terms of the Kähler potentials in terms of the component fields reads

$$\mathcal{L} \supset K_{\mathrm{SM}}\Big|_D + K_{\mathrm{DM}}\Big|_D \tag{5.51}$$

$$\supset g_1 V_\mu^{(\mathrm{M})} \left[\left(\overline{f}_i \overline{\sigma}^\mu f_i + 2\mathrm{i}\tilde{f}_i^\dagger \partial_\mu \tilde{f}_i\right) + g_2\left(\overline{\chi}\,\overline{\sigma}^\mu \chi + 2\mathrm{i}\varphi^\dagger \partial_\mu \varphi\right)\right]$$
$$+ \sqrt{2} g_1 \left(\tilde{f}_i \overline{\lambda}\,\overline{f}_i + \tilde{f}_i^\dagger \lambda f_i\right) + \sqrt{2} g_2 \left(\varphi \overline{\lambda}\overline{\chi} + \varphi^\dagger \lambda \chi\right) + \begin{pmatrix} \chi \leftrightarrow \chi' \\ \varphi \leftrightarrow \varphi' \\ g_2 \leftrightarrow -g_2 \end{pmatrix} . \tag{5.52}$$

Each of these terms gives rise to a three-point vertex that connects either two dark matter particles or two standard model fields with a mediator. Moreover, there exists also a four-scalar vertex that arises from the auxiliary field of the mediator. In particular, the relevant part of the Lagrangian that contains the auxiliary field D_{M} in the mediator multiplet $V^{(\mathrm{M})}$ reads

$$\mathcal{L}_{(D_{\mathrm{M}})} = \frac{1}{2} D_{\mathrm{M}}^2 + g_1 D_{\mathrm{M}} |\tilde{f}_i|^2 + g_2 D_{\mathrm{M}} |\varphi|^2 - g_2 D_{\mathrm{M}} |\varphi'|^2 + \ldots \,. \tag{5.53}$$

As usual, the auxiliary field D_{M} has algebraic equations of motion. If it is set on-shell, the Lagrangian yields

$$\mathcal{L}_{(D_{\mathrm{M}})} = -\frac{1}{2}\left(g_1 |\tilde{f}_i|^2 + g_2 |\varphi|^2 - g_2 |\varphi'|^2\right)^2 + \ldots \,, \tag{5.54}$$

and hence, we obtain a four-scalar vertex for the bosonic components of the dark matter and the standard model multiplets with a coupling $g_1 g_2$. Let us briefly discuss the conceivable range of values for the couplings g_1 and g_2. In supersymmetric gauge theories, each gauge coupling is given by a gauge kinetic function f. For example, in the case of the U(1) associated with the mediator field $V_\mu^{(\mathrm{M})}$ we have

$$f_{\mathrm{U}(1)} \; = \; S + \Delta_{\mathrm{U}(1)}(T_i, U_i) \;, \tag{5.55}$$

where S is the heterotic axio-dilaton and the threshold correction $\Delta_{\mathrm{U}(1)}$ is a stringy one-loop contribution [158] that is in general a complicated function of the geometric moduli T_i and U_i, see ref. [159]. In particular, the form of the threshold depends on the Wilson line configuration and the underlying lattice [160]. However, for the non-factorizable orbifold with arbitrary Wilson lines we are considering, the precise form of $\Delta_{\mathrm{U}(1)}$ is unknown. Still, we expect that by varying the geometric moduli, one can generate wide ranges of effective couplings for the mediator U(1). The couplings of the standard model gauge group follow a similar pattern

$$f_{G_{\mathrm{SM}},a} \; = \; S + \Delta_{G_{\mathrm{SM}},a}(T_i, U_i) \;, \tag{5.56}$$

where a labels the $\mathrm{SU}(3)_\mathrm{C}, \mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{U}(1)_\mathrm{Y}$ gauge factors in the standard model gauge group. However, the threshold corrections $\Delta_{G_{\mathrm{SM}},a}$ for the standard model have in general a different functional dependence on the geometric moduli than $\Delta_{\mathrm{U}(1)}$. Hence, it is conceivable that the mediator couplings can be varied without spoiling the unification of the standard model gauge couplings. This is always the case when varying the geometric moduli

$$\begin{pmatrix} T_i \\ U_i \end{pmatrix} \; \longmapsto \; \begin{pmatrix} T_i' \\ U_i' \end{pmatrix} \tag{5.57}$$

induces a large variation in the threshold corrections of the coupling of the mediator U(1), which can be formulated as

$$\Big|\Delta_{\mathrm{U}(1)}(T_i', U_i') - \Delta_{\mathrm{U}(1)}(T_i, U_i)\Big| \; \gg \; \Big|\Delta_{\mathrm{U}(1)}(T_i, U_i)\Big| \;, \tag{5.58}$$

but at the same time the threshold corrections for the standard model gauge couplings, that need to be unified at or around the traditional GUT scale remain practically constant

$$\Delta_{G_{\mathrm{SM}},a}(T_i', U_i') \; \sim \; \Delta_{G_{\mathrm{SM}},a}(T_i, U_i) \;. \tag{5.59}$$

Even without studying the threshold corrections for a general string theory compactification in detail, it seems conceivable that a large range of couplings for the massive U(1) can be generated by this mechanism. Hence, we treat the couplings g_1 and g_2 as free parameters of our model. As our results will show, extreme values for the couplings have indeed the power to make the model unrealistic. However, we will also see that the required cosmological parameters do not change drastically for wide ranges of the couplings. Therefore, we believe that the lack of a precise prediction for the values of the couplings does not affect the predictivity of our approach.

5.4.2 Superpotential terms

The couplings via the the Kähler potential terms are not the only way to couple the dark sector to the standard model (or, more generally, to the thermal bath). Specifically, one can write down couplings of the mediator to the standard model superpotential via Higgs portal and neutrino-portal-like terms. From a stringy point of view, these couplings can only exist if the Higgs field is localized, as otherwise the selection rules cannot be satisfied. Moreover, for the neutrino portal, the Higgs field has to live at the same fixed point as one of the three lepton doublets. In principle, it could live at a different fixed point in the same sector, but then the couplings would be suppressed exponentially by the Kähler modulus. At the same time, they would be forbidden if the Higgs doublets and all lepton doublets would live in different twisted sectors. Similarly, ternary superpotential couplings of the mediator to the dark matter candidate can be constructed. As in the Kähler potential case we make again the assumption that the mediator couples to all SM states with the same coupling constant. Then the terms containing the mediator in the corresponding superpotential read

$$\mathcal{W} = \mathcal{W}_{\mathrm{M}} + \mathcal{W}_{\mathrm{DM}} + \mathcal{W}_{\text{Higgs-portal}} + \mathcal{W}_{\text{neutrino-portal}} \tag{5.60}$$

$$= \frac{m_{\mathrm{M}}}{2}\,\Phi_{\mathrm{M}}^2 + \frac{\lambda_{\mathrm{M}}}{3}\,\Phi_{\mathrm{M}}^3 + \frac{m_{\mathrm{DM}}}{2}\,\Phi_{\mathrm{DM}}\,\Phi'_{\mathrm{DM}} + \lambda_{\mathrm{DM}}\,\Phi_{\mathrm{M}}\,\Phi_{\mathrm{DM}}\,\Phi'_{\mathrm{DM}} + \lambda_{\mathrm{H}}\,\Phi_{\mathrm{M}}\,\widehat{H}_u\,\widehat{H}_d + \lambda_{\mathrm{N}}\,\Phi_{\mathrm{M}}\,\widehat{H}_u\,\widehat{L}\ . \tag{5.61}$$

Here, we denote the SM Higgs superfields by $\widehat{H}_{u,d} = (\widetilde{H}_{u,d}, H_{u,d})$ the lepton doublet by $\widehat{L} = (\ell, \tilde{\ell})$. As in the Kähler potential case, we are interested in three-point interactions involving two dark matter or standard model states. Again, there appear also four-boson interactions if the auxiliary field of the mediator multiplet is set on-shell. If we collect only the relevant terms from the F terms of this choice for the superpotential, we find the following terms in the Lagrangian

$$\begin{aligned}\mathcal{L} \supset\ & - m_{\mathrm{M}} M^\dagger \left[\lambda_{\mathrm{DM}}\left(1 + \frac{m_{\mathrm{DM}}}{m_{\mathrm{M}}}\right)\varphi\varphi' + \lambda_{\mathrm{H}} H_u H_d + \lambda_{\mathrm{N}} H_u \tilde{\ell}\right] \\ & - \lambda_{\mathrm{DM}}\varphi^\dagger\varphi'^\dagger\left(\lambda_{\mathrm{H}} H_u H_d + \lambda_{\mathrm{N}} H_u\tilde{\ell}\right) + \lambda_{\mathrm{DM}}\left[\chi_{\mathrm{M}}\chi\varphi' + \chi_{\mathrm{M}}\varphi\chi' + M\chi\chi'\right] \\ & + \lambda_{\mathrm{H}}\left[\chi_{\mathrm{M}}\widetilde{H}_u H_d + \chi_{\mathrm{M}} H_u\widetilde{H}_d + M\widetilde{H}_u\widetilde{H}_d\right] + \lambda_{\mathrm{N}}\left[\chi_{\mathrm{M}}\widetilde{H}_u\tilde{\ell} + \chi_{\mathrm{M}} H_u\ell + M\widetilde{H}_u\ell\right] \\ & + \text{h.c.}\ ,\end{aligned} \tag{5.62}$$

where the first two terms arise from the scalar potential of the mediator multiplet. If the couplings $\lambda_{\mathrm{N,H}}$ are chosen to be of the same order of magnitude as the U(1) couplings $g_{1,2}$ in the Kähler potential, it turns out that the superpotential couplings happen to contribute numerically a little less to the dark matter production rate than the Kähler terms. As a result, the superpotential terms do not allow to probe a much larger portion of the parameter space and hence do not provide much insight. On the other hand, their existence is rather model-dependent (because of the need of a specific localization of the Higgses), and therefore we will ignore this possibility by putting the Higgs into the bulk (thereby forbidding all superpotential couplings for DM production) and concentrate on the Kähler terms only.

