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Optimal Investment Strategies for Pension Funds

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Abstract

This doctoral thesis studies stochastic portfolio optimization problems with finite horizon in a complete continuous-time financial market model that consists of multiple asset classes, where a pension fund investor desires to maximize her expected utility assigned to the terminal wealth and the intertemporal consumption/pension or buffer rate. We consider various model extensions to traditional approaches such as a more realistic behavioral model for the investor's risk preferences, age-dependent risk aversion and the involvement of buffer mechanisms and pension adjustment rules. Moreover, economic interpretations and justifications of the proposed models are provided. The associated portfolio selection problems are solved by applying case-specific optimization methodologies; in particular we employ the Martingale method and Merton's approach, combined with suitable transformations of the optimization problems. The solutions in form of optimal quantitative dynamic asset allocation and consumption strategies as well as corresponding optimal replicating wealth processes are achieved for the flexible class of hyperbolic absolute risk aversion (HARA) utility functions. The relevance of the proposed models is illustrated in numerical optimization and simulation case studies which are carried out to elaborate on the characteristics and impact of the determined optimal investment strategies.

Zusammenfassung

Die vorliegende Dissertation befasst sich mit stochastischen Portfoliooptimierungsproblemen eines Pensionsfonds-Investors, welcher den erwarteten Nutzen seines Endvermögens und seines intertemporalen Konsums oder seiner Pensions-/Pufferrate maximieren möchte, in einem vollständigen, zeitstetigen und aus mehreren risikobehafteten Anlageklassen bestehenden Finanzmarktmodell. Die betrachteten verschiedenartigen Modelle erweitern bisher bekannte Ansätze hinsichtlich eines realitätsnäheren verhaltensgesteuerten Modells für die Risikopräferenzen von Investoren, altersabhängiger Risikoaversion sowie der Einbindung von Pufferungsmechnismen und Regelungen zur Pensionsanpassung. Darüber hinaus werden ökonomische Interpretationen und Begründungen der vorgeschlagenen Modelle diskutiert. Zur Lösung der zugehörigen Portfolioselektionsprobleme werden anwendungsfallspeizische Optimierungsmethoden verwendet; insbesondere werden die Martingalmethode und Merton's Methode, in Kombination mit passenden Transformationen, angewandt. Die Lösungen in Gestalt der optimalen quantitativen dynamischen Vermögensanlage- und Konsumstrategien sowie den zugehörigen replizierenden Vermögensprozessen werden für die flexible Klasse der „hyperbolic absolute risk aversion (HARA)" Nutzenfunktionen hergeleitet. Die Relevanz und Bedeutung der vorgeschlagenen Modelle werden durch numerische Optimierungs- und Simulationsstudien veranschaulicht, welche zum Zweck der Analyse der Eigenschaften und Auswirkungen der bestimmten optimalen Investmentstrategien durchgeführt werden.

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Contents

1 Introduction

We motivate in Section 1.1 the general topic and objective of this thesis and outline the relevance of the general research subjects carried out in Chapters 3–5. Furthermore, Section 1.2 provides an overview of the structure of this thesis. Section 1.3 contains a description of the applied optimization techniques on a structural basis.

1.1 Motivation and objective

Private pension insurance and generally pension funds are of a particular interest not only because of their economically important role for investors but also for the entire financial industry itself. The Organisation for Economic Co-operation and Development (OECD), with exemplary member countries Australia, Canada, France, Germany, Japan, Netherlands, Norway, United Kingdom and the United States of America, published in 2019 that "Pension assets accumulated through pension funds, pension insurance contracts and other retirement savings products amounted to USD 44.1 trillion at the end of 2018" (Organisation for Economic Co-operation and Development (2019)). According to Insitutional Investor (2019) and Willis Towers Watson (2019), the Government Pension Investment Fund of Japan with pension assets amounting to more than USD 1.3 trillion is the world's largest pension fund in 2018 in terms of total assets under management. Moreover, the European Central Bank (ECB) reported an amount of EUR 723.1 billion that corresponds to pension fund assets in Germany at the end of the year 2019 (European Central Bank (2020)). These large numbers of assets under management show the relevance of institutional pension funds. Meaningful pension fund strategies are therefore important and can generate significant benefits to customers by enhancing portfolio returns. Hensel et al. (1991) and Beath (2014) highlight the impact of the asset allocation decision on the performance of a pension fund. Empirical studies of the performance and the asset allocation dynamics of pension funds is provided by Blake et al. (1999) for the U.K., Beath (2014) for the U.S. and Bams et al. (2016a,b) for a global perspective. According to Tirimba (2013), one role of pension funds is the "provision of ways to manage uncertainty and control risk". The various roles of pension funds are examined and discussed in Davis (2002), Meng and Pfau (2010) and Tirimba (2013).

Pension funds usually provide some portfolio insurance mechanism to their investors. Portfolio insurance strategies, that ensure a certain minimum portfolio value or floor, are economically important and vital in particular for pension funds which have to meet certain payments or liabilities in the future. The well-known *Constant Proportion Portfolio Insurance (CPPI)* and the *Option-based Portfolio Insurance (OBPI)* are two prominent examples that are widely spread among the banking and insurance industries. The CPPI strategy was introduced by Perold (1986) for the fixed income asset class (see also Perold and Sharpe (1988)) and Black and Jones (1987) for equities. Additional analysis on CPPI can be found in Black and Rouhani (1989) and Black and Perold (1992). Basically, the CPPI implements a strategy that allocates assets dynamically over time and provides a certain downside protection (lower bound on the terminal wealth) in bearish times but also allows for some performance participation in bullish markets. The OBPI strategy, introduced by Leland and Rubinstein (1976), simultaneously invests in a portfolio that consists of risky assets and buys a put option written on it. One might have a look at Bertrand and Prigent (2005) or Kraus and Zagst (2011) for a comparison between CPPI and OBPI.

In this thesis we consider the asset allocation strategy of a general investor and in particular of a pension fund investor. A suitable management of an investor's wealth is strongly connected with the application of a proper dynamic investment strategy that generates desired return profiles such as a performance seeking attribute while limiting the downside risk. As every investor has an individual risk appetite or risk attitude, professional decision making under uncertainty needs to consider an adequate modeling of a certain risk-reward-tradeoff. A more risk-averse investor generally prefers a portfolio with a lower risk in terms of some risk measure, coming at the cost of smaller returns on average. The question arises how the pension fund's wealth is to be invested such that the benefits for the investors are maximized. To address this question, we derive the optimal investment strategies by maximizing the expected utility of an investor's wealth, first studied by Merton (1969) and Merton (1971) in a continuous-time Black-Scholes framework, under three innovative models. Inside every model, we consider different aspects that improve the findings in the existing literature and generate meaningful benefits to investors. At this stage, we forego naming the relevant literature, it is provided in the three main chapters of this thesis. We contribute to the literature by addressing the following relevant questions in the field of sophisticated active asset management:

First, we introduce a pension plan that considers behavioral treats of the investor in form of a behavioral finance conform utility function and individual probability weighting functions. In consequence, the model for the investor's preferences towards risk comes closer to the true behavior which leads to a more sophisticated investment strategy in line with the observed behavioral aspects. Furthermore, the structural form of the derived strategy can be classified into the optimal asset allocation for funded and underfunded portfolios; hence providing a solution for funded as well as underfunded pension funds.

The second innovation addresses the issue that risk aversion naturally changes with the age of a pension fund investor which typically goes hand in hand with a stepwise reduction of the relative risky asset allocation over time. As most pension plans studied so far consider a constant coefficient of risk aversion over all ages, determining and analyzing optimal asset allocation in a time-varying risk preference model can lead to substantial portfolio improvements.

Last but not least, we model a new innovative pension product that comes with a buffer and a pension adjustment mechanism to allow for higher expected returns on the investments particularly within low interest rate environments. The upside potential and thus future annuity payments are usually limited due to guarantees that can often be met only if a high proportion of the wealth is allocated to defensive asset classes such as government bonds. Therefore, alternative strategies without guarantees but with a certain limitation of the downside risk can provide a significant contribution. The political starting point for this pension scheme, named “Nahles-Rente” or “Sozialpartnermodell”, is the so-called “Betriebsrentenstärkungsgesetz (BRSG)” which came into force on January 1st, 2018. It allows company pension schemes to only make contribution-related promises but forbids performance-related guarantees.

In all considered models we determine and analyze the theoretically founded optimal asset allocation decision which moreover provides some portfolio insurance character in each case. The achieved innovative investment strategies can be understood as single-investor specific or cohort-specific if a pension fund cohort, commonly grouped by the age of the cohort members, is considered.

1.2 Thesis structure

In what follows we provide an overview over the structure of the thesis and give a brief summary of the conducted research and contributions to the existing literature.

Chapter 2 contains the basic financial market model and the considered utility function concept. Further it covers useful mathematical results needed for later derivations.

Chapter 3 on *Behavioral Portfolio Choice under Hyperbolic Absolute Risk Aversion* deals with the investment problem for a pension fund investor with probability distortion functions and an S-shaped utility function whose utility on gains satisfies the Inada condition at infinity, albeit not necessarily at zero, in a complete continuous-time financial market model. In particular, a piece-wise utility function with *hyperbolic absolute risk aversion (HARA)* is applied. The considered behavioral framework, *Cumulative Prospect Theory (CPT)*, was originally introduced by Tversky and Kahneman (1992). The utility model allows for increasing, constant or decreasing relative risk aversion. The continuous-time portfolio selection problem under the S-shaped HARA utility function in combination with probability distortion functions on gains and losses is solved theoretically for the first time, the optimal terminal wealth and its replicating wealth process and investment strategy are stated. In addition, conditions on the utility and the probability distortion functions for well-posedness and closed-form solutions are provided. A specific probability distortion function family is presented which fulfills all those requirements. This generalizes the work by Jin and Zhou (2008). Finally, a numerical case study is carried out to illustrate the impact of the utility function and the probability distortion functions. The general solution to the behavioral portfolio selection problem particularly provides an optimal asset allocation strategy that covers both initially well-funded and underfunded setups. This is due to the CPT approach which comprises a gain and a loss area relative to some benchmark that could be interpreted as present value of outstanding future liabilities of the pension fund.

In Chapter 4 on *Optimal Life-Cycle Consumption and Investment Decisions under Age-Dependent Risk Preferences* we solve the problem of maximizing the expected utility of future consumption and terminal wealth to determine the optimal pension or life-cycle fund strategy for a cohort of pension fund investors. The setup is strongly related to a DC pension plan where additionally (individual) consumption is taken into account. The consumption rate is subject to a time-varying minimum level and terminal wealth is subject to a terminal floor. Moreover, the preference between consumption and terminal wealth as well as the intertemporal coefficient of risk aversion are time-varying and therefore depend on the age of the considered pension cohort. The optimal consumption and investment policies are calculated in the case of a Black-Scholes financial market framework and hyperbolic absolute risk aversion (HARA) utility functions. We generalize Ye (2008) by adding an age-dependent coefficient of risk aversion and extend Steffensen (2011), Hentschel (2016) and Aase (2017) by considering consumption in combination with terminal wealth and allowing for consumption and terminal wealth floors via an application of HARA utility functions. A case study on fitting several models to realistic, time-dependent life-cycle consumption and relative investment profiles shows that only our extended model with time-varying preference parameters provides sufficient flexibility for an adequate fit. This is of particular interest to life-cycle products for (private) pension investments or pension insurance in general.

In Chapter 5 on *A new German Pension Product: "Nahles-Rente"/"Sozialpartnermodell"* we study a new, innovative pension product in Germany, called "Nahles-Rente" or "Sozialpartnermodell", that is regulated by “Bundesanstalt für Finanzdienstleistungsaufsicht (BaFin)”. The product generally consists of two phases: the pre-retirement or accumulation phase and the post-retirement or decumulation phase. The main novelty compared to usual pension plans is that the “Nahles-Rente” product comes without any guarantee to enhance expected returns in a low interest rate environment. In turn, annuity cash flows paid out in the post-retirement period will potentially have to be decreased (or increased), depending on the funding ratio of the pension fund. Therefore, i.e. for safety and stability reasons, there exist buffer processes in both phases to stabilize the wealth and pension evolution. Due to the nature of the product, we consider the accumulation and decumulation phase separately. In the accumulation phase, the pension fund investor generally aims for maximizing her expected utility from wealth at retirement-entry time. We determine the optimal investment strategy under a portfolio smoothing mechanism by solving the continuous-time *Hamilton-Jacobi-Bellman (HJB)* equation in closed-form. In the decumulation phase, the investor wants to maximize her expected utility from future pension cash flows; we solve the corresponding discrete-time Bellman equation for the finite- and infinite-horizon problem and by this receive the optimal investment strategy under a buffer mechanism and pension adjustments. Extensive numerical case studies are carried out to illustrate the optimal controls and characteristics.

Chapter 6 finally concludes with a summary of the most important findings and possible future research.

The detailed structures, targets and more in-depth details and motivation on the conducted research, for instance on the behavioral concept in Chapter 3, the life-cycle model in Chapter 4, or the dynamics and mechanism of the pension product in Chapter 5, are presented in the relevant chapters.

The content in this thesis is based on the following research papers:

- Chapter 3:

  Escobar-Anel et al. (2020a):
  Escobar-Anel, M., A. Lichtenstern, and R. Zagst (2020). Behavioral Portfolio Choice under Hyperbolic Absolute Risk Aversion. *International Journal of Theoretical and Applied Finance.* `https://doi.org/10.1142/S0219024920500454`.

  Escobar-Anel et al. (2020b):
  Escobar-Anel, M., A. Lichtenstern, and R. Zagst (2020). Behavioral Portfolio Insurance Strategies. *Financial Markets and Portfolio Management.* `https://doi.org/10.1007/s11408-020-00353-5`.

- Chapter 4:

  Lichtenstern et al. (2020):
  Lichtenstern, A., P. V. Shevchenko, and R. Zagst (2020). Optimal Life-Cycle Consumption and Investment Decisions under Age-Dependent Risk Preferences. *Mathematics and Financial Economics.* `https://doi.org/10.1007/s11579-020-00276-9`.

- Chapter 5:

  Lichtenstern and Zagst (2020):
  Lichtenstern, A. and R. Zagst (2020). “Nahles-Rente”/“Sozialpartnermodell” - Optimal Investment Strategies in the Accumulation and Decumulation Phase. *Research report. ERGO Center of Excellence in Insurance, Technical University of Munich.*

1.3 Applied optimization methods

We finish this introductory chapter with a short description of the applied optimization concepts in Chapters 3–5, which are of a diverse nature and highly application-specific. The general portfolio optimization problem is about maximizing expected utility coming from terminal wealth and intermediate consumption across all admissible investment strategies given an initial budget. We apply the following two famous approaches for continuous-time dynamic portfolio optimization:

The *martingale method* decomposes the portfolio problem into a static optimization problem, where one needs to determine the optimal consumption rate and the optimal terminal wealth, and a representation problem, where one has to derive the replicating portfolio process or asset allocation strategy. This approach is based on the completeness of the financial market model. Further details can be found for instance in Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), Korn (1997) and Karatzas and Shreve (1998).

In *Merton's approach*, the portfolio selection problem is interpreted as a stochastic control problem; this method remains applicable if the market is incomplete. The approach uses the so-called Hamilton-Jacobi-Bellman (HJB) equation based on the Bellman principle of optimality. For further readings we exemplarily refer to Bellman (1957), Merton (1969), Merton (1971), Merton (1992), Korn (1997), Yong and Zhou (1999) and Nisio (2015).

In Chapter 3 we apply the martingale method to solve the respective problem of maximizing expected utility from terminal wealth subject to a CPT-conform S-shaped utility function and probability distortions. The martingale method is also used in Chapter 4, where we maximize expected utility from consumption and terminal wealth with age-dependent risk preferences. In contrast, Chapter 5 applies Merton's approach in continuous time as well as the discrete-time setting to tackle the two considered problems (maximizing expected utility from terminal wealth; maximizing expected utility from pension cash flows) with buffer mechanisms.

In Chapter 3 we moreover apply the following technique: First, following the line of Jin and Zhou (2008), the problem with an S-shaped utility function is split into two sub-problems, one that covers the concave and one that covers the convex part of the utility. Afterwards, those two sub-problems are solved individually using the martingale method in a second step, where the static terminal wealth problems are transferred into problems on the quantile function (quantile transformation). At the end, the two individual solutions need to be merged in an optimal fashion. In Chapter 4 we employ a similar technique, in line with Karatzas and Shreve (1998) and Lakner and Nygren (2006), that divides the consumption-investment problem into a terminal wealth problem and a separate consumption problem. Those two sub-problems can be solved individually and the respective solutions need to be merged optimally at the end.

2 Mathematical Preliminaries

In this chapter we first introduce the basic financial market model in Section 2.1. Afterwards we present some mathematical tools applied later (Section 2.2) and the replicating portfolio of claims with a certain payoff structure (Section 2.3). Finally, we conclude this chapter with providing useful definitions and properties of utility functions in Section 2.4.

2.1 The basic financial market model

In this section we provide the basic underlying financial market model, based on Karatzas and Shreve (1998), Zagst (2002) and Björk (2009). Specific deviations from this model, for instance the notion of a consumption- and an income-rate process in Chapters 4 and 5, are introduced in the relevant chapters.

Let $W = (W(t))_{t\in[0,T]}$, $W(t) := (W_1(t), \ldots, W_N(t))'$, $N \in \mathbb{N}$, be a standard $\mathcal{F}_t$-adapted N-dimensional Brownian motion that is defined on a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The time T denotes a fixed terminal time or investment horizon. It is assumed that $\mathcal{F}_t$ is the natural filtration generated by $W(s)$, $0 \leqslant s \leqslant t$, augmented by all the null sets. Ω is the sample space and $\mathbb{P}$ denotes the real-world probability measure. Following Karatzas and Shreve (1998), a continuous-time financial market is defined that consists of $N+1$ continuously traded assets: one risk-free asset or bank account P_0 and N risky assets or asset classes P_i, $i = 1, \ldots, N$, which might include e.g. stocks, real estate or commodities. We denote the price processes by $P = (P_0, \hat{P}')'$ for all assets in the market and $\hat{P} = (P_1, \ldots, P_N)'$ for the risky assets. The financial market is supposed to be frictionless and satisfy the usual standard assumptions. Uncertainty in the continuous-time financial market is introduced by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The price process $P_0(t)$, $t \in [0,T]$, of the bank account is subject to the equation

$$dP_0(t) = rP_0(t)dt, \ P_0(0) = 1, \tag{2.1}$$

with constant riskless interest rate $r \geqslant 0$. It is well-known that the solution to this ordinary differential equation (ODE) is given by $P_0(t) = e^{rt}$. The price processes $P_i(t)$, $i = 1, \ldots, N$, $t \in [0,T]$, of the remaining N risky assets are subject to the stochastic differential equations (SDE)

$$dP_i(t) = P_i(t)\left(\mu_i dt + \sigma_i dW(t)\right) = P_i(t)\left(\mu_i dt + \sum_{j=1}^{N} \sigma_{ij} dW_j(t)\right), \ P_i(0) = p_i > 0, \tag{2.2}$$

where $\mu := (\mu_1, \ldots, \mu_N)' \in \mathbb{R}^N$ with $\mu - r\mathbf{1} > \mathbf{0}$ is the constant drift and $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{iN}) \in \mathbb{R}_+^{1\times N}$ denotes the constant volatility vector of assets $i = 1, \ldots, N$. The volatility matrix is defined as $\sigma = (\sigma_{ij})_{i,j=1,\ldots,N}$ with corresponding covariance matrix $\Sigma = \sigma\sigma'$ of the log-returns which is assumed to be strongly positive definite, i.e. there exists $K > 0$ such that $\mathbb{P}$-a.s. it holds $x'\Sigma x \geqslant Kx'x$,

$\forall x \in \mathbb{R}^N$. From Eq. (3.2), p. 45, in Zagst (2002) it follows for all $i = 1, \ldots, N$:

$$\sup_{0 \leqslant t \leqslant T} P_i(t) < \infty, \ \mathbb{P} - \text{a.s.}.$$

Within this framework, let $\gamma := \sigma^{-1} (\mu - r\mathbf{1})$ denote the market price of risk. According to Karatzas and Shreve (1998), in this case of Black-Scholes market dynamics, there exists a unique risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := e^{-\frac{1}{2}\|\gamma\|^2 t - \gamma' W(t)}.$$

Moreover, the financial market is complete which allows to value stochastic cash flow or payment streams under the measure $\mathbb{Q}$ as expected discounted values, meaning that the cost of a portfolio replicating the contract is given by its expected discounted value under $\mathbb{Q}$. The associated pricing kernel or state price deflator, which we denote by $\tilde{Z}(t)$, $t \in [0, T]$, is defined as

$$\tilde{Z}(t) := e^{-\left(r + \frac{1}{2}\|\gamma\|^2\right)t - \gamma' W(t)} \tag{2.3}$$

and can be used for the valuation of cash flow streams under the real-world probability measure $\mathbb{P}$. Its dynamics are subject to the stochastic differential equation

$$d\tilde{Z}(t) = -\tilde{Z}(t) \left(r dt + \gamma' dW(t)\right), \ \tilde{Z}(0) = 1.$$

For the ease of notation we denote $\tilde{Z} := \tilde{Z}(T)$ in what follows. It is $\int_0^T \|\gamma\|^2 ds = \|\gamma\|^2 T \neq 0$ which implies that the pricing kernel $\tilde{Z}$ admits no atom and is a non-degenerate log-normal random variable, i.e. $\ln \tilde{Z} \sim \mathcal{N}\left(\mu_{\tilde{Z}}, \sigma^2_{\tilde{Z}}\right)$, with $\mu_{\tilde{Z}} := -\left(r + \frac{1}{2}\|\gamma\|^2\right) T$ and $\sigma^2_{\tilde{Z}} := \|\gamma\|^2 T$. Let $F_{\tilde{Z}}$ denote the distribution function of the pricing kernel and $F^{-1}_{\tilde{Z}}$ its inverse. Further define $\mu_{\tilde{Z}}(t) := -\left(r + \frac{1}{2}\|\gamma\|^2\right)(T - t)$ and $\sigma^2_{\tilde{Z}}(t) := \|\gamma\|^2 (T - t)$ the respective parameters of the log-normal variable $\tilde{Z}(t, T) := \tilde{Z}/\tilde{Z}(t)$.

In what follows we consider $\mathcal{F}_t$-progressively measurable self-financing trading strategies or portfolio processes $\varphi = (\varphi_0, \hat{\varphi})'$, $\hat{\varphi} = (\varphi_1, \ldots, \varphi_N)'$, such that $\mathbb{P}$-a.s. it holds $\int_0^T |\varphi_0(t)| dt < \infty$ and $\int_0^T \|\hat{\varphi}(t)\|^2 dt < \infty$, $\mathbb{P}$-a.s.. $\varphi_i(t)$ represents the number of individual shares of asset i held by the investor at time t. The associated wealth process $V = (V(t))_{t \in [0,T]}$, $V(t) = V(t, \varphi)$, with initial wealth $V(0) = v_0$ is defined by

$$V(t) := \sum_{i=0}^{N} \varphi_i(t) P_i(t).$$

The self-financing property of φ further leads to

$$V(t) = v_0 + \sum_{i=0}^{N} \int_0^t \varphi_i(s) dP_i(s).$$

A trading strategy φ is called admissible if it fulfills the aforementioned conditions, summarized by the following set $\Lambda^\star(v_0)$ that contains all admissible strategies φ to the initial wealth v_0:

$$\Lambda^\star(v_0) := \Bigg\{\varphi \;\Bigg|\; \forall t \in [0,T]:\ \varphi(t) \in \mathbb{R}^{N+1},\ V(0) = v_0,\ V(t) \text{ is } \mathbb{P}-\text{a.s. bounded from below}, \\ \varphi \text{ is progressively measurable and self-financing}, \\ \int_0^T |\varphi_0(t)|dt < \infty \text{ and } \int_0^T \|\hat{\varphi}(t)\|^2 dt < \infty,\ \mathbb{P}-\text{a.s.}\Bigg\}. \tag{2.4}$$

The corresponding $\mathcal{F}_t$-progressively measurable self-financing relative portfolio process is denoted by $\pi = (\pi_0, \hat{\pi}')'$ with risky relative investment $\hat{\pi} = (\pi_1, \dots, \pi_N)'$ and risk-free relative investment $\pi_0(t) = 1 - \hat{\pi}(t)'\mathbf{1}$, where $\pi_i(t)$ denotes the fraction or proportion of wealth allocated to asset i at time t. The relation between π and φ is described by

$$\pi_i(t) = \begin{cases} \frac{\varphi_i(t)P_i(t)}{V(t)}, & \text{if } V(t) \neq 0, \\ 0 & , \text{ if } V(t) = 0. \end{cases}$$

Accordingly, $\pi_i(t)V(t)$ denotes the time-t exposure to the asset i. The wealth process $V = (V(t))_{t\in[0,T]}$, $V(t) = V(t,\pi)$, associated with π satisfies the stochastic differential equation

$$dV(t) = V(t)\left[\left(r + \hat{\pi}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}(t)'\sigma dW(t)\right],\ V(0) = v_0. \tag{2.5}$$

A relative portfolio process π is called admissible to the initial wealth v_0 if it fulfills the following conditions, summarized by the set $\tilde{\Lambda}(v_0)$ that is defined next:

$$\tilde{\Lambda}(v_0) := \left\{\pi \;\middle|\; \varphi \in \Lambda^\star(v_0) \text{ with } \varphi_i(t) := \frac{\pi_i(t)V(t)}{P_i(t)}\ \forall t \in [0,T]\right\}. \tag{2.6}$$

Note that the following characteristics of $\varphi \in \Lambda^\star(v_0)$ are directly passed on to $\pi \in \tilde{\Lambda}(v_0)$: π is progressively measurable and self-financing, with $\forall t \in [0,T]:\ \pi(t) \in \mathbb{R}^{N+1}$ such that $V(t)$ is $\mathbb{P}$-a.s. bounded from below and $V(0) = v_0$. Furthermore, $\pi \in \tilde{\Lambda}(v_0)$ is supposed to fulfill the conditions $\int_0^T |\pi_0(t)V(t)|dt < \infty$ and $\int_0^T \|\hat{\pi}(t)V(t)\|^2 dt < \infty$, $\mathbb{P}$-a.s., since

$$\int_0^T |\pi_0(t)V(t)|dt = \int_0^T |\varphi_0(t)P_0(t)|dt \leqslant \underbrace{(\sup_{0\leqslant s\leqslant T} P_0(s))}_{<\infty} \underbrace{\int_0^T |\varphi_0(t)|dt}_{\overset{\varphi\in\Lambda^\star(v_0)}{<}\infty} < \infty,\ \mathbb{P}-\text{a.s.},$$

$$\begin{aligned}\int_0^T \|\hat{\pi}(t)V(t)\|^2 dt &= \sum_{i=1}^N \int_0^T |\pi_i(t)V(t)|^2 dt = \sum_{i=1}^N \int_0^T |\varphi_i(t)P_i(t)|^2 dt \leqslant \sum_{i=1}^N (\sup_{0\leqslant s\leqslant T} P_i(s))^2 \int_0^T |\varphi_i(t)|^2 dt \\ &\leqslant \underbrace{\left(\sup_{i=1,\dots,N} (\underbrace{\sup_{0\leqslant s\leqslant T} P_i(s)}_{<\infty})^2\right)}_{<\infty} \underbrace{\sum_{i=1}^N \int_0^T |\varphi_i(t)|^2 dt}_{=\int_0^T \|\hat{\varphi}(t)\|^2 dt \overset{\varphi\in\Lambda^\star(v_0)}{<}\infty} < \infty,\ \mathbb{P}-\text{a.s.}.\end{aligned}$$

2.2 Basic mathematical tools

In this section we summarize for the convenience of the reader some basic mathematical tools that will be used later. First, exchangeability of differentiation and integration or expectation is justified by the well-known Leibniz integral rule for continuous and continuously differentiable functions.

Theorem 2.1 (Leibniz integral rule (Protter and Morrey, Jr. (1985), Theorem 3, Chapter 8))**.** *Let g, h, F be continuous and continuously differentiable functions. It then holds*

$$\frac{d}{dt}\left(\int_{g(t)}^{h(t)} F(x,t)dx\right) = F(h(t),t)h'(t) - F(g(t),t)g'(t) + \int_{g(t)}^{h(t)} \frac{\partial F(x,t)}{\partial t}dx\,.$$

For further readings we refer to Flanders (1973). The continuity and differentiability conditions are fulfilled in all cases where the rule is applied. Moreover, we have the well-known mean value result for integrals.

Theorem 2.2 (First mean value theorem for integrals (Trench (2003), Theorem 3.3.7))**.** *For two integrable functions $f(x)$ and $g(x)$ on the interval (a,b), where $f(x)$ is continuous and $g(x)$ does not change sign on (a,b), there exists $d \in (a,b)$ such that*

$$\int_a^b f(x)g(x)dx = f(d)\int_a^b g(x)dx.$$

Furthermore, we later need the concept of diagonally dominant matrices and a certain property that is linked to such matrices. Therefore, we define the notion of a (strictly) diagonally dominant matrix next.

Definition 2.3 (Diagonally dominant matrix (Horn and Johnson (2013), Definition 6.1.9))**.** *A matrix $A = (a_{ij})_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, is said to be diagonally dominant if*

$$|a_{ii}| \geqslant \sum_{j=1, j\neq i}^{n} |a_{ij}|\,, \ \forall i \in \{1,\ldots,n\}.$$

If the inequality is strict for all $i \in \{1,\ldots,n\}$, the matrix A is called strictly diagonally dominant.

It can be shown that every strictly diagonally dominant matrix is invertible.

Theorem 2.4 (Horn and Johnson (2013), Theorem 6.1.10, part (a))**.** *Let $A = (a_{ij})_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ be strictly diagonally dominant. Then A is non-singular.*

Moerover, we later need a specific eigenvalue result for stochastic matrices which are also known as probability or transition matrices.

Definition 2.5 (Stochastic matrix (Horn and Johnson (2013), p. 547))**.** *A matrix $A = (a_{ij})_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, $a_{ij} \geqslant 0$, is a (row) stochastic matrix if $A\mathbf{1} = \mathbf{1}$, i.e. if all row sums of A are equal to one.*

Definition 2.6 (Eigenvalue and eigenvector (Horn and Johnson (2013), Definition 1.1.2))**.** *Let $A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$. If a scalar $\lambda \in \mathbb{R}$ and a vector $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \not\equiv \mathbf{0}$, satisfy the equation*

$$A\mathbf{v} = \lambda\mathbf{v},$$

then $\lambda =: \lambda(A)$ is called an eigenvalue of A and $\mathbf{v}$ is called an eigenvector of A associated with λ.

We have the following result for the eigenvalues of a stochastic matrix.

Theorem 2.7. *The maximal absolute eigenvalue of a stochastic matrix A is equal to one, i.e.* $\max|\lambda(A)| = 1$.

Proof. We prove that any stochastic matrix A has the eigenvalue $\lambda(A) = 1$ and that the absolute value of any eigenvalue $\lambda(A)$ of A is less than or equal to one.

1. Existence of eigenvalue $\lambda(A) = 1$:

   The vector $\mathbf{1}$ that consists of ones is an eigenvector to the eigenvalue $\lambda(A) = 1$ for any stochastic matrix $A = (a_{ij})_{i,j=1,\dots,n}$ because the rows of A sum up to one:

$$A\mathbf{1} = \begin{pmatrix} a_{ij} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{ij} + a_{12} + \ldots + a_{1n} \\ \vdots \\ a_{n1} + a_{n2} + \ldots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \cdot \mathbf{1}.$$

2. Eigenvalue bound $|\lambda(A)| \leqslant 1$:

   Let $\lambda(A)$ be an eigenvalue of the stochastic matrix A and let $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \mathbf{0}$ be the corresponding eigenvector, i.e. $A\mathbf{v} = \lambda(A)\mathbf{v}$. When we compare the i-th row of both sides of the equality, we obtain

$$\sum_{j=1}^{n} a_{ij} v_j = \lambda(A) v_i, \ i = 1, \ldots, n. \tag{2.7}$$

   Further let

$$m := \underset{j \in \{1,\ldots,n\}}{\arg\max} \{|v_j|\}$$

   and thus v_m denotes the entry of the eigenvector $\mathbf{v}$ with the maximal absolute value: $|v_m| \geqslant |v_j|$ $\forall j \in \{1, \ldots, n\}$. Due to $\mathbf{v} \not\equiv \mathbf{0}$ it is $|v_m| > 0$. Inserting $i = m$ in Eq. (2.7) while considering the absolute value leads to

$$|\lambda(A)| \cdot |v_m| = |\lambda(A) v_m| \overset{(2.7):\ i=m}{=} \left| \sum_{j=1}^{n} a_{mj} v_j \right| \overset{\text{triangle inequality}}{\leqslant} \sum_{j=1}^{n} |a_{mj} v_j| \overset{a_{mj} \geqslant 0}{=} \sum_{j=1}^{n} a_{mj} |v_j|$$
$$\overset{|v_j| \leqslant |v_m|}{\leqslant} \sum_{j=1}^{n} a_{mj} |v_m| = |v_m| \sum_{j=1}^{n} a_{mj} = |v_m| \,.$$

Hence, as $|v_m| > 0$, we must have $|\lambda(A)| \leqslant 1$ for any arbitrary eigenvalue $\lambda(A)$.

In total, this shows that $\max |\lambda(A)| = 1$ for any stochastic matrix A. □

For further readings on this topic we refer to Landau and Odlyzko (1981), Asmussen (2003) and the Perron-Frobenius theory.

Finally, we need the formula for truncated moments of a log-normally distributed random variable for replication purposes, where Φ denotes the distribution function of a standard normal random variable.

Theorem 2.8. *Let the random variable Z be log-normally distributed with $\ln Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$ and $\eta \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^+$ with $z_1 \leqslant z_2$. The formula for the truncated log-normal moment is:*

$$\mathbb{E}\left[Z^\eta \mathbb{1}_{Z\in(z_1,z_2)}\right] = e^{\mu_Z\eta + \frac{1}{2}\sigma_Z^2\eta^2}\left[\Phi\left(\frac{\ln z_2 - \mu_Z}{\sigma_Z} - \sigma_Z\eta\right) - \Phi\left(\frac{\ln z_1 - \mu_Z}{\sigma_Z} - \sigma_Z\eta\right)\right].$$

Proof. The density function f_Z, that is associated with Z, is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z z} e^{-\frac{1}{2}\left(\frac{\ln z - \mu_Z}{\sigma_Z}\right)^2}.$$

This implies

$$\mathbb{E}\left[Z^\eta \mathbb{1}_{Z\in(z_1,z_2)}\right] = \int_{z_1}^{z_2} z^\eta f_Z(z)dz = \int_{z_1}^{z_2} \frac{z^{\eta-1}}{\sqrt{2\pi}\sigma_Z} e^{-\frac{1}{2}\left(\frac{\ln z - \mu_Z}{\sigma_Z}\right)^2} dz.$$

Applying the substitution $z = e^{\mu_Z + x\sigma_Z}$ leads to

$$\begin{aligned}
\mathbb{E}\left[Z^\eta \mathbb{1}_{Z\in(z_1,z_2)}\right] &= \int_{z_1}^{z_2} \frac{z^{\eta-1}}{\sqrt{2\pi}\sigma_Z} e^{-\frac{1}{2}\left(\frac{\ln z - \mu_Z}{\sigma_Z}\right)^2} dz \\
&= \int_{\frac{\ln z_1 - \mu_Z}{\sigma_Z}}^{\frac{\ln z_2 - \mu_Z}{\sigma_Z}} \frac{e^{\mu_Z(\eta-1) + x\sigma_Z(\eta-1)}}{\sqrt{2\pi}\sigma_Z} e^{-\frac{1}{2}\left(\frac{\mu_Z + x\sigma_Z - \mu_Z}{\sigma_Z}\right)^2} e^{\mu_Z + x\sigma_Z}\sigma_Z dx \\
&= \int_{\frac{\ln z_1 - \mu_Z}{\sigma_Z}}^{\frac{\ln z_2 - \mu_Z}{\sigma_Z}} \frac{e^{\mu_Z\eta + x\sigma_Z\eta}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = e^{\mu_Z\eta} \int_{\frac{\ln z_1 - \mu_Z}{\sigma_Z}}^{\frac{\ln z_2 - \mu_Z}{\sigma_Z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + x\sigma_Z\eta} dx \\
&= e^{\mu_Z\eta} \int_{\frac{\ln z_1 - \mu_Z}{\sigma_Z}}^{\frac{\ln z_2 - \mu_Z}{\sigma_Z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2 - 2x\sigma_Z\eta + \sigma_Z^2\eta^2 - \sigma_Z^2\eta^2\right)} dx \\
&= e^{\mu_Z\eta + \frac{1}{2}\sigma_Z^2\eta^2} \int_{\frac{\ln z_1 - \mu_Z}{\sigma_Z}}^{\frac{\ln z_2 - \mu_Z}{\sigma_Z}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - \sigma_Z\eta)^2}}_{\text{density of a } \mathcal{N}(\sigma_Z\eta, 1)\text{-random variable}} dx \\
&= e^{\mu_Z\eta + \frac{1}{2}\sigma_Z^2\eta^2}\left[\Phi\left(\frac{\ln z_2 - \mu_Z}{\sigma_Z} - \sigma_Z\eta\right) - \Phi\left(\frac{\ln z_1 - \mu_Z}{\sigma_Z} - \sigma_Z\eta\right)\right].
\end{aligned}$$

□

2.3 Replication of Selected Payoffs

The following Theorem 2.9 states the formulas for the replicating strategy and the replicating wealth process corresponding to a digital or binary option with payoff $\tilde{Z}^\eta \mathbb{1}_{\tilde{Z}\in(z_1,z_2)}$ for $0 \leqslant z_1 < z_2 \leqslant \infty$ and $\eta \in \mathbb{R}$, with Φ and ψ denoting the distribution and density function of a standard normal random variable:

$$\psi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \ \Phi(x) := \int_{-\infty}^{x} \psi(y)dy.$$

Theorem 2.9. *Let a payoff of the form $\tilde{Z}^\eta \mathbb{1}_{\tilde{Z}\in(z_1,z_2)}$ be given with $0 \leqslant z_1 < z_2 \leqslant \infty$ and $\eta \in \mathbb{R}$.*

1. *If $z_2 < \infty$, the replicating trading strategy $\pi = (\pi_0, \hat{\pi}')'$ of the payoff and its corresponding wealth process $V(t)$ are given by*

$$\begin{aligned}
\hat{\pi}(t)V(t) &= \Bigg\{ \frac{1}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)} \left[z_2^{\eta+1}\psi\left(\frac{\ln z_2 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) - z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)\right] \\
&\quad - \eta V(t) \Bigg\} \left(\sigma\sigma'\right)^{-1} (\mu - r\mathbf{1}), \\
\pi_0(t) &= 1 - \hat{\pi}(t)'\mathbf{1}, \\
V(t) &= \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)} \int_{\frac{z_1}{\tilde{Z}(t)}}^{\frac{z_2}{\tilde{Z}(t)}} y^\eta \psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) dy \\
&= \tilde{Z}(t)^\eta e^{\mu_{\tilde{Z}}(t)(\eta+1)+\frac{1}{2}\sigma^2_{\tilde{Z}}(t)(\eta+1)^2} \Bigg[\Phi\left(\frac{\ln z_2 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)(\eta+1)\right) \\
&\quad - \Phi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)(\eta+1)\right)\Bigg].
\end{aligned}$$

2. *If $z_2 = \infty$, the replicating trading strategy $\pi = (\pi_0, \hat{\pi}')'$ of the payoff and its corresponding wealth process $V(t)$ are given by*

$$\begin{aligned}
\hat{\pi}(t)V(t) &= \left[-\frac{1}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)} z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) - \eta V(t)\right] \left(\sigma\sigma'\right)^{-1} (\mu - r\mathbf{1}), \\
\pi_0(t) &= 1 - \hat{\pi}(t)'\mathbf{1}, \\
V(t) &= \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)} \int_{\frac{z_1}{\tilde{Z}(t)}}^{\infty} y^\eta \psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) dy \\
&= \tilde{Z}(t)^\eta e^{\mu_{\tilde{Z}}(t)(\eta+1)+\frac{1}{2}\sigma^2_{\tilde{Z}}(t)(\eta+1)^2} \left[1 - \Phi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)(\eta+1)\right)\right].
\end{aligned}$$

Proof. The pricing kernel is subject to the equations

$$\tilde{Z}(t) = e^{-\left(r+\frac{1}{2}\|\gamma\|^2\right)t-\gamma' W(t)}, \ d\tilde{Z}(t) = -\tilde{Z}(t)\left(rdt + \gamma' dW(t)\right),$$

for $t \in [0,T]$ with $\gamma = \sigma^{-1}(\mu - r\mathbf{1})$ and $\tilde{Z}(0) = 1$. Furthermore, define $\tilde{Z}(t,T) := \tilde{Z}(T)/\tilde{Z}(t)$. In the considered market it holds $\ln \tilde{Z}(t,T)|_{\mathcal{F}_t} \sim \mathcal{N}\left(\mu_{\tilde{Z}}(t), \sigma^2_{\tilde{Z}}(t)\right)$ with $\mu_{\tilde{Z}}(t) \in \mathbb{R}$ and $\sigma_{\tilde{Z}}(t) > 0 \; \forall t \in [0,T]$ such that $\mu_{\tilde{Z}}(t) := -\left(r + \frac{1}{2}\|\gamma\|^2\right)(T-t)$ and $\sigma^2_{\tilde{Z}}(t) := \|\gamma\|^2(T-t)$.

First assume $z_2 < \infty$. Due to the completeness of the considered market, the claim of the form $\tilde{Z}^\eta \mathbb{1}_{\tilde{Z}\in(z_1,z_2)}$ is replicable and it is well-known (Karatzas and Shreve (1998); Björk (2009)) that in this case, by employing Bayes' Formula, the replicating wealth process, due to uniqueness of the equivalent martingale measure $\mathbb{Q}$, is uniquely determined by

$$\begin{aligned}
V(t) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{P_0(t)}{P_0(T)}\tilde{Z}(T)^\eta \mathbb{1}_{\tilde{Z}(T)\in(z_1,z_2)}|\mathcal{F}_t\right] \overset{\text{Bayes' formula}}{=} \mathbb{E}\left[\frac{\tilde{Z}(T)}{\tilde{Z}(t)}\tilde{Z}(T)^\eta \mathbb{1}_{\tilde{Z}(T)\in(z_1,z_2)}|\mathcal{F}_t\right] \\
&= \tilde{Z}(t)^\eta \int_{\frac{z_1}{\tilde{Z}(t)}}^{\frac{z_2}{\tilde{Z}(t)}} y^{\eta+1}\frac{1}{\sqrt{2\pi}\sigma_{\tilde{Z}}(t)}\frac{1}{y}e^{-\frac{1}{2}\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)^2} dy \\
&= \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)} \int_{\frac{z_1}{\tilde{Z}(t)}}^{\frac{z_2}{\tilde{Z}(t)}} y^{\eta}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)^2} dy \\
&= \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)} \int_{\frac{z_1}{\tilde{Z}(t)}}^{\frac{z_2}{\tilde{Z}(t)}} y^{\eta}\psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) dy = f(t,\tilde{Z}(t))
\end{aligned} \tag{2.8}$$

with

$$f(t,z) := \frac{z^\eta}{\sigma_{\tilde{Z}}(t)} \int_{\frac{z_1}{z}}^{\frac{z_2}{z}} y^{\eta}\psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) dy.$$

Applying Itô's formula and comparing the diffusion part to the diffusion part of the general dynamics of a wealth process $dV(t)$ in Eq. (2.5) leads to

$$-\tilde{Z}(t)f_z(t,\tilde{Z}(t))\gamma' = V(t)\hat{\pi}(t)'\sigma.$$

Together with $\gamma = \sigma^{-1}(\mu - r\mathbf{1})$ we conclude

$$\hat{\pi}(t)V(t) = -\tilde{Z}(t)f_z(t,\tilde{Z}(t))\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}). \tag{2.9}$$

This implies that the investment strategy corresponding to $V(t)$ is given by Eq. (2.9) and $\pi_0(t) = 1 - \hat{\pi}(t)'\mathbf{1}$. Let us additionally have a look at the expression for $f_z(t,\tilde{Z}(t))$. Using Leibniz integral rule in Theorem 2.1, we receive:

$$\begin{aligned}
f_z(t,z) &= \frac{\partial f(t,z)}{\partial z} = \frac{\partial}{\partial z}\frac{z^\eta}{\sigma_{\tilde{Z}}(t)}\int_{\frac{z_1}{z}}^{\frac{z_2}{z}} y^{\eta}\psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) dy \\
&= \frac{\eta}{z}f(t,z) + \frac{1}{\sigma_{\tilde{Z}}(t)z^2}\left[-z_2^{\eta+1}\psi\left(\frac{\ln z_2 - \ln z - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) + z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln z - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)\right].
\end{aligned}$$

Hence, using Eq. (2.8),

$$f_z(t,\tilde{Z}(t)) = \frac{\eta}{\tilde{Z}(t)}V(t) - \frac{1}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)^2}\left[z_2^{\eta+1}\psi\left(\frac{\ln z_2 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) - z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)\right]$$

and one can write Eq. (2.9) as

$$\hat{\pi}(t)V(t) = \left\{\frac{1}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)}\left[z_2^{\eta+1}\psi\left(\frac{\ln z_2 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) - z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)\right] - \eta V(t)\right\}\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}). \tag{2.10}$$

Now assume $z_2 = \infty$. Then $V(t) = f(t,\tilde{Z}(t))$ becomes to

$$f(t,\tilde{Z}(t)) = \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)}\int_{\frac{z_1}{z}}^{\infty} y^\eta \psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)dy$$

and therefore

$$\hat{\pi}(t)V(t) = \left[-\frac{1}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)}z_1^{\eta+1}\psi\left(\frac{\ln z_1 - \ln \tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right) - \eta V(t)\right]\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}). \tag{2.11}$$

In view of $\tilde{Z}(t)$ being given at t and $\tilde{Z}(t,T)$ being independent of $\mathcal{F}_t$ and log-normally distributed with parameters $\mu_{\tilde{Z}}(t)$ and $\sigma_{\tilde{Z}}(t)$, in view of Theorem 2.8 one can additionally reformulate $V(t)$ as

$$\begin{aligned} V(t) &\overset{(2.8)}{=} \frac{\tilde{Z}(t)^\eta}{\sigma_{\tilde{Z}}(t)}\int_{\frac{z_1}{\tilde{Z}(t)}}^{\frac{z_2}{\tilde{Z}(t)}} y^\eta\psi\left(\frac{\ln y - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)}\right)dy = \tilde{Z}(t)^\eta\mathbb{E}\left[\tilde{Z}(t,T)^{\eta+1}\mathbb{1}_{\frac{z_1}{\tilde{Z}(t)}\leqslant\tilde{Z}(t,T)\leqslant\frac{z_2}{\tilde{Z}(t)}}\middle|\mathcal{F}_t\right] \\ &\overset{\text{Thm. 2.8}}{=} \tilde{Z}(t)^\eta e^{\mu_{\tilde{Z}}(t)(\eta+1)+\frac{1}{2}\sigma^2_{\tilde{Z}}(t)(\eta+1)^2}\left[\Phi\left(\frac{\ln z_2 - \ln\tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)(\eta+1)\right)\right. \\ &\qquad \left. - \Phi\left(\frac{\ln z_1 - \ln\tilde{Z}(t) - \mu_{\tilde{Z}}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)(\eta+1)\right)\right] \end{aligned} \tag{2.12}$$

for general z_2. The result for $z_2 = \infty$ follows straightforward. Equations (2.8), (2.10), (2.11) and (2.12) state the desired result. □

2.4 Utility functions

In this section we introduce some usual characteristics of utility functions and define the applied class of utility functions.

The risk preferences of an investor are characterized by a utility function $U : D \to \mathbb{R}$, $v \mapsto U(v)$. In accordance with Korn (1997), the usual assumptions on utility functions are: U is strictly increasing, continuously differentiable and concave in its argument. Therefore, the investor generally prefers more to less, is risk-averse and her increase in the degree of happiness or satisfaction gets smaller as wealth increases. In particular, Pratt (1964) showed that an investor is (strictly) risk-averse if and only if U is (strictly) concave. In contrast, a convex utility function implies a risk-seeking behavior of the investor. Partially convex and concave utility functions are applied in Cumulative Prospect Theory (CPT), see Chapter 3. Globally concave utility functions are applied in *Expected Utility Theory (EUT)*, see Chapters 4 and 5. In addition, we assume U to be twice continuously differentiable.

We define the *Arrow-Pratt measure of absolute and relative risk aversion* (also called coefficient of absolute and relative risk aversion) as measures for risk aversion associated with utility functions, developed by Pratt (1964) and Arrow (1970).

Definition 2.10 (Arrow-Pratt measure of absolute risk aversion)**.** *The Arrow-Pratt measure of absolute risk aversion of a utility function U is defined by*

$$\mathcal{A}(v) := -\frac{U''(v)}{U'(v)} = -\frac{\partial}{\partial v}\ln U'(v).$$

Definition 2.11 (Arrow-Pratt measure of relative risk aversion)**.** *The Arrow-Pratt measure of relative risk aversion of a utility function U is defined by*

$$\mathcal{R}(v) := -\frac{vU''(v)}{U'(v)} = v\mathcal{A}(v).$$

A positive/zero/negative Arrow-Pratt measure generally indicates a risk-averse/risk-neutral/risk-seeking attitude. Strictly increasing and concave utility functions provide positive Arrow-Pratt measures.

Utility functions can then be classified in terms of having an increasing, constant or decreasing Arrow-Pratt measure of absolute or relative risk aversion. In this thesis, we consider a popular and flexible class or family of utility functions, the so-called *hyperbolic absolute risk aversion (HARA)* utility functions, in line with the parameterization in Merton (1971) and Ingersoll (1987).

Definition 2.12 (HARA utility function)**.** *The HARA utility function is defined by*

$$U(v) := a\frac{1-b}{b}\left(\frac{1}{1-b}(v-F)\right)^b, \ v \in (F, \infty)$$

with $b < 1$, $b \neq 0$, $a > 0$ and $F \geqslant 0$.

b is called coefficient of risk aversion, F denotes a floor. From the definition one can infer that HARA utility function is increasing and strictly concave in the argument v. The associated Arrow-Pratt measures are

$$\mathcal{A}(v) = (1-b)\frac{1}{v-F} > 0,$$
$$\mathcal{R}(v) = (1-b)\frac{v}{v-F} > 0.$$

The Arrow-Pratt measure of absolute risk aversion $\mathcal{A}(v)$ admits a hyperbolic form, which represents the name-giving characteristic of this utility class. Furthermore, both Arrow-Pratt measures are positive due to $b < 1$ (risk aversion) and a smaller value for b is related to a higher degree of risk aversion. The HARA utility function covers the special case of a *linear* utility as limiting case for $b \to 1$, the *Logarithmic* utility for $b \to 0$ and the *constant relative risk aversion (CRRA)* or *Power* utility for $F = 0$:

1. Linear utility function ($b \to 1$):

$$U(v) = a\,(v - F)$$

2. Logarithmic utility function ($b \to 0$):

$$U(v) = a \ln(v - F)$$

3. CRRA utility function ($F = 0$):

$$U(v) = a\frac{1-b}{b}\left(\frac{1}{1-b}v\right)^b$$

   which can be reduced to the usual form when selecting $a := \frac{1}{(1-b)^{1-b}}$:

$$U(v) = \frac{v^b}{b}$$

The associated Arrow-Pratt measures of risk aversion in these cases are given by:

1. Linear utility function ($b \to 1$):

$$\mathcal{A}(v) = 0,\ \mathcal{R}(v) = 0.$$

2. Logarithmic utility function ($b \to 0$):

$$\mathcal{A}(v) = \frac{1}{v-F} > 0,\ \mathcal{R}(v) = \frac{v}{v-F} > 0.$$

3. CRRA utility function ($F = 0$):

$$\mathcal{A}(v) = (1-b)\frac{1}{v} > 0,\ \mathcal{R}(v) = 1 - b > 0.$$

This particularly shows that an investor applying a linear utility function has a risk-neutral attitude, while an investor with a CRRA utility function has a constant relative risk aversion.

A time-dependent HARA utility function is defined next analogically to the HARA utility function given by Definition 2.12.

Definition 2.13 (Time-dependent HARA utility function)**.** *The time-dependent HARA utility function is defined by*

$$U(t,v) := \left(e^{-\beta t}a(t)\right)\frac{1-b(t)}{b(t)}\left(\frac{1}{1-b(t)}\left(v-F(t)\right)\right)^{b(t)},\ v \in (F(t),\infty)$$

with $\beta \geqslant 0$*,* $b:[0,T] \to (-\infty,1)\backslash\{0\}$*,* $a(t) > 0$ *and* $F(t) \geqslant 0$*.*

$b(t)$ is the coefficient of risk aversion that varies by time, $F(t)$ denotes a time-depending floor. A time-dependent utility function is exemplarily used for assigning a utility to an intermediate consumption-rate process. The time-dependent HARA utility function in Definition 2.13 is a usual, time-independent HARA utility function in Definition 2.12 for every fixed $t \in [0,T]$. By this, analogue implications on the shape of the utility function and the measures of risk aversion as for the time-independent HARA utility apply.

3 Behavioral Portfolio Choice under Hyperbolic Absolute Risk Aversion

> Human beings cannot comprehend very large or very small numbers. It would be useful for us to acknowledge that fact.
>
> Daniel Kahneman

This chapter considers the stochastic portfolio optimization problem for a pension fund investor under the behavioral finance concept Cumulative Prospect Theory (CPT) and is a reproduction of Escobar-Anel et al. (2020a), with minor changes, and few parts of Escobar-Anel et al. (2020b). The motivation is two-fold: First, portfolio selection problems that maximize the expected utility of a globally concave utility function, postulating a globally risk-averse behavior of investors, are extensively studied in the literature. However, studies by Tversky and Kahneman (1992) and Abdellaoui et al. (2013) have shown that investors do not behave globally risk-averse. Ariely (2008) posed the legitimate question: "Wouldn't economics make a lot more sense if it were based on how people actually behave, instead of how they should behave?". For this sake, we consider a behavioral model that reflects investors' true reactions more adequately; hence the considered model comes closer to the true behavior of investors compared to the traditional expected utility theory model. Another piece of motivation arise from a practical problem that pension funds have to face. According to Escobar-Anel et al. (2020b), pension funds have a need to insure their portfolios against downside risk to meet their future liabilities especially in distress and volatile market environments. Non-anticipated shocks or negative interest rates, jumps, crashes or overnight trading restrictions in asset prices could drop pension fund portfolios below desired levels (present value of pension obligations) making them underfunded with pension assets–to–pension liabilities ratio below 100%. In particular within the current low interest rate environment, a high number of pension funds happen to be underfunded which is a severe practical problem. Because of such scenarios there is a need for an investment strategy which covers both the case of funded and underfunded portfolios. In underfunded scenarios, the well-known Constant Proportion Portfolio Insurance (CPPI) strategies cannot be applied anymore. Hence there is a need for alternative suitable investment strategies for pension funds which work on both sides of the obligations, i.e. for funded and underfunded funds. Escobar-Anel et al. (2020b) study this practical problem of underfunded pension funds to achieve their investment goals (becoming well-funded) in the specialized framework of a CRRA utility function and provide economic insights into the properties and performance of the optimal investment strategy that arise from the behavioral portfolio selection problem and its deviation from a classical CPPI. In what follows we offer an optimal investment strategy for well-funded as well as underfunded funds independently of the reason for falling in the underfunded area, that is founded on Cumulative Prospect Theory (CPT).

CPT was introduced by Tversky and Kahneman (1992) and overcomes some drawbacks of Expected Utility Theory (see von Neumann and Morgenstern (1944); Merton (1969); Allais (1953); Ellsberg (1961); Friedman and Savage (1948); Mehra and Prescott (1985)). The experimental study on the behavior of financial professionals by Abdellaoui et al. (2013) supports CPT as decision making model. The novelties compared to the traditional Expected Utility Theory (EUT) are the following:

1. People evaluate assets in terms of gains and losses (with respect to a reference wealth) and not on total positions.
2. People behave risk-averse on gains and risk-seeking on losses, and not uniformly risk-averse.
3. People overweight small and underweight large probabilities.
4. People are more sensitive to losses than to gains, known as "loss aversion".

If the reference wealth is interpreted as the value of the liabilities of a pension fund, then the above-mentioned gains can be viewed as wealths of well-funded portfolios (sum of pension assets exceeds liabilities), whereas losses are associated with wealths of underfunded portfolios (sum of pension assets fails liabilities).

For academic literature on Cumulative Prospect Theory and related work we refer to Kahneman and Tversky (1979), Tversky and Wakker (1993), Fennema and Wakker (1997), Chateauneuf and Wakker (1999), Wu and Gonzalez (1999), Wakker and Zank (2002), Baucells and Heukamp (2006), Abdellaoui et al. (2007), Schmidt and Zank (2008) and Kontek and Lewandowski (2018). For some published work on Cumulative Prospect Theory related to portfolio selection problems we refer to Jin et al. (2008), Jin and Zhou (2008), Bernard and Ghossoub (2010), Zhou (2010), He and Zhou (2011a), He and Zhou (2011b), Jin et al. (2011), Rásonyi and Rodrigues (2012), Jin and Zhou (2013), Rásonyi and Rodrigues (2014) and Xu (2016). In particular, Jin and Zhou (2008) provide a solution to the behavioral portfolio selection problem in the special case of an infinite right-hand limit of the utility function's derivative at the point where the utility function turns from concavity to convexity, i.e. the utility function on gains satisfies both Inada conditions. This solution, since the utility function is of a general form, still depends on two unknown variables, called c and v_+. c can be regarded as the threshold for distinguishing good states of the market from bad ones, v_+ can be interpreted as initial budget distributed to a gain part problem, the remainder is distributed to a loss part problem, both arising from separating the S-shaped utility function. For a specific utility function, namely a CRRA or Power utility function, Jin and Zhou (2008) provide a closed-form solution. The resulting general model, which consists of an S-shaped utility and probability distortion functions, is described in Section 3.1. Jin and Zhou (2008) were the first to present a general solution to this kind of behavioral portfolio selection problem. Since they impose several conditions in order to solve the problem, the possible choices for utility and probability distortion functions are limited.

In particular, utility functions on gains which do not satisfy the Inada condition at zero, for instance those with globally finite hyperbolic absolute risk aversion such as HARA utility, are not included in their framework. Relaxing the Inada condition at zero leads to an increased complexity in obtaining the optimal solution to the behavioral portfolio problem. Moreover, closed-form solutions are only presented for CRRA utilities. In this chapter, this is extended to a broader class of investor's risk preferences, a utility function with positive but not necessarily infinite derivative at the arbitrary reference wealth in general, and HARA utility in particular. The latter covers the constant as well as the non-constant relative risk aversion coefficient case. More explicitly, the considered HARA utility function represents a utility function with increasing relative risk aversion, where the coefficient of relative risk aversion increases and changes from negative (within losses) to positive (within gains). Under an expected utility treatment of risk aversion, increasing relative risk aversion is known to model a decrease in the fraction of the wealth or portfolio held in the risky asset when the (pension fund) investor experiences an increase in the fund wealth. In addition to this expected utility view on risk aversion, behavioral objects such as probability distortion functions and a reference wealth are added which also affect risk aversion. In a standard Black-Scholes financial market model, the main contributions of this chapter comprise

1. the main theoretical results in Corollary 3.6 and Theorems 3.8 and 3.12, which state a parametric representation of and provide a closed-form solution to the portfolio optimization problem in CPT in general and for HARA utility,

2. a full validation of a setting in Section 3.2.3, i.e. utility and probability distortion function, where a CRRA and HARA utility setup is to be shown feasible and closed-form solutions for the optimal terminal fund wealth and replicating strategy in CPT are provided,

3. and the insight from the numerical case study in Section 3.3 that HARA (CPT) investors outperform CRRA (CPT) and HARA (EUT) investors in bull markets, but underperform in bear markets when a concave probability distortion on the gains is applied. When the pension fund applies a convex probability distortion on the gains, then she can benefit from her behavioral trait within bear markets and suffers within bull markets compared to the HARA (EUT) model. Moreover, the empirical study shows that CPT can lead to a mispricing or even bubbles in stock markets when a concave distortion on the gains is considered, while investors applying HARA utility stronger contribute to such mispricing or bubbles than CRRA utility investors, both within CPT. In opposite, a convex distortion on the gains can help explaining situations in which investors are underinvested in risky assets in their portfolios.

In summary, our work contributes compared to Jin and Zhou (2008) especially in the way that the Inada condition at zero imposed on the gains utility is relaxed which allows for an application of HARA utility. Thus the framework is generalized. The statements and proofs are to be modified in a non-trivial way and we additionally apply a certain probability distortion function which is analytically tractable and leads to a well-posed problem for some parameter choices when it is applied on the positive and negative part. This is in contrast to the distortion proposed by Jin and Zhou (2008) which leads to an ill-posed problem when applied on both parts.

Furthermore, we would like to mention the paper by Xu (2016) which uses another ansatz for solving the standalone positive part problem (concave utility with probability distortion function) that allows for some further relaxations of assumptions. For our specifications, HARA utility function and a certain probability distortion function family, this does not make a difference to our HARA solution as all conditions are automatically fulfilled in both cases, i.e. Xu (2016) and our technical assumptions. Instead we do not stay with the theoretical concept but focus on a HARA utility and a special distortion giving us existence and an easier closed-form solution. In addition, we also solve the negative part. In particular, the consideration of the negative part with convex utility function and therefore an overall S-shaped utility function with probability weighting functions on both gains and losses is claimed by CPT. Moreover, we demonstrate how the separate or individual solutions for the positive and negative part have to be glued together optimally and provide conditions under which the pension fund always stays in the gain area. Additionally, we also provide the optimal solution in terms of investment strategy and fund wealth process in case the pension fund does not stay in the positive part area for sure, which is not covered by Xu (2016). Finally, we also give a comparison to other strategies plus interpretation.

The remainder of this chapter is organized as follows. In Section 3.1 the considered Black-Scholes financial market model, which consists of one risk-free asset and N risky assets, the general behavioral model, the resulting behavioral portfolio optimization problem and its mathematical objectives are introduced. Section 3.2 represents the main part of this chapter. It covers the presentation of the optimal form of the solution for a general S-shaped utility function which depends on two remaining and unknown parameters, called c and v_+. Moreover, explicit formulas are provided for a HARA utility function. Sufficient conditions for well-posedness are stated, and additionally sufficient conditions on the initial wealth and the probability distortion functions are given such that the final solution is obtained analytically. In other cases, the solution is to be found numerically. Finally, a special probability distortion function family is shown to satisfy any placed assumption and therefore leads to an analytical, closed-form solution of the behavioral portfolio selection problem. In addition, a numerical case study in Section 3.3 empirically examines and visualizes the optimal portfolios. The Appendix A to this chapter covers the proofs: the proofs of the main results for general probability distortion functions are to be found in Appendix A.1, the proofs with respect to the special applied probability distortion function family are summarized in Appendix A.2.

3.1 CPT Model and Framework

Let us consider the financial market model introduced in Section 2.1. The CPT objectives are modeled in what follows. Let the reference point $B = B(T)$ at terminal time T be a bounded and $\mathcal{F}_T$-measurable random variable which satisfies $\mathbb{E}[\tilde{Z}B] < \infty$. B determines the threshold between gains and losses[1] at time T. Pension funds with a wealth that is above/below the present value of B (liabilities) can be interpreted as well-funded/underfunded portfolios. Next we are given two utility functions U_+ and U_- and two distortions in probability w_+ and w_-, one for the gains and one for the losses, which are subject to the following assumption (cf. Rásonyi and Rodrigues (2012); Jin and Zhou (2008)):

Assumption 3.1.

1. $U_+, U_- : \mathbb{R}^+ \to \mathbb{R}^+$ *are strictly increasing and concave, with* $U_+(0) = U_-(0) = 0$. *Moreover,* U_- *is strictly concave at* 0, U_+ *is strictly concave and twice differentiable with* $U'_+(\infty) := \lim_{v \nearrow \infty} U'_+(v) = 0$.
2. $w_+, w_- : [0,1] \to [0,1]$ *are differentiable and strictly increasing, with* $w_+(0) = w_-(0) = 0$ *and* $w_+(1) = w_-(1) = 1$.

The positive utility U_+ does not need to satisfy the Inada condition $U'_+(0+) := \lim_{v \searrow 0} U'_+(v) = \infty$. The overall utility function $U : \mathbb{R} \to \mathbb{R}$ is S-shaped and defined by

$$v \mapsto U(v) := U_+(v^+) - U_-(v^-)\,.$$

[1] Terminal wealths above B are called gains, terminal wealths below B losses. Therefore, a gain is not always associated with a positive return and a loss not always with a negative return, it depends on the reference point.

This utility describes an investor who is risk-averse in the gains ($v > 0$, later $v > B$) and risk-seeking in the losses ($v < 0$, later $v < B$). In accordance with Rásonyi and Rodrigues (2012) and Jin and Zhou (2008) the objective functional for any contingent claim X is $\mathcal{J}(X) := \mathcal{J}_+(X^+) - \mathcal{J}_-(X^-)$ with

$$\mathcal{J}_\pm(X^\pm) := \int_0^\infty w_\pm(\mathbb{P}(U_\pm(X^\pm) > x))dx = \mathbb{E}\left[U_\pm(X^\pm)w'_\pm(1 - F_{X^\pm}(X^\pm))\right] . \tag{3.1}$$

The meaning of the functions $w_\pm$ can be seen from Eq. (3.1). They distort the probabilities to create the effect of overweighting extreme events while underweighting average events. The portfolio selection problem then is maximizing $\mathcal{J}(V(T) - B)$ to the initial wealth v_0 under the condition of admissible relative portfolio processes π:

$$\mathcal{V}(v_0) = \sup_{\pi\in\Lambda} \mathcal{J}(\pi; v_0),\ \mathcal{J}(\pi; v_0) = \mathcal{J}(V(T) - B). \tag{3.2}$$

Λ denotes the set of admissible investment strategies π that satisfy the following set of conditions[2]:

$A^{(3)}$: $\pi \in \tilde{\Lambda}(v_0)$.

$B^{(3)}$: π admits a unique solution to Eq. (2.5).

$C^{(3)}$: π fulfills the associated budget constraint

$$\mathbb{E}\left[\tilde{Z}(T)V(T)\right] = v_0. \tag{3.3}$$

Problem (3.2) is tackled by the Martingale approach. Without loss of generality[3] assume $B = 0$ throughout when not expressed otherwise. The associated terminal wealth problem is

$$\begin{aligned} &\mathcal{V}(v_0) = \sup_X \mathcal{J}(X) \\ &\text{subject to } \begin{cases} \text{Initial wealth } v_0, \text{ i.e. } \mathbb{E}[\tilde{Z}X] = v_0, \\ X \text{ is } \mathcal{F}_T\text{-measurable and almost surely bounded from below.} \end{cases} \end{aligned} \tag{P}$$

The optimal final payoff arises from solving Problem (P), replicating this payoff leads to the optimal investment strategy. X to be a.s. lower bounded is required to obtain an admissible replicating trading strategy.

HARA Utility Function. Let U_+ and U_- be selected out of the family of HARA utility functions, defined in line with Section 2.4, such that

$$\begin{aligned} &U_+ : \mathbb{R}^+ \to \mathbb{R}^+,\ v \mapsto U_+(v) := \frac{1-b_H}{b_H}\left[\frac{a_H}{1-b_H}v + d_H\right]^{b_H} - \frac{1-b_H}{b_H}d_H^{b_H}, \\ &U_- : \mathbb{R}^+ \to \mathbb{R}^+,\ v \mapsto U_-(v) := k_{H-}U_+(v), \end{aligned} \tag{3.4}$$

[2] The superscript (here: $\cdot^{(3)}$) marks the chapter, or later also the section, in which the condition is introduced.