5.5 Dark matter production from Kähler potential terms

Let us now study the dark matter production via freeze-in through $2 \to 2$ scattering by Kähler potential terms. Using the dark matter–mediator and standard model–mediator couplings in the Lagrangian (5.51) and (5.54), at tree-level, we find the relevant channels for the non-gravitational interactions of dark matter with the standard model are given by the processes shown in fig.s 5.2–5.7. Note that we only show production channels for the dark matter multiplet Φ_{DM}, of course analogous diagrams exist also for its mass partner multiplet Φ'_{DM}.

Ultimately, we are interested in the total number of dark sector particles produced, i.e. in the number density

$$n \;=\; n_\chi + n_{\chi^\dagger} + n_\varphi + n_{\varphi^\dagger} + n_{\chi'} + n_{\chi'^\dagger} + n_{\varphi'} + n_{\varphi'^\dagger}\ , \tag{5.63}$$

and hence we can apply the methods introduced above to deal with the coannihilations e.g. in fig. 5.7. Considering these processes, the non-vanishing cross sections σ_{ij} for the production of dark sector states are given by

$$\sigma_{\chi\overline{\chi}} \;= \sigma_{\chi\overline{\chi}\to f_i\overline{f}_i} + \sigma_{\chi\overline{\chi}\to \tilde{f}_i\tilde{f}_i^\dagger} \tag{5.64}$$

$$\sigma_{\varphi\varphi^\dagger} \;= \sigma_{\varphi\varphi^\dagger\to f_i\overline{f}_i} + \sigma_{\varphi\varphi^\dagger\to \tilde{f}_i\tilde{f}_i^\dagger} \tag{5.65}$$

$$\sigma_{\chi\varphi^\dagger} \;= \sigma_{\chi\varphi^\dagger\to f_i\tilde{f}_i^\dagger} \tag{5.66}$$

$$\sigma_{\overline{\chi}\varphi} \;= \sigma_{\chi\varphi^\dagger}\ , \tag{5.67}$$

plus the corresponding terms for χ', φ'. Together with the condition that $\sigma_{ij} = \sigma_{ji}$, these are all non-zero contributions to the total cross section. With these preparations in place, one can perform the thermal averaging eq. (5.20) numerically (cf. figure 5.8). Using this result for the effective thermally averaged cross section, one can observe that the dark matter candidate is too heavy to attain thermal equilibrium even for largish couplings as the reaction rate

$$\Gamma_{\mathrm{DM}} \;=\; \langle \sigma_{\mathrm{eff}} v\rangle\, n \tag{5.68}$$

is too small compared to any conceivable value of the Hubble rate in the very early universe. Therefore, one can only hope to produce the various particles in the dark sector via freeze-in.

Modeling (pre)heating. Unlike the freeze-out production, a freeze-in scenario depends on the chosen model for inflation, especially on the reheating at the end of inflation. Different scenarios for inflation and reheating lead to different maximal temperatures $T_{\max}$ during reheating. As one can see from the temperature dependence of the integrand for the relic abundance X_∞ (i.e. the cross section in figure 5.8 times the equilibrium density squared), the production of dark matter quickly becomes inefficient for temperatures below the dark matter mass scale. As a result, the maximal value for the relic abundance is obtained if the maximal temperature $T_{\max}$ is as high as possible, or, in other words, if

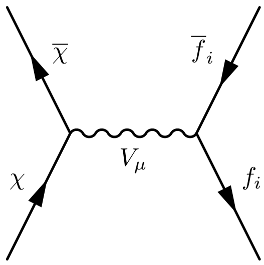


Figure 5.2: $\chi\overline{\chi} \leftrightarrow f_i\overline{f}_i$

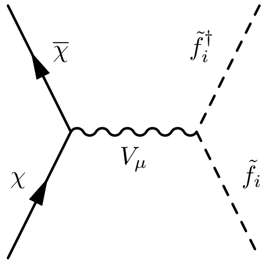


Figure 5.3: $\chi\overline{\chi} \leftrightarrow \tilde{f}_i^\dagger\tilde{f}_i$

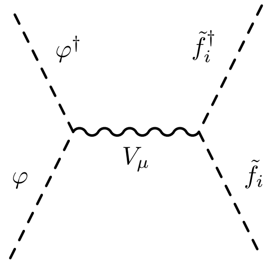


Figure 5.4: $\varphi^\dagger\varphi \leftrightarrow \tilde{f}_i^\dagger\tilde{f}_i$

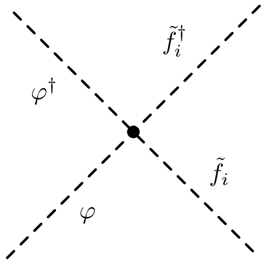


Figure 5.5: $\varphi^\dagger\varphi \leftrightarrow \tilde{f}_i^\dagger\tilde{f}_i$

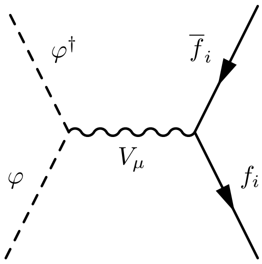


Figure 5.6: $\varphi^\dagger\varphi \leftrightarrow f_i\overline{f}_i$

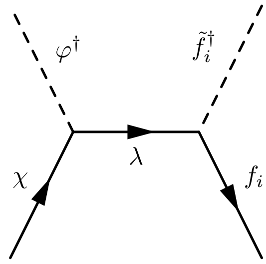


Figure 5.7: $\varphi^\dagger\chi \leftrightarrow \tilde{f}_i^\dagger f_i$

reheating phase after inflation is as short as possible. Realizing a near-instantaneous reheating requires

$$\frac{H_i}{\Gamma} \;\sim\; 1 \,, \tag{5.69}$$

where H_i is the Hubble rate at the end of inflation (i.e. just before reheating starts) and Γ denotes the inflaton decay rate. If this instantaneous scenario is realized, the reheating temperature $T_{\rm rh}$ (i.e. the temperature at the end of reheating) and the highest temperature attained during reheating $T_{\rm max}$ coincide and are given in terms of the Planck mass and the Hubble rate H_i by

$$T_{\rm rh} \;\approx\; 0.25\sqrt{m_{\rm Pl}\, H_i} \,. \tag{5.70}$$

The most straightforward way to achieve a near-instantaneous reheating is by employing non perturbative scenarios [161]. There, even Planck scale particles may be produced non-thermally during (pre)heating, a feature that is absent in perturbative models for reheating. Moreover, these scenarios are certainly not the most generic ones. Instead, we will assume that a near-instantaneous reheating is achieved perturbatively (along the lines of [135], where it is shown that this can indeed be realized), which is followed by a radiation-dominated phase during which the thermal production of dark matter takes place. Among all perturbative scenarios, the near-instantaneous scenario sets an upper limit on the amount of thermally produced dark matter for a given Hubble rate after inflation H_i. On the other hand, this can yield a lower bound on the Hubble rate H_i needed in order to explain the observed relic density $\Omega_X h^2$ by our dark matter candidate only.

While a certain value for the Hubble rate is needed to produce enough dark matter, the non-observation of tensor modes in the cosmic microwave background (CMB) by the

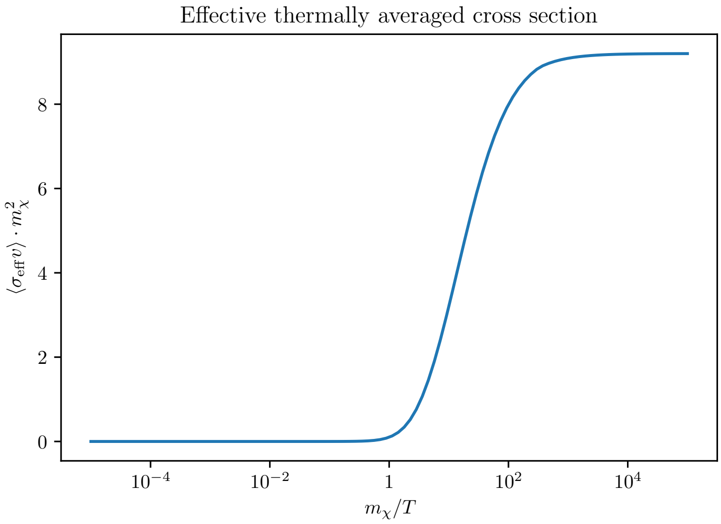


Figure 5.8: *The effective thermally averaged cross section for the* $2 \to 2$ *production of dark matter, for a mediator mass* $m_{\text{M}} = 1.8\, m_\chi$ *and all couplings set to unity. The cross section approaches a constant value for* $T \ll m_\chi$*. It should also be noted that the typical reheating temperatures lie at values up to* $0.7\, m_\chi$ *(corresponding to* $x = 1.4$*), so that the cross section cannot be approximated as being constant at the relevant temperature scales.*

Planck satellite combined with constraints from Bicep2 and Keck set an upper bound on the Hubble rate H_i. In particular, there is an upper bound on the tensor-to-scalar ratio $r < 0.056$ [162] using the latest available data. The resulting bound on the Hubble rate constrains the maximally allowed reheating temperature to lie a little below the GUT scale

$$T_{\text{rh}} \;<\; 5.8 \cdot 10^{-4} m_{\text{Pl}} \;\approx\; 7 \cdot 10^{15}\,\text{GeV}\,. \tag{5.71}$$

It is expected that this bound becomes much tighter in the next few years [163]. If we adopt the convention that the scale factor after inflation a_i can be set to 1, the temperature and the Hubble rate depend on the scale factor as

$$T(a) \;=\; \frac{T_{\text{rh}}}{a}\,, \quad H(a) \;=\; \frac{H_i}{a^2} \tag{5.72}$$

during the radiation-dominated phase after reheating. As a result, the relic abundance eq. (5.31) can be seen as a function $X_\infty(H_i, g_1 g_2, m_\chi, m_{\text{M}})$ of the Hubble rate at the end of inflation H_i, the (product of the) involved couplings $g_1 g_2$, the dark matter mass m_χ and the mediator mass m_{M}.

A "self-tuning" mechanism of freeze-in production. Before we dive into the explicit results, let us note that the Hubble rate H_i required to obtain the observed relic abundance remains almost constant even if vectorlike SM exotics are added due to the nature of freeze-in production. The required value of $X_\infty^{\text{crit.}}$ grows with the number of degrees of freedom in the thermal bath g_*, as they eventually produce entropy. On the other hand, more states in the thermal bath also lead to more production channels for the production of dark matter. Therefore, if the couplings of all contributing chiral multiplets are roughly equal, the Hubble rate needed to match the correct final abundance is determined by the contribution $x_\infty = X_\infty/N_\Phi$ of a single multiplet to the final abundance. Consequently, the critical contribution per chiral multiplet can be expressed as

$$x_\infty^{\text{crit.}} = \frac{g_*}{N_\Phi} R \,, \tag{5.73}$$

where g_* counts the number of degrees of freedom in the thermal bath at T_{rh} and N_Φ is the number of contributing chiral multiplets. For the case of the MSSM with three right handed neutrinos, we have $g_* = 240$ and $N_\Phi = 48$, leading to the value (5.32). Now, if a given model has vectorlike exotics, these values change. In particular, adding n_V vectorlike pairs of exotics leads to

$$g_* \mapsto g_* + 7.5\, n_V \quad \text{and} \quad N_\Phi \mapsto N_\Phi + 2\, n_V \,. \tag{5.74}$$

Hence, adding an arbitrary number of vectorlike exotics, i.e. sending $n_V \to \infty$, lowers $x_\infty^{\text{crit.}}$ by at most 25%. For the Hubble rate H_i, this entails only a small adjustment (up to a few percent) and hence our results are largely insensitive to the full particle content of a given model.

5.6 Results

Let us now discuss the resulting relic abundances that arise from our setup. In order to determine the final abundances X_∞, we solve the integral (5.31) numerically, with the aid of methods developed in [164]. Because we chose the simplified reheating scenario, the only cosmological parameter that appears in the final relic abundance is the Hubble rate H_i after inflation. As our model does not provide us with a preferred value for H_i (only with an upper bound), we will use the other parameters to infer the value H_i has to attain in order to obtain the observed relic density. On the particle physics side, the relic abundance predominantly depends on the dark matter mass m_χ and the product of the couplings $g_1 g_2$. In addition, there is also a mild dependence on the mediator mass m_{M}, however it turns out that this parameter has only a minor effect, especially for largish values of the couplings g_1 and g_2. As the dark matter mass lies necessarily around the GUT scale (at most a little above), the product of the couplings is the only free parameter of the model, and we will use it to make predictions on the required Hubble rate. In particular, we are interested in determining for which cases the required Hubble rate comes into conflict with the existing and projected bounds.

We display our results for two values of the dark matter mass in figure 5.9. In order to probe the allowed range of values for the Hubble rate, we vary the product of the couplings $g_1 g_2$ over a broad range. For each value of the product of the couplings, we determine the value of H_i that is required to produce the critical dark matter relic abundance $X_\infty^{\text{crit.}}$, eq. (5.32), that matches the observed dark matter relic density. We perform this scan using two values of the mediator mass in order to demonstrate the relative insensitivity of the results on this parameter.

Examining the trajectory of the critical Hubble rate, one first notices that the required value for H_i lies within the same order of magnitude for any value of the product of the couplings g_1g_2 greater than 10^{-8}, namely around 10^{11} to 10^{12} GeV for a GUT scale dark matter particle. It becomes apparent that the Hubble rate required to match the observed dark matter relic density exceeds the CMB bounds [162] if the couplings are made sufficiently small. However, if the couplings are that small, the thermal production of dark matter is in fact dominated by the exchange of gravitons (presented in ref. [14, 135]) rather than the stringy mediators, so that our initial assumption (namely that the gravitational component can be merely neglected) is violated. As a result, we can only probe the cosmological parameter space if the product of the couplings g_1g_2 is greater than 10^{-6}. Our results show that for any sensible dark matter mass, the present CMB bounds are out of reach because the corresponding value for H_i would imply an overproduction of dark matter through graviton exchange already. The current bounds may at most be probed with a less efficient reheating scenario. On the other hand, it is possible to probe the projected bounds with a dark matter mass of $m_\chi \sim 3 \cdot 10^{16}$GeV or larger. In this case, the smallest possible value for the couplings is $g_1g_2 = 10^{-4}$, which corresponds to a Hubble rate of $H_i \sim 10^{12}$ GeV. Note that for this choice of parameters, the contribution of graviton exchange is sufficiently suppressed so that it can be neglected. Moreover, for this mass range also the gravitational production through variations of the metric (cf. [131]) is inefficient and plays no role.

While the mediator mass has some influence on the precise required value of the Hubble rate at the end of inflation (and makes the bounds become tighter for lower values of m_{M}), we observe that the induces change in the required value for H_i is at most 20%, which—given the crudeness of all our assumptions—is not a large effect.

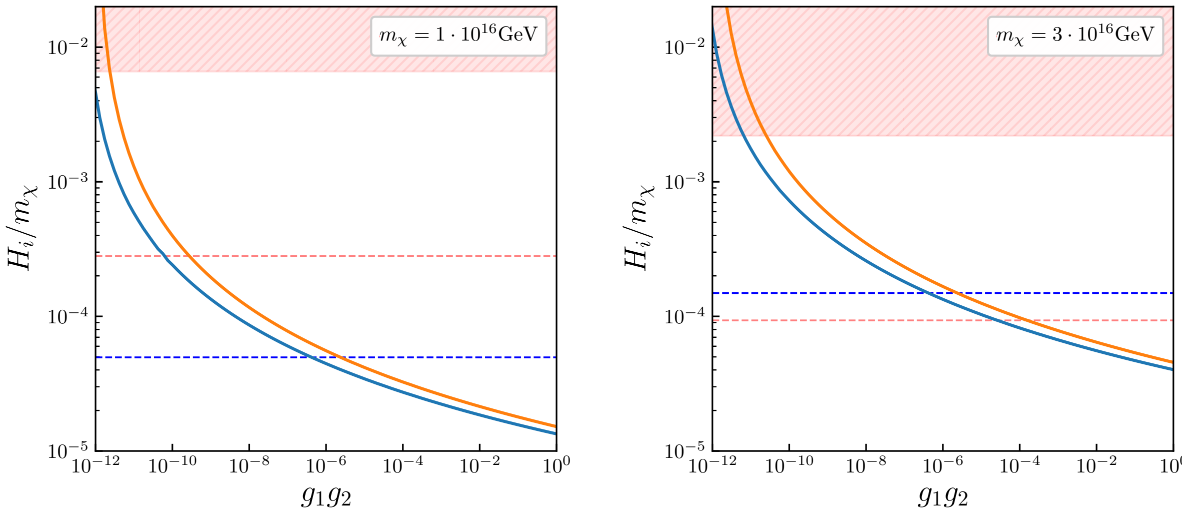


Figure 5.9: *Relation between the critical Hubble rate at the end of inflation and the product of the couplings. We display the critical Hubble rate for a dark matter mass of* 10^{16}GeV *(left) and* $3 \cdot 10^{16}$GeV *(right). The blue and orange curves indicate a mediator mass of* $1.9\,m_\chi$ *and* $1.0\,m_\chi$*, respectively. The red area at the top is excluded by the currently observed bound for the tensor-to-scalar ratio in the CMB. Additionally, the projected sensitivity of CMB experiments is shown as the dashed red line. The critical Hubble rate of graviton exchange* [14] *is indicated by the blue dashed line.*