[3] Adding B and replacing v_0 by $v_0 - \mathbb{E}[\tilde{Z}B]$ in the solution for $B = 0$ is the solution for a non-zero B. Due to the completeness of the market, there exist a replicating wealth process and trading strategy for any stochastic B. We will include a general B if necessary and helpful when it comes to interpretation, for instance in the discussion about risk aversion in the following paragraph.

with parameters $0 < b_H < 1$, $a_H > 0$, $d_H \geqslant 0$ and $k_{H-} > 1$. The latter parameter accounts for the phenomenon of loss aversion. As mentioned in Section 2.4, the HARA utility covers the special case of CRRA utility for $d_H = 0$ and the Logarithmic utility as limiting case $b_H \to 0$. Moreover, HARA utility clearly satisfies Assumption 3.1 on the utility function. For the first derivative of U_+ ($U'_- = k_{H-}U'_+$) and its inverse it generally holds

$$U'_+ : \mathbb{R}^+ \to (0, a_H d_H^{-(1-b_H)}),\ v \mapsto U'_+(v) = a_H \left[\frac{a_H}{1-b_H} v + d_H\right]^{-(1-b_H)},$$
$$(U'^{-1}_+) : (0, a_H d_H^{-(1-b_H)}) \to \mathbb{R}^+,\ y \mapsto (U'_+)^{-1}(y) = \frac{1-b_H}{a_H}\left[\left(\frac{y}{a_H}\right)^{-\frac{1}{1-b_H}} - d_H\right].$$

In accordance with Definition 2.10, the associated Arrow-Pratt measure $\mathcal{A}$ of absolute risk aversion for the gain and loss part admits the following hyperbolic representation

$$\mathcal{A}_\pm(v) := -\frac{U''_\pm(v)}{U'_\pm(v)} = -\frac{\partial}{\partial v} \ln U'_\pm(v) = \frac{1}{\frac{1}{1-b_H} v + \frac{d_H}{a_H}} = \frac{1-b_H}{v + \frac{1-b_H}{a_H} d_H} > 0,\ v \in \mathbb{R}^+.$$

The overall Arrow-Pratt measure $\mathcal{A}$ which corresponds to U can be written as

$$\mathcal{A}(v) := \mathcal{A}_+(v^+)\mathbb{1}_{v\geqslant 0} - \mathcal{A}_-(v^-)\mathbb{1}_{v<0} = sgn(v)\frac{1-b_H}{|v| + \frac{1-b_H}{a_H} d_H}, \tag{3.5}$$

where *sgn* denotes the standard sign function. One can observe $\mathcal{A} > 0$ on gains (risk-averse behavior) and $\mathcal{A} < 0$ on losses (risk-seeking behavior). Moreover, $\mathcal{A}$ decreases for both gains and losses because

$$\mathcal{A}'(v) = -\frac{1-b_H}{\left(|v| + \frac{1-b_H}{a_H} d_H\right)^2} < 0,\ v \in \mathbb{R}\backslash\{0\}.$$

The coefficient of relative risk aversion $\mathcal{R}$, according to Definition 2.11, is given by

$$\mathcal{R}(v) = v\mathcal{A}(v) = v\mathcal{A}_+(v^+)\mathbb{1}_{v\geqslant 0} - v\mathcal{A}_-(v^-)\mathbb{1}_{v<0} = (1-b_H)\frac{|v|}{|v| + \frac{1-b_H}{a_H} d_H}. \tag{3.6}$$

The first derivative of $\mathcal{R}(v)$ with respect to v can be calculated as

$$\mathcal{R}'(v) = sgn(v)(1-b_H)\frac{\frac{1-b_H}{a_H} d_H}{\left(|v| + \frac{1-b_H}{a_H} d_H\right)^2},\ v \in \mathbb{R}\backslash\{0\}.$$

This shows that HARA utility inside the CPT concept provides increasing relative risk aversion $\mathcal{R}(v)$ on gains ($v \geqslant 0$) and decreasing relative risk aversion $\mathcal{R}(v)$ on losses ($v < 0$). The economic intuition behind $\mathcal{R}$ is that increasing relative risk aversion means a decrease in the fraction of the wealth allocated to the risky asset when the investor experiences an increase in wealth (see Pratt (1964), Eeckhoudt et al. (2005)). Inside the loss area, the decreasing $\mathcal{R}$ value therefore implies that the relative risky exposure is raised if one comes closer to the gains in order to jump into the gain area. This is also reflected by $\mathcal{A}$ which is negative and smallest in the loss region close to the gain region. Moreover, notice that $\mathcal{R} > 0$ on gains and losses. For losses, this is only due to the

multiplication with $v < 0$ which changes the sign of $\mathcal{R}$. Power utility ($d_H = 0$) is referred to as constant relative risk aversion (CRRA) because relative risk aversion is constant within gains and within losses since $\mathcal{R}(v) \equiv 1 - b_H$. The corresponding coefficient of absolute risk aversion is given by $\mathcal{A}(v) = \frac{1-b_H}{v}$.
Note that the general objective function to be maximized is $\mathcal{J}(V(T) - B)$. If now U is evaluated at $v - B$ instead of v, which we denote by $U^{(B)}$, i.e.

$$U^{(B)}(v) := U_+^{(B)}(v) - U_-^{(B)}(v)$$

with

$$\begin{aligned} U_+^{(B)}(v) &:= U_+((v-B)^+) = U_+(v-B)\mathbb{1}_{v \geqslant B}, \\ U_-^{(B)}(v) &:= U_-((v-B)^-) = U_-(B-v)\mathbb{1}_{v<B} = k_{H-}U_+(B-v)\mathbb{1}_{v<B}, \end{aligned}$$

then $\mathcal{A}_\pm$ simply generalizes to

$$\begin{aligned} \mathcal{A}_+^{(B)}(v) &:= -\frac{\left(U_+^{(B)}\right)''(v)}{\left(U_+^{(B)}\right)'(v)} = \frac{1-b_H}{v - B + \frac{1-b_H}{a_H} d_H} > 0, \ v \geqslant B, \\ \mathcal{A}_-^{(B)}(v) &:= -\frac{\left(U_-^{(B)}\right)''(v)}{\left(U_-^{(B)}\right)'(v)} = \frac{1-b_H}{B - v + \frac{1-b_H}{a_H} d_H} > 0, \ v < B. \end{aligned}$$

The overall Arrow-Pratt measure $\mathcal{A}^{(B)}$ which corresponds to $U^{(B)}$ can then be formulated as

$$\mathcal{A}^{(B)}(v) := \mathcal{A}_+^{(B)}(v)\mathbb{1}_{v \geqslant B} - \mathcal{A}_-^{(B)}(v)\mathbb{1}_{v<B} = sgn(v-B)\frac{1-b_H}{|v-B| + \frac{1-b_H}{a_H} d_H}. \tag{3.7}$$

We again observe $\mathcal{A} > 0$ on gains (risk-averse behavior) and $\mathcal{A} < 0$ on losses (risk-seeking behavior). Likewise, $\mathcal{A}$ decreases for both gains and losses since

$$\left(\mathcal{A}^{(B)}\right)'(v) = -\frac{1-b_H}{\left(|v-B| + \frac{1-b_H}{a_H} d_H\right)^2} < 0, \ v \in \mathbb{R}\backslash\{B\}.$$

Furthermore, the coefficient of relative risk aversion $\mathcal{R}$ becomes

$$\mathcal{R}^{(B)}(v) := v\mathcal{A}^{(B)}(v) = sgn(v-B)(1-b_H)\frac{v}{|v-B| + \frac{1-b_H}{a_H} d_H}. \tag{3.8}$$

We now want to examine the slope of $\mathcal{R}^{(B)}(v)$. We obtain the following for the gain part ($v \geqslant B$) and the loss part ($v < B$):

$$\begin{aligned} \left(\mathcal{R}^{(B)}\right)'(v) &= -(1-b_H)\frac{B - \frac{1-b_H}{a_H} d_H}{\left(v - B + \frac{1-b_H}{a_H} d_H\right)^2}, \ v > B, \\ \left(\mathcal{R}^{(B)}\right)'(v) &= -(1-b_H)\frac{B + \frac{1-b_H}{a_H} d_H}{\left(B - v + \frac{1-b_H}{a_H} d_H\right)^2}, \ v < B. \end{aligned}$$

From this we infer that $\left(\mathcal{R}^{(B)}\right)'(v) \gtreqless 0$ if $B \lesseqgtr \frac{1-b_H}{a_H} d_H$ for the gain part and $\left(\mathcal{R}^{(B)}\right)'(v) \gtreqless 0$ if $B \lesseqgtr -\frac{1-b_H}{a_H} d_H$ for the loss part. Thus, we detect that the applied HARA utility in the CPT framework can lead to either an increasing, constant or decreasing relative risk aversion coefficient $\mathcal{R}^{(B)}$ in the gain area. The same applies to the loss region independently of the gain area. In addition, multiple pairwise combinations of different directions (different sign of the slopes) for the gain and loss part are possible. Suppose exemplarily $-\frac{1-b_H}{a_H} d_H < B < \frac{1-b_H}{a_H} d_H$ for the moment (this covers the special case $B = 0$). Then $\mathcal{R}^{(B)}(v)$ increases on gains ($v \geqslant B$), but decreases on losses ($v < B$). In opposite, CRRA utility inside CPT ($d_H = 0$) always provides either globally increasing, constant or decreasing relative risk aversion since $\left(\mathcal{R}^{(B)}\right)'(v) \gtreqless 0$ if $B \lesseqgtr 0$ on both the gains and losses. A sign change in $\left(\mathcal{R}^{(B)}\right)'(v)$ from gains to losses is ruled out by CRRA utility function. This shows the flexibility of HARA over CRRA.

3.2 Main Results

We derive the general solution to the terminal wealth problem (P) for a general utility function satisfying $U_+'(0+) > 0$, followed by stating the general solution, providing conditions for well-posedness and closed-form solutions for HARA utility. In particular, we employ the classical martingale method and adapt it to our special setup; for orientation we refer to Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989). In addition, a feasible probability distortion function family is presented thereafter, which under some condition leads to a closed-form solution when HARA utility is applied.

3.2.1 General Solution for $\mathbf{U_+'(0+) > 0}$

The solution for the special case where $U_+'(0+) = \infty$ is already given in Jin and Zhou (2008). For instance, CRRA satisfies this condition. In opposite, HARA utility as defined in (3.4) for $d_H \neq 0$ violates $U_+'(0+) = \infty$, but still fulfills $U_+'(0+) = a_H d_H^{-(1-b_H)} > 0$. Therefore, let $U_+'(0+) \in (0, \infty)$ from now on (the derived general solution turns out to cover $U_+'(0+) = \infty$ as a limiting case). Let us place the following two technical assumptions.

Assumption 3.2.

1. *$z \mapsto \frac{F_{\tilde{Z}}^{-1}(z)}{w_+'(z)}$ is a continuous function in $z \in (0, 1)$.*
2. $\lim\limits_{z \searrow 0} \frac{F_{\tilde{Z}}^{-1}(z)}{w_+'(z)} = 0.$
3. $\lim\limits_{z \nearrow 1} \frac{F_{\tilde{Z}}^{-1}(z)}{w_+'(z)} = \infty.$

Assumption 3.3.

1. *$F_{\tilde{Z}}^{-1}(z)/w_+'(z)$ is non-decreasing in $z \in (0, 1]$ (monotonicity condition).*
2. $\mathbb{E}\left[U_+\left(\left(U_+'\right)^{-1}\left(\frac{\eta \tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\right) w_+'(F_{\tilde{Z}}(\tilde{Z}))\right] < \infty \ \forall \eta > 0.$

3. *There exists* $z \in (0,1)$ *with* $f(z) = 0$ *for any arbitrary* $a > 0$ *and* $0 < c \leqslant \infty$, *where* $f : (0,1) \to \mathbb{R}$ *is defined by*

$$z \mapsto f(z) := \lambda(z) - U'_+(0+)\frac{w'_+(1-z)}{F_{\tilde{Z}}^{-1}(1-z)},$$

with $\lambda(z) \geqslant 0$ *solving the equation*

$$\mathbb{E}\left[\tilde{Z}(U'_+)^{-1}\left(\frac{\lambda(z)\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right] = a\,.$$

Assumptions 3.2 and 3.3.1 ensure continuity, suitable limits and monotonicity of the term $\frac{F_{\tilde{Z}}^{-1}(z)}{w'_+(z)}$. Both assumptions are especially needed to obtain a closed-form representation to Problem (A.1) in Appendix A.1, obtaining an analytical expression would be an open problem otherwise. Assumption 3.3.2 ensures that the problem is well-posed, 3.3.3 is a technical assumption.

Let $\mathcal{J}_+(X) < \infty$ for any non-negative, $\mathcal{F}_T$-measurable random variable X satisfying $\mathbb{E}[\tilde{Z}X] < \infty$. In the following we modify and generalize the approach by Jin and Zhou (2008) which splits the terminal wealth problem (P) into three sub-problems; two of them have to be re-solved due to the more general condition $U'_+(0+) > 0$. Let $c \geqslant 0$ and $v_+ \geqslant v_0^+$. The main general result states the optimal terminal wealth solution to Problem (P) in dependence of the solutions to the following two sub-problems:

$$\begin{aligned}&\mathcal{Y}_+(c, v_+) = \sup_X \mathcal{J}_+(X)\\ &\text{subject to} \begin{cases}\mathcal{J}_+(X) = \mathbb{E}\left[U_+(X)w'_+(1-F_X(X))\right],\\ \mathbb{E}[\tilde{Z}X] = v_+,\ X \geqslant 0 \text{ a.s.},\ X = 0 \text{ a.s. on } \{\omega : \tilde{Z} > c\}.\end{cases}\end{aligned} \tag{P$_+$}$$

Let $\mathcal{Y}_+(c, v_+)$ denote the optimal value of (P$_+$) for given c and v_+.

$$\begin{aligned}&\sup_{c,v_+} \mathcal{Y}(c, v_+)\\ &\text{subject to} \begin{cases}\mathcal{Y}(c, v_+) = \mathcal{Y}_+(c, v_+) - U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1 - F_{\tilde{Z}}(c)),\\ 0 \leqslant c \leqslant \infty,\ v_+ \geqslant v_0^+,\\ v_+ = 0 \text{ when } c = 0,\ v_+ = v_0 \text{ when } c = \infty,\end{cases}\end{aligned} \tag{P*}$$

where the convention

$$U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1 - F_{\tilde{Z}}(c)) := 0$$

when $c = \infty$ and $v_+ = v_0$ applies. The next theorem provides the solution to the terminal wealth problem (P) when Problems (P$_+$) and (P*) are solved. Its proof is analogue to the proof of Theorem 4.1(ii) in Jin and Zhou (2008), the required arguments still hold in our setup.

Theorem 3.4. *Let* $(c^\star, v_+^\star)$ *be optimal for Problem* (P*) *and* $X_+^\star$ *optimal for Problem* (P$_+$) *with parameters* $(c^\star, v_+^\star)$, *then* $X^\star := X_+^\star \mathbb{1}_{\tilde{Z}\leqslant c^\star} - \frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c^\star}]}\mathbb{1}_{\tilde{Z}>c^\star}$ *is optimal for Problem* (P).

We remind the reader that all proofs to this chapter are stored in Appendix A. Theorem 3.4 shows that only $c^\star$, $v_+^\star$ to Problem (P*) and $X_+^\star$ to Problem (P$_+$) with parameters $(c^\star, v_+^\star)$ need to be determined. The solution to the last objective, $X_+^\star$, is given by the upcoming theorem.

Theorem 3.5. *Let $0 \leqslant c \leqslant \infty$, and $v_+ \geqslant v_0^+$.*

1. *If $v_+ = 0$, then the optimal solution of Problem* (P$_+$) *is $X_+^\star = 0$ and $\mathcal{Y}_+(c, v_+) = 0$.*
2. *If $v_+ > 0$ and $c = 0$, then there is no feasible solution to Problem* (P$_+$) *and $\mathcal{Y}_+(c, v_+) = -\infty$.*
3. *If $v_+ > 0$ and $0 < c \leqslant \infty$, then the optimal solution to Problem* (P$_+$) *is*

$$X_+^\star = X_+^\star(\lambda) = (U_+')^{-1}\left(\lambda \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))},$$

with the optimal value

$$\mathcal{Y}_+(c, v_+) = \mathbb{E}\left[U_+\left((U_+')^{-1}\left(\lambda \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right) w_+'(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z} \leqslant c}\right],$$

where $\lambda > 0$ is the real number satisfying $\mathbb{E}[\tilde{Z}X_+^\star(\lambda)] = v_+$ and $\hat{z} \in (0,1)$ such that $\lambda \frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})} = U_+'(0+)$.

The next corollary summarizes our findings for Problems (P*) and (P) in the non-trivial Case 3 in Theorem 3.5.

Corollary 3.6.

1. *Let $v_+ > 0$ and $0 < c \leqslant \infty$. Then the objective function of Problem* (P*) *specializes to*

$$\begin{aligned}\mathcal{Y}(c, v_+) = \mathbb{E}&\left[U_+\left((U_+')^{-1}\left(\lambda \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right) w_+'(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z} \leqslant c}\right] \\ &- U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1 - F_{\tilde{Z}}(c)).\end{aligned}$$

2. *Let $(c^\star, v_+^\star)$ denote the solution to Problem* (P*)*, and let $v_+^\star > 0$ and $0 < c^\star \leqslant \infty$. Then the solution to Problem* (P) *is*

$$X^\star = (U_+')^{-1}\left(\lambda^\star \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c^\star)(1-\hat{z}^\star))} - \frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c^\star}]}\mathbb{1}_{\tilde{Z}>c^\star},$$

where $\lambda^\star > 0$ and $\hat{z}^\star \in (0,1)$ are defined accordingly.

$\hat{z}^\star$ in Corollary 3.6 determines the probability of ending in the gain, neutral or loss area to be

$$\begin{aligned}&\mathbb{P}(X^\star > 0) = \mathbb{P}(\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c^\star)(1-\hat{z}^\star))) = F_{\tilde{Z}}(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c^\star)(1-\hat{z}^\star))) = F_{\tilde{Z}}(c^\star)(1-\hat{z}^\star),\\ &\mathbb{P}(X^\star = 0) = F_{\tilde{Z}}(c^\star)\hat{z}^\star,\\ &\mathbb{P}(X^\star < 0) = 1 - F_{\tilde{Z}}(c^\star).\end{aligned}$$

It is worth to mention that forcing $\hat{z}^\star := 0$, which is equivalent to $U'_+(0+) = \infty$ and means that the utility on gains satisfies the Inada condition at zero, solves the problem in the Jin and Zhou (2008) world. When instead forcing $c^\star := \infty$, this coincides with the Xu (2016) framework of a globally concave utility function and we obtain the corresponding special solution. Hence, Corollary 3.6 indeed is an extension of Jin and Zhou (2008) and Xu (2016), but covers both situations as special cases.

3.2.2 HARA Utility Function

Within this section an S-shaped HARA utility function as defined in (3.4) is applied. We provide the general solution to Problem (P), state conditions for well-posedness and provide conditions on the probability distortion functions such that the solution can be expressed in closed-form. But firstly, the following lemma shows how HARA utility simplifies placed assumptions.

Lemma 3.7. *For the HARA utility function defined by* (3.4)*, Assumption 3.3.3 is fulfilled under Assumption 3.2, and Assumption 3.3.2 is fulfilled if* $\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\right] < \infty$.

General Solution (HARA Utility). The following theorem summarizes the formulas for the objective value function of Problem (P*) and the optimal terminal wealth to Problem (P) under HARA utility.

Theorem 3.8.

1. *The value function $\mathcal{Y}(c, v_+)$ of Problem* (P*) *which is to be maximized is given by*

$$\mathcal{Y}(c, v_+) = \mathcal{Y}_+(c, v_+) - U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1 - F_{\tilde{Z}}(c)),$$

with

$$\begin{aligned}\mathcal{Y}_+(c, v_+) = {} & (1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left(v_+ + \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{b_H} \\ & \times\left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{1-b_H} \\ & - \frac{1-b_H}{b_H}d_H^{b_H}\mathbb{E}\left[w'_+(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\end{aligned}$$

and

$$\begin{aligned}& U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1 - F_{\tilde{Z}}(c)) \\ & = k_{H-}(1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left[\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]} + \frac{1-b_H}{a_H}d_H\right]^{b_H} w_-(1 - F_{\tilde{Z}}(c)) \\ & - k_{H-}\frac{1-b_H}{b_H}d_H^{b_H}w_-(1 - F_{\tilde{Z}}(c)).\end{aligned}$$

2. *The solution* $X_+^\star = X_+^\star(c, v_+)$ *to Problem* (P_+) *for general* c *and* v_+*, with* $v_+ > 0$ *and* $0 < c \leqslant \infty$*, is given by*

$$X_+^\star = \left[\left(\frac{v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]}{\mathbb{E}\left[\tilde{Z} \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]} \right) \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} - \frac{1-b_H}{a_H} d_H \right] \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}. \tag{3.9}$$

Let the optimal pair $(c^\star, v_+^\star)$ *to Problem* $(\mathrm{P}^\star)$ *be given with* $v_+^\star > 0$ *and* $0 < c^\star \leqslant \infty$*. Then the optimal terminal wealth to Problem* (P) *is*

$$X^\star = X_+^\star(c^\star, v_+^\star) \mathbb{1}_{\tilde{Z} \leqslant c^\star} - \frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z} \mathbb{1}_{\tilde{Z} > c^\star}]} \mathbb{1}_{\tilde{Z} > c^\star}.$$

Replicating the optimal terminal wealth $X^\star$ in Theorem 3.8 gives the optimal asset allocation strategy of the pension fund investor for a well-funded ($v_0 \geqslant 0$) and an underfunded ($v_0 < 0$) portfolio under HARA utility. Notice that the optimal investment strategy can easily be determined by Section 2.3 (cf. distortion function selection in Section 3.2.3) if the expression $\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}$ is of the form $\tilde{Z}^\eta$ for some $\eta \in \mathbb{R}$ because $X^\star$ then becomes the sum of digital options.

Well-Posedness (HARA Utility). After stating the final main result for HARA utility, we address the issue of well-posedness of Problem (P). Define

$$v_H(c, v_+) := \left(v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}[\tilde{Z}]\right)^{b_H} \left(\mathbb{E}\left[\tilde{Z} \left(\frac{w'_+(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}}\right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant c}\right]\right)^{1-b_H} - k_{H-} \frac{w_-(1 - F_{\tilde{Z}}(c))}{\left(\mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} > c}\right]\right)^{b_H}} (v_+ - v_0)^{b_H}$$

for $0 \leqslant c \leqslant \infty$, $v_+ \geqslant v_0^+$.

Lemma 3.9. *If* $\sup_{c \geqslant 0, v_+ \geqslant v_0^+} v_H(c, v_+) < \infty$*, then Problem* (P) *is well-posed.*

Now define

$$k_H(c) := k_{H-} \frac{w_-(1 - F_{\tilde{Z}}(c))}{\left(\mathbb{E}\left[\tilde{Z} \left(\frac{w'_+(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}}\right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant c}\right]\right)^{1-b_H} \left(\mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} > c}\right]\right)^{b_H}}, \quad c \geqslant 0. \tag{3.10}$$

Then

$$v_H(c, v_+) = \left(\mathbb{E}\left[\tilde{Z}\left(\frac{w'_+(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}}\right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z}\leqslant c}\right]\right)^{1-b_H} \times \left\{\left(v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}[\tilde{Z}]\right)^{b_H} - k_H(c)\,(v_+ - v_0)^{b_H}\right\}.$$

The next theorem, where the cases $v_0 \geqslant 0$ and $v_0 < 0$ are separated, provides sufficient conditions for Problem (P) to be well-posed.

Theorem 3.10.

1. *Let* $v_0 \geqslant 0$. *If* $\inf_{c>0} k_H(c) \geqslant 1$, *then the original Problem* (P) *is well-posed.*
2. *Let* $v_0 < 0$. *If* $\inf_{c>0} k_H(c) > 1$, *then the original Problem* (P) *is well-posed.*

Conditions $\inf_{c>0} k_H(c) \geqslant 1$ and $\inf_{c>0} k_H(c) > 1$ ensure that $v_H(c, v_+) < \infty$ and therefore $\mathcal{Y}(c, v_+) < \infty$. Note that the defined $k_H(c)$ coincides with the function $k(c)$ in Jin and Zhou (2008). But its characterization in Theorem 3.10 only leads to sufficient instead of equivalent conditions for well-posedness since $k_H(c)$ is derived via an upper bound for the value function.

Closed-Form Solution (HARA Utility). In what follows we analytically prove $(c^\star, v_+^\star) = (\infty, v_0)$ for Problem (P*) for a sufficiently wealthy investor, hence derive the solution to the portfolio selection problem in closed form. For tractability reasons we concentrate on the well-funded investor for studying closed-form solutions in what follows and place the following additional assumption.

Assumption 3.11.

1. $v_0 > \frac{1-b_H}{a_H} d_H e^{-rT}$.[4]
2. $\inf_{c>0} k_H(c) > 1$.

Assumption 3.11.1 means that the investor needs to start off with a sufficiently high level of wealth – determined by the preferences but non-depending on the perception of probabilities, Assumption 3.11.2 ensures well-posedness. Notice that Assumption 3.11 does not demand for an isolated condition on w_-. Further note that both conditions in Assumption 3.11 are very similar to the ones derived in Theorem 9.1 by Jin and Zhou (2008) for a CRRA utility ($d_H = 0$). Moreover, under Assumption 3.11 we are able to show that the value function $\mathcal{Y}(c, v_+)$ of Problem (P*) is strictly monotone decreasing in v_+, and $\mathcal{Y}(c, v_0)$ is monotone increasing in c.

Theorem 3.12. *Let Assumption 3.11 hold. Then the value function* $\mathcal{Y}(c, v_+)$ *of Problem* (P*) *is strictly monotone decreasing in* v_+ *for any* $c > 0$. *Moreover,* $\mathcal{Y}(c, v_0)$ *is monotone increasing in* c, *strictly monotone increasing for almost every* c, *and Problem* (P*) *has the unique solution* $c^\star = \infty$ *and* $v_+^\star = v_0$. *Moreover, Problem* (P) *is well-posed and the optimal terminal wealth* ($B = 0$) *is*

[4] The condition reads $v_0 - \mathbb{E}[\tilde{Z}B] > \frac{1-b_H}{a_H} d_H e^{-rT}$ when a general B is considered.

$$X^\star = \left[\left(\frac{v_0 + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}\right]}\right)\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} - \frac{1-b_H}{a_H} d_H\right]\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))},$$

where $\hat{z}(\infty, v_0) \in (0,1)$ *is such that the budget constraint* $\mathbb{E}\left[\tilde{Z}X^\star\right] = v_0$ *holds and the relation to* $\lambda(\infty, v_0) > 0$ *is*

$$\lambda(\infty, v_0)\frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}{w'_+(1-\hat{z}(\infty,v_0))} = U'_+(0+) = a_H d_H^{b_H-1}\,.$$

Theorem 3.12 implies that an investor will always stay in the well-funded area when starting well-funded. Assumption 3.11 describes sufficient but not necessarily required conditions on the involved utility and probability distortion functions such that $c^\star = \infty$ and $v_+^\star = v_0$ is optimal to Problem (P*). Notice that Theorem 3.12 implies that only the distortion function on the positive part, i.e. w_+, and not w_- affects the final outcome, when the distortion function w_- on the negative part is suitably selected according to Assumption 3.11.

3.2.3 A Feasible Probability Distortion Function

Assumption 3.11 introduces an additional assumption made on the probability distortion functions w_+ and w_- in order to prove $(c^\star, v_+^\star) = (\infty, v_0)$ optimal for Problem (P*) for a sufficiently wealthy investor who applies an S-shaped HARA utility function. In this work, a special probability distortion function family is considered and analyzed, which is constructed in a way such that the monotonicity condition in Assumption 3.3.1 is fulfilled. The distortion family is introduced by Brummer et al. (2018) and is inspired by the distortion in Wang (2000). We analyze this distortion family in view of Assumption 3.11, and show that the assumption is satisfied for some choice of parameters.

Definition, Characteristics and Properties. Let Φ denote the distribution function of a standard normal, Φ^{-1} its inverse. Let $r = 0$ and set $\sigma_L := 0$ in the definition by Brummer et al. (2018). The considered distortion is defined by

$$w(p) := \left(\Phi\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)\right)^\alpha, \ p \in [0,1], \tag{3.11}$$

with parameters $\alpha \in (0,1]$, $\delta \in \mathbb{R}$. The distortion in (3.11) can be reduced to the distortion in Wang (2000) when $\alpha = 1$. Brummer et al. (2018) show that w with $\alpha \in (0,1]$ and $\delta \leqslant 1$ fulfills the monotonicity condition in Assumption 3.3.1. Moreover, w with $\alpha \in (0,1)$ and $\delta > 0$ is reverse-S shaped. Altogether, w with $\alpha \in (0,1]$ and $\delta \in (0,1]$ satisfies the monotonicity condition in Assumption 3.3.1 and is furthermore reverse-S shaped if $\alpha \in (0,1)$. As described in Brummer et al. (2018), for a fixed α, a decrease in δ results in a mean shift to the left, hence increasing the left tail probabilities (downside risk). A decrease in the parameter α then leads to an increase in the upper tail of the distribution. We further find[5]:

$$\begin{aligned} w'(p) &= \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-(1-\delta)\sigma_{\tilde{Z}}\Phi^{-1}(p)}\,(w(p))^{\frac{\alpha-1}{\alpha}}\,F_{\tilde{Z}}^{-1}(p), \\ w'(F_{\tilde{Z}}(\tilde{Z})) &= \alpha e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde{Z}}^2}\left(\Phi\left(\frac{\ln\tilde{Z}}{\sigma_{\tilde{Z}}}-\left(\delta-\frac{1}{2}\right)\sigma_{\tilde{Z}}\right)\right)^{\alpha-1}\tilde{Z}^{\delta}\,. \end{aligned} \tag{3.12}$$

For simplicity, the case $\alpha = 1$ is considered later. $\alpha = 1$ immediately implies that $w'(p)$ and $w'(F_{\tilde{Z}}(\tilde{Z}))$ reduce to

$$w'(p) = e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde{Z}}^2}F_{\tilde{Z}}^{-1}(p)^{\delta},\ w'(F_{\tilde{Z}}(\tilde{Z})) = e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde{Z}}^2}\tilde{Z}^{\delta}\,. \tag{3.13}$$

Let w_+, w_- be selected according to (3.11) with parameters α_+, δ_+ and α_-, δ_-. It can be shown that the placed Assumptions 3.1, 3.2 and 3.3.1 and 3.3.3 are generally fulfilled when $\delta_+ \leqslant 1$. The next theorem shows conditions on the distortion parameters such that Assumption 3.11 holds for this special probability distortion in the case where $\alpha_+ = 1$. Notice that under $\alpha_+ = 1$, additionally 3.3.2 holds for HARA utility, thus any placed assumption is satisfied.

Theorem 3.13. *Let the probability distortion functions w_+, w_- be defined according to* (3.11) *and let $\alpha_+ = 1$. If $v_0 > \frac{1-b_H}{a_H}d_H e^{-rT}$, $\alpha_- \leqslant b_H$, $\delta_- \leqslant -1$*[6] *and $\delta_+ \in \mathbb{R}$ such that $\frac{k_{H-}}{e^{\frac{1}{2}\sigma_{\tilde{Z}}^2\left[\frac{b_H}{1-b_H}(1-\delta_+)^2\right]}} > 1$, then Assumption 3.11 is satisfied.*

Theorem 3.13 therefore shows that the distortion defined in (3.11) is a proper choice and satisfies all made assumptions under some parameter conditions.

Further note that there exist a variety of other probability distortion families besides the one in (3.11), for instance the one by Tversky and Kahneman (1992) or Jin and Zhou (2008). The first distortion family leads to undesired shapes of the distortion for some choice of parameters and is not selected here for tractability reasons. The latter one by Jin and Zhou (2008) does not lead to a well-posed problem when applied on both the positive and negative part. Our applied distortion is analytically tractable and leads to a well-posed problem for suitably chosen parameters when it is applied on both the positive and negative part.

Replicating Wealth and Relative Portfolio Process. Let the reference wealth B be a constant at terminal time T, not necessarily zero, and additionally $r = 0$ as before, then $\mathbb{E}[\tilde{Z}] = 1$ and $\mathbb{E}[\tilde{Z}B] = B$. Moreover, let Assumption 3.11 hold, and let $\delta_+ \leqslant 1$ and $\alpha_+ = 1$ be the distortion parameters of w_+, δ_- and α_- the distortion parameters of w_-. The next theorem presents the formulas for the optimal final payoff $X^\star$, the wealth process $V^\star(t)$ and the corresponding relative portfolio process $\pi^\star(t)$ which replicates $X^\star$.

Theorem 3.14. *Let B be constant at time T and $r = 0$. Additionally, let Assumption 3.11 hold, and let $\delta_+ \leqslant 1$ and $\alpha_+ = 1$. Define*

[5] The detailed calculations to Eq. (3.12) and (3.13) are given at the beginning of Appendix A.2.

[6] $\delta_- < 0$ implies a globally concave probability distortion function on the losses, i.e. the small probabilities of large losses are overweighted.

$$\tilde{\chi}_{A+} := \frac{v_0 - B + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right]}{\mathbb{E}\left[\tilde{Z}^{\frac{\delta_+ - b_H}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right]}$$

and let ψ denote the density function of a standard normal random variable. Then:

1. *The optimal terminal wealth is given by:*

$$X^\star = \left[\tilde{\chi}_{A+} \tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} - \frac{1-b_H}{a_H} d_H\right] \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))} + B\,.$$

2. *The expected value of the optimal terminal wealth is:*

$$\begin{aligned}\mathbb{E}\left[X^\star\right] = \tilde{\chi}_{A+} e^{\frac{1}{2}\sigma_{\tilde{Z}}^2\left[\left(\frac{1-\delta_+}{1-b_H}\right) + \left(\frac{1-\delta_+}{1-b_H}\right)^2\right]} \Phi\left(\Phi^{-1}\left(1 - \hat{z}(\infty, v_0 - B)\right) + \sigma_{\tilde{Z}}\left(\frac{1-\delta_+}{1-b_H}\right)\right) \\ - \frac{1-b_H}{a_H} d_H \Phi\left(\Phi^{-1}\left(1 - \hat{z}(\infty, v_0 - B)\right) - \sigma_{\tilde{Z}}\right) + B\,.\end{aligned}$$

3. *The replicating wealth process $V^\star(t)$, $t \in [0,T]$, is given by:*

$$V^\star(t) = \tilde{\chi}_{A+} V_1^\star(t) - \frac{1-b_H}{a_H} d_H V_2^\star(t) + B,$$

with

$$\begin{aligned}V_1^\star(T) &= \tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}, \\ V_1^\star(t) &= \tilde{Z}(t)^{-\frac{1-\delta_+}{1-b_H}} \mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\delta_+ - b_H}{1-b_H}} \mathbb{1}_{\tilde{Z}(t,T) \leqslant \frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}{\tilde{Z}(t)}} \middle| \mathcal{F}_t\right] \\ &= \tilde{Z}(t)^{-\frac{1-\delta_+}{1-b_H}} e^{\frac{1}{2}\sigma_{\tilde{Z}}^2(t)\left[\left(\frac{\delta_+ - b_H}{1-b_H}\right)^2 - \left(\frac{\delta_+ - b_H}{1-b_H}\right)\right]} \\ &\quad \times \Phi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1 - \hat{z}(\infty, v_0 - B))\right) - \ln \tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)\left(\frac{\delta_+ - b_H}{1-b_H} - \frac{1}{2}\right)\right)\end{aligned}$$

and

$$\begin{aligned}V_2^\star(T) &= \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}, \\ V_2^\star(t) &= \mathbb{E}\left[\tilde{Z}(t,T) \mathbb{1}_{\tilde{Z}(t,T) \leqslant \frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}{\tilde{Z}(t)}} \middle| \mathcal{F}_t\right] \\ &= \Phi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1 - \hat{z}(\infty, v_0 - B))\right) - \ln \tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} - \frac{1}{2}\sigma_{\tilde{Z}}(t)\right).\end{aligned}$$

4. *The replicating relative portfolio process* $\pi^\star(t)$, $t \in [0,T]$, *is given by:*

$$\begin{aligned}\hat{\pi}^\star(t)V^\star(t) = &\left[\tilde{\chi}_{A+}\frac{1-\delta_+}{1-b_H}V_1^\star(t) + \frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)}\right.\\ &\times \psi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0-B))\right) - \ln \tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} + \frac{1}{2}\sigma_{\tilde{Z}}(t)\right)\\ &\left.\times\left\{\tilde{\chi}_{A+}\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0-B))\right)^{-\frac{1-\delta_+}{1-b_H}} - \frac{1-b_H}{a_H}d_H\right\}\right]\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}),\\ \pi_0^\star(t) = 1 - \hat{\pi}^\star(t)'\mathbb{1}\,.\end{aligned}$$

The formula for $X^\star$ shows that the optimal payoff is the sum of shifted digital or binary options. A similar form for the optimal terminal payoff is frequently achieved under an expected utility maximizing framework. The formula on the expected final payoff can be used for performance/return evaluation. From the replicating wealth process $V^\star(t)$, with $V^\star(T) = X^\star$, we learn how the portfolio wealth evolves over time and reacts to market changes. The last formula on the optimal risky exposure $\hat{\pi}^\star(t)V^\star(t)$ shows how the optimal final payout $X^\star$ and its corresponding process $V^\star(t)$ can be replicated continuously in time.

3.2.4 Applications: two special cases

Under the setting and assumptions of Section 3.2.3 we elaborate on two special cases where the optimal investment strategy turns out to be a traditional Constant Proportion Portfolio Insurance (CPPI) strategy that can easily be implemented.

Special case 1: CRRA utility within CPT. When d_H is set to zero, HARA utility ends in a simple CRRA utility. The definition of HARA utility within CPT in (3.4) evolves to

$$\begin{aligned}&U_+ : \mathbb{R}^+ \to \mathbb{R}^+,\, v \mapsto U_+(v) := \frac{1-b_H}{b_H}\left[\frac{a_H}{1-b_H}v\right]^{b_H} = (1-b_H)^{1-b_H}\,\frac{a_H^{b_H}}{b_H}v^{b_H},\\ &U_- : \mathbb{R}^+ \to \mathbb{R}^+,\, v \mapsto U_-(v) := k_{H-}U_+(v),\end{aligned}$$

with utility function $U : \mathbb{R} \to \mathbb{R}$, $v \mapsto U(v) := U_+(v^+)\mathbb{1}_{v \geqslant 0} - U_-(v^-)\mathbb{1}_{v<0}$. This definition implies $\hat{z}(\infty, v_0 - B) = 0$. Hence, the corresponding optimal terminal wealth, its expectation and its replicating processes directly follow by setting $d_H = 0$ and $\hat{z}(\infty, v_0 - B) = 0$ in Theorem 3.14.

Corollary 3.15. *Let* $\tilde{\chi}_{A+} = \frac{v_0 - B}{\mathbb{E}\left[\tilde{Z}^{\frac{\delta_+ - b_H}{1-b_H}}\right]} > 0$. *Then:*

1. *The optimal terminal wealth is given by:*

$$X^\star = \tilde{\chi}_{A+}\tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} + B\,.$$

2. *The expected value of the optimal terminal wealth is:*

$$\mathbb{E}\left[X^\star\right] = (v_0 - B)\frac{\mathbb{E}\left[\tilde{Z}^{-\frac{1-\delta_+}{1-b_H}}\right]}{\mathbb{E}\left[\tilde{Z}^{\frac{\delta_+-b_H}{1-b_H}}\right]} + B = \tilde{\chi}_{A+}\, e^{\frac{1}{2}\sigma^2_{\tilde{Z}}\left[\left(\frac{1-\delta_+}{1-b_H}\right)+\left(\frac{1-\delta_+}{1-b_H}\right)^2\right]} + B\,.$$

3. *The replicating wealth process $V^\star(t)$, $t \in [0,T]$, is given by:*

$$V^\star(t) = \tilde{\chi}_{A+} V_1^\star(t) + B,$$

with

$$V_1^\star(t) = \tilde{Z}(t)^{-\frac{1-\delta_+}{1-b_H}}\,\mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\delta_+-b_H}{1-b_H}}\middle|\mathcal{F}_t\right] = \tilde{Z}(t)^{-\frac{1-\delta_+}{1-b_H}}\, e^{\frac{1}{2}\sigma^2_{\tilde{Z}}(t)\left[\left(\frac{\delta_+-b_H}{1-b_H}\right)^2-\left(\frac{\delta_+-b_H}{1-b_H}\right)\right]}.$$

4. *The replicating relative portfolio process $\pi^\star(t)$, $t \in [0,T]$, is given by:*

$$\hat{\pi}^\star(t)V^\star(t) = \frac{1-\delta_+}{1-b_H}\left(V^\star(t)-B\right)\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}),$$
$$\pi_0^\star(t) = 1 - \hat{\pi}^\star(t)'\mathbb{1}\,.$$

Therefore $X^\star$ is larger than B almost surely, i.e. $\mathbb{P}(X^\star > B) = 1$ and $\mathbb{P}(X^\star = B) = 0$. Moreover, the resulting optimal allocation is a CPPI strategy to the floor B.

Special case 2: HARA utility within EUT. Let HARA utility function within Expected Utility Theory (EUT) be defined by

$$U : [B,\infty) \to \mathbb{R}^+,\ v \mapsto U(v) := \frac{1-b_H}{b_H}\left[\frac{a_H}{1-b_H}(v-B)\right]^{b_H} = (1-b_H)^{1-b_H}\,\frac{a_H^{b_H}}{b_H}(v-B)^{b_H}\ .$$

HARA within EUT is globally concave and no longer S-shaped. It coincides with CRRA utility function (CPT) of the gains U_+, shifted to B instead of 0. The corresponding Arrow-Pratt measure is given by $\mathcal{A}(v) = \frac{1-b_H}{v-B}$, $v \in [B,\infty)$, and is positive. It is well-known that the solution to the Expected Utility Theory problem with a globally concave HARA utility function (and no probability distortion functions) is a CPPI strategy. The associated final wealth and its replicating processes immediately follow from the above behavioral CRRA utility solution in Corollary 3.15 when forcing the probability distortion function to the identity function ($\alpha_+ = 1$, $\delta_+ = 0$).

Corollary 3.16. *Let* $\tilde{\chi}_{A+} = \frac{v_0-B}{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}}\right]} > 0$. *Then:*

1. *The optimal terminal wealth is given by:*

$$X^\star = \tilde{\chi}_{A+}\tilde{Z}^{-\frac{1}{1-b_H}} + B\,.$$

2. *The expected value of the optimal terminal wealth is:*

$$\mathbb{E}\left[X^\star\right] = (v_0 - B)\frac{\mathbb{E}\left[\tilde{Z}^{-\frac{1}{1-b_H}}\right]}{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}}\right]} + B = \tilde{\chi}_{A+} e^{\frac{1}{2}\sigma^2_{\tilde{Z}}\left[\left(\frac{1}{1-b_H}\right)+\left(\frac{1}{1-b_H}\right)^2\right]} + B\,.$$

3. *The replicating wealth process $V^\star(t)$, $t \in [0,T]$, is given by:*

$$V^\star(t) = \tilde{\chi}_{A+} V_1^\star(t) + B,$$

with

$$V_1^\star(t) = \tilde{Z}(t)^{-\frac{1}{1-b_H}} \mathbb{E}\left[\tilde{Z}(t,T)^{-\frac{b_H}{1-b_H}}\middle|\mathcal{F}_t\right] = \tilde{Z}(t)^{\frac{1}{1-b_H}} e^{\frac{1}{2}\sigma^2_{\tilde{Z}}(t)\left[\left(-\frac{b_H}{1-b_H}\right)^2+\left(\frac{b_H}{1-b_H}\right)\right]}\,.$$

4. *The replicating relative portfolio process $\pi^\star(t)$, $t \in [0,T]$, is given by:*

$$\hat{\pi}^\star(t)V^\star(t) = \frac{1}{1-b_H}\left(V^\star(t) - B\right)\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}),$$
$$\pi_0^\star(t) = 1 - \hat{\pi}^\star(t)'\mathbb{1}\,.$$

Corollary 3.16 gives $\mathbb{P}(X^\star > B) = 1$ and $\mathbb{P}(X^\star = B) = 0$. In addition, as already mentioned, the relative portfolio process describes a CPPI strategy. Notice that the investment strategy for CRRA utility within CPT is also a CPPI strategy to the same floor B. The difference between both lies in the respective multiplier. The relation between the multipliers, let $m_{CRRA(CPT)}$ denote the CPPI multiplier for the CRRA (CPT) and $m_{HARA(EUT)}$ for the HARA (EUT) strategy, is the following:

$$m_{CRRA(CPT)} = m_{HARA(EUT)}\left(1 - \delta_+\right)\,. \tag{3.14}$$

This means that the probability distortion function w_+ does not change the structure or characteristics of the optimal investment strategy, it is still of a CPPI type, but distorts the underlying CPPI multiplier. When $\delta_+ > 0$ (w_+ convex, i.e. underweighting of the probability of large gains), the probability distortion reduces the multiplier and thus reduces the magnitude of the risky investments. When $\delta_+ < 0$ (w_+ concave, i.e. overweighting of the probability of large gains), the opposite is the case.

To conclude with, from the formulas in Corollaries 3.15 and 3.16 we learn the following: The CPT problem with CRRA utility leads to a CPPI strategy as optimal portfolio. The EUT problem with HARA utility also leads to a CPPI strategy as optimal portfolio. The difference between the two optimal CPPI strategies is in the multiple, the CPT strategy's multiple is a function of the probability distortion parameter. Finally, the CPT problem with HARA utility does not lead to a CPPI strategy as optimal investment strategy in general. The optimal portfolio can be regarded as kind of a "distorted" CPPI strategy which reduces to a pure CPPI strategy in the two above special cases.

Remark 3.17 (Underfunded setup)**.** *From Assumption 3.11 onward, we focused on an initially well-funded investor with* $v_0 > 0$*, or* $v_0 > \mathbb{E}[\tilde{Z}B]$ *respectively, which allows for deriving the optimal variables* $c^\star$*,* $v_+^\star$ *and thus the optimal investment strategy explicitly. Nonetheless, we would like to emphasize that the CPT approach allows for starting with an underfunded portfolio, i.e. with initially* $v_0 < 0$*, or* $v_0 < \mathbb{E}[\tilde{Z}B]$ *respectively.* $\mathbb{E}[\tilde{Z}B]$ *can exemplarily denote the present value of all outstanding future liabilities of the pension fund.*

Theorem 3.8 provides the optimal terminal wealth $V^\star(T) = X^\star = X^\star(c^\star, v_+^\star)$ *for both the initially well-funded and underfunded setup, given the optimal pair* $(c^\star, v_+^\star)$ *to Problem* $(\mathrm{P}^\star)$ *with* $v_+^\star > 0$ *and* $0 < c^\star \leqslant \infty$*.*

If one now starts underfunded with $v_0 < 0$*, i.e.* $\mathbb{E}[\tilde{Z}X^\star] = v_0 < 0$*, it must hold* $c^\star < \infty$*, because otherwise it would be* $X^\star = X_+^\star(c^\star, v_+^\star) \geqslant 0$*,* $\mathbb{P}$*-a.s., with then* $\mathbb{E}[\tilde{Z}X^\star] \geqslant 0$ *which is a contradiction. For this reason, it holds* $\mathbb{P}(\tilde{Z} > c^\star) > 0$ *and the investor could possibly end up with a loss (wealth below the benchmark). In such a case, the loss is fixed and is given by* $\frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c^\star}]}$*. Note* $v_+^\star > 0 > v_0$ *and thus* $v_+^\star - v_0 > 0$*. This further shows that* $c^\star = \infty$ *is only possible if* $v_0 \geqslant 0$ *(well-funded setting), where the fixed loss part then vanishes, also compare* $X^\star$ *in Theorem 3.8 (solution for general* $c^\star$*) and Theorem 3.12 (solution for* $c^\star = \infty$ *for a sufficiently wealthy investor).*

In summary, Theorem 3.8 states the optimal terminal wealth $V^\star(T) = X^\star = X^\star(c^\star, v_+^\star)$ *for general probability distortion functions and general initial wealth* v_0*. The associated optimal investment strategy can be found by replication of* $V^\star(T)$*. We already mentioned that Section 2.3 can be used for replication if the term* $\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}$ *is a digital option of the form* $\tilde{Z}^\eta$ *for some* $\eta \in \mathbb{R}$*.*

Highlighting particularly the difference between the funded and the underfunded setup, the optimal terminal wealth can structurally only deviate in the presence of the fixed loss part $-\frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c^\star}]}\mathbb{1}_{\tilde{Z}>c^\star}$*, which can be present in the well-funded but must be present in the underfunded setup. However, this loss part admits the form of a simple digital option (*$\tilde{Z}^\eta$ *with* $\eta = 0$*) which can easily be replicated by applying Theorem 2.9 in Section 2.3. Hence, the hedging strategy in the underfunded setup consists of two parts: the first part for a general* $c^\star$ *value structurally coincides with the hedging strategy from the well-funded setup with* $c^\star = \infty$ *(cf. Theorem 3.14) and can be determined analogously. The second part (loss part), which vanishes in the situation where* $c^\star$ *is infinite, can simply be replicated using Theorem 2.9 in Section 2.3.*

If $d_H = 0$ *(CRRA utility within CPT), Corollary 3.15 shows the optimal asset allocation strategy for a well-funded investor. Escobar-Anel et al. (2020b) study the special case of a CRRA utility within CPT for both an initially well-funded as well as an underfunded portfolio and provide an economic sound interpretation of the resulting optimal investment strategies. For further details we refer to Escobar-Anel et al. (2020b), additionally including a detailed derivation of the hedging strategy also in the underfunded case.*

3.3 Numerical Case Study

The last section has shown that Assumption 3.11 allows for the application of the probability distortion function defined in Eq. (3.11) on both the positive and the negative part. In what follows, the optimal terminal fund wealth $X^\star$, the corresponding replicating wealth process $V^\star(t)$ and the replicating investment strategy or relative portfolio process $\pi^\star(t)$ are examined numerically.

In particular we intend to empirically compare HARA and CRRA utility within behavioral finance (variation in the utility parameter d_H; $d_H = 0$ is CRRA utility), and to compare the resulting optimal portfolio and allocation between the behavioral finance concept (S-shaped utility function and distortions on the probabilities) and the solution for HARA utility in an Expected Utility Theory framework (variation in the probability distortion parameters).

In the case study two different probability distortion functions on the gains are considered which allow for a closed-form solution to the optimal terminal fund wealth, its replicating wealth and relative portfolio process. Therefore, a globally concave as well as convex probability distortion function on the gains together with a globally concave one on the losses are considered. The concave distortion on the losses implies overweighting of tail probabilities in the losses (extremely large and unlikely losses). The concave distortion on the gains implies overweighting of tail probabilities in the gains (extremely large and unlikely gains), whereas the convex distortion on the gains implies underweighting of tail probabilities in the gains. Hence the investor with a concave distortion on the gains has an optimistic view about the future whereas the investor applying a convex distortion has a pessimistic view and overweights the probability of a crash.

Throughout the whole numerical case study, one risky asset ($N = 1$) is considered. The solution for the price process of the risky asset for general constant interest rate r is simply given by

$$P_1(t) = p_1 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}, \ t \in [0, T] .$$

Then

$$e^{-\gamma W(t)} = \left(\frac{P_1(t)}{p_1} e^{-\left(\mu - \frac{1}{2}\sigma^2\right)t} \right)^{-\frac{\gamma}{\sigma}} = \left(\frac{P_1(t)}{p_1} \right)^{-\frac{\gamma}{\sigma}} e^{\frac{\gamma}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right)t}$$

with $\gamma = \frac{\mu - r}{\sigma} > 0$. The relation between the pricing kernel $\tilde{Z}$ and the underlying stock price $P_1(T)$ at terminal time T is then given as follows:

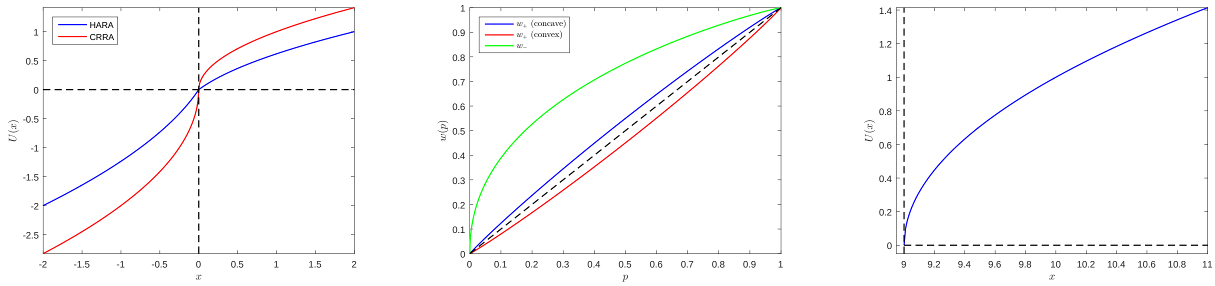
$$\tilde{Z} = e^{-\left(r + \frac{1}{2}\gamma^2\right)T - \gamma W(T)} = e^{\frac{1}{2}(\mu + r)\left(\frac{\mu - r}{\sigma^2} - 1\right)T} \left(\frac{P_1(T)}{p_1} \right)^{-\frac{\mu - r}{\sigma^2}} ,$$

$$P_1(T) = p_1 e^{\frac{1}{2}(\mu + r)\left(1 - \frac{\sigma^2}{\mu - r}\right)T} \tilde{Z}^{-\frac{\sigma^2}{\mu - r}} .$$

Hence, $\tilde{Z}$ can be regarded as a strictly monotone decreasing function in $P_1(T)$ since $\mu > r$ is assumed and $e^{\frac{1}{2}(\mu + r)\left(\frac{\mu - r}{\sigma^2} - 1\right)T} > 0$ is a positive constant, and vice versa. Therefore, with good states of the world or market we mean high stock prices $P_1(T)$ and thus equivalently low values for the pricing kernel $\tilde{Z}$.

The explicit setting to be considered in this numerical case study is as follows.

1. Market: $r = 0\%$, $\mu = 5\%$, $\sigma = 20\%$, $T = 1$.
2. Wealth: $v_0 = 10$; $B = 9$.
3. Utility function: $b_H = 0.5$, $a_H = 0.5$, $d_H = 0.25$, $k_{H-} = 2$.
4. Probability distortion functions:
   a) Setting 1: w_+ concave with $\alpha_+ = 1$, $\delta_+ = -0.5$, $\alpha_- = 0.5$, $\delta_- = -1$.



(a) HARA and CRRA utility functions U within CPT *(blue = HARA, red = CRRA)*.

(b) Probability distortion functions w_+, w_- within CPT *(blue|red = w_+ concave|convex, green = w_-)*.

(c) HARA utility function U within EUT.

Figure 3.1: S-shaped utility functions and probability distortion functions within CPT, globally concave utility function within EUT.

b) Setting 2: w_+ convex with $\alpha_+ = 1$, $\delta_+ = 0.5$, $\alpha_- = 0.5$, $\delta_- = -1$.

Figure 3.1 graphically visualizes the considered utility and probability distortion functions. Both described setups fulfill Assumption 3.11 since Theorem 3.13 can be applied as

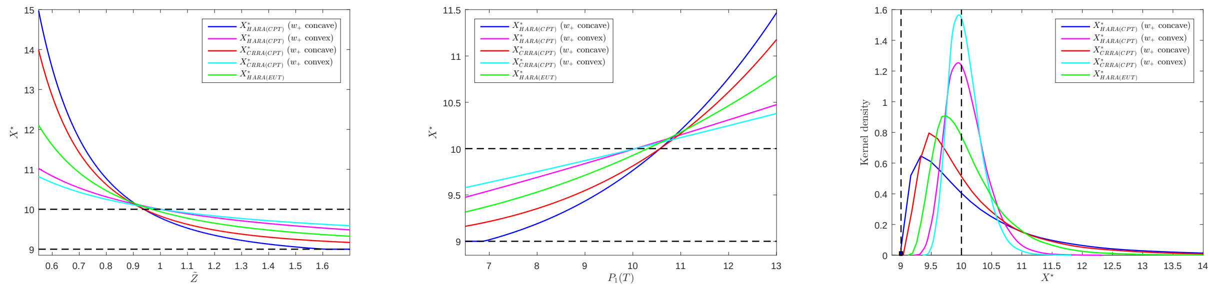
- $v_0 - Be^{-rT} = 1 > 0.25 = \frac{1-b_H}{a_H} d_H e^{-rT}$ and
- $\frac{k_{H-}}{e^{\frac{1}{2}\sigma_{\tilde{Z}}^2\left[\frac{b_H}{1-b_H}(1-\delta_+)^2\right]}} = 1.8642 > 1$ (w_+ concave) and $\frac{k_{H-}}{e^{\frac{1}{2}\sigma_{\tilde{Z}}^2\left[\frac{b_H}{1-b_H}(1-\delta_+)^2\right]}} = 1.9844 > 1$ (w_+ convex).

In summary, Assumption 3.11 is satisfied for both probability distortion cases and hence Theorems 3.12 and 3.14 can be applied.

The CRRA utility (CPT) case can be obtained by setting $d_H = 0$. The HARA utility function (EUT) coincides with the positive part CRRA utility function (CPT) shifted[7] to $B = 9$ instead of 0. According to Eq. (A.5) in Appendix A.1, $\hat{z}(\infty, v_0 - B)$ is the root of the function

$$\begin{aligned} f(z) = \; & (1-b_H)^{1-b_H}\, a_H^{b_H} \left(\frac{v_0 - B + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-z)}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_{A+}(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H - 1}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-z)}\right]} \right)^{-(1-b_H)} \\ & - a_H d_H^{-(1-b_H)} \frac{w'_+(1-z)}{F_{\tilde{Z}}^{-1}(1-z)} \\ = \; & (1-b_H)^{1-b_H}\, a_H^{b_H}\, e^{\frac{1}{2}(\delta_+ - \delta_+^2)\sigma_{\tilde{Z}}^2} \left(\frac{v_0 - B + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-z)}\right]}{\mathbb{E}\left[\tilde{Z}^{\frac{\delta_+ - b_H}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-z)}\right]} \right)^{-(1-b_H)} \\ & - a_H d_H^{-(1-b_H)} e^{\frac{1}{2}(1-\delta_+^2)\sigma_{\tilde{Z}}^2 - (1-\delta_+)\sigma_{\tilde{Z}}\Phi^{-1}(1-z)} . \end{aligned}$$

[7] In CPT the S-shaped utility function is centered around 0, but the optimization is done on $X - B$. In EUT, the optimization is under X. Therefore, the CPT CRRA utility function on the gains is equivalent to the EUT HARA utility function.



(a) Optimal terminal wealth $X^\star$ and pricing kernel $\tilde{Z}$ at T.

(b) Optimal terminal wealth $X^\star$ and final stock price $P_1(T)$ at T.

(c) Kernel density estimate of the optimal terminal wealth $X^\star$ at T (normal kernel).

Figure 3.2: Dependence between the optimal terminal wealth $X^\star$ and the pricing kernel $\tilde{Z}$ or respectively the stock price $P_1(T)$ at terminal time T *(blue|magenta = HARA (CPT) with w_+ concave|convex, red|cyan = CRRA (CPT) with w_+ concave|convex, green = HARA (EUT))*.

Numerically, we obtain $\hat{z}(\infty, v_0 - B) = \hat{z}(\infty, 1) = 0.0218$, thus $F_{\tilde{Z}}^{-1}(1 - \hat{z}(\infty, 1)) = 1.6054$ when w_+ is concave, and $\hat{z}(\infty, 1) = 2.6273 \times 10^{-11}$, $F_{\tilde{Z}}^{-1}(1 - \hat{z}(\infty, 1)) = 5.0010$ when w_+ is convex. In view of Theorem 3.14 and Corollaries 3.15 and 3.16 the optimal terminal wealths for the three different models turn out to be

$$
\begin{aligned}
X^\star_{HARA(CPT)} &= \left[1.0344 \times \tilde{Z}^{-3} - 0.25\right] \mathbb{1}_{\tilde{Z} \leqslant 1.6054} + 9, \\
X^\star_{CRRA(CPT)} &= 0.8290 \times \tilde{Z}^{-3} + 9, \\
X^\star_{HARA(EUT)} &= 0.9394 \times \tilde{Z}^{-2} + 9,
\end{aligned}
$$

when w_+ is concave, and

$$
\begin{aligned}
X^\star_{HARA(CPT)} &= \left[1.25 \times \tilde{Z}^{-1} - 0.25\right] \mathbb{1}_{\tilde{Z} \leqslant 5.0010} + 9, \\
X^\star_{CRRA(CPT)} &= \tilde{Z}^{-1} + 9, \\
X^\star_{HARA(EUT)} &= 0.9394 \times \tilde{Z}^{-2} + 9,
\end{aligned}
$$

when w_+ is convex. Notice that for the CRRA (CPT) and HARA (EUT) case it holds $\mathbb{P}(X^\star > B) = 1$, whereas for HARA (CPT) we have $\mathbb{P}(X^\star > B) = 1 - \hat{z}(\infty, 1) = 97.82\%$, $\mathbb{P}(X^\star = B) = \hat{z}(\infty, 1) = 2.18\%$ for the concave and $\mathbb{P}(X^\star > B) = (100 - 2.6273 \times 10^{-9})\%$, $\mathbb{P}(X^\star = B) = 2.6273 \times 10^{-9}\%$ for the convex w_+.

Figure 3.2 examines the three final payoffs $X^\star$ in dependence of the pricing kernel $\tilde{Z}$ as well as the stock price $P_1(T)$ and further illustrates the kernel density estimates at the end of the investment period T. Figure 3.2(b) shows that the final payoff is a strictly convex function of the final stock price for the CRRA (CPT) and the HARA (EUT) case. For HARA (CPT), the final portfolio value is a convex, but no longer strictly convex function of the terminal stock price, since the final value equals B for $\tilde{Z} \geqslant 1.6054$ ($\tilde{Z} \geqslant 5.0010$) which is equivalent to $P_1(T) \leqslant 6.8818$ ($P_1(T) \leqslant 2.7729$). Figure 3.2(c) shows the asymmetric, positively skewed density functions of the final portfolio values. Table 3.1 additionally provides the simulated values for the probabilities of a gain and a neutral situation, as well as the empirical mean, standard deviation, Sharpe Ratio and Adjusted Sharpe

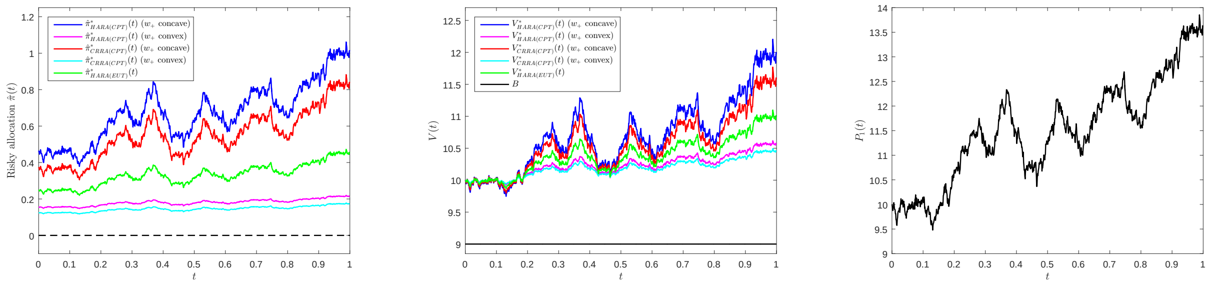
	HARA (CPT) (w_+ concave)	HARA (CPT) (w_+ convex)	CRRA (CPT) (w_+ concave)	CRRA (CPT) (w_+ convex)	HARA (EUT)
Probability of a gain situation $\mathbb{P}(X^\star > B)$:	97.82%	$\lessapprox$ 100%	100%	100%	100%
Probability of a neutral situation $\mathbb{P}(X^\star = B)$:	2.18%	$\gtrapprox$ 0%	0%	0%	0%
Expectation $\mathbb{E}\left[r_{X^\star}\right]$:	2.68%	0.83%	2.16%	0.67%	1.38%
Standard deviation $Sd\left(r_{X^\star}\right)$:	13.22%	3.40%	10.60%	2.72%	6.10%
Sharpe Ratio $SR\left(r_{X^\star}\right)$:	20.27%	24.51%	20.34%	24.51%	22.66%
Adjusted Sharpe Ratio $ASR\left(r_{X^\star}\right)$:	21.78%	25.25%	21.86%	25.25%	23.89%

Table 3.1: Estimated values for the probabilities of gain and neutral situations, average, standard deviation, Sharpe Ratio and Adjusted Sharpe Ratio of the corresponding optimal portfolios' return for HARA and CRRA utility in a CPT and HARA utility in an EUT setting.

Ratio of the final portfolio returns. The Adjusted Sharpe Ratio (see Leland (1999)) accounts for skewness and kurtosis of a return distribution. The return is defined by $r_{X^\star} := \frac{X^\star - v_0}{v_0}$, the number of simulated paths is $10,000$.

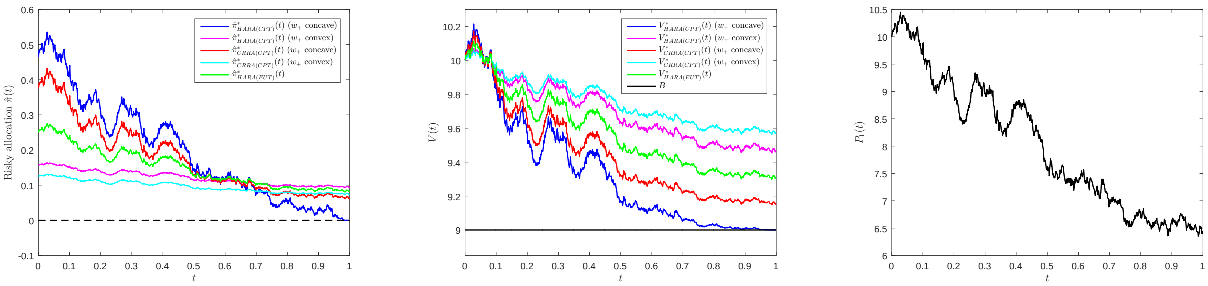
After examining the objectives at terminal time T, the focus lies on the behavior of the corresponding replicating wealth and relative portfolio processes during the investment period. In general, the initial relative capital allocation at $t = 0$ is 46.51% (15.62%) in the stock and 53.49% (84.38%) in the bank account for HARA (CPT), 37.50% (12.50%) in the stock and 62.50% (87.50%) in the bank account for CRRA (CPT) for the concave (convex) w_+, and 25.00% in the stock and 75.00% in the bank account for HARA (EUT). Hence, the HARA (EUT) strategy starts most defensive among the considered models, the HARA (CPT) strategy starts more aggressive than the CRRA (CPT) strategy. Note that, as already mentioned, only the HARA (CPT) strategy is not of a CPPI type. Figure 3.3 illustrates the objectives under an upward movement of the stock, Figure 3.4 presents the same objectives under a downward movement of the underlying. It can be observed that the CPT strategies with concave distortion outperform the CPT strategies with convex distortion in uptrend and underperform them in downtrend markets. The optimal EUT strategy, for both $\hat{\pi}^\star(t)$ and $V^\star(t)$, lies in the middle with the optimal CPT strategies for concave distortion on the one and for convex distortion on the other side.

In summary, the empirical study shows the following message; notice that within the four different CPT settings, same utility functions for different probability distortions as well as same probability



(a) Risky relative portfolio processes $\hat{\pi}^\star(t)$. (b) Replicating wealth processes $V^\star(t)$. (c) Stock price process $P_1(t)$.

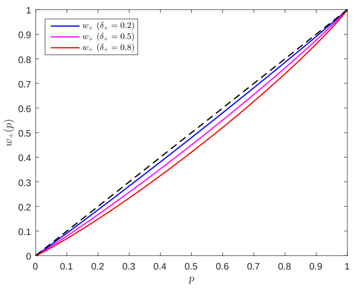
Figure 3.3: Risky relative portfolio processes $\hat{\pi}^\star(t)$ and replicating wealth processes $V^\star(t)$ under an upward movement of the stock price process $P_1(t)$ *(blue|magenta = HARA (CPT) with* w_+ *concave|convex, red|cyan = CRRA (CPT) with* w_+ *concave|convex, green = HARA (EUT)).*



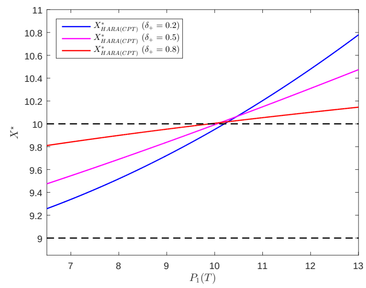
(a) Risky relative portfolio processes $\hat{\pi}^\star(t)$. (b) Replicating wealth processes $V^\star(t)$. (c) Stock price process $P_1(t)$.

Figure 3.4: Risky relative portfolio processes $\hat{\pi}^\star(t)$ and replicating wealth processes $V^\star(t)$ under an downward movement of the stock price process $P_1(t)$ *(blue|magenta = HARA (CPT) with* w_+ *concave|convex, red|cyan = CRRA (CPT) with* w_+ *concave|convex, green = HARA (EUT)).*

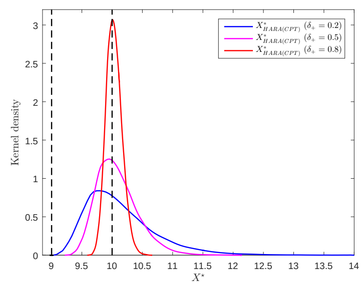
distortions for different utility functions are considered: Figure 3.2 illustrates path-independently that for a concave w_+, the HARA (CPT) strategy outperforms within bull markets while it underperforms within bear markets. Therefore, the investor benefits from her behavioral trait when the market is bullish. Compared to the CRRA (CPT) strategy with concave w_+ this is due to the less risk-averse selected utility function; compared to the HARA (EUT) strategy this is because of the less risk-averse utility and the concave probability distortion function on the gains ($\delta_+ < 0$ selected), which lets the investor bet on larger probabilities of extreme positive events. The CPT strategies with a convex w_+ behave in an opposite manner. When $\delta_+ > 0$ (w_+ convex) is selected, then the investor benefits from her behavior within bear markets and suffers within bull markets. This characteristics can directly be seen in the formulas for the wealth and replicating strategy in Theorem 3.14 and Corollaries 3.15 and 3.16: First, when $\delta_+ < 0$ (w_+ concave), then the exponent of $\tilde{Z}$ which corresponds to the CPT strategies is more negative than the EUT exponent. Moreover, a negative δ_+ leads to a higher multiplier of the resulting optimal CPPI strategy for the CRRA



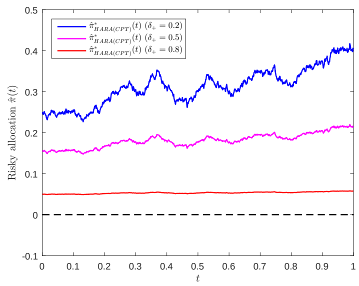
(a) Convex probability distortion functions w_+.



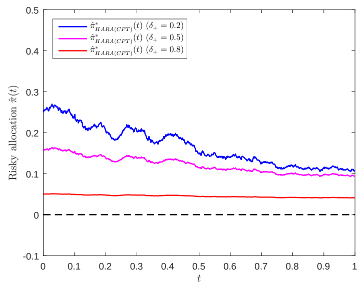
(b) Optimal terminal wealth $X^\star$ and final stock price $P_1(T)$.



(c) Kernel density estimate of the optimal terminal wealth $X^\star$ (normal kernel).



(d) Risky relative portfolio processes $\hat{\pi}^\star(t)$ under an increasing stock price.



(e) Risky relative portfolio processes $\hat{\pi}^\star(t)$ under a decreasing stock price.

Figure 3.5: Terminal wealth and risky relative portfolio process under HARA (CPT) utility function model and concave probability distortion w_- (default parameters) for three varying convex probability distortions w_+.

(CPT) setup, cf. Eq. (3.14). Second, when $\delta_+ > 0$ the picture turns to the opposite: the exponent of $\tilde{Z}$ which corresponds to the CPT strategies is now less negative than the EUT exponent, and the CPPI multiplier of the CRRA (CPT) strategy is lower, thus less is invested risky. Moreover, the CPT strategies with a convex w_+ provide the highest (Adjusted) Sharpe Ratios, followed by the HARA (EUT) strategy and thereafter the CPT strategies with a concave w_+.

In addition, Figure 3.5 provides the target objectives under three different convex probability distortion functions w_+, i.e. for different δ_+ values (analogue for the concave w_+ case). For increasing δ_+, the convexity of the distortion w_+ becomes more pronounced with a lower slope close to 0 and a higher slope close to 1 for w_+. This has the effect that tail probabilities of large gains are underweighted (lower slope close to 0), but small probabilities for large losses are overweighted (higher slope close to 1). Thus, by increasing δ_+, the degree of convexity of w_+ is increased and therefore the subjectively perceived probability of large gains decreases and the probability of large losses increases. The implication on the optimal portfolio is that more downside protection is needed, but with a lower upside potential. Figure 3.5 quantifies this impact. It can be seen that a stronger convexity leads to a less risky and volatile portfolio with a higher downside protection, which implies a better performance in bearish markets with a reduced probability for big losses, and a worse performance in bullish markets compared to a less convex w_+.

Finally, Appendix A.3 provides some more figures and discussion on explaining mispricing of share prices or unreasonable over-/underweighting of risky assets in investor portfolios.

4 Optimal Life-Cycle Consumption and Investment Decisions under Age-Dependent Risk Preferences

> The special sphere of finance within economics is the study of allocation and deployment of economic resources, both spatially and across time, in an uncertain environment. To capture the influence and interaction of time and uncertainty effectively requires sophisticated mathematical and computational tools.
>
> Robert C. Merton

A suitable management of pensions needs to consider earnings/contributions and investment, but should also account for the required consumption during the accumulation and/or decumulation phase. For this sake, in this chapter, which is a reproduction of Lichtenstern et al. (2020) with minor changes, we consider the finite-horizon portfolio problem of maximizing expected utility of future consumption and terminal wealth to determine the optimal pension or life-cycle fund strategy for a cohort of pension fund investors. The setup is strongly related to a DC pension plan where additionally (individual) consumption is taken into account. Within this framework, Lakner and Nygren (2006) describe the trade-off the investor faces as a compromise between "living well" (consumption) and "becoming rich" (terminal wealth). Classical consumption-investment problems consider constant risk aversion in the intertemporal utility functions for consumption besides a personal discount rate or impatience factor, see Merton (1969) or Merton (1971). Within classical models (where constant relative risk aversion (CRRA) utilities are applied), optimal portfolio policies turn out to be constant over the life-cycle, meaning time and wealth independent. According to Aase (2017) this is "against empirical evidence, and against the typical recommendations of portfolio managers". Furthermore, Aase (2017) and Yang et al. (2014) argue that the tendency of stocks to outperform bonds over long horizons in the past is one of the reasons why people at a younger age are advised to allocate a higher proportion of wealth to equities compared to older people. In addition, Benzoni et al. (2007) examine the impact of labor income on life-cycle strategies and show the importance of dynamic strategies that change over time, thus motivate dynamic and age-dependent strategies. Evidence for changing risk aversion over the life-cycle is reported in the literature, although there is no broad agreement on its behavior: Morin and Suarez (1983), Bakshi and Chen (1994), Palsson (1996), Bellante and Green (2004), Al-Ajmi (2008), Ho (2009), Yao et al. (2011) and Albert and Duffy (2012) observe increasing risk aversion by age, Bellante and Saba (1986) and Wang and Hanna (1997) find risk aversion decreasing by age and Riley and Chow (1992) detect different behavior between the pre- and post-retirement phase. Age-depending risk preferences can economically be motivated by the observed behavior of people to stepwise reduce their investment risk the closer they are to retirement. This behavior is reflected in many life-cycle fund allocation policies, see for instance Gebler and Matterson (2010) or Erickson and Cunniff (2015). An important economic reasoning behind this is that the older the person, the less time to retirement entrance is left and therefore the less likely it is for her to overcome a potential market crash, strongly connected to the fear of having an insufficient wealth left for retirement. Moreover, it is reasonable that the closer to retirement time, the more satisfaction is connected with savings, i.e. with a lower consumption surplus, which yields a higher initial wealth for the decumulation phase. Based on these economic reasons, it is meaningful to consider age-varying preference parameters (dependent on the age of the pension cohort or the individual investor) in form of a coefficient of risk aversion, later called $b(t)$, and a weighting factor, later referred to as $a(t)$, that governs the relative importance of consumption at different points in time. The latter has no impact on risk aversion but can control for the varying preference between consumption and terminal wealth over time. In an analysis of the optimal controls in Section 4.3 we show that our proposed model can explain and describe people's observed behavior of reducing relative risky investments over time while simultaneously targeting a certain function for the consumption rate on average. In opposite, we find that the previously described existing models are not able to capture this behavior. Therefore, particularly Section 4.3 shows that it is economically important to have separate functions or parameters for risk aversion and preference of consumption over terminal wealth, $a(t)$ and $b(t)$.