5.7 Conclusions

The goal of this chapter is to demonstrate that once one assumes string theory to be a valid solution for quantum gravity, it can also solve other problems of fundamental physics along the way (without being specifically designed to do so). One of these problems that may be solved implicitly is a microscopic explanation for the existence of dark matter, which has been tackled in the broader context of string theory [165, 166, 167]. In this chapter, we build upon two interlinked observations regarding the properties of generic string models, namely that (i) the appearance of (heavy) standard model singlets with masses at or above the GUT scale is very common, and (ii) the topological properties of the compactification space (be it an orbifold or a Calabi–Yau) yield a stabilizing symmetry for at least a subset of these heavy singlets. Hence, as far as particle physics is concerned, these heavy singlets provide a viable dark matter candidate *if* there exists a mechanism that produces them in large enough quantities. Because it is generically heavy, our stringy dark matter candidate never attains thermal equilibrium and hence cannot be produced via the usual freeze-out process. However, building on the initial observations in ref. [134], it has been shown in ref. [14] that the thermal production of dark matter with masses almost up to the Planck scale can be realized via freeze-in production, even with suppressed couplings of the dark matter sector to the particles in the thermal bath. Hence, one has reasons to believe that the stringy dark matter candidate may be produced in sufficient quantities as well.

In order to prove and quantify these statements, we have to pick a specific model, which, however, is constructed in such a way that its generalization to a generic string model is more or less clear. Our specific model is based on an orbifold with non-trivial fundamental group π_1, and the dark matter candidate is then a winding string around a non-contractible cycle (whose existence is guaranteed by the non-trivial fundamental group). An analysis of the allowed string couplings reveals that this special class of winding strings is stabilized against decay. The stringy selection rules also reveal that the dark matter candidate couples to thermal bath particles (like, e.g. the standard model) both via gravity (as in the original PIDM scenario), but also via stringy mediators that are themselves winding strings and that are uncharged under the SM gauge symmetries. These stringy states give rise to $2 \to 2$ scattering amplitudes that have the potential to dominate over graviton exchange channels. In order to study the freeze-in production of dark matter via these channels, we avoid the in-depth calculation of the stringy $2 \to 2$ scattering amplitudes but instead build an effective $\mathcal{N} = 1$ supersymmetric field theory that captures all necessary couplings and charges. In this effective field theory, we find that contributions to the production of dark matter can arise from the Kähler potential and—if the localization of the standard model Higgs doublet is chosen right—also from superpotential terms. In order to keep our results most general, we opt for the more model-independent contributions from the Kähler potential. We find that the Kähler potential terms stem from a massive U(1) gauge symmetry under which both the dark matter candidate and the standard model matter are charged. Unlike for the dark matter candidate, there exist points in moduli space where the U(1) gauge bosons become massless. We observe that the U(1) gauge couplings depend on the precise values of the geometric moduli and can be varied almost at will without spoiling the unification of the standard model gauge couplings at the traditional GUT scale. Hence, we view these couplings as free parameters of the theory. If we insist that strictly the observed amount of dark matter is produced, a Hubble rate at the end of inflation of order 10^{12} GeV is necessary to do so for most of the sensible part of the parameter space. In principle, future CMB experiments are sensitive enough to restrict the possible range of the gauge couplings under the massive U(1) gauge symmetry by restricting the maximally allowed value of the Hubble rate after inflation due to the non-observation of tensor modes in the spectrum. Hence, that means that the paradigm of string scale interacting dark matter can in principle be ruled out as soon as future experiments become sensitive enough and place tighter bounds on the tensor-to-scalar ratio in the CMB. For now, we observe that the precise value of the gauge couplings only has a minor effect on the outcome, as long as these charges do not take extreme values. Moreover, also the (moduli-dependent) mass of the mediator fields has no qualitative influence on the overall outcome. This leads us to believe that our results are on a certain level model-independent. On the other hand, also the spectrum of our model, i.e. the possible appearance of vector-like standard model exotics is shown to affect the necessary value of the Hubble rate after inflation only up to a few percent. Together with the fact that the existence of non-contractible cycles is rather common in semi-realistic string models, we believe that our findings carry over very well to other explicit constructions, for example Calabi–Yau compactifications with freely-acting Wilson lines [168, 157, 169, 170].

Outlook. What we have presented here is a rough proof of concept that can be expanded in many ways, on the string theory, the particle physics and the cosmology side of the problem. Within the context of string theory, we restrict ourselves to models where the ten-dimensional $E_8 \times E_8$ gauge symmetry was broken *locally*, i.e. where the Wilson line associated with the non-contractible cycle is not responsible for the breaking of some intermediate GUT to the SM gauge group. This was necessary to keep the dark matter candidate a complete SM singlet. It would be interesting to see if it is possible to construct an explicit orbifold model where the gauge symmetry is broken *non-locally* without giving the dark matter candidate a charge under the standard model gauge group. Moreover, the stringy origin of the four-point amplitudes from which the $2 \to 2$ production of dark matter arises deserves attention. On the particle physics side, an interesting class of models may arise when the assumption of the dark matter candidate being a complete standard model singlet is dropped and it is placed in an electrically neutral SU(2) doublet instead. Then, it is interesting to study how the $2 \to 2$ production channels by the exchange of SU(2) and massive stringy U(1) gauge bosons interfere. By taking things a step further, one may even allow for a multi-component dark matter scenario where dark matter candidates that are SM singlets are produced alongside an SU(2) charged dark matter component. Another important generalization would be to allow for non-Abelian stringy mediators instead of simply a U(1). On the cosmological side, we make the simplification of near-instantaneous reheating. It is known that more realistic models (at least the more generic ones) allow for a finite duration of reheating, which leads to models where the maximal temperature reached during reheating is lower than in the instantaneous case. It is tempting to see how the allowed parameter space for the stringy gauge couplings is further reduced in these scenarios.

6

Discussion

In this thesis, we explore the physics of orbifold compactifications of heterotic string theory. Especially in chapters 4 and 5, we put an emphasis on *string-inspired* field theory models. These string-inspired models are designed such that they reflect the collective properties of wide ranges of *string-derived* models (i.e. low energy field theories of *one* particular superstring vacuum), and may therefore count as generic.

For many years, retaining $\mathcal{N} = 1$ supersymmetry in four dimensions has been the leading paradigm in string model building. However, in the light of the fact that current experiments have not yet provided any evidence for the existence of low-scale supersymmetry, the study of non-supersymmetric string theories has gained renewed interest in the past years [55]. Chapter 3 deals with the consequences of breaking supersymmetry already at the string level. Of particular interest in the context of heterotic constructions is the prediction for the cosmological constant at one loop. As it is proportional to the dilaton tadpole, keeping it small is strictly necessary to obtain a stable vacuum. While the cosmological constant (and with it the dilaton tadpole) vanishes identically in supersymmetric models, it may attain large values once SUSY is broken at the compactification scale. In this context, the logical question arises whether or not a SUSY-breaking compactification necessarily implies a non-vanishing cosmological constant, or whether there are certain models with no target-space supersymmetry but with a zero cosmological constant. At one-loop level, the relevant object to study is the modular integral over the one-loop string partition function. The partition function on an orbifold can be organized as a sum in terms of boundary conditions (the so-called sectors), and each sector in turn factorizes into a contribution coming from left- and rightmoving modes of the string. The vanishing of the cosmological constant can now be realized in various ways, ranging from a non-zero string partition function that integrates to zero over the domain of integration (which is the most involved case) to the most simplistic case, where the integrand vanishes sector per sector already. As it turns out, the more involved ways to obtain a vanishing partition function are difficult (if not impossible) to realize, and even if they existed, they would be extremely model-dependent, so that obtaining realistic particle physics would turn out to be very difficult. Hence, we concentrate on the simplest case where the partition function is supposed to vanish sector per sector. The starting point for our analysis is to study which of the factors of a sector can potentially vanish. By deriving a set of general Riemann identities for the number-theoretic functions that appear in the partition function, we are able to show that only the rightmover partition function can vanish in any sensible (heterotic) string model. In particular, we show that the vanishing of the partition function depends on how the discrete rotations on the orbifold act on target space spinors. Hence, the problem of vanishing or non-vanishing sectors in the partition function can in fact be understood by studying the group theoretical properties of the spinor embedding of the orbifold point group. In our study, we show that the partition function in a non-supersymmetric string model cannot vanish sector per sector, and hence it has to be assumed that the same is true for the cosmological constant and the dilaton tadpole. We provide a proof for this statement in two ways: First, by constructing the spinor embeddings for each admissible space group explicitly and then showing that none of the possible spinor representations fulfills the necessary conditions. A second, more abstract way of proving the no-go statement can be obtained by re-formulating the problem in terms of the representation theory of discrete groups and then showing that none of the point groups possesses any representation (no matter if it corresponds to an actual orbifold geometry) that has the desired properties. These properties mainly concern the branching of the (four-dimensional) spinor representation into the various $\mathbb{Z}_N$ subgroups of the point group. In order to break SUSY, the representation must not contain a trivial singlet of the point group, while it has to branch into at least one trivial singlet of each $\mathbb{Z}_N$ subgroup in order for the partition function to vanish. While we show the non-existence of representations with these properties by simply enumerating all four-dimensional representations of all admissible point groups, we also find evidence that the statement might hold for any discrete group, not just the ones that may act as orbifold point groups. The corresponding statement is formulated as a conjecture that, once proven, may prove useful in model building with discrete groups. On the other hand, the no-go result for vanishing partition functions in non-supersymmetric compactifications of the heterotic string may be interpreted to suggest that a generic, stable superstring vacuum necessarily descends from a supersymmetric theory, and hence motivates the existence of supersymmetry beyond the TeV scale in string-inspired models.

Chapter 4 is devoted to the study of the emergence of discrete symmetries as remnants of higher-dimensional gauge theories. While discrete symmetries are ubiquitous in bottom-up model building, there exist arguments from quantum gravity that any symmetry must be ultimately gauged [13]. Hence, any successful top-down approach should give rise to discrete remnants of broken gauge symmetries in its low-energy limit. As opposed to the unbroken *continuous* gauge symmetries, the rigorous derivation of the discrete remnants is not as straightforward, and it easily happens that a surviving symmetry is overlooked. In this chapter, we use a typical string-inspired setup, namely a higher-dimensional gauge theory that gets compactified to four dimensions on an orbifold. In this setting, the higher-dimensional gauge symmetry is broken by orbifold boundary conditions of the gauge fields. Whenever possible, we put an emphasis on the connection between this field-theoretic model and a full string-derived setup, which can be achieved by choosing the orbifold boundary conditions accordingly. In order to study the unbroken symmetries in the low-energy regime, we provide a general framework that can be used to identify discrete remnants of the broken higher-dimensional gauge symmetry. We are able to show that indeed the situation for discrete remnants is more complex than for continuous symmetries, because the discrete symmetries have to fulfill only a more liberal condition than the continuous ones. If we parametrize the gauge action of the orbifold boundary conditions by P, an element U of the higher-dimensional gauge group $\mathcal{G}$ survives the orbifolding if

$$[P, U] \in Z(\mathcal{G}) , \tag{6.1}$$

where $Z(\mathcal{G})$ denotes the center of the group $\mathcal{G}$. Notably, the usual conditions only capture those transformations U that strictly commute with the boundary conditions and hence leave many surviving transformations aside. We demonstrate the application of this condition in various setups. There, we observe in how far our condition gives rise to discrete symmetries that can be useful in the context of CP and flavor model building. Along these lines, we observe that finding solutions to our condition can be arbitrarily difficult if the group $\mathcal{G}$ is large and the boundary condition P does not take a particularly simple form, because in general one has to check an infinite number of candidate transformations. However, we provide an ansatz to reduce the set of possibly surviving transformations to a finite number of candidates by making use of the Weyl reflections in the root lattice of the group $\mathcal{G}$. In particular, we show that every discrete remnant symmetry found in our examples can be traced back to an element of the Weyl group of $\mathcal{G}$ that is no longer an element of the Weyl group of the unbroken low-energy continuous gauge symmetry. While our general approach is independent of the matter content of the theory, we shortly comment on the inclusion of matter in our framework, again in such a way that the connection to a full-fledged string construction is clear. All in all, our approach provides the model building community with a powerful tool to construct realistic top-down models of flavor and CP, where both the connection to the UV-complete theory (i.e. string theory) and bottom-up models are straightforward to perform.

Finally, the topic of chapter 5 is to explore how a generic string model can give rise to a viable candidate for dark matter. In our approach, we make use of the fact that heavy standard model singlets frequently appear in generic string models. Moreover, large classes of compactification spaces M (in particular Calabi–Yau manifolds, but also certain orbifolds) automatically guarantee the stability of these singlets. This stability originates from a discrete $\mathbb{Z}_N$ symmetry which in turn is the result of certain topological properties of the compactification space, namely that the so-called fundamental group $\pi_1(M)$ is non-trivial. Hence, the singlet becomes a natural candidate for dark matter. However, as this dark matter candidate has masses at or above the GUT scale, it is not clear a priori whether it can be produced in sufficient quantities in order to explain the dark matter relic density observed today. In particular, the high mass prevents this dark matter candidate from ever attaining thermal equilibrium, so that its production via the traditional freeze-out mechanism is infeasible. On the other hand, it has been demonstrated previously that—given the right conditions in the early universe—the production of massive dark matter candidates with masses almost up to the Planck scale seems possible without them ever being in thermal equilibrium, via freeze-in production, as shown in particular in the context of Planckian Interacting Dark Matter (PIDM) [14]. While the PIDM program only takes the exchange of gravitons as mediators into account, other channels may appear in a string theory setup. In order to study these additional contributions (and their relative importance compared to graviton exchange), we picked an explicit model derived from heterotic strings on a special class of orbifolds. In this explicit model, we realize the stable dark matter candidate as a winding string that winds around a non-contractible cycle on the orbifold. If the thermal bath states are taken to be brane fields, we show that different stringy couplings of the dark matter candidate to the standard model fields are present, ranging from gravitons and hidden gauge bosons to purely stringy couplings via the exchange of other heavy string states. The latter ones are shown to dominate over graviton exchange in the absence of hidden gauge bosons under which *both* the DM candidate and the SM fields are charged, which we take as the generic case. We make use of the analysis of the couplings allowed by string theory in order to build an effective $\mathcal{N} = 1$ supersymmetric field theory that captures all relevant fields and their couplings. In this string-inspired field theory, we then analyze the thermal production of dark matter through $2 \to 2$ scattering via freeze-in. While there can be contributions from the superpotential, we find that the contributions from the Kähler potential terms are more generic, and furthermore that the superpotential terms do not provide a greater insight and we therefore consider only the Kähler terms in the remainder of our analysis. These Kähler terms are then shown to correspond to the exchange of a massive U(1) gauge boson, that is massive at generic points in the moduli space, but may become massless at special points. We find that the relevant *particle physics* parameters of our model are then only the mass of the dark matter candidate and mediator, which are both functions of the geometric moduli, as well as the couplings of the dark matter candidate and the standard model fields to the massive U(1). While the dark matter mass is shown to lie around the GUT scale, we demonstrate that there is no prediction for the couplings from first principles, i.e. they can be varied almost at will without spoiling the unification of the standard model gauge couplings. These particle physics parameters can be used to infer the necessary values of the *cosmological* parameters, which—if we make the simplifying assumption of a near-instantaneous reheating scenario—consist solely of the Hubble rate at the end of inflation. On the other hand, the Hubble rate after inflation is constrained by observations of the cosmic microwave background. In effect, if we insist that the stringy dark matter candidate makes up all of dark matter in the present universe, this allows us to directly probe the string theory parameter space. We observe that it is indeed possible to achieve the observed dark matter relic density for a high enough Hubble rate, which makes our heavy string state a viable dark matter candidate. All in all, in contrast to other proposals for stringy dark matter candidates, our approach provides both a stable string state as a dark matter candidate *and* a set of stringy interactions through which the candidate is produced.

The results derived in this work allow for a better characterization of string-inspired models of particle physics: We can motivate the existence of supersymmetry even at high scales in order to retain computational control over the cosmological constant. In effect, supersymmetry may serve a purpose even if it does not address the hierarchy problem as effectively as it was once believed. We also provide a rather complete toolbox for top-down flavor model building with discrete gauge symmetries. Finally, we show that string-inspired models give rise to a vast playground for dark matter model building. It is interesting to compare the results of this thesis to the study of the heterotic orbifold landscape using techniques from machine learning. In [20], the intuition that phenomenologically viable models cluster together in a number of fertile islands in the landscape was verified using an autoencoder neural network. It certainly deserves to be studied if the fertile patches can be characterized by physical properties, e.g. the presence of promising flavor groups.