In addition, consumption and wealth floors are introduced which have an economic meaning as minimum required levels of consumption and wealth. This motivates the development of a dynamic life-cycle model with time-varying risk preferences such as coefficient of risk aversion and consumption and wealth floors which can capture age-depending consumption and investment behavior of investors.

Related literature to this topic consider stochastic income and unemployment risks, see Bodie et al. (1992), Koo (1998), Munk (2000), Viceira (2001), Huang and Milevsky (2008), Jang et al. (2013), Bensoussan et al. (2016), Wang et al. (2016) or Chen et al. (2018). Setups where the investor faces uncertain lifetime, mortality and optimal life insurance are considered in Yaari (1965), Pliska and Ye (2007), Menoncin and Regis (2017), Zou and Cadenillas (2014), Duarte et al. (2014), Huang et al. (2012), Kronborg and Steffensen (2015), Shen and Wei (2016), Guasoni and Huang (2019) and Ye (2008). Optimal consumption and investment under insurer default risk is studied by Jang et al. (2019). Kraft and Munk (2011), Kraft et al. (2018), Andréasson et al. (2017), Cuoco and Liu (2000) and Damgaard et al. (2003) analyze optimal housing as a durable good. Constraints in the optimization problem are considered in Cvitanic and Karatzas (1992), Cuoco (1997), Elie and Touzi (2008) and Grandits (2015). Moreover, Akian et al. (1996), Altarovici et al. (2017), Dai et al. (2009) and Hobson et al. (2019) analyze the portfolio problem under transaction costs. The application of HARA utility functions in a life-cycle context can be found in Huang and Milevsky (2008), Ye (2008), Chang and Rong (2014), Chang and Chang (2017) and Wang et al. (2017). Moreover, Back et al. (2019) study a life-cycle consumption problem for HARA utility with time-independent, increasing risk aversion and examine the relation between age and portfolio risk by using Monte Carlo analysis. Tang et al. (2018) study an optimal consumption-investment problem under CRRA utility function with age-independent risk aversion, but examine the impact of hyperbolic discounting, where the rate of time preference is a function of time. We generalize this approach by considering general $a(t)$ or $e^{-\beta t}a(t)$, respectively, and by introducing age-varying risk aversion.

In this chapter we apply HARA utility functions on both the consumption and terminal wealth and consider time-varying preferences: an age-depending preference between consumption and terminal wealth and an age-depending coefficient of risk aversion in the intertemporal consumption utility. For simplicity, income is treated as a deterministic process. Furthermore, we do not model mortality and consider a fixed time horizon T that corresponds to a retirement age, thus we assume the agent to survive up to the age of retirement. A positive, fixed floor in the terminal utility ensures a minimum liquid asset wealth level at the age of retirement, which is meaningful as the retiree needs wealth to live from and could possibly afford housing from this wealth. In addition, a positive, time-varying floor in the consumption utility guarantees a minimum (time-dependent) consumption rate. This is essential during the accumulation phase as for instance living expenses, rental payments when home is rented or mortgage payments and maintenance costs when home is bought and financed by debt or only maintenance costs when the agent already fully owns a house (e.g. inherited) need to be covered. All of these needs motivate the economic demand for both a positive minimum level of consumption and terminal wealth.

In summary, previous studies on age-depending risk preferences such as risk aversion have particularly shown the need of such age-varying preference coefficients. The studies moreover illustrate that, depending on the investigated situation and use case, there exist real-world situations where increasing as well as decreasing preferences over time can occur and can be meaningful. Beyond that, even non-monotonic changes in risk aversion can be reasonable. This motivates the consideration of a mathematical model that covers a very general structure for age-depending preferences: represented by $a(t)$ and $b(t)$ in our proposed model. Building upon this we show that there is a lack of a sound mathematical model for age-varying risk preferences in the existing literature and therefore no solution is provided yet. Formulated differently, the existing models and solutions to a consumption-investment problem are only provided in very restrictive settings. Most related to our work are Ye (2008), Steffensen (2011), Hentschel (2016) and Aase (2017). The difference of our approach to these papers is as follows. Ye (2008) considers income, mortality and HARA utilities for both consumption and terminal wealth under a constant coefficient of risk aversion, i.e. constant $b(t)$, but where the age-dependent preference between consumption and wealth $a(t)$ is incorporated. We generalize the results by introducing a time-dependent coefficient of risk aversion $b(t)$. Steffensen (2011) provides a first insight into the optimal policy when the utility parameters of the intertemporal utility, which is of a CRRA type, are time-varying; thus $a(t)$ and $b(t)$ are captured. But the model disregards terminal wealth, consumption floor and labor income. In a similar fashion, Hentschel (2016) studies the consumption problem for CRRA utility with habit formation and considers $a(t)$ and $b(t)$. Similar to Steffensen (2011), neither terminal wealth nor consumption floor nor income are included in their model. Finally, Aase (2017) uses the martingale method (that allows to reformulate the optimal stochastic control problem to a simpler maximization problem with constraint) to determine optimal consumption and investment under mortality risk and a CRRA utility with age-depending risk aversion $b(t)$. But the model does not consider terminal wealth, consumption floor, income or time-varying preference $a(t)$.

The main contributions and innovations of this chapter can be summarized into two parts: a theoretical and a practical part. First, in the theoretical part we consider all the "ingredients" of the models in the above mentioned papers ($a(t)$, $b(t)$, terminal wealth, floors for consumption and terminal wealth via HARA utilities, income process) that lead to a novel, very flexible and more realistic dynamic life-cycle model framework. We extend or generalize Ye (2008) by adding an age-dependent coefficient of risk aversion $b(t)$ and Steffensen (2011), Hentschel (2016) and Aase (2017) by considering terminal wealth and allowing for consumption and terminal wealth floors via an application of HARA utility functions. The corresponding consumption-investment problem is solved in closed-form and interpretations are given. A closed-form solution is particularly beneficial for interpretation and implementation purposes. Second, in the practical part, we carry out an exemplary numerical case study. Here, we first fit realistic predetermined target policies for consumption and relative allocation to several models, showing how our proposed model can be calibrated. Moreover, we realize that only our proposed and most general model is sufficiently flexible to describe human preferences on consumption and investment in a suitable fashion. This implies that modeling the agent's preferences in an age-depending fashion is inevitable. In addition, we provide interpretation and graphical visualization of the optimal controls under a bullish and bearish market scenario. Therein we elaborate on the influence of $a(t)$, $b(t)$ and the wealth floor F on the optimal controls. In summary, we mathematically improve existing models and solutions to the consumption-investment problem and furthermore demonstrate that our proposed model and its solution can be used in real-world applications.

To solve the respective portfolio problem, we follow a separation approach similar to the ones developed by Karatzas and Shreve (1998) and Lakner and Nygren (2006). It divides the original consumption-terminal wealth optimization problem into two sub-problems, the corresponding consumption problem and the terminal wealth problem. These separate problems are to be solved individually. Due to time-dependent preference parameters we apply the martingale method in line with Aase (2017) to solve the individual problems in closed form. Afterwards, we show how the individual solutions have to be glued together in order to obtain the general solution to the original consumption-terminal wealth problem.

The remainder of this chapter is organized as follows. Section 4.1 introduces the financial market and the portfolio problem of interest, Section 4.2 shows the separation approach and the solution to the problem. A fit of the analytic strategy to suitable consumption and investment curves is conducted in Section 4.3, followed by an investigation of the optimal controls and corresponding wealth process. Appendix B summarizes all proofs of the claimed statements: the proofs for Section 4.2.1 on the consumption problem can be found in Appendix B.1, the proofs related to Section 4.2.2 on the terminal wealth problem in Appendix B.2, and for the proofs associated with Section 4.2.3 on merging both individual solutions, see Appendix B.3.

4.1 The financial market model and consumption-investment problem

Let us consider the financial market model introduced in Section 2.1. Additionally to Chapter 3, we consider a consumption process. Let $(c(t))_{t\in[0,T]}$ denote a non-negative, progressively measurable, real-valued stochastic consumption-rate process with $\int_0^T c(t)dt < \infty$, $\mathbb{P}$-a.s., and $(y(t))_{t\in[0,T]}$ a non-negative, deterministic income-rate process with $\int_0^T y(t)dt < \infty$. Those technical conditions are assumed to ensure a solution for the subsequently formulated stochastic problem. The dynamics of the investor's wealth process $V = (V(t))_{t\in[0,T]}$ under the strategy (π, c) to initial wealth $V(0) = v_0 > 0$, including liquid assets, consumption and income, is then given by

$$dV(t) = V(t)\left[\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}(t)'\sigma dW(t)\right] - c(t)dt + y(t)dt. \tag{4.1}$$

The relative investment in the risk-free asset is $\pi_0(t) = 1 - \hat{\pi}(t)'\mathbf{1}$. We consider the objective of maximizing expected utility of future terminal wealth and consumption, starting at time 0 and ending at T. Hence the objective function to be maximized is

$$\mathcal{J}(\pi, c; v_0) = \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(V(T))\right], \tag{4.2}$$

where $v_0 > 0$ denotes the initial endowment of the investor. All expectations in this chapter are with respect to the real-world measure $\mathbb{P}$. The general portfolio optimization problem with initial wealth $V(0) = v_0 > 0$ to be solved is then given by

$$\mathcal{V}(v_0) = \sup_{(\pi,c)\in\Lambda} \mathcal{J}(\pi, c; v_0). \tag{4.3}$$

$\mathcal{V}(v_0)$ is the value function of the problem. Let $\tilde{\Lambda}'(v_0)$ be defined as the set which deviates from $\tilde{\Lambda}(v_0)$ only in the self-financing property that is adjusted for consumption and income:

$$V(t) = v_0 + \sum_{i=0}^{N} \int_0^t \varphi_i(s)dP_i(s) - \int_0^T c(t)dt + \int_0^T y(t)dt.$$

Λ then denotes the set of admissible investment and consumption strategies (π, c) that satisfy the following conditions:

$A^{(4)}$: $(\pi, c) \in \tilde{\Lambda}'(v_0)$.

$B^{(4)}$: (π, c) admits a unique solution to Eq. (4.1).

$C^{(4)}$: (π, c) fulfills the associated budget constraint

$$\mathbb{E}\left[\int_0^T \tilde{Z}(t)c(t)dt + \tilde{Z}(T)V(T)\right] \leqslant v_0 + \mathbb{E}\left[\int_0^T \tilde{Z}(t)y(t)dt\right] = v_0 + \int_0^T e^{-rt}y(t)dt. \tag{4.4}$$

$D^{(4)}$: (π, c) is such that $V(t) + \int_t^T e^{-r(s-t)}y(s)ds \geqslant 0$, $\mathbb{P}$-a.s., $\forall t \in [0, T]$.

$E^{(4)}$: $(c(t))_{t\in[0,T]} \geqslant 0$ is progressively measurable with $\int_0^T c(t)dt < \infty$, $\mathbb{P}$-a.s., and satisfies the integrability condition $\mathbb{E}\left[\int_0^T |U_1(t, c(t))|dt\right] < \infty$.

$F^{(4)}$: $(y(t))_{t\in[0,T]} \geqslant 0$ is deterministic with $\int_0^T y(t)dt < \infty$.

We briefly compare the above conditions covered by Λ with the corresponding conditions in Chapter 3 that belong to the set Λ in Problem (3.2): First, Condition $A^{(3)}$ ($\pi \in \tilde{\Lambda}(v_0)$) is replaced by $A^{(4)}$ ($(\pi, c) \in \tilde{\Lambda}'(v_0)$), where the self-financing property is adjusted for consumption and income as mentioned above. Moreover, Conditions $E^{(4)}$ and $F^{(4)}$ generally introduce the consumption and income process which were absent in Chapter 3, hence are newly added here. Conditions $B^{(4)}$ and $C^{(4)}$ substitute $B^{(3)}$ and $C^{(3)}$ by the inclusion of consumption and income. Condition $D^{(4)}$ is added and introduces an explicit lower bound on the wealth.

The budget constraint (4.4) describes the requirement that today's value of future consumption and terminal wealth, less income, must not exceed the initial endowment. It can be shown that for the optimal investment and consumption strategy $(\hat{\pi}^\star, c^\star)$ to Problem (4.3), Eq. (4.4) holds with equality. We consider a preference utility model given by the HARA utility functions

$$
\begin{aligned}
U_1(t,c) &= \begin{cases} \left(e^{-\beta t}a(t)\right)\frac{1-b(t)}{b(t)}\left(\frac{1}{1-b(t)}\left(c-\bar{c}(t)\right)\right)^{b(t)}, & \text{for } b(t)\neq 0 \\ \left(e^{-\beta t}a(t)\right)\ln\left(c-\bar{c}(t)\right) & , \text{ for } b(t)=0, \end{cases} \\
U_2(v) &= e^{-\beta T}\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(v-F)\right)^{\hat{b}},
\end{aligned}
\tag{4.5}
$$

for $\beta \geqslant 0$, $b:[0,T]\to(-\infty,1)$ continuous[1], $\hat{b}<1$, $\hat{b}\neq 0$, $a(t)>0$, $\hat{a}>0$, $c(t)>\bar{c}(t)$, $\bar{c}(t)\geqslant 0$ deterministic, and $v>F$ with $F\geqslant 0$. U_2 is a continuously differentiable and strictly concave terminal utility function, U_1 denotes a continuous (intertemporal consumption) utility function which is continuously differentiable and strictly concave in the second argument. This utility model accounts for several desired aspects: minimum liquid asset wealth level $F\geqslant 0$ at the age of retirement T, minimum consumption rate $\bar{c}(t)\geqslant 0$ and time-varying preference of consumption over terminal wealth in terms of $a(t)$. Moreover, the coefficient of risk aversion $b(t)$ in the consumption utility is now a continuous function in time. In the following, we omit the case where $b(t)=0$ in Eq. (4.5) and concentrate on $b(t)\neq 0$ for ease of exposition. But we would like to highlight that all expressions and statements are still valid when inserting $b(t)=0$. This is due to the fact that log-utility describes the limiting case of a HARA utility function when $b(t)\to 0$ with

$$
\frac{\partial}{\partial c}\left(U_1(t,c)|_{b(t)=0}\right) = \left(e^{-\beta t}a(t)\right)\frac{1}{c-\bar{c}(t)} = \lim_{b(t)\to 0}\left(\frac{\partial}{\partial c}U_1(t,c)|_{b(t)\neq 0}\right).
$$

Therefore, the definition of $U_1(t,c)$ in Eq. (4.5) describes a continuous extension in the first derivative of $U_1(t,c)$ with respect to argument c. Since this derivative (and its inverse) is the only term that enters the optimal consumption and investment strategy later, the solution derived for $b(t)\neq 0$ is also valid for the case $b(t)=0$ and derived by simply inserting $b(t)=0$ at the very last step.

Remark 4.1. *Notice that according to Section 2.4 the associated Arrow-Pratt measure of absolute risk aversion $\mathcal{A}(v)$ admits the following hyperbolic representation*

$$
\mathcal{A}_1(t,c) = \frac{1-b(t)}{c-\bar{c}(t)} > 0,\ \ \mathcal{A}_2(v) = \frac{1-\hat{b}}{v-F} > 0.
$$

For this reason, we use the notation of an increasing $b(t)$ as a synonym for a decreasing coefficient of risk aversion and vice versa. Further note that $a(t)$ does not appear in $\mathcal{A}_1(t,c)$. Therefore we have two input functions $a(t)$ and $b(t)$ where $a(t)$ has no influence on risk aversion, but $b(t)$ determines it; hence a very flexible model.

Since we have $c(t)>\bar{c}(t)$ and $V(T)>F$ by definition of the utility functions in (4.5), we restrict

$$
v_0 > \int_0^T e^{-rs}\left(\bar{c}(s)-y(s)\right)ds + e^{-rT}F =: F(0) \tag{4.6}
$$

on the initial endowment in (4.4). It is useful to define

$$
F(t) := \mathbb{E}\left[\int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)}\bar{c}(s)ds + \frac{\tilde{Z}(T)}{\tilde{Z}(t)}F - \int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)}y(s)ds \middle| \mathcal{F}_t\right]
$$

[1] For completeness, we would like to emphasize that the continuity assumption is only required to find a nice representation of the solution for the optimal investment strategy later (cf. Theorem 4.2). A generalized version in the case of a discontinuous $b(t)$ can be found at the end of the proof of Theorem 4.2.

$$= \int_t^T \mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)}\middle|\mathcal{F}_t\right](\bar{c}(s) - y(s))\,ds + F\mathbb{E}\left[\frac{\tilde{Z}(T)}{\tilde{Z}(t)}\middle|\mathcal{F}_t\right]$$
$$= \int_t^T e^{-r(s-t)}\,(\bar{c}(s) - y(s))\,ds + e^{-r(T-t)}F. \tag{4.7}$$

$F(t)$ can be interpreted as the time t value of all future minimal liabilities less income. $F(t)$ equals the sum of the time t wealth necessary to meet all the future minimum living expenses and expenditures $\bar{c}(s)$, $s \in [t, T]$, during the remaining time and the time t value of the minimum desired terminal wealth level F; future salary income is subtracted as it reduces the time t value of the minimum required capital.

4.2 Solution: Separation technique

In the sequel we follow the separation technique approach by Karatzas and Shreve (1998) and Lakner and Nygren (2006) for solving the consumption-terminal wealth problem as defined by (4.3). We split the problem into two sub-problems: the consumption-only and terminal wealth-only problem. Both individual problems are separately solved via the martingale method, similar to the approach by Aase (2017). We would like to mention that Steffensen (2011) employed the idea of the above-named separation technique to find an appropriate ansatz for the value function of the consumption-only problem with age-dependent risk aversion. In contrast, we apply this technique to separate consumption from terminal wealth. The individual problem solutions are optimally merged at the end. For this sake, let us consider the two individual problems first.

4.2.1 The consumption problem

The consumption-only problem is

$$\begin{aligned} \mathcal{J}_1(\pi, c; v_1) &= \mathbb{E}\left[\int_0^T U_1(t, c(t))dt\right], \\ \mathcal{V}_1(v_1) &= \sup_{(\pi,c)\in\Lambda_1} \mathcal{J}_1(\pi, c; v_1), \end{aligned} \tag{4.8}$$

where Λ_1 covers all admissible investment and consumption strategies (π, c), where the conditions coincide with the ones in Λ that belong to the general Problem (4.3) of this Chapter 4, except for $A^{(4)}$ and $C^{(4)}$ which are replaced by $A^{(4.2.1)}$ and $C^{(4.2.1)}$, due to the splitting of Problem (4.3) into a consumption and a terminal wealth problem:

$A^{(4.2.1)}$: $(\pi, c) \in \tilde{\Lambda}'(v_1)$.

$C^{(4.2.1)}$: (π, c) fulfills the associated budget constraint

$$\mathbb{E}\left[\int_0^T \tilde{Z}(t)c(t)dt\right] \leqslant v_1 + \mathbb{E}\left[\int_0^T \tilde{Z}(t)y(t)dt\right] = v_1 + \int_0^T e^{-rt}y(t)dt. \tag{4.9}$$

In more detail, the initial wealth v_0 to the original problem is exchanged with v_1 which denotes the initial wealth to the consumption-only problem. Furthermore, the terminal wealth $V(T)$ in the budget constraint is removed since we deal with the consumption-only problem.

Steffensen (2011) provides a proof for CRRA utility functions by solving the associated Hamilton-Jacobi-Bellman (HJB) equation. We follow the approach by Aase (2017), likewise for a HARA utility function. We extend the findings of Aase (2017) by introducing a time-varying, deterministic consumption floor $\bar{c}(t)$, a time-varying preference function $a(t)$ of consumption over terminal wealth and an income-rate process $y(t)$.

In order to guarantee the consumption rate floor, note $c(t) > \bar{c}(t)$, let us assume the following lower boundary for v_1 which equals the integral over the discounted consumption floor rate minus income rate over the whole horizon of interest:

$$\begin{aligned} v_1 &> \int_0^T e^{-rs} \left(\bar{c}(s) - y(s)\right) ds =: F_1(0), \\ F_1(t) &:= \mathbb{E}\left[\int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)} \left(\bar{c}(s) - y(s)\right) ds \middle| \mathcal{F}_t\right] = \int_t^T e^{-r(s-t)} \left(\bar{c}(s) - y(s)\right) ds. \end{aligned} \tag{4.10}$$

Notice that $v_1 < 0$ is possible since a sufficiently large positive income stream can be high enough to finance consumption. Using the martingale method we solve the problem as summarized by the theorem below.

Theorem 4.2. *The solution to the optimal stochastic control problem* (4.8) *with intertemporal utility function* U_1 *in* (4.5) *is*

$$\begin{aligned} \hat{\pi}_1(t; v_1) &= \frac{1}{1 - b(\tilde{t}_1)} \Sigma^{-1}(\mu - r\mathbf{1}) \frac{V_1(t; v_1) - F_1(t)}{V_1(t; v_1)}, \\ c_1(t; v_1) &= g(t, t; v_1) \tilde{Z}(t)^{\frac{1}{b(t)-1}} + \bar{c}(t) = (1 - b(t)) \left(\lambda_1 \frac{e^{\beta t}}{a(t)} \tilde{Z}(t)\right)^{\frac{1}{b(t)-1}} + \bar{c}(t), \\ V_1(t; v_1) &= \int_t^T g(s, t; v_1) \tilde{Z}(t)^{\frac{1}{b(s)-1}} ds + F_1(t), \\ V_1(T; v_1) &= 0, \end{aligned}$$

for all $t \in [0, T]$, *where*

$$g(s, t; v_1) = (1 - b(s)) \left(\frac{e^{\beta s - b(s)\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(s-t)}}{a(s)}\right)^{\frac{1}{b(s)-1}} \lambda_1^{\frac{1}{b(s)-1}}.$$

$\lambda_1 = \lambda_1(v_1) > 0$ *satisfies the budget constraint uniquely and is subject to the equation*

$$\int_0^T g(t, 0; v_1) dt = v_1 - F_1(0). \tag{4.11}$$

$\tilde{t}_1 = \tilde{t}_1(v_1) \in (t, T)$ *is the solution to the equation*

$$\int_t^T \frac{1}{b(s) - 1} g(s, t; v_1) \tilde{Z}(t)^{\frac{1}{b(s)-1}} ds = \frac{1}{b(\tilde{t}_1) - 1} \int_t^T g(s, t; v_1) \tilde{Z}(t)^{\frac{1}{b(s)-1}} ds. \tag{4.12}$$

For the optimal $c_1(t; v_1)$, *Eq.* (4.9) *is fulfilled with equality.*

We remind the reader that all proofs to this chapter can be found in Appendix B. It is clear that

$c_1(t; v_1) > \bar{c}(t)$, a.s.. We now aim to interpret the optimal investment strategy as *Proportional Portfolio Insurance (PPI)* strategy. The CPPI strategy family corresponds to a constant multiple, the PPI is more general and also covers proportional strategies with time-varying or even state-dependent multiples. Zieling et al. (2014) evaluate the performance of such strategies. Theorem 4.2 shows that the optimal investment strategy generally is a PPI strategy with time-varying floor $F_1(t)$ at time t, equal to the time t value of the accumulated outstanding future consumption floor minus income. Notice that $\tilde{t}_1$ can firstly be determined at time t, since the value depends on the stochastic $\tilde{Z}(t)$ which is not known before time t. Hence, $\tilde{t}_1$ is time- and also state-dependent and thus the optimal PPI strategy itself is time- and state-dependent through its PPI multiple. The PPI multiple in summary is time-varying, state-dependent and depends on all future coefficients of risk aversion via $b(\tilde{t}_1)$.

Furthermore, $V_1(0; v_1) > F_1(0)$ holds by the assumption in (4.10). In addition, $\hat{\pi}_1(t; v_1)$ converges to 0 when $V_1(t; v_1)$ approaches $F_1(t)$. Thus, $V_1(t; v_1) > F_1(t)$ a.s., which additionally follows directly from the formula for $V_1(t; v_1)$ in Theorem 4.2. This further implies that $(\hat{\pi}_1, c_1)$ is an admissible pair, i.e. $(\hat{\pi}_1, c_1) \in \Lambda_1$. The next remark provides the solution under time-independent risk aversion.

Remark 4.3. *When $b(t) \equiv b$, then*

$$\hat{\pi}_1(t; v_1) = \frac{1}{1-b}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V_1(t; v_1) - F_1(t)}{V_1(t; v_1)}$$

which is a conventional CPPI strategy with constant multiple. Moreover, if $\bar{c}(t) - y(t) \equiv 0$, i.e. the minimum consumption is eating up the whole income, then

$$\hat{\pi}_1(t; v_1) = \frac{1}{1-b}\Sigma^{-1}(\mu - r\mathbf{1}),$$

which is a constant-mix strategy and represents the standard, well-known result for CRRA utility with constant risk-aversion parameter.

Some comments on the initial capital v_1 and the sign of the risky investments come next. As already pointed out, a start with a negative initial capital $V_1(0; v_1) = v_1 < 0$ to Problem (4.8) is possible and might be reasonable in a sense that accumulated income over the life-cycle is expected to exceed total consumption. Hence, there is no need to require positive capital to this problem. For this reason, $V_1(t; v_1) < 0$ can happen and might be reasonable, too.

Theorem 4.2 tells that the optimal relative investment strategy is given by

$$\hat{\pi}_1(t; v_1) = \frac{1}{1-b(\tilde{t}_1)}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V_1(t; v_1) - F_1(t)}{V_1(t; v_1)},$$

where $V_1(t; v_1) > F_1(t)$ a.s.. Let $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ for $i \in \{1, \dots, N\}$, which for instance is the case when there is only one risky asset ($N = 1$) because then $\Sigma^{-1}(\mu - r) = \frac{\mu - r}{\sigma^2} > 0$ since $\mu - r\mathbf{1} > 0$ was assumed. Then

$$\begin{aligned} (\hat{\pi}_1(t; v_1))_i > 0 &\Leftrightarrow V_1(t; v_1) > 0, \\ (\hat{\pi}_1(t; v_1))_i < 0 &\Leftrightarrow V_1(t; v_1) < 0. \end{aligned}$$

Even if we previously argued that $V_1(t;v_1) < 0$ is a meaningful case, the conclusion $(\hat{\pi}_1(t;v_1))_i < 0$ under $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ sounds odd at a first glance. But when looking at the optimal exposure to risky asset i, one finds that

$$(\hat{\pi}_1(t;v_1)V_1(t;v_1))_i = \frac{1}{1-b(\tilde{t}_1)} \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i (V_1(t;v_1) - F_1(t)),$$

which, under $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$, is positive no matter if $V_1(t;v_1) < 0$ or $V_1(t;v_1) > 0$. Therefore, the amount of money invested in the risky assets is always positive. The opposite inequalities and conclusions for $(\hat{\pi}_1(t;v_1))_i$ and $(\hat{\pi}_1(t;v_1)V_1(t;v_1))_i$ apply if $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i < 0$. In summary, the sign of the optimal exposure to the single risky assets is determined by

$$(\hat{\pi}_1(t;v_1)V_1(t;v_1))_i > 0 \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0.$$

Thus, $(\hat{\pi}_1(t;v_1)V_1(t;v_1))_i > 0$ is possible although it might be $(\hat{\pi}_1(t;v_1))_i < 0$.

Finally, let $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ for all $i \in \{1,\dots,N\}$. When $V_1(t;v_1) < 0$, the optimal exposure to the risk-free asset is negative because

$$\underbrace{V_1(t;v_1)}_{<0}\left(1 - \underbrace{\hat{\pi}_1(t;v_1)'\mathbf{1}}_{<0}\right) < V_1(t;v_1) < 0.$$

This in turn implies that in case of $V_1(t;v_1) < 0$, the investor takes leverage by borrowing from the risk-free account to achieve her investment goals. Leverage at this point can make sense as future income provides some security; note that $V_1(t;v_1) < 0$ immediately implies that the time t value of accumulated future income exceeds the expected value of consumption.

Some more properties of $\hat{\pi}_1(t;v_1)$ can be found analytically as follows. The first and second derivative of $(\hat{\pi}_1(t;v_1))_i$, $i = 1,\dots,N$, with respect to wealth $V_1(t;v_1)$ are

$$\begin{aligned}
\frac{\partial}{\partial V_1(t;v_1)} (\hat{\pi}_1(t;v_1))_i &= \frac{1}{1-b(\tilde{t}_1)} \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{F_1(t)}{V_1(t;v_1)^2},\\
\frac{\partial^2}{\partial V_1(t;v_1)^2} (\hat{\pi}_1(t;v_1))_i &= -2\frac{1}{1-b(\tilde{t}_1)} \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{F_1(t)}{V_1(t;v_1)^3}.
\end{aligned}$$

Let $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ for $i \in \{1,\dots,N\}$, then

1. $\frac{\partial}{\partial V_1(t;v_1)} (\hat{\pi}_1(t;v_1))_i \overset{(>)}{\geqslant} 0 \;\Leftrightarrow\; F_1(t) \overset{(>)}{\geqslant} 0.$

2. $\frac{\partial^2}{\partial V_1(t;v_1)^2} (\hat{\pi}_1(t;v_1))_i \overset{(<)}{\leqslant} 0 \;\Leftrightarrow$ either $F_1(t) \overset{(>)}{\geqslant} 0$ and $V_1(t;v_1) > 0$ or $F_1(t) \overset{(<)}{\leqslant} 0$ and $V_1(t;v_1) < 0$.

This implies that at time t:

1. $(\hat{\pi}_1(t;v_1))_i$ is increasing in $V_1(t;v_1)$ if and only if $F_1(t) \geqslant 0$, and decreasing in $V_1(t;v_1)$ otherwise.

2. $(\hat{\pi}_1(t;v_1))_i$ is concave in $V_1(t;v_1)$ if and only if

   a) either $F_1(t) \geqslant 0$ and $V_1(t;v_1) > 0$

   b) or $F_1(t) \leqslant 0$ and $V_1(t;v_1) < 0$, and convex in $V_1(t; v_1)$ otherwise.

The opposite inequalities and conclusions for $(\hat{\pi}_1(t; v_1))_i$ and its derivatives apply if $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i < 0$.

The optimal controls in Theorem 4.2 determine the value function and the value for λ_1 as follows.

Theorem 4.4. *The optimal value function $\mathcal{V}_1(v_1)$ to Problem (4.8) is strictly increasing and concave in v_1. Its value and first and second derivative with respect to the initial budget v_1 are given by*

$$\mathcal{V}_1(v_1) = \int_0^T \frac{1-b(t)}{b(t)} \left(\frac{e^{\left[\beta - b(t)\left(r - \frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)} \right)^{\frac{1}{b(t)-1}} \lambda_1^{\frac{b(t)}{b(t)-1}} dt,$$
$$\mathcal{V}_1'(v_1) = \lambda_1 > 0,$$
$$\mathcal{V}_1''(v_1) = \frac{\partial}{\partial v_1}\lambda_1 = -\left(\int_0^T \left(\frac{e^{\left[\beta - b(t)\left(r - \frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)} \right)^{\frac{1}{b(t)-1}} \lambda_1^{-\frac{b(t)-2}{b(t)-1}} dt \right)^{-1} < 0.$$

4.2.2 The terminal wealth problem

The terminal wealth-only problem is

$$\begin{aligned} \mathcal{J}_2(\pi, c; v_2) &= \mathbb{E}\left[U_2(V(T))\right], \\ \mathcal{V}_2(v_2) &= \sup_{(\pi,c)\in\Lambda_2} \mathcal{J}_2(\pi, c; v_2), \end{aligned} \tag{4.13}$$

with Λ_2 denoting the set of admissible investment and consumption strategies (π, c), where the following conditions deviate from the conditions in Λ to the general Problem (4.3) of Chapter 4:

$A^{(4.2.2)}$: $(\pi, c) \in \tilde{\Lambda}'(v_2)$.

$B^{(4.2.2)}$: (π, c) admits a unique solution to Eq. (4.1) for $y(t) \equiv 0$.

$C^{(4.2.2)}$: (π, c) fulfills the associated budget constraint

$$\mathbb{E}\left[\tilde{Z}(T)V(T)\right] \leqslant v_2, \ v_2 \geqslant 0. \tag{4.14}$$

$D^{(4.2.2)}$: (π, c) is such that $V(t) \geqslant 0$, $\mathbb{P}$-a.s., $\forall t \in [0, T]$.

Conditions $B^{(4.2.2)}$ and $D^{(4.2.2)}$ replace $B^{(4)}$ and $D^{(4)}$ by simply forcing $y(t) \equiv 0$ because the income-rate process was already used in the consumption problem in Section 4.2.1. For the very same reason, Condition $F^{(4)}$ is removed. In addition, the budget constraint in $C^{(4)}$ is exchanged with the one in $C^{(4.2.2)}$, since the terminal wealth-only problem is considered with initial wealth v_2. Therefore, also $A^{(4)}$ gets replaced by $A^{(4.2.2)}$.

In order to guarantee the terminal wealth floor, note $V(T) > F$, let us assume the following lower bound for v_2 which equals the discounted terminal floor:

$$v_2 > e^{-rT}F =: F_2(0), \ F_2(t) := \mathbb{E}\left[\frac{\tilde{Z}(T)}{\tilde{Z}(t)} F \middle| \mathcal{F}_t \right] = e^{-r(T-t)}F \geqslant 0. \tag{4.15}$$

Applying the martingale approach leads to the solution to the terminal wealth problem according to the upcoming theorem.

Theorem 4.5. *The solution to Problem* (4.13) *with terminal utility function* U_2 *in* (4.5) *is*

$$\begin{aligned}
\hat{\pi}_2(t;v_2) &= \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V_2(t;v_2)-F_2(t)}{V_2(t;v_2)},\\
c_2(t;v_2) &= 0,\\
V_2(t;v_2) &= \left(v_2 - e^{-rT}F\right)e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t),\\
V_2(T;v_2) &= \left(v_2 - e^{-rT}F\right)e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)^{\frac{1}{\hat{b}-1}} + F,
\end{aligned}$$

for all $t \in [0,T]$*. For the optimal* $\hat{\pi}_2(t;v_2)$*, Eq.* (4.14) *is fulfilled with equality.*

Theorem 4.5 shows that the optimal fraction of wealth allocated to the risky assets follows a CPPI strategy with floor $F_2(t) \geqslant 0$ at time t, with constant multiple. Moreover, $V_2(0;v_2) > F_2(0) = e^{-rT}F$ by the assumption in (4.15). In addition, $\hat{\pi}_2(t;v_2)$ converges to 0 when $V_2(t;v_1)$ approaches $F_2(t)$. Thus, it follows $V_2(t;v_2) > F_2(t)$ a.s., which additionally yields that $(\hat{\pi}_2, 0)$ is an admissible pair, i.e. $(\hat{\pi}_2, 0) \in \Lambda_2$. The characteristic $V_2(t;v_2) > F_2(t)$ a.s. also directly follows from the formula for $V_2(t;v_2)$ in Theorem 4.5. The next remark shows that the optimal proportion allocated to the risky assets is constant over time if one disregards the floor F.

Remark 4.6. *When* $F = 0$*, then*

$$\hat{\pi}_2(t;v_2) = \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})$$

which is a constant-mix strategy and equals the standard result for CRRA utility with constant risk-aversion parameter, where the optimal fraction of wealth allocated to the single risky assets does not depend on time or wealth.

In what follows we analyze some characteristics of the optimal strategy $\hat{\pi}_2(t;v_2)$. The first and second derivative of $(\hat{\pi}_2(t;v_2))_i$, $i = 1,\dots,N$, with respect to wealth $V_2(t;v_2)$ are

$$\begin{aligned}
\frac{\partial}{\partial V_2(t;v_2)}\left(\hat{\pi}_2(t;v_2)\right)_i &= \frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i\frac{F_2(t)}{V_2(t;v_2)^2},\\
\frac{\partial^2}{\partial V_2(t;v_2)^2}\left(\hat{\pi}_2(t;v_2)\right)_i &= -2\frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i\frac{F_2(t)}{V_2(t;v_2)^3}.
\end{aligned}$$

Let $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ for $i \in \{1,\dots,N\}$. Then, it follows $\frac{\partial}{\partial V_2(t;v_2)}\left(\hat{\pi}_2(t;v_2)\right)_i \geqslant 0$ as well as $\frac{\partial^2}{\partial V_2(t;v_2)^2}\left(\hat{\pi}_2(t;v_2)\right)_i \leqslant 0$, where the inequalities hold strictly when $F > 0$. Hence, $\left(\hat{\pi}_2(t;v_2)\right)_i$ increases and is concave in the wealth $V_2(t;v_2)$. Otherwise, if $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i < 0$ for $i \in \{1,\dots,N\}$, then $\left(\hat{\pi}_2(t;v_2)\right)_i$ decreases and is convex in the wealth $V_2(t;v_2)$. For the optimal exposure to the risky assets it therefore holds

$$\left(\hat{\pi}_2(t;v_2)V_2(t;v_2)\right)_i > 0 \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0.$$

Thus, either it is $(\hat{\pi}_2(t;v_2))_i > 0$ and $(\hat{\pi}_2(t;v_2)V_2(t;v_2))_i > 0$ or $(\hat{\pi}_2(t;v_2))_i < 0$ and $(\hat{\pi}_2(t;v_2)V_2(t;v_2))_i < 0$.

The optimal controls in Theorem 4.5 determine the value function and the value for λ_2.

Theorem 4.7. *The optimal value function $\mathcal{V}_2(v_2)$ to Problem (4.13) is strictly increasing and concave in v_2. Its value and first and second derivative with respect to the initial budget v_2 are given by*

$$\begin{aligned}
\mathcal{V}_2(v_2) &= e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T} \frac{\left(1-\hat{b}\right)^{1-\hat{b}}}{\hat{b}} \hat{a}\left(v_2 - F_2(0)\right)^{\hat{b}}, \\
\mathcal{V}_2'(v_2) &= \lambda_2 > 0, \\
\mathcal{V}_2''(v_2) &= \frac{\partial}{\partial v_2}\lambda_2 = -e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T} \left(1-\hat{b}\right)^{2-\hat{b}} \hat{a}\left(v_2 - F_2(0)\right)^{\hat{b}-2} < 0.
\end{aligned}$$

The Lagrange multiplier is given by (B.8) *as*

$$\lambda_2 = e^{-\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T} \left(1-\hat{b}\right)^{1-\hat{b}} \hat{a}\left(v_2 - F_2(0)\right)^{\hat{b}-1} > 0.$$

4.2.3 Optimal merging of the individual solutions

Let $(\pi_1(t;v_1), c_1(t;v_1))$ denote the optimal controls to Problem (4.8) with optimal wealth process $V_1(t;v_1)$ to the initial wealth $v_1 \geqslant \int_0^T e^{-rt}\left(\bar{c}(t) - y(t)\right) dt = F_1(0)$ and $(\pi_2(t;v_2), c_2(t;v_2))$ the optimal controls to Problem (4.13) with optimal wealth process $V_2(t;v_2)$ to the initial wealth $v_2 \geqslant e^{-rT}F = F_2(0)$. Then merging the two solutions to solve Problem (4.3) is based on the following theorem.

Theorem 4.8. *The connection between the value functions is*

$$\mathcal{V}(v_0) = \sup_{v_1 \geqslant F_1(0),\ v_2 \geqslant F_2(0),\ v_1+v_2=v_0} \left\{\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2)\right\}.$$

Notice that $F(t) = F_1(t) + F_2(t)$, hence (4.6) ensures that $v_0 = v_1 + v_2 > F_1(0) + F_2(0)$ is claimed. When discounted future income exceeds consumption over the considered period, i.e. when the initial budget to the consumption problem is negative ($v_1 < 0$), then $v_2 > v_0$ and a higher amount of money v_2 is invested according to the terminal wealth problem at initial time as the initial endowment v_0 of the investor.

Theorem 4.8 shows that an optimal allocation to consumption and terminal wealth at $t = 0$ together with the solution to the two separate problems equals the solution to the original optimization problem. The optimal initial budgets are denoted by $v_1^\star$ and $v_2^\star$. The next lemma provides the solution for $v_1^\star$ and $v_2^\star$ within our specified setup.

Lemma 4.9. *The optimal $v_1^\star$ generally solves*

$$\mathcal{V}_1'(v_1) - \mathcal{V}_2'(v_0 - v_1) = 0. \tag{4.16}$$

The optimal $v_2^\star$ is given by $v_2^\star = v_0 - v_1^\star$. Moreover, $v_1^\star$ exists uniquely and satisfies the boundary condition $F_1(0) \leqslant v_1^\star \leqslant v_0 - F_2(0)$. $v_1^\star$ is the solution to the equation

$$v_1 - \int_0^T \chi(t) \left(v_0 - v_1 - F_2(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt = F_1(0) \tag{4.17}$$

with

$$\chi(t) = (1-b(t)) \left(1-\hat{b}\right)^{\frac{1-\hat{b}}{b(t)-1}} \left(\frac{\hat{a}}{a(t)}\right)^{\frac{1}{b(t)-1}} \left(\frac{e^{\left[\beta - b(t)\left(r - \frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{e^{\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}\right)^{\frac{1}{b(t)-1}} > 0. \tag{4.18}$$

Moreover, the optimal Lagrange multiplier $\lambda_1^\star = \lambda_1(v_1^\star)$ is given by

$$\lambda_1^\star = \left(1-\hat{b}\right)^{1-\hat{b}} \hat{a} e^{-\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T} \left(v_0 - v_1^\star - F_2(0)\right)^{\hat{b}-1}.$$

For general $a(t)$ and $b(t)$, v_1^* as the unique solution to Eq. (4.17) can for instance be determined numerically. Denote by $v_1^\star \geqslant F_1(0)$, $v_2^\star \geqslant F_2(0)$ with $v_1^\star + v_2^\star = v_0$ the optimal allocation of the initial wealth according to Lemma 4.9 in what follows and denote $\lambda_1^\star = \lambda_1(v_1^\star)$ and $\tilde{t}_1^\star = \tilde{t}_1(v_1^\star)$. We use the individual solutions to the two separate Problems (4.8) and (4.13) and merge both solutions optimally according to Lemma 4.9 to obtain the solution to the original Problem (4.3).

Theorem 4.10. *The optimal wealth process is given by $V^\star(t; v_0) = V_1(t; v_1^\star) + V_2(t; v_2^\star)$. The optimal controls to Problem (4.3) are*

$$c^\star(t; v_0) = c_1(t; v_1^\star), \;\; \hat{\pi}^\star(t; v_0) = \frac{\hat{\pi}_1(t; v_1^\star)V_1(t; v_1^\star) + \hat{\pi}_2(t; v_2^\star)V_2(t; v_2^\star)}{V_1(t; v_1^\star) + V_2(t; v_2^\star)}.$$

The optimal controls and the optimal wealth process to Problem (4.3) under the utility function setup (4.5) are given by

$$\begin{aligned}
\hat{\pi}^\star(t; v_0) &= \Sigma^{-1}(\mu - r\mathbf{1}) \frac{\frac{1}{1-b(\tilde{t}_1^\star)}\left(V_1(t; v_1^\star) - F_1(t)\right) + \frac{1}{1-\hat{b}}\left(V_2(t; v_2^\star) - F_2(t)\right)}{V^\star(t; v_0)}, \\
c^\star(t; v_0) &= g(t, t; v_1^\star)\tilde{Z}(t)^{\frac{1}{b(t)-1}} + \bar{c}(t) = (1-b(t)) \left(\lambda_1^\star \frac{e^{\beta t}}{a(t)} \tilde{Z}(t)\right)^{\frac{1}{b(t)-1}} + \bar{c}(t), \\
V^\star(t; v_0) &= \int_t^T g(s, t; v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds + \left(v_2^\star - F_2(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F(t), \\
V^\star(T; v_0) &= \left(v_2^\star - F_2(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T} \tilde{Z}(T)^{\frac{1}{\hat{b}-1}} + F, \\
V_1(t; v_1^\star) &= \int_t^T g(s, t; v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds + F_1(t), \\
V_2(t; v_2^\star) &= \left(v_2^\star - F_2(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t), \; \forall t \in [0, T], \text{ with}
\end{aligned}$$

$g(s, t; v_1^\star) = \chi(s) e^{\frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)t} \left(v_0 - v_1^\star - F_2(0)\right)^{\frac{\hat{b}-1}{b(s)-1}}$*, and* $\tilde{t}_1^\star = \tilde{t}_1(v_1^\star) \in (t, T)$ *solves*

$$b(\tilde{t}_1^\star) = 1 + \frac{\int_t^T g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds}{\int_t^T \frac{1}{b(s)-1} g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds}.$$

For the optimal $(\hat{\pi}^\star(t;v_0), c^\star(t;v_0))$, Eq. (4.4) holds with equality.

It follows immediately that $c_1(t;v_1) > \bar{c}(t)$, a.s.. Theorem 4.10 furthermore proves that the general optimal relative investment strategy can be written as a mixture of a PPI and a CPPI strategy, but is not necessarily of a PPI or even CPPI type itself. The PPI comes from the consumption-only problem, see Theorem 4.2, the CPPI arises as the solution to the terminal wealth-only problem, see Theorem 4.5. The way which of the two strategies dominates the overall optimal investment policy is initially determined by the wealth distribution through $v_1^\star$ and $v_2^\star$ and later through $V_1(t;v_1^\star)$ and $V_2(t;v_2^\star)$. The special case where the coefficient of risk aversion $b(t)$ from consumption equals the one from terminal wealth $\hat{b}$ at any time is covered by the next remark.

Remark 4.11. *Assume $b(t) \equiv \hat{b}$ constant. Then the optimal controls turn into*

$$\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0) = \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)}{V^\star_{(b(t)\equiv\hat{b})}(t;v_0)},$$
$$c^\star_{(b(t)\equiv\hat{b})}(t;v_0) = \zeta(t)\left(V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)\right) + \bar{c}(t),$$

with

$$\zeta(t) = \frac{\chi(t)}{\int_t^T \chi(s)ds + 1} > 0,$$

where

$$\chi(t) = \left(\frac{\hat{a}}{a(t)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right](T-t)} > 0.$$

The optimal investment strategy $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ now is a traditional CPPI strategy with floor $F(t)$ and constant multiple vector $\frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})$. The optimal consumption rate $c^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ is the sum of the consumption floor $\bar{c}(t)$ and the time-varying proportion $\zeta(t)$ of the cushion $V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)$ at time t. The fraction between the risky exposure (vector) and consumption is time-varying and it holds

$$\frac{\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)V^\star_{(b(t)\equiv\hat{b})}(t;v_0)}{c^\star_{(b(t)\equiv\hat{b})}(t;v_0)} = \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\left(\zeta(t) + \frac{\bar{c}(t)}{V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)}\right)^{-1}. \tag{4.19}$$

Optimal consumption $c^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ as well as, under $\Sigma^{-1}(\mu - r\mathbf{1}) > \mathbf{0}$, optimal risky exposure $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)V^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ linearly increase in the cushion $V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)$. Hence, the higher the surplus $V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)$, the more is invested risky and the more is consumed. The formula (4.19) shows that, under $\Sigma^{-1}(\mu - r\mathbf{1}) > \mathbf{0}$, an increase in the cushion $V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)$ leads to a stronger increase in the risky exposure $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)V^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ than in consumption $c^\star_{(b(t)\equiv\hat{b})}(t;v_0)$. Therefore, for a larger surplus $V^\star_{(b(t)\equiv\hat{b})}(t;v_0) - F(t)$, also the relative increase in the *risky exposure is larger than the relative increase in consumption, thus investing money in stocks is preferred to consuming.*

The associated optimal wealth process is given as a function of the pricing kernel

$$V^{\star}_{(b(t)\equiv\hat{b})}(t;v_0) = \frac{1}{\zeta(t)}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\frac{\chi(t)}{\int_0^T \chi(t)dt + 1} + F(t).$$

This special case result coincides with the findings by Ye (2008), who used the HJB approach, extended by additionally providing the optimal wealth process $V^{\star}_{(b(t)\equiv\hat{b})}(t;v_0)$.

We aim to interpret the optimal $\hat{\pi}^{\star}(t;v_0)$ for time-varying $b(t)$ and particularly to point out the difference to constant $b(t)$ in Remark 4.11. Writing $V_1(t;v_1^{\star}) = V^{\star}(t;v_0) - V_2(t;v_2^{\star})$ where $V_2(t;v_2^{\star})$ follows the wealth process of a standard CPPI strategy with floor $F_2(t)$ at time t to the initial endowment $v_2^{\star}$ and constant multiplier vector $\frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})$, we obtain the following representation of the optimal investment decision

$$\begin{aligned}
\hat{\pi}^{\star}(t;v_0) &= \Sigma^{-1}(\mu - r\mathbf{1})\frac{\frac{1}{1-b(\tilde{t}_1^{\star})}\left(V^{\star}(t;v_0) - V_2(t;v_2^{\star}) - F_1(t)\right) + \frac{1}{1-\hat{b}}\left(V_2(t;v_2^{\star}) - F_2(t)\right)}{V^{\star}(t;v_0)} \\
&= \Sigma^{-1}(\mu - r\mathbf{1})\left\{\frac{1}{1-b(\tilde{t}_1^{\star})}\frac{V^{\star}(t;v_0) - F_1(t)}{V^{\star}(t;v_0)} + \frac{\hat{b} - b(\tilde{t}_1^{\star})}{(1-\hat{b})\left(1-b(\tilde{t}_1^{\star})\right)}\frac{V_2(t;v_2^{\star}) - \frac{1-b(\tilde{t}_1^{\star})}{\hat{b}-b(\tilde{t}_1^{\star})}F_2(t)}{V^{\star}(t;v_0)}\right\} \\
&= \Sigma^{-1}(\mu - r\mathbf{1})\left\{\frac{1}{1-b(\tilde{t}_1^{\star})}\frac{V^{\star}(t;v_0) - F(t)}{V^{\star}(t;v_0)} + \frac{\hat{b} - b(\tilde{t}_1^{\star})}{(1-\hat{b})\left(1-b(\tilde{t}_1^{\star})\right)}\frac{V_2(t;v_2^{\star}) - F_2(t)}{V^{\star}(t;v_0)}\right\} \\
&= \Sigma^{-1}(\mu - r\mathbf{1}) \\
&\quad\times\left\{\frac{1}{1-b(\tilde{t}_1^{\star})}\frac{V^{\star}(t;v_0) - F(t)}{V^{\star}(t;v_0)} + \frac{\hat{b} - b(\tilde{t}_1^{\star})}{(1-\hat{b})\left(1-b(\tilde{t}_1^{\star})\right)}\frac{V_2(t;v_2^{\star})}{V^{\star}(t;v_0)}\frac{V_2(t;v_2^{\star}) - F_2(t)}{V_2(t;v_2^{\star})}\right\},
\end{aligned} \tag{4.20}$$

which can be implemented easily; $F(t)$ is defined in (4.7). Formula (4.20) shows that the optimal relative allocation $\hat{\pi}^{\star}(t;v_0)$ can be written as a PPI strategy in $V^{\star}(t;v_0)$ with floor $F(t)$ plus a PPI strategy in $V_2(t;v_2^{\star})$ with floor $F_2(t)$. Alternatively, write $V_2(t;v_2^{\star}) = V^{\star}(t;v_0) - V_1(t;v_1^{\star})$, where $V_1(t;v_1^{\star})$ is the replicating wealth process of a PPI strategy with floor $F_1(t)$ to the initial wealth $v_1^{\star}$ and now time- and state-varying multiplier vector $\frac{1}{1-b(\tilde{t}_1)}\Sigma^{-1}(\mu - r\mathbf{1})$ and, in contrast to $V_2(t;v_2^{\star})$, a non-zero consumption-rate process. Then $\hat{\pi}^{\star}(t;v_0)$ can be reformulated as

$$\begin{aligned}
\hat{\pi}^{\star}(t;v_0) &= \Sigma^{-1}(\mu - r\mathbf{1})\left\{\left(\frac{1}{1-b(\tilde{t}_1^{\star})} - \frac{1}{1-\hat{b}}\right)\frac{V_1(t;v_1^{\star}) - F_1(t)}{V^{\star}(t;v_0)} + \frac{1}{1-\hat{b}}\frac{V^{\star}(t;v_0) - F(t)}{V^{\star}(t;v_0)}\right\} \\
&= \Sigma^{-1}(\mu - r\mathbf{1})\left\{\frac{1}{1-\hat{b}}\frac{V^{\star}(t;v_0) - F(t)}{V^{\star}(t;v_0)} - \frac{\hat{b} - b(\tilde{t}_1^{\star})}{(1-\hat{b})(1-b(\tilde{t}_1^{\star}))}\frac{V_1(t;v_1^{\star}) - F_2(t)}{V^{\star}(t;v_0)}\right\} \\
&= \Sigma^{-1}(\mu - r\mathbf{1})\left\{\frac{1}{1-\hat{b}}\frac{V^{\star}(t;v_0) - F(t)}{V^{\star}(t;v_0)} - \frac{\hat{b} - b(\tilde{t}_1^{\star})}{(1-\hat{b})(1-b(\tilde{t}_1^{\star}))}\frac{V_1(t;v_1^{\star})}{V^{\star}(t;v_0)}\frac{V_1(t;v_1^{\star}) - F_1(t)}{V_1(t;v_1^{\star})}\right\}.
\end{aligned} \tag{4.21}$$

This formula shows that the optimal relative investment $\hat{\pi}^{\star}(t;v_0)$ is the sum of a conventional CPPI strategy on $V^{\star}(t;v_0)$ with floor $F(t)$ and a PPI strategy on $V_1(t;v_1^{\star})$ with floor $F_1(t)$.

Recall from Remark 4.11 that $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ for constant $b(t)\equiv\hat{b}$ follows a traditional CPPI strategy $\frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V^\star_{(b(t)\equiv\hat{b})}(t;v_0)-F(t)}{V^\star_{(b(t)\equiv\hat{b})}(t;v_0)}$ to the floor $F(t)$. The formula for $\hat{\pi}^\star(t;v_0)$ in (4.21) shows that the optimal strategy $\hat{\pi}^\star(t;v_0)$ for time-varying $b(t)$ consists of two parts:

1. The first part coincides with $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)$ and is a traditional CPPI strategy $\frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V^\star(t;v_0)-F(t)}{V^\star(t;v_0)}$ in $V^\star(t;v_0)$ to the floor $F(t)$.
2. The second, additional part is a time- and state-varying term which can be either positive, negative or zero; hence it can reduce or increase risky investments or can leave it unmodified in comparison with $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)$.

It is the second part which leads to a deviation in $\hat{\pi}^\star(t;v_0)$ compared to $\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)$. For this sake, we analyze this second piece in what follows. Note that by Theorem 4.2 it holds $V_1(t;v_1) > F_1(t)$ a.s..

1. If $V^\star(t;v_0) > 0$, for instance this is reasonable for $v_0 > 0$ and an income rate that outweighs or exceeds consumption, then it follows

$$\frac{V_1(t;v_1^\star)}{V^\star(t;v_0)}\frac{V_1(t;v_1^\star)-F_1(t)}{V_1(t;v_1^\star)} = \frac{V_1(t;v_1^\star)-F_1(t)}{V^\star(t;v_0)} > 0.$$

   This implies for $i = 1,\dots,N$ at time t:

   a) $\hat{b} > b(\tilde{t}_1^\star)$:

$$(\hat{\pi}^\star(t;v_0))_i < \frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{V^\star(t;v_0)-F(t)}{V^\star(t;v_0)} \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0.$$

   b) $\hat{b} = b(\tilde{t}_1^\star)$:

$$\hat{\pi}^\star(t;v_0) = \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V^\star(t;v_0)-F(t)}{V^\star(t;v_0)}.$$

   c) $\hat{b} < b(\tilde{t}_1^\star)$:

$$(\hat{\pi}^\star(t;v_0))_i > \frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{V^\star(t;v_0)-F(t)}{V^\star(t;v_0)} \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i < 0.$$

2. If $V^\star(t;v_0) < 0$, for instance this is reasonable for $v_0 < 0$ and a high demand for consumption in the past, then it follows

$$\frac{V_1(t;v_1^\star)}{V^\star(t;v_0)}\frac{V_1(t;v_1^\star)-F_1(t)}{V_1(t;v_1^\star)} = \frac{V_1(t;v_1^\star)-F_1(t)}{V^\star(t;v_0)} > 0.$$

   This in turn implies for $i = 1,\dots,N$ at time t:

a) $\hat{b} > b(\tilde{t}_1^\star)$:

$$(\hat{\pi}^\star(t;v_0))_i > \frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{V^\star(t;v_0) - F(t)}{V^\star(t;v_0)} \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0.$$

b) $\hat{b} = b(\tilde{t}_1^\star)$:

$$\hat{\pi}^\star(t;v_0) = \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V^\star(t;v_0) - F(t)}{V^\star(t;v_0)}.$$

c) $\hat{b} < b(\tilde{t}_1^\star)$:

$$(\hat{\pi}^\star(t;v_0))_i < \frac{1}{1-\hat{b}}\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i \frac{V^\star(t;v_0) - F(t)}{V^\star(t;v_0)} \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i < 0.$$

In particular, consider the situation $V^\star(t;v_0) > 0$ and let $\left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0$ hold for risky asset i. Under $\hat{b} > b(\tilde{t}_1^\star)$, the optimal relative investment in stock i, which is $(\hat{\pi}^\star(t;v_0))_i$, is reduced compared to the relative investment decision $\left(\hat{\pi}^\star_{(b(t)\equiv\hat{b})}(t;v_0)\right)_i$ under $b(t) \equiv \hat{b}$. Since $\hat{b} > b(\tilde{t}_1^\star)$ can be interpreted as higher risk aversion for consumption than terminal wealth, this is meaningful.

In the situation $V^\star(t;v_0) < 0$ the interpretation seems counterintuitive at first glance. But when looking at risky exposures rather than risky relative investments, analogue conclusions hold. The same approach shall be used when considering $V^\star(t;v_0) = 0$.

Furthermore, it is worth to mention that $\hat{\pi}^\star(t;v_0)$ approaches 0 when $V^\star(t;v_0)$ approaches $F(t)$, which can be observed in (4.20); the argument is the following: When $V^\star(t;v_0)$ falls towards $F(t)$, then automatically $V_1(t;v_1^\star)$ approaches $F_1(t)$ and $V_2(t;v_2^\star)$ converges towards $F_2(t)$ simultaneously, since $V^\star(t;v_0) = V_1(t;v_1^\star) + V_2(t;v_2^\star)$, $F(t) = F_1(t) + F_2(t)$ and $V_1(t;v_1^\star) > F_1(t)$, $V_2(t;v_2^\star) > F_2(t)$ a.s. which was already shown in Sections 4.2.1 and 4.2.2. We moreover proved that in this case $\hat{\pi}_1(t;v_1^\star)$ and $\hat{\pi}_2(t;v_2^\star)$ approach 0. By Theorem 4.10 it follows that also $\hat{\pi}^\star(t;v_0)$ must converge to 0. Therefore, as $v_0 > F(0)$ is assumed, it follows that $V^\star(t;v_0) > F(t)$ a.s., which can additionally be seen in the respective formula in Theorem 4.10, and the optimal decision rules provide portfolio insurance over the whole life-cycle. $F(t)$ is called the minimum asset wealth level, it holds $F(T) = F$.

The optimal exposure to the risky assets equals the sum of the optimal risky exposures of the two sub-problems

$$\hat{\pi}^\star(t;v_0)V^\star(t;v_0) = \hat{\pi}_1(t;v_1^\star)V_1(t;v_1^\star) + \hat{\pi}_2(t;v_2^\star)V_2(t;v_2^\star)$$

and by the findings in Sections 4.2.1 and 4.2.2 it holds

$$(\hat{\pi}^\star(t;v_0)V^\star(t;v_0))_i > 0 \;\Leftrightarrow\; \left(\Sigma^{-1}(\mu - r\mathbf{1})\right)_i > 0.$$

For the ease of exposition we so far assumed that the income process is deterministic. The following remark shows the solution for a stochastic income process.

Remark 4.12 (Stochastic income process). *Let* $(y(t))_{t\in[0,T]}$ *be a non-negative, stochastic income-rate process with* $\int_0^T y(t)dt < \infty$, $\mathbb{P}$*-a.s., that is perfectly hedgeable and measurable with respect to the filtration* $(\mathcal{F}_t)_{t\in[0,T]}$ *generated by the market. The stated results are still valid after replacing integrals of the form* $\int_t^T e^{-r(s-t)}y(s)ds$ *by the more general conditional expectation* $\mathbb{E}\left[\int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)}y(s)ds\middle|\mathcal{F}_t\right] = \int_t^T \mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)}y(s)\middle|\mathcal{F}_t\right]ds = \int_t^T e^{-r(s-t)}\mathbb{E}_{\mathbb{Q}}\left[y(s)\middle|\mathcal{F}_t\right]ds$ *by the Bayes formula for arbitrary* $t \in [0,T]$*, in particular in the definition of* $F_1(t)$ *and* $F(t)$*. If* $(y(t))_{t\in[0,T]}$ *is supposed to be independent to* $\mathcal{F}$*, i.e. independent to the market stochastics, then the conditional expectation* $\mathbb{E}\left[\int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)}y(s)ds\middle|\mathcal{F}_t\right]$ *can be reduced to* $\int_t^T e^{-r(s-t)}\mathbb{E}_{\mathbb{Q}}[y(s)]\,ds$*. For the lower bounds of* v_0 *and* v_1*,* (4.6) *and* (4.10) *need to be replaced by*

$$v_0 > \int_0^T e^{-rs}\left(\bar{c}(s) - \bar{y}(s)\right)ds + e^{-rT}F,$$
$$v_1 > \int_0^T e^{-rs}\left(\bar{c}(s) - \bar{y}(s)\right)ds,$$

where $\bar{y}(s) = \sup\{x \geqslant 0 : \mathbb{P}(y(s) \geqslant x) = 1\}$ *denotes the minimal level of income;* $\bar{y}(s) > 0$ *is meaningful due to unemployment benefits paid by the government.*

Remark 4.12 demonstrates that our entire framework and the derived solution to the consumption-investment problem under a fully hedgeable stochastic uncertain inflow stream, such as (parts of) labor income risk, are still valid after simple modifications. Therefore, the remark addresses income risk as an important source of risk when studying life-cycle strategies in practice. Nevertheless, labor income risk in real world is not generally hedgeable and life-cycle strategies depend on non-hedgeable income risk. Optimal consumption and investment strategies within our proposed model in such a situation remain an open problem. To overcome this problem, we consider a cohort of pension fund investors: we can implicitly assume that the uncertain aggregated labor income of the entire cohort can better be predicted than the uncertain labor income of a single agent. Furthermore, it could be assumed that the pension fund investor commits to pay a certain amount every year (or a certain payment stream) that is set upfront. This would reduce or even remove the influence of the non-hedgeable part of the labor income risk in the pension fund. For more discussion on the relation between uncertain labor income and life-cycle strategies we refer the interested reader to Benzoni et al. (2007) and Polkovnichenko (2007).

Moreover, when mortality risk in the form of an intensity model is integrated, the planning horizon becomes the minimum of the terminal time T and the uncertain lifetime τ. As a result, one needs to adjust the discount rate or discount factor by the survival probability for simple processes with a constant or deterministic force of mortality, cf. the later Section 5.2.1.3 or Ye (2008). For more advanced models, particularly with stochastic mortality, we refer to Escobar et al. (2016) or Shen and Wei (2016).

4.3 Analysis of optimal controls and wealth process: A case study

This section targets to calibrate the life-cycle model to realistic time-dependent structures for consumption and investment observed in practice and outline the difference between our presented solution with age-depending $a(t)$ and $b(t)$ functions and the models with either only $a(t)$ or $b(t)$

time-varying or none. Hence, we not only estimate $\hat{b}$, $a(t)$ and $b(t)$ for our model, but additionally provide the respective estimates when $a(t)$ or $b(t)$, or both, are assumed to be constants. A comparison of the fit of the different models allows for making a statement on the accuracy of the models in describing the agent's behavior. For notational convenience we call the three benchmark models as follows:

- $M_{a,b(t)}$: $a(t) \equiv a$ constant, $b(t)$ time-varying
- $M_{a(t),b}$: $a(t)$ time-varying, $b(t) \equiv b$ constant
- $M_{a,b}$: $a(t) \equiv a$ and $b(t) \equiv b$ constant

The subscript thus indicates whether $a(t)$ or $b(t)$ are age-varying. Therefore, our model is denoted by $M_{a(t),b(t)}$. As already indicated before, $M_{a,b(t)}$ is (partially) covered by Steffensen (2011), Hentschel (2016) and Aase (2017), $M_{a(t),b}$ and $M_{a,b}$ are covered by Ye (2008).

In the later Subsection 4.3.3, we additionally analyze the impact of the floors $\bar{c}(t)$ and F, where our model $M_{a(t),b(t)}$ is compared to the same model but with CRRA utility functions, i.e. $\bar{c}(t) \equiv 0$ and $F \equiv 0$. The CRRA model is denoted by $M^{CRRA}_{a(t),b(t)}$ and is (partially) considered by Steffensen (2011), Hentschel (2016) and Aase (2017).

4.3.1 Assumptions

We assume an exemplary agent with average income, liabilities etc. A similar case study can be carried out for a pension cohort, but for simplicity and data availability we consider an individual client. In detail, we make the following (simplifying) assumptions:

Let the market consist of one risk-free and one risky asset ($N = 1$) with parameters $r = 0.5\%$, $\mu = 5\%$, and $\sigma = 20\%$; these values correspond approximately to the EURONIA Overnight Rate and the performance of the DAX 30 Performance Index as an equity index over the 11 year period from 17 October 2007 to 17 October 2018. The risky asset can coincide with, but is not restricted to a pure equity portfolio. In general it can be any arbitrary given portfolio which consists of risky assets. The price process of the risky asset is assumed to be $P(t) = p_1 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} = p_1 e^{\frac{1}{2}(\mu+r)(1-\frac{\sigma}{\gamma})} \tilde{Z}(t)^{-\frac{\sigma}{\gamma}}$ with initial price $P(0) = p_1 = 100$. Furthermore, let $T = 40$ years be the time to retirement, $t = 25$ years the current age of the investor and 65 years the age of retirement. For the net salary function it is assumed $y(t) = \frac{\tilde{r}}{e^{\tilde{r}}-1} y_0 e^{\tilde{r}t}$ with $y_0 = 26,200$ EUR and $\tilde{r} = 2.07\%$. This corresponds to a net annual starting salary approximately equal to the average for a graduate in Germany in 2017 (cf. online portals Absolventa GmbH (2018) or StepStone (2017)), with an annual increase equal to the average for a household's net salary in Germany over years 2011 to 2016 according to Statistisches Bundesamt (2018). Net income accumulated over the first year is $\int_0^1 y(t)dt = y_0$ and income accumulated within the year from time s to $s+1$ is $\int_s^{s+1} y(t)dt = \frac{\tilde{r}}{e^{\tilde{r}}-1} y_0 \frac{e^{\tilde{r}(s+1)} - e^{\tilde{r}s}}{\tilde{r}} = y_0 e^{\tilde{r}s}$.

For the agent's utility functions, let $\beta = 3\%$ (cf. Ye (2008)) and $\hat{a} = 1$. Let the terminal wealth floor be $F = 435,125$ EUR which is motivated by the following argument: According to Statistisches Bundesamt (2017), Deutsche Aktuarvereinigung (DAV) e.V. (2017) or Wirtschaftskammer Österreich (2016) a lifetime around 81 years can be expected for a currently 25 year old person in Germany. Thus survival of $81-65 = 16$ years are expected after retirement at the age of 65. We assume that the agent secures the income inflow during retirement to be 75% of the last wage paid from year 64 to 65 (replacement ratio of 75%), which is $\int_{39}^{40} y(t)dt = y_0 e^{39\tilde{r}} = 58,736$ EUR. Assume that every year, half

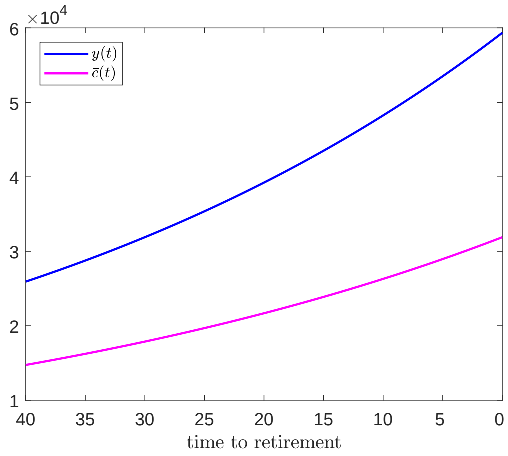


Figure 4.1: Income rate $y(t)$ and consumption floor rate $\bar{c}(t)$ (in EUR).

Market:	$r = 0.5\%$	$\mu = 5\%$	$\sigma = 20\%$	$T = 40$
Utility function $U_1(t, c)$:	$\bar{c}(t) = \frac{\bar{r}}{e^{\bar{r}}-1}\bar{c}_0 e^{\bar{r}t}$	$\bar{c}_0 = 14,880$	$\bar{r} = 1.93\%$	$\beta = 3\%$
Utility function $U_2(v)$:	$\hat{a} = 1$	$F = 435,125$		
Labor income $y(t)$:	$y(t) = \frac{\tilde{r}}{e^{\tilde{r}}-1}y_0 e^{\tilde{r}t}$	$y_0 = 26,200$	$\tilde{r} = 2.07\%$	

Table 4.1: Parameter assumptions.

of this amount is covered by a separate pension account or plan, e.g. provided by the government. In addition, the agent wants to secure against longevity risk, hence considers $16 \times (100+30)\% = 20.8$ years instead of 16 years for the remaining lifetime after the age of retirement. Thus, F as value at time T is chosen to be $F = \int_0^{20.8} \frac{0.75 \times 58{,}736 \text{ EUR}}{2} e^{-rt} dt = \frac{0.75 \times 58{,}736 \text{ EUR}}{2} \left(\frac{1-e^{-20.8 \times r}}{r} \right) = 435,125$ EUR. Finally, the function for the net consumption floor is supposed to take the form $\bar{c}(t) = \frac{\bar{r}}{e^{\bar{r}}-1}\bar{c}_0 e^{\bar{r}t}$ with $\bar{c}_0 = 14,880$ EUR and $\bar{r} = 1.93\%$. This corresponds to a starting value equal to approximately 50% of the average household consumption in Germany in 2016 as starting point, with an annual increase equal to the increase in average household consumption in Germany over years 2011 to 2016 (published by Statistisches Bundesamt (2018)). Minimum consumption expenses incurred within the first year is $\int_0^1 \bar{c}(t)dt = \bar{c}_0$, within year s to $s+1$ is $\int_s^{s+1} \bar{c}(t)dt = \bar{c}_0 e^{\bar{r}s}$. The assumed income and consumption floor rates are visualized in Figure 4.1. Table 4.1 summarizes the parameters that are set upfront.

4.3.2 Fitting/Calibration under exponential preferences and discussion

In what follows we calibrate the remaining utility parameters $\hat{b}$, $a(t)$ and $b(t)$ to suitable curves for consumption and relative allocation. The targeted curves for parameter fitting are summarized by Table 4.2. The consumption rate $c^\star(t; v_0)$ is calibrated with respect to the hump-shaped type observed by Carroll (1997), Gourinchas and Parker (2002), Jensen and Steffensen (2015) and Tang et al. (2018). The relative risky investment $\hat{\pi}^\star(t; v_0)$ is calibrated towards the $(100 - \text{age})\%$ rule of

$\hat{\pi}^\star(t; v_0)$	$c^\star(t; v_0)$
$\hat{\pi}(t) = \frac{100-(t+25)}{100}$, $t \in [0, T]$ (100 − age)% rule (total stock ratio)	$c(t) = -25(t-26)^2 + 37{,}732$ in EUR, $t \in [0, T]$ thus $c(0) = 20{,}832$ EUR (= 70% of average household consumption in Germany in 2016, cf. Statistisches Bundesamt (2018), as starting consumption rate), turning point at $t = 26$ (age 51) with a maximum targeted consumption of $37{,}732$ EUR.

Table 4.2: Target curves for calibration.

thumb; a similar structure is frequently applied by financial advisors and asset management companies for life-cycle funds (see Malkiel (1990), Bodie and Crane (1997), Shiller (2005), Minderhoud et al. (2011), Gebler and Matterson (2010), Shafir (2013)). Following this popular rule, the client at age 25 years starts with a 75% equity investment, linearly decreases it by her age such that she ends with a 35% investment in equities at the age of retirement with 65 years[2]. We would like to mention that in particular relative risky investment curves or products provided by asset management companies are to be understood deterministic, i.e. wealth-/state-independent. Therefore, we calibrate the remaining unknown parameters with respect to the expected values for consumption and risky relative investment. In more detail, we fit the expected value for consumption, which is $\mathbb{E}[c^\star(t; v_0)]$, to the given consumption curve. For $\mathbb{E}[\hat{\pi}^\star(t; v_0)]$ we apply the following estimate: we estimate the risky exposure $\mathbb{E}[\hat{\pi}^\star(t; v_0)V^\star(t; v_0)]$ without any bias and then replace $V^\star(t; v_0)$ by its unbiased expectation $\mathbb{E}[V^\star(t; v_0)]$ to obtain the estimate $\frac{\mathbb{E}[\hat{\pi}^\star(t;v_0)V^\star(t;v_0)]}{\mathbb{E}[V^\star(t;v_0)]}$ for $\mathbb{E}[\hat{\pi}^\star(t; v_0)]$. By doing this we replace $\mathbb{E}[\hat{\pi}^\star(t; v_0)]$ by $\frac{\mathbb{E}[\hat{\pi}^\star(t;v_0)V^\star(t;v_0)]}{\mathbb{E}[V^\star(t;v_0)]}$ and fit the latter expression to the given linear relative investment curve. For further readings on deterministic investment strategies we refer to Christiansen and Steffensen (2013) and Christiansen and Steffensen (2018). In summary, we have unbiased estimates for the expected values of optimal consumption, risky exposure and wealth process, and a modified estimate for the expectation of the optimal relative risky investment.

Let $a(t)$ and $b(t)$ take the form of an exponential function, i.e. $a(t) = a_0 e^{\lambda_a t}$ and $b(t) = b_0 e^{\lambda_b t}$. Moreover, let $v_0 = 250{,}000$ EUR. The estimation is carried out via the *Matlab* function `lsqcurvefit` which solves nonlinear curve-fitting (data-fitting) problems in a least-squares sense and minimizes the sum of the squared relative distances $\left(\frac{y_i - f(x_i)}{f(x_i)}\right)^2$ of a function value $f(x_i)$ to a value y_i. The underlying time points for target consumption and allocation are set weekly on an equidistant grid which yields $2{,}080$ points in the time interval $[0, T]$ with $T = 40$. Thus, y_i and $f(x_i)$ describe both, target consumption and allocation.

[2]We would like to point out that for the sake of calibration and illustration we select this specific exemplary setting that of course does not picture all potential use cases. We fit our model to those exemplary average target curves such that we have control and orientation on the expectations (consumption and investment). Based upon this, the portfolio is managed optimally and dynamically over time. In general, the target curves can be chosen differently for each specific application. If one is interested in explaining some specific behavior patterns such as described by the non-participation or the moderate equity holdings puzzle, a different choice for the average target curves (if data is available) could be used, or different utility functions could be applied such as an S-shaped function that arises from Cumulative Prospect Theory (cf. Chapter 3).

	Sum of squared relative distances	$\hat{b}$	$a(t)$	$b(t)$
$M_{a(t),b(t)}$	6.0425	−0.9849	$a_0 = 5.2864 \times 10^7$, $\lambda_a = -0.6673$	$b_0 = -4.9731$, $\lambda_b = -0.0340$
$M_{a,b(t)}$	31.3157	−0.8325	$a_0 = 0.7997 \times 10^7$, $\lambda_a := 0$	$b_0 = -4.0243$, $\lambda_b = 0.0012$
$M_{a(t),b}$	31.1801	−0.8344	$a_0 = 1.8187 \times 10^7$, $\lambda_a = -0.0363$	$b_0 = -4.1441$, $\lambda_b := 0$
$M_{a,b}$	33.5350	−0.8247	$a_0 = 0.3425 \times 10^7$, $\lambda_a := 0$	$b_0 = -3.9697$, $\lambda_b := 0$

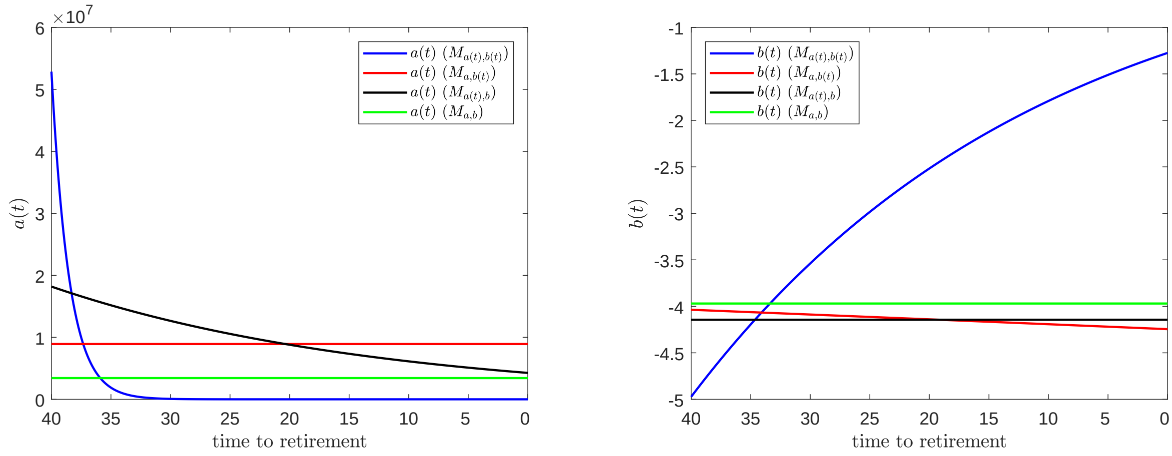
Table 4.3: Calibrated parameters and sum of squared relative residuals.

Table 4.3 gives an overview of the estimated utility parameters and provides the sum of squared relative errors as a quality criterion. The errors show that considering age-depending functions $a(t)$ and $b(t)$ simultaneously in model $M_{a(t),b(t)}$ leads to a comparatively huge improvement in accuracy of the fit compared to any of the three benchmark models: model $M_{a(t),b(t)}$ sum of squared relative distances is only 19.38% of the respective sum for model $M_{a(t),b}$ which provides the second best fit in terms of sum of squared relative residuals.

Figure 4.2 visualizes the fitted parameters and preference functions $\hat{b}$, $a(t)$, $b(t)$. The table and figure show that the estimated coefficient of risk aversion $\hat{b}$ for our model $M_{a(t),b(t)}$ is more negative, which means a higher risk aversion, compared to the three benchmark models $M_{a,b(t)}$, $M_{a(t),b}$, $M_{a,b}$. Furthermore, $a(t)$ is decreasing both within model $M_{a(t),b(t)}$ and $M_{a(t),b}$. In contrast, $b(t)$ increases in model $M_{a(t),b(t)}$ over time whereas it decreases in the comparison model $M_{a,b(t)}$. $b(t)$ in models $M_{a,b(t)}$, $M_{a(t),b}$, $M_{a,b}$ stay very close over the whole life-cycle whereas $b(t)$ in $M_{a(t),b(t)}$ starts more negative and ends less negative. In summary, this means that in model $M_{a(t),b(t)}$ the risk aversion decreases through increasing $b(t)$, but preference of the investor between consumption and terminal wealth is shifted more and more to terminal wealth through decreasing $a(t)$.

Figure 4.3 illustrates the expected optimal consumption rate and relative risky investment for the fitted parameters in comparison with the given target policies or average profile. In addition to Table 4.3 the figure illustrates that, under exponential preferences $a(t)$ and $b(t)$, only the most flexible model $M_{a(t),b(t)}$ provides an accurate and precise fit for both consumption rate and risky relative allocation. We realize that the benchmark models $M_{a,b(t)}$, $M_{a(t),b}$, $M_{a,b}$ apparently do not provide enough flexibility to simultaneously describe the predetermined consumption and relative allocation curves. Whereas the fits for the relative investment $\hat{\pi}^\star(t; v_0)$ look acceptable, all three benchmark models fail in explaining the targeted consumption rate $c^\star(t; v_0)$. We further notice that $c^\star(t; v_0)$ and $\hat{\pi}^\star(t; v_0)$ for the models $M_{a,b(t)}$ and $M_{a(t),b}$ are very similar (red and black lines in the respective figures).

In summary, Table 4.3 and Figure 4.3 demonstrate that model $M_{a(t),b(t)}$ is the only one among our considered models which provides enough flexibility to model a hump-shaped consumption decision curve besides a linear risky allocation curve. All three benchmark models, which disregard time-



(a) $a(t)$. (b) $b(t)$.

Figure 4.2: Estimated preference functions $a(t)$ and $b(t)$.

dependency of $a(t)$ or $b(t)$ or both, do not lead to a satisfactory fit. In addition, fitting optimal consumption of the four models to the given consumption curve, while ignoring relative investments, shows the same picture. The result is that the sum of the squared distances associated with model $M_{a(t),b(t)}$ is only 21.26% of the respective sum associated with the second best model $M_{a,b(t)}$. This supports our findings and conclusion that time-varying preference parameters are indeed needed to model the given time-dependent hump-shaped consumption and linear risky allocation in an accurate way.

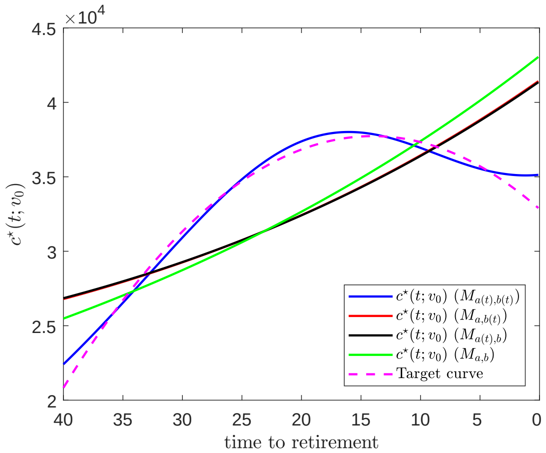
In addition to the parameter estimation for the expected path, we provide the figures for optimal consumption, risky relative portfolio and wealth process of all four models under two representative scenarios: a mostly upward (see Figure 4.4) and a mostly downward (see Figure 4.5) moving path for the underlying stock. The corresponding expected paths for the consumption rate, the relative risky investment and the wealth process can be found in Figure 4.3.

In the increasing stock price case optimal consumption and risky relative allocation for model $M_{a(t),b(t)}$ stay very close to the targeted curve since the corresponding wealth stays close to its expected path and shows some reverting behavior. For a stronger increasing underlying price process, consumption exceeds the given consumption curve for the expected path. When the stock price decreases, then optimal consumption and risky allocation for model $M_{a(t),b(t)}$ fall below the target curves after approximately 15 to 20 years. In particular higher consumption can no longer be afforded due to a poorly performing equity market. This goes hand in hand with a reduction on the relative risky allocation.

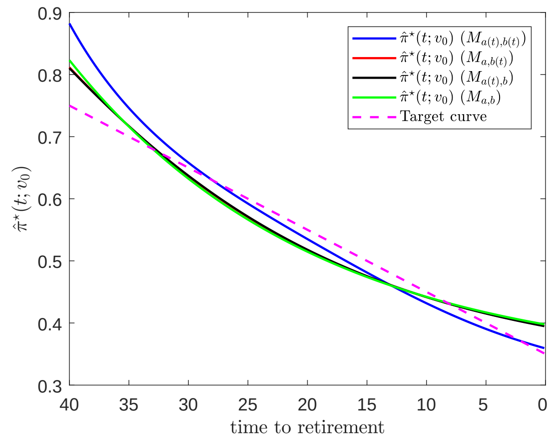
At first glance, it seems that there is a big difference in optimal consumption between our model $M_{a(t),b(t)}$ and the three benchmark models $M_{a,b(t)}$, $M_{a(t),b}$ and $M_{a,b}$ while optimal risky investments and wealth paths for all four models remain in a quite narrow area, although deviation of risky investments from its target curve can be high. This is due to different scales for wealth and consumption. Figure 4.6 visualizes the differences, denoted by Δ, in the fitted consumption and relative risky investment and the corresponding wealth process for the three benchmark models to our model within the expected path situation. It can be observed that relative risky allocation $\hat{\pi}^\star(t; v_0)$ of model $M_{a(t),b(t)}$ exceeds the ones associated with the three benchmark models in the first half of the considered period of 40 years by up to eight percentage points, and falls below in the second half. Moreover, the difference looks monotone decreasing in age. Furthermore, the wealth process which corresponds to model $M_{a(t),b(t)}$ outperforms the three benchmark models in the first half, but provides a lower wealth in the second half due to a higher consumption rate from approx. year 8 to 30, with a certain recovery in the wealth close to retirement.

The two exemplary scenarios and the expected development situation which was used for fitting show that the benchmark models $M_{a,b(t)}$, $M_{a(t),b}$ and $M_{a,b}$ overestimate the given consumption curve in early and older years (close to $t = 0$ and $t = 40$) and underestimate it in between. For our model $M_{a(t),b(t)}$, the optimal consumption rate stays very close to its target curve until consumption cannot be afforded anymore because of a low wealth as result of a strong market decline. We conclude that especially within phases of poor stock performance, both $c^\star(t; v_0)$ and $\hat{\pi}^\star(t; v_0)$ can deviate a lot from their given curves.

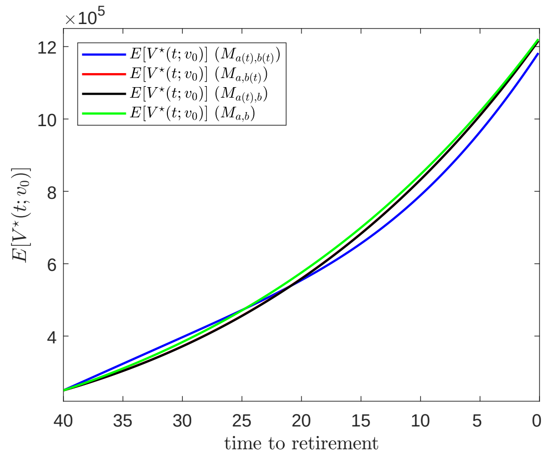
Finally, we would like to comment the behavior of the fitted functions $a(t)$ and $b(t)$ and their impact on optimal consumption and investment. From Figure 4.2 on the fitted $a(t)$ and $b(t)$ and from Figures 4.3 (b), 4.4 (b), 4.5 (b) we infer the following: First, if we have an isolated look on the impact of the risk aversion $b(t)$ on $\hat{\pi}^\star(t; v_0)$ through model $M_{a,b(t)}$, we observe that a decreasing $b(t)$ (= increasing risk aversion) leads to the desired decline in $\hat{\pi}^\star(t; v_0)$ with age which is very intuitive. Second, if we have an isolated look on the impact of $a(t)$ on $\hat{\pi}^\star(t; v_0)$ through model $M_{a(t),b}$, we see that a decline in $a(t)$ accounts for decreasing risky relative portfolio holdings $\hat{\pi}^\star(t; v_0)$ with age, too. Therefore, if we turn to model $M_{a(t),b(t)}$, both $a(t)$ and $b(t)$ affect the optimal investment strategy $\hat{\pi}^\star(t; v_0)$. As additionally we have a hump-shaped target consumption curve, there is a tradeoff between $a(t)$ and $b(t)$ in calibrating the respective expected expressions towards decreasing risky relative investment and hump-shaped consumption simultaneously. Within our specific setup, one can observe that $a(t)$ decreases and $b(t)$ increases (see Figure 4.2 for model $M_{a(t),b(t)}$), i.e. for model $M_{a(t),b(t)}$ the decreasing $a(t)$ takes over the effect of decreasing risky portfolio weights $\hat{\pi}^\star(t; v_0)$, whereas $b(t)$ balances the hump-shaped form of the target consumption. For this reason, in summary, it is more the interplay between $a(t)$ and $b(t)$ that drives the risky investment and consumption.



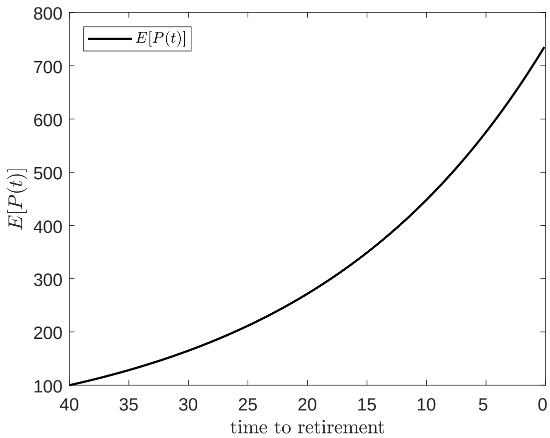
(a) Fitted consumption rate.



(b) Fitted relative risky investment.

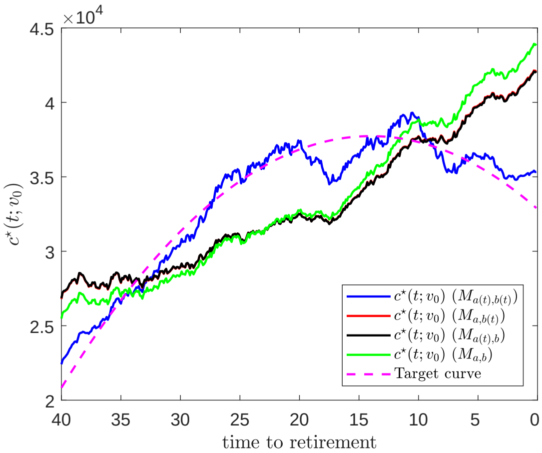


(c) $\mathbb{E}[V^\star(t;v_0)]$.

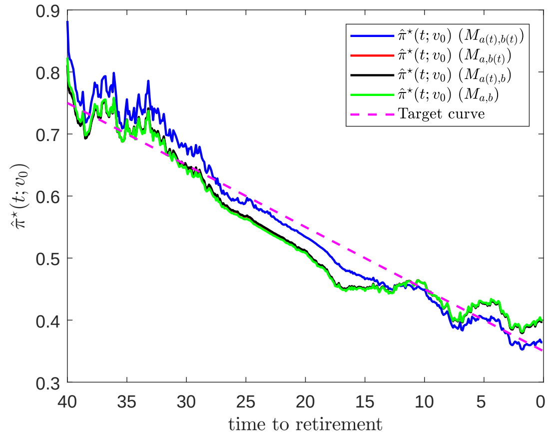


(d) $\mathbb{E}[P(t)] = p_1 e^{\mu t}$.

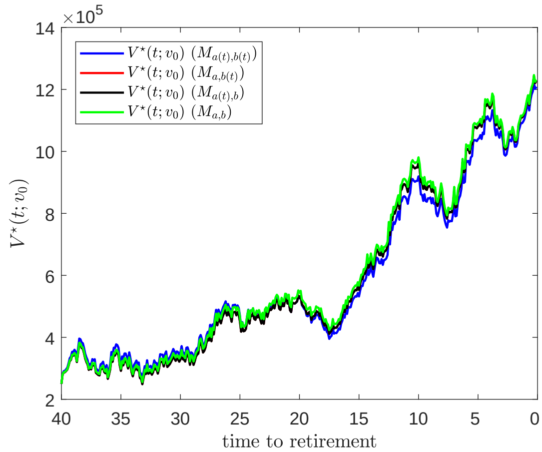
Figure 4.3: Fitted expected consumption rate $c^\star(t;v_0)$ and relative risky investment $\hat{\pi}^\star(t;v_0)$, expected wealth process $\mathbb{E}[V^\star(t;v_0)]$ and stock price process $\mathbb{E}[P(t)]$.



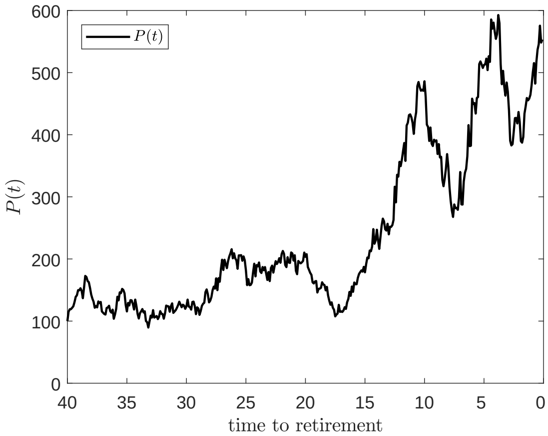
(a) Optimal consumption rate $c^\star(t;v_0)$.



(b) Optimal risky relative investment $\hat{\pi}^\star(t;v_0)$.

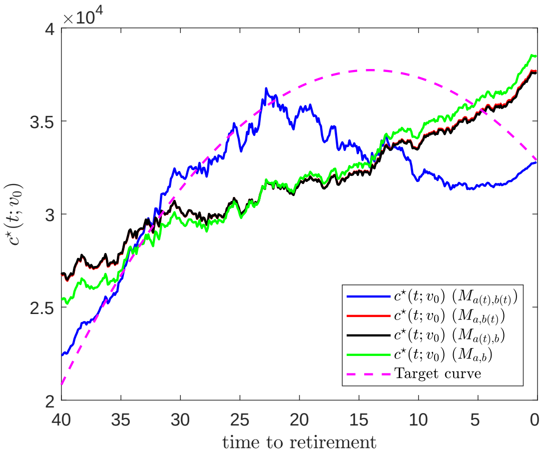


(c) Optimal wealth $V^\star(t;v_0)$.

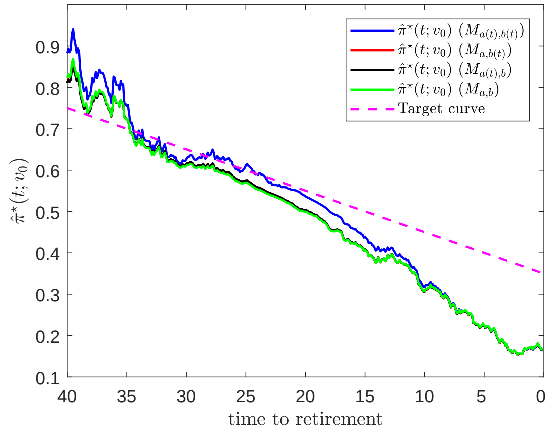


(d) $P(t)$.

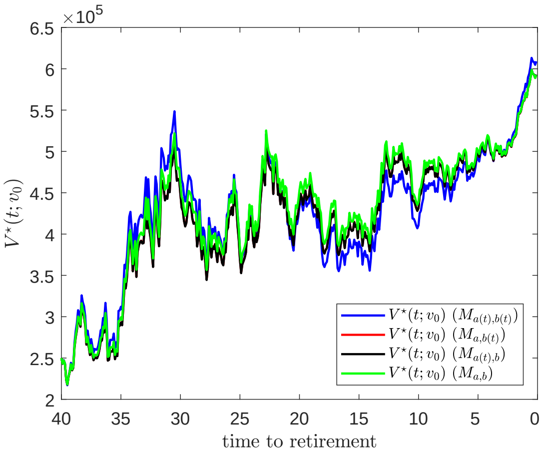
Figure 4.4: Optimal consumption, risky relative investment strategy and wealth under an increasing risky asset price process.



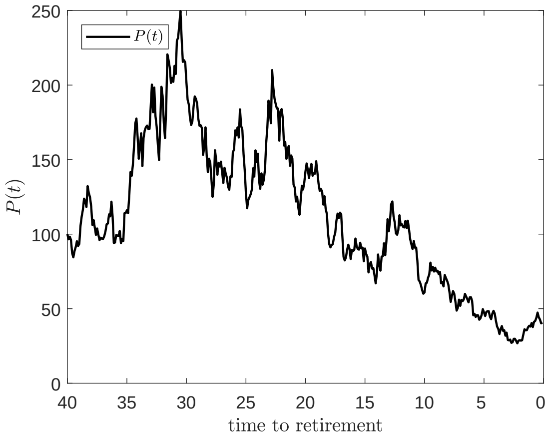
(a) Optimal consumption rate $c^\star(t;v_0)$.



(b) Optimal risky relative investment $\hat{\pi}^\star(t;v_0)$.

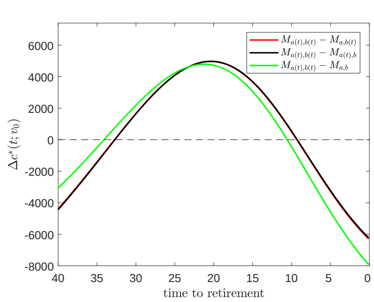


(c) Optimal wealth $V^\star(t;v_0)$.

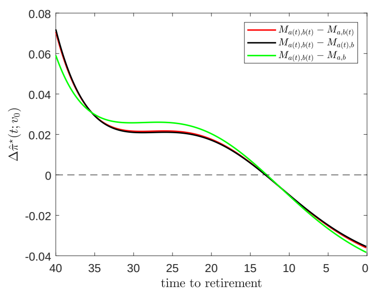


(d) $P(t)$.

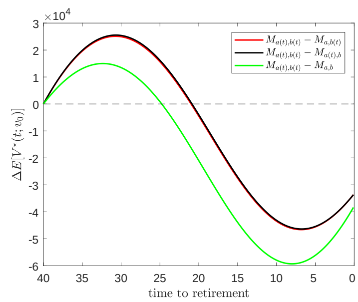
Figure 4.5: Optimal consumption, risky relative investment strategy and wealth under a decreasing risky asset price process.



(a) Difference in fitted consumption rate.



(b) Difference in fitted relative risky investment.



(c) Difference in expected wealth $\mathbb{E}[V^\star(t;v_0)]$.

Figure 4.6: Difference in fitted expected consumption rate, relative risky investment and expected wealth.

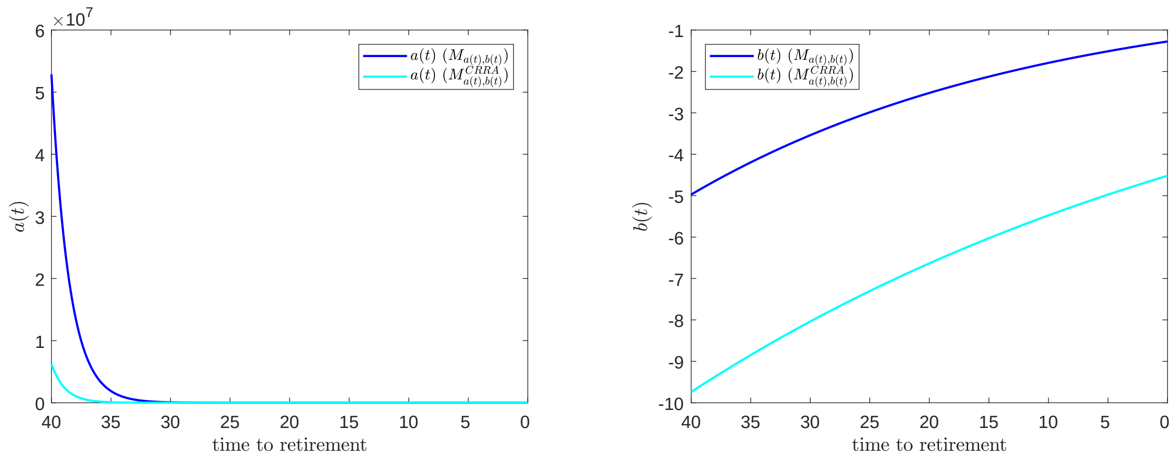
4.3.3 Comparison with CRRA

We conclude the case study section by exploring the impact of minimum consumption and wealth floors on calibration and optimal controls. For this sake, we fit the model $M_{a(t),b(t)}$ to the very same parameters and target curves as before, but now enforce $\bar{c}(t) \equiv 0$ and $F \equiv 0$. This CRRA model is referred to as $M^{CRRA}_{a(t),b(t)}$. Table 4.4 provides the estimated parameters and the sum of the squared relative residuals. In terms of this sum, it is clear that model $M_{a(t),b(t)}$ provides a more adequate fit than model $M^{CRRA}_{a(t),b(t)}$, its sum is only 4.82% of the sum which corresponds to $M^{CRRA}_{a(t),b(t)}$. Going even further, all three benchmark models $M_{a,b(t)}$, $M_{a(t),b}$ and $M_{a,b}$ from the previous subsection, which all consider minimum levels for consumption and wealth, provide a more precise fit than $M^{CRRA}_{a(t),b(t)}$ in view of the sum of squared relative residuals. This shows that the introduction of floors for consumption and wealth in the model is essential.

Figure 4.7 visualizes the estimated input functions, Figure 4.8 provides the graphics about the fitted consumption and relative risky portfolio process with the expected wealth and stock price path. Besides a larger sum of the squared relative distances for model $M^{CRRA}_{a(t),b(t)}$, especially the fitted risky investments $\hat{\pi}^{\star}(t; v_0)$ in Figure 4.8 show that zero floors for consumption and wealth ($\bar{c}(t) \equiv 0$ and $F \equiv 0$) leads to an imprecise calibration and a large deviation from its given target curve due to a drop in model flexibility. Table 4.4 suggests that this drop in flexibility is attempted to be compensated by a higher risk aversion in terms of more negative estimated values for $\hat{b}$ and $b(t)$, see also Figure 4.7.

	Sum of squared relative distances	$\hat{b}$	$a(t)$	$b(t)$
$M_{a(t),b(t)}$	6.0425	-0.9849	$a_0 = 5.2864 \times 10^7$, $\lambda_a = -0.6673$	$b_0 = -4.9731$, $\lambda_b = -0.0340$
$M^{CRRA}_{a(t),b(t)}$	125.3497	-4.4867	$a_0 = 0.6238 \times 10^7$, $\lambda_a = -0.8689$	$b_0 = -9.7397$, $\lambda_b = -0.0192$

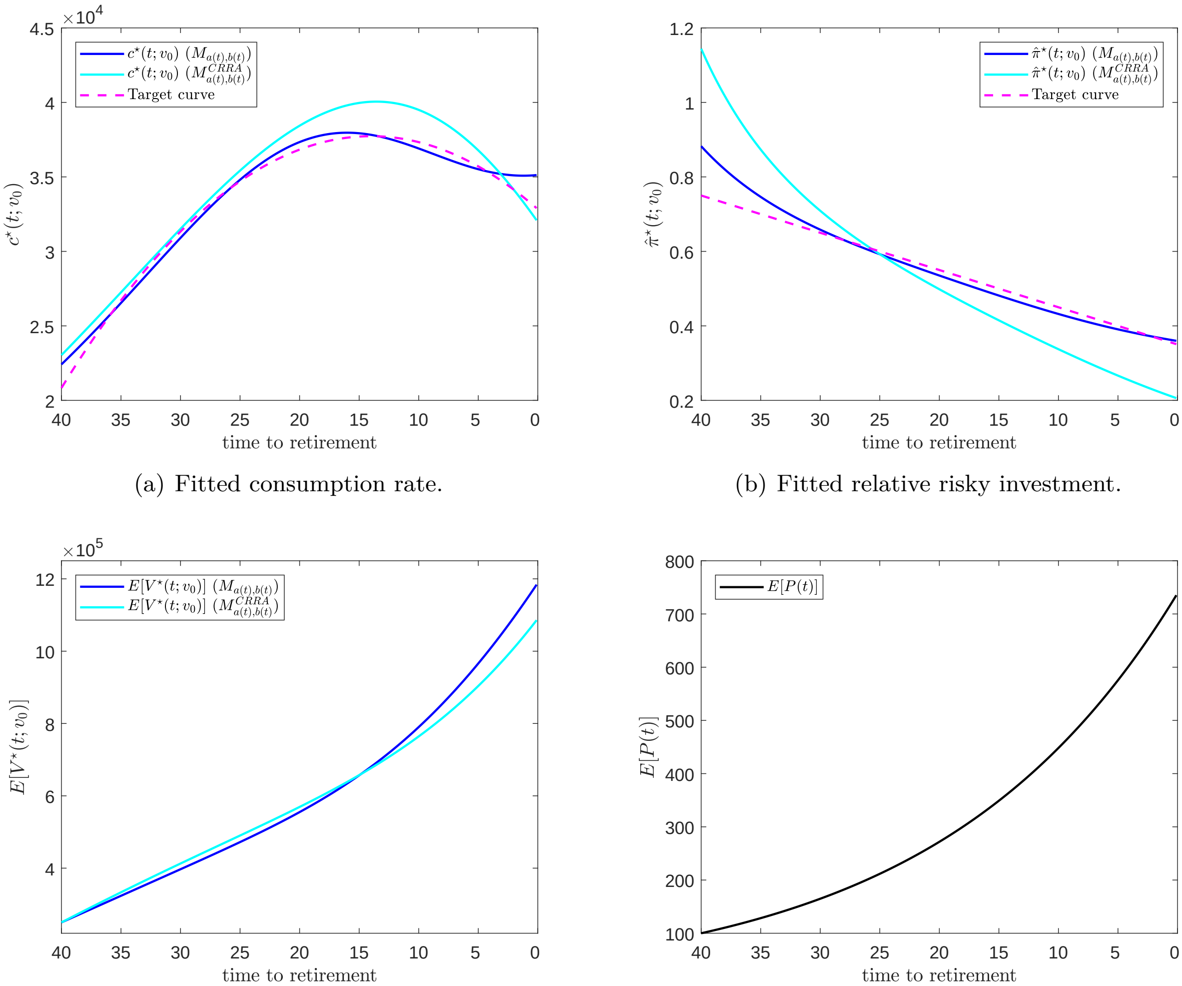
Table 4.4: Calibrated parameters and sum of squared relative residuals for CRRA.



(a) $a(t)$.

(b) $b(t)$.

Figure 4.7: Estimated preference functions $a(t)$ and $b(t)$ for CRRA.



(a) Fitted consumption rate.

(b) Fitted relative risky investment.

(c) $\mathbb{E}[V^\star(t;v_0)]$.

(d) $\mathbb{E}[P(t)] = p_1 e^{\mu t}$.

Figure 4.8: Fitted expected consumption rate $c^\star(t;v_0)$ and relative risky investment $\hat\pi^\star(t;v_0)$, expected wealth process $\mathbb{E}[V^\star(t;v_0)]$ and stock price process $\mathbb{E}[P(t)]$ for CRRA.

5 A new German Pension Product: "Nahles-Rente"/"Sozialpartnermodell"

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.
>
> Richard Bellman

In the recent low interest rate environment, traditional pension products allocate a high fraction of wealth to defensive assets and thus offer only a relatively small expected return on the investments due to promised guarantees. By this, the pension fund wealth of a client grows at a very small rate and thus the future pension payments will be rather low. Generally, clients seek for and desire a stable evolution of their reported wealth (and their pension) at a high expected return and with a limited downside. To allow for a performance or return seeking characteristic, the new "Nahles-Rente" pension product basically comes with no pension cash flow guarantee at all. The wealth accumulation phase shows similarities to a defined contribution (DC) plan, but with an additional smoothing process to stabilize the reported wealth evolution. The wealth decumulation phase can be regarded as a generalization of a defined benefit (DB) plan, where the pensions stay constant as long as the wealth remains inside a pre-defined corridor. Some more details, information and current status can be found in aba and IVS (2017) and Pohl (2019). For a proposal and some discussion of an alternative model formulation that designs a pension product without guarantees we refer to Boado-Penas et al. (2020), where the modeling is related to but differs from our approach both on the accumulation and decumulation part. Within the accumulation phase prior to retirement, the "Nahles-Rente" thus provides a flexible setup that allows seeking for higher returns compared to products with guarantees and managing the total fund's wealth with two accounts: a reported wealth balance (primary account) and a buffer balance (secondary account). The latter aims to smooth the reported wealth and to accumulate some buffer amount that can be used during the post-retirement phase to decrease the probability of pension shortenings as clients generally fear reductions in pension payments. As this new pension product is currently in a development stage and is being built up, we study the impact of the associated model. This chapter is a reproduction of Lichtenstern and Zagst (2020) with minor changes. In Section 5.1 the portfolio problem associated with the accumulation phase is studied, whereas Section 5.2 deals with the decumulation phase. Generally we refer to Chapter 4 for a detailed literature overview on optimal investment management over the entire life-cycle. Additional specific publications for this chapter are cited at the relevant places.

5.1 Accumulation phase

In this section we present an innovative optimal investment strategy for the accumulation phase of a private pension insurance plan or pension saving scheme under a portfolio smoothing mechanism. The pension fund consists of two accounts: the primary account or investment portfolio is reported to the customers, the secondary account or buffer account acts in the background. Reporting a smoothed, but steadily increasing portfolio wealth or primary account balance is quite important for insurance companies and their clients, for instance due to planning stability and to avoid surprising drops in the level of future expected annuity payments. In general, it is unclear how an optimal portfolio and investment strategy look for a given buffer scheme. The main findings and contributions comprise the closed-form solution to the continuous-time optimal investment problem under a wealth-dependent buffer scheme, the derivation of risk and return figures and the presentation of an explicit parametric setting such that the terminal buffer balance is always non-negative. Finally, within a numerical case study, we illustrate the optimal portfolio policy and the stochastic accumulation of a buffer account and its benefits compared to asset allocation strategies without a buffer rule.

The accumulation phase can be modeled on a single client or on a cohort basis with individual buffer accounts. We assume that customers belonging to the same age-cohort can be managed as a group and that the group specific collective portfolios and buffer processes can be managed separately. A cohort-specific account or portfolio and a cohort-specific buffer account is assigned to each cohort. This allows to apply a dynamic, age-dependent (besides a market-dependent) investment strategy, that can invest the pension fund's asset into the $N + 1$ assets, which may add value to the clients wealth compared to a constant capital allocation mix. The number of cohorts m is specified as the oldest age cohort of the participants and m_j is the number of people in age cohort-j for $j = 1, 2, ..., m$. We assume that plan members in cohort-j are endowed with a salary process $y_j(t)$ which has deterministic dynamics. A certain proportion λ of the salary is paid into the plan continuously restricting us to the inflow model $\lambda y_j(t)$ (employee) and $\eta(\lambda y_j(t))$ (employer), i.e. the total contribution rate is given by $(1 + \eta)\lambda y_j(t)$. The standard example is $\lambda = 4\%$ and $\eta = 15\%$ for all ages during pre-retirement. It is important to note that intertemporal changes to other (“Nahles-Rente”) pension products are excluded in this work for the sake of simplicity. Within our model we implicitly assume all customers to survive until retirement entry time T. Another reason for not considering mortality in the accumulation phase is the specific product design: if a client dies prior to retirement time T, the fund does not pay out any bequest payments. The contributions are then invested in the financial market following custom-tailored dynamic investment strategies for each cohort. The specific investment strategy is obtained by solving the expected utility maximization of the investment portfolio accumulated until retirement. This problem is explained and solved in what follows. The problem needs to be solved for every cohort. Because of keeping the notation simple and understandable we use a quite general notation. For instance, the inflow process (premium or contribution to the pension fund) of one cohort is simply denoted by $y(t)$ instead of $(1 + \eta)\lambda y_j(t)$ for cohort j.

The part on the accumulation phase is structured as follows: Section 5.1.1 introduces the considered financial market model, formulates the pension fund dynamics of this new German pension product and states the associated portfolio optimization problem under a general buffer rule. In Section 5.1.2 we propose a specific wealth-dependent buffer mechanism and determine the corresponding optimal asset allocation policy, the optimal reported and total wealth accounts and buffer balance in closed-form. As the applied wealth-dependent buffer system can lead to a negative total buffer amount, we further elaborate on a flexible parametric setup that allows for a smoothing benefit besides a positive terminal buffer balance. Moreover, analytic formulas for risk and return measures are derived. The obtained results for the optimal strategy and balances are visualized and analyzed in a numerical case study. All proofs are stored in Appendix C.

5.1.1 The financial market model and investment problem

Compared to the basic financial market model introduced in Section 2.1, there are not only investment decisions that have to be made, but also consumption decisions throughout the whole life-cycle (later interpreted as buffer decisions). Therefore, let $(c(t))_{t\in[0,T]} \in \mathbb{R}$ denote the consumption-rate process that is supposed to be progressively measurable such that $\int_0^T |c(t)|dt < \infty$, $\mathbb{P}$-a.s.. Moreover, let $(y(t))_{t\in[0,T]} \geqslant 0$ stand for the income-rate process that is continuously paid into the fund, with $\int_0^T y(t)dt < \infty$, $\mathbb{P}$-a.s.. For reasons of simplicity, we assume $y(t)$ to be deterministic for all $t \in [0,T]$[1]. Under the strategy (π, c) by considering the liquid assets, consumption and income, the dynamics of the wealth process $V = (V(t))_{t\in[0,T]} = (V(t,\pi))_{t\in[0,T]}$ to the initial wealth $V(0) = v_0 > 0$[2] is

$$dV(t) = V(t)\left[\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}(t)'\sigma dW(t)\right] - c(t)dt + y(t)dt. \tag{5.1}$$

In what follows we propose a suitable, economically meaningful process for $c(t)$, hence $c(t)$ is a pre-specified given process and not to be optimized. The objective is to maximize expected utility coming from future terminal wealth at time T, given a suitable consumption strategy $c(t)$, hence is an investment problem under a given (stochastic) consumption strategy. Therefore, the objective function that needs to be maximized reads

$$\mathcal{J}(\pi; v_0, c) = \mathbb{E}\left[U(V(T))\right]. \tag{5.2}$$

If nothing more is specified, then all expectations are again understood to be under the real-world measure $\mathbb{P}$. The portfolio selection problem that arises is then

$$\mathcal{V}(v_0, c) = \sup_{\pi\in\Lambda} \mathcal{J}(\pi; v_0, c), \tag{5.3}$$

where the initial endowment v_0 and the (stochastic) consumption policy c are given. $\mathcal{V}(v_0, c)$ is the value function of this problem. The set Λ covers the admissible investment and consumption strategies (π, c) which deviate to the conditions in Λ to Problem (4.3) of Chapter 4 as follows:

$B^{(5.1)}$: (π, c) admits a unique solution to Eq. (5.1).

$C^{(5.1)}$: (π, c) fulfills the associated budget constraint[3]

[1] This might look restrictive for a single client, but becomes reasonable when we consider cohorts of customers which are grouped by their age.

[2] In our setup the private pension product or contract starts with the payment of the first premium, thus $v_0 > 0$ equals the first discrete-time premium that is paid by the cohort.

[3] The integral $\int_0^T e^{-rt}y(t)dt$ represents the present value of the (expected) future labor income from time 0 to time T, commonly referred to as human capital (cf. Duarte et al. (2014)), which we denote by $HC(0)$ and that grows at the rate $y(t)$. In our case, $HC(0)$ actually represents the proportion $(1+\eta)\lambda$ of the entire human capital because this proportion flows into the pension fund in form of contributions. Generally we define $HC(t) := \int_t^T e^{-r(s-t)}y(s)ds$ as the time-t present value of future income from time t to time T and call it human capital at time t. We further define the later needed notion $Y(t) := \int_0^t e^{-r(s-t)}y(s)ds$, it denotes the time-$t$ present value of all previously accumulated inflows from time 0 to time t, and call it accumulated time-t human capital. It holds $HC(t) + Y(t) = e^{rt}HC(0)$.

$$\mathbb{E}\left[\int_0^T \tilde{Z}(t)c(t)dt + \tilde{Z}(T)V(T)\right] \leqslant v_0 + \mathbb{E}\left[\int_0^T \tilde{Z}(t)y(t)dt\right] = v_0 + \int_0^T e^{-rt}y(t)dt. \quad (5.4)$$

$E^{(5.1)}$: $(c(t))_{t\in[0,T]} \in \mathbb{R}$ is progressively measurable with $\int_0^T |c(t)|dt < \infty$, $\mathbb{P}$-a.s., and satisfies the integrability condition $\mathbb{E}\left[\int_0^T |U_1(t, c(t))|dt\right] < \infty$.

We briefly compare and provide the connection between the conditions in Λ and the ones that correspond to Λ of the general Problem (4.3) in Chapter 4, where we considered a quite similar model: First note that Conditions $A^{(4)}$, $D^{(4)}$ and $F^{(4)}$ remain untouched. Moreover, since $c(t)$ is allowed to turn negative later on for smoothing purposes, Condition $E^{(5.1)}$ substitutes $E^{(4)}$ by allowing for $c(t) \in \mathbb{R}$ and adjusting the integrability condition to $\int_0^T |c(t)|dt < \infty$, $\mathbb{P}$-a.s.. Lastly, Conditions $B^{(5.1)}$ and $C^{(5.1)}$ replace $B^{(4)}$ and $C^{(4)}$, although the involved formulas did not change, i.e. Eq. (5.1) and (4.1) as well as Eq. (5.4) and (4.4) coincide. However, at this point we wanted to emphasize that $c(t)$ is given or determined beforehand in the current Chapter 5, whereas $c(t)$ was a control variable that needed to be optimized in Chapter 4. For this reason, we posed the Conditions $B^{(5.1)}$ and $C^{(5.1)}$ which are conditions rather on π given a certain consumption or buffer rule c instead of requirements on (π, c), and removed $B^{(4)}$ and $C^{(4)}$ which are conditions on (π, c).

Note again that the process c can be stochastic, but is not a control variable. Nevertheless it can depend on V and thus on the control π. Hence, no utility is assigned to the buffer, but we aim to maximize utility from terminal fund wealth with a given buffer mechanism. Let us consider the following HARA utility function for the terminal wealth (cf. Definition 2.12 in Section 2.4):

$$U(v) = \hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(v-F)\right)^{\hat{b}}, \quad (5.5)$$

for $\hat{b} < 1$, $\hat{b} \neq 0$, $\hat{a} > 0$ and $v > F$ with $F \geqslant 0$. U is a continuously differentiable and strictly concave terminal utility function. This utility model accounts for the desired aspect of a minimum liquid asset wealth level[4] $F \geqslant 0$ at the age of retirement T. $\hat{b}$ is the coefficient of risk aversion[5]. The associated Arrow-Pratt measure $\mathcal{A}(v)$ of absolute risk aversion is given by Section 2.4 as

$$\mathcal{A}(v) = \frac{1-\hat{b}}{v-F} > 0.$$

Because $V(T) > F$ by definition of the HARA utility function in (5.5), it is inevitable that

$$v_0 + \int_0^T e^{-rt}y(t)dt > e^{-rT}F \quad (5.6)$$

holds true, i.e. the balance between v_0, $y(t)$ and F has to be set appropriately.

[4]This "internal" wealth floor can be far below guarantees in traditional life insurance products and hence, a higher fraction of wealth can be allocated to risky assets which leads to a higher average return of the investment portfolio. For instance, F can be set as the expected time-T value of all required minimum pension payments that have to be made in the future (for times $t > T$).

[5]As we model cohort-specific pension funds, the coefficient of risk aversion can initially be determined for each cohort independently in view of the cohort-dependent remaining time to retirement.

5.1.2 Buffer rate process as a proportion of the change in the fund surplus

In general, there exist three types of buffers: individual buffer, collective buffer and buffers built from safety contributions that are paid by the company ("Puffer aus Sicherungsbeiträgen"). Within our considerations we neglect the latter one and assume it to be an additional exogenously given buffer that provides extra safety to the customers. A buffer is built by taking money out of the pension fund for some (safety) reason. The motivation for the creation of a buffer is (at least) two-fold: First, it is reasonable to take some money out of the portfolio in good times, i.e. in times of appreciation in portfolio wealth, and put this into a buffer account that is invested riskless (or in practice allocated to very defensive assets such as AAA-rated government bonds). By doing this, money can be accumulated in a safety account which decreases the downside risk of the total wealth, i.e. investment portfolio ($V(t)$) plus buffer account (later called $C(t)$), at retirement (profit lock-in). Moreover, the safety buffer can be used for the decumulation phase to reduce the probability of shortenings in the pension payments. Second, it also makes sense to use the buffer to smooth the wealth process within the accumulation phase.

In the following we concentrate on the motivation of a wealth process smoothing character of a buffer rule during the accumulation phase. A positive terminal buffer account adds additional benefit, but is of secondary interest for the pre-retirement phase. Therefore, during the accumulation phase the fund manager generally wants to smooth the portfolio around some benchmark, called $B(t)$. In what follows we present two appropriate symmetric smoothing rules. When there is a symmetric buffer rate process, then not only a safety buffer (for the pension phase) is built, but also investment returns are smoothed. We now view and re-interpret the consumption rate $c(t)$ as a buffer rate. As we invest the buffer rate in the risk-free asset, the buffer account $C(t)$ at time t is equal to

$$C(t) = \int_0^t c(s)e^{r(t-s)}ds.$$

This definition leads to

$$dC(t) = (rC(t) + c(t))\,dt.$$

We propose a buffer scheme that smooths the portfolio return around the benchmark return $dB(t)$[6]. We consider a buffer rate process $c(t) = c(t, V)$ that distributes a proportion $\alpha(t)$ of the change in the fund surplus $V(t) - B(t)$ for some deterministic and continuously differentiable benchmark function $B(t)$:

$$c(t)dt = \alpha(t)d(V(t) - B(t)), \;\; \alpha(t) \in [0, 1]. \tag{5.7}$$

Hence, the buffer rate equals the proportion[7] $\alpha(t)$ of the change in the surplus $V(t) - B(t)$. In other words, the portfolio return is artificially decreased (by an increase in the buffer account) when it exceeds the benchmark return, i.e. of $dV(t) > dB(t)$, and vice versa. The buffer rule in (5.7) smooths the portfolio returns compared to the benchmark $B(t)$ as follows: If the portfolio return outperforms the return or change of the benchmark, then a buffer is set aside for bad times.

[6]In Appendix C.4 we supplementary provide the solution for an alternative buffer rule that smooths the wealth around the benchmark wealth $B(t)$.

[7]$\alpha(t)$ can for instance increase with time t, that means the proportion of the change in the wealth surplus that is put into the safety buffer increases with the age of the client or decreases with time to retirement ("Saving and smoothing become more important when closer to retirement").

In contrast, if the portfolio underperforms the benchmark, then money from the buffer is used to compensate for this underperformance and to increase the reported portfolio value. However, applying this rule, the time-T value of the accumulated buffer account can turn negative. We assume that in this case, the “Puffer aus Sicherungsbeiträgen” or some other collected risk capital cover the respective amount[8]. Moreover, Guillén et al. (2006) motivate such a mechanism with an actual Danish pension product (“TidsPension” or “TimePension”, launched by the Danish Life Insurer Codan in 2002) that has a surplus distribution rule and a return smoothing mechanism. Therefore, in this subsection we establish a continuous version of this product explained in Guillén et al. (2006).

The introduced setting is related to Battocchio et al. (2007), but is more general in terms of the utility and benchmark framework. In detail, we improve and generalize the findings of Battocchio et al. (2007) (excluding mortality and post-retirement phase) by considering a general floor F in the utility function and by allowing for a flexible benchmark function $B(t)$.

When we insert $c(t)$ from (5.7) into the formula for $dV(t)$, then (5.1) becomes

$$dV(t) = V(t)\left[\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}(t)'\sigma dW(t)\right] - \alpha(t)d(V(t) - B(t)) + y(t)dt$$

which turns into

$$\begin{aligned} dV(t) &= V(t)\left[\frac{1}{1+\alpha(t)}\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right)dt + \frac{1}{1+\alpha(t)}\hat{\pi}(t)'\sigma dW(t)\right] \\ &\quad + \frac{1}{1+\alpha(t)}\left(\alpha(t)dB(t) + y(t)dt\right) \\ &= V(t)\left[\frac{1}{1+\alpha(t)}\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right)dt + \frac{1}{1+\alpha(t)}\hat{\pi}(t)'\sigma dW(t)\right] \\ &\quad + \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t) + y(t)\right)dt. \end{aligned} \tag{5.8}$$

5.1.2.1 General solution

The next theorem provides the general closed-form results for the optimal control $\hat{\pi}^{\star}_{\alpha,B}(t)$ and the corresponding optimal fund wealth process $V^{\star}_{\alpha,B}(t) = V(t, \hat{\pi}^{\star}_{\alpha,B})$. Let $v_0 > \tilde{F}_{\alpha,B}(0)$ be assumed from now on[9].

Theorem 5.1 (General solution). *The optimal portfolio process is given by*

$$\hat{\pi}^{\star}_{\alpha,B}(t) = \frac{1+\alpha(t)}{1-\hat{b}}\frac{V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^{\star}_{\alpha,B}(t)}\Sigma^{-1}\left(\mu - r\mathbf{1}\right),$$

[8] In the later Section 5.1.2.4 we introduce a framework, where the accumulated buffer balance can be shown to end up with a non-negative terminal value under a suitable selection of the relevant parameters.

[9] $\tilde{F}_{\alpha,B}(0)$ is defined in the upcoming Theorem 5.1. This extends (5.6) by the buffer scheme and is required to obtain meaningful results for the optimal fund wealth and portfolio allocation. If $dB(t) = B'(t)dt$ is very high, then with a high probability the buffer rate $c(t)$ will be negative, hence money is put from the buffer into the portfolio to cover some part of the underperformance. At the same time the value of $\tilde{F}_{\alpha,B}(0)$ is reduced and less initial capital is sufficient to cover the guarantee F because of the most likely negative buffer rate $c(t)$.

which is of a Proportional Portfolio Insurance (PPI) type with cohort-age dependent but state- or market-independent multiple, and where we define

$$\begin{aligned}\tilde{F}_{\alpha,B}(t) &:= Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T \frac{\alpha(s)B'(s)+y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds \\ &= \left(F - \frac{\alpha(T)}{1+\alpha(T)}B(T)\right) e^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} + \frac{\alpha(t)}{1+\alpha(t)}B(t) \\ &\quad - \int_t^T \frac{1}{1+\alpha(s)} \left(\frac{\alpha(s)}{1+\alpha(s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right) B(s) + y(s)\right) e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds,\end{aligned}$$

with $\tilde{F}_{\alpha,B}(T) = F$ *and* $\tilde{F}'_{\alpha,B}(t) = \frac{1}{1+\alpha(t)}\left(r\tilde{F}_{\alpha,B}(t) + \alpha(t)B'(t) + y(t)\right)$. *The associated optimal wealth process* $V^\star_{\alpha,B}(t)$ *of the pension fund follows the SDE*

$$dV^\star_{\alpha,B}(t) = (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] + \tilde{F}'_{\alpha,B}(t)dt,$$

the fund surplus $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$ *is log-normally distributed and follows the SDE*

$$d\left(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right],$$

hence

$$\begin{aligned}V^\star_{\alpha,B}(t) &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^t \frac{1}{1+\alpha(s)}ds + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t\right] + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} \\ &> \tilde{F}_{\alpha,B}(t).\end{aligned}$$

The optimal accumulated buffer account is equal to

$$\begin{aligned}C^\star_{\alpha,B}(t) &= \int_0^t c(s)e^{r(t-s)}ds \\ &= \alpha(t)\left(V^\star_{\alpha,B}(t) - B(t)\right) - e^{rt}\alpha(0)(v_0 - B(0)) \\ &\quad + \int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(V^\star_{\alpha,B}(s) - B(s)\right)ds.\end{aligned}$$

We would like to remind the reader that all proofs to this chapter are postponed to Appendix C. One can observe that higher future inflows $y(s)$, $s \in [t,T]$, decrease the deterministic guarantee $\tilde{F}_{\alpha,B}(t)$ of $V^\star_{\alpha,B}(t)$, but increase its risky position through a higher value for $v_0 - \tilde{F}_{\alpha,B}(0)$, increasing the riskiness of the strategy. Furthermore, the formula for the investment strategy $\hat{\pi}^\star_{\alpha,B}(t)$ reflects that under a higher value for $\alpha(t)$ in the buffer rule, the age-dependent multiplier of the PPI gets larger. But in opposite, an increase in $\alpha(t)$ leads to a different value for the cushion $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$. Hence, the buffer scheme impacts the optimal portfolio weights through both, the PPI multiplier and the floor or cushion. Moreover, $C^\star_{\alpha,B}(t)$ is given in closed-form, depends on the current wealth $V^\star_{\alpha,B}(t)$ and all previous portfolio wealths $V^\star_{\alpha,B}(s)$, $s \leqslant t$, thus is path-dependent. It can observed that this path-dependency is removed when $\frac{\alpha'(t)}{\alpha(t)} \equiv r$ which holds true iff $\alpha(t) = \alpha_0 e^{rt}$, $\alpha(0) = \alpha_0$, with the special case of $\alpha(t)$ being constant while $r = 0$. Furthermore, the formulas show how the portfolio is managed optimally and dynamically over time, i.e. with decreasing time to retirement of the cohort. We would like to mention that the choice for $B(t)$ determines $\tilde{F}_{\alpha,B}(t)$.

In the following we look at the characteristics expectation, variance, Value-at-Risk and shortfall probability of the fund wealth and the expectation of the buffer account. The Value-at-Risk for a random variable X with level $\beta \in [0,1]$ is generally defined by $VaR_\beta(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant \beta\}$.

Theorem 5.2 (Fund characteristics). *Let $\Phi(\cdot)$ denote the cumulative distribution function of a standard normal random variable.*

- *Expected fund wealth:*

$$\mathbb{E}\left[V^\star_{\alpha,B}(t)\right] = \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)+\|\gamma\|^2 t\right]}$$

- *Variance of the fund wealth:*

$$Var\left(V^\star_{\alpha,B}(t)\right) = (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{\frac{2}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)+\|\gamma\|^2 t\right]}\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)$$

- *Value-at-Risk/Quantiles of the fund wealth distribution with level $\beta \in [0,1]$:*

$$VaR_\beta\left(V^\star_{\alpha,B}(t)\right) = \tilde{F}_{\alpha,B}(t) + e^{\mathbb{E}\left[V^\star_{\alpha,B}(t)\right]-\tilde{F}_{\alpha,B}(t)+\Phi^{-1}(\beta)\sqrt{Var\left(V^\star_{\alpha,B}(t)\right)}}$$

- *Shortfall probability of the fund wealth with threshold $s > \tilde{F}_{\alpha,B}(t)$:*

$$\mathbb{P}\left(V^\star_{\alpha,B}(t) \leqslant s\right) = \Phi\left(\frac{\ln(s - \tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^\star_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^\star_{\alpha,B}(t)\right)}}\right)$$

- *Expected accumulated buffer account:*

$$\begin{aligned}\mathbb{E}\left[C^\star_{\alpha,B}(t)\right] &= \alpha(t)\left(\mathbb{E}\left[V^\star_{\alpha,B}(t)\right] - B(t)\right) - e^{rt}\alpha(0)(v_0 - B(0)) \\ &\quad + \int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(\mathbb{E}\left[V^\star_{\alpha,B}(s)\right] - B(s)\right)ds \\ &= \alpha(t)\left(\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)+\|\gamma\|^2 t\right]} - B(t)\right) \\ &\quad - e^{rt}\alpha(0)(v_0 - B(0)) + \int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s) \\ &\quad \times\left(\tilde{F}_{\alpha,B}(s) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^s \frac{1-\hat{b}}{1+\alpha(u)}du\right)+\|\gamma\|^2 s\right]} - B(s)\right)ds.\end{aligned}$$

The results in Theorem 5.2 can be used for risk management purposes[10], for calibrating the model parameters such as the coefficient of risk aversion $\hat{b}$ or the smoothing parameter $\alpha(t)$.

5.1.2.2 Special cases

We now look at some special cases that arise from Theorem 5.1. First, if the buffer rule parameter is constant and thus time-independent, $\alpha(t) \equiv \alpha$, then the formulas reduce to the following:

Corollary 5.3 (Constant buffer rule parameter: $\alpha(t) \equiv \alpha$)**.** *If $\alpha(t) \equiv \alpha$, then the optimal portfolio process is given by*

$$\hat{\pi}^{\star}_{\alpha,B}(t) = \frac{1+\alpha}{1-\hat{b}} \frac{V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^{\star}_{\alpha,B}(t)} \Sigma^{-1} (\mu - r\mathbf{1}),$$

which is a Constant Proportion Portfolio Insurance (CPPI) strategy with constant multiple and cohort-age dependent but state- or market-independent cushion floor

$$\begin{aligned} \tilde{F}_{\alpha,B}(t) &= Fe^{-\frac{1}{1+\alpha}r(T-t)} - \int_t^T \frac{\alpha B'(s) + y(s)}{1+\alpha} e^{-\frac{1}{1+\alpha}r(s-t)} ds \\ &= \left(F - \frac{\alpha}{1+\alpha}B(T)\right) e^{-\frac{1}{1+\alpha}r(T-t)} + \frac{\alpha}{1+\alpha}B(t) \\ &\quad - \frac{1}{1+\alpha}\int_t^T \left(\frac{\alpha}{1+\alpha}rB(s) + y(s)\right) e^{-\frac{1}{1+\alpha}r(s-t)} ds, \end{aligned}$$

with $\tilde{F}_{\alpha,B}(T) = F$ and $\tilde{F}'_{\alpha,B}(t) = \frac{1}{1+\alpha}\left(r\tilde{F}_{\alpha,B}(t) + \alpha B'(t) + y(t)\right)$. Moreover, it holds

$$\begin{aligned} dV^{\star}_{\alpha,B}(t) &= (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] \\ &\quad + \tilde{F}'_{\alpha,B}(t)dt, \\ d\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) &= (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right], \end{aligned}$$

and

$$\begin{aligned} V^{\star}_{\alpha,B}(t) &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\left[\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right]t + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{1}{1-\hat{b}}\left[\frac{\alpha+\hat{b}}{1+\alpha}r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right]t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}. \end{aligned}$$

Finally, the formula for the accumulated buffer account simplifies to

$$C^{\star}_{\alpha,B}(t) = \alpha\left[\left(V^{\star}_{\alpha,B}(t) - B(t)\right) - e^{rt}(v_0 - B(0)) + r\int_0^t e^{r(t-s)}\left(V^{\star}_{\alpha,B}(s) - B(s)\right) ds\right].$$

[10] Note that all formulas and measures could be transferred into formulas for the portfolio return $\frac{V^{\star}_{\alpha,B}(t) - v_0}{v_0}$ instead of the portfolio wealth $V^{\star}_{\alpha,B}(t)$ without any computational difficulty.

In view of $V^\star_{\alpha,B}(t) \geqslant \tilde{F}_{\alpha,B}(t)\ \forall t \in [0,T]$ and $r \geqslant 0$, Corollary 5.3 gives the natural lower boundary for $C^\star_{\alpha,B}(t)$ that depends on the selection of $B(t)$:

$$C^\star_{\alpha,B}(t) \geqslant \alpha \left[\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - e^{rt}(v_0 - B(0)) + r \int_0^t e^{r(t-s)} \left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds \right].$$

If there is no buffer mechanism at all, i.e. if $\alpha(t)$ is forced to zero, then the next corollary states the specific results.

Corollary 5.4 (No buffer: $\alpha(t) \equiv 0$)**.** *In the special case where $\alpha(t) \equiv 0$ (i.e. no buffer account) the formula for $\hat{\pi}^\star_{0,B}(t)$ reduces to*

$$\begin{aligned}\hat{\pi}^\star_{0,B}(t) &= \frac{1}{1-\hat{b}} \frac{V^\star_{0,B}(t) - \tilde{F}_{0,B}(t)}{V^\star_{0,B}(t)} \Sigma^{-1}(\mu - r\mathbf{1}) \\ &= \frac{1}{1-\hat{b}} \frac{V^\star_{0,B}(t) - \left(Fe^{-r(T-t)} - \int_t^T y(s)e^{-r(s-t)}ds\right)}{V^\star_{0,B}(t)} \Sigma^{-1}(\mu - r\mathbf{1})\end{aligned}$$

which is a CPPI strategy with constant multiplier and time-t floor

$$\tilde{F}_{0,B}(t) = Fe^{-r(T-t)} - \int_t^T y(s)e^{-r(s-t)}ds,$$

with $\tilde{F}_{0,B}(T) = F$ and $\tilde{F}'_{0,B}(t) = r\tilde{F}_{0,B}(t) + y(t)$, being equal to the discounted terminal wealth guarantee F minus the time-t present value of accumulated future inflows. Moreover, it is

$$\begin{aligned} dV^\star_{0,B}(t) &= (V^\star_{0,B}(t) - \tilde{F}_{0,B}(t)) \left[\left(r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right] + \tilde{F}'_{0,B}(t)dt, \\ d\left(V^\star_{0,B}(t) - \tilde{F}_{0,B}(t)\right) &= (V^\star_{0,B}(t) - \tilde{F}_{0,B}(t)) \left[\left(r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right], \end{aligned}$$

and

$$\begin{aligned} V^\star_{0,B}(t) &= \tilde{F}_{0,B}(t) + (v_0 - \tilde{F}_{0,B}(0)) e^{\left[r + \frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right] t + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{0,B}(t) + (v_0 - \tilde{F}_{0,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r + \frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} \end{aligned}$$

with $V^\star_{0,B}(t) \geqslant \tilde{F}_{0,B}(t)$ because condition (5.6) *coincides with $v_0 \geqslant \tilde{F}_{0,B}(0)$. For completeness, $C^\star_{\alpha,B}(t) \equiv 0$.*

Some more special case results can be found in Appendix C.1.1.

We complement our theoretical analysis with a numerical case study that considers a constant buffer parameter $\alpha(t) \equiv \alpha$ (cf. Corollary 5.3) and a very general functional setup to demonstrate applicability for very flexible functionals.

5.1.2.3 Scenario generation and numerical analysis of the optimal pension fund strategy: general setting

Within this numerical case study we illustrate the evolution of the objects of interest within a bullish, a bearish and a volatile non-directional market scenario. We consider and compare the performance of the following portfolio processes:

- $V^{\star}_{\alpha,B}(t)$: optimal *reported* wealth process with buffer scheme that corresponds to the optimal investment strategy $\hat{\pi}^{\star}_{\alpha,B}(t)$
- $V^{\star}_{\alpha,B}(t) + C^{\star}_{\alpha,B}(t)$: optimal *total* wealth process with buffer scheme that corresponds to the optimal investment strategy $\hat{\pi}^{\star}_{\alpha,B}(t)$
- $V^{\star}_{0,B}(t)$: optimal *reported and total* wealth process without buffer scheme that corresponds to the optimal investment strategy $\hat{\pi}^{\star}_{0,B}(t)$
- $\tilde{V}(t)$: *reported and total* wealth process without buffer scheme that corresponds to the same investment strategy $\hat{\pi}^{\star}_{\alpha,B}(t)$.

$\tilde{V}(t)$ uses the very same investment strategy as $V^{\star}_{\alpha,B}(t)$, but does not apply any buffer mechanism.

We show that the reported wealth $V^{\star}_{\alpha,B}(t)$ for the trading strategy with smoothing is less volatile and closer to the benchmark $B(t)$ than the reported wealth processes $V^{\star}_{0,B}(t)$ and $\tilde{V}(t)$ without any smoothing mechanism. In addition, we show that the total wealth $V^{\star}_{\alpha,B}(t) + C^{\star}_{\alpha,B}(t)$ for the trading strategy with smoothing does not lose much in comparison with the non-smoothed total wealth processes.

To quantify the smoothing effect of the reported wealth process, we additionally introduce two measures:

1. Wealth distance measure:

$$SM_1(t) := V(t) - B(t)$$

2. Return distance measure:

$$\begin{aligned} SM_2(t) := \ & \ln\left(\frac{V(t)}{V(t-\Delta)}\right) - \ln\left(\frac{B(t)}{B(t-\Delta)}\right) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right) \\ = \ & \ln\left(\frac{V(t)/B(t)}{V(t-\Delta)/B(t-\Delta)}\right) = \ln\left(\frac{V(t)}{B(t)}\right) - \ln\left(\frac{V(t-\Delta)}{B(t-\Delta)}\right) \end{aligned}$$

The first measure $SM_1(t)$ measures the distance between the reported $V(t)$ and the benchmark $B(t)$. The second measure $SM_2(t)$ measures the distance between the log-returns of the reported $V(t)$ and $B(t)$ in the previous investment period.

Setting. For this sake, we place the following structural assumptions: First, let

$$y(t) = \frac{\tilde{r}}{e^{\tilde{r}} - 1} y_0 e^{\tilde{r}t}, \ \ y_0 > 0, \tilde{r} > 0.$$

y_0 denotes the initial income, $y(t)$ increases with the rate $\tilde{r}$ (expected increase). This definition of $y(t)$ coincides with the assumed parameterization of the income-rate process in Section 4.3.1. It follows

$$\int_0^1 y(t)dt = y_0, \quad \int_s^{s+1} y(t)dt = \frac{\tilde{r}}{e^{\tilde{r}} - 1} y_0 \frac{e^{\tilde{r}(s+1)} - e^{\tilde{r}s}}{\tilde{r}} = y_0 e^{\tilde{r}s}.$$

For the wealth buffer benchmark we propose

$$B(t) = B(0)e^{r_B t} + \int_0^t e^{r_B(t-s)} y(s)ds, \ \ r_B \in \mathbb{R}, B(0) \in \mathbb{R}.$$

It is reasonable to set $B(0) = v_0$. If $r_B = 0$, then $B(t)$ coincides with the collected inflow amount. If $r_B \neq 0$, then $B(t)$ is the value that is obtained when the collected inflow amount was invested at the target return r_B. Moreover, we have

$$B'(t) = r_B B(t) + y(t).$$

Furthermore, let the parameters be selected as follows: The financial market consists of one risky asset or equity fund ($N = 1$) denoted by $P(t)$ with risk and return parameters $\mu = 5\%, \sigma = 20\%$. The interest rate in the market is $r = 0.5\%$, the planning horizon is $T = 40$. We assume that there are 500 clients in one cohort that has $T = 40$ years to retirement entry time. The cohort’s average or expected contribution process follows the parameters $y_0 = 500 \times (1 + \eta)\lambda \cdot 40,000$ EUR $= 920,000$ EUR ($\lambda = 4\%$, $\eta = 15\%$), where 40,000 EUR is assumed to be the average initial gross income of a client in the cohort and $\tilde{r} = 1\%$ the average increase in the income of the cohort. The initial capital is $v_0 = 10y_0 = 9,200,000$ EUR. For the buffer scheme we suppose $\alpha = 40\%$ and $B(0) = v_0 = 9,200,000$ EUR, $r_B = r = 0.5\%$, i.e. no additional return target in excess of r. The internal guarantee is set to $F = 80\% \times \left(v_0 e^{rT} + \int_0^T e^{r(T-t)} y(t)dt\right) = 48,596,935$ EUR, which is equal to 80% of the compounded wealth coming from initial capital and contributions. Finally, the coefficient of risk aversion is selected as $\hat{b} = -1$.

Simulation results. Within this setting we receive the following results for three exemplary, representative market trajectories. We perform weekly rebalancing, hence the investment strategy $\pi(t)$ stays constant along the interval $[t, t + \Delta)$ for $\Delta = 1/52$ and is updated at every grid point.

Figures 5.1 and 5.2 provide the stock price process, the wealth processes, the risky relative portfolios, the smoothing measures, the buffer rate and account in a bullish market environment. Over the entire investment period, the trajectory for $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ follows the one for $\tilde{V}(t)$ rather closely. The significant differences between $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ and $V^\star_{0,B}(t)$ in this scenario are due to the higher participation of the buffer rule strategy in the stock earnings because of a higher risky portfolio weight.

Furthermore, the graphics show that the buffer mechanism indeed smooths the reported wealth process compared to the non-smoothed processes (see Figure 5.1 (b)). This is also reflected by the smoothing measures in Figure 5.1 (e), (f). Moreover, $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ does not lose much compared to $\tilde{V}(t)$. Hence the accumulation of a safety buffer with a terminal share of around 15% of the total portfolio value does not reduce the total performance a lot, but smooths the reported wealth.

Figures 5.3 and 5.4 provide the analogue pictures within a mostly bearish market environment. The buffer account terminates in the negative area with an absolute value of approximately 10% of the total fund wealth. The reported fund balance is closer to the benchmark $B(t)$ due to the buffer scheme.

Furthermore, Figures 5.5 and 5.6 provide the plots under a volatile stock price process with no direction. In particular in this scenario, when comparing $V^\star_{\alpha,B}(t)$ with $\tilde{V}(t)$, one can see that the presence of a buffer mechanism truly smooths the reported portfolio balance when the very same investment strategy is applied. Moreover, the mechanism of saving money in good times (profit lock-in) and returning it in bad times leads to a slightly higher total wealth and thus a better performance at the end of the investment period in such scenarios. As the buffer account is positive but close to zero in this case, one can really see that the buffer is used to smooth out the dramatic changes in the market return from very positive to very negative. This can also be observed in the respective subfigures.