A

Jacobi theta-functions: Definitions and useful identities

The Dedekind eta- and the Jacobi theta-functions are the main building blocks of (heterotic) string partition functions. They arise from the study of number-theoretic problems (like, e.g. counting the number of integer partitions). This appendix lists their definitions and basic properties, especially their behavior under modular transformations. For more references see e.g. [171, 172, 173]. The Jacobi theta-functions are defined as

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,z) \;=\; \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2}\mathrm{e}^{2\pi\mathrm{i}(n+a)\,(b+z)}\;, \tag{A.1}$$

where $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$, $z\in\mathbb{C}$ and a,b are called characteristics. Based on these definitions one observes that

$$\vartheta\begin{bmatrix}a+1\\b\end{bmatrix} = \vartheta\begin{bmatrix}a\\b\end{bmatrix} \tag{A.2}$$

$$\vartheta\begin{bmatrix}a\\b+1\end{bmatrix} = \mathrm{e}^{2\pi\mathrm{i}a}\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}\;. \tag{A.3}$$

Furthermore, one can trade a non-zero lower characteristic for a shift in the variable z

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,z) \;=\; \vartheta\begin{bmatrix}a\\0\end{bmatrix}(\tau,z+b)\;. \tag{A.4}$$

In our string partition functions, we can usually set $z=0$ and in this case simply write $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau,0)\equiv\vartheta\begin{bmatrix}a\\b\end{bmatrix}(\tau)$. The theta functions quite often appear together with Dedekind eta-functions. In this case, one has the very useful identity

$$\frac{\vartheta\begin{bmatrix}\alpha\\\alpha'\end{bmatrix}(\tau)}{\eta(\tau)} \;=\; \mathrm{e}^{2\pi\mathrm{i}\alpha\alpha'}\,q^{\frac{\alpha^2}{2}-\frac{1}{24}}\prod_{n=0}^{\infty}\left(1+q^{n+\alpha-\frac{1}{2}}\mathrm{e}^{2\pi\mathrm{i}\alpha'}\right)\left(1+q^{n-\alpha-\frac{1}{2}}\mathrm{e}^{-2\pi\mathrm{i}\alpha'}\right)\;. \tag{A.5}$$

Moreover, characteristics of ten dimensional string partition functions are always $\alpha, \alpha' \in \{0, \frac{1}{2}\}$ and one defines

$$\vartheta_1(\tau) = \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \mathrm{e}^{2\pi\mathrm{i}(n+\frac{1}{2})\frac{1}{2}} = 0 \tag{A.6}$$

$$\vartheta_2(\tau) = \vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} \tag{A.7}$$

$$\vartheta_3(\tau) = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \tag{A.8}$$

$$\vartheta_4(\tau) = \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(\tau,0) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \mathrm{e}^{2\pi\mathrm{i}\frac{n}{2}} = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} (-1)^n \ . \tag{A.9}$$

In terms of these functions, there exist two useful identities. The first one is the "aequatio identica satis abstrusa" [174], the abstruse identity by Jacobi

$$\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 = 0 \ , \tag{A.10}$$

which in our string theory application translates to a vanishing rightmover partition function in ten-dimensional supersymmetric theories. The second one is the so-called triple product identity

$$\vartheta_2 \, \vartheta_3 \, \vartheta_4 = 2\,\eta^3 \ , \tag{A.11}$$

which obviously identifies the Dedekind eta-function as a product of theta-functions. Under modular transformations, one finds that the Dedekind eta-function transforms as

$$\eta \xrightarrow{T} \mathrm{e}^{\frac{\mathrm{i}\pi}{12}}\eta \qquad\qquad \eta \xrightarrow{S} \sqrt{-\mathrm{i}\tau}\,\eta \tag{A.12}$$

and the Jacobi theta-functions as

$$\vartheta_2 \xrightarrow{T} \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\vartheta_2 \qquad\qquad \vartheta_2 \xrightarrow{S} \sqrt{-\mathrm{i}\tau}\,\vartheta_4 \tag{A.13}$$

$$\vartheta_3 \xrightarrow{T} \vartheta_4 \qquad\qquad \vartheta_3 \xrightarrow{S} \sqrt{-\mathrm{i}\tau}\,\vartheta_3 \tag{A.14}$$

$$\vartheta_4 \xrightarrow{T} \vartheta_3 \qquad\qquad \vartheta_4 \xrightarrow{S} \sqrt{-\mathrm{i}\tau}\,\vartheta_2 \ . \tag{A.15}$$

B

A more general Riemann identity

In this appendix, we derive a generalization to the Riemann identity (3.38). In particular, we study products of d Jacobi theta-functions (corresponding to d complex fermions in physics applications), where d is taken to be an even number. We assume that the d fermions obey $\mathbb{Z}_{d/2}$ spin structures, i.e. they show non-trivial boundary conditions parametrized by a $\mathbb{Z}_{d/2}$ action. What we will show here is that the modular invariant sum over $d/2$ products of d theta-functions can be recast to a single theta-function.

As in the special case with $d = 4$, we build our approach on an orthogonal matrix S

$$S^T \;=\; S \;, \qquad S^T S \;=\; \mathbb{1}_d \;. \tag{B.1}$$

which we insert in any appearing Euclidean inner product. A possible, systematic choice for S is

$$S \;=\; \frac{2}{d}\, w\, w^T - \mathbb{1}_d \;, \tag{B.2}$$

where w is a d-dimensional vector with non-zero integer entries that fulfills $w^2 = d$. The most straightforward choice is to take $w = e_d$, i.e. the d-dimensional vector with all entries set to one. Let us use the shorthand $\nu = d/2$ as it appears frequently. As we chose d to be even, ν is always an integer. In what follows, we are interested in summations over integer lattices. In that regard, it is useful to observe that a general integer vector n after multiplication with S is in general no longer integer. However, it can be split

$$\tilde{n} \;=\; S\, n \;=\; \frac{1}{\nu}\Big(\sum_i n_i\Big)\, e_d - n \;\stackrel{!}{=}\; \frac{t}{\nu}\, e_d + m \;, \tag{B.3}$$

into a part that is still an integer vector m plus a fractional part, in which we take $t = 0, \ldots, \nu - 1$. We observe that, because of the special form of S, m and t are related

$$e_d^T m + t \in \nu\, \mathbb{Z} \;. \tag{B.4}$$

In all that follows, we work with d-dimensional theta-functions that are products of the usual ones

$$\vartheta\begin{bmatrix} \tilde{\alpha} \\ \tilde{\alpha}' \end{bmatrix}_{(d)} \;=\; \prod_{i=0}^{d-1} \vartheta\begin{bmatrix} \tilde{\alpha}_i \\ \tilde{\alpha}'_i \end{bmatrix} \;. \tag{B.5}$$

We now rewrite this expression in terms of characteristics $\alpha = S\widetilde{\alpha}$ by exploiting the expression (B.3). We find

$$\vartheta\begin{bmatrix}\widetilde{\alpha}\\ \widetilde{\alpha}'\end{bmatrix}_{(d)} = \sum_{t=0}^{\nu-1}\sum_{m\in\mathbb{Z}^d}\delta_{\nu\mathbb{Z}}(e_d^T m + t)\, e^{2\pi\mathrm{i}\left\{\frac{\tau}{2}\left(m+\frac{t}{\nu}e_d-\alpha\right)^2+\left(z-\alpha'\right)^T\left(m+\frac{t}{\nu}e_d-\alpha\right)\right\}}\ , \tag{B.6}$$

where we enforce the constraint (B.4) by inserting an appropriate projector

$$\delta_{\nu\mathbb{Z}}(\beta) = \frac{1}{\nu}\sum_{t'=0}^{\nu-1} e^{2\pi\mathrm{i}\,\frac{t'}{\nu}\beta}\ . \tag{B.7}$$

By writing out the projector, we can rewrite the summand in terms of d-dimensional theta-functions again, which brings us to the desired connection

$$\vartheta\begin{bmatrix}\widetilde{\alpha}\\ \widetilde{\alpha}'\end{bmatrix}_{(d)} = \frac{1}{\nu}\sum_{t',t=0}^{\nu-1} e^{-2\pi\mathrm{i}\left(\frac{t't}{\nu}-\frac{t'}{\nu}e_d^T\alpha\right)}\,\vartheta\begin{bmatrix}\alpha-\frac{t}{\nu}\,e_d\\ \alpha'-\frac{t'}{\nu}\,e_d\end{bmatrix}_{(d)}\ . \tag{B.8}$$

The result can be brought in an even more useful form, namely by noticing that one can equivalently make the replacements $\alpha \mapsto \frac{1}{2}\,e_d - \alpha$ and $\widetilde{\alpha} \mapsto \frac{1}{2}\,e_d - \widetilde{\alpha}$. With this shifts in the characteristics, our final Riemann identity reads

$$\frac{1}{\nu}\sum_{t',t=0}^{\nu-1} e^{-2\pi\mathrm{i}\left\{\frac{t't}{\nu}+\frac{t'}{\nu}e_d^T\alpha\right\}}\,\vartheta\begin{bmatrix}\frac{1}{2}\,e_d-\frac{t}{\nu}\,e_d-\alpha\\ \frac{1}{2}\,e_d-\frac{t'}{\nu}\,e_d-\alpha'\end{bmatrix}_{(d)} = \vartheta\begin{bmatrix}\frac{1}{2}\,e_d-\widetilde{\alpha}\\ \frac{1}{2}\,e_d-\widetilde{\alpha}'\end{bmatrix}_{(d)}\ . \tag{B.9}$$

C

Group-theoretical appendix

C.1 Vector and spinor representations

In this appendix, we collect all necessary definitions for the construction of the vector representation $D_{\mathbf{v}}$, the spinor representation $D_{\mathbf{s}}$ and the embedding of the latter into SU(4).