Moreover, we generally observe across all three considered scenarios that the existence of a symmetric buffer process with smoothing character seems to increase the relative risky investment. In a scenario where the stock price decreases, the relative portfolio for the smoothed version approaches the one for the unsmoothened process.

Finally, Figure 5.7 provides the Value-at-Risk numbers of the different wealth processes and Table 5.1 shows risk and return numbers of the wealth processes and the buffer portfolio. The numbers for $VaR_\beta\left(C^\star_{\alpha,B}(T)\right)$ can be regarded as β-worst case losses for the product offering insurance company. The Sharpe Ratio, defined by $SR(X) := \frac{\mathbb{E}[X]}{Sd(X)}$, is a risk-adjusted performance measure frequently used for performance evaluation and comparison. Figure 5.8 provides the kernel density estimates for the terminal buffer balance $C^\star_{\alpha,B}(T)$, the reported wealth $V^\star_{\alpha,B}(T)$ and the total wealths $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$, $V^\star_{0,B}(t)$ and $\tilde{V}(t)$. It shows that all distributions are skewed to the right.

The average smoothing measures $\overline{SM_1(t)}$, $\overline{SM_2(t)}$ average $SM_1(t)$, $SM_2(t)$ over all time steps and across all simulated scenarios; the numbers are given in Table 5.2.

To conclude, the figures and tables show the following picture:

First, when comparing $V^\star_{\alpha,B}(t)$ or $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ with $\tilde{V}(t)$, where the same investment strategy is applied, but the first considers a buffer whereas the latter does not, we find:

1. Reported wealth: The optimal reported wealth process $V^\star_{\alpha,B}(t)$ with buffer mechanism provides the better smoothing feature around $B(t)$ compared to the other wealth process. Table 5.2 showing the average smoothing measures supports this conclusion besides the pathwise figures.

2. Total wealth: The performance of the total wealth $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ of the optimal portfolio with smoothing mechanism does not fall below the total performance of the wealth process $\tilde{V}(t)$. In particular, although $\mathbb{E}\left[\tilde{V}(T)\right] > \mathbb{E}\left[V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)\right]$, for the Sharpe Ratio as a risk-adjusted performance measure it holds $SR\left(\tilde{V}(T)\right) < SR\left(V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)\right)$ (cf. Table 5.1). Moreover, the Value-at-Risk numbers show that the smoothed total portfolio $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$ provides similar worst case losses than $\tilde{V}(t)$, see also Figure 5.7. Hence, the performance of both total wealths is at a comparable level.

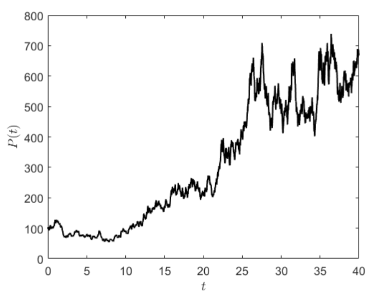
Second, when we compare $V^\star_{\alpha,B}(t)$ or $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$ with $V^\star_{0,B}(t)$, where not the same investment strategy is applied, but the latter applies the optimal strategy when no buffer is considered, we find:

1. Reported wealth: The optimal reported wealth process $V^{\star}_{\alpha,B}(t)$ with buffer mechanism provides a slightly better smoothing feature around $B(t)$ compared to the other wealth process. Table 5.2 showing the average smoothing measures supports this conclusion besides the pathwise figures.

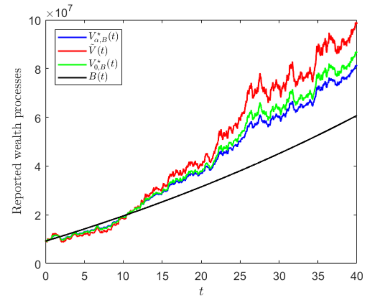
2. Total wealth: The performance of the total wealth $V^{\star}_{\alpha,B}(t) + C^{\star}_{\alpha,B}(t)$ of the optimal portfolio with smoothing mechanism does not fall behind the total performance of the wealth process $V^{\star}_{0,B}(t)$. Although the Sharpe Ratio of $V^{\star}_{0,B}(T)$ slightly exceeds the one of $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, i.e. $SR\left(V^{\star}_{0,B}(T)\right) > SR\left(V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)\right)$, the expected total fund wealth for $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$ is higher: $\mathbb{E}\left[V^{\star}_{0,B}(T)\right] < \mathbb{E}\left[V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)\right]$ (cf. Table 5.1). Additionally, the Value-at-Risk numbers show, especially at very small levels β, that $V^{\star}_{0,B}(T)$ provides better worst case losses than the smoothed total portfolio $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, see also Figure 5.7. This is due to a much more defensive investment strategy applied for $V^{\star}_{0,B}(T)$, for instance cf. Figures 5.1 (d) and 5.3 (d). At higher β values, the relation turns. Furthermore, the reported wealth $V^{\star}_{\alpha,B}(T)$ provides the best Value-at-Risk numbers, exceeding the ones for $V^{\star}_{0,B}(T)$.

Notice that due to smoothing, the reported wealth $V^{\star}_{\alpha,B}(T)$ provides the lowest volatility but largest Sharpe Ratio among all considered portfolios.

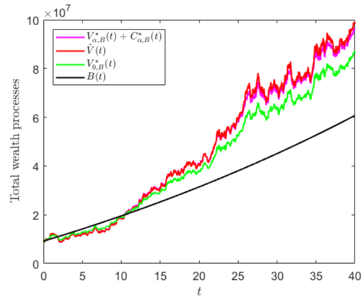
Therefore, the selling point of the strategy with buffer mechanism is that the reported wealth process is smoothed with respect to the wealth benchmark $B(t)$ and the performance of the total wealth process is comparable and on a very similar level than the wealth processes without a smoothing mechanism.



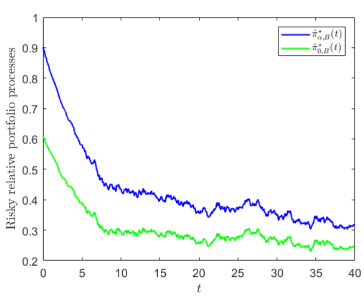
(a) Stock price $P(t)$.



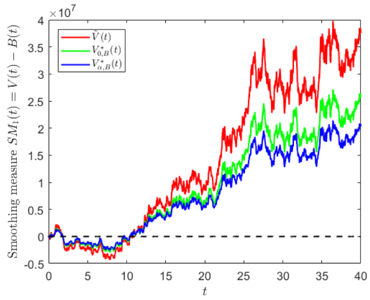
(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.



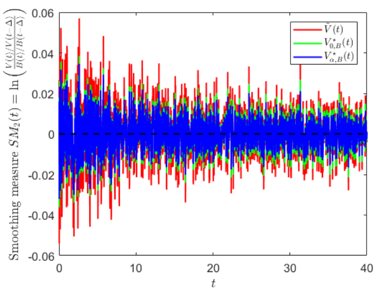
(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.



(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

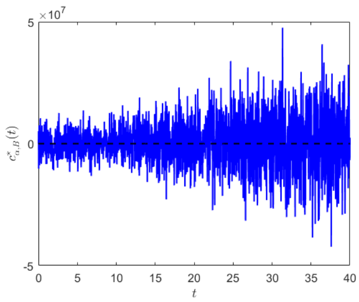


(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

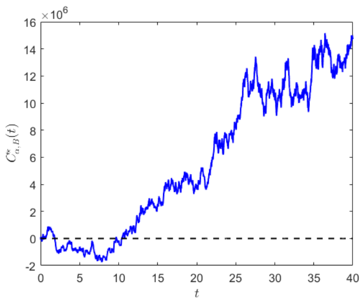


(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

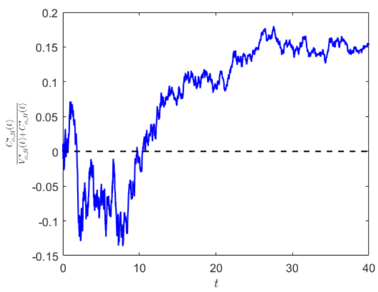
Figure 5.1: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bull market.



(a) Buffer rate $c^\star_{\alpha,B}(t)$.

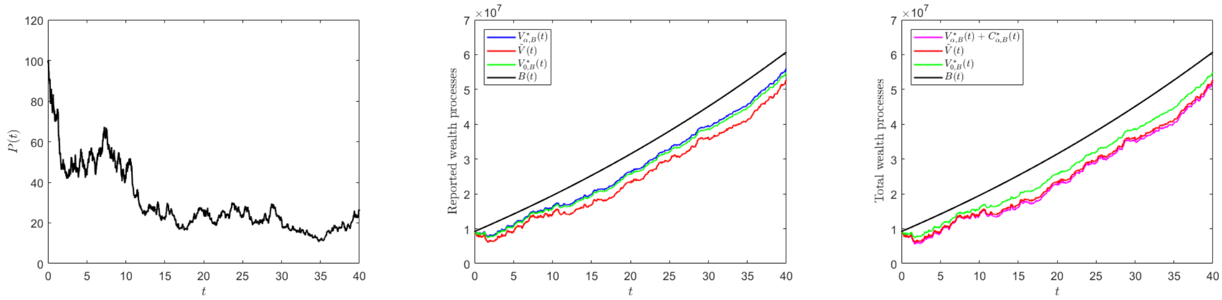


(b) Buffer account $C^\star_{\alpha,B}(t)$.



(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t)+C^\star_{\alpha,B}(t)$.

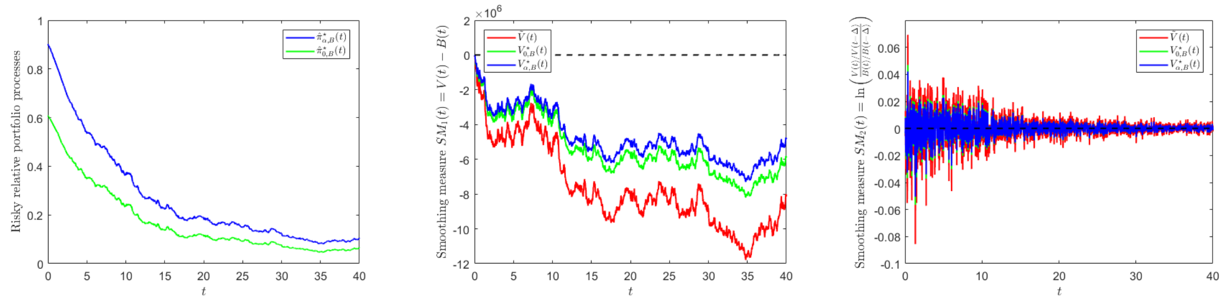
Figure 5.2: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bull market.



(a) Stock price $P(t)$.

(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

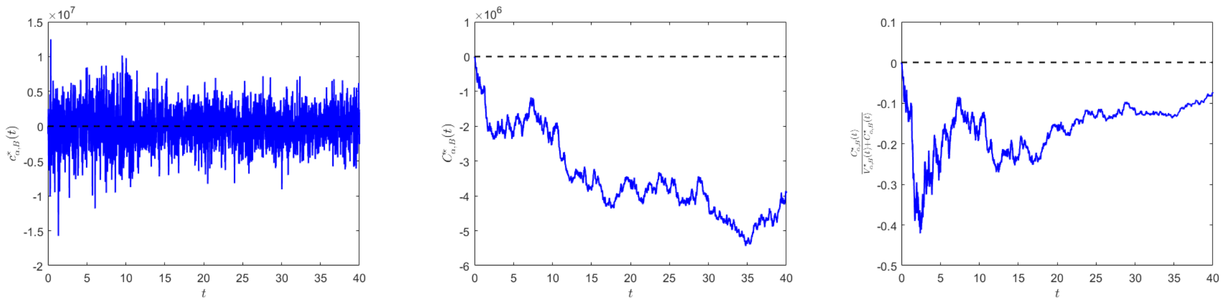


(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

Figure 5.3: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bear market.

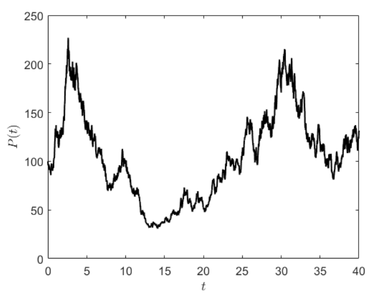


(a) Buffer rate $c^\star_{\alpha,B}(t)$.

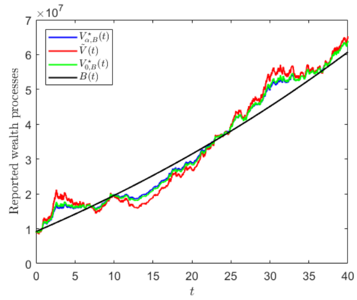
(b) Buffer account $C^\star_{\alpha,B}(t)$.

(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$.

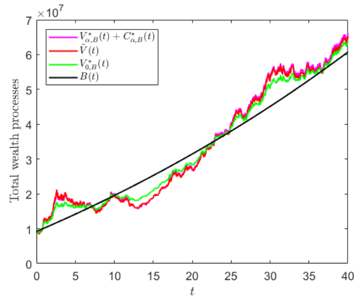
Figure 5.4: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bear market.



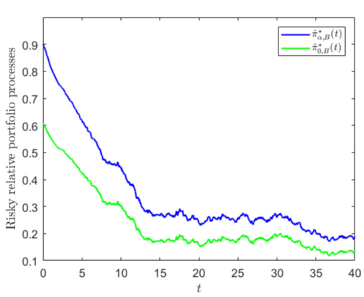
(a) Stock price $P(t)$.



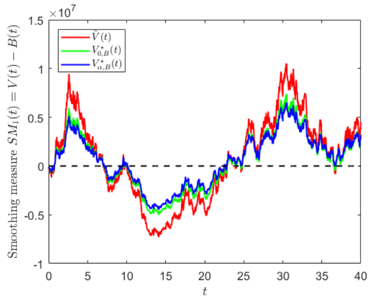
(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.



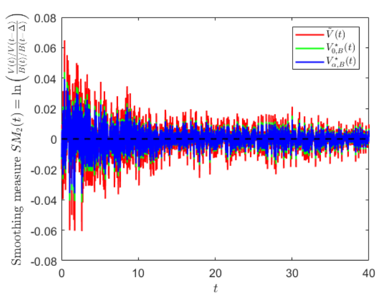
(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.



(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

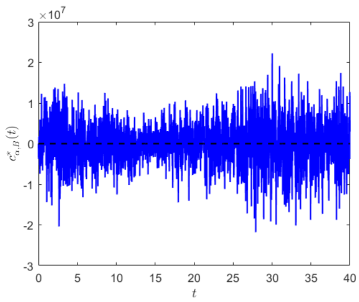


(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

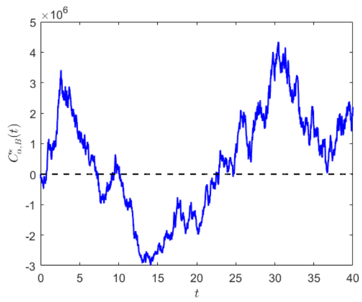


(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

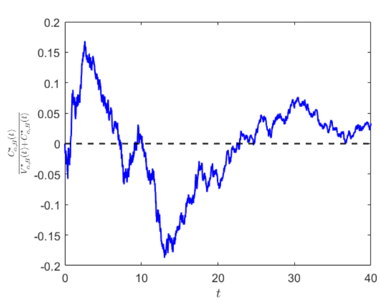
Figure 5.5: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a non-directional market.



(a) Buffer rate $c^\star_{\alpha,B}(t)$.

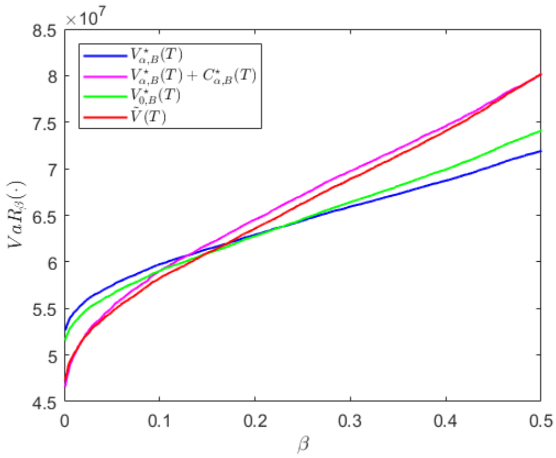


(b) Buffer account $C^\star_{\alpha,B}(t)$.



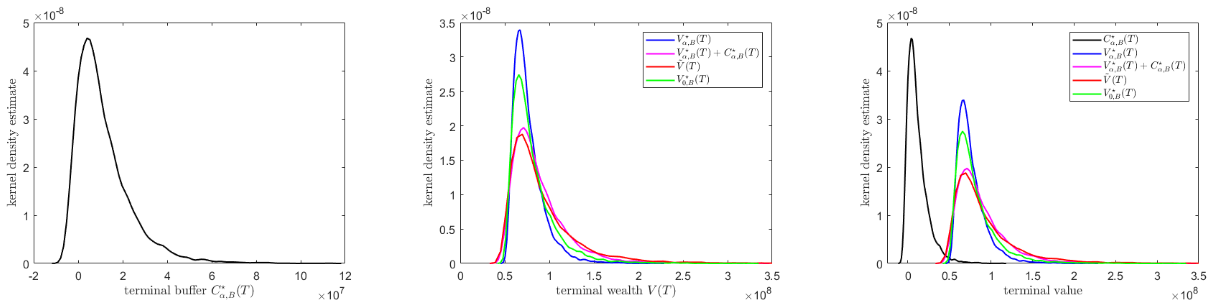
(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t)+C^\star_{\alpha,B}(t)$.

Figure 5.6: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a non-directional market.



(a) $VaR_{\beta}(\cdot)$.

Figure 5.7: $VaR_{\beta}(\cdot)$ vs. β for the terminal portfolio values $V^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,0}(T)$ and $\tilde{V}(T)$.



(a) Buffer account $C^{\star}_{\alpha,B}(T)$.

(b) Wealths $V^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, $V^{\star}_{0,B}(t)$ and $\tilde{V}(t)$.

(c) Buffer account and wealths.

Figure 5.8: Kernel density estimates of $C^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, $V^{\star}_{0,B}(t)$ and $\tilde{V}(t)$.

	$\mathbb{E}[\cdot]$	$Sd(\cdot)$	$SR(\cdot)$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$
$V^{\star}_{\alpha,B}(T)$	7.6342	1.7471	4.3696	5.7436	5.4548
$V^{\star}_{\alpha,B}(T)+C^{\star}_{\alpha,B}(T)$	8.7613	2.9950	2.9253	5.5079	4.9895
$V^{\star}_{0,B}(T)$	8.1160	2.5283	3.2101	5.6530	5.3440
$\tilde{V}(T)$	9.2339	4.2491	2.1731	5.4645	5.0126
	$\mathbb{P}(\cdot < 0)$	$\mathbb{E}[\cdot]$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$	
$C^{\star}_{\alpha,B}(T)$	12.95%	1.1271	-0.23517	-0.46093	

Table 5.1: Terminal performance numbers (values $\cdot 10^7$ except for $\mathbb{P}(C^{\star}_{\alpha,B}(T) < 0)$ and $SR(\cdot)$) under the optimal and the comparative investment strategies under 10,000 simulations and annual rebalancing.

	$\overline{SM_1(t)}$	$\overline{SM_2(t)}$
$V^{\star}_{\alpha,B}(t)$	$6.5759 \cdot 10^{6}$	$5.1629 \cdot 10^{-3}$
$V^{\star}_{0,B}(t)$	$8.1992 \cdot 10^{6}$	$6.3059 \cdot 10^{-3}$
$\tilde{V}(t)$	$1.2665 \cdot 10^{7}$	$8.6574 \cdot 10^{-3}$

Table 5.2: Average smoothing measures $\overline{SM_1(t)}$, $\overline{SM_2(t)}$ for the reported wealth processes under 10,000 simulations and annual rebalancing.

5.1.2.4 Purely accumulated buffer

Let $\alpha(t) \equiv \alpha$ and $C^{\star}_{\alpha,B}(t) = \int_0^t c(s)ds$, $dC^{\star}_{\alpha,B}(t) = c(t)dt = \alpha d(V^{\star}_{\alpha,B}(t) - B(t))$. Therefore, we suppose that no interest is paid on the buffer account (= time-t present value of so far accumulated buffer cash flows), i.e. the buffer money is not invested at some rate but is only parked at an account with zero interest rate (pure deposit); hence the buffer is purely accumulated. We provide closed-form solutions and characteristics of the buffer account $C^{\star}_{\alpha,B}(t)$. In the situation where $r = 0$, both worlds coincide: $\int_0^t e^{r(t-s)}c(s)ds = \int_0^t c(s)ds$. It follows

$$\begin{aligned} C^{\star}_{\alpha,B}(t) &= \int_0^t \alpha d(V^{\star}_{\alpha,B}(s) - B(s)) \\ &= \alpha\left[(V^{\star}_{\alpha,B}(t) - B(t)) - (V^{\star}_{\alpha,B}(0) - B(0))\right] \\ &= \alpha\left[(V^{\star}_{\alpha,B}(t) - B(t)) - (v_0 - B(0))\right] \\ &\overset{\text{if } B(0)=v_0}{=} \alpha(V^{\star}_{\alpha,B}(t) - B(t)) \end{aligned} \tag{5.9}$$

with of course $C^{\star}_{\alpha,B}(0) = 0$. Therefore, $C^{\star}_{\alpha,B}(t)$ now only depends on the current wealth $V^{\star}_{\alpha,B}(t)$ and not longer on the realized path $V^{\star}_{\alpha,B}(s)$, $s \leqslant t$. Based on Eq. (5.9) we can infer the following properties:

Theorem 5.5 (Properties of the purely accumulated buffer account $C^{\star}_{\alpha,B}(t)$)**.** *Let $\alpha(t) \equiv \alpha$ and $C^{\star}_{\alpha,B}(t) = \alpha \int_0^t d(V^{\star}_{\alpha,B}(s) - B(s))ds$. Then:*

1. *Distribution of the accumulated buffer account $C^{\star}_{\alpha,B}(t)$:*

$$\begin{aligned} F_{C^{\star}_{\alpha,B}(t)}(x) &= \mathbb{P}\left(C^{\star}_{\alpha,B}(t) \leqslant x\right) = \mathbb{P}\left(V^{\star}_{\alpha,B}(t) \leqslant \frac{x}{\alpha} + [B(t) + (v_0 - B(0))]\right) \\ &= F_{V^{\star}_{\alpha,B}(t)}\left(\frac{x}{\alpha} + [B(t) + (v_0 - B(0))]\right) \\ &= F_{V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}\left(\frac{x}{\alpha} + [B(t) + (v_0 - B(0))] - \tilde{F}_{\alpha,B}(t)\right), \end{aligned}$$

where $V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$ is log-normally distributed.

2. *Expected accumulated buffer account $C^{\star}_{\alpha,B}(t)$:*

$$\mathbb{E}\left[C^{\star}_{\alpha,B}(t)\right] = \alpha\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - \alpha\left[B(t) + (v_0 - B(0))\right]$$

3. *Variance of the accumulated buffer account $C^{\star}_{\alpha,B}(t)$:*

$$Var\left(C^{\star}_{\alpha,B}(t)\right) = \alpha^2 Var\left(V^{\star}_{\alpha,B}(t)\right)$$

4. *Value-at-Risk/Quantiles of the accumulated buffer account $C^{\star}_{\alpha,B}(t)$ with level $\beta \in [0,1]$:*

$$VaR_\beta\left(C^{\star}_{\alpha,B}(t)\right) = \alpha VaR_\beta\left(V^{\star}_{\alpha,B}(t)\right) - \alpha\left[B(t) + (v_0 - B(0))\right]$$

5. *Shortfall probability of the accumulated buffer account $C^{\star}_{\alpha,B}(t)$ with threshold $\tilde{s} > \alpha\left(\tilde{F}_{\alpha,B}(t) - [B(t) + (v_0 - B(0))]\right)$:*

$$\mathbb{P}\left(C^{\star}_{\alpha,B}(t) \leqslant \tilde{s}\right) = \mathbb{P}\left(V^{\star}_{\alpha,B}(t) \leqslant \frac{\tilde{s}}{\alpha} + [B(t) + (v_0 - B(0))]\right)$$

The numbers $\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]$, $Var\left(V^{\star}_{\alpha,B}(t)\right)$, $VaR_{\beta}\left(V^{\star}_{\alpha,B}(t)\right)$ *and* $\mathbb{P}\left(V^{\star}_{\alpha,B}(t) \leqslant s\right)$ *are given by Theorem 5.2.*

In the following we examine constraints on $C^{\star}_{\alpha,B}(t)$, a possible structural choice for the buffer wealth benchmark $B(t)$ and the internal, terminal guarantee F that fits to those constraints. The structure of the proposed $B(t)$ and F can be interpreted as a proportion of the already accumulated human capital. In particular, we aim to achieve a non-negative buffer balance, i.e. $C^{\star}_{\alpha,B}(t) \geqslant 0$, but with a smoothing feature, i.e. $c(t)dt = \alpha d(V^{\star}_{\alpha,B}(t) - B(t)) < 0$ shall be possible. We summarize the specific setting in the next assumption:

Assumption 5.6. *Let* $\alpha(t) \equiv \alpha$, $r = 0$ *and recall*

$$Y(t) = \int_0^t y(s)ds > 0$$

for the accumulated time-t human capital inside the interval $[0, t]$ *when* $r = 0$ *according to Footnote 3 in Chapter 5 with* $dY(t) = y(t)dt$. *Moreover, suppose*

$$B(t) := B(0) + \delta \int_0^t y(s)ds = B(0) + \delta Y(t), \;\; \delta \geqslant 0,$$

with $dB(t) = \delta y(t)dt$ *proportional to the inflows, and for instance* $B(0) = v_0$. *Furthermore, we consider the following form for the internal, terminal guarantee:*

$$F := v_0 + \tilde{\delta} \int_0^T y(s)ds = v_0 + \tilde{\delta} Y(T), \;\; \tilde{\delta} \geqslant 0.$$

Under Assumption 5.6, it is

$$\begin{aligned}
\tilde{F}_{\alpha,B}(t) &= F - \frac{1+\alpha\delta}{1+\alpha}\left(Y(T) - Y(t)\right) \\
&= v_0 + \frac{1+\alpha\delta}{1+\alpha} Y(t) + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right) Y(T), \\
\tilde{F}_{\alpha,B}(0) &= F - \frac{1+\alpha\delta}{1+\alpha} Y(T) \\
&= v_0 + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right) Y(T), \\
\tilde{F}_{\alpha,B}(t) - B(t) &= F - B(0) - \frac{1+\alpha\delta}{1+\alpha} Y(T) + \frac{1-\delta}{1+\alpha} Y(t) \\
&= v_0 - B(0) + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right) Y(T) + \frac{1-\delta}{1+\alpha} Y(t), \\
d(\tilde{F}_{\alpha,B}(t) - B(t)) &= \frac{1-\delta}{1+\alpha} y(t)dt.
\end{aligned}$$

Hence,

$$\begin{aligned} c(t)dt &\overset{(5.7)}{=} \alpha d(V^\star_{\alpha,B}(t) - B(t)) \\ &= \alpha\left((V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\frac{1}{1-\hat{b}}\|\gamma\|^2 dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] + \frac{1-\delta}{1+\alpha} y(t)dt \right) \end{aligned} \tag{5.10}$$

which can take any value in $\mathbb{R}$ due to its stochastic component. Additionally, the accumulated buffer becomes

$$\begin{aligned} C^\star_{\alpha,B}(t) &= \alpha\left[(V^\star_{\alpha,B}(t) - B(t)) - (v_0 - B(0))\right] \\ &= \alpha\Bigg[\left(v_0 - F + \frac{1+\alpha\delta}{1+\alpha} Y(T)\right) e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2 t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - v_0 + F - \frac{1+\alpha\delta}{1+\alpha} Y(T) \\ &\quad + \frac{1-\delta}{1+\alpha} Y(t)\Bigg] \\ &= \alpha\left[\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T)\right) e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2 t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right) Y(T) + \frac{1-\delta}{1+\alpha} Y(t)\right]. \end{aligned} \tag{5.11}$$

For the remainder of this section, we relax the strictness of the following two inequalities and allow for equality. First, from the constraint on v_0, $y(t)$ and F in Eq. (5.6) it follows

$$v_0 + \int_0^T e^{-rt} y(t)dt \geqslant e^{-rT} F \overset{r=0}{\Leftrightarrow} v_0 + Y(T) \geqslant v_0 + \tilde{\delta} Y(T) \Leftrightarrow \tilde{\delta} \leqslant 1,$$

and second, from the assumption $v_0 \geqslant \tilde{F}_{\alpha,B}(0)$ prior to Theorem 5.1 we obtain

$$v_0 \geqslant \tilde{F}_{\alpha,B}(0) \Leftrightarrow \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) \geqslant 0 \Leftrightarrow \tilde{\delta} \leqslant \frac{1+\alpha\delta}{1+\alpha}.$$

However, in view of Theorem 5.1, the optimal investment strategy in the equality case breaks down to a pure riskless investment with all stochasticity being removed:

$$\hat{\pi}^\star_{\alpha,B}(t) \equiv 0\%,\ V^\star_{\alpha,B}(t) = \tilde{F}_{\alpha,B}(t),\ C^\star_{\alpha,B}(t) = \alpha\frac{1-\delta}{1+\alpha} Y(t).$$

Putting the two above conditions on $\tilde{\delta}$ together, we have the feasibility restriction

$$\tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right]. \tag{5.12}$$

Note that $\frac{1+\alpha\delta}{1+\alpha}$ can be rewritten as follows:

$$\frac{1+\alpha\delta}{1+\alpha} = \delta + \frac{1-\delta}{1+\alpha} = 1 + \frac{\alpha(\delta-1)}{1+\alpha}.$$

We detect the following lower and upper boundaries for $\frac{1+\alpha\delta}{1+\alpha}$ (that are not necessarily binding):

1. Let $0 \leqslant \delta \leqslant 1$:

   - Lower bound:

     Since $\frac{1+\alpha\delta}{1+\alpha} = \delta + \frac{1-\delta}{1+\alpha} \geqslant \delta$ and $\frac{1+\alpha\delta}{1+\alpha} \geqslant \frac{1}{1+\alpha}$, we obtain as lower bound

     $$\frac{1+\alpha\delta}{1+\alpha} \geqslant \max\left\{\delta, \frac{1}{1+\alpha}\right\}$$

     and especially $\frac{1+\alpha\delta}{1+\alpha} \geqslant \delta$.

   - Upper bound:

     Due to $\frac{1+\alpha\delta}{1+\alpha} = 1 + \frac{\alpha(\delta-1)}{1+\alpha} \leqslant 1$ we obtain the upper bound

     $$\frac{1+\alpha\delta}{1+\alpha} \leqslant 1.$$

2. Let $\delta > 1$:

   - Lower bound:

     From $\frac{1+\alpha\delta}{1+\alpha} = 1 + \frac{\alpha(\delta-1)}{1+\alpha} > 1$ it follows

     $$\frac{1+\alpha\delta}{1+\alpha} > 1$$

     for the lower bound.

   - Upper bound:

     Because of $\frac{1+\alpha\delta}{1+\alpha} = \delta + \frac{1-\delta}{1+\alpha} < \delta$ we obtain the upper bound

     $$\frac{1+\alpha\delta}{1+\alpha} < \delta.$$

The detailed calculations below Assumption 5.6 can be found in Appendix C.2. Under Assumption 5.6, we examine non-negativity conditions on $C^{\star}_{\alpha,B}(t)$ and find the following closed-form results:

Theorem 5.7 (Non-negativity of $C^{\star}_{\alpha,B}(t)$)**.** *Under the setup of Assumption 5.6, it follows*

$$\begin{aligned} C^{\star}_{\alpha,B}(t) &\geqslant \alpha\left[\left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right)Y(T) + \frac{1-\delta}{1+\alpha}Y(t)\right], \\ C^{\star}_{\alpha,B}(T) &\geqslant \alpha\left(\tilde{\delta} - \delta\right)Y(T), \end{aligned}$$

and the following claims hold:

1. $C^{\star}_{\alpha,B}(t) \geqslant 0$ for some $t \in [0,T]$:

$$C^{\star}_{\alpha,B}(t) \geqslant 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right].$$

2. $C^\star_{\alpha,B}(t) \geqslant 0$ *for all* $t \in [0,T]$*:*

$$C^\star_{\alpha,B}(t) \geqslant 0 \ \forall t \in [0,T] \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}.$$

3. $C^\star_{\alpha,B}(T) \geqslant 0$*:*

$$C^\star_{\alpha,B}(T) \geqslant 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta, \frac{1+\alpha\delta}{1+\alpha}\right].$$

This theorem shows that a throughout non-negative buffer balance is only possible if $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$ which leads to a pure riskless investment $\hat{\pi}^\star_{\alpha,B}(t) \equiv 0\%$. However, the model allows for a final non-negative buffer account $C^\star_{\alpha,B}(T) \geqslant 0$, or a non-negative buffer account for some end-of-the-year times $C^\star_{\alpha,B}(t) \geqslant 0$ for $t = t_1, t_2, t_3, \ldots$, for other choices of the parameters. Moreover, we find the following interesting characteristics that arise from (the proof of) Theorem 5.7.

Remark 5.8 (Comments on the non-negativity of $C^\star_{\alpha,B}(t)$)**.** *Let Assumption 5.6 hold true. Then:*

1. From Theorem 5.7 we know that

$$C^\star_{\alpha,B}(t) \geqslant 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right].$$

Let $\delta \in [0,1)$*, the case* $\delta = 1$ *is trivial because* $\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}$ *as well as* $\frac{1+\alpha\delta}{1+\alpha}$ *break down to one. As for* $\delta \in [0,1)$ *the term* $\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}$ *decreases in* t*, we find:*

a) $C^\star_{\alpha,B}(t) \geqslant 0$ *for all* $t \in \{t_1, \ldots, t_n\}$*,* $t_i < t_{i+1}$ $\forall i = 1, \ldots, n-1$*,* $n \in \mathbb{N}$*:*

$$\begin{aligned}
&C^\star_{\alpha,B}(t) \geqslant 0 \ \forall t \in \{t_1, \ldots, t_n\} \\
&\quad \Leftrightarrow \delta \in [0,1),\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(\min_{i=1,\ldots,n}\{t_i\})}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right] \\
&\quad \Leftrightarrow \delta \in [0,1),\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t_1)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right] \\
&\quad \Leftrightarrow C^\star_{\alpha,B}(t_1) \geqslant 0 \\
&\quad \Leftrightarrow C^\star_{\alpha,B}\left(\min_{i=1,\ldots,n}\{t_i\}\right) \geqslant 0.
\end{aligned}$$

b) $C^\star_{\alpha,B}(s) \geqslant 0$ *for all* $s \geqslant t \in (0,T]$*:*

$$\begin{aligned}
C^\star_{\alpha,B}(s) \geqslant 0 \ \forall s \geqslant t \in (0,T] &\Leftrightarrow C^\star_{\alpha,B}\left(\min_{s\in[t,T]}\{s\}\right) \geqslant 0 \\
&\Leftrightarrow C^\star_{\alpha,B}(t) \geqslant 0.
\end{aligned}$$

Thus, “the smallest time wins”, i.e. is crucial. It generally follows that if the parameters are selected such that $C^\star_{\alpha,B}(t) \geqslant 0$ $\mathbb{P}$*-a.s., then it also is* $C^\star_{\alpha,B}(s) \geqslant 0$ *for all times* s *after time* t*,* $s > t$*.*

2. *The limiting case $t \searrow 0$ yields the special case $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$:*

$$C^\star_{\alpha,B}(t) \geqslant 0 \text{ for } t \searrow 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\underbrace{\delta + \frac{1-\delta}{1+\alpha} \underbrace{\frac{Y(T)-Y(t)}{Y(T)}}_{\nearrow 1 \text{ for } t \searrow 0}}_{\nearrow \frac{1+\alpha\delta}{1+\alpha} \text{ for } t \searrow 0}, \frac{1+\alpha\delta}{1+\alpha} \right]$$
$$\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}.$$

As already mentioned, for the choice $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$, we obtain $v_0 = \tilde{F}_{\alpha,B}(0)$ and therefore

$$V^\star_{\alpha,B}(t) = \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{1}{1-\hat{b}}\left[\frac{\alpha+\hat{b}}{1+\alpha} r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right]t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} = \tilde{F}_{\alpha,B}(t)$$

is deterministic. In accordance with Corollary 5.3 this implies $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) \equiv 0$ and $\hat{\pi}^\star_{\alpha,B}(t) \equiv \mathbf{0}$, i.e. there is a full 100% riskless investment. Furthermore, from Eq. (5.11)

$$\begin{aligned} C^\star_{\alpha,B}(t) &= \alpha\Bigg[\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta} \right) Y(T) \right) e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2 t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) Y(T) \\ &\quad + \frac{1-\delta}{1+\alpha} Y(t) \Bigg] \\ &= \alpha \frac{1-\delta}{1+\alpha} Y(t) = \alpha \left(\tilde{\delta} - \delta \right) Y(t) \geqslant 0 \end{aligned}$$

with

$$c(t)dt = dC^\star_{\alpha,B}(t) = \alpha \frac{1-\delta}{1+\alpha} y(t)dt = \alpha \left(\tilde{\delta} - \delta \right) y(t)dt \geqslant 0.$$

Thus, all important numbers are deterministic, non-stochastic ($V^\star_{\alpha,B}(t)$, $\pi^\star_{\alpha,B}(t)$, $C^\star_{\alpha,B}(t)$, $c(t)$), and a buffer is only accumulated at a deterministic rate, but never transferred back to the portfolio for smoothing purposes. If additionally $\delta = 1$, then $c(t) \equiv 0$ and $C^\star_{\alpha,B}(t) \equiv 0$.

3. *Parts 1. and 2. show that*

$$\begin{aligned} C^\star_{\alpha,B}(s) \geqslant 0\ \forall s \geqslant t \in (0,T] &\Leftrightarrow C^\star_{\alpha,B}(t) \geqslant 0 \text{ for } t \in (0,T] \\ &\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha} \right] \neq \varnothing \end{aligned}$$

and

$$C^\star_{\alpha,B}(t) \geqslant 0 \text{ for } t \searrow 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}.$$

Therefore, if we require $C^\star_{\alpha,B}(t) \geqslant 0$ for some t, then $t > 0$ must be either strictly positive, or the stochasticity needs to be removed (case $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$). The economic argument is the following: If $C^\star_{\alpha,B}(t) \geqslant 0$ is to be guaranteed, then a sufficient amount of capital needs to be *collected until time t (here: accumulated time-t human capital $Y(t)$) with a sufficiently large deterministic drift rate $y(t)$ that overweighs the potential, stochastic losses in the wealth. From the calculations below Assumption 5.6:*

$$\begin{aligned} c(t)dt &= \alpha d(V^{\star}_{\alpha,B}(t) - B(t)) \\ &= \alpha \left(\underbrace{(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\frac{1}{1-\hat{b}} \|\gamma\|^2 dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right]}_{= \text{ investment part, possibly } <0} + \underbrace{\frac{1-\delta}{1+\alpha} y(t)dt}_{= \text{ deterministic drift part } >0} \right). \end{aligned}$$

This result is also in line with the following interpretation: Let $W(t) \searrow -\infty$ for $t \searrow 0$ (equivalent with $P_i(t) \searrow 0$ for $t \searrow 0$), then the accumulated deterministic drift part $\alpha \frac{1-\delta}{1+\alpha} Y(t)$ with rate $\alpha \frac{1-\delta}{1+\alpha} y(t)dt$ cannot compensate for the loss in the investment part because it is deterministic and thus bounded, which results in $C^{\star}_{\alpha,B}(t) < 0$ for $t \searrow 0$ in this situation. The investment risk can only be ruled out if the loss in the investment part is capped by 0 which is the case if and only if $V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) \equiv 0$ (case $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$). But then, as described in part 2., all numbers $V^{\star}_{\alpha,B}(t)$, $\pi^{\star}_{\alpha,B}(t)$, $C^{\star}_{\alpha,B}(t)$, $c(t)$ are deterministic.

One could further aim to limit the downside risk, quantified by the risk measure Value-at-Risk (quantile), that relaxes the absolute non-negative condition $C^{\star}_{\alpha,B}(t) \geqslant 0$. Analogical conditions for such a Value-at-Risk constraint $VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0$ for some level $\beta \in (0,1]$ instead of $C^{\star}_{\alpha,B}(t) \geqslant 0$ can be found in Appendix C.1.2.

We conclude our analysis under Assumption 5.6 with a numerical case study. Note that the first case study in Section 5.1.2.3 considered a general setup different to the one in Assumption 5.6 to demonstrate applicability beyond this assumption and for different and more flexible functionals. Now, the second case study deals with the parameterization in Assumption 5.6 and as a special case uses parameters such that $C^{\star}_{\alpha,B}(T) \geqslant 0$. It demonstrates that the selected setup is economically reasonable.

5.1.2.5 Scenario generation and numerical analysis of the optimal pension fund strategy: $C^{\star}_{\alpha,B}(T) \geqslant 0$

We now study a setup that lies within the scope of Assumption 5.6 and that leads to an accumulated buffer balance with $C^{\star}_{\alpha,B}(T) \geqslant 0$, $\mathbb{P}$-a.s.. In order to keep the numerical findings comparable with the previous Section 5.1.2.3, we change the setup as little as possible: Different to 5.1.2.3, we now consider $r = 0\%$, $\delta = \tilde{\delta} = 0.7$ with $B(t) = v_0 + \delta Y(t)$, $F = v_0 + \tilde{\delta} Y(T)$ and

$$Y(t) = \int_0^t y(s)ds = \int_0^t \frac{\tilde{r}}{e^{\tilde{r}} - 1} y_0 e^{\tilde{r}s} ds = y_0 \sum_{s=0}^{t-1} e^{\tilde{r}s} = y_0 \frac{e^{\tilde{r}t} - 1}{e^{\tilde{r}} - 1},$$

all other parameters and functionals beyond Assumption 5.6 being equal to the setting in Section 5.1.2.3. In line with the parameter selection, the internal terminal guarantee F then becomes $F = v_0 + \tilde{\delta} Y(T) = 9{,}200{,}000 + 0.7 \int_0^{40} y(t)dt = 40{,}715{,}407$ EUR. According to Theorem 5.7, within this setting it holds $C^{\star}_{\alpha,B}(T) \geqslant 0$.

Simulation results. Within this setting under Assumption 5.6 we receive the numerical results and now briefly present the differences between the findings in Section 5.1.2.3:

1. Smoothing with respect to the benchmark return $B(t)$ is more pronounced, cf. for instance SM_2 in Tables 5.2 and 5.4.

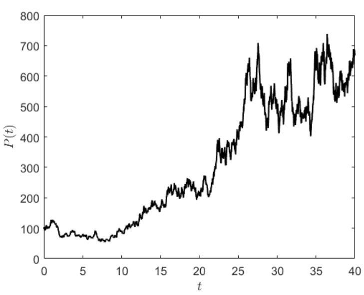
2. The total buffer balance $C^\star_{\alpha,B}(T)$, in particular in Figure 5.12 (b), always ends in the non-negative area due to the positive deterministic drift component $\alpha\frac{1-\delta}{1+\alpha}y(t)dt > 0$ in $dC^\star_{\alpha,B}(t)$. This is also made visible by the kernel density estimate of $C^\star_{\alpha,B}(T)$ in Figure 5.16. Together with a low $\hat{\pi}^\star_{\alpha,B}(t)$ for a decreasing stock price, $C^\star_{\alpha,B}(t)$ has a stochastic part that is close to zero due to the low $\hat{\pi}^\star_{\alpha,B}(t)$, and a positive deterministic part that drives $C^\star_{\alpha,B}(t)$ into the positive area.

3. Moreover, Figure 5.15 shows that, in contrast to Figure 5.7 in Section 5.1.2.3, the Value-at-Risk numbers of all total wealths exceed the Value-at-Risk for the reported wealth of the buffer strategy which is obvious as the buffer $C^\star_{\alpha,B}(T) \geqslant 0$.

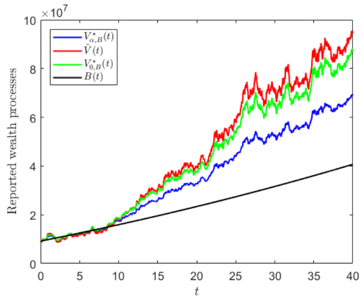
4. In accordance with Figures 5.9–5.15 and Table 5.3, the total wealths $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$ and $V^\star_{0,B}(T)$ lead to comparable risk-return numbers and behavior, with the difference that the $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$ strategy benefits from the buffer part feature and even provides a higher Sharpe Ratio due to a lower standard deviation. Note that both are optimal strategies, one with a buffering process, one without. This shows that one can follow an optimal investment strategy with a buffer rule without losing in performance.

5. Finally, the reported $V^\star_{\alpha,B}(T)$ provides the highest Sharpe Ratio by far. In addition, all Value-at-Risk numbers of $C^\star_{\alpha,B}(T)$ are positive as $C^\star_{\alpha,B}(T) \geqslant 0$ in the setup.

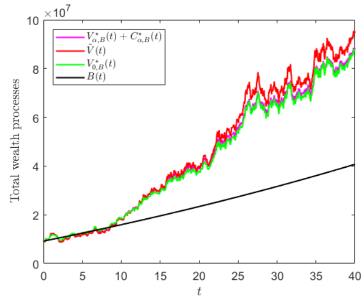
In summary, the case study demonstrates that a pension fund can invest according to an optimal strategy with a certain buffer rule and by this does not underperform compared to an optimal strategy without a buffer rule. In opposite, the buffer strategy even provides a higher Sharpe Ratio and simultaneously has a smoothing effect and accumulates a safety buffer.



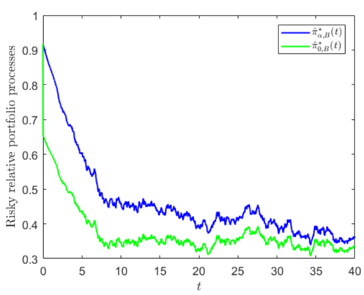
(a) Stock price $P(t)$.



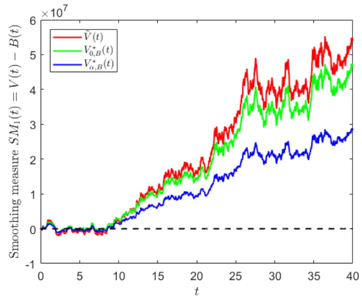
(b) Reported wealths $V^{\star}_{\alpha,B}(t)$, $V^{\star}_{0,B}(t)$, $\tilde{V}(t)$.



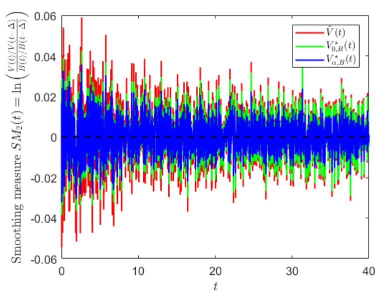
(c) Total wealths $V^{\star}_{\alpha,B}(t) + C^{\star}_{\alpha,B}(t)$, $V^{\star}_{0,B}(t)$, $\tilde{V}(t)$.



(d) Relative risky portfolio $\hat{\pi}^{\star}_{\alpha,B}(t)$, $\hat{\pi}^{\star}_{0,B}(t)$.

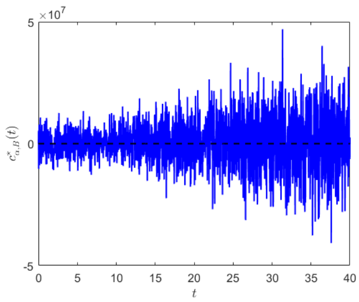


(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

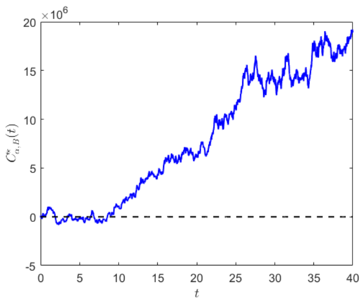


(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

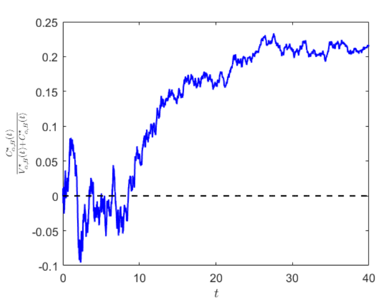
Figure 5.9: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bull market.



(a) Buffer rate $c^{\star}_{\alpha,B}(t)$.

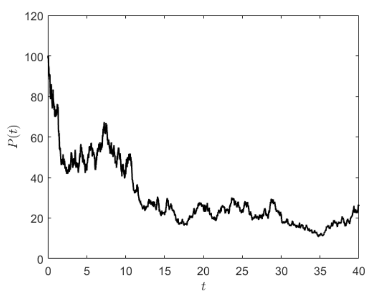


(b) Buffer account $C^{\star}_{\alpha,B}(t)$.

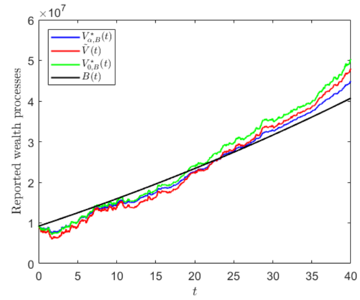


(c) Ratio between the buffer account $C^{\star}_{\alpha,B}(t)$ and the total wealth $V^{\star}_{\alpha,B}(t)+C^{\star}_{\alpha,B}(t)$.

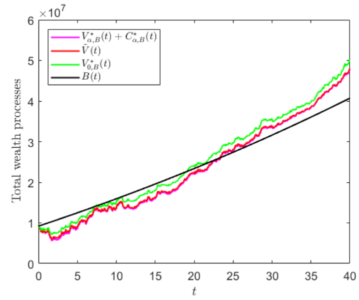
Figure 5.10: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bull market.



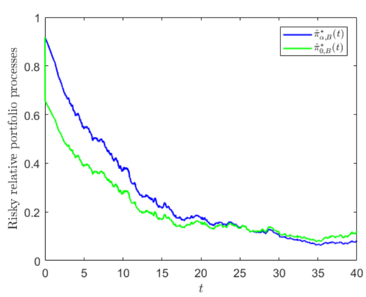
(a) Stock price $P(t)$.



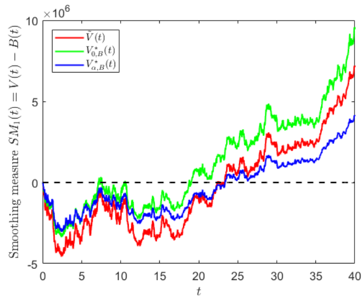
(b) Reported wealths $V^{\star}_{\alpha,B}(t)$, $V^{\star}_{0,B}(t)$, $\tilde{V}(t)$.



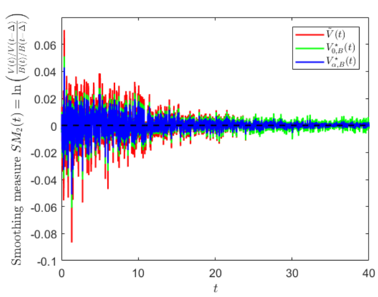
(c) Total wealths $V^{\star}_{\alpha,B}(t) + C^{\star}_{\alpha,B}(t)$, $V^{\star}_{0,B}(t)$, $\tilde{V}(t)$.



(d) Relative risky portfolio $\hat{\pi}^{\star}_{\alpha,B}(t)$, $\hat{\pi}^{\star}_{0,B}(t)$.

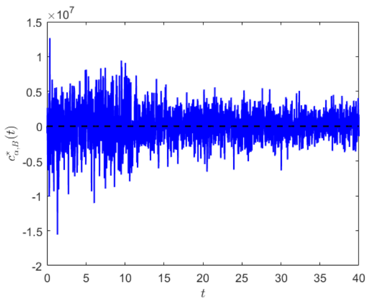


(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

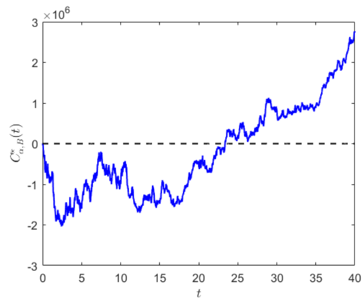


(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

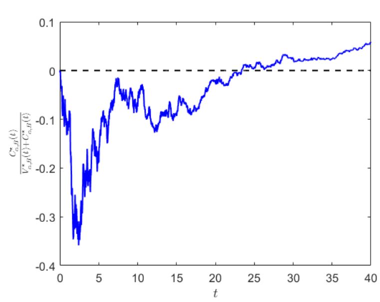
Figure 5.11: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bear market.



(a) Buffer rate $c^{\star}_{\alpha,B}(t)$.

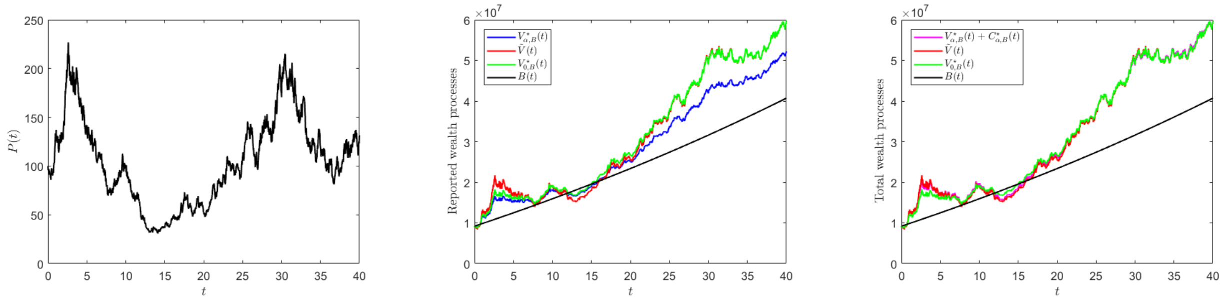


(b) Buffer account $C^{\star}_{\alpha,B}(t)$.



(c) Ratio between the buffer account $C^{\star}_{\alpha,B}(t)$ and the total wealth $V^{\star}_{\alpha,B}(t)+C^{\star}_{\alpha,B}(t)$.

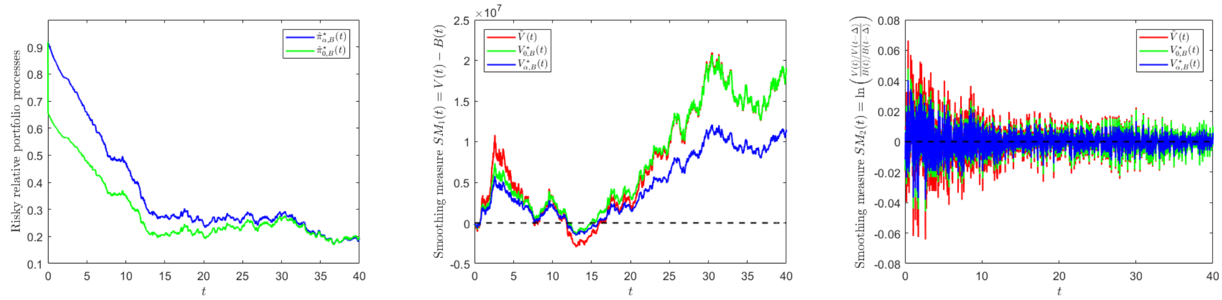
Figure 5.12: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bear market.



(a) Stock price $P(t)$.

(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

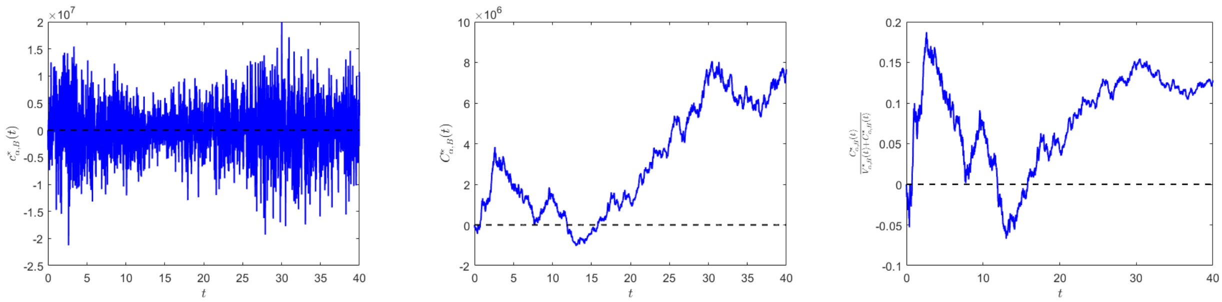


(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

(e) Smoothing measure $SM_1(t) = V(t) - B(t)$.

(f) Smoothing measure $SM_2(t) = \ln\left(\frac{V(t)/V(t-\Delta)}{B(t)/B(t-\Delta)}\right)$.

Figure 5.13: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 40\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a non-directional market.

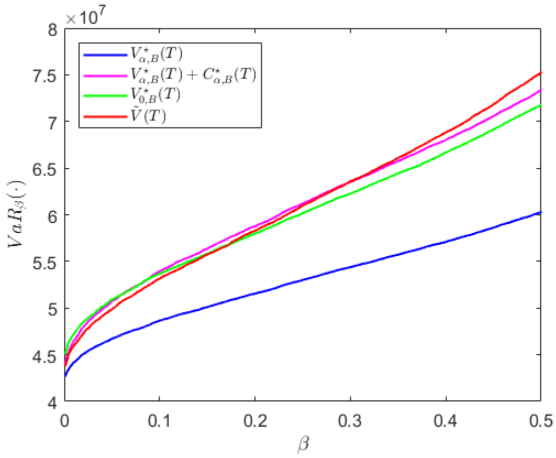


(a) Buffer rate $c^\star_{\alpha,B}(t)$.

(b) Buffer account $C^\star_{\alpha,B}(t)$.

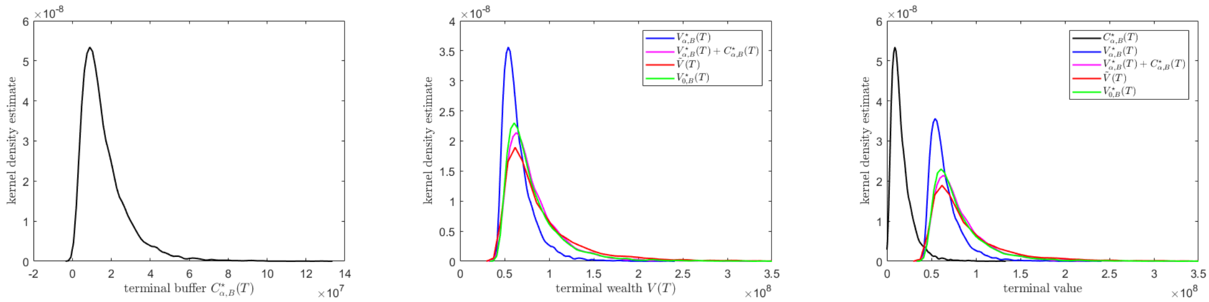
(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$.

Figure 5.14: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a non-directional market.



(a) $VaR_\beta\,(\cdot)$.

Figure 5.15: $VaR_\beta\,(\cdot)$ vs. β for the terminal portfolio values $V^\star_{\alpha,B}(T)$, $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$, $V^\star_{\alpha,0}(T)$ and $\tilde{V}(T)$.



(a) Buffer account $C^\star_{\alpha,B}(T)$.

(b) Wealths $V^\star_{\alpha,B}(T)$, $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$, $V^\star_{0,B}(t)$ and $\tilde{V}(t)$.

(c) Buffer account and wealths.

Figure 5.16: Kernel density estimates of $C^\star_{\alpha,B}(T)$, $V^\star_{\alpha,B}(T)$, $V^\star_{\alpha,B}(T) + C^\star_{\alpha,B}(T)$, $V^\star_{0,B}(t)$ and $\tilde{V}(t)$.

	$\mathbb{E}[\cdot]$	$Sd(\cdot)$	$SR(\cdot)$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$
$V^{\star}_{\alpha,B}(T)$	6.5275	1.8477	3.5328	4.6721	4.4184
$V^{\star}_{\alpha,B}(T)+C^{\star}_{\alpha,B}(T)$	8.1649	3.0795	2.6513	5.0724	4.6497
$V^{\star}_{0,B}(T)$	8.1334	3.3145	2.4539	5.0892	4.7230
$\tilde{V}(T)$	9.0335	5.0954	1.7729	4.9864	4.5955
	$\mathbb{P}(\cdot < 0)$	$\mathbb{E}[\cdot]$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$	
$C^{\star}_{\alpha,B}(T)$	0%	1.6373	0.4004	0.2313	

Table 5.3: Terminal performance numbers (values $\cdot 10^7$ except for $\mathbb{P}(C^{\star}_{\alpha,B}(T) < 0)$ and $SR(\cdot)$) under the optimal and the comparative investment strategies under $10,000$ simulations and annual rebalancing.

	$\overline{SM_1(t)}$	$\overline{SM_2(t)}$
$V^{\star}_{\alpha,B}(t)$	$1.0967 \cdot 10^{7}$	$1.1004 \cdot 10^{-2}$
$V^{\star}_{0,B}(t)$	$1.7211 \cdot 10^{7}$	$1.5796 \cdot 10^{-2}$
$\tilde{V}(t)$	$2.1025 \cdot 10^{7}$	$1.7490 \cdot 10^{-2}$

Table 5.4: Average smoothing measures $\overline{SM_1(t)}$, $\overline{SM_2(t)}$ for the reported wealth processes under $10,000$ simulations and annual rebalancing.

5.2 Decumulation phase

This section considers the decumulation phase optimization problem for a “Nahles-Rente” or “Sozialpartnermodell” pension product in Germany. The continuous-time optimization problem is defined consisting of two specialties: first, we have a product-specific pension adjustment mechanism based on a certain capital coverage ratio which stipulates compulsory pension adjustments if the pension fund is underfunded or significantly overfunded, and second, due to the retiree’s fear of and aversion against pension reductions, we introduce a total wealth distribution to an investment portfolio and a buffer portfolio to lower the probability of future potential pension shortenings. Due to the inherent complexity of the continuous-time framework, the discrete-time version of the optimization problem is considered and solved via the Bellman principle. In addition, for computational reasons, a policy function iteration algorithm is introduced to find a stationary solution to the problem in a computationally efficient and elegant fashion. A numerical case study on optimization and simulation completes the work with highlighting the clients’ benefits (superior relative performance) in the proposed model, which lies within the scope of the regulations.

The part on the decumulation phase is organized as follows: Section 5.2.1 introduces the continuous-time mathematical framework that builds the basis for the decumulation phase and the resulting portfolio optimization problem. Further, in Section 5.2.2 we solve the portfolio selection problem in the single-client and cohort model discrete in time. Due to implementation reasons, Section 5.2.3 provides an approximate solution to the original discrete-time problem in form of a stationary solution. An extensive numerical case study visualizes the optimal asset allocation strategy and highlights the benefits.

5.2.1 The continuous-time mathematical model

We present the mathematical modeling of the pension plan dynamics associated with the decumulation phase, starting with the transition from accumulation to decumulation phase. Afterwards we elaborate on the continuous-time decumulation model and the optimization problem.

5.2.1.1 The post-retirement pension fund setup at starting time T: transition at retirement entry time

We establish the continuous-time framework. Time T denotes the initial time where the post-retirement pension fund is started. The total individual wealth $V_{ij}^{(\text{total})}(T)$ of client i in cohort j at time T is given by

$$V_{ij}^{(\text{total})}(T) := V_{ij}^{(\text{acc})}(T) + B_{ij}^{(\text{acc})}(T),$$

with individual primary fund account wealth $V_{ij}^{(\text{acc})}(T)$ and individual buffer account value $B_{ij}^{(\text{acc})}(T)$ at time T coming from the accumulation phase[11].

The pension payments (or rate) P_{ij} are defined at time T such that a certain capital coverage ratio (“Kapitaldeckungsgrad”)

[11] The buffer balance $B_{ij}^{(\text{acc})}(T)$ at time T coincides with the optimal terminal buffer account $C_{\alpha,B}^{\star}(T)$ in the accumulation phase.

$$CCR_{ij}^{(\text{total})}(T) := \frac{V_{ij}^{(\text{total})}(T)}{E_{ij}(T)}$$

is met, where $E_{ij}(T) := \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij} dt\right]$ denotes the time-T present value of all outstanding future pension payments to this specific client under a constant assumption and $\tau_{ij}^x(T)$ denotes the uncertain total lifetime of client i in cohort j who is aged x at time T. Regulations of “Bundesanstalt für Finanzdienstleistungsaufsicht (BaFin)” force

$$CCR_{ij}^{(\text{total})}(T) \in [100\%, 125\%] \tag{5.13}$$

for the initial post-retirement setup.

The total wealth that belongs to client i in cohort j is internally divided into an investment portfolio $V_{ij}^{(\text{inv})}(T)$ (invested into a portfolio mix of riskless and risky assets) and a buffer portfolio $V_{ij}^{(\text{buffer})}(T)$ (purely riskless investment by definition) such that

$$V_{ij}^{(\text{total})}(T) = V_{ij}^{(\text{inv})}(T) + V_{ij}^{(\text{buffer})}(T). \tag{5.14}$$

We propose

$$V_{ij}^{(\text{buffer})}(T) := \alpha \left(V_{ij}^{(\text{total})}(T) - E_{ij}(T)\right) \tag{5.15}$$

for some $\alpha \in [0, 1]$; the remainder builds the investment portfolio

$$\begin{aligned} V_{ij}^{(\text{inv})}(T) &:= V_{ij}^{(\text{total})}(T) - V_{ij}^{(\text{buffer})}(T) = V_{ij}^{(\text{total})}(T) - \alpha \left(V_{ij}^{(\text{total})}(T) - E_{ij}(T)\right) \\ &= \alpha E_{ij}(T) + (1-\alpha) V_{ij}^{(\text{total})}(T) \\ &= E_{ij}(T) + (1-\alpha)\left(V_{ij}^{(\text{total})}(T) - E_{ij}(T)\right). \end{aligned} \tag{5.16}$$

Thus we define the initial buffer balance to be the proportion α of the cushion $V_{ij}^{(\text{total})}(T) - E_{ij}(T)$, the remaining fund flows into the initial investment portfolio.

We would like to control the ratio for the investment portfolio such that all pension payments can be made by the investment portfolio and where besides there exists a buffer account that can help out in bad scenarios. For this sake, let us therefore denote

$$\bar{p} := CCR_{ij}^{(\text{inv})}(T) := \frac{V_{ij}^{(\text{inv})}(T)}{E_{ij}(T)} = \frac{\alpha E_{ij}(T) + (1-\alpha) V_{ij}^{(\text{total})}(T)}{E_{ij}(T)} = \alpha + (1-\alpha) CCR_{ij}^{(\text{total})}(T). \tag{5.17}$$

For instance, we could set $\bar{p} = 112.5\%$ identical for any client which is the center of the $CCR_{ij}^{(\text{total})}(T)$ corridor. From (5.17) the $CCR_{ij}^{(\text{total})}(T)$ can be reformulated to be

$$CCR_{ij}^{(\text{total})}(T) = \frac{\bar{p} - \alpha}{1 - \alpha}. \tag{5.18}$$

Moreover, from (5.17) it follows

$$E_{ij}(T) = \frac{(1-\alpha)V_{ij}^{(\text{total})}(T)}{\bar{p}-\alpha} \;\Leftrightarrow\; P_{ij} = \frac{1-\alpha}{\bar{p}-\alpha}\frac{V_{ij}^{(\text{total})}(T)}{\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}dt\right]}. \tag{5.19}$$

This means that when $\bar{p}$ and $V_{ij}^{(\text{total})}(T)$ are given, then the single control is on α that determines the pension payments P_{ij}. Moreover, we define the term

$$\begin{aligned} CCR_{ij}^{(\text{buffer})}(T) &:= \frac{V_{ij}^{(\text{buffer})}(T)}{E_{ij}(T)} = \frac{\alpha\left(V_{ij}^{(\text{total})}(T) - E_{ij}(T)\right)}{E_{ij}(T)} = \alpha CCR_{ij}^{(\text{total})}(T) - \alpha \\ &= \alpha\frac{\bar{p}-\alpha}{1-\alpha} - \alpha = \alpha\frac{\bar{p}-1}{1-\alpha}. \end{aligned} \tag{5.20}$$

Notice that as we define $\bar{p} := CCR_{ij}^{(\text{inv})}(T)$ in (5.17) to coincide for any customer, so do $CCR_{ij}^{(\text{total})}(T)$ and $CCR_{ij}^{(\text{buffer})}(T)$ which we learn from (5.18) and (5.20).

As $CCR_{ij}^{(\text{total})}(T) \in [100\%, 125\%]$ is required, i.e. it has to stay inside the boundaries, we must have $\frac{\bar{p}-\alpha}{1-\alpha} \in [100\%, 125\%]$. For economical reasons, suppose $\bar{p} \in [100\%, 125\%]$ and $\alpha \in [0, 1]$. Therefore we have the regulatory condition

$$\alpha \in \left[0\%, \frac{125\% - \bar{p}}{125\% - 100\%}\right] \tag{5.21}$$

on the control variable α. In particular, when $\bar{p} = 112.5\%$, then α can be selected out of the interval $[0\%, 50\%]$.

Remark 5.9 (Minimum pension payment)**.** *If there is a minimum pension payment P_{min} expected from this pension product, in view of* (5.19) *it must hold $\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}P_{min}dt\right] \leqslant \frac{V_{ij}^{(total)}(T)}{\bar{p}}$. The control variable α has to be chosen such that from* (5.19)*:*

$$\frac{(1-\alpha)V_{ij}^{(total)}(T)}{\bar{p}-\alpha} = E_{ij}(T) = \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}P_{ij}dt\right] \geqslant \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}P_{min}dt\right].$$

Notice that the left-hand side of the inequality is maximal under $\alpha = 0$, since $\frac{\partial\frac{1-\alpha}{\bar{p}-\alpha}}{\partial\alpha} = \frac{1-\bar{p}}{(\bar{p}-\alpha)^2} \leqslant 0$. Hence, the above condition introduces an extra upper constraint

$$\alpha \leqslant \frac{V_{ij}^{(total)}(T) - \bar{p}\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}P_{min}dt\right]}{V_{ij}^{(total)}(T) - \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)}P_{min}dt\right]}$$

on α that additionally depends on the initial wealth $V_{ij}^{(total)}(T)$ and the level P_{min}.

Let m denote the number of cohorts and m_j the number of clients in cohort j. The initial collective total wealth, investment portfolio and buffer account are given by

$$
\begin{aligned}
V_c^{(\text{total})}(T) &:= \sum_{j=1}^{m}\sum_{i=1}^{m_j} V_{ij}^{(\text{total})}(T),\\
V_c^{(\text{buffer})}(T) &:= \sum_{j=1}^{m}\sum_{i=1}^{m_j} V_{ij}^{(\text{buffer})}(T) = \sum_{j=1}^{m}\sum_{i=1}^{m_j} \alpha\left(V_{ij}^{(\text{total})}(T) - E_{ij}(T)\right) = \alpha\left(V_c^{(\text{total})}(T) - E_c(T)\right),\\
V_c^{(\text{inv})}(T) &:= \sum_{j=1}^{m}\sum_{i=1}^{m_j} V_{ij}^{(\text{inv})}(T) = \sum_{j=1}^{m}\sum_{i=1}^{m_j}\left(V_{ij}^{(\text{total})}(T) - V_{ij}^{(\text{buffer})}(T)\right) = V_c^{(\text{total})}(T) - V_c^{(\text{buffer})}(T)\\
&= E_c(T) + (1-\alpha)\left(V_c^{(\text{total})}(T) - E_c(T)\right),\\
CCR_c^{(\text{total})}(T) &:= \frac{V_c^{(\text{total})}(T)}{E_c(T)},\ CCR_c^{(\text{inv})}(T) := \frac{V_c^{(\text{inv})}(T)}{E_c(T)},\ CCR_c^{(\text{buffer})}(T) := \frac{V_c^{(\text{buffer})}(T)}{E_c(T)}
\end{aligned}
\tag{5.22}
$$

with $E_c(T) := \sum_{j=1}^{m}\sum_{i=1}^{m_j} E_{ij}(T) = \sum_{j=1}^{m}\sum_{i=1}^{m_j} \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}dt\right]$ denotes the time-T present value of all future pension payments to all clients in the pension fund under the assumption that all P_{ij} stay constant.

The properties $CCR_{ij}^{(\text{inv})}(T) \equiv \bar{p}$ and $CCR_{ij}^{(\text{total})}(T) \equiv \frac{\bar{p}-\alpha}{1-\alpha}$ are passed to the collective objects:

$$
\begin{aligned}
CCR_c^{(\text{inv})}(T) &= \frac{V_c^{(\text{inv})}(T)}{E_c(T)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{m_j} V_{ij}^{(\text{inv})}(T)}{\sum_{j=1}^{m}\sum_{i=1}^{m_j} E_{ij}(T)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{m_j} \bar{p}E_{ij}(T)}{\sum_{j=1}^{m}\sum_{i=1}^{m_j} E_{ij}(T)} = \bar{p} = CCR_{ij}^{(\text{inv})}(T),\\
CCR_c^{(\text{total})}(T) &= \frac{V_c^{(\text{total})}(T)}{E_c(T)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{m_j} V_{ij}^{(\text{total})}(T)}{\sum_{j=1}^{m}\sum_{i=1}^{m_j} E_{ij}(T)} = \frac{\sum_{j=1}^{m}\sum_{i=1}^{m_j} \frac{\bar{p}-\alpha}{1-\alpha}E_{ij}(T)}{\sum_{j=1}^{m}\sum_{i=1}^{m_j} E_{ij}(T)}\\
&= \frac{\bar{p}-\alpha}{1-\alpha} = CCR_{ij}^{(\text{total})}(T)
\end{aligned}
\tag{5.23}
$$

for all $j = 1, \ldots m$, $i = 1, \ldots, m_j$. Hence, the requested initial regulatory constraint $CCR_c^{(\text{total})}(T) \in [100\%, 125\%]$ on the collective total fund is satisfied iff it is satisfied for any single customer. In summary, under the proposed framework, both initial collective ratios $CCR_c^{(\text{total})}(T)$ as well as $CCR_c^{(\text{inv})}(T)$ coincide with the individual initial ratios $CCR_{ij}^{(\text{total})}(T)$ and $CCR_{ij}^{(\text{total})}(T)$.

5.2.1.2 Mechanism for times $t > T$

We place the following simplifying assumptions:

1. We consider a single client, i.e. $m = 1$, $m_1 = 1$, that is x years old at time T.
2. We consider a constant force of mortality, i.e. a constant rate $\lambda_x = \lambda_{x(ij)}$ with maximal possible total lifetime $T^\star$ (for instance $T^\star = 120$ years). Therefore, the survival probability of a client aged x at time T to survive from time T until time $t > T$ is given by $\mathbb{P}(\tau_{ij}^x(T) \geqslant t) = e^{-\lambda_x(t-T)}$, $\lambda_x > 0$. Moreover, we assume $\tau_{ij}^x(T)$ (uncertain total lifetime) to be independent of the

filtration $\mathbb{F}$. Within this model[12], we have for $s \geqslant t$:

$$\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s \middle| \tau_{ij}^x(T) \geqslant t\right) = \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s, \tau_{ij}^x(T) \geqslant t\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} = \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} = \frac{e^{-\lambda_x(s-T)}}{e^{-\lambda_x(t-T)}}$$
$$= e^{-\lambda_x(s-t)}.$$

3. We do not consider fund inflows, i.e. we assume that no new pensioners enter the fund.

To start with, we can derive the development of $E_c(t)$ under the assumption of constant P_{ij} (or until the first pension adjustment):

$$\begin{aligned}
E_c(T) = E_{ij}(T) = \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij} dt\right] &\overset{P_{ij}\text{ constant}}{=} P_{ij}\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} dt\right] \\
= P_{ij}\mathbb{E}\left[\int_T^{T^\star} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} \mathbb{1}_{\tau_{ij}^x(T)\geqslant t} dt\right] &\overset{\text{Fubini}}{=} P_{ij}\int_T^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(t)}{\tilde{Z}(T)} \mathbb{1}_{\tau_{ij}^x(T)\geqslant t}\right] dt \\
= P_{ij}\int_T^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(t)}{\tilde{Z}(T)} \middle| \tau_{ij}^x(T) \geqslant t\right] \mathbb{P}(\tau_{ij}^x(T) \geqslant t) dt \\
\overset{\tau_{ij}^x(T)\text{ independent of }\mathbb{F}}{=} P_{ij}\int_T^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(t)}{\tilde{Z}(T)}\right] \mathbb{P}(\tau_{ij}^x(T) \geqslant t) dt &= P_{ij}\int_T^{T^\star} e^{-r(t-T)} \mathbb{P}(\tau_{ij}^x(T) \geqslant t) dt \\
= P_{ij}\int_T^{T^\star} e^{-r(t-T)} e^{-\lambda_x(t-T)} dt &= P_{ij}\int_T^{T^\star} e^{-(r+\lambda_x)(t-T)} dt \\
= P_{ij}\frac{e^{-(r+\lambda_x)(t-T)}}{-(r+\lambda_x)}\Bigg|_{t=T}^{t=T^\star} &= \frac{P_{ij}}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star-T)}\right), \\
E_c(t) := E_{ij}(t) := \mathbb{E}\left[\int_t^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t)} P_{ij} ds \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right] \\
= P_{ij}\int_t^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)} \mathbb{1}_{\tau_{ij}^x(T)\geqslant s} \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right] ds \\
= P_{ij}\int_t^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)} \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right] \mathbb{P}\left(\tau_{ij}^x(T) \geqslant s \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right) ds
\end{aligned}$$

[12]A relaxation of a constant λ_x can be a time-dependent λ_x. Then, for $s \geqslant t$:

$$\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s \middle| \tau_{ij}^x(T) \geqslant t\right) = \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s, \tau_{ij}^x(T) \geqslant t\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} = \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} = \frac{e^{-\int_T^s \lambda_x(u)du}}{e^{-\int_T^t \lambda_x(u)du}} = e^{-\int_t^s \lambda_x(u)du}.$$

If $\lambda_x(t)$ increases in t, then we obtain the following reasonable relation for $t < s$ and $h > 0$:

$$\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t+h \middle| \tau_{ij}^x(T) \geqslant t\right) = e^{-\int_t^{t+h} \lambda_x(u)du} > e^{-\int_s^{s+h} \lambda_x(u)du} = \mathbb{P}\left(\tau_{ij}^x(T) \geqslant s+h \middle| \tau_{ij}^x(T) \geqslant s\right).$$

This is in contrast to

$$\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t+h \middle| \tau_{ij}^x(T) \geqslant t\right) = e^{-\lambda_x h} = \mathbb{P}\left(\tau_{ij}^x(T) \geqslant s+h \middle| \tau_{ij}^x(T) \geqslant s\right)$$

when λ_x is constant.

$$
\begin{aligned}
&= P_{ij} \int_t^{T^\star} \mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)} \middle| \mathcal{F}_t \right] \mathbb{P}\left(\tau_{ij}^x(T) \geqslant s \middle| \tau_{ij}^x(T) \geqslant t \right) ds \\
&= P_{ij} \int_t^{T^\star} e^{-r(s-t)} \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s, \tau_{ij}^x(T) \geqslant t\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} ds \\
&= P_{ij} \int_t^{T^\star} e^{-r(s-t)} \frac{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant s\right)}{\mathbb{P}\left(\tau_{ij}^x(T) \geqslant t\right)} ds \\
&= P_{ij} \int_t^{T^\star} e^{-r(s-t)} \frac{e^{-\lambda_x(s-T)}}{e^{-\lambda_x(t-T)}} ds = P_{ij} \int_t^{T^\star} e^{-r(s-t)} e^{-\lambda_x(s-t)} ds \\
&= \frac{P_{ij}}{r+\lambda_x} \left(1 - e^{-(r+\lambda_x)(T^\star - t)} \right), \ t > T.
\end{aligned}
$$

One can observe that $E_c(t)$, under constant P_{ij}, decreases in t; $E_c(t)$ denotes the time-t present value of all future pension payments after time t to this client under constant P_{ij}.

From time T on, the fund wealth is invested into a portfolio that consists of riskless and risky investments, the buffer account is purely invested riskless.

For $t > T$, the ratios under constant P_{ij} are given by

$$
CCR_c^{(\text{total})}(t) := \frac{V_c^{(\text{total})}(t)}{E_c(t)}, \ CCR_c^{(\text{inv})}(t) := \frac{V_c^{(\text{inv})}(t)}{E_c(t)}, \ CCR_c^{(\text{buffer})}(t) := \frac{V_c^{(\text{buffer})}(t)}{E_c(t)},
$$

analogously to the definitions in Eq. (5.22) for time T. Note that the ratios depend on the selected α value.

On top, the proposed buffer rate mechanism for $t \geqslant T$, which drives $V_c^{(\text{buffer})}(t)$, works as follows:

$$
\begin{aligned}
& V_c^{(\text{buffer})}(t) := \alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right), \\
\Leftrightarrow\ & V_c^{(\text{buffer})}(T) := \alpha \left(V_c^{(\text{total})}(T) - E_c(T) \right), \\
& dV_c^{(\text{buffer})}(t) := c_c^{(\text{buffer})}(t)dt := \alpha d \left(V_c^{(\text{total})}(t) - E_c(t) \right), \ t > T.
\end{aligned}
\tag{5.24}
$$

Implicitly, we assume that the buffer account is a simple account that pays no interest, i.e.

$$
\begin{aligned}
V_c^{(\text{buffer})}(t) &= V_c^{(\text{buffer})}(T) + \int_T^t dV_c^{(\text{buffer})}(s) = V_c^{(\text{buffer})}(T) + \int_T^t c_c^{(\text{buffer})}(s) ds \\
&= \alpha \left(V_c^{(\text{total})}(T) - E_c(T) \right) + \int_T^t \alpha d \left(V_c^{(\text{total})}(s) - E_c(s) \right) \\
&= \alpha \left(V_c^{(\text{total})}(T) - E_c(T) \right) + \left[\alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right) - \alpha \left(V_c^{(\text{total})}(T) - E_c(T) \right) \right] \\
&= \alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right).
\end{aligned}
\tag{5.25}
$$

A very beneficial feature of this buffer account and process is:

$$
CCR_c^{(\text{total})}(t) \searrow 100\% \ \Leftrightarrow \ V_c^{(\text{total})}(t) \searrow E_c(t) \ \Leftrightarrow \ V_c^{(\text{buffer})}(t) \searrow 0
$$

Moreover, the time-t wealth that corresponds to the investment portfolio is given by

$$V_c^{(\mathrm{inv})}(t) := V_c^{(\mathrm{total})}(t) - V_c^{(\mathrm{buffer})}(t) \overset{(5.25)}{=} E_c(t) + (1-\alpha)\left(V_c^{(\mathrm{total})}(t) - E_c(t)\right). \tag{5.26}$$

The pension rate P_{ij} is continuously withdrawn from the fund. The fund keeps the pension payments constant until time

$$t_1 := \inf\left\{t \in [T, \tau_{ij}^x(T)] : CCR_c^{(\mathrm{total})}(t) \notin [100\%, 125\%]\right\},$$

which is the first time when the funding ratio leaves the regulatory corridor. If time t_1 exists, i.e. if $t_1(\omega) < \tau_{ij}^x(T)$, then the fund adjusts the pension payment such that

$$CCR_c^{(\mathrm{total})}(t_1) = \frac{\bar{p} - \alpha}{1-\alpha}$$

holds again after the adjustment. Thus, we define

$$P_{ij}^{t_1^+} = \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\mathrm{total})}(t_1)}{\mathbb{E}\left[\int_{t_1}^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t_1)} ds \middle| \mathcal{F}_{t_1}, \tau_{ij}^x(T) \geqslant t_1\right]} = \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\mathrm{total})}(t_1)}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t_1)}\right)}$$

which means that we set back (restart)

$$\frac{V_c^{(\mathrm{total})}(t_1)}{E_c(t_1)} := \frac{\bar{p}-\alpha}{1-\alpha}$$

with $P_{ij}^{t_1^+}$ inside $E_c(t_1)$; $P_{ij}^{t_1^+}$ denotes the pension rate that is paid after time t_1 (decision made at time t_1). In general, we define the stopping times

$$t_n := \inf\left\{t \in (t_{n-1}, \tau_{ij}^x(T)] : CCR_c^{(\mathrm{total})}(t) \notin [100\%, 125\%] \middle| \tau_{ij}^x(T) \geqslant t_{n-1}\right\} \tag{5.27}$$

and adjust the pension rate at any t_n as for t_1. Between t_n and t_{n+1}, the pension rate is constant at $P_{ij}^{t_n^+}$. Further notice that the reset at time t_n not only implies $CCR_c^{(\mathrm{total})}(t_n) = \frac{\bar{p}-\alpha}{1-\alpha}$ but also

$$\begin{aligned} CCR_c^{(\mathrm{inv})}(t_n) &= \frac{V_c^{(\mathrm{inv})}(t_n)}{E_c(t_n)} = \frac{V_c^{(\mathrm{total})}(t_n) - V_c^{(\mathrm{buffer})}(t_n)}{E_c(t_n)} \\ &= CCR_c^{(\mathrm{total})}(t_n) - \frac{\alpha\left(V_c^{(\mathrm{total})}(t_n) - E_c(t_n)\right)}{E_c(t_n)} \\ &= (1-\alpha) CCR_c^{(\mathrm{total})}(t_n) + \alpha = (1-\alpha)\frac{\bar{p}-\alpha}{1-\alpha} + \alpha \\ &= \bar{p}. \end{aligned} \tag{5.28}$$

One could define $t_0 := T$. The (highly path-dependent) pension rate at time t is an $\mathcal{F}_t$-measurable random variable, w.l.o.g. $n(t) = \sup\{n \in \mathbb{N}_0 : t_n \leqslant t\}$, and is defined to be

$$
\begin{aligned}
P_{ij}(t) = P_{ij}^{t_{n(t)}^+} &= \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t_{n(t)})}{\mathbb{E}\left[\int_{t_{n(t)}}^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t_{n(t)})} ds \middle| \mathcal{F}_{t_{n(t)}}, \tau_{ij}^x(T) \geqslant t_{n(t)}\right]} \\
&= \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t_{n(t)})}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t_{n(t)})}\right)}.
\end{aligned} \tag{5.29}
$$

Due to a possible adjustment in the pension rate P_{ij} we need to adjust $E_c(t)$ at t_n as well:

$$
\begin{aligned}
E_c(t) &= \mathbb{E}\left[\int_t^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t)} P_{ij}(t) ds \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right] \\
&\overset{P_{ij}(t) \text{ is } \mathcal{F}_t\text{-measurable}}{=} P_{ij}(t)\mathbb{E}\left[\int_t^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t)} ds \middle| \mathcal{F}_t, \tau_{ij}^x(T) \geqslant t\right] \\
&= \frac{P_{ij}(t)}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t)}\right).
\end{aligned} \tag{5.30}
$$

Note that $P_{ij}(t)$ is a known constant under $\mathcal{F}_t$. Hence at time t we assume that the currently paid pension is going to be constant up to the uncertain death time. In other words, we assume that the pension fund is controlled at time t under the assumptions that time-t pension rate is applied during the entire remaining uncertain lifetime. This is also due to regulatory requirements as $P_{ij}(t)$ is used to evaluate the $CCR_c^{(\text{total})}(t)$.

Hence, the discretized version of the development of $P_{ij}(t)$ is as follows: If

$$
\begin{aligned}
\frac{V_c^{(\text{total})}(t+\Delta)}{E_c(t+\Delta|P_{ij}(t))} &= \frac{V_c^{(\text{total})}(t+\Delta)}{\mathbb{E}\left[\int_{t+\Delta}^{\tau_{ij}^x(T)} \frac{\tilde{Z}(u)}{\tilde{Z}(t+\Delta)} P_{ij}(t) du \middle| \mathcal{F}_{t+\Delta}, \tau_{ij}^x(T) \geqslant t+\Delta\right]} \\
&= \frac{V_c^{(\text{total})}(t+\Delta)}{\frac{P_{ij}(t)}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - (t+\Delta))}\right)} \\
&\in [100\%, 125\%],
\end{aligned}
$$

then $P_{ij}(t+\Delta) := P_{ij}(t)$; otherwise

$$
\begin{aligned}
P_{ij}(t+\Delta) &= \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t+\Delta)}{\mathbb{E}\left[\int_{t+\Delta}^{\tau_{ij}^x(T)} \frac{\tilde{Z}(u)}{\tilde{Z}(t+\Delta)} du \middle| \mathcal{F}_{t+\Delta}, \tau_{ij}^x(T) \geqslant t+\Delta\right]} \\
&= \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t+\Delta)}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - (t+\Delta))}\right)}.
\end{aligned}
$$

In summary,

$$
P_{ij}(t+\Delta) = \begin{cases} P_{ij}(t), \text{ if } V_c^{(\text{total})}(t+\Delta) \in [100\%, 125\%] \times \frac{P_{ij}(t)}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - (t+\Delta))}\right) \\ \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t+\Delta)}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - (t+\Delta))}\right)}, \text{ otherwise.} \end{cases} \tag{5.31}
$$

Eq. (5.31) tells that if the past performance of the total wealth is very high, then the pension for the next period gets larger. In opposite, if the performance of the total wealth in the preceding period was very low, the pension for the upcoming period gets reduced. Finally, the pension payment remains unchanged if the total wealth stays within some lower and upper boundary.

For the buffer portfolio, it follows $V_c^{(\text{buffer})}(t) \geqslant 0$ by construction as there is an immediate system reset whenever $CCR_c^{(\text{total})}(t)$ falls short 100%. In summary, we transfer money from the investment to the buffer portfolio if the surplus $V_c^{(\text{total})}(t) - E_c(t)$ increases, and vice versa. Hence, more stable pension payments P_{ij} are targeted with a lower probability of a decrease in P_{ij}.

Since $V_c^{(\text{buffer})}(t) = \alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right)$ for any $t \geqslant T$, we have

$$\begin{aligned} V_c^{(\text{inv})}(t) &\overset{(5.26)}{=} V_c^{(\text{total})}(t) - V_c^{(\text{buffer})}(t) = E_c(t) + (1-\alpha) \left(V_c^{(\text{total})}(t) - E_c(t) \right) \\ \Leftrightarrow V_c^{(\text{total})}(t) &= \frac{1}{1-\alpha} \left(V_c^{(\text{inv})}(t) - \alpha E_c(t) \right). \end{aligned} \tag{5.32}$$

Something similar holds for the buffer portfolio relation:

$$V_c^{(\text{buffer})}(t) = \alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right) \Leftrightarrow V_c^{(\text{total})}(t) = E_c(t) + \frac{1}{\alpha} V_c^{(\text{buffer})}(t).$$

The SDE for the total wealth

$$\begin{aligned} dV_c^{(\text{total})}(t) &= dV_c^{(\text{inv})}(t) + dV_c^{(\text{buffer})}(t) \\ &= dV_c^{(\text{inv})}(t) + d\left(\alpha \left(V_c^{(\text{total})}(t) - E_c(t) \right) \right) \\ &= dV_c^{(\text{inv})}(t) + \alpha dV_c^{(\text{total})}(t) - \alpha dE_c(t), \end{aligned}$$

therefore becomes

$$\begin{aligned} dV_c^{(\text{total})}(t) &= \frac{1}{1-\alpha} \left(dV_c^{(\text{inv})}(t) - \alpha dE_c(t) \right), \\ d\left(V_c^{(\text{total})}(t) - E_c(t) \right) &= \frac{1}{1-\alpha} \left(dV_c^{(\text{inv})}(t) - dE_c(t) \right). \end{aligned} \tag{5.33}$$

The SDE for the investment portfolio looks then like this, $t \geqslant t_{n(t)}$ and $n(t) = \sup\{n \in \mathbb{N}_0 : t_n \leqslant t\}$, i.e. $t_{n(t)}$ = biggest time point of pension adjustment $\leqslant t$:

$$\begin{aligned} dV_c^{(\text{inv})}(t) &= V_c^{(\text{inv})}(t) \left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1}) \right) dt + \hat{\pi}^{(\text{inv})}(t)' \sigma dW(t) \right] - P_{ij}(t)dt - c_c^{(\text{buffer})}(t)dt \\ &= V_c^{(\text{inv})}(t) \left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1}) \right) dt + \hat{\pi}^{(\text{inv})}(t)' \sigma dW(t) \right] - P_{ij}(t)dt \\ &\quad - \alpha d\left(V_c^{(\text{total})}(t) - E_c(t) \right) \\ &\overset{(5.33)}{=} V_c^{(\text{inv})}(t) \left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1}) \right) dt + \hat{\pi}^{(\text{inv})}(t)' \sigma dW(t) \right] - P_{ij}(t)dt \\ &\quad - \frac{\alpha}{1-\alpha} \left(dV_c^{(\text{inv})}(t) - dE_c(t) \right). \end{aligned}$$

Solving for $dV_c^{(\text{inv})}(t)$ leads to

$$\begin{aligned} dV_c^{(\text{inv})}(t) = (1-\alpha)V_c^{(\text{inv})}(t)\left[\left(r+\hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] \\ - (1-\alpha)P_{ij}(t)dt + \alpha dE_c(t). \end{aligned} \tag{5.34}$$

Notice that therefore

$$\begin{aligned} dV_c^{(\text{total})}(t) &\overset{(5.33)}{=} \frac{1}{1-\alpha}\left(dV_c^{(\text{inv})}(t) - \alpha dE_c(t)\right) \\ &= V_c^{(\text{inv})}(t)\left[\left(r+\hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] - P_{ij}(t)dt \end{aligned}$$

which visualizes that only the investment portfolio wealth is allocated to the risk-free and risky assets. Notice that by definition

$$\hat{\pi}^{(\text{buffer})}(t) \equiv 0. \tag{5.35}$$

Therefore, the total time-t risky exposure is given by $\hat{\pi}^{(\text{inv})}(t)V_c^{(\text{inv})}(t)$ which determines the following relative risky investment of the total wealth $\hat{\pi}^{(\text{total})}$:

$$\begin{aligned} &\hat{\pi}^{(\text{total})}(t)V_c^{(\text{total})}(t) = \hat{\pi}^{(\text{inv})}(t)V_c^{(\text{inv})}(t) \\ \Leftrightarrow\ &\hat{\pi}^{(\text{total})}(t) = \frac{V_c^{(\text{inv})}(t)}{V_c^{(\text{total})}(t)}\hat{\pi}^{(\text{inv})}(t) = \frac{CCR_c^{(\text{inv})}(t)}{CCR_c^{(\text{total})}(t)}\hat{\pi}^{(\text{inv})}(t) \\ &\qquad = \frac{V_c^{(\text{total})}(t) - V_c^{(\text{buffer})}(t)}{V_c^{(\text{total})}(t)}\hat{\pi}^{(\text{inv})}(t) \overset{(5.17)}{=} \frac{(1-\alpha)CCR_c^{(\text{total})}(t)+\alpha}{CCR_c^{(\text{total})}(t)}\hat{\pi}^{(\text{inv})}(t). \end{aligned} \tag{5.36}$$

Since $CCR_c^{(\text{total})}(t) \in [100\%, 125\%]$ by definition (otherwise reset of the system), we obtain $\frac{(1-\alpha)CCR_c^{(\text{total})}(t)+\alpha}{CCR_c^{(\text{total})}(t)} \in \left[\frac{(1-\alpha)1.25+\alpha}{1.25}, 1\right]$ (the ratio is monotone decreasing in $CCR_c^{(\text{total})}(t)$) and thus

$$\hat{\pi}^{(\text{total})}(t) = \frac{(1-\alpha)CCR_c^{(\text{total})}(t)+\alpha}{CCR_c^{(\text{total})}(t)}\hat{\pi}^{(\text{inv})}(t) \in \left[\frac{(1-\alpha)1.25+\alpha}{1.25}, 1\right] \times \hat{\pi}^{(\text{inv})}(t).$$

To prevent from leverage (i.e. $(\hat{\pi}^{(\text{total})}(t))'\mathbf{1} > 1$) for a given α, one has to restrict $(\hat{\pi}^{(\text{total})}(t))'\mathbf{1} \leqslant 1$. To exclude short-selling (i.e. $(\hat{\pi}^{(\text{total})}(t))_i < 0$ for some risky asset $i = 1, \ldots, N$), one has to restrict $\hat{\pi}^{(\text{inv})}(t) \geqslant \mathbf{0}$.

Note that at every reset time t_n, we have $CCR_c^{(\text{total})}(t_n) = \frac{\bar{p}-\alpha}{1-\alpha}$ by definition, in particular at initial time T. Thus,

$$\hat{\pi}^{(\text{total})}(t_n) = \frac{(1-\alpha)\frac{\bar{p}-\alpha}{1-\alpha}+\alpha}{\frac{\bar{p}-\alpha}{1-\alpha}}\hat{\pi}^{(\text{inv})}(t_n) = \frac{\bar{p}(1-\alpha)}{\bar{p}-\alpha}\hat{\pi}^{(\text{inv})}(t_n) = \underbrace{\frac{\bar{p}-\alpha\bar{p}}{\bar{p}-\alpha}}_{<1}\hat{\pi}^{(\text{inv})}(t_n)$$

which implies component-wise

$$\left|\hat{\pi}^{(\text{total})}(t_n)\right| = \left|\frac{\bar{p}-\alpha\bar{p}}{\bar{p}-\alpha}\right|\left|\hat{\pi}^{(\text{inv})}(t_n)\right| < \left|\hat{\pi}^{(\text{inv})}(t_n)\right|$$

if $\alpha > 0$. Notice that $\left|\hat{\pi}^{(\text{total})}(t_n)\right| < \left|\hat{\pi}^{(\text{inv})}(t_n)\right|$ is equivalent to

$$V_c^{(\text{buffer})}(t_n) > 0$$

because

$$1 > \frac{\left|\hat{\pi}^{(\text{total})}(t_n)\right|}{\left|\hat{\pi}^{(\text{inv})}(t_n)\right|} \overset{(5.36)}{=} \left|\frac{V_c^{(\text{total})}(t) - V_c^{(\text{buffer})}(t)}{V_c^{(\text{total})}(t)}\right| = \left|1 - \frac{V_c^{(\text{buffer})}(t)}{V_c^{(\text{total})}(t)}\right| \Rightarrow V_c^{(\text{buffer})}(t_n) > 0.$$

$V_c^{(\text{buffer})}(t_n) > 0$ also follows directly from construction of the system. Moreover,

$$\begin{aligned} dV_c^{(\text{total})}(t) &= V_c^{(\text{inv})}(t)\left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] - P_{ij}(t)dt \\ &\overset{(5.32)}{=} \left[E_c(t) + (1-\alpha)\left(V_c^{(\text{total})}(t) - E_c(t)\right)\right] \\ &\quad \times \left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] - P_{ij}(t)dt. \end{aligned} \tag{5.37}$$

Hence, the discretized version of the SDE for the total wealth $V_c^{(\text{total})}(t)$ is

$$\begin{aligned} V_c^{(\text{total})}(t+\Delta) = V_c^{(\text{total})}(t) &+ \left[E_c(t) + (1-\alpha)\left(V_c^{(\text{total})}(t) - E_c(t)\right)\right] \\ &\times \left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)\Delta + \hat{\pi}^{(\text{inv})}(t)'\sigma\sqrt{\Delta}Z\right] - P_{ij}(t)\Delta, \end{aligned} \tag{5.38}$$

where $Z \sim \mathcal{N}(0,1)$ is an N-dimensional vector of independent standard normal random variables.

Within our setup, we have the following for any $t \neq t_n$:

First,

$$dE_c(t) = E_c'(t)dt \overset{(5.30)}{=} -P_{ij}(t)e^{-(r+\lambda_x)(T^\star - t)}dt.$$

At $t = t_n$ there is a discontinuous up- or downwards jump in $E_c(t)$ due to the pension rate adjustment. Second, for any $t \neq t_n$, $dV_c^{(\text{inv})}(t)$ therefore becomes

$$\begin{aligned} dV_c^{(\text{inv})}(t) &\overset{(5.34)}{=} (1-\alpha)V_c^{(\text{inv})}(t)\left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] \\ &\quad - (1-\alpha)P_{ij}(t)dt + \alpha dE_c(t) \\ &\overset{(5.30)}{=} (1-\alpha)V_c^{(\text{inv})}(t)\left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] \\ &\quad - (1-\alpha)P_{ij}(t)dt - \alpha P_{ij}(t)e^{-(r+\lambda_x)(T^\star - t)}dt \\ &= (1-\alpha)V_c^{(\text{inv})}(t)\left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] \\ &\quad - \left(1 - \alpha + \alpha e^{-(r+\lambda_x)(T^\star - t)}\right)P_{ij}(t)dt. \end{aligned} \tag{5.39}$$

When we insert formula (5.29) for $P_{ij}(t)$ in (5.39), we obtain

$$\begin{aligned} dV_c^{(\text{inv})}(t) = (1-\alpha)V_c^{(\text{inv})}(t)&\left[\left(r + \hat{\pi}^{(\text{inv})}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}^{(\text{inv})}(t)'\sigma dW(t)\right] \\ &- \left(1 - \alpha + \alpha e^{-(r+\lambda_x)(T^\star - t)}\right)\frac{1-\alpha}{\bar{p} - \alpha}\frac{V_c^{(\text{total})}(t_{n(t)})}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t_{n(t)})}\right)}dt \end{aligned}$$

$$
\begin{aligned}
&\overset{(5.32)}{=} (1-\alpha)V_c^{(\mathrm{inv})}(t)\left[\left(r+\hat{\pi}^{(\mathrm{inv})}(t)'(\mu-r\mathbf{1})\right)dt+\hat{\pi}^{(\mathrm{inv})}(t)'\sigma dW(t)\right]\\
&\quad-\left(1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}\right)\frac{1-\alpha}{\bar{p}-\alpha}\frac{\frac{1}{1-\alpha}\left(V_c^{(\mathrm{inv})}(t_{n(t)})-\alpha E_c(t_{n(t)})\right)}{\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}dt\\
&=(1-\alpha)V_c^{(\mathrm{inv})}(t)\left[\left(r+\hat{\pi}^{(\mathrm{inv})}(t)'(\mu-r\mathbf{1})\right)dt+\hat{\pi}^{(\mathrm{inv})}(t)'\sigma dW(t)\right]\\
&\quad-V_c^{(\mathrm{inv})}(t_{n(t)})\frac{1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}}{(\bar{p}-\alpha)\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}dt\\
&\quad+E_c(t_{n(t)})\frac{\alpha\left(1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}\right)}{(\bar{p}-\alpha)\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}dt\\
&\overset{(5.28)}{=} (1-\alpha)V_c^{(\mathrm{inv})}(t)\left[\left(r+\hat{\pi}^{(\mathrm{inv})}(t)'(\mu-r\mathbf{1})\right)dt+\hat{\pi}^{(\mathrm{inv})}(t)'\sigma dW(t)\right]\\
&\quad-V_c^{(\mathrm{inv})}(t_{n(t)})\frac{1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}}{(\bar{p}-\alpha)\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}dt\\
&\quad+V_c^{(\mathrm{inv})}(t_{n(t)})\frac{\alpha\left(1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}\right)}{\bar{p}(\bar{p}-\alpha)\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}dt\\
&=(1-\alpha)V_c^{(\mathrm{inv})}(t)\left[\left(r+\hat{\pi}^{(\mathrm{inv})}(t)'(\mu-r\mathbf{1})\right)dt+\hat{\pi}^{(\mathrm{inv})}(t)'\sigma dW(t)\right]\\
&\quad-V_c^{(\mathrm{inv})}(t_{n(t)})\left\{\frac{1-\alpha+\alpha e^{-(r+\lambda_x)(T^\star-t)}}{(\bar{p}-\alpha)\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star-t_{n(t)})}\right)}\left[1-\frac{\alpha}{\bar{p}}\right]\right\}dt.
\end{aligned}
$$

Just for a better understanding let us consider the time point t_n for a pension shortening (the analogue applies for a pension increase): As already shown, the buffer account $V_c^{(\mathrm{buffer})}(t)$ tends to zero in such a situation. Moreover, at the time of the reset, the total wealth $V_c^{(\mathrm{total})}(t_n)$ is redistributed such that the investment wealth $V_c^{(\mathrm{inv})}(t_n)$ gets decreased while the buffer wealth $V_c^{(\mathrm{buffer})}(t_n)$ gets increased by the same amount. The total wealth $V_c^{(\mathrm{total})}(t_n)$ therefore follows a continuous process.

5.2.1.3 The continuous-time optimization problem

We target to maximize the client's expected utility coming from the sum of the received stochastic future cash flows, i.e. we optimize from the client's perspective as this is inevitable to create and design useful products and to attract customers. The risk-return tradeoff in the optimization depends on the type of applied utility function. Generally, the client desires higher expected cash flows given some risk; the buffer portfolio aims to decrease and limit the probability of a pension rate reduction within the decumulation phase. Therefore, the objective function, for a given buffer rule, that is to be maximized is given by

$$
\mathcal{J}(\pi^{(\mathrm{inv})}; v_0, c_c^{(\mathrm{buffer})}) = \mathbb{E}\left[\beta_1\int_T^{\tau_{ij}^x(T)\wedge\tilde{T}} U_1(t, P_{ij}(t))dt + \beta_2 U_2(V_c^{(\mathrm{total})}(\tau_{ij}^x(T)\wedge\tilde{T}))\right] \tag{5.40}
$$

for weights $\beta_1, \beta_2 \geqslant 0$, subject to

$$P_{ij}(t) \overset{(5.29)}{=} \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t_{n(t)})}{\mathbb{E}\left[\int_{t_{n(t)}}^{\tau_{ij}^x(T)} \frac{\tilde{Z}(s)}{\tilde{Z}(t_{n(t)})} ds \middle| \mathcal{F}_{t_{n(t)}}, \tau_{ij}^x(T) \geqslant t_{n(t)}\right]} = \frac{1-\alpha}{\bar{p}-\alpha} \frac{V_c^{(\text{total})}(t_{n(t)})}{\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star - t_{n(t)})}\right)}.$$

$\tilde{T} \in (T, T^\star]$ denotes the maximal planning horizon, $\tau_{ij}^x(T) \wedge \tilde{T}$ the minimum between this maximal horizon and the uncertain death time of the client. U_1, U_2 are utility functions.

Notice that $V_c^{(\text{total})}(t)$ depends on $\pi^{(\text{inv})}$ and $P_{ij}(t)$ is a function in $V_c^{(\text{total})}(t)$. The portfolio selection problem that arises is then

$$\mathcal{V}(v_0, c_c^{(\text{buffer})}) = \sup_{\pi^{(\text{inv})} \in \Lambda} \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}), \tag{5.41}$$

where the set Λ comprises all admissible investment strategies $\pi^{(\text{inv})}$ which fulfill the following requirements:

$A^{(5.2)}$: $\pi^{(\text{inv})} \in \tilde{\Lambda}'(v_0)$.

$B^{(5.2)}$: $\pi^{(\text{inv})}$ admits a unique solution to Eq. (5.37).

$C^{(5.2)}$: $\pi^{(\text{inv})}$ fulfills the associated budget constraint

$$\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)\wedge\tilde{T}} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t)dt + \frac{\tilde{Z}(\tau_{ij}^x(T)\wedge\tilde{T})}{\tilde{Z}(T)} V_c^{(\text{total})}(\tau_{ij}^x(T)\wedge\tilde{T})\right] \leqslant v_0, \tag{5.42}$$

where $V_c^{(\text{total})}(0) = v_0 > 0$ denotes the initial wealth of the post-retirement fund.

$E^{(5.2)}$: $\pi^{(\text{inv})}$ satisfies $\mathbb{E}\left[\beta_1 \int_T^{\tilde{T}} |U_1(t, P_{ij}(t))|dt + \beta_2 |U_2(V_c^{(\text{total})}(\tilde{T}))|\right] < \infty$.

It is noteworthy that all conditions are now on $\pi^{(\text{inv})}$, which is meaningful as this became the decision variable. Notice that neither the buffer $c_c^{(\text{buffer})}$ nor the pension $P_{ij}(t)$ are control variables; in particular the buffer rule and the pension adjustment mechanism were explicitly defined in the previous subsection. Therefore, both the buffer and the pension are omitted in the conditions summarized in Λ. Furthermore, it is striking that all conditions seem new. We briefly compare them with Λ from Problem (5.3) in the accumulation phase in Section 5.1: First, $F^{(4)}$ gets absent because no intermediate inflow process is considered. Further $D^{(4)}$, which describes a lower bound on the wealth, is removed as the definition of the pension and buffer rules automatically introduce a lower bound on the wealth by construction (pension gets reduced if the level of the wealth falls short the present value of the pension liabilities). In the extreme case where the total wealth approaches zero from above, so do the investment and buffer wealths, while the pension is reduced to a zero value. Moreover, $A^{(5.2)}$ replaces $A^{(4)}$ with the simply change to $\pi^{(\text{inv})}$, where the pension rate P_{ij} is taken as the consumption rate c in the self-financing property. $B^{(5.2)}$, $C^{(5.2)}$ and $E^{(5.2)}$ exchange with $B^{(5.1)}$, $C^{(5.1)}$ and $E^{(5.1)}$ because of the new SDE for the wealth, the new budget constraint and the new integrability condition, all due to the inclusion of an additional pension rate process which is different to the accumulation phase setting.

Generally, the objective function in (5.40) can be reformulated as

$$
\begin{aligned}
\mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) &= \mathbb{E}\left[\beta_1 \int_T^{\tau_{ij}^x(T)\wedge\tilde{T}} U_1(t, P_{ij}(t))dt + \beta_2 U_2(V_c^{(\text{total})}(\tau_{ij}^x(T)\wedge\tilde{T}))\right] \\
&= \beta_1 \mathbb{E}\left[\int_T^{\tilde{T}} U_1(t, P_{ij}(t))\mathbb{1}_{\tau_{ij}^x(T)\geqslant t} dt\right] \\
&\quad + \beta_2 \mathbb{E}\left[U_2(V_c^{(\text{total})}(\tilde{T}))\mathbb{1}_{\tau_{ij}^x(T)\geqslant\tilde{T}} + U_2(V_c^{(\text{total})}(\tau_{ij}^x(T)))\mathbb{1}_{\tau_{ij}^x(T)<\tilde{T}}\right] \\
&= \beta_1 \int_T^{\tilde{T}} \mathbb{E}\left[U_1(t, P_{ij}(t))\middle|\tau_{ij}^x(T)\geqslant t\right]\mathbb{P}\left(\tau_{ij}^x(T)\geqslant t\right)dt \\
&\quad + \beta_2 \mathbb{E}\left[U_2(V_c^{(\text{total})}(\tilde{T}))\middle|\tau_{ij}^x(T)\geqslant\tilde{T}\right]\mathbb{P}\left(\tau_{ij}^x(T)\geqslant\tilde{T}\right) \\
&\quad + \beta_2 \mathbb{E}\left[\int_T^{\tilde{T}} U_2(V_c^{(\text{total})}(t))d\mathbb{P}\left(\tau_{ij}^x(T)\leqslant t\right)\right] \\
&= \beta_1 \int_T^{\tilde{T}} \mathbb{E}\left[U_1(t, P_{ij}(t))\right]\mathbb{P}\left(\tau_{ij}^x(T)\geqslant t\right)dt \\
&\quad + \beta_2 \mathbb{E}\left[U_2(V_c^{(\text{total})}(\tilde{T}))\right]\mathbb{P}\left(\tau_{ij}^x(T)\geqslant\tilde{T}\right) \\
&\quad + \beta_2 \mathbb{E}\left[\int_T^{\tilde{T}} U_2(V_c^{(\text{total})}(t))\frac{d\mathbb{P}\left(\tau_{ij}^x(T)\leqslant t\right)}{dt}dt\right] \\
&= \beta_1 \int_T^{\tilde{T}} e^{-\lambda_x(t-T)}\mathbb{E}\left[U_1(t, P_{ij}(t))\right]dt + \beta_2 e^{-\lambda_x(\tilde{T}-T)}\mathbb{E}\left[U_2(V_c^{(\text{total})}(\tilde{T}))\right] \\
&\quad + \beta_2 \mathbb{E}\left[\int_T^{\tilde{T}} \lambda_x e^{-\lambda_x(t-T)} U_2(V_c^{(\text{total})}(t))dt\right] \\
&= \beta_1 \int_T^{\tilde{T}} e^{-\lambda_x(t-T)}\mathbb{E}\left[U_1(t, P_{ij}(t))\right]dt + \beta_2 \int_T^{\tilde{T}} \lambda_x e^{-\lambda_x(t-T)}\mathbb{E}\left[U_2(V_c^{(\text{total})}(t))\right]dt \\
&\quad + \beta_2 e^{-\lambda_x(\tilde{T}-T)}\mathbb{E}\left[U_2(V_c^{(\text{total})}(\tilde{T}))\right] \\
&= \mathbb{E}\left[\int_T^{\tilde{T}} \left\{\beta_1 e^{-\lambda_x(t-T)} U_1(t, P_{ij}(t)) + \beta_2 \lambda_x e^{-\lambda_x(t-T)} U_2(V_c^{(\text{total})}(t))\right\} dt\right] \\
&\quad + \mathbb{E}\left[\beta_2 e^{-\lambda_x(\tilde{T}-T)} U_2(V_c^{(\text{total})}(\tilde{T}))\right].
\end{aligned}
$$

If $\beta_1 = 0, \beta_2 > 0$, then there is only a bequest motive and the client desires a maximal possible wealth at death time respectively at $\tilde{T}$. In opposite, if $\beta_1 > 0, \beta_2 = 0$, then no bequest motive is considered and the agent tries to consume all the fund's wealth prior to her death in an optimal fashion. We focus on this setup ($\beta_2 := 0$, $\beta_1 := 1$ w.l.o.g.) in what follows and assume that in case of death no money is paid out to heirs. Therefore the objective function becomes

$$
\mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) = \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)} U_1(t, P_{ij}(t))dt\right].
$$

If we select U_1 to be an increasing concave utility function, this means that the client prefers a larger pension rate $P_{ij}(t)$, but an increase in the pension would lead to less additional satisfaction the larger the pension already is.

The budget constraints (5.42) in general reads

$$\begin{aligned}
v_0 &\geqslant \mathbb{E}\left[\int_T^{\tau_{ij}^x(T)\wedge\tilde{T}} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t)dt + \frac{\tilde{Z}(\tau_{ij}^x(T)\wedge\tilde{T})}{\tilde{Z}(T)} V_c^{(\text{total})}(\tau_{ij}^x(T)\wedge\tilde{T})\right] \\
&= \mathbb{E}\left[\int_T^{\tilde{T}} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t)\mathbb{1}_{\tau_{ij}^x(T)\geqslant t} dt\right] \\
&\quad + \mathbb{E}\left[\frac{\tilde{Z}(\tilde{T})}{\tilde{Z}(T)} V_c^{(\text{total})}(\tilde{T})\mathbb{1}_{\tau_{ij}^x(T)\geqslant\tilde{T}} + \frac{\tilde{Z}(\tau_{ij}^x(T))}{\tilde{Z}(T)} V_c^{(\text{total})}(\tau_{ij}^x(T))\mathbb{1}_{\tau_{ij}^x(T)<\tilde{T}}\right] \\
&= \mathbb{E}\left[\int_T^{\tilde{T}} \left\{e^{-\lambda_x(t-T)}\frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t) + \lambda_x e^{-\lambda_x(t-T)}\frac{\tilde{Z}(t)}{\tilde{Z}(T)} V_c^{(\text{total})}(t)\right\} dt + e^{-\lambda_x(\tilde{T}-T)}\frac{\tilde{Z}(\tilde{T})}{\tilde{Z}(T)} V_c^{(\text{total})}(\tilde{T})\right]
\end{aligned}$$

and becomes

$$v_0 \geqslant \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)}\frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t)dt\right].$$

if there is no bequest payment.

5.2.2 Discrete-time dynamic optimization

We now tackle the optimization problem from Section 5.2.1.3 for $\beta_2 = 0$ in discrete time. Recall that the continuous version is:

$$\begin{aligned}
&\mathcal{V}(v_0, c_c^{(\text{buffer})}) = \sup_{\pi^{(\text{inv})}\in\Lambda} \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) \\
&\text{s.t.} \begin{cases} \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) = \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)} U_1(t, P_{ij}(t))dt\right] \\ v_0 \geqslant \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)}\frac{\tilde{Z}(t)}{\tilde{Z}(T)} P_{ij}(t)dt\right] \end{cases}
\end{aligned} \tag{5.43}$$

The general target is to maximize the (expected utility of the) accumulated pension cash flows received by the customer. The buffer portfolio is established to reduce the probability of undesired pension shortenings and thus to keep the pension more stable.

Let the intertemporal utility function $U_1(t, p)$ admit the following form:

$$U_1(t, p) = e^{-\beta(t-T)}\tilde{U}_1(p), \tag{5.44}$$

where $\tilde{U}_1$ is a strictly increasing and concave utility function and $\beta \geqslant 0$ denotes the subjective discount rate[13] with utility discount factor $e^{-\beta(t-T)}$.

5.2.2.1 The discrete-time optimization problem

We now translate the problem into the corresponding problem that is discrete in time. For this sake, we divide the investment period $[T, \tilde{T}]$ into an equidistant grid with a distance of $\Delta > 0$ between

[13]We discount to time T which is in our case the initial time of the decumulation phase.

every grid point

$$t^{(k)} := T + \Delta \cdot k, \;\; k = 0, \dots, N_\Delta \tag{5.45}$$

with $N_\Delta := \frac{\tilde{T}-T}{\Delta}$, such that $t^{(0)} = T$ and $t^{(N_\Delta)} = \tilde{T}$ with $t^{(k+1)} - t^{(k)} \equiv \Delta$. We assume $N_\Delta = \frac{\tilde{T}-T}{\Delta} \overset{!}{\in} \mathbb{N}$ which for instance holds if $\tilde{T} - T \in \mathbb{N}$ is in full years and $\Delta \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{12}, \frac{1}{52}, \frac{1}{250}, \dots\}$, i.e. pension adjustments and rebalancing of the portfolio take place annually, semi-annually, quarterly, monthly, weekly, daily, etc..

Within the discrete optimization setting, the decision variable $\pi^{(\text{inv})}$ is determined at any grid time point $t^{(k)}$, but then kept constant for the period $[t^{(k)}, t^{(k+1)}) = [t^{(k)}, t^{(k)} + \Delta)$, i.e. $\pi^{(\text{inv})}(t^{(k)})$ is applied on the entire interval $[t^{(k)}, t^{(k+1)})$ and is updated again at time $t^{(k+1)}$. We further suppose that given time $t^{(k)}$, we are given $V_c^{(\text{total})}(t^{(k)})$ and thus furthermore know $P_{ij}(t^{(k)})$ that is constant in the time period $[t^{(k)}, t^{(k+1)})$. This implies that we know $E_c(t^{(k)})$ at time $t^{(k)}$.

Within this discrete framework, the objective function $\mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})})$ that is to be maximized translates to

$$\begin{aligned}
\mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) &= \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)} U_1(t, P_{ij}(t))dt\right] \\
&= \mathbb{E}\left[\int_T^{\tilde{T}} e^{-\lambda_x(t-T)} e^{-\beta(t-T)} \tilde{U}_1(P_{ij}(t))dt\right] \\
&= \mathbb{E}\left[\int_T^{\tilde{T}} e^{-(\lambda_x+\beta)(t-T)} \tilde{U}_1(P_{ij}(t))dt\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N_\Delta - 1} \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(t-T)} \tilde{U}_1(P_{ij}(t))dt\right] \\
&\overset{P_{ij}(t) \equiv P_{ij}(t^{(k)}) \text{ on } [t^{(k)}, t^{(k+1)})}{=} \mathbb{E}\left[\sum_{k=0}^{N_\Delta - 1} \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(t-T)} \tilde{U}_1(P_{ij}(t^{(k)}))dt\right].
\end{aligned}$$

Let $\mathbb{A} := \{a \in [0,1]^N : \; a'\mathbf{1} \leqslant 1\}$ denote the set that includes all possible portfolio weights at a given time point, contrary to Λ denoting the set that consists of all admissible continuous-time investment strategies. The definition of $\mathbb{A}$ ensures the following: $a \geqslant \mathbf{0}$ prevents from short-selling of a risky asset, $a'\mathbf{1} \leqslant 1$ rules out leverage. In the case of a single asset class ($N = 1$), the set reduces to $\mathbb{A} = [0,1]$. The optimization problem in discrete time then reads[14]

$$\begin{aligned}
&\mathcal{V}(v_0, c_c^{(\text{buffer})}) = \sup_{\hat{\pi}^{(\text{inv})}(t^{(0)}), \dots, \hat{\pi}^{(\text{inv})}(t^{(N_\Delta)}) \in \mathbb{A}} \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) \\
&\text{s.t. } \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}) = \mathbb{E}\left[\sum_{k=0}^{N_\Delta - 1} \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(t-T)} \tilde{U}_1(P_{ij}(t^{(k)}))dt\right].
\end{aligned} \tag{5.46}$$

[14] We consider $\hat{\pi}^{(\text{inv})}$ as decision variable instead of $\pi^{(\text{inv})}$, since there is a unique relation between $\hat{\pi}^{(\text{inv})}$ and $\pi^{(\text{inv})}$: $\pi^{(\text{inv})} = \left(\pi_0^{(\text{inv})}, \left(\hat{\pi}^{(\text{inv})}\right)'\right) = \left(1 - \left(\hat{\pi}^{(\text{inv})}\right)' \mathbf{1}, \left(\hat{\pi}^{(\text{inv})}\right)'\right)$.

Now define for $k \in \{0, \ldots, N_\Delta\}$:

$$\begin{aligned} V_{(k)} &:= V_c^{(\text{total})}(t^{(k)}), \\ P_{(k)} &:= P_{ij}(t^{(k)}), \\ S_{(k)} &:= \left(V_{(k)}, P_{(k)}\right), \\ a_k &:= \hat{\pi}^{(\text{inv})}(t^{(k)}) = \hat{\pi}^{(\text{inv})}(t^{(k)}; S_{(k)}), \\ \mathcal{F}_k &:= \mathcal{F}_{t^{(k)}}. \end{aligned}$$

$S_{(k)}$ denotes the two-dimensional state space with $S_{(k)} \in \mathbb{R}_+^2$. a_k is the action (or control variable) for period $[t^{(k)}, t^{(k+1)})$, it is the risky relative investment strategy of the investment portfolio, with $a_k \in \mathbb{A}$. $\mathcal{F}_k$ contains all the information accumulated from time $t = 0$ to time $t = t^{(k)}$, which also includes the information $\left(V_c^{(\text{total})}(t^{(k)}), P_{ij}(t^{(k)})\right) = \left(V_{(k)}, P_{(k)}\right) = S_{(k)}$.

We now address the stochastic control problem in (5.46); we assume a Markov model, i.e. the objects at time $t^{(k+1)}$ depend only on the respective objects at time $t^{(k)}$ but not on all preceding times $t^{(0)}, \ldots, t^{(k-1)}$. Hence, the information $S_{(k)}$ at time $t^{(k)}$ is sufficient, $\mathcal{F}_k$ contains additional but unnecessary information. In view of the dynamic programming principle (or Bellman's principle) (cf. Bellman (1952), Bellman (1957), Bellman (1958), Nisio (2015), or Escobar et al. (2019) for an application), we first define the following time-t objective functional for $t \in \{t^{(k)} : k = 0, \ldots, N_\Delta\}$ that is to be maximized; for convenience let $t = t^{(k)}$ for some $k \in \{0, \ldots, N_\Delta - 1\}$:

$$\begin{aligned} &\mathcal{V}_k(S_{(k)}; c_c^{(\text{buffer})}) = \sup_{a_k, \ldots, a_{N_\Delta - 1} \in \mathbb{A}} \mathcal{J}_k(a; S_{(k)}, c_c^{(\text{buffer})}) \\ &\text{s.t. } \mathcal{J}_k(a; S_{(k)}, c_c^{(\text{buffer})}) = \mathbb{E}\left[\sum_{i=k}^{N_\Delta - 1} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x + \beta)(u - t^{(k)})} \tilde{U}_1(P_{(i)}) du \middle| S_{(k)}\right] \end{aligned} \tag{5.47}$$

with

$$\mathcal{J}_{N_\Delta}(a; S_{(N_\Delta)}, c_c^{(\text{buffer})}) = 0. \tag{5.48}$$

As we have a Markov model, we search for the optimal asset allocation decision rule $a_k^\star = \hat{\pi}^{\star(\text{inv})}(t^{(k)}) = f_k(S_{(k)})$ at time $t^{(k)}$. Note

$$\mathcal{J}_0(a; S_{(0)}, c_c^{(\text{buffer})}) = \mathcal{J}(\pi^{(\text{inv})}; v_0, c_c^{(\text{buffer})}),$$

where $S_{(0)} = (V_{(0)}, P_{(0)}) \overset{(5.29)}{=} \left(v_0, \frac{1-\alpha}{\bar{p}-\alpha} \frac{v_0}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - T)}\right)}\right)$. We further find that

$$\begin{aligned} \mathcal{J}_k(a; S_{(k)}, c_c^{(\text{buffer})}) &\overset{(5.47)}{=} \mathbb{E}\left[\sum_{i=k}^{N_\Delta - 1} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x + \beta)(u - t^{(k)})} \tilde{U}_1(P_{(i)}) du \middle| S_{(k)}\right] \\ &= \mathbb{E}\left[\int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x + \beta)(u - t^{(k)})} \tilde{U}_1(P_{(k)}) du \middle| S_{(k)}\right] \\ &\quad + \mathbb{E}\left[\sum_{i=k+1}^{N_\Delta - 1} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x + \beta)(u - t^{(k)})} \tilde{U}_1(P_{(i)}) du \middle| S_{(k)}\right] \end{aligned}$$

$$
\begin{aligned}
&\overset{P_{(k)}\text{ known at }t^{(k)}}{=} \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du \\
&\quad + \mathbb{E}\left[\sum_{i=k+1}^{N_\Delta-1}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(i)})du \middle| S_{(k)}\right] \\
&= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du \\
&\quad + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})}\mathbb{E}\left[\sum_{i=k+1}^{N_\Delta-1}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-t^{(k+1)})}\tilde{U}_1(P_{(i)})du \middle| S_{(k)}\right] \\
&\overset{\text{tower rule}}{=} \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \\
&\quad \times \mathbb{E}\left[\underbrace{\mathbb{E}\left[\sum_{i=k+1}^{N_\Delta-1}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-t^{(k+1)})}\tilde{U}_1(P_{(i)})du \middle| S_{(k+1)}\right]}_{=\mathcal{J}_{k+1}(a;S_{(k+1)},c_c^{(\text{buffer})})} \middle| S_{(k)}\right] \\
&= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \\
&\quad \times \mathbb{E}\left[\mathcal{J}_{k+1}(a;S_{(k+1)},c_c^{(\text{buffer})}) \middle| S_{(k)}\right].
\end{aligned}
$$

Hence, for the value function it holds

$$
\begin{aligned}
\mathcal{V}_k(S_{(k)};c_c^{(\text{buffer})}) &= \sup_{a_k,\dots,a_{N_\Delta-1}\in\mathbb{A}} \mathcal{J}_k(a;S_{(k)},c_c^{(\text{buffer})}) \\
&= \sup_{a_k,\dots,a_{N_\Delta-1}\in\mathbb{A}} \Bigg\{ \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du \\
&\quad + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})}\mathbb{E}\left[\mathcal{J}_{k+1}(a;S_{(k+1)},c_c^{(\text{buffer})}) \middle| S_{(k)}\right] \Bigg\} \\
&= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du \\
&\quad + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \sup_{a_k,\dots,a_{N_\Delta-1}\in\mathbb{A}} \left\{\mathbb{E}\left[\mathcal{J}_{k+1}(a;S_{(k+1)},c_c^{(\text{buffer})}) \middle| S_{(k)}\right]\right\} \\
&= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \\
&\quad \times \sup_{a_k\in\mathbb{A}}\left\{ \sup_{a_{k+1},\dots,a_{N_\Delta-1}\in\mathbb{A}} \left\{\mathbb{E}\left[\mathbb{E}\left[\mathcal{J}_{k+1}(a;S_{(k+1)},c_c^{(\text{buffer})}) \middle| S_{(k+1)}\right] \middle| S_{(k)}\right]\right\}\right\} \\
&= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du \\
&\quad + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \sup_{a_k\in\mathbb{A}}\left\{\mathbb{E}\left[\mathcal{V}_{k+1}(S_{(k+1)};c_c^{(\text{buffer})}) \middle| S_{(k)}\right]\right\},
\end{aligned}
\qquad (5.49)
$$

,

where the last equality (exchangeability of sup and $\mathbb{E}$) follows from Bellman (1952). Let $Z \sim \mathcal{N}(0,1)$ be a multi-dimensional vector of independent standard normal random variables of dimension N (= number of asset classes). Z represents the stochastic part of the fund return in period $[t^{(k)}, t^{(k+1)})$ (independent in every period), i.e. Z is the risk driver or risk factor that drives the fund's performance besides the deterministic drift part. According to Equations (5.38) and (5.31), the transition function T_B for $S_{(k)} \mapsto S_{(k+1)}$ is

$$\begin{aligned} &T_B : \mathbb{R}_+^2 \times \mathbb{A} \times \mathbb{R} \to \mathbb{R}_+^2, \\ &\big(S_{(k)}, a_k, Z\big) \mapsto S_{(k+1)} = T_B(S_{(k)}, a_k, Z) = \begin{pmatrix} V_{(k+1)}(S_{(k)}, a_k, Z) \\ P_{(k+1)}(S_{(k)}, a_k, Z) \end{pmatrix} = \begin{pmatrix} T_B^{(V)}(S_{(k)}, a_k, Z) \\ T_B^{(P)}(S_{(k)}, a_k, Z) \end{pmatrix}, \end{aligned} \tag{5.50}$$

where

$$\begin{aligned} V_{(k+1)} &= T_B^{(V)}(S_{(k)}, a_k, Z) \\ &\overset{(5.38)}{=} V_{(k)} + \Big[E_c(t^{(k)}|S_{(k)}) + (1-\alpha)\Big(V_{(k)} - E_c(t^{(k)}|S_{(k)})\Big)\Big]\Big[\big(r + a_k'(\mu - r\mathbf{1})\big)\Delta + a_k'\sigma\sqrt{\Delta}Z\Big] \\ &\quad - P_{(k)}\Delta \end{aligned}$$

and

$$P_{(k+1)} = T_B^{(P)}(S_{(k)}, a_k, Z) \overset{(5.31)}{=} \begin{cases} P_{(k)} & , \text{ if } \frac{T_B^{(V)}(S_{(k)}, a_k, Z)}{E_c(t^{(k+1)}|S_{(k)})} \in [100\%, 125\%] \\ \frac{1-\alpha}{\bar{p}-\alpha} \frac{T_B^{(V)}(S_{(k)}, a_k, Z)}{\frac{1}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t^{(k+1)})}\right)}, & \text{otherwise} \end{cases}$$

with

$$E_c(t^{(j)}|S_{(i)}) = \frac{P_{(i)}}{r + \lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star - t^{(j)})}\right)$$

for $j \geqslant i$. Notice that this is only an approximation as we use the discretized (and not the original continuous) formulas. Due to the stochastic risk driver of the fund return Z being a standard normal random variable, Eq. (5.38) directly implies that

$$V_{(k+1)}|S_{(k)}, a_k \overset{\text{approx.}}{\sim} \mathcal{N}\left(\mu_k, \sigma_k^2\right) \tag{5.51}$$

with

$$\begin{aligned} \mu_k &= V_{(k)} + \Big\{\Big[E_c(t^{(k)}|S_{(k)}) + (1-\alpha)\Big(V_{(k)} - E_c(t^{(k)}|S_{(k)})\Big)\Big]\big(r + a_k'(\mu - r\mathbf{1})\big) - P_{(k)}\Big\}\Delta, \\ \sigma_k^2 &= \Big[E_c(t^{(k)}|S_{(k)}) + (1-\alpha)\Big(V_{(k)} - E_c(t^{(k)}|S_{(k)})\Big)\Big]^2 \|a_k'\sigma\|^2\Delta. \end{aligned}$$

We further have

$$
\begin{aligned}
V_c^{(\text{buffer})}(t^{(k)}) &= \alpha\left(V_{(k)} - E_c(t^{(k)}|S_{(k)})\right),\\
V_c^{(\text{inv})}(t^{(k)}) &= V_{(k)} - V_c^{(\text{buffer})}(t^{(k)}) = E_c(t^{(k)}|S_{(k)}) + (1-\alpha)\left(V_{(k)} - E_c(t^{(k)}|S_{(k)})\right),\\
CCR_c^{(\text{total})}(t^{(k)}) &= \frac{V_{(k)}}{E_c(t^{(k)}|S_{(k)})},\\
\hat{\pi}^{(\text{total})}(t^{(k)}) &\overset{(5.36)}{=} \frac{V_c^{(\text{inv})}(t^{(k)})}{V_{(k)}}a_k = \frac{V_{(k)} - V_c^{(\text{buffer})}(t^{(k)})}{V_{(k)}}a_k \overset{(5.36)}{=} \frac{(1-\alpha)CCR_c^{(\text{total})}(t^{(k)}) + \alpha}{CCR_c^{(\text{total})}(t^{(k)})}a_k.
\end{aligned}
\tag{5.52}
$$

5.2.2.2 Bellman equation

In view of Eq. (5.49), the corresponding Bellman equation to Problem (5.47) for the finite-horizon $\tilde{T} = t^{(N_\Delta)} < \infty$ (note: $\tilde{T} \in (T, T^\star]$) is given by

$$
\begin{aligned}
\mathcal{V}_{N_\Delta}(S_{(N_\Delta)}; c_c^{(\text{buffer})}) &= 0,\ k = N_\Delta,\\
\mathcal{V}_k(S_{(k)}; c_c^{(\text{buffer})}) &= \sup_{a_k \in \mathbb{A}}\left\{r_k(S_{(k)}, a_k) + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})}\mathbb{E}\left[\mathcal{V}_{k+1}(S_{(k+1)}; c_c^{(\text{buffer})})\middle| S_{(k)}\right]\right\},\\
&\quad k \in \{N_\Delta - 1, \dots, 0\}
\end{aligned}
\tag{5.53}
$$

with one-period or one-stage reward function in period $[t^{(k)}, t^{(k+1)})$

$$
\begin{aligned}
r_k(S_{(k)}, a_k) \equiv r_k(S_{(k)}) &= \int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)(u-t^{(k)})}\tilde{U}_1(P_{(k)})du\\
&= e^{(\lambda_x+\beta)t^{(k)}}\tilde{U}_1(P_{(k)})\int_{t^{(k)}}^{t^{(k+1)}} e^{-(\lambda_x+\beta)u}du\\
&= e^{(\lambda_x+\beta)t^{(k)}}\tilde{U}_1(P_{(k)})\left[\frac{e^{-(\lambda_x+\beta)u}}{-(\lambda_x+\beta)}\bigg|_{u=t^{(k)}}^{u=t^{(k+1)}}\right]\\
&= e^{(\lambda_x+\beta)t^{(k)}}\tilde{U}_1(P_{(k)})\left[\frac{e^{-(\lambda_x+\beta)t^{(k+1)}} - e^{-(\lambda_x+\beta)t^{(k)}}}{-(\lambda_x+\beta)}\right]\\
&= \tilde{U}_1(P_{(k)})\left[\frac{e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} - 1}{-(\lambda_x+\beta)}\right]\\
&= \frac{1}{\lambda_x+\beta}\left(1 - e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})}\right)\tilde{U}_1(P_{(k)})\\
&= \frac{1}{\lambda_x+\beta}\left(1 - e^{-(\lambda_x+\beta)\Delta}\right)\tilde{U}_1(P_{(k)}),
\end{aligned}
\tag{5.54}
$$

that describes the contribution or reward to the client's satisfaction in the period $[t^{(k)}, t^{(k+1)})$ linked to the pension $P_{(k)}$ that is paid out in $[t^{(k)}, t^{(k+1)})$ independently of the action or applied relative risky investment strategy $a_k = \hat{\pi}^{(\text{inv})}(t^{(k)})$. Since the value for $r_k(S_{(k)})$ is already known at time

$t^{(k)}$, i.e. it is deterministic and independent of the decision a_k, we obtain

$$\mathcal{V}_k(S_{(k)}; c_c^{(\text{buffer})}) = r_k(S_{(k)}) + e^{-(\lambda_x+\beta)\Delta} \sup_{a_k \in \mathbb{A}} \left\{ \mathbb{E}\left[\mathcal{V}_{k+1}(S_{(k+1)}; c_c^{(\text{buffer})}) \middle| S_{(k)} \right] \right\}, \ k \in \{N_\Delta - 1, \ldots, 0\} \tag{5.55}$$

in the second part of the Bellman equation (5.53).

The original discrete-time dynamic optimization problem can then be solved by backwards induction of the Bellman equation (5.53). The optimal decision rule or policy $a_k^\star = f_k(S_{(k)})$ needs to be determined in any step and for every possible state $S_{(k)}$ backwards in time. By this, we further receive the optimal total risky relative portfolio process $\hat{\pi}^{\star(\text{total})} = \hat{\pi}^{(\text{total})}(a_k^\star)$ through (5.52).

5.2.2.3 Extension to a single cohort

Let us consider a cohort of clients grouped by age ($x = x(j)$ years old at time T) that has m_j members and aims to solve the discrete-time dynamic optimization problem in Section 5.2.2.1. We manage the total cohort portfolio and the pension collectively and thus define

$$P_j(t) := \sum_{i=1}^{m_j} P_{ij}(t)$$

to be the sum of all pension payments $P_{ij}(t)$ connected to all members i in cohort j at time t. Since we consider one cohort, there are no intertemporal inflows into the model. We assume that there is no bequest paid out in the case of a cohort member's death. Further we re-interpret the mortality model: The survival probability

$$\mathbb{P}\left(\tau_{ij}^{x(j)}(T) \geqslant s \middle| \tau_{ij}^{x(j)}(T) \geqslant t \right) = e^{-\lambda_{x(j)}(s-t)}, \ s \geqslant t,$$

for a single client is now regarded as the average relative survival frequency of the cohort, i.e. we assume $e^{-\lambda_{x(j)}(s-t)}$ to be the average proportion of clients in cohort j that survive from time t to time s. This comes from the following observation: Let $\tau_{ij}^{x(j)}(T)$ denote the uncertain remaining lifetime of client i in cohort j which is identically distributed among all clients in one cohort. Then the uncertain proportion of survivors from time t to s in the cohort is described by the random variable $\frac{\sum_{i=1}^{m_j} \mathbb{1}_{\{\tau_{ij}^{x(j)}(T) \geqslant s | \tau_{ij}^{x(j)}(T) \geqslant t\}}}{m_j}$. Its expectation is

$$\begin{aligned}
\mathbb{E}\left[\frac{\sum_{i=1}^{m_j} \mathbb{1}_{\{\tau_{ij}^{x(j)}(T) \geqslant s | \tau_{ij}^{x(j)}(T) \geqslant t\}}}{m_j} \right] &= \frac{\sum_{i=1}^{m_j} \mathbb{E}\left[\mathbb{1}_{\{\tau_{ij}^{x(j)}(T) \geqslant s | \tau_{ij}^{x(j)}(T) \geqslant t\}} \right]}{m_j} \\
&= \frac{\sum_{i=1}^{m_j} \mathbb{P}(\tau_{ij}^{x(j)}(T) \geqslant s | \tau_{ij}^{x(j)}(T) \geqslant t)}{m_j} \\
&\overset{\tau_{ij}^{x(j)}(T) \text{ identically distributed } \forall i \in \{1,\ldots,m_j\}}{=} \frac{\sum_{i=1}^{m_j} \mathbb{P}(\tau_{1j}^{x(j)}(T) \geqslant s | \tau_{1j}^{x(j)}(T) \geqslant t)}{m_j}
\end{aligned}$$

$$= \frac{m_j \mathbb{P}(\tau_{1j}^{x(j)}(T) \geqslant s | \tau_{1j}^{x(j)}(T) \geqslant t)}{m_j}$$
$$= \mathbb{P}(\tau_{1j}^{x(j)}(T) \geqslant s | \tau_{1j}^{x(j)}(T) \geqslant t) = e^{-\lambda_{x(j)}(s-t)}.$$

In other words, the average cohort proportion of surviving clients equals the survival probability of a single client in this cohort. Moreover, since the number of customers in cohort j reduces continuously in time due to deaths of cohort members, the average pension cash flows $P_j(t)$ needs to be adjusted to

$$P_j(s) := e^{-\lambda_{x(j)}(s-t)} P_j(t), \;\; s \geqslant t,$$

assuming that all single-client pensions remain constant, and only those connected to a client's death are removed. We define $P_{(k)} := P_j(t^{(k)})$ in the state $S_{(k)} = \big(V_{(k)}, P_{(k)}\big)$, where $V_{(k)}$ denotes the total collective wealth of cohort j. For this reason, we have to modify the transition function $T_B^{(P)}$ for the pension $P_{(k)}$ as follows:

$$P_{(k+1)} = T_B^{(P)}(S_{(k)}, a_k, Z) = \begin{cases} e^{-\lambda_{x(j)}(t^{(k+1)} - t^{(k)})} P_{(k)}, \text{ if } \frac{T_B^{(V)}(S_{(k)}, a_k, Z)}{E_c(t^{(k+1)} | S_{(k)})} \in [100\%, 125\%] \\ \frac{1-\alpha}{\bar{p}-\alpha} \frac{T_B^{(V)}(S_{(k)}, a_k, Z)}{\frac{1}{r+\lambda_{x(j)}} \left(1 - e^{-(r+\lambda_{x(j)})(T^\star - t^{(k+1)})}\right)} \quad\quad , \text{ otherwise} \end{cases}$$

with

$$E_c(t^{(j)} | S_{(i)}) = \frac{e^{-\lambda_{x(j)}(t^{(j)} - t^{(i)})} P_{(i)}}{r + \lambda_{x(j)}} \left(1 - e^{-(r+\lambda_{x(j)})(T^\star - t^{(j)})}\right)$$

for $j \geqslant i$. If $CCR_c^{(\text{total})}(t^{(k+1)})$ stays inside its pre-defined corridor, the collective cohort pension $P_{(k+1)}$ at time $t^{(k+1)}$ decreases with rate $\lambda_{x(j)}$ (on average) due to the deaths of cohort members. At the same time, the individual pensions of clients that survived until time $t^{(k+1)}$ remain untouched, i.e. stable. Thus, $P_{(k+1)} = e^{-\lambda_{x(j)}(t^{(k+1)} - t^{(k)})} P_{(k)}$ indicates a stable, constant individual pension $P_{ij}(t^{(k+1)}) = P_{ij}(t^{(k)})$ for those clients in the cohort that are still alive at time $t^{(k+1)}$. Using this notation, the Bellman equation in (5.53) and the one-period reward function in (5.54) remain the same[15]. Finally, due to this definition, E_c decreases in time (death of cohort members). As there are no bequest payments, this implies that the $CCR_c^{(\text{total})}$ is more likely to cross the 125%-border and less likely to fall short the 100%-border compared to the single-client model if the same investment strategy is applied.

Remark 5.10. *It is remarkable that the probabilities for future reductions of individual customer pensions in the cohort model are smaller than the ones in the single-client model, whereas the probability of future pension enhancements in the cohort model are larger than in the single-client model, if the same investment strategy is applied. The economic reason is that the wealth of a client in the cohort that died in the previous period remains in the collective portfolio and is not paid out*

[15]Note: The rate $\lambda_{x(j)}$ decreases the average total sum of pensions $P_{(k+1)}$ that is to be paid in the next period because some fraction of the clients in the cohort died during the previous period. One could use this rate to increase the individual pensions $P_{ij}(t^{(k+1)})$ of the clients that survived step-by-step while keeping $P_{(k+1)}$ of the cohort constant. In this case, we find ourselves in the optimization framework (single-client perspective) of Section 5.2.2.1 and 5.2.2.2.

to heirs, while the cohort-related collective pension declines. Therefore, the survivors in the cohort benefit from the death of a cohort member.

5.2.2.4 Implementation

We now describe the general implementation steps of the solution procedure of the Bellman equation in (5.53) and thereafter apply a specific setting to find the optimal decision variables a_k, $k = 0, \dots, N_\Delta$.