The vector representation. The vector representation is generated by the $\mathfrak{so}(6)$ Lie algebra with generators

$$(J_{ij})_{kl} \;=\; \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} \quad i,j = 1,\ldots,6\;. \tag{C.1}$$

In principle it would be sufficient to restrict oneself to $i < j$ in order to obtain a basis for antisymmetric 6×6 matrices. In terms of these generators, the vector representation of a generic point group element θ is given by

$$D_{\mathbf{v}}(\theta) \;=\; \exp\left(\tfrac{1}{2}\,\omega(\theta)_{ij}\,J_{ij}\right)\;, \tag{C.2}$$

where the sum over i, j is implicit and the factor 1/2 accounts for double counting.

The spinor representation. For the spin embedding, we need the six-dimensional (Euclidean) Clifford algebra which consists of eight-by-eight matrices Γ_i, $i = 1,\ldots,6$. Together with the chirality operator $\tilde{\Gamma} = \mathrm{i}\,\Gamma_1\,\Gamma_2\ldots\Gamma_6$, they fulfill the following anticommutation relations

$$\left\{\Gamma_i, \Gamma_j\right\} \;=\; 2\,\delta_{ij}\,\mathbb{1}_8\;, \qquad \tilde{\Gamma}^2 \;=\; \mathbb{1}_8\;, \qquad \left\{\Gamma_i, \tilde{\Gamma}\right\} \;=\; 0\;. \tag{C.3}$$

For our purposes, we need the charge conjugation matrix C

$$C\,\Gamma_i\,C^{-1} \;=\; \Gamma_i^T\;, \qquad C\,\tilde{\Gamma}\,C^{-1} \;=\; -\,\tilde{\Gamma}^T\;, \qquad C^\dagger \;=\; -\,C^T \;=\; C\;. \tag{C.4}$$

Then, the spinor representation of an element θ of the point group is

$$D_{\mathbf{s}}(\theta) \;=\; \exp\left(\tfrac{1}{2}\,\omega_{ij}\;\Sigma_{ij}\right) \quad \text{with} \quad C\,D_{\mathbf{s}}(\theta)\,C^{-1} \;=\; D_{\mathbf{s}}(\theta^{-1})^T = D_{\mathbf{s}}(\theta)^*\;, \tag{C.5}$$

where the spin generators Σ_{ij} are defined as

$$\Sigma_{ij} \;=\; -\,\Sigma_{ji} \;=\; \tfrac{1}{4}\Big[\Gamma_i, \Gamma_j\Big] \;, \qquad C\,\Sigma_{ij}\,C^{-1} \;=\; -\,\Sigma_{ij}^T \;, \quad \text{and} \quad \mathrm{Tr}(\Sigma_{ij}) = 0 \;. \tag{C.6}$$

In detail, the relation between the vector and the spinor representation is

$$D_{\mathbf{s}}(\theta)^T\,C\Gamma_i\,D_{\mathbf{s}}(\theta) \;=\; \Big[D_{\mathbf{v}}(\theta)\Big]_{ij}\,C\Gamma_j \;. \tag{C.7}$$

While the spinor representation determines the vector representation without any ambiguity

$$\Big[D_{\mathbf{v}}(\theta)\Big]_{ij} \;=\; \tfrac{1}{8}\,\mathrm{Tr}\Big[D_{\mathbf{s}}(\theta^{-1})\,\Gamma_i\,D_{\mathbf{s}}(\theta)\,\Gamma_j\Big] \;, \tag{C.8}$$

one observes that $D_{\mathbf{s}}(\theta)$ and $-D_{\mathbf{s}}(\theta)$ give rise to the same vector representation, reflecting the double cover property.

Chiral spinor representations. The eight-dimensional spinor representation $D_{\mathbf{s}}$ still contains redundant degrees of freedom because it is reducible. Using projectors

$$P^{(\pm)} \;=\; \frac{\mathbb{1} \pm \tilde{\Gamma}}{2} \tag{C.9}$$

we can define irreducible chiral representations $D_{\mathbf{s}}^{(\pm)}(\theta) = P^{(\pm)} D_{\mathbf{s}}(\theta)$

$$D_{\mathbf{s}}^{(+)}(\theta) \;=\; \begin{pmatrix} D_{\mathbf{4}}(\theta) & 0 \\ 0 & 0 \end{pmatrix} \;, \qquad D_{\mathbf{s}}^{(-)}(\theta) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & D_{\bar{\mathbf{4}}}(\theta) \end{pmatrix} \;, \tag{C.10}$$

where

$$D_{\mathbf{4}}(\theta) \;=\; \exp\left(\tfrac{1}{2}\,\omega_{ij}\,\sigma_{ij}\right) \qquad D_{\bar{\mathbf{4}}}(\theta) \;=\; \exp\left(\tfrac{1}{2}\,\omega_{ij}\,\bar{\sigma}_{ij}\right) \tag{C.11}$$

are elements of SU(4). Here, the generators σ_{ij} are defined in terms of $\mathfrak{su}(4)$ generators as

$$\sigma_{ij} \;=\; \tfrac{1}{4}(\bar{\gamma}_i\gamma_j - \bar{\gamma}_j\gamma_i) \quad \text{and} \quad \bar{\sigma}_{ij} \;=\; \tfrac{1}{4}(\gamma_i\bar{\gamma}_j - \gamma_j\bar{\gamma}_i) \;. \tag{C.12}$$

We can work with either $D_{\mathbf{4}}(\theta)$ or $D_{\bar{\mathbf{4}}}(\theta)$. If we pick, $D_{\mathbf{4}}(\theta)$, its relation t the vector representation is

$$D_{\mathbf{4}}(\theta)^T c\gamma_i\,D_{\mathbf{4}}(\theta) = [D_{\mathbf{v}}(\theta)]_{ij}\,c\gamma_j \;, \tag{C.13}$$

which exhibits the same double cover properties as $D_{\mathbf{s}}$.

C.2 An example with non-isomorphic spinor and vector groups

In this section, we provide an explicit example for a spinor action that is not isomorphic to the action on vectors.

The quaternion group Q_8. The quaternion group has the following presentation

$$\langle \theta_1, \theta_2 \,|\, \theta_1^4 = \theta_2^4 = \mathbb{1}\,, \theta_2^2 = \theta_1^2\,, \theta_1\,\theta_2\,\theta_1 = \theta_2 \rangle\ . \tag{C.14}$$

The group Q_8 has eight elements in five conjugacy classes. If we introduce $\theta_3 = \theta_1\theta_2$ for notational simplicity, these conjugacy classes are

$$[\mathbb{1}] \;=\; \{\mathbb{1}\}\,,\quad [\theta_1] \;=\; \{\theta_1, \theta_1^3\}\,,\quad [\theta_2] \;=\; \{\theta_2, \theta_2^3\}\,,\quad [\theta_3] \;=\; \{\theta_3, \theta_3^3\}\,,\quad [\theta_1^2] \;=\; \{\theta_1^2\}\,. \tag{C.15}$$

The quaternion group has five irreducible representations. There is the trivial singlet representation $\mathbf{1}_{++}$ as well as three non-trivial singlets $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$, $\mathbf{1}_{--}$. Moreover, there is a doublet representation $\mathbf{2}$. It is noteworthy that the doublet representation is faithful, i.e. it generates a matrix group isomorphic to Q_8. The explicit matrices of the generators in the various representations can be chosen to be

$$D_{\mathbf{1}_{\text{AB}}}(\theta_1) \;=\; \text{A}\,1\ ,\ D_{\mathbf{1}_{\text{AB}}}(\theta_2) \;=\; \text{B}\,1\ ,\ D_{\mathbf{1}_{\text{AB}}}(\theta_3) \;=\; \text{A}\cdot\text{B}\,1,\ D_{\mathbf{2}}(\theta_a) \;=\; -\,\mathrm{i}\,\sigma_a\ , \tag{C.16}$$

where $\text{A}, \text{B} = \pm$ and σ_a are the Pauli matrices. For our example, we need the tensor products between the various irreps which read (see [175, 176] for a discussion of the two-times antisymmetrization)

$$\mathbf{1}_{\text{AB}} \otimes \mathbf{1}_{\text{CD}} \;=\; \mathbf{1}_{\text{A}\cdot\text{C}\ \text{B}\cdot\text{D}}\ ,\quad \mathbf{1}_{\text{AB}} \otimes \mathbf{2} \;=\; \mathbf{2}\ ,\quad \mathbf{2} \otimes \mathbf{2} \;=\; \bigoplus_{\text{A,B}=\pm} \mathbf{1}_{\text{AB}}\ ,\quad [\mathbf{2}]_2 \;=\; \mathbf{1}_{++}\ . \tag{C.17}$$

A $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with a Q_8 spinor action. In this example, we will show that an Abelian $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold can have an action on spinor space that is if fact a representation of Q_8 and therefore not isomorphic to the vector representation. We start by choosing the spinor action

$$D_{\mathbf{4}} \;=\; \mathbf{2} \oplus \mathbf{2}\,, \tag{C.18}$$

which can be realized by choosing the following matrices for the generators

$$D_{\mathbf{4}}(\theta_1) \;=\; \begin{pmatrix} -\mathrm{i}\sigma_1 & 0 \\ 0 & -\mathrm{i}\sigma_1 \end{pmatrix}\ ,\qquad D_{\mathbf{4}}(\theta_2) \;=\; \begin{pmatrix} -\mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}\ . \tag{C.19}$$