We choose a utility function with hyperbolic absolute risk aversion (HARA), cf. Definition 2.12 in Section 2.4, as intertemporal consumption utility function (consumption = pension):

$$\tilde{U}_1(p) := \hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(p-F)\right)^b, \; U_1(t,p) = e^{-\beta(t-T)}\tilde{U}_1(p) = e^{-\beta(t-T)}\hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(p-F)\right)^b \tag{5.56}$$

with coefficient of risk aversion $b < 1$, $b \neq 0$ and $\hat{a} > 0$, $p > F$ with $F \geqslant 0$. This utility function is increasing and strictly concave in the argument p.

We discretize time with grid points $t^{(k)}$ and step size Δ as before. We assume $N = 1$ from now on whenever it comes to implementation, i.e. the financial market consists of a single risky asset class that can be interpreted as a mutual fund. The stochastic return or shock is discretized by the following equidistant partition of the probability space $[0,1]$ for the risk factor $Z \in \mathbb{R} = (-\infty, \infty)$: Let $q \in (0,1)$; for instance $q = 5\%$. The corresponding cumulative probabilities are

$$q^{(i)} := q^{(0)} + \Delta^{(q)} \cdot i, \; i = 0, \dots, N_q \tag{5.57}$$

with $\Delta^{(q)} := q$, $N_q := \frac{1-q}{q} \overset{!}{\in} \mathbb{N}$ because then $\sum_{i=0}^{N_q} q = (1+N_q)q = 1$. For instance, one could set $q^{(0)} = 5\%$, $\Delta^{(q)} = 5\%$ ($N_q = 19$), then $q^{(i)} = 5\%, 10\%, \dots, 95\%, 100\%$. The corresponding values or representatives for Z with probability $\mathbb{P}(Z = z(q^{(i)})) = q$ and quantile probabilities $q^{(i)}$ are obtained by

$$\begin{aligned} z(q^{(0)}) &:= \Phi^{-1}\left(\frac{0+q^{(0)}}{2}\right), \\ z(q^{(i)}) &:= \Phi^{-1}\left(\frac{q^{(i)}+q^{(i+1)}}{2}\right), \; i = 1, \dots, N_q - 1, \\ z(q^{(N_q)}) &:= \Phi^{-1}\left(\frac{q^{(N_q)}+1}{2}\right). \end{aligned} \tag{5.58}$$

With this definition, the z values are stronger centered around zero, with a larger step size for large positive and negative values. Lastly, we discretize the decision set for the control variable a_k. Since $a_k \in \mathbb{A}$ with $\mathbb{A} = [0,1]$, we split the interval $\mathbb{A} = [0,1]$ into a grid with equidistant distances and representatives

$$\lambda^{(i)} := \lambda^{(0)} + \Delta^{(\lambda)} \cdot i, \; i = 0, \dots, N_\lambda \tag{5.59}$$

with $N_\lambda := \frac{\lambda^{(N_\lambda)} - \lambda^{(0)}}{\Delta^{(\lambda)}} \overset{!}{\in} \mathbb{N}$. It is natural to select $\lambda^{(0)} = 0$ and $\lambda^{(N_\lambda)} = 1$, or apply lower and upper bound constraints on the relative risky investment if present. Thus, if for instance $\Delta^{(\lambda)} = 1\%$, a_k can take any integer percentage value, i.e. $a_k \in \{\lambda^{(0)}, \ldots, \lambda^{(N_\lambda)}\} = \{0\%, 1\%, 2\%, \ldots, 98\%, 99\%, 100\%\}$.

We now explain the backwards recursion mechanism to solve the Bellman equation and thus to find the optimal action a_k in any period $[t^{(k)}, t^{(k+1)})$. First of all, the HARA utility function in (5.56) implies the following period-k reward function:

$$\begin{aligned} r_k(S_{(k)}) &\overset{(5.54)}{=} \frac{1}{\lambda_x + \beta} \left(1 - e^{-(\lambda_x + \beta)\Delta}\right) \tilde{U}_1(P_{(k)}) \\ &= \frac{1}{\lambda_x + \beta} \left(1 - e^{-(\lambda_x + \beta)\Delta}\right) \hat{a} \frac{1-b}{b} \left(\frac{1}{1-b}(P_{(k)} - F)\right)^b . \end{aligned}$$

The optimal a_k can then be found by the following algorithm:

1. $k = N_\Delta$:

   It is $\mathcal{V}_{N_\Delta}(S_{(N_\Delta)}; c_c^{(\text{buffer})}) = 0$ for any state $S_{(N_\Delta)}$ at terminal time $t^{(N_\Delta)} = \tilde{T}$.

2. $k = N_\Delta - 1, \ldots, 0$:

   The Bellman equation for $k = N_\Delta - 1, \ldots, 0$ is given by

$$\begin{aligned} \mathcal{V}_k(S_{(k)}; c_c^{(\text{buffer})}) &= r_k(S_{(k)}) + e^{-(\lambda_x + \beta)(t^{(k+1)} - t^{(k)})} \\ &\quad \times \max_{a_k \in \{\lambda^{(0)}, \ldots, \lambda^{(N_\lambda)}\}} \left\{ \mathbb{E}\left[\mathcal{V}_{k+1}(S_{(k+1)}; c_c^{(\text{buffer})}) \middle| S_{(k)} \right] \right\} \\ &\overset{S_{(k+1)} = T_B(S_{(k)}, a_k, Z)}{=} r_k(S_{(k)}) + e^{-(\lambda_x + \beta)(t^{(k+1)} - t^{(k)})} \\ &\quad \times \max_{a_k \in \{\lambda^{(0)}, \ldots, \lambda^{(N_\lambda)}\}} \left\{ \mathbb{E}\left[\mathcal{V}_{k+1}(T_B(S_{(k)}, a_k, Z); c_c^{(\text{buffer})}) \middle| S_{(k)} \right] \right\} \\ &= r_k(S_{(k)}) + e^{-(\lambda_x + \beta)(t^{(k+1)} - t^{(k)})} \\ &\quad \times \max_{a_k \in \{\lambda^{(0)}, \ldots, \lambda^{(N_\lambda)}\}} \left\{ \sum_{i=0}^{N_z} q(z^{(i)}) \mathcal{V}_{k+1}(T_B(S_{(k)}, a_k, z^{(i)}); c_c^{(\text{buffer})}) \middle| S_{(k)} \right\}, \end{aligned}$$

   where $\mathcal{V}_{k+1}(T_B(S_{(k)}, a_k, z^{(i)}); c_c^{(\text{buffer})})$ is given from the backwards recursion. Then,

$$a_k^\star := \underset{a_k \in \{\lambda^{(0)}, \ldots, \lambda^{(N_\lambda)}\}}{\arg\max} \left\{ \sum_{i=0}^{N_z} q(z^{(i)}) \mathcal{V}_{k+1}(T_B(S_{(k)}, a_k, z^{(i)}); c_c^{(\text{buffer})}) \middle| S_{(k)} \right\}$$

   denotes the optimal decision for the relative risky investment[16] at time $t^{(k)}$ for the investment period $[t^{(k)}, t^{(k+1)})$ with then

$$\begin{aligned} \mathcal{V}_k(S_{(k)}; c_c^{(\text{buffer})}) &= r_k(S_{(k)}) + e^{-(\lambda_x + \beta)(t^{(k+1)} - t^{(k)})} \\ &\quad \times \left\{ \sum_{i=0}^{N_z} q(z^{(i)}) \mathcal{V}_{k+1}(T_B(S_{(k)}, a_k^\star, z^{(i)}); c_c^{(\text{buffer})}) \middle| S_{(k)} \right\}. \end{aligned}$$

[16]If $a_k^\star$ is not unique, then we select the smallest value among all maximizers, thus we follow the most defensive strategy.

a) In particular: $k = N_\Delta - 1$

The Bellman equation reduces to

$$\begin{aligned}\mathcal{V}_{N_\Delta-1}(S_{(N_\Delta-1)};c_c^{(\text{buffer})}) &= r_{N_\Delta-1}(S_{(N_\Delta-1)}) + e^{-(\lambda_x+\beta)(t^{(k+1)}-t^{(k)})} \\ &\quad \times \max_{a_{N_\Delta-1}\in\{\lambda^{(0)},\ldots,\lambda^{(N_\lambda)}\}} \left\{ \mathbb{E}\left[\underbrace{\mathcal{V}_{N_\Delta}(S_{(N_\Delta)};c_c^{(\text{buffer})})}_{=0} \middle| S_{(N_\Delta-1)} \right] \right\} \\ &= r_{N_\Delta-1}(S_{(N_\Delta-1)}).\end{aligned}$$

Thus, we find that in the last full period $[t^{(N_\Delta-1)}, t^{(N_\Delta)})$, the decision variable $a_{N_\Delta-1}$ has no impact on the one-period reward function $r_{N_\Delta-1}(S_{(N_\Delta-1)})$ and furthermore no impact on the future satisfaction as $\mathcal{V}_{N_\Delta}(S_{(N_\Delta)};c_c^{(\text{buffer})}) = 0$ for all states $S_{(N_\Delta)}$.

b) In particular: $k = N_\Delta - 2$

In this case, the Bellman equation can be reduced to

$$\begin{aligned}&\mathcal{V}_{N_\Delta-2}(S_{(N_\Delta-2)};c_c^{(\text{buffer})}) \\ &\quad = r_{N_\Delta-2}(S_{(N_\Delta-2)}) + e^{-(\lambda_x+\beta)(t^{(N_\Delta-1)}-t^{(N_\Delta-2)})} \\ &\quad \times \max_{a_{N_\Delta-2}\in\{\lambda^{(0)},\ldots,\lambda^{(N_\lambda)}\}} \left\{ \mathbb{E}\left[\underbrace{\mathcal{V}_{N_\Delta-1}(S_{(N_\Delta-1)};c_c^{(\text{buffer})})}_{=r_{N_\Delta-1}(S_{(N_\Delta-1)})} \middle| S_{(N_\Delta-2)} \right] \right\} \\ &\quad = r_{N_\Delta-2}(S_{(N_\Delta-2)}) + e^{-(\lambda_x+\beta)(t^{(N_\Delta-1)}-t^{(N_\Delta-2)})} \\ &\quad \times \max_{a_{N_\Delta-2}\in\{\lambda^{(0)},\ldots,\lambda^{(N_\lambda)}\}} \left\{ \mathbb{E}\left[r_{N_\Delta-1}(S_{(N_\Delta-1)}) \middle| S_{(N_\Delta-2)} \right] \right\}.\end{aligned}$$

From the backwards recursion, we particularly learn that $a_{N_\Delta-1}$ has no influence at all. Therefore, we omit the very last period from time $t^{(N_\Delta-1)}$ to $t^{(N_\Delta)}$ in what follows since $t^{(N_\Delta-2)}$ is the last decision time; thus last period to be considered is $[t^{(N_\Delta-2)}, t^{(N_\Delta-1)})$.

In total, the discretization then leads to a grid of $[(N_z+1)(N_\lambda+1)]^i$ nodes or possible states at time $t^{(i)}$ with a total number of grid points (= number of paths or scenarios) of $[(N_z+1)(N_\lambda+1)]^{N_\Delta-1}$ at time $t^{(N_\Delta-1)}$. Therefore, in summary there are $N_{\text{states}} := \sum_{i=0}^{N_\Delta-1} [(N_z+1)(N_\lambda+1)]^i$ possible states in the entire grid. After the optimal decisions $a_k^\star$ for every state are determined, then the full tree with total number of states N_{states} reduces to a tree with $N_{\text{states}}^\star := \sum_{i=0}^{N_\Delta-1}(N_z+1)^i$ points, where all decision paths besides the optimal $a_k^\star$ are removed.

We implement the above algorithm in *Matlab* by generating a tree from the grid that covers all possible state spaces $S_{(k)}$. The algorithm shows that we do an optimization via evaluation on the grid and selection of the decision variable that provides the maximum at each time step. The more dense the grid, i.e. the smaller the time step Δ, return step $\Delta^{(z)}$ and decision step $\Delta^{(\lambda)}$, the closer the optimal controls come to the continuous counterpart. But the density of the discretization grids, i.e. the step sizes, are limited by the computational power and running time.

5.2.2.5 Case study: two decision periods for a cohort of clients

We focus on the cohort perspective and consider a cohort of clients with an initial age of $T = 65$ years (retirement entry time). First, we introduce the general setting and the parameter choices: As we would like to consider two decision periods in a first approach, we select $\tilde{T} = T + 3\Delta$ as planning horizon; then an investment decision has to be made at times T and $T + \Delta$ for periods $[T, T + \Delta)$ and $[T + \Delta, T + 2\Delta)$. Moreover, let $T^\star = 120$ years (maximal total lifetime), $\lambda_x = \lambda_{x(j)} = 1.18\%$ (mortality rate of the cohort: determined such that the survival probability of a 65-year old client to survive one more year coincides with the average survival probability of 99.202956% (female) and 98.457889% (male) in Germany[17], cf. Statistisches Bundesamt (2019)), $\bar{p} = 112.5\%$ (CCR of the investment portfolio at initial time and at every reset) and $v_0 = 10,000$ (initial post-retirement wealth at time $t^{(0)} = T$). For the market we suppose $r = 1\%$, $\mu = 2.97\%$, $\sigma = 11.75\%$ (market parameters: risk-free interest rate; drift and volatility of one risky asset which is interpreted as a buy & hold portfolio that consists of the three asset classes government bonds, corporate bonds and equity with initial weights $\frac{1}{N} = \frac{1}{3}$ each, where we used the numbers from the parameter estimation in Escobar et al. (2019)). The HARA utility function parameters are assumed to be $\beta = 3\%$ (cf. Ye (2008)), $b = -1$, $a = 1$ and $F = 25.8$[18] which gives

$$U_1(t, p) = e^{-\beta(t-T)}\tilde{U}_1(p) = -e^{-0.03(t-T)}\frac{4}{p - 25.8}.$$

For the discretization we suppose:

1. time t:

   $N_\Delta = 3$ with a step size of $\Delta = 1$ (annual rebalancing and adjustments of the pension payments) which implies $t^{(i)} = T + i$, $i = 0, \ldots, N_\Delta$ and thus $t^{(i)} \in \{T, T+1, \ldots, \tilde{T} - 1, \tilde{T}\} = \{T, T+1, T+2, T+3\}$ in the grid.

2. risk driver Z:

   We suppose probability intervals of length $q = 2.5\%$, i.e. $q^{(i)} = q^{(0)} + \Delta^{(q)} \cdot i,\ i = 0, \ldots, N_q$ with $\Delta^{(q)} = q = 2.5\%$, $N_q = \frac{1-q}{q} = 39$. The corresponding representatives for Z start with $z(q^{(0)}) = -2.2414$ and end with $z(q^{(39)}) = 2.2414$.

3. decision interval $\mathbb{A}$:

   For the sake of simplicity, we consider steps of five percentage points for a_k, i.e. $\lambda^{(0)} = 0\%$, $\lambda^{(N_\lambda)} = \lambda^{(20)} = 100\%$, $N_\lambda = 20$ (equivalent to $\Delta^{(\lambda)} = 0.05$) which translates to $a_k \in \{0\%, 5\%, 10\%, \ldots, 90\%, 95\%, 100\%\}$.

These discretization parameters lead to a grid of $[(39+1)(20+1)]^i = 840^i$ states at time $t^{(i)} \in \{T, T+1, T+2\}$ and therefore in total of $N_{\text{states}} = \sum_{i=0}^{2} 840^i = 1 + 840 + 840^2 = 706,441$ nodes in the entire grid. Having determined and selected the optimal $a_k^\star = \hat{\pi}^{\star(\text{inv})}(t^{(k)})$, this tree reduces to a tree with $N_{\text{states}}^\star = \sum_{i=0}^{2} 40^i = 1 + 40 + 40^2 = 1,641$ as the total number of nodes.

Let us consider three different values for the buffer parameter α: $\alpha = 0\%$ (no buffer), $\alpha = 20\%$ (moderate buffer) and $\alpha = 40\%$ (pronounced buffer). This choice implies the initial pension rates

[17] This calibration of the mortality rate is reasonable since we consider two decision periods (= two years) within this analysis. Therefore, the focus of the mortality model calibration needs to be in the very near future.

[18] It has to hold $P_{(k)} > F$ for all applied $P_{(k)}$. We select $F = 10\% \times P_{(0)}$ which will result in $F = 25.8$.

$P_{(0)} = (277|270|258)$ (for $\alpha = (0\%|20\%|40\%)$) through[19]

$$P_{(0)} \overset{(5.19)}{=} \frac{1-\alpha}{\bar{p}-\alpha} \frac{v_0}{\mathbb{E}\left[\int_T^{\tau_{ij}^x(T)} \frac{\tilde{Z}(t)}{\tilde{Z}(T)} dt\right]} = \frac{1-\alpha}{\bar{p}-\alpha} \frac{v_0}{\frac{1}{r+\lambda_x}\left(1-e^{-(r+\lambda_x)(T^\star - T)}\right)}.$$

Intuitively, the higher the buffer parameter, the lower is the initial pension. Moreover, the initial distribution to the investment and the buffer portfolio is $\frac{V_c^{(\text{buffer})}(T)}{v_0} = (0\%|2.7\%|6.9\%)$, $\frac{V_c^{(\text{inv})}(T)}{v_0} = (100\%|97.3\%|93.1\%)$ due to Eq. (5.52):

$$V_c^{(\text{buffer})}(T) = \alpha\left(v_0 - E_c(t^{(0)}|S_{(0)})\right),$$
$$V_c^{(\text{inv})}(T) = E_c(t^{(0)}|S_{(0)}) + (1-\alpha)\left(v_0 - E_c(t^{(0)}|S_{(0)})\right).$$

The initial capital coverage ratio is by definition $CCR_c^{(\text{total})}(T) \overset{(5.23)}{=} \frac{\bar{p}-\alpha}{1-\alpha} = (112.5\%|115.6\%|120.8\%)$.

If we apply the Bellman equation, we receive the following outcome summarized by Table 5.5:

1. The optimal investment decision $a_0^\star$ at initial time $t^{(0)} = T$ for the investment portfolio and its corresponding optimal initial investment decision $\hat{\pi}^{\star(\text{total})}(t^{(0)})$ for the total portfolio for the period $[t^{(0)}, t^{(1)}) = [T, T+1)$, independent of Z.

2. The evolution of the state $S_{(0)} = (V_{(0)}, P_{(0)})$, the proportion of the total wealth $V_{(0)}$ that is allocated or distributed to the buffer account $V_c^{(\text{buffer})}(t)$ and the investment portfolio $V_c^{(\text{inv})}(t)$, and the capital coverage ratio $CCR_c^{(\text{total})}(t)$ (for $\alpha = (0\%|20\%|40\%)$) from initial time $t = T$ to the next observation time $t = T + \Delta = T + 1$ under all possible underlying traded fund return realizations determined by Z in period $[t^{(0)}, t^{(1)}) = [T, T+1)$.

3. The optimal investment decision $a_1^\star$ at time $t^{(1)}$ for the investment portfolio and its corresponding optimal initial investment decision $\hat{\pi}^{\star(\text{total})}(t^{(1)})$ for the total portfolio for the period $[t^{(1)}, t^{(2)}) = [T+1, T+2)$, depending on the realization of the random variable Z from period $[t^{(0)}, t^{(1)}) = [T, T+1)$.

As we have $N_q + 1 = 40$ different possible realizations of Z in period $[t^{(0)}, t^{(1)}) = [T, T+1)$, we only consider five exemplary representatives, each with an assigned probability of 2.5%: $z(q^{(0)}) = -2.2414$, $z(q^{(10)}) = -0.6357$, $z(q^{(20)}) = 0.0313$, $z(q^{(30)}) = 0.7144$ and $z(q^{(39)}) = 2.2414$. Those values correspond to the one-period returns $return(Z) = \mu\Delta + \sigma\sqrt{\Delta}Z$ of the traded underlying fund of $return(z(q^{(0)})) = -23.4\%$, $return(z(q^{(10)})) = -4.5\%$, $return(z(q^{(20)})) = 3.3\%$, $return(z(q^{(30)})) = 11.4\%$ and $return(z(q^{(39)})) = 29.3\%$. In general, if a pension adjustment has to be made at some time t, it follows $CCR_c^{(\text{total})}(t) = \frac{\bar{p}-\alpha}{1-\alpha} = (112.5\%|115.6\%|120.8\%)$.

We summarize our optimization results in Tables 5.5 and 5.6. Due to

$$P_{(k+1)} = e^{-\lambda_{x(j)}(t^{(k+1)} - t^{(k)})} P_{(k)} < P_{(k)}$$

for the total cohort pension indicating a stable individual pension $P_{ij}(t^{(k+1)}) = P_{ij}(t^{(k)})$, we use the fraction $\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta} P_{(0)}}$ in Table 5.5. We already explained before that due to the cohort view,

[19] We round percentage numbers to one decimal digit and total value numbers ($S_{(k)}$) without any decimal place.

	$Z = z(q^{(0)}) = -2.2414$	$Z = z(q^{(10)}) = -0.6357$	$Z = z(q^{(20)}) = 0.0313$	$Z = z(q^{(30)}) = 0.7144$	$Z = z(q^{(39)}) = 2.2414$
$V_{(0)}$:	10, 000	(independent of Z)			
$P_{(0)}$:	277|270|258	(independent of Z)			
$a_0^\star$:	100%|100%|95%	(independent of Z)			
$\hat{\pi}^{\star(\text{total})}(t^{(0)})$:	100%|97.3%|88.4%	(independent of Z)			
$V_{(1)}$:	7, 386|7, 457|7, 680	9, 273|9, 292|9, 348	10, 056|10, 055|10, 042	10, 859|10, 836|10, 751	12, 653|12, 582|12, 338
$P_{(1)}$:	207|203|200	274|267|255	274|267|255	274|295|280	354|343|322
$\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta}P_{(0)}}$:	75.5%|76.2%|78.5%	100%|100%|100%	100%|100%|100%	100%|110.7%|109.8%	129.3%|128.5%|126.0%
$\frac{V_c^{(\text{buffer})}(t^{(1)})}{V_{(1)}}$:	0%|2.7%|6.9%	0%|1.8%|5.3%	0%|3.2%|7.7%	0%|2.7%|6.9%	0%|2.7%|6.9%
$\frac{V_c^{(\text{inv})}(t^{(1)})}{V_{(1)}}$:	100%|97.3%|93.1%	100%|98.2%|94.7%	100%|96.8%|92.3%	100%|97.3%|93.1%	100%|97.3%|93.1%
$CCR_c^{(\text{total})}(t^{(1)})$:	112.5%|115.6%|120.8%	106.6%|109.8%|115.4%	115.6%|118.8%|124.0%	124.8%|115.6%|120.8%	112.5%|115.6%|120.8%
$a_1^\star$:	80%|60%|60%	85%|80%|65%	90%|85%|70%	85%|90%|75%	5%|0%|90%
$\hat{\pi}^{\star(\text{total})}(t^{(1)})$:	80%|58.4%|55.9%	85%|78.6%|61.5%	90%|82.3%|64.6%	85%|87.6%|69.8%	5%|0%|83.8%

Table 5.5: Optimal decision variables and evolution of processes for one cohort for $\alpha = (0\%|20\%|40\%)$.

$\mathbb{P}(P_{(1)} < e^{-\lambda_{x(j)}\Delta}P_{(0)})$	$\mathbb{P}(P_{(1)} = e^{-\lambda_{x(j)}\Delta}P_{(0)})$	$\mathbb{P}(P_{(1)} > e^{-\lambda_{x(j)}\Delta}P_{(0)})$	$\mathbb{P}(P_{(1)} < e^{-\lambda_{x(j)}\Delta}P_{(0)}$ or $P_{(2)} < e^{-\lambda_{x(j)}\Delta}P_{(1)})$
12.5%|7.5%|2.5%	65%|62.5%|52.5%	22.5%|30%|45%	17.4%|12.3%|5.1%

Table 5.6: Probabilities of pension rate changes for one cohort for $\alpha = (0\%|20\%|40\%)$.

$\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta}P_{(0)}} < 100\%$ indicates an individual pension reduction, $\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta}P_{(0)}} = 100\%$ a stable individual pension development and $\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta}P_{(0)}} > 100\%$ an enhancement of the individual pension of the client members from time $t^{(0)} = T$ to $t^{(1)} = T+1$. For this reason, we compute the probabilities $\mathbb{P}(P_{(1)} \gtreqless e^{-\lambda_{x(j)}\Delta}P_{(0)})$ and $\mathbb{P}(P_{(1)} < e^{-\lambda_{x(j)}\Delta}P_{(0)}$ or $P_{(2)} < e^{-\lambda_{x(j)}\Delta}P_{(1)})$ instead of $\mathbb{P}(P_{(1)} \gtreqless P_{(0)})$ and $\mathbb{P}(P_{(1)} < P_{(0)}$ or $P_{(2)} < P_{(1)})$.

We draw the following conclusions:

Table 5.5 provides the optimal asset allocation variables and its corresponding wealth and pension evolution. The table shows that the initial optimal relative risky allocation $\hat{\pi}^{\star(\text{total})}(t^{(0)})$ decreases with α. After one period, the table shows a similar picture: The optimal relative portfolio $\hat{\pi}^{\star(\text{total})}(t^{(1)})$ of the total wealth mostly decreases with α. The exception is for an extremely well-performing underlying asset (case $Z = z(q^{(39)}) = 2.2414$), where there is a profit lock-in mechanism for $\alpha = (0\%|20\%)$ with $\hat{\pi}^{\star(\text{total})}(t^{(1)})$ being equal or close to zero, but a high relative risky allocation $\hat{\pi}^{\star(\text{total})}(t^{(1)})$ for $\alpha = 40\%$. This performance seeking behavior for $\alpha = 40\%$ after an exceptionally high risky asset return ($Z = z(q^{(39)}) = 2.2414$) follows from a very high $CCR_c^{(\text{total})}(t^{(1)})$ value (and thus a high buffer portfolio value) connected with a very low probability of a potential pension reduction after the next investment period. Furthermore, we detect that a higher total wealth $V_{(1)}$ mostly leads to a higher optimal relative risky investment $a_1^\star$ and $\hat{\pi}^{\star(\text{total})}(t^{(1)})$ for underlying risky asset returns below the profit lock-in barrier.

Table 5.6 provides the probabilities of individual pension shortenings and enhancements after the first period and the probability of at least one pension reduction in the two considered consecutive periods, under consideration of all return scenarios Z in both periods. According to Table 5.6, the optimal trading strategy with the higher buffer parameter α simultaneously decreases the probability of a pension reduction while it increases the probability of a pension enhancement. Table 5.5 further shows that in the case of a reduction (in times of a poor market performance), the client benefits from a lower proportional reduction of the pension (i.e. larger $\frac{P_{(1)}}{e^{-\lambda_{x(j)}\Delta}P_{(0)}}$) for a higher buffer parameter α. We further observe that the $CCR_c^{(\text{total})}(t^{(1)})$ values for higher α values remain larger in almost all considered Z scenarios. For the Z values considered in the table, the individual pension rate $P_{(1)}$ is increased for the two best Z values when $\alpha > 0\%$, but only for the best Z value when $\alpha = 0\%$. This shows that a higher α value leads to an earlier (but smaller proportional) pension enhancement in terms of market performance. In summary, a more pronounced buffer system can on the one hand significantly reduce the probability of future pension shortenings and on the other hand lower the proportional pension reduction if a shortening cannot be avoided. Therefore, the proposed buffer system can considerably improve the retiree's satisfaction. But the benefit from a particularly lower probability for a possible pension reduction in the future comes at the cost of an initially lower pension $P_{(0)}$. Moreover, a larger probability of a pension increase for larger α values can be explained by a larger CCR value at initial time and at every reset time, which makes it more likely that the CCR outgoes its corridor forcing an increase in the pension. Further notice that the wealth, that corresponds to a client who died in the previous period, remains in the collective portfolio (see Remark 5.10 at the end of Section 5.2.2.3) and thus also lowers the probability of pension shortenings while it increases the probability of a pension raise. Therefore, the survivors in the cohort profit from the death of a cohort member.

Remark 5.11. *The above case study with two decision periods requires a high additional computational effort if, besides a large grid size for Z and a_k, the number of decision periods is increased. By this, the number of states increases exponentially as shown earlier. Furthermore, the optimization (recursive solution of the Bellman equation) particularly needs to be performed for each initial state $S_{(0)} = (V_{(0)}, P_{(0)})$ separately; thus for every single client or cohort. To find a computationally efficient solution for an arbitrary planning horizon, an arbitrary number of decision periods and an arbitrary initial state, we present an elegant approximate solution in form of a stationary solution next.*

5.2.3 A stationary solution

In general, we are now looking for a faster and more efficient algorithm to find the optimal investment decision, motivated by Remark 5.11. Let $\tilde{T} = T^\star = \infty$ hold. The idea is that $a_k^\star(S) \equiv a^\star(S) = \hat{\pi}^{\star(\text{inv})}(S)$ for all states S, i.e. the optimal asset allocation decision depends on the current state and is independent of time; we seek for a stationary solution. This leads to the infinite-horizon discrete-time optimization problem

$$\begin{aligned} \mathcal{V}(S) = \mathcal{V}(S; c_c^{(\text{buffer})}) &= \sup_{a(S)\in\mathbb{A}} \mathcal{J}(a(S); S, c_c^{(\text{buffer})}) \\ \text{s.t. } \mathcal{J}(a(S); S, c_c^{(\text{buffer})}) &= \mathbb{E}\left[\sum_{i=0}^{\infty}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)}\tilde{U}_1(P(t^{(i)}))du\right] \end{aligned} \tag{5.60}$$

with $t^{(0)} = T$, state $S = (V, P)$ and stochastic pension $P(t^{(i)})$ at time $t^{(i)}$ that depends on $a(S)$ through the balance V. Recall that $\mathbb{A} = \{a \in [0,1]^N : a'\mathbf{1} \leqslant 1\}$. Due to $\tilde{T} = T^\star = \infty$, the corresponding Bellman equation to this problem is as follows:

$$\begin{aligned}\mathcal{V}(S) &= \sup_{a(S)\in\mathbb{A}} \left\{ r(S,a(S)) + e^{-(\lambda_x+\beta)\Delta}\mathbb{E}\left[\mathcal{V}(T_B(S,a(S),Z))\middle| S\right]\right\} \\ &= r(S) + e^{-(\lambda_x+\beta)\Delta} \sup_{a(S)\in\mathbb{A}} \left\{\mathbb{E}\left[\mathcal{V}(T_B(S,a(S),Z))\middle| S\right]\right\},\end{aligned} \tag{5.61}$$

where the last equality holds due to $r(S,a(S)) \equiv r(S)$ for the one-period reward under HARA utility (cf. (5.56)) with

$$r(S,a(S)) \equiv r(S) = r((V,P)) \equiv r(P) = \frac{1}{\lambda_x+\beta}\left(1-e^{-(\lambda_x+\beta)\Delta}\right)\hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(P-F)\right)^b. \tag{5.62}$$

It is obvious that the problem is independent of time and falls into the class of stationary, infinite-horizon Markovian decision problems, also known as *Markovian dynammic programming (MDP) problems.* The transition function T_B for $S \mapsto T_B(S,a,Z)$ is given by

$$T_B : \mathbb{R}_+^2 \times \mathbb{A} \times \mathbb{R} \to \mathbb{R}_+^2,\ (S,a,Z) \mapsto T_B(S,a,Z) = \begin{pmatrix} T_B^{(V)}(S,a,Z) \\ T_B^{(P)}(S,a,Z)\end{pmatrix},$$

where $S = (V,P)$ and

$$T_B^{(V)}(S,a,Z) = V + \left[E_c(S) + (1-\alpha)\left(V - E_c(S)\right)\right]\left[\left(r + a'(\mu - r\mathbf{1})\right)\Delta + a'\sigma\sqrt{\Delta}Z\right] - P\Delta$$

and[20]

$$T_B^{(P)}(S,a,Z) = \begin{cases} P, \text{ if } \frac{T_B^{(V)}(S,a,Z)}{E_c(S)} \in [100\%, 125\%] \\ \frac{1-\alpha}{\bar{p}-\alpha}\frac{T_B^{(V)}(S,a,Z)}{\frac{1}{r+\lambda_x}} & , \text{ otherwise}\end{cases}$$

with

$$E_c(S) = \frac{P}{r+\lambda_x}.$$

$E_c(S)$ can be regarded as perpetual annuity because of $T^\star = \infty$. The gap between a usual $T^\star$ value such as $T^\star = 120$ years and infinity has a very low probability, hence is negligible. But it simplifies calculations a lot. In more detail, the crucial object is the absolute error in the infinite-horizon problem (5.60) compared to the finite-horizon problem (5.46):

[20] The definition corresponds to the transition for a single client in Section 5.2.2.1. If we aim for determining the optimal investment decisions for a cohort of customers, then we define analogically to Section 5.2.2.3:

$$T_B^{(P)}(S,a,Z) = \begin{cases} e^{-\lambda_{x(j)}\Delta}P, \text{ if } \frac{T_B^{(V)}(S,a,Z)}{e^{-\lambda_{x(j)}\Delta}E_c(S)} \in [100\%, 125\%] \\ \frac{1-\alpha}{\bar{p}-\alpha}\frac{T_B^{(V)}(S,a,Z)}{\frac{1}{r+\lambda_{x(j)}}} & , \text{ otherwise}\end{cases}$$

with

$$E_c(S) = \frac{P}{r+\lambda_{x(j)}}.$$

$$\begin{aligned}
error_{T^\star,\infty} &:= \left| \mathbb{E}\left[\sum_{i=0}^{\infty} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)} \tilde{U}_1(P(t^{(i)}))du \right] \right. \\
&\quad \left. - \mathbb{E}\left[\sum_{i=0}^{N_\Delta-1} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)} \tilde{U}_1(P(t^{(i)}))du \right] \right| \\
&= \left| \mathbb{E}\left[\sum_{i=N_\Delta}^{\infty} \int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)} \tilde{U}_1(P(t^{(i)}))du \right] \right| \\
&= \left| \mathbb{E}\left[\int_{T^\star}^{\infty} e^{-(\lambda_x+\beta)(u-T)} \tilde{U}_1(P(u))du \right] \right| \\
&= e^{-(\lambda_x+\beta)(T^\star-T)} \left| \mathbb{E}\left[\int_{T^\star}^{\infty} e^{-(\lambda_x+\beta)(u-T^\star)} \tilde{U}_1(P(u))du \right] \right| \\
&\leqslant e^{-(\lambda_x+\beta)(T^\star-T)} \mathbb{E}\left[\int_{T^\star}^{\infty} e^{-(\lambda_x+\beta)(u-T^\star)} \left|\tilde{U}_1(P(u))\right| du \right]. \qquad (5.63)
\end{aligned}$$

The approximation is more reliable if the absolute error is small. If one desires to use the solution to the infinite-horizon problem as an approximation for the solution to the finite-horizon problem, one needs to ensure that $error_{T^\star,\infty}$ is sufficiently small to justify the approach. Notice that the probability to survive from initial time T to time $T^\star = 120$ (which is $e^{-\lambda_x(T^\star-T)}$) and even longer is close to zero, hence the probability of living longer than to time $T^\star = 120$ is neglectable. In the above error, the additional β in the exponential function further gives $e^{-(\lambda_x+\beta)(T^\star-T)} < e^{-\lambda_x(T^\star-T)}$. Moreover, the value for $E_c(S)$ stays very close to its former value $E_c(T|S) = \frac{P}{r+\lambda_x}\left(1 - e^{-(r+\lambda_x)(T^\star-T)}\right) = E_c(S) - \frac{P}{r+\lambda_x}e^{-(r+\lambda_x)(T^\star-T)} \approx E_c(S)$ because, as already mentioned, $e^{-(r+\lambda_x)(T^\star-T)}$ is even closer to zero than the survival probability if $r > 0$.

For ease of exposition, we place the following assumption on the utility function that is to hold from now on.

Assumption 5.12. *Let us consider HARA utility $\tilde{U}_1(P)$ (parameterization in Eq. (5.56)) for $P_{min} \leqslant P \leqslant P_{max}$ with $F < P_{min} < P_{max} < \infty$[21].*

Assumption 5.12 introduces lower and upper bounds for the pension payment that is to be paid out. We then have to adjust the transition function $T_B^{(P)}$ for the pension to become

$$T_B^{(P)}(S,a,Z) = \begin{cases} P & , \text{ if } \frac{T_B^{(V)}(S,a,Z)}{E_c(S)} \in [100\%, 125\%] \\ \max\left\{ \min\left\{ \frac{1-\alpha}{\bar{p}-\alpha} \frac{T_B^{(V)}(S,a,Z)}{\frac{1}{r+\lambda_x}}, P_{max} \right\}, P_{min} \right\}, & \text{otherwise} \end{cases}$$

in the single-client model[22]. Based on Assumption 5.12, we can further conclude that

[21] In the case of a positive coefficient of risk aversion $0 < b < 1$, the lower bound P_{min} can be neglected ($P_{min} := F$). In the case of a negative coefficient of risk aversion $b < 0$, the upper bound P_{max} can be neglected ($P_{max} := \infty$).

[22] For the cohort model, the adjusted transition function for the pension needs to be modified to

$$T_B^{(P)}(S,a,Z) = \begin{cases} \max\left\{ e^{-\lambda_{x(j)}\Delta} P, P_{min} \right\}, \text{ if } \frac{T_B^{(V)}(S,a,Z)}{e^{-\lambda_{x(j)}\Delta} E_c(S)} \in [100\%, 125\%] \\ \max\left\{ \min\left\{ \frac{1-\alpha}{\bar{p}-\alpha} \frac{T_B^{(V)}(S,a,Z)}{\frac{1}{r+\lambda_{x(j)}}}, P_{max} \right\}, P_{min} \right\} & , \text{ otherwise.} \end{cases}$$

$\tilde{U}_1(p) \overset{(5.56)}{=} \hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(p-F)\right)^b$ is bounded:

1. $b < 0$:

$$\tilde{U}_1(p) \leqslant 0,\ \tilde{U}_1(p) \overset{p \geqslant P_{min}}{\geqslant} \underbrace{\hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(P_{min}-F)\right)^b}_{<0}$$

$$\Rightarrow \exists 0 < K_{\tilde{U}_1} < \infty : \left|\tilde{U}_1(p)\right| < K_{\tilde{U}_1}.$$

2. $0 < b < 1$:

$$\tilde{U}_1(p) \geqslant 0,\ \tilde{U}_1(p) \overset{p \leqslant P_{max}}{\leqslant} \underbrace{\hat{a}\frac{1-b}{b}\left(\frac{1}{1-b}(P_{max}-F)\right)^b}_{>0}$$

$$\Rightarrow \exists 0 < K_{\tilde{U}_1} < \infty : \left|\tilde{U}_1(p)\right| < K_{\tilde{U}_1}.$$

In view of Eq. (5.63) we find the following result on $error_{T^\star,\infty}$:

Remark 5.13. *Let $K_{\tilde{U}_1}$ be an upper bound for $\tilde{U}_1(p)$ (to be supposed as small as possible). Then it holds*

$$error_{T^\star,\infty} \overset{(5.63)}{\leqslant} e^{-(\lambda_x+\beta)(T^\star-T)}\mathbb{E}\left[\int_{T^\star}^{\infty} e^{-(\lambda_x+\beta)(u-T^\star)} \underbrace{\left|\tilde{U}_1(P(u))\right|}_{<K_{\tilde{U}_1}} du\right]$$
$$< K_{\tilde{U}_1} e^{-(\lambda_x+\beta)(T^\star-T)} \int_{T^\star}^{\infty} e^{-(\lambda_x+\beta)(u-T^\star)} du = \frac{K_{\tilde{U}_1}}{\lambda_x+\beta} e^{-(\lambda_x+\beta)(T^\star-T)}.$$

If we define $K_{error_{T^\star,\infty}} := \frac{K_{\tilde{U}_1}}{\lambda_x+\beta}e^{-(\lambda_x+\beta)(T^\star-T)}$ and further denote $\mathcal{V}_\infty$ the value function to the infinite-horizon problem and $\mathcal{V}_{T^\star}$ the value function to the finite-horizon problem with $\bar{T} = T^\star$, we end up with

$$\mathcal{V}_\infty \in \left(\mathcal{V}_{T^\star} - K_{error_{T^\star,\infty}}, \mathcal{V}_{T^\star} + K_{error_{T^\star,\infty}}\right).$$

Hence, the deviation in the value functions can be estimated and controlled by $K_{error_{T^\star,\infty}}$.

Furthermore, since the proposed algorithm performs the optimization on a finite set for the action $a(S)$ (discretization grid for $\mathbb{A}$ with $a(S) \in \mathbb{A}$), i.e. on a set with a finite number of elements, the following assumption is said to hold true from now on.

Assumption 5.14. *$\mathbb{A}$ has a finite number of elements.*

For instance, one could assume 1%-point steps in the set $\mathbb{A} := \left\{a \in \mathbb{A}_1^N : a'\mathbf{1} \leqslant 1\right\}$, $\mathbb{A}_1 := \{0\%, 1\%, 2\%, \ldots, 98\%, 99\%, 100\%\}$, with $\mathbb{A} \equiv \mathbb{A}_1$ in the situation of one risky asset class ($N = 1$). Further note that every function $f : \mathbb{A} \to \mathbb{R}$ attains its maximum on the finite set $\mathbb{A}$.

Hence, from Assumption 5.14 it immediately follows that the supremum over $a(S) \in \mathbb{A}$ turns into its maximum:

$$\begin{aligned}\mathcal{V}(S) &\overset{(5.61)}{=} r(S) + e^{-(\lambda_x+\beta)\Delta} \sup_{a(S)\in\mathbb{A}} \left\{ \mathbb{E}\left[\mathcal{V}(T_B(S,a(S),Z)) \middle| S \right] \right\} \\ &\overset{\text{Assumption 5.14}}{=} r(S) + e^{-(\lambda_x+\beta)\Delta} \max_{a(S)\in\mathbb{A}} \left\{ \mathbb{E}\left[\mathcal{V}(T_B(S,a(S),Z)) \middle| S \right] \right\}.\end{aligned}$$

Thus, the maximum is attained. In what follows we present an approach to solve Problem (5.60) under Assumptions 5.12 and 5.14 via the Bellman equation (5.61). The theoretical justification and foundation of the approach, in particular of the proposed policy function iteration algorithm, can be found in Section 5.2.3.5.

5.2.3.1 Definition of the grid

We now solve the problem, i.e. the Bellman equation for every possible state S, on a grid. For this sake, we define grids for the state space S, the risk factor Z (for $N = 1$) and the action a.

1. Discretization of the state space $S = (V, P)$:

   We build up the state space grid following the three steps below:

   a) One-dimensional total wealth grid V:

      V_{min} (minimal value for V), V_{max} (maximal value for V), n_V (number of values for V with equidistant distances):

      $$V^{(i)} \in \text{Grid}(V) := \{V_{min}, \ldots, V_{max}\},\ i = 1, \ldots, n_V$$

      with cardinality n_V.

   b) One-dimensional capital coverage ratio grid CCR:

      Let Grid(CCR) be the grid of the capital coverage ratio that ranges from $CCR_{min} = 100\%$ to $CCR_{max} = 125\%$ with n_{CCR} number of values in the equidistant grid, for instance Grid(CCR) $= \{1, 1.01, 1.02, \ldots, 1.23, 1.24, 1.25\}$.

   c) Two-dimensional state space S:

      For every $V^{(i)} \in \text{Grid}(V)$ and every $CCR^{(j)} \in \text{Grid}(CCR)$, the pair $(V^{(i)}, P^{(ij)}) \in \text{Grid}(S)$ is in the grid Grid(S) for the state space S, where

      $$P^{(ij)} := \frac{V^{(i)}}{CCR^{(j)} \frac{1}{r+\lambda_x}} \Leftrightarrow CCR^{(j)} = \frac{V^{(i)}}{P^{(ij)} \frac{1}{r+\lambda_x}}.$$

      Thus, the size of Grid(S) is $n_S = n_V \cdot n_{CCR}$. Notice that in view of Assumption 5.12 it must hold $F < \min_{i,j} P^{(ij)}$ with $\min_{i,j} P^{(ij)} = \frac{\min_i V^{(i)}}{\max_j CCR^{(j)} \frac{1}{r+\lambda_x}} \geqslant \frac{V_{min}}{1.25 \frac{1}{r+\lambda_x}}$ as well as $\max_{i,j} P^{(ij)} < \infty$ with $\max_{i,j} P^{(ij)} = \frac{\max_i V^{(i)}}{\min_j CCR^{(j)} \frac{1}{r+\lambda_x}} \leqslant \frac{V_{max}}{\frac{1}{r+\lambda_x}}$.

2. Discretization of the risk driver Z (cf. 5.2.2.4):

   According to (5.57), the cumulative probabilities are

   $$q^{(i)} := q^{(0)} + \Delta^{(q)} \cdot i, \ i = 0, \dots, N_q$$

   with $\Delta^{(q)} := q \in (0,1)$, $N_q := \frac{1-q}{q} \overset{!}{\in} \mathbb{N}$ because then $\sum_{i=0}^{N_q} q = (1+N_q)q = 1$. The corresponding values or representatives for Z with probability q and quantile probabilities $q^{(i)}$ are obtained by (5.58):

   $$\begin{aligned} z(q^{(0)}) &:= \Phi^{-1}\left(\frac{0 + q^{(0)}}{2}\right), \\ z(q^{(i)}) &:= \Phi^{-1}\left(\frac{q^{(i)} + q^{(i+1)}}{2}\right), \ i = 1, \dots, N_q - 1, \\ z(q^{(N_q)}) &:= \Phi^{-1}\left(\frac{q^{(N_q)} + 1}{2}\right). \end{aligned}$$

   With this definition, the z values are stronger centered around zero, with a larger step size for large positive and negative values. Then

   $$Z \in \text{Grid}(Z) := \{z(q^{(0)}), \dots, z(q^{(N_q)})\} = \{Z_1, \dots, Z_{n_Z}\}$$

   with cardinality $n_Z := N_q + 1$ and probabilities q, i.e. $Z_j = z(q^{(j-1)})$, $j = 1, \dots, n_Z$.

3. Discretization of the investment decision $a \in \mathbb{A}$ (cf. 5.2.2.4):

   In accordance with (5.59), we split the interval $\mathbb{A} = [0,1]$ into a grid with equidistant distances and representatives

   $$a_{(i)} := a_{(0)} + \Delta^{(a)} \cdot i, \ i = 0, \dots, N_a$$

   with $N_a := \frac{a_{(N_a)} - a_{(0)}}{\Delta^{(a)}} \overset{!}{\in} \mathbb{N}$. It is natural to select $a_{(0)} = 0$ and $a_{(N_a)} = 1$, or apply lower and upper bound constraints on the relative risky investment if present. Therefore,

   $$a \in \text{Grid}(a) := \{a_{(0)}, \dots, a_{(N_a)}\}$$

   with cardinality $n_a := N_a + 1$. It is clear that the discretization Grid(a) for $\mathbb{A}$ fulfills Assumption 5.14 on $\mathbb{A}$.

We would like to mention that the construction of Grid(S) is very efficient since it consists of admissible (V,P)-pairs only and rules out non-admissible (V,P)-pairs; admissible pairs fulfill $\frac{V}{\frac{P}{r+\lambda_x}} \in [100\%, 125\%]$. Furthermore, by construction we ensure that the CCR values are uniformly spread over the entire corridor $[100\%, 125\%]$. If now $T_B(S^{(i)}, a_{(k)}, Z_j) \notin$ Grid(S), we select the grid node in the state space grid that provides the smallest sum of squared relative distances to $T_B(S^{(i)}, a_{(k)}, Z_j) \notin$ Grid(S) (relative with respect to the center of the respective grid) as method for interpolation between grid points. One should additionally be aware that V_{min} and V_{max} need to be chosen carefully (i.e. sufficiently small and large) and that the number of grid points in between needs to be sufficiently large to capture a representative number of states. Otherwise, the grid would not be dense enough and the upcoming optimization can lead to strange results. We provide a proposal for a grid in Section 5.2.3.4.

5.2.3.2 Stationary grid solution

The definition of the grids allows us to rewrite the expectation in the Bellman equation for any state $S^{(l)} \in \mathrm{Grid}(S)$:

$$\begin{aligned}\mathcal{V}(S^{(l)}) &= r(S^{(l)}) + e^{-(\lambda_x+\beta)\Delta} \max_{a(S^{(l)})\in\mathrm{Grid}(a)} \left\{ \sum_{j=1}^{n_Z} \underbrace{\mathbb{P}(Z=Z_j)}_{\equiv q} \mathcal{V}(T_B(S^{(l)}, a(S^{(l)}), Z_j)) \right\} \\ &= r(S^{(l)}) + e^{-(\lambda_x+\beta)\Delta} q \max_{a(S^{(l)})\in\mathrm{Grid}(a)} \left\{ \sum_{j=1}^{n_Z} \mathcal{V}(T_B(S^{(l)}, a(S^{(l)}), Z_j)) \right\}.\end{aligned}$$

The optimal policy $a^\star(S^{(l)})$ is the maximizer:

$$a^\star(S^{(l)}) := \operatorname*{arg\,max}_{a\in\mathrm{Grid}(a)} \left\{ \sum_{j=1}^{n_Z} \mathcal{V}(T_B(S^{(l)}, a, Z_j)) \right\}.$$

In the following we aim to solve this Bellman equation for every $S^{(l)} \in \mathrm{Grid}(S)$ and by this determine the optimal decisions or policies $a^\star(S^{(l)})$ for all states in the grid.

We now treat every $a = a(S^{(l)}) \in \mathrm{Grid}(a)$ as if it was the maximizer of the Bellman equation, and select the optimal $a^\star(S^{(l)})$ at the very end by choosing the one that maximizes $\mathcal{V}(S^{(l)})$. Hence, for all $S^{(l)} \in \mathrm{Grid}(S)$ and all $a(S^{(l)}) \in \mathrm{Grid}(a)$ we consider

$$\mathcal{V}(S^{(l)}) = r(S^{(l)}) + e^{-(\lambda_x+\beta)\Delta} q \sum_{j=1}^{n_Z} \mathcal{V}(T_B(S^{(l)}, a(S^{(l)}), Z_j)) \tag{5.64}$$

which is a linear system of equations since $T_B(S^{(l)}, a(S^{(l)}), Z_j) \in \mathrm{Grid}(S)$ according to the applied interpolation rule if not already in the grid.

Given a specific $a(S^{(l)}) = a_{(i(l))} \in \mathrm{Grid}(a)$ for some $i(l) \in \{0, \ldots, N_a\}$, we solve this linear system in $\mathcal{V}(S^{(l)})$ for all $l = 1, \ldots, n_S$ by rewriting the right-hand sum using matrix notation with $S = (S^{(1)}, \ldots, S^{(n_S)})'$ the vector that consists of all state grid points in $\mathrm{Grid}(S)$. Define the matrix $Q \in \mathbb{N}^{n_S \times n_S}$ by

$$Q := Q_1 Q_2$$

with block matrix $Q_1 \in \{0,1\}^{n_S \times (n_S \cdot n_Z)}$ such that

$$Q_1 := \left(\underbrace{I_{n_S} \cdots I_{n_S} \cdots I_{n_S}}_{n_Z \text{ times } I_{n_S}} \right),$$

where $I_{n_S} \in \{0,1\}^{n_S \times n_S}$ is the identity matrix with dimension n_S. Furthermore, $Q_2 \in \{0,1\}^{(n_S \cdot n_Z) \times n_S}$ is defined as a block matrix such that

$$Q_2 := \begin{pmatrix} A_1 \\ \vdots \\ A_j \\ \vdots \\ A_{n_Z} \end{pmatrix}$$

with $A_j \in \{0,1\}^{n_S \times n_S}$ defined by

$$(A_j)_{lm} := \begin{cases} 1, & \text{if } T_B(S^{(l)}, a_{(i(l))}, Z_j) = S^{(m)} \\ 0 & , \text{ otherwise.} \end{cases}$$

Then it follows

$$Q_{l\cdot} = (Q_1 Q_2)_{l\cdot} = (Q_1)_{l\cdot} Q_2 = \left(I_{n_S} \cdots I_{n_S} \cdots I_{n_S}\right)_{l\cdot} \begin{pmatrix} A_1 \\ \vdots \\ A_j \\ \vdots \\ A_{n_Z} \end{pmatrix} = d_l$$

with $d_l \in \mathbb{N}^{1 \times n_S}$ such that $(d_l)_m$ is the number of Z_j, $j = 1, \ldots, n_Z$, which leads to a transition from state $S^{(l)}$ to state $S^{(m)}$. Consequently, it follows for the vector $Q\mathcal{V}(S) \in \mathbb{R}^{n_S}$, where $\mathcal{V}(S) = (\mathcal{V}(S^{(1)}), \ldots, \mathcal{V}(S^{(n_S)}))' \in \mathbb{R}^{n_S}$:

$$(Q\mathcal{V}(S))_l = Q_{l\cdot}\mathcal{V}(S) = d_l \mathcal{V}(S) = \sum_{m=1}^{n_S} (d_l)_m \mathcal{V}(S^{(m)}) = \sum_{j=1}^{n_Z} \mathcal{V}(T_B(S^{(l)}, a_{(i(l))}, Z_j))$$

Note $Q = Q(a)$ with $a = (a_{(i(1))}, \ldots, a_{(i(n_S))})'$, and therefore,

$$\mathcal{V}(S^{(l)}) \overset{(5.64)}{=} r(S^{(l)}) + e^{-(\lambda_x+\beta)\Delta} q \sum_{j=1}^{n_Z} \mathcal{V}(T_B(S^{(l)}, a_{(i(l))}, Z_j)) = r(S^{(l)}) + e^{-(\lambda_x+\beta)\Delta} q \left(Q(a)\mathcal{V}(S)\right)_l$$

which allows us to rewrite the linear system in the value function in matrix-vector form:

$$\mathcal{V}(S) = r(S) + e^{-(\lambda_x+\beta)\Delta} qQ(a)\mathcal{V}(S) \Leftrightarrow \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)\mathcal{V}(S) = r(S), \tag{5.65}$$

where $r(S) = (r(S^{(1)}), \ldots, r(S^{(n_S)}))' \in \mathbb{R}^{n_S}$. This is a linear equation system in $\mathcal{V}(S)$ and can easily be solved; theoretically the solution reads

$$\mathcal{V}(S) = \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)^{-1} r(S). \tag{5.66}$$

Notice that $I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)$ is a strictly diagonally dominant matrix according to the Definition 2.3 in Section 2.2: First, the matrix $qQ(a)$ is a (row) stochastic matrix (all rows sum up to one and all entries are non-negative) in line with Definition 2.5 as it represents the transition matrix which contains the transition probabilities from one state to another as entries. Hence, by construction we have $(qQ(a))_{ij} \geqslant 0$, $\forall i,j \in \{1,\dots,n_S\}$, and

$$\sum_{j=1}^{n_S} (qQ(a))_{ij} = 1, \ \forall i \in \{1,\dots,n_S\}.$$

Together with $(I_{n_S})_{ii} = 1$ and $(I_{n_S})_{ij} = 0$ for $i \neq j$, this automatically implies that

$$\begin{aligned}
\left|\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)_{ii}\right| &= \left|\underbrace{(I_{n_S})_{ii}}_{=1} - e^{-(\lambda_x+\beta)\Delta} \underbrace{(qQ(a))_{ii}}_{=1-\sum_{j=1,j\neq i}^{n_S}(qQ(a))_{ij}}\right| \\
&= \left|1 - e^{-(\lambda_x+\beta)\Delta}\left(1 - \sum_{j=1,j\neq i}^{n_S} (qQ(a))_{ij}\right)\right| \\
&= \left|\underbrace{1 - e^{-(\lambda_x+\beta)\Delta} + e^{-(\lambda_x+\beta)\Delta} \sum_{j=1,j\neq i}^{n_S} (qQ(a))_{ij}}_{\geqslant 0}\right| \\
&= 1 - e^{-(\lambda_x+\beta)\Delta} + \left|e^{-(\lambda_x+\beta)\Delta} \sum_{j=1,j\neq i}^{n_S} (qQ(a))_{ij}\right| \\
&\overset{(qQ(a))_{ij}\geqslant 0}{=} 1 - e^{-(\lambda_x+\beta)\Delta} + \sum_{j=1,j\neq i}^{n_S} \left|e^{-(\lambda_x+\beta)\Delta} (qQ(a))_{ij}\right|,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{j=1,j\neq i}^{n_S} \left|e^{-(\lambda_x+\beta)\Delta} (qQ(a))_{ij}\right| &= \sum_{j=1,j\neq i}^{n_S} \left|-e^{-(\lambda_x+\beta)\Delta} (qQ(a))_{ij}\right| \\
&\overset{(I_{n_S})_{ij}=0,\ j\neq i}{=} \sum_{j=1,j\neq i}^{n_S} \left|\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)_{ij}\right|.
\end{aligned}$$

Thus it holds

$$\begin{aligned}
\left|\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)_{ii}\right| &= \underbrace{1 - e^{-(\lambda_x+\beta)\Delta}}_{>0} + \sum_{j=1,j\neq i}^{n_S} \left|\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)_{ij}\right| \\
&> \sum_{j=1,j\neq i}^{n_S} \left|\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)\right)_{ij}\right|
\end{aligned}$$

which implies that the matrix $I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)$ is strictly diagonally dominant. Therefore, the inverse of $I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a)$ always exists due to Theorem 2.4.

Eq. (5.65) shows that in contrast to Section 5.2.2, where the Bellman equation needs to be solved recursively in every time step $t^{(k)}$, we could reduce the problem to finding the fixed point to Bellman's equation

$$\mathcal{V}(S) = \Gamma(\mathcal{V}(S)) := r(S) + e^{-(\lambda_x+\beta)\Delta} q \max_{a \in \text{Grid}(a)} \{Q(a)\mathcal{V}(S)\}$$

with function Γ, often called Bellman operator, defined accordingly.

Now we would like to draw attention to the fact that $Q = Q(a)$ with $a = (a_{(i(1))}, \ldots, a_{(i(n_S))})'$: For this reason, we need to repeat the above for all possible combinations $i(l) \in \{0, \ldots, N_a\}$, $l = 1, \ldots, n_S$, and finally select the combination $a^\star(S) = (a^\star(S^{(1)}), \ldots, a^\star(S^{(n_S)}))'$ that maximizes $\mathcal{V}(S)$ across all $a_{(i)}$ combinations in Grid(a). The total number of combinations equals $n_a^{n_S}$. Thus, we have to calculate $n_a^{n_S}$ times an $n_S \times n_S$ transition matrix $Q(a)$ (plus additionally the inverse of $I_{n_S} - e^{-(\lambda_x+\beta)\Delta} q Q(a)$). If we have $n_V = 1,000$ grid points for V and $n_{CCR} = 26$ for CCR[23], then $n_S = 26,000$. If additionally the allocation grid is divided into steps of 5%, i.e. $n_a = 21$, we would have to calculate $21^{26,000} \approx 10^{34,378}$ times an $26,000 \times 26,000$ matrix, which is a vast number. To overcome this computational problem, we present an alternative by using a policy function iteration algorithm in the following subsection.

5.2.3.3 Policy function iteration: the algorithm

The algorithm is a tailored version of Howard's improvement algorithm, and iterates the policy $a(S)$ until it converges towards its optimal value. For further readings on the policy iteration we refer to Bellman (1955), Bellman (1957), Howard (1960), Puterman (1977), Puterman and Brumelle (1979), Puterman (1981) and Santos and Rust (2004). As mentioned earlier, the theoretical justification and foundation of the policy function iteration algorithm is provided in Section 5.2.3.5. Notice that the one-period total discount factor to this problem is $0 < e^{-(\lambda_x+\beta)\Delta} < 1$ and is a composite of the one-period utility discount factor $e^{-\beta\Delta}$ and the mortality discount factor $e^{-\lambda_x\Delta}$. For n periods, the total discount factor is $e^{-(\lambda_x+\beta)n\Delta}$ and converges to zero as $n \to \infty$. In what follows we describe the policy function iterating mechanism:

Let $a^{(i)} = a^{(i)}(S)$ denote the decision value at iteration step i. Let n_{iter} denote the number of iterations until the algorithm stops. The terminal $a^{(n_{iter})} = a^{(n_{iter})}(S)$ is regarded as the optimal final decision variable $a^\star = a^\star(S)$. Inside the algorithm we repeat the policy improvement and policy evaluation until a sufficient, prescribed level of convergence or solution tolerance is achieved.

1. $i = 0$:
   a) Select initially $a^{(0)}(S) \in \text{Grid}(a)$ for all states in the grid, for instance $a^{(0)}(S^{(l)}) := 0$ $\forall l \in \{1, \ldots, n_S\}$ if $0 \in \text{Grid}(a)$.
   b) Define initially $\mathcal{V}(S) = \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} q Q(a^{(0)}(S))\right)^{-1} r(S)$ for all states in the grid according to Eq. (5.66).
   c) Select a convergence criterion $\epsilon > 0$.
2. Iteration $i = 1, 2, \ldots$:
   a) *Policy improvement*: For all $S^{(l)} \in \text{Grid}(S)$, $l = 1, \ldots, n_S$, find a new policy rule $a^{(i)}(S^{(l)}) \in \text{Grid}(a)$, such that[24]

[23] This coincides with the applied grid in Section 5.2.3.4.

[24] If $a^{(i)}(S^{(l)})$ is not unique, then we select the smallest value among all maximizers and thereby follow the most defensive strategy.

$$\begin{aligned} a^{(i)}(S^{(l)}) &:= \underset{a \in \text{Grid}(a)}{\arg\max} \left\{ r(S^{(l)}, a) + e^{-(\lambda_x+\beta)\Delta} q \sum_{j=1}^{n_Z} \mathcal{V}^{(i-1)}(T_B(S^{(l)}, a, Z_j)) \right\} \\ &\overset{r(S^{(l)},a)\equiv r(S^{(l)})}{=} \underset{a \in \text{Grid}(a)}{\arg\max} \left\{ \sum_{j=1}^{n_Z} \mathcal{V}^{(i-1)}(T_B(S^{(l)}, a, Z_j)) \right\} \end{aligned} \tag{5.67}$$

with $\mathcal{V}^{(i-1)}(T_B(S^{(l)}, a, Z_j)) = \mathcal{V}^{(i-1)}(S^{(m)})$ given from the previous iteration step and according to applied interpolation rule if $S^{(m)}$ not already in the grid.

b) *Policy evaluation*: Having determined $a^{(i)}(S^{(l)})$ for each state $S^{(l)} \in \text{Grid}(S)$, $a^{(i)}(S) = (a^{(i)}(S^{(1)}), \ldots, a^{(i)}(S^{(n_S)}))'$, we update the value function according to Eq. (5.66):

$$\mathcal{V}^{(i)}(S) = \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} q Q(a^{(i)}(S)) \right)^{-1} r(S) \tag{5.68}$$

Although it looks like only P is crucial for the development of the value function $\mathcal{V}^{(i)}(S)$ at first glance (recall $r(S) = r(P)$), we want to mention that the transition from P to the next period P' highly depends on V and its transition.

3. Check the convergence criterion: If $\max_{S \in \text{Grid}(S)} \left\{ \left| a^{(i)}(S) - a^{(i-1)}(S) \right| \right\} \leqslant \epsilon$, then stop and set $n_{iter} := i$ and $a^\star(S) := a^{(n_{iter})}(S)$[25]. Otherwise, repeat Step 2. for iteration $i + 1$.

When the algorithm stops after n_{iter} iterations, i.e. when the convergence criterion after iteration step n_{iter} is met, the stationary solution to the problem is defined as $a^\star = a^\star(S) = a^{(n_{iter})}(S)$ for all grid states.

Before we derive the optimal allocations in a case study next, we briefly summarize the benefits that are associated with this policy function iteration procedure:

- One can determine and thereafter use the optimal strategy for all states independently of time; thus the iterative approach as a very elegant method enhances the speed and efficiency of the numerical optimization.

- The optimal control is independent of the initial state. Therefore, the derived optimals can be used for different initial states. One only has to make sure that the considered initial state lies approximately in the center of the grid, such that a sufficient number of grid nodes still are above and below the starting state. Otherwise it could happen that one remains at the edge of the grid (due to the applied interpolation rule) which would lead to a suboptimal strategy.

In addition, we provide a comment on the speed of convergence of the algorithm. As already mentioned before, the total discount factor equals $e^{-(\lambda_x+\beta)n\Delta}$ for n periods. In the previous case study we used $\beta = 3\%$ and $\lambda_x = 1.18\%$. Then the convergence factor has an approximate size of $e^{-(\lambda_x+\beta)\Delta} = 0.9591$ after one iteration (step size $\Delta = 1$ year), $e^{-(\lambda_x+\beta)10\Delta} = 0.6584$ after ten iterations and $e^{-(\lambda_x+\beta)100\Delta} = 0.0153$ after hundred iterations. This shows that a rather low number

[25] As long as $\epsilon < \Delta^{(\lambda)}$, it holds $a^{(n_{iter})}(S) = a^{(n_{iter}-1)}(S)$.

of iterations is necessary[26]. In particular, Santos and Rust (2004) further argue that policy iteration commonly converges to its stationary solution after a small number of iterations.

Every iteration requires the calculation of $Q(a^{(i)}(S))$ plus the calculation of the inverse of $I_{n_S} - e^{-(\lambda_x+\beta)\Delta} q Q(a^{(i)}(S))$[27] which are both matrices of dimension $n_S \times n_S$; in total the algorithm requires the calculation of n_{iter} times an $n_S \times n_S$ (plus an inverse). This is usually dramatically faster than calculating $n_a^{n_S}$ times an $n_S \times n_S$ matrix (plus an inverse) in Subsection 5.2.3.2. Furthermore, one can use and exploit the property of Q, Q_1, Q_2 to be very sparse matrices; in *Matlab* the functions `sparse(m,n)` and `speye(n)` generate the required sparse matrices which saves memory. Additionally the command `x = A\b` is recommended for solving systems of linear equations of the form $Ax = b$ efficiently.

If one looks for a collective investment strategy for multiple cohorts, one can aggregate the optimal single-cohort policies and apply a mixture strategy weighted according to the total cohort-specific wealths.

5.2.3.4 Case study: policy function iteration for a cohort of clients

We finally demonstrate the presented policy function iteration algorithm in a numerical case study, where we concentrate on the cohort perspective. First, the optimal investment decision variables are determined for every state under the infinite-horizon problem (stationary solution). Afterwards, a simulation analysis in the finite-horizon model, where the approximate optimal stationary solution to the infinite-horizon model is applied, provides the most relevant numbers and probabilities and compares the considered strategies.

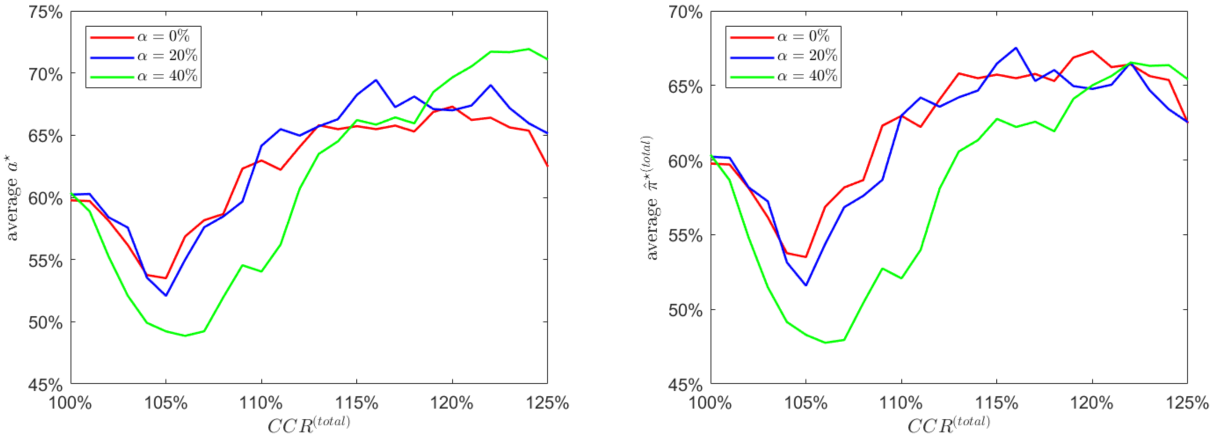
For ease of comparison, we select the very same grids and parameters as in the former case study in Section 5.2.2.5. Furthermore, we determine and consider the optimal investment strategy for $\alpha = (0\%|20\%|40\%)$ (no | moderate | pronounced buffer).

Optimization. Let $V_0 = 10,000$ as before and let us define the state space grid by $V_{min} = 20\% \times V_0$, $V_{max} = 500\% \times V_0$, $n_V = 1,000$ for Grid(V). We further select a step size of 1% for Grid(CCR), thus $n_{CCR} = 26$. This leads to a total grid size of $n_S = 26,000$ states. The grids for Z and a remain unchanged. In particular, Assumptions 5.12 and 5.14 are fulfilled. We seek for a fixed-point solution to the value function according to the policy function iteration algorithm in Section 5.2.3.3. We would like to comment that it only takes seven iterations maximal ($n_{iter} \leqslant 7$) to find the fixed point for $\alpha = (0\%|20\%|40\%)$ and with that the stationary solution to the infinite-horizon optimization problem. Thus, the algorithm converges very quickly (takes $\approx 2h$ for each α).

Figure 5.17 visualizes the average optimal risky relative asset allocations $\hat{\pi}^{\star(\text{inv})} = a^\star$ (investment portfolio) and $\hat{\pi}^{\star(\text{total})}$ (total cohort portfolio) for all $CCR_c^{(\text{total})}$ values in the grid. The grid of the state space was constructed such that there are $n_V = 1,000$ different V values and $n_{CCR} = 26$

[26] Notice that $\lambda_x = 1.18\%$ is suitable if one looks at very short planning horizons. Since in our case the planning horizon is artificially infinity, we can alternatively calibrate the mortality rate λ_x towards the remaining life expectancy of a 65-year old client such that it coincides with the average remaining expected lifetime of 21.06 years (female) and 17.87 years (male) in Germany, cf. Statistisches Bundesamt (2019). We receive $\lambda_x = 5.14\%$. Then the convergence factor drops to $e^{-(\lambda_x+\beta)\Delta} = 0.9218$ after one iteration (step size $\Delta = 1$ year), $e^{-(\lambda_x+\beta)10\Delta} = 0.4431$ after ten iterations and $e^{-(\lambda_x+\beta)100\Delta} = 0.0003$ after hundred iterations. This shows that a higher mortality rate leads to a faster convergence of the algorithm.

[27] In practice, instead of calculating the inverse matrix one typically rather solves the system of linear equations associated with Eq. (5.68) to obtain $\mathcal{V}^{(i)}(S)$ in a more efficient fashion.



(a) $a^{\star}$. (b) $\hat{\pi}^{\star(\text{total})}$.

Figure 5.17: Average $a^{\star}$ and $\hat{\pi}^{\star(\text{total})}$ for a given $CCR_c^{(\text{total})}$ value in the grid.

different P values for every V value selected such that the corresponding grid for the CCR has a step size of one percentage point. Hence, for every $CCR \in \{100\%, 101\%, \ldots, 124\%, 125\%\}$, we build the average over the n_V values for $a^{\star}$ and $\hat{\pi}^{\star(\text{total})}$ that have an equal CCR value. The pattern comes close to an S-shaped form: It can be seen that a higher buffer parameter α, in particular for $\alpha = 40\%$, leads to a lower relative risky investment for small $CCR_c^{(\text{total})}$ values, but catches up for large $CCR_c^{(\text{total})}$ values. This is a desired behavior, since it implies a lower risk of a pension shortening for small $CCR_c^{(\text{total})}$ values within the range $[100\%, 110\%]$, without losing the upside potential of a pension enhancement for $CCR_c^{(\text{total})}$ values close to 125%. Furthermore, except for the region $CCR_c^{(\text{total})} \in [100\%, 105\%]$, the average optimal risky relative investment increases with the $CCR_c^{(\text{total})}$ value. This is meaningful since with a higher $CCR_c^{(\text{total})}$ value, one is less exposed to the risk of falling outside the left boundary of the $CCR_c^{(\text{total})}$ corridor (pension reduction risk). The higher risky investment close to 100% is also reasonable. Imagine the $CCR_c^{(\text{total})}$ is close to 100%; if now the risky allocation is very small, even some positive return of the underlying asset class cannot compensate for the outflows (cohort-related pensions), which pushes the $CCR_c^{(\text{total})}$ below 100% with a high probability.