This representation is in SU(4) and hence is guaranteed to be a spinor representation. However, the analysis of the tensor product $\mathbf{6} = [\mathbf{4}]_2$ shows that the associated six-dimensional vector representation consists of Q_8 singlets only

$$[\mathbf{2} \oplus \mathbf{2}']_2 \;=\; [\mathbf{2}]_2 \oplus [\mathbf{2}']_2 \oplus (\mathbf{2} \otimes \mathbf{2}') \;=\; \mathbf{1}_{++} \oplus \mathbf{1}_{++} \oplus \mathbf{1}_{++} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{--}\ . \tag{C.20}$$

In effect, the representation matrices can be chosen to be

$$D_{\mathbf{v}}(\theta_1) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\ ,\qquad D_{\mathbf{v}}(\theta_2) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\ , \tag{C.21}$$

which do *not* generate a Q_8 but only a $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the spinor representation, the defining relations of Q_8 are realized as

$$D_{\mathbf{4}}(\theta_1)^2 \;=\; -\,\mathbb{1}_4\;, \quad D_{\mathbf{4}}(\theta_2)^2 \;=\; -\,\mathbb{1}_4\;, \quad D_{\mathbf{4}}(\theta_1)\,D_{\mathbf{4}}(\theta_2) \;=\; -\,D_{\mathbf{4}}(\theta_2)\,D_{\mathbf{4}}(\theta_1)\;, \tag{C.22}$$

which shows that $-\mathbb{1}_8 \in P_{\mathbf{s}}$ and hence all supersymmetries are broken. Note that the same relations look differently in the non-isomorphic vector representation

$$D_{\mathbf{v}}(\theta_1)^2 \;=\; \mathbb{1}_6\;, \quad D_{\mathbf{v}}(\theta_2)^2 \;=\; \mathbb{1}_6\;, \quad D_{\mathbf{v}}(\theta_1)\,D_{\mathbf{v}}(\theta_2) \;=\; D_{\mathbf{v}}(\theta_2)\,D_{\mathbf{v}}(\theta_1)\;. \tag{C.23}$$

which makes it clear where the two representations differ on a group-theoretical level.

D

D-parity in Pati–Salam from orbifolding

The goal of this appendix is to demonstrate the action of the D-parity derived in chapter 4 on an explicit representation of SO(4).

In section 4.4.1, we showed that among the unbroken continuous gauge symmetries, also a $\mathbb{Z}_2$ survives. In the SO(4) block, the D-parity acts by conjugation with the matrix

$$U_{\mathbb{Z}_2} = \mathrm{diag}(1,1,1,-1) \ . \tag{D.1}$$

we will now study the consequences of this transformation on an explicit representation, starting from the Lie algebra $\mathfrak{so}(4)$. The elements of the $\mathfrak{so}(4)$ algebra are generated by six antisymmetric matrices $M_i, i = 1,2,3$ and $N_i, i = 1,2,3$ which fulfill the following relations

$$[M_i, M_j] \ = \ \mathrm{i}\,\varepsilon_{ijk}\,M_k \ , \quad [N_i, N_j] \ = \ \mathrm{i}\,\varepsilon_{ijk}\,M_k \ , \quad [M_i, N_j] \ = \ \mathrm{i}\,\varepsilon_{ijk}\,N_k \ . \tag{D.2}$$

An explicit set of matrices that fulfills these relations is

$$M_1 \ = \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ M_2 \ = \ \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \ M_3 \ = \ \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{D.3a}$$

$$N_1 \ = \ \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \ , \ N_2 \ = \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} \ , \ N_3 \ = \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \ . \tag{D.3b}$$

As the $\mathfrak{so}(4)$ Lie algebra is the direct sum of two $\mathfrak{su}(2)$ Lie algebras, one can make the basis change $W_i^{\pm} := \frac{1}{2}\,(M_i \pm N_i)$, which yields the two orthogonal $\mathfrak{su}(2)$ algebras

$$\left[W_i^+, W_j^+\right] \ = \ \mathrm{i}\,\varepsilon_{ijk}\,W_k^+ \ , \quad \left[W_i^-, W_j^-\right] \ = \ \mathrm{i}\,\varepsilon_{ijk}\,W_k^- \ , \quad \left[W_i^+, W_j^-\right] \ = \ 0 \ , \tag{D.4}$$

by which we separate $\mathfrak{so}(4)$ into $\mathfrak{su}(2)_\mathrm{L} \oplus \mathfrak{su}(2)_\mathrm{R}$. However, one observes that the action of conjugation by $U_{\mathbb{Z}_2}$ is

$$W_i^+ \ \mapsto \ U_{\mathbb{Z}_2}\,W_i^+\,U_{\mathbb{Z}_2}^{-1} \ = \ W_i^- \ , \quad W_i^- \ \mapsto \ U_{\mathbb{Z}_2}\,W_i^-\,U_{\mathbb{Z}_2}^{-1} \ = \ W_i^+ \ . \tag{D.5}$$

This proves our claim that our remnant $\mathbb{Z}_2$ exchanges $\mathfrak{su}(2)_{\mathrm{L}}$ and $\mathfrak{su}(2)_{\mathrm{R}}$, and can hence be identified with the D-parity.

E

Massive U(1) gauge bosons from string theory

In this appendix, we study the realization of massive U(1) gauge bosons acting as mediators between a winding dark matter candidate and localized standard model fields in more depth. Following the discussion in [34, 149], a coupling needs to fulfill the following condition if it is allowed

$$(\mathbb{1}, 0) \;\in\; [g_1] \cdot [g_2] \cdot [g_3] \;, \tag{E.1}$$

where each $[g_i]$ is the conjugacy class of g_i. As discussed, the mediator needs to have trivial $\mathbb{Z}_4$ charge, which, for winding strings reduces to $n_2 + n_4 + n_6 = 0 \bmod 2$. Moreover, the mediators need to have $p_{\text{sh}} = 0$ for gauge invariance. We observe that there are several winding strings that fulfill the space group selection rule

1. There are winding strings that couple to both the dark matter candidate and localized SM states
   - On the dark matter side, we find that e.g. $\left(\mathbb{1} \,\middle|\, \frac{1}{2}(e_2 - e_4 - e_6)\right) \in [(\mathbb{1}\,|\,\tau)]$ and $\left(\mathbb{1} \,\middle|\, -\frac{1}{2}(e_2 + e_4 + e_6)\right) \in [(\mathbb{1}\,|\,-\tau)]$. Hence, $(\mathbb{1}\,|\,0) \in [(\mathbb{1}\,|\,\tau)] \cdot [(\mathbb{1}\,|\,-\tau)] \cdot (\mathbb{1}\,|\,e_4 + e_6)$ and the coupling is allowed.
   - On the standard model side, $(\mathbb{1}\,|\,-\tau)\,(\theta\,|\,0)\,(\mathbb{1}\,|\,\tau) = (\theta\,|\,-e_4 - e_6) \in [(\theta\,|\,0)]$. Hence, $(\mathbb{1}\,|\,0) \in (\mathbb{1}\,|\,e_4 + e_6) \cdot [(\theta\,|\,0)] \cdot [(\theta\,|\,0)]$, allowing the coupling.
2. Using the properties of the Wilson lines in this geometry [151], their local shift is a lattice vector, so that $p_{\text{sh}} = 0$ can be chosen.

We have just seen that there are winding heavy winding mediators (which can be chosen to have space-time spin 1) that couple both to the dark matter candidate and to localized strings in the θ-twisted sector.

In a field theory interpretation, these states are gauge bosons of a massive U(1) gauge symmetry. This can be understood by noting that for special values of the geometric moduli, these states can become massless, provided that the winding strings also carry appropriate Kaluza–Klein numbers. Therefore, one can compare these states very well to a gauge symmetry that is broken spontaneously if certain fields attain a non-trivial VEV. In our case, the gauge symmetry is broken by a VEV of the Kähler moduli. Following the discussion in [100], one can argue that both the dark matter candidate and localized SM matter should be charged under the massive U(1). The argument goes as follows: When the U(1) gauge bosons become massless at special points in moduli space, they become the ladder operators of a larger, non-Abelian gauge symmetry through symmetry enhancement. In a string theory setup, it is apparent that there exist winding and twisted strings that transform in non-trivial representation of the enhanced gauge group. Hence, in a generic setting, they will have also non-trivial charges under the U(1)s generated by the particular ladder operators.

Similar mediators also exist for the ω- and $\theta\omega$-twisted sector. In total, we find the winding numbers for the mediator states displayed in table E.1. Note that these winding strings in general carry KK momenta besides the winding, which makes them truly stringy states that cannot be explained in a purely field-theoretic setup.

sector of SM	θ	ω	$\theta\omega$
$g \in \mathcal{S}$ of mediator	$(\mathbb{1}\,\|\,e_4 + e_6)$	$(\mathbb{1}\,\|\,e_2 + e_6)$	$(\mathbb{1}\,\|\,e_2 + e_4)$

Table E.1: *Constructing elements of mediator strings fulfilling the space group selection rules. Note that the physical states usually also carry KK numbers.*

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