Simulation study. We next carry out a brief simulation study with a longer time horizon of $\tilde{T} = T + 10\Delta = T + 10$ years $(\Delta = 1)$, i.e. simulation over 10 years compared to the two years in Section 5.2.2.5. We start with the same initial states $S_0 = (V_0, P_0) = (10{,}000, (277|270|258))$ for $\alpha = (0\%|20\%|40\%)$; thus the initial $CCR_c^{(\text{total})}(T)$ and the initial buffer-to-wealth ratio $\frac{V_c^{(\text{buffer})}(T)}{V_0}$ as well as the investment-to-wealth ratio $\frac{V_c^{(\text{inv})}(T)}{V_0}$ keep the same. We simulate 10,000 paths of the relevant processes where we use the optimal stationary solution as asset allocation that corresponds to the closest grid point.

We assume that the average mortality (explained in Section 5.2.2.3) for the cohort is realized. We look at the optimal relative pension evolution $\frac{P^{\star}(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}$, where $P^{\star}(t)$ denotes the cohort pension at time t under the optimal stationary asset allocation strategy $a^{\star} = a^{\star}(S)$. Notice that

$\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0} = \frac{P^\star(t+\Delta)}{e^{-\lambda_{x(j)}(t+\Delta-T)}P_0}$ indicates a stable individual pension for the customers in the cohort from time t to $t+\Delta$, i.e. if $\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}$ is stable then the individual pensions are stable. In what follows we always look at the individual pension perspective in the cohort. Moreover, let $V^\star(t)$ denote the total cohort wealth at time t under $a^\star = a^\star(S)$, in what follows we look at the optimal relative total wealth evolution $\frac{V^\star(t)}{V_0}$.

We would like to point out that the pension $P^\star(t)$ and the asset allocation decisions $a^\star(t)$, $\hat{\pi}^{\star(\text{total})}(t)$ are constant on every annual interval $[t^{(i)}, t^{(i)} + \Delta)$ and are only changed at the evaluation times $t^{(i)}$.

Table 5.7 illustrates the probabilities of pension shortenings and enhancements. Table 5.8 provides risk and reward numbers for the relative pension and the total wealth. In general, we observe that a higher buffer parameter α significantly improves the probabilities in Table 5.7 from a client's perspective. In particular, the probability that the average individual pension that is to be paid out over the entire period is larger than the initial pension level P_0 and the probability that there are more pension enhancements than reductions are quite high, especially for $\alpha = 40\%$. However, both the (relative) risk in terms of volatility and Value-at-Risk and the (relative) reward in terms of expected value do not suffer, which is remarkable. Actually the opposite is the case: A higher buffer parameter α leads to a higher average of the relative pension level and a lower standard deviation (lower standard deviation of relative pension means a more stable pension development); moreover, the worst case relative pensions in the tail (Value-at-Risk) also exceed the ones for smaller α. The single exception is the volatility of the pension, where $\alpha = 20\%$ shows a slightly smaller number than $\alpha = 40\%$. Those benefits of the $\alpha > 0\%$ portfolios comes at the cost of an initially lower pension level $P_0 = P_0(\alpha)$, which represents a tradeoff between the initial pension level and future pension properties. The selection of the case-specific optimal α value, named $\alpha^\star$, depends on the respective target or criterion. If for instance the probability of at least one pension shortening shall coincide with a pre-defined probability p_{red}, $\alpha^\star$ can be selected such that the corresponding probability comes closest to p_{red}. Alternatively, $\alpha^\star$ could be selected such that the expectation of the sum of pension cash flows gets maximized.

In summary in terms of the individual cohort pension (relative to the initial pension level P_0), one can see that $\alpha = 40\%$ outperforms the $\alpha = (0\%|20\%)$ strategies, and the $\alpha = 20\%$ outperforms the $\alpha = 0\%$ strategy. The higher the buffer parameter α, the more the downside risk is limited, and even the upside potential is enhanced.

We draw the conclusion that our proposed model, where we divide our total wealth into an investment and a buffer portfolio, leads to a sophisticated optimal dynamic asset allocation policy that is performance seeking while reducing downside risks and improving probabilities; hence provides remarkable and meaningful benefits to clients.

Finally, we simulate the optimal strategy $a^\star$, the pension $P^\star$ and the wealth $V^\star$ evolution under three different scenarios: a bullish, a bearish and a non-directional market. In each simulation we need to generate the risk driver Z for every period. Figure 5.18 provides the corresponding underlying risky asset class price processes, denoted by $V_Z(t)$, that correspond to the development of Z. Next, Figure 5.19 illustrates the evolution of the relative pension, Figure 5.20 visualizes the very same but for the total wealth. From Figure 5.19 we infer that

1. the individual pensions increase more often for higher α and even end up with a higher terminal pension (relative to P_0) in a bullish market,

	Probability[28]
$\mathbb{P}$("at least one pension reduction"):	49.5%|36.4%|25.5%
$\mathbb{P}$("path-wise average pension $\geqslant P_0$"):	74.1%|80.8%|86.1%
$\mathbb{P}$("number of pension enhancements $\geqslant$ number of pension reductions"):	84.4%|91.3%|96.8%

Table 5.7: Probabilities of pension rate changes for $\alpha = (0\%|20\%|40\%)$.

[29]	$\alpha = 0\%$	$\alpha = 20\%$	$\alpha = 40\%$
$\mathbb{E}\left[\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}\right]$:	107.5%	108.4%	110.9%
$Sd\left(\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}\right)$:	18.1%	16.8%	17.0%
$VaR_{0.05}\left(\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}\right)$:	83.3%	84.8%	86.5%
$VaR_{0.01}\left(\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}\right)$:	71.9%	75.1%	77.0%
$\mathbb{E}\left[\frac{V^\star(t)}{V_0}\right]$:	96.6%	96.2%	96.5%
$Sd\left(\frac{V^\star(t)}{V_0}\right)$:	15.7%	14.3%	14.6%

Table 5.8: Relative performance numbers for $\alpha = (0\%|20\%|40\%)$ under $10,000$ simulations.

2. the individual pensions decrease only once for $\alpha = 40\%$ but twice for the remaining ($\alpha = (0\%|20\%)$) in a bearish market,

3. and the individual pensions do not decline for $\alpha = 40\%$ but do decrease and behave very unstable and volatile for the remaining ($\alpha = (0\%|20\%)$) in a non-directional market.

In total, the number of pension reductions for $\alpha > 0\%$ (with buffer) never exceeds the respective number for $\alpha = 0\%$ (no buffer) in the considered representative scenarios.

Figure 5.21 complements the former figures on the pension and wealth evolution with a visualization of the $CCR_c^{(\text{total})}(t)$ development. While the $CCR_c^{(\text{total})}(t)$ values for $\alpha > 0\%$ (with buffer) do not generally fall short the respective values for $\alpha = 0\%$ (no buffer), the $\alpha > 0\%$ portfolios need less pension shortenings to keep the $CCR_c^{(\text{total})}(t)$ inside its target corridor. Therefore, with selecting a higher $\alpha\%$ value, one can improve the management of the wealth such that the $CCR_c^{(\text{total})}(t)$ remains more stable in its corridor without reducing the pension.

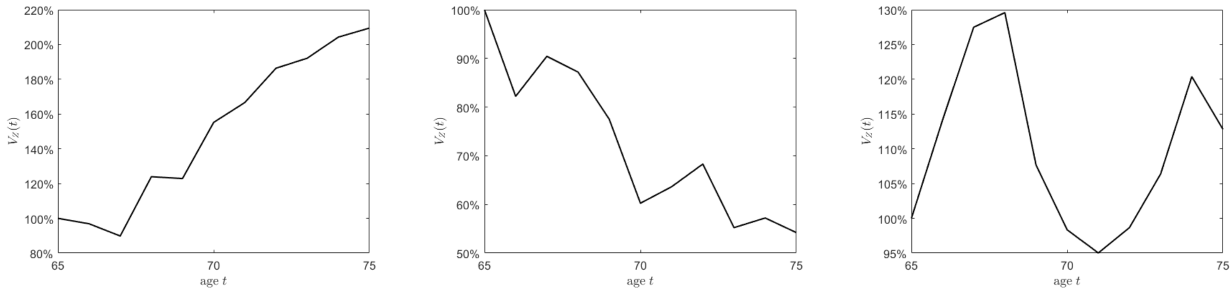
In addition, Figures 5.22 and 5.23 show the optimal asset allocation policies $a^\star(t)$ for the investment wealth and $\hat{\pi}^{\star(\text{total})}(t)$ for the total wealth. One can observe that the optimal strategy for $\alpha = 40\%$ frequently behaves opposed to the optimal strategy for $\alpha = 0\%$. Moreover, Figure 5.24 illustrates

[28] We count the number of paths which fulfill the statement in $\mathbb{P}(\cdot)$. The relative frequency then serves as an estimation for the probability.

[29] Note that $\frac{P^\star(t)}{e^{-\lambda_{x(j)}(t-T)}P_0} = 1$ and $\frac{V^\star(t)}{V_0} = 1$ for initial time $t = T$.

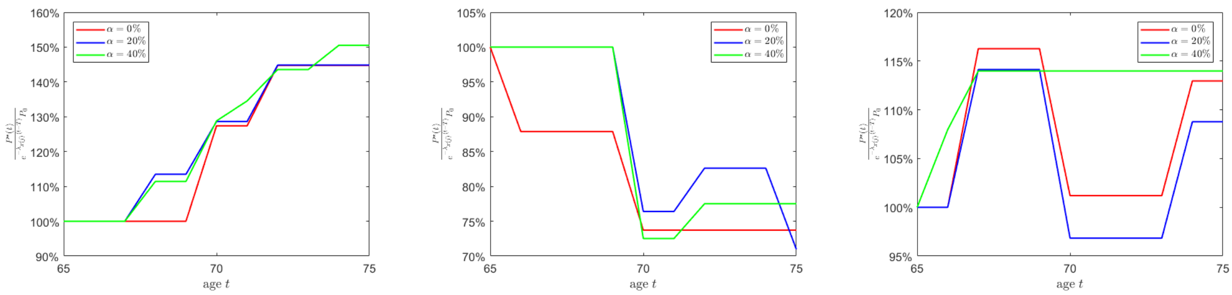
the kernel density estimates for the path-wise average pensions and wealths. Note that for one path, a higher path-wise average pension automatically implies a higher total sum of pension cash flows received by the customer. The figure points out that although the distributions of the wealths are rather close among all considered α values (see also expected values and volatilities in Table 5.8), the distributions of the relative pensions differ. The pension distribution for $\alpha = 40\%$ has lower probability on the left end and is more shifted to the right; this is also reflected in Table 5.8. Thus, a pension fund client that follows the $\alpha = 40\%$ strategy benefits in terms of the pension distribution since lower pensions compared to the initial pension level P_0 are on average less likely. However, as already explained, these benefits come at the cost of an initially lower pension level P_0. We would like to comment that the averages over all simulated $a^\star(t)$ and $\hat{\pi}^{\star(\text{total})}(t)$ values are very close to each other among the three considered buffer parameters α. However, as analyzed above, the relative performance and characteristics of the optimal portfolios with a buffer ($\alpha > 0\%$) are superior over the optimal portfolio without a buffer ($\alpha = 0\%$). This shows that the dynamics and the structure of the asset allocation plays a crucial role.

Compared to the case study in Section 5.2.2.5 we now discover a generally lower level for the optimal risky relative allocations $a^\star(t)$ and $\hat{\pi}^{\star(\text{total})}(t)$. However, the probability of one or more pension shortenings is now higher in comparison with Section 5.2.2.5. Both observations follow from the longer planning horizon: ten instead of two decision periods with a length of one year for every period. It is reasonable that the overall level of the risky relative allocation drops and the investment strategy is thus a bit more defensive if a longer investment horizon is considered because the investment return distribution becomes wider for a larger investment horizon and thus the probability that some barrier is crossed increases. If one aims for similar probabilities of pension shortenings, the risk thus needs to be reduced.



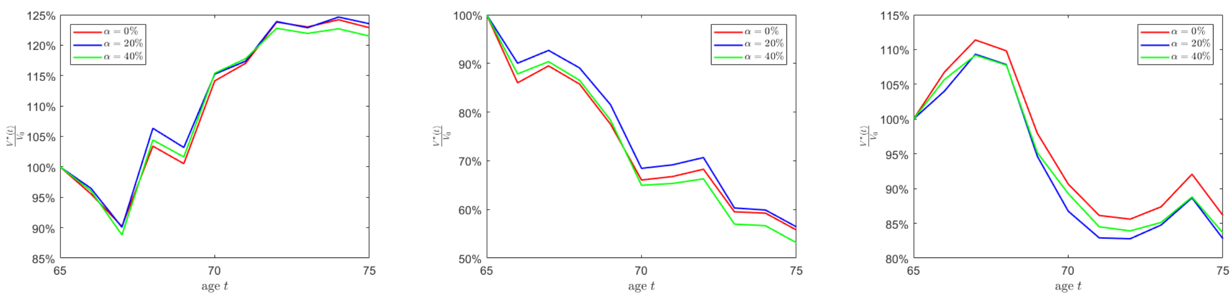
(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.18: Underlying risky asset class price processes $V_Z(t)$ that correspond to risk factor evolution Z in a bullish (left), bearish (center) and non-directional (right) market.



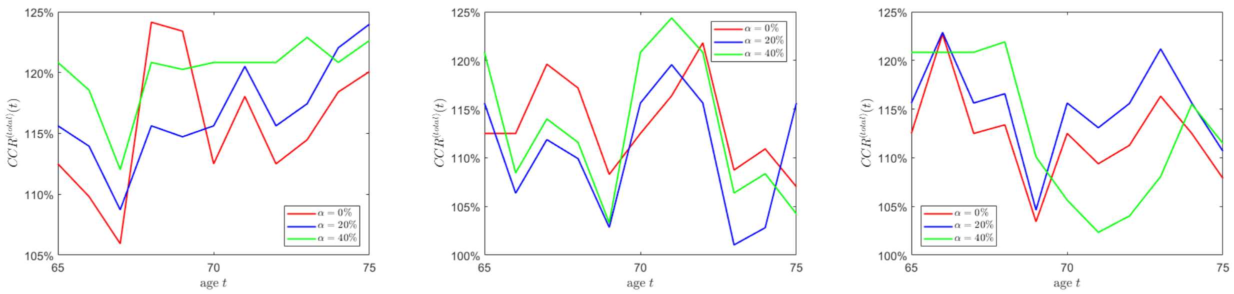
(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.19: Optimal relative pension process $\frac{P^{\star}(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}$ in a bullish (left), bearish (center) and non-directional (right) market.



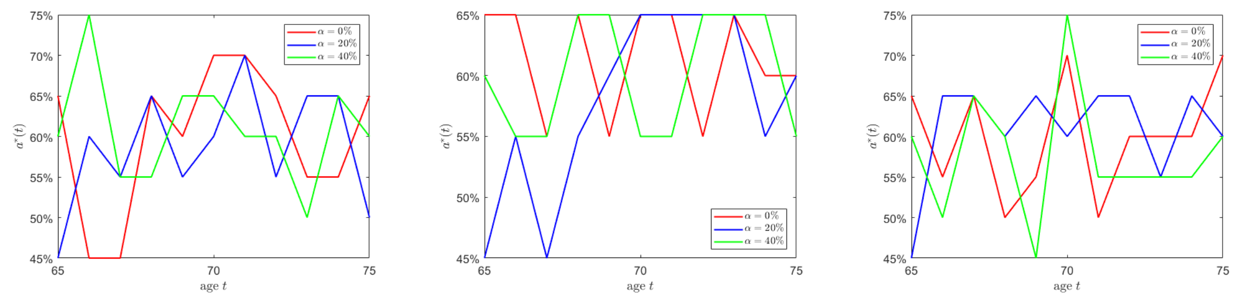
(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.20: Optimal relative total wealth process $\frac{V^{\star}(t)}{V_0}$ in a bullish (left), bearish (center) and non-directional (right) market.



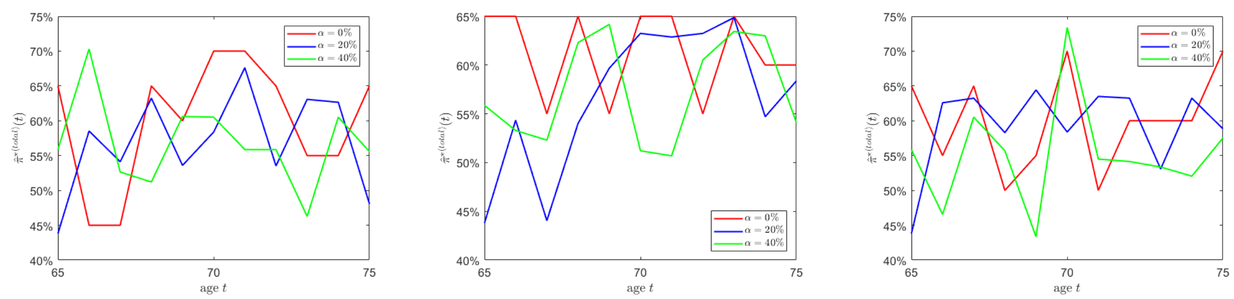
(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.21: Evolution of $CCR_c^{(\text{total})}(t)$ in a bullish (left), bearish (center) and non-directional (right) market.



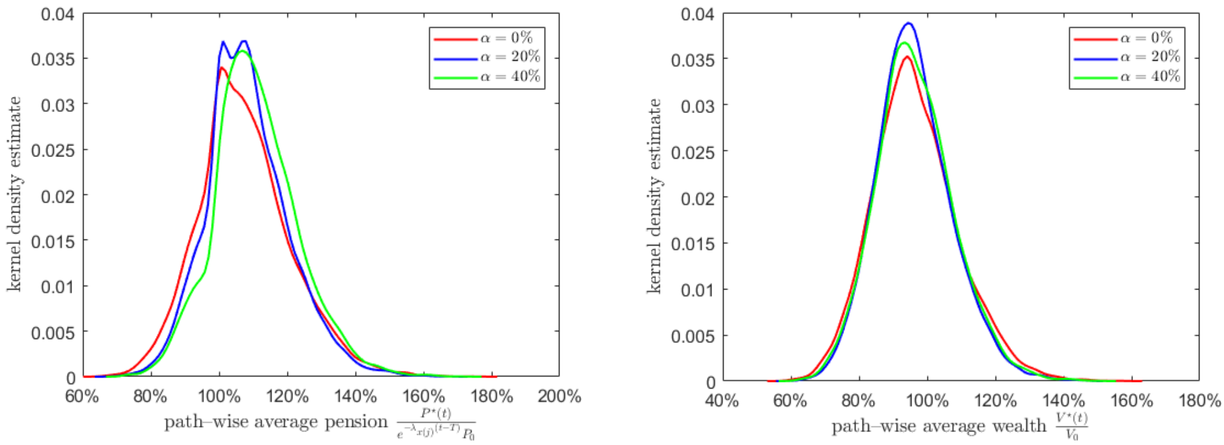
(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.22: Optimal asset allocation decision $a^{\star}(t)$ in a bullish (left), bearish (center) and non-directional (right) market.



(a) Bull market. (b) Bear market. (c) Non-directional market.

Figure 5.23: Optimal asset allocation decision $\hat{\pi}^{\star(\text{total})}(t)$ of the total wealth in a bullish (left), bearish (center) and non-directional (right) market.



(a) Pension. (b) Wealth.

Figure 5.24: Kernel density estimates of path-wise averages of $\frac{P^{\star}(t)}{e^{-\lambda_{x(j)}(t-T)}P_0}$ and $\frac{V^{\star}(t)}{V_0}$.

5.2.3.5 Theoretical foundation

Problem (5.60) is a stationary, infinite-horizon Markovian dynammic programming (MDP) problem in line with the definition in Santos and Rust (2004). We now theoretically justify our policy iteration approach in Section 5.2.3.3 for solving Problem (5.60) under Assumptions 5.12 and 5.14, where we used that the value function is a fixed point. It is necessary to prove the existence and optimality of a unique fixed point for our policy function iteration algorithm and monotone convergence to such a solution.

First, we prove existence of a unique fixed point and optimality of the stationary solution. For this, we introduce the notion of a metric and a metric space.

Definition 5.15 (Metric space (Searcóid (2007), Definition 1.1.1, p. 2; Stokey and Lucas, Jr. (1999), p. 44, 47))**.** *Suppose X is a set and d is a real function defined on the Cartesian product $X \times X$. Then d is called a metric on X if, and only if, for each $f, g, h \in X$,*

1. *Positive property: $d(f, g) \geqslant 0$ with equality if, and only if, $f = g$;*
2. *Symmetric property: $d(f, g) = d(g, f)$; and*
3. *Triangle inequality: $d(f, g) \leqslant d(f, h) + d(h, f)$.*

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to an element in X.

The number $d(f, g)$ is called the distance between f and g with respect to the metric d. (X, d) is called a metric space. Usually, X is simply called metric space.

In what follows we denote S the state space, $s \in S$ a certain state. Further, X is the set of functions that map from S to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, i.e. $X = \{f : S \to \overline{\mathbb{R}}\}$, or a suitable subset of this set.

Example 5.16 (Uniform metric)**.** *Let $f, g \in X$ be real-valued bounded functions that map from the set S to $\mathbb{R}$. Then, the uniform metric is defined as*

$$d(f, g) := \|f - g\|_\infty = \sup_{s \in S} \{|f(s) - g(s)|\},$$

where $\|f\|_\infty := \sup_{s \in S} \{|f(s)|\}$ is called uniform or sup norm of f.

In accordance with Searcóid (2007) and Stokey and Lucas, Jr. (1999) we further define a contraction mapping and a fixed point.

Definition 5.17 (Contraction mapping (Searcóid (2007), Definition 9.9.1, p. 160; Stokey and Lucas, Jr. (1999), p. 50))**.** *Suppose (X, d) is a metric space. A map $\Gamma : X \to X$ is called a (strong) contraction mapping with modulus β if, and only if, there exists $\beta \in [0, 1)$ such that $d(\Gamma(f), \Gamma(g)) \leqslant \beta \cdot d(f, g)$ for all $f, g \in X$.*

Definition 5.18 (Fixed point (Searcóid (2007), Definition 10.10.1, p. 180))**.** *Suppose X is a non-empty set and $\Gamma : X \to X$. A point $f \in X$ is called a fixed point for Γ if and only if $\Gamma(f) = f$.*

Searcóid (2007) argues that "Strong contractions on a metric space, when iterated, tend to pull all the points of the space together into a single point". The underlying theory is the Contraction Mapping Theorem (or Banach's Fixed-Point Theorem).

Theorem 5.19 (Contraction Mapping Theorem (Searcóid (2007), Theorem 10.10.3, p. 181; Stokey and Lucas, Jr. (1999), Theorem 3.2, p. 50))**.** *Suppose (X, d) is a non-empty complete metric space and $\Gamma : X \to X$ is a (strong) contraction mapping with modulus $\beta \in (0, 1)$. Then:*

1. *Γ has a unique fixed point $f \in X$; and*
2. *for any $f_0 \in X$, the sequence $(\Gamma^n(f_0))$ converges to f with*

$$d(\Gamma^n(f_0), f) \leqslant \beta^n \cdot d(f_0, f),\ n = 0, 1, 2, \ldots$$

Theorem 5.19 particularly ensures existence of a unique fixed point for a (strong) contraction mapping.

Santos and Rust (2004) comment that MDP problems (see also beginning of Section 5.2.3) are mathematically equivalent to computing the fixed point to the Bellman equation

$$\mathcal{V} = \Gamma\mathcal{V}$$

with Bellman operator of interest Γ (defined in line with Eq. (5.61))

$$(\Gamma f)(s) := r(s) + \overline{\beta} \sup_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s, a(s), Z)) \middle| s \right] \right\} \tag{5.69}$$

for some function f, where $\overline{\beta} := e^{-(\lambda_x+\beta)\Delta} \in (0, 1)$. We now follow the line of Rieder (1988) and prove the existence of a unique fixed point of the Bellman operator Γ and optimality of the stationary policy for the infinite-horizon problem. The definition of a stationary policy is given next.

Definition 5.20 (Stationary policy (Wakuta (1992), p. 150; Hinderer et al. (2016), p. 214))**.** *A sequence $a^\infty := \{a, a, a, \ldots\}$ for some decision rule $a = (a(s))_{s\in S} \in \mathbb{A}$ is called stationary policy.*

As $\mathcal{V}(s)$, respectively the reward $r(s)$ or the utility $\tilde{U}_1(p)$, is not necessarily bounded in general, we need the notion of an upper barrier function, adjusted to our framework.

Definition 5.21 (Upper barrier function (Rieder (1988), Chapter 1))**.** *A measurable function $b_u : S \to \mathbb{R}_+$ is called upper barrier function if there exist constants $c_1, c_2 \geqslant 0$ such that*

1. *$r(s) \leqslant c_1 b_u(s)$ for all $s \in S$.*
2. *$\mathbb{E}\left[b_u(T_B(s, a, Z))|s\right] = \int Q(s, a; dz) b_u(T_B(s, a, z)) \leqslant c_2 b_u(s)$ for all $s \in S$ and $a \in \mathbb{A}$.*

Q denotes the transition probability measure with $Q(s, a; \cdot)$ being a probability measure for all $(s, a) \in S \times \mathbb{A}$.

Furthermore, we later need the notion of a maximisator which we define next.

Definition 5.22 (Maximisator (Rieder (1988), Chapter 1))**.** *Let $f \in X$. The policy $a_f = a_f(s)$ is called a maximisator for f if it maximizes*

$$(\Gamma f)(s) \overset{(5.69)}{=} r(s) + \overline{\beta} \max_{a(s) \in \mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s, a(s), Z)) \middle| s \right] \right\}$$

for all $s \in S$, i.e. if

$$a_f(s) = \underset{a(s) \in \mathbb{A}}{\arg\max} \left\{ \mathbb{E}\left[f(T_B(s, a(s), Z)) \middle| s \right] \right\}.$$

We already explained earlier that, due to Assumption 5.14, the maximum of any arbitrary function over the finite set $\mathbb{A}$ is attained, for this reason in particular for the function $g : \mathbb{A} \to \mathbb{R}$, $a \mapsto g(a) = \mathbb{E}\left[f(T_B(s, a(s), Z)) \middle| s \right]$ for any given $s \in S$, and thus that the maximisator a_f for every $f \in X$ exists. Moreover, from Assumption 5.12 we already inferred that the utility $\tilde{U}_1(p) \overset{(5.56)}{=} \hat{a} \frac{1-b}{b} \left(\frac{1}{1-b}(p - F) \right)^b$ is bounded, i.e. $\exists 0 < K_{\tilde{U}_1} < \infty$ with $\left| \tilde{U}_1(p) \right| < K_{\tilde{U}_1}$. This immediately implies that also $r(s)$ is bounded with

$$|r(s)| \overset{(5.62)}{=} \frac{1}{\lambda_x + \beta} \left(1 - e^{-(\lambda_x + \beta)\Delta} \right) \underbrace{\left| \tilde{U}_1(p) \right|}_{< K_{\tilde{U}_1}} < \frac{1}{\lambda_x + \beta} \left(1 - e^{-(\lambda_x + \beta)\Delta} \right) K_{\tilde{U}_1}$$

$$\Rightarrow \exists 0 < K_r < \infty : \; |r(s)| < K_r.$$

In view of Definition 5.21, it is thus clear that $b_u \equiv 1$ is an upper barrier function. In line with Rieder (1988) we further define the set

$$\mathbb{B}_{b_u} := \left\{ f : S \to \overline{\mathbb{R}} : \; f \text{ measurable}, \; f(s) \leqslant c b_u(s) \text{ for all } s \in S, \; \text{for one } c \in \mathbb{R}_+ \right\} \subset \{ f : S \to \overline{\mathbb{R}} \}$$

and the weighted sup norm

$$\|f\|_{b_u} := \sup_{s \in S} \frac{|f(s)|}{b_u(s)}$$

which turns in our setting ($b_u \equiv 1$) to the usual sup norm

$$\|f\|_{b_u} = \sup_{s \in S} |f(s)| = \|f\|_\infty .$$

Then $\mathbb{B}_{b_u}$ becomes

$$\mathbb{B}_{b_u} = \left\{ f : S \to \overline{\mathbb{R}} : \; f \text{ measurable}, \; \left\| f^+ \right\|_\infty < \infty \right\}, \tag{5.70}$$

where f^+ denotes the positive part of f. Hence $\mathbb{B}_{b_u}$ denotes the measurable functions mapping from S to $\overline{\mathbb{R}}$ that have an upper bound.

In addition, the boundedness of $\tilde{U}_1(p)$ not only implies boundedness of $r(s)$ but also of $\mathcal{J}(a(s); s, c_c^{(\text{buffer})})$ and $\mathcal{V}(s)$ in Problem (5.60), since

$$\begin{aligned}
\left|\mathcal{J}(a(s); s, c_c^{(\text{buffer})})\right| &= \left|\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)}\tilde{U}_1(P(t^{(i)}))du\right]\right| \\
&\leqslant \mathbb{E}\left[\sum_{i=0}^{\infty}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)} \underbrace{\left|\tilde{U}_1(P(t^{(i)}))\right|}_{<K_{\tilde{U}_1},\ 0<K_{\tilde{U}_1}<\infty} du\right] \\
&< K_{\tilde{U}_1}\mathbb{E}\left[\sum_{i=0}^{\infty}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)}du\right] = K_{\tilde{U}_1}\sum_{i=0}^{\infty}\int_{t^{(i)}}^{t^{(i+1)}} e^{-(\lambda_x+\beta)(u-T)}du \\
&= K_{\tilde{U}_1}\int_{T}^{\infty} e^{-(\lambda_x+\beta)(u-T)}du = K_{\tilde{U}_1}\left[\frac{e^{-(\lambda_x+\beta)(u-T)}}{-(\lambda_x+\beta)}\Bigg|_{u=T}^{u=\infty}\right] \\
&= \frac{K_{\tilde{U}_1}}{\lambda_x+\beta} \\
&\Rightarrow \exists 0 < K_{\mathcal{J}} < \infty:\ \left|\mathcal{J}(a(s); s, c_c^{(\text{buffer})})\right| < K_{\mathcal{J}}
\end{aligned}$$

which leads to

$$\begin{aligned}
|\mathcal{V}(s)| = \left|\sup_{a(s)\in\mathbb{A}} \mathcal{J}(a(s); s, c_c^{(\text{buffer})})\right| &\overset{\mathbb{A}\text{ finite}}{=} \left|\max_{a(s)\in\mathbb{A}} \mathcal{J}(a(s); s, c_c^{(\text{buffer})})\right| \\
&\leqslant \max_{a(s)\in\mathbb{A}} \underbrace{\left|\mathcal{J}(a(s); s, c_c^{(\text{buffer})})\right|}_{<K_{\mathcal{J}}} < \max_{a(s)\in\mathbb{A}} K_{\mathcal{J}} = K_{\mathcal{J}} \\
&\Rightarrow \exists 0 < K_{\mathcal{V}} < \infty:\ |\mathcal{V}(s)| < K_{\mathcal{V}}.
\end{aligned}$$

In line with these observations, we define the set $X_{b_u} \subset \mathbb{B}_{b_u}$ to contain bounded functions only:

$$X_{b_u} := \left\{f : S \to \overline{\mathbb{R}} :\ f \text{ measurable},\ \|f\|_\infty < \infty\right\}. \tag{5.71}$$

From the above calculations it clearly follows $\mathcal{V}(s) \in X_{b_u}$ as well as $\mathcal{J}(a(s); s, c_c^{(\text{buffer})}) \in X_{b_u}$ for all $a(s) \in \mathbb{A}$. We now show that the Bellman operator Γ is a contraction mapping on X_{b_u} equipped with the sup norm $d := \|\cdot\|_\infty$, i.e. on the metric space (X_{b_u}, d).

Theorem 5.23. *Let Assumptions 5.12 and 5.14 be fulfilled. Then, Γ is a contraction mapping on X_{b_u} with modulus $\overline{\beta} \in (0, 1)$.*

Proof. Notice that Γ clearly maps from X_{b_u} to X_{b_u} because for any arbitrary $f \in X_{b_u}$ with $|f(s)| < K_f$ for some $0 < K_f < \infty$, there exists $0 < K_{\Gamma f} < \infty$ such that $|(\Gamma f)(s)| < K_{\Gamma f}$ for all $s \in S$:

$$\begin{aligned}
|(\Gamma f)(s)| &= \left|r(s) + \overline{\beta}\sup_{a(s)\in\mathbb{A}}\left\{\mathbb{E}\left[f(T_B(s, a(s), Z))\Big| s\right]\right\}\right| \\
&\overset{\mathbb{A}\text{ finite}}{=} \left|r(s) + \overline{\beta}\max_{a(s)\in\mathbb{A}}\left\{\mathbb{E}\left[f(T_B(s, a(s), Z))\Big| s\right]\right\}\right|
\end{aligned}$$

$$\leqslant \underbrace{|r(s)|}_{<K_r} + \overline{\beta} \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[\underbrace{|f(T_B(s,a(s),Z))|}_{<K_f} \middle| s \right] \right\} < K_{\Gamma f}$$

for every constant $0 < K_{\Gamma f} < \infty$ with $K_{\Gamma f} \geqslant K_r + \overline{\beta} K_f$. Consequently, $\Gamma f \in X_{b_u}$. We further prove the claim that $d(\Gamma f, \Gamma g) \leqslant \overline{\beta} \cdot d(f,g)$ for all $f, g \in X_{b_u}$. We deduce

$$\begin{aligned} d(\Gamma f, \Gamma g) &= \|\Gamma f - \Gamma g\|_\infty = \sup_{s\in S} |(\Gamma f - \Gamma g)(s)| \\ &= \sup_{s\in S} \left\{ \left| r(s) + \overline{\beta} \sup_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} \right.\right. \\ &\qquad \left.\left. - r(s) - \overline{\beta} \sup_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\} \right| \right\} \\ &= \overline{\beta} \sup_{s\in S} \left\{ \left| \sup_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} - \sup_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\} \right| \right\} \\ &\overset{\mathbb{A}\text{ finite}}{=} \overline{\beta} \sup_{s\in S} \left\{ \left| \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} - \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\} \right| \right\}. \end{aligned}$$

Denote $a_f(s) := \arg\max_{a(s)\in\mathbb{A}} \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right]$ the corresponding maximisator. For the function inside the first supremum we use the following inequality, where we assume w.l.o.g. that $\max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} \geqslant \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\}$:

$$\begin{aligned} &\left| \underbrace{\max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} - \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\}}_{\geqslant 0} \right| \\ &= \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} - \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\} \\ &= \mathbb{E}\left[f(T_B(s,a_f(s),Z)) \middle| s \right] - \underbrace{\max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\}}_{\geqslant \mathbb{E}\left[g(T_B(s,a_f(s),Z)) \middle| s \right]} \\ &\leqslant \mathbb{E}\left[f(T_B(s,a_f(s),Z)) \middle| s \right] - \mathbb{E}\left[g(T_B(s,a_f(s),Z)) \middle| s \right] \\ &= \mathbb{E}\left[f(T_B(s,a_f(s),Z)) - g(T_B(s,a_f(s),Z)) \middle| s \right]. \end{aligned}$$

Inserting this back, gives

$$d(\Gamma f, \Gamma g) = \overline{\beta} \sup_{s\in S} \left\{ \left| \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[f(T_B(s,a(s),Z)) \middle| s \right] \right\} - \max_{a(s)\in\mathbb{A}} \left\{ \mathbb{E}\left[g(T_B(s,a(s),Z)) \middle| s \right] \right\} \right| \right\}$$

$$
\begin{aligned}
&\leqslant \overline{\beta} \sup_{s \in S} \left\{ \mathbb{E}\left[\underbrace{f(T_B(s, a_f(s), Z)) - g(T_B(s, a_f(s), Z))}_{\leqslant \sup_{s\in S} |f(s)-g(s)|} \middle| s \right] \right\} \\
&\leqslant \overline{\beta} \sup_{s \in S} |(f(s) - g(s))| = \overline{\beta} \|f - g\|_\infty \\
&= \overline{\beta} \cdot d(f, g)
\end{aligned}
$$

which was to be shown. □

We now come to the main result in Rieder (1988) about existence of a unique fixed point of Γ and optimality of the stationary policy. We copy the theorem and omit the unnecessary conditions (due to $b_u \equiv 1$).

Theorem 5.24 (Rieder (1988), Satz 1.5)**.** *Let (X, d) be a non-empty complete metric space with $X \subset \mathbb{B}_{b_u}$ and let the following hold:*

1. *For all $f \in X$ there exists a maximisator a_f of f.*
2. *Γ is a contraction on X.*
3. *$0 \in X$.*

Then the following claims hold true:

a) $\mathcal{V} \in X$, $\Gamma\mathcal{V} = \mathcal{V}$ and $\mathcal{V}$ is the unique fixed point of Γ in X.

b) The stationary policy $(a_{\mathcal{V}})^\infty$ is the optimal solution to the infinite-horizon discrete-time optimization problem.

Using this result, we infer the following outcome:

Theorem 5.25. *Let Assumptions 5.12 and 5.14 hold true. Further, let $X = X_{b_u}$ and d be the uniform metric. Then the value function $\mathcal{V}$ is the unique fixed point of Γ in X and the stationary policy $(a_{\mathcal{V}})^\infty$ is the optimal solution to the infinite-horizon discrete-time optimization problem.*

Proof. We prove that the conditions 1.-3. in Theorem 5.24 are fulfilled on (X, d). Hence, the conclusions a) and b) in Theorem 5.24 hold true and the statement in Theorem 5.25 is verified.

Let Assumptions 5.12 and 5.14 be satisfied and let $X = X_{b_u}$ and d be the uniform metric. First, (X, d) is clearly non-empty and also complete as every Cauchy sequence converges within X. We prove 1.-3.:

1. For all $f \in X$ there exists a maximisator a_f of f:

   It was already shown below Definition 5.22 that the maximisator a_f for every $f \in X_{b_u}$ exists due to $\mathbb{A}$ being finite according to Assumption 5.14.

2. Γ is a contraction on X:

   This was proven in Theorem 5.23.

3. $0 \in X$:

   The zero function clearly satisfies $0 \in X = X_{b_u}$.

□

From Theorem 5.25 we infer that, under Assumptions 5.12 and 5.14, there exists a fixed point of Γ in X_{b_u} and if one can find a fixed point of Γ in X_{b_u}, this fixed point is unique and coincides with the value function $\mathcal{V}$. Moreover, the stationary policy $(a_{\mathcal{V}})^\infty$ is the optimal solution to the infinite-horizon discrete-time optimization problem.

Remark 5.26. *In Assumption 5.14 it was assumed that $\mathbb{A}$ consists of finite elements only. A generalization to a continuous set $\mathbb{A}$ is generally possible if the requirements by Rieder (1988), summarized in Theorem 5.24, are fulfilled. In detail, a suitable set X_{b_u} (class of functions) has to exist such that there is a maximisator a_f of f for all $f \in X_{b_u}$ according to Definition 5.22, with Γ mapping from X_{b_u} to X_{b_u}.*

In view of the policy function iteration algorithm in Section 5.2.3.3, it remains to show that this algorithm indeed converges to a fixed point $\mathcal{V}^{(i)} \to \mathcal{V}$ with corresponding optimal stationary policy $a^{(i)} \to (a_{\mathcal{V}})^\infty$.

Santos and Rust (2004) explain that the policy function iteration algorithm as presented in Section 5.2.3.3 can be shown to generate a sequence with $\mathcal{V}^{(i+1)} \geqslant \mathcal{V}^{(i)}$ under fairly general conditions. In our setup where the state space S and the values for the risk driver Z come from finite sets or grids (cf. Section 5.2.3.3), we have the following general monotonicity result for the iterated value function:

Theorem 5.27 (Monotonicity of $\mathcal{V}^{(i)}$)**.** *The iteration in the policy function algorithm (cf. Section 5.2.3.3) leads to a monotone increasing sequence $\left(\mathcal{V}^{(i)}\right)_{i=0,1,2,\ldots}$:*

$$\mathcal{V}^{(i+1)}(S) - \mathcal{V}^{(i)}(S) \geqslant \mathbf{0}.$$

Proof. First, we have

$$\begin{aligned}
&\mathcal{V}^{(i+1)}(S) - \mathcal{V}^{(i)}(S) \\
&\overset{(5.68)\text{: policy evaluation step}}{=} \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^{-1} r(S) - \mathcal{V}^{(i)}(S) \\
&= \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^{-1} \left[r(S) - \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right) \mathcal{V}^{(i)}(S)\right] \\
&= \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^{-1} \left(r(S) + e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\mathcal{V}^{(i)}(S) - \mathcal{V}^{(i)}(S)\right) \\
&\overset{(5.67)\text{: policy improvement step}}{=} \left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^{-1} \\
&\qquad\qquad \times \left(\max_{a(S)\in\mathrm{Grid}(a)} \left\{r(S) + e^{-(\lambda_x+\beta)\Delta} qQ(a(S))\mathcal{V}^{(i)}(S)\right\} - \mathcal{V}^{(i)}(S)\right).
\end{aligned}$$

For the second term we observe

$$
\begin{aligned}
&\max_{a(S)\in \mathrm{Grid}(a)} \left\{ r(S) + e^{-(\lambda_x+\beta)\Delta} qQ(a(S))\mathcal{V}^{(i)}(S) \right\} - \mathcal{V}^{(i)}(S) \\
&\geqslant \left\{ r(S) + e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i)}(S))\mathcal{V}^{(i)}(S) \right\} - \mathcal{V}^{(i)}(S) \overset{(5.68)\text{: policy evaluation step}}{=} \mathcal{V}^{(i)}(S) - \mathcal{V}^{(i)}(S) = \mathbf{0}.
\end{aligned}
$$

Hence every entry in the second term is non-negative. The same holds for the first term by the following argument: First, it is $(I_n - H)^{-1} = \sum_{k=0}^{\infty} H^k = I_n + \sum_{k=1}^{\infty} H^k$ for any matrix $H \in \mathbb{R}^{n\times n}$ such that the power series $\sum_{k=1}^{\infty} H^k$ converges due to

$$
(I_n - H)\sum_{k=0}^{\infty} H^k = \sum_{k=0}^{\infty} H^k - \sum_{k=1}^{\infty} H^k = I_n = (I_n - H)\,(I_n - H)^{-1}.
$$

According to Young (1981), the matrix power series $\sum_{k=1}^{\infty} H^k$ (also called "Neumann series") converges if for every eigenvalue $\lambda(H)$ of the matrix H it holds $|\lambda(H)| < 1$, i.e. $\max|\lambda(H)| < 1$.

Set $G := qQ(a^{(i+1)}(S))$ and $H := e^{-(\lambda_x+\beta)\Delta}G$. The matrix G is a (row) stochastic matrix (all rows sum up to one and all entries are non-negative) in line with Definition 2.5 since it represents the transition matrix that contains the transition probabilities from one state to another as entries. With

$$
Gx = \lambda(G)x \Leftrightarrow Hx = \lambda(H)x,\ \ \lambda(H) := e^{-(\lambda_x+\beta)\Delta}\lambda(G),
$$

it follows that $\lambda(G)$ is an eigenvalue to matrix G if and only if $\lambda(H) = e^{-(\lambda_x+\beta)\Delta}\lambda(G)$ is an eigenvalue to matrix H. From Theorem 2.7 it follows that $\max|\lambda(G)| = 1$ as G is a stochastic matrix. Thus,

$$
\max|\lambda(H)| = \max|e^{-(\lambda_x+\beta)\Delta}\lambda(G)| = e^{-(\lambda_x+\beta)\Delta}\underbrace{\max|\lambda(G)|}_{=1} = e^{-(\lambda_x+\beta)\Delta} < 1.
$$

We conclude that

$$
\left(I_{n_S} - e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^{-1} = I_{n_S} + \sum_{k=1}^{\infty}\left(e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^k,
$$

where all entries of I_{n_S} and $\left(e^{-(\lambda_x+\beta)\Delta} qQ(a^{(i+1)}(S))\right)^k$ are non-negative since $e^{-(\lambda_x+\beta)\Delta} > 0$ and $G = qQ(a^{(i+1)}(S)) \geqslant 0$ is a stochastic matrix.

In summary, we multiply a matrix of non-negative entries (first term) with a vector of non-negative entries (second term) and therefore finally receive a vector of non-negative entries which implies monotonicity in the value function:

$$
\mathcal{V}^{(i+1)}(S) - \mathcal{V}^{(i)}(S) \geqslant \mathbf{0}.
$$

□

In view of Theorem 5.27,, that tells that the iterated value function does not cycle, we conclude the following:

Theorem 5.28 (Convergence of the policy function iteration algorithm)**.** *Let us consider a finite state space Grid*(S) *and a finite action space Grid*(a)*. Then the policy function iteration algorithm* *converges to the true fixed point for the contraction* Γ*, which is the optimal value function* $\mathcal{V}$ *of the problem, within a finite number of iteration steps.*

The argument is clear due to monotonicity in Theorem 5.27 and the finite cardinality of the state and action space, see also Santos and Rust (2004). Further readings on convergence results can also be found in Puterman and Brumelle (1979) and Puterman (1981). In summary, we have proven that, under Assumptions 5.12 and 5.14, our problem admits a unique fixed point solution and our presented algorithm in Section 5.2.3.3 converges to this solution. This finishes the theoretical foundation section.

6 Conclusion

In the context of utility maximization we studied three improvements of current models in the academic literature and determined the optimal investment strategies. In all proposed frameworks we aimed to model the pension fund investor's behavior and preferences towards risk in a more realistic way and by this targeted to create meaningful benefits for investors. Here come some chapter-specific conclusions and comments on possible future research:

Chapter 3 dealt with optimal portfolio choice for a pension fund investor within the behavioral finance concept of Cumulative Prospect Theory, and therefore considered an S-shaped utility function and investor-specific distortions on the probabilities in modeling investor's risk preferences and views. The general solution for both a utility function with arbitrary positive and not necessarily infinite derivative at the reference wealth as well as HARA utility function in particular was derived, for well-funded as well as underfunded pension funds. Sufficient conditions for well-posedness of the optimization problem and conditions for obtaining a closed-form solution when considering HARA utility function were provided. Moreover, a suitable probability distortion function family was shown to satisfy the imposed conditions. This work therefore extended existing results for special utility and probability distortion functions and significantly widened the class of utility functions which can be applied by investors such that closed-form solutions, together with the analytical expression of the optimal investment strategy, are achieved. Moreover, the novel application of the HARA utility in the behavioral S-shaped utility model allowed for modeling a utility function which can admit an increasing, constant or decreasing relative risk aversion representation. It turned out that the solution for HARA utility involves a higher complexity via a constraint in form of a non-linear equation (see introduced $\hat{z}$). This complicated not only the proof of optimality, but also led to a structural break in the final optimal terminal wealth solution compared to the CRRA case. Whereas a sufficiently wealthy investor applying CRRA will always remain above the pre-specified benchmark B at the end of the investment period, a sufficiently wealthy investor applying HARA can either end above B with a high probability or end at B with a small probability. There is a one-to-one relationship between the probability of ending at B and $\hat{z}$. It can be observed that for instance the probability distortion function on the gains has an impact on the value of $\hat{z}$ and thus on the probability of ending at B. When the gains distortion underestimates probabilities of small gains and overestimates the probabilities of large gains, i.e. when the investor's view on future returns is optimistic, then the value for $\hat{z}$ increases and so does the probability of ending at B because the investor bets more on increasing stock prices, and vice versa. The numerical case study finally showed and visualized that the considered model and model parameters can lead to a suitable optimal investment strategy and thus an improvement of the CRRA utility model. In addition, CPT can help to explain mispricing of share prices, the creation of bubbles in financial markets or situations of underinvestments in risky assets. Possible future research could deal with exploring further utility or probability distortion functions beyond our framework, or the integration of a suitable contribution inflow process to the pension fund.

Chapter 4 studied the optimal quantitative and dynamic consumption and investment strategies for a pension fund investor under age-dependent risk preferences (coefficient of risk aversion $b(t)$

and preference between consumption and terminal wealth $a(t)$). The findings demonstrated that strategies applied for life-cycle pension funds or pension insurance could significantly be improved by taking age-dependent risk preferences into account. For this reason, our work combined the elements terminal wealth with a minimum level and consumption under time-varying risk preferences and minimum level of consumption into a dynamic life-cycle consumption-investment model. A sound economic understanding of the model parts was provided. In Section 4.2 the corresponding portfolio optimization problem was solved analytically with a separation approach which allowed to solve the consumption and the terminal wealth part of the original consumption-investment problem separately. The formulas showed that age-depending risk preferences in combination with terminal wealth considerations and minimum levels for consumption and wealth have a significant impact on the optimal controls. Section 4.3 investigated the optimal controls and provided a comparison with already existing and solved benchmark models. The analysis was divided into two parts. In the first part the risk preferences were calibrated towards given realistic curves for consumption and investment. The result emphasized that only our proposed flexible model, in comparison with the other considered benchmark models, provides an adequate fit of the agent's behavior. We draw the conclusion that time-varying preferences (risk aversion $b(t)$ and preference between consumption and terminal wealth $a(t)$) are necessary to provide a sufficient degree of flexibility to accurately fit the two control variables consumption and investment. The very same result was obtained when time-dependent preference functions were considered, but the consumption and wealth floors were omitted. The second part focused on the behavior analysis of the optimal consumption, investment and wealth under a positive and negative market environment. Future research on this topic could deal with generalizations of the dynamic life-cycle model. For instance, investment constraints could be included to make the whole setup more applicable as budgets in practice are commonly exposed to constraints on allocation or risk. Furthermore, since unemployment risk and uncertain future income are essential for individuals, those risks and impacts on the optimal controls and wealth process could be further explored. Finally, including mortality and a life insurance product into the model could help people in determining their optimal individual life insurance investment embedded in a more realistic, flexible framework.

Chapter 5 studied a possible implementation of the innovative pension plan known as "Nahles-Rente". We transferred the product rules into a mathematical model and solved the resulting portfolio selection problem in closed-form (Accumulation phase in Section 5.1) and via the discrete-time Bellman equation (Decumulation phase in Section 5.2). We draw the following conclusions:

Section 5.1 proposed a suitable wealth-dependent buffer rate process and provided analytical closed-form solutions for the complex optimization problem that corresponds to the accumulation phase of a new German pension product. We demonstrated that the solution can be implemented and reasonable results can be achieved. Moreover, a numerical analysis showed that the optimal policy indeed provides beneficial characteristics such as a smoothing mechanism of the reported wealth process and an increase in the Sharpe Ratio. In summary, it was shown that the optimal reported portfolio with buffer provides a smoothed evolution compared to optimal portfolios without a buffer rule. Furthermore, although the smoothing mechanism lowered the downside risk and the volatility of the wealth, it did not much lower the upside potential. Hence, we proposed a buffer rule that is beneficial to clients seeking for pension products with a more stable reported wealth development, and further contributed in finding the optimal controls and wealths in closed-form such that it could be implemented in practice. Finally, we provided a reasonable setting where the terminal buffer balance remains always non-negative. Future research could address and study this desirable characteristic in multiple directions such as elaborating on other parameterizations for $B(t)$ and F, different to Assumption 5.6, or settings that allow for $r \neq 0$.

Section 5.2 modeled the complex mechanism of the product with ingredients buffer balance and pension adjustments in the post-retirement phase. In particular we proposed a special buffer rule. The resulting optimization problem with finite-horizon was solved via Bellman's equation. The stationary asset allocation solution approach to the infinite-horizon problem further provided an elegant approximate solution to the problem. Therein, we introduced an efficient policy function iteration algorithm that converged to the unique stationary solution. A case study showed several meaningful benefits to customers. The following conclusions apply in the scope of the tested buffer parameters: First, a more pronounced buffer parameter can significantly improve the probabilities of interest (pension shortenings and pension level evolution). Furthermore, the higher the buffer parameter, the more the downside risk was limited while even the upside potential was enhanced, both relative to the initial pension. Of course, these benefits came at the cost of an initially smaller pension payment. In summary, there was a tradeoff between relative outperformance (more pronounced buffer) and initial pension level (less pronounced buffer).

Overall, we detected that our proposed model leads to a sophisticated optimal dynamic asset allocation policy that is performance seeking while reducing downside risks and improving the probabilities of interest; hence provides remarkable and meaningful benefits to clients. Possible future research studies on "Nahles-Rente" pension products could generally elaborate on alternative buffer processes inside the accumulation and the decumulation phase or consider a more advanced mortality model with a mortality rate that is exposed to (unexpected) shocks such as the paper by Escobar et al. (2016) which studies a mortality model with mortality improvement ratio in the framework of pricing variable annuities with guaranteed minimum repayments. Furthermore, future research could deal with the design, the modeling and the optimal management of a pension fund plan that belongs to an entire collective of investors, where the wealth is managed identically for all clients instead of a cohort-specific treatment.

In summary, in all of the three Chapters 3–5 we derived the optimal asset allocation policy and illustrated that the application can lead to meaningful benefits for pension fund investors.

A Appendix to Chapter 3

A.1 Proof of Main Results

Proof of Theorem 3.5. Cases 1 and 2 are clear, we prove 3. For this sake, let us consider the following Choquet maximization problem:

$$\begin{aligned} &\sup_X \mathcal{J}_+(X) \\ &\text{subject to } \begin{cases} \mathcal{J}_+(X) = \int_0^\infty w_+(\mathbb{P}(U_+(X) > y))dy, \\ \mathbb{E}[\tilde{Z}X] = a \geqslant 0, \ X \geqslant 0. \end{cases} \end{aligned} \tag{A.1}$$

Applying a quantile transformation and using the Lagrange method to remove the budget constraint according to Jin and Zhou (2008), Problem (A.1) can be transformed into the following problem on the quantile, for given $\lambda \in \mathbb{R}$:

$$\begin{aligned} &\sup_g \mathcal{J}_+^\lambda(g) \\ &\text{subject to } \begin{cases} \mathcal{J}_+^\lambda(g) = \mathbb{E}\left[U_+\left(g(Z)\right) w'_+\left(1-Z\right) - \lambda g(Z) F_{\tilde{Z}}^{-1}\left(1-Z\right)\right], \\ g \in \Gamma := \{g : [0,1) \to \mathbb{R}^+ \text{ non-decreasing, left-continuous, with } g(0) = 0\}. \end{cases} \end{aligned} \tag{A.2}$$

Here $Z := 1 - F_{\tilde{Z}}(\tilde{Z}) \sim U(0,1)$, and Γ is the set of quantile functions of all the non-negative random variables. Let us ignore the implicit and complex constraint $g \in \Gamma$ in the first step. For each fixed $z \in (0,1)$ we maximize

$$U_+(g(z))w'_+(1-z) - \lambda g(z) F_{\tilde{Z}}^{-1}(1-z)$$

over $g(z) \in \mathbb{R}^+$. We can make two observations:

First, maximizing $U_+(g(z))w'_+(1-z) - \lambda g(z)F_{\tilde{Z}}^{-1}(1-z)$ over $g(z)$ is equivalent with maximizing $U_+(g(z)) - g(z)\lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w'_+(1-z)}$ over $g(z)$ since $w'_+(1-z) > 0 \ \forall z \in (0,1)$ is independent of g.

Second, the term $\lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w'_+(1-z)}$ is non-increasing in z by Assumption 3.3.1. Further, due to Assumption 3.2, it is a continuous function in z for $z \in (0,1)$ mapping to $(0,\infty)$ hence there is at least one $\hat{z} = \hat{z}(\lambda) \in (0,1)$ such that $\lambda \frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w'_+(1-\hat{z})} = U'_+(0+)$. Then, by monotonicity and continuity, we have

$$\begin{aligned} \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w'_+(1-z)} &< U'_+(0+), \text{ for } z \in (\hat{z},1), \\ \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w'_+(1-z)} &\geqslant U'_+(0+), \text{ for } z \in (0,\hat{z}] \,. \end{aligned}$$

Let us consider the appearing two cases $z \in (\hat{z}, 1)$ and $z \in (0, \hat{z}]$ separately.

1. $z \in (\hat{z}, 1) \Leftrightarrow \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} < U_+'(0+)$:

   The zero first derivative condition leads to

$$U_+'(g(z)) = \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} .$$

   Since $0 < \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} < U_+'(0+)$, $(U_+')^{-1}$ is well-defined and we can solve for $g(z)$. As long as $(U_+')^{-1}(z)$ and $\frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)}$ are non-increasing in $z \in [0,1)$, then $g(z)$ is non-decreasing in $z \in (\hat{z}, 1)$. Moreover, $g(\hat{z}) = 0$, $g(z) > 0$ and thus g solves Problem (A.2).

2. $z \in (0, \hat{z}] \Leftrightarrow \lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} \geqslant U_+'(0+)$:

   In this case we have

$$U_+(g(z)) - g(z)\lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} \leqslant U_+(g(z)) - g(z)U_+'(0+),$$

   since $g(z) \geqslant 0$. The right hand side is maximal when $U_+'(g(z)) - U_+'(0+) \stackrel{!}{=} 0$ which is only the case when $g(z) = 0$. As for $g(z) = 0$ it is $U_+(g(z)) - g(z)\lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)} = 0$, $g(z) = 0$ solves Problem (A.2) for $z \in (0, \hat{z}]$.

Hence, the optimal quantile function $g^\star$ to Problem (A.2) is given by

$$g^\star(z) = \begin{cases} 0, & z \in [0, \hat{z}], \\ (U_+')^{-1}\left(\lambda \frac{F_{\tilde{Z}}^{-1}(1-z)}{w_+'(1-z)}\right), & z \in (\hat{z}, 1), \end{cases}$$

where $\lambda > 0$ satisfies

$$\mathbb{E}\left[g^\star(Z) F_{\tilde{Z}}^{-1}(1-Z)\right] = a$$

and $\hat{z} \in (0,1)$ such that

$$\lambda \frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})} = U_+'(0+) .$$

After some transformation, Appendix C in Jin and Zhou (2008) now tells that $X^\star = g^\star(Z)$ is optimal to Problem (A.1). Inserting $Z = 1 - F_{\tilde{Z}}(\tilde{Z})$ leads to

$$X^\star = X^\star(\lambda) = (U_+')^{-1}\left(\lambda \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z})} \tag{A.3}$$

being an optimal solution for Problem (A.1) if $\mathbb{E}\left[U_+\left(\left(U_+'\right)^{-1}\left(\frac{\eta\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\right) w_+'(F_{\tilde{Z}}(\tilde{Z}))\right] < \infty$ $\forall \eta > 0$. $\lambda > 0$ is the one satisfying $\mathbb{E}[\tilde{Z}X^\star] = a \geqslant 0$. In addition, the form of $X^\star$ directly implies

$$\begin{aligned}\mathbb{P}(X^\star = 0) &= \mathbb{P}(\tilde{Z} > F_{\tilde{Z}}^{-1}(1-\hat{z})) = 1 - \mathbb{P}(\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z})) = 1 - F_{\tilde{Z}}(F_{\tilde{Z}}^{-1}(1-\hat{z})) \\ &= 1 - (1-\hat{z}) = \hat{z}\,.\end{aligned}$$

We now turn to Problem (P$_+$). Let $c > 0$ (the case $c = 0$ is clear), $A = \{\omega : \tilde{Z} \leqslant c\}$, and define $w_A(x) := w_+(x\mathbb{P}(A))/w_+(\mathbb{P}(A))$, $x \in [0,1]$. w_A is strictly increasing and differentiable with $w_A(0) = 0$ and $w_A(1) = 1$. For any feasible X to Problem (P$_+$) and $y \geqslant 0$ it is

$$w_+(\mathbb{P}(U_+(X) > y)) = w_+(\mathbb{P}(U_+(X) > y|A)\mathbb{P}(A)) = w_+(\mathbb{P}(A))w_A(\mathbb{P}(U_+(X) > y|A)\,.$$

Next, we study Problem (P$_+$) in the space $(\Omega \cap A, \mathcal{F} \cap A, \mathbb{P}_A := \mathbb{P}(\cdot|A))$:

$$\begin{aligned}&\sup_Y \mathcal{J}_+(Y) \\ &\text{subject to } \begin{cases}\mathcal{J}_+(Y) = w_+(\mathbb{P}(A))\int_0^\infty w_A(\mathbb{P}_A(U_+(Y) > y))dy, \\ \mathbb{E}_A[\tilde{Z}Y] = v_+/\mathbb{P}(A) \geqslant 0,\ Y \geqslant 0.\end{cases}\end{aligned} \tag{A.4}$$

By following the analogous steps in Jin and Zhou (2008) on pp. 401–402 and applying Eq. (A.3), we can conclude that the optimal solution for (A.4) is given by

$$Y^* = (U_+')^{-1}\left(\bar{\lambda}\frac{\tilde{Z}}{w_A'(F_A(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z} \leqslant F_A^{-1}(1-\hat{z})}$$

for some $\bar{\lambda} > 0$ (Lagrange multiplier of corresponding problem), where $F_A(x) := \mathbb{P}_A(\tilde{Z} \leqslant x)$, $x \geqslant 0$. Defining $\lambda := \frac{w_+(\mathbb{P}(A))}{\mathbb{P}(A)}\bar{\lambda} \geqslant 0$ and observing that $\mathbb{P}(A) = \mathbb{P}(\tilde{Z} \leqslant c) = F_{\tilde{Z}}(c)$, we obtain the optimality of $X_+^* := Y^*\mathbb{1}_{\tilde{Z} \leqslant c}$ in view of the relation between Problem (A.4) and Problem (P$_+$) with then

$$\begin{aligned}X_+^* &= Y^*\mathbb{1}_{\tilde{Z}\leqslant c} = (U_+')^{-1}\left(\bar{\lambda}\frac{\tilde{Z}}{w_A'(F_A(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z} \leqslant F_A^{-1}(1-\hat{z})}\mathbb{1}_{\tilde{Z}\leqslant c} \\ &= (U_+')^{-1}\left(\lambda\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z} \leqslant \min\{c, F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))\}}\,.\end{aligned}$$

By monotonicity of $F_{\tilde{Z}}^{-1}$ and due to $\hat{z} \in (0,1)$, we have

$$X_+^* = (U_+')^{-1}\left(\lambda\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\,.$$

Finally, the optimal value, i.e. objective functional, of Problem (P$_+$) can be calculated via $\mathbb{E}_{\mathbb{P}_A}[\cdot] = \mathbb{E}[\cdot|A] = \frac{\mathbb{E}[\cdot\mathbb{1}_A]}{\mathbb{P}(A)}$ for $\mathbb{P}(A) > 0$ as follows:

$$\begin{aligned}\mathcal{Y}_+(c, v_+) &= w_+(\mathbb{P}(A))\mathbb{E}_{\mathbb{P}_A}\left[U_+(Y^*)w_A'(F_A(\tilde{Z}))\right] = w_+(\mathbb{P}(A))\mathbb{E}\left[U_+(Y^*)w_A'(F_A(\tilde{Z}))|A\right] \\ &= \mathbb{E}\left[U_+(Y^*)w_+'(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant c}\right]\end{aligned}$$

which completes the proof. □

Proof of Lemma 3.7. First, recall Assumption 3.3.2:

$$\mathbb{E}\left[U_+\left(\left(U_+'\right)^{-1}\left(\frac{\eta\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\right)w_+'(F_{\tilde{Z}}(\tilde{Z}))\right] < \infty,\ \forall\eta > 0\,.$$

Inserting HARA utility U_+ leads to

$$\begin{aligned}
&\mathbb{E}\left[U_+\left(\left(U_+'\right)^{-1}\left(\frac{\eta\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\right)w_+'(F_{\tilde{Z}}(\tilde{Z}))\right]\\
&= \mathbb{E}\left[U_+\left(\frac{1-b_H}{a_H}\left[\left(\frac{\frac{\eta\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}}{a_H}\right)^{-\frac{1}{1-b_H}} - d_H\right]\right)w_+'(F_{\tilde{Z}}(\tilde{Z}))\right]\\
&= \frac{1-b_H}{b_H}a_H^{\frac{b_H}{1-b_H}}\,\mathbb{E}\left[\left(\frac{\eta\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{b_H}{1-b_H}} w_+'(F_{\tilde{Z}}(\tilde{Z}))\right] - \frac{1-b_H}{b_H}d_H^{b_H}\,\mathbb{E}\left[\underbrace{w_+'(F_{\tilde{Z}}(\tilde{Z}))}_{>0}\right]\\
&< \frac{1-b_H}{b_H}\left(\frac{a_H}{\eta}\right)^{\frac{b_H}{1-b_H}}\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\right]\,.
\end{aligned}$$

Therefore, under HARA utility it is sufficient that $\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\right] < \infty$.

Second, Assumption 3.3.3 is automatically satisfied under Assumption 3.2 when applying HARA utility. Recall Assumption 3.3.3:

There exists $z \in (0,1)$ with $f(z) = 0$ for any arbitrary $a > 0$ and $0 < c \leqslant \infty$, where $f : (0,1) \to \mathbb{R}$ is defined by

$$z \mapsto f(z) := \lambda(z) - U_+'(0+)\frac{w_+'(1-z)}{F_{\tilde{Z}}^{-1}(1-z)},$$

with $\lambda(z) \geqslant 0$ solving $\mathbb{E}\left[\tilde{Z}(U_+')^{-1}\left(\frac{\lambda(z)\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right] = a.$

To prove this claim, we insert HARA utility in the budget equation and solve for $\lambda(z)$, which yields

$$\lambda(z) = (1-b_H)^{1-b_H}\,a_H^{b_H}\left(\frac{a + \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right]}\right)^{-(1-b_H)}\,.$$

Thus

$$f(z) = (1-b_H)^{1-b_H}\, a_H^{b_H} \left(\frac{a + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-z))}\right]} \right)^{-(1-b_H)} - a_H d_H^{-(1-b_H)} \frac{w'_+(1-z)}{F_{\tilde{Z}}^{-1}(1-z)}\,. \tag{A.5}$$

We now show that there exists $z \in (0,1)$ such that $f(z) = 0$ for general probability distortion function w_+ satisfying Assumption 3.2. First, f is continuous. Next, the limits are

$$\lim_{z\searrow 0} f(z) = (1-b_H)^{1-b_H}\, a_H^{b_H} \left(\frac{a + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant c}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant c}\right]} \right)^{-(1-b_H)} \in (0,\infty],$$

$$\lim_{z\nearrow 1} f(z) = -\infty\,.$$

In summary, we have that f is continuous with $f(0+) > 0$ and $f(1-) < 0$. Hence, there exists $z \in (0,1)$ such that $f(z) = 0$. □

Proof of Theorem 3.8. 1: For the value function, HARA utility implies

$$\begin{aligned}
&\mathcal{Y}_+(c, v_+) \\
&\overset{(3.9)}{=} \mathbb{E}\left[U_+\left(\left(\frac{v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]} \right) \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} - \frac{1-b_H}{a_H} d_H \right) \right. \\
&\qquad \left. \times\, w'_+(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right] \\
&= (1-b_H)^{1-b_H} \frac{a_H^{b_H}}{b_H} \left(v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] \right)^{b_H} \\
&\quad \times \left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] \right)^{1-b_H} \\
&\quad - \frac{1-b_H}{b_H} d_H^{b_H} \mathbb{E}\left[w'_+(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right].
\end{aligned}$$

Since

$$\begin{aligned}
&U_-\left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]}\right) w_-(1-F_{\tilde{Z}}(c)) \\
&= k_{H-}(1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left[\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]} + \frac{1-b_H}{a_H}d_H\right]^{b_H} w_-(1-F_{\tilde{Z}}(c)) \\
&\quad - k_{H-}\frac{1-b_H}{b_H}d_H^{b_H} w_-(1-F_{\tilde{Z}}(c)),
\end{aligned}$$

the total objective function $\mathcal{Y}(c, v_+)$ is then

$$\begin{aligned}
\mathcal{Y}(c, v_+) = (1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}&\left(v_+ + \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{b_H} \\
&\times\left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{1-b_H} \\
&-\frac{1-b_H}{b_H}d_H^{b_H}\mathbb{E}\left[w_+'(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] \\
&- k_{H-}(1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left[\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}]} + \frac{1-b_H}{a_H}d_H\right]^{b_H} w_-(1-F_{\tilde{Z}}(c)) \\
&+ k_{H-}\frac{1-b_H}{b_H}d_H^{b_H} w_-(1-F_{\tilde{Z}}(c))\,.
\end{aligned}$$

2: For HARA utility defined in (3.4) the formula for $X_+^\star = X_+^\star(c, v_+)$ in the final payoff reads

$$\begin{aligned}
X_+^\star &= (U_+')^{-1}\left(\lambda\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \\
&= \left[(1-b_H)a_H^{\frac{b_H}{1-b_H}}\lambda^{-\frac{1}{1-b_H}}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} - \frac{1-b_H}{a_H}d_H\right]\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\,.
\end{aligned}$$

The budget constraint gives

$$\begin{aligned}
v_+ &= \mathbb{E}\left[\tilde{Z}X_+^\star\right] \\
&= (1-b_H)a_H^{\frac{b_H}{1-b_H}}\lambda^{-\frac{1}{1-b_H}}\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] \\
&\quad - \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\,.
\end{aligned}$$

Solving for λ and plugging into the formula for $X_+^\star$ in turn yields

$$X_+^\star = \left[\left(\frac{v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]}{\mathbb{E}\left[\tilde{Z} \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]} \right) \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{-\frac{1}{1-b_H}} - \frac{1-b_H}{a_H} d_H \right] \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} .$$

Given the optimal pair $(c^\star, v_+^\star)$ to Problem (P*), the optimal terminal wealth to Problem (P) under HARA utility then is

$$X^\star = X_+^\star(c^\star, v_+^\star) \mathbb{1}_{\tilde{Z} \leqslant c^\star} - \frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z} \mathbb{1}_{\tilde{Z} > c^\star}]} \mathbb{1}_{\tilde{Z} > c^\star} .$$

□

Proof of Lemma 3.9. In accordance with Theorem 3.8, the objective function of Problem (P*) satisfies

$$\begin{aligned}
\mathcal{Y}(c, v_+) &\leqslant (1-b_H)^{1-b_H} \frac{a_H^{b_H}}{b_H} \left(v_+ + \frac{1-b_H}{a_H} d_H \mathbb{E}[\tilde{Z}] \right)^{b_H} \\
&\quad \times \left(\mathbb{E}\left[\tilde{Z} \left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))} \right)^{-\frac{1}{1-b_H}} \leqslant \mathbb{1}_{\tilde{Z} \leqslant c} \right] \right)^{1-b_H} - k_{H-}(1-b_H)^{1-b_H} \\
&\quad \times \frac{a_H^{b_H}}{b_H} \left(\frac{v_+ - v_0}{\mathbb{E}[\tilde{Z} \mathbb{1}_{\tilde{Z} > c}]} \right)^{b_H} w_-(1 - F_{\tilde{Z}}(c)) + k_{H-} \frac{1-b_H}{b_H} d_H^{b_H} \\
&= (1-b_H)^{1-b_H} \frac{a_H^{b_H}}{b_H} v_H(c, v_+) + k_{H-} \frac{1-b_H}{b_H} d_H^{b_H} .
\end{aligned}$$

Therefore, Problem (P*) is well-posed, i.e. it holds $\sup_{c \geqslant 0, v_+ \geqslant v_0^+} \mathcal{Y}(c, v_+) < \infty$, if we have $\sup_{c \geqslant 0, v_+ \geqslant v_0^+} v_H(c, v_+) < \infty$, and thus Problem (P) is also well-posed. □

Proof of Theorem 3.10. 1: Let $v_0 \geqslant 0$ and $\inf_{c>0} k_H(c) \geqslant 1$.

Define $\theta(c) := \mathbb{E}\left[\tilde{Z} \left(\frac{w'_+(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}} \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant c} \right] > 0$ and $\delta := \frac{1-b_H}{a_H} d_H \mathbb{E}[\tilde{Z}]$. It holds $0 < \delta < \infty$ and $\theta(c)$ increases in c. Then

$$v_H(c, v_+) = \theta(c)^{1-b_H} \left[(v_+ + \delta)^{b_H} - k_H(c) (v_+ - v_0)^{b_H} \right] .$$

We consider the problem $\max_{x \geqslant v_0} f(x)$ where $f(x) := (x+\delta)^{b_H} - k(x - v_0)^{b_H}$ and $k \geqslant 0$ fixed. Since $f'(x) = b_H \left[(x+\delta)^{b_H - 1} - k(x - v_0)^{b_H - 1} \right]$, we conclude that:

(a) If $k < 1$, then $f(x) = x^{b_H}\left[\left(1+\frac{\delta}{x}\right)^{b_H} - k\left(1-\frac{v_0}{x}\right)^{b_H}\right] \to \infty$ as $x \to \infty$, which implies that $\sup_{x\geqslant v_0} f(x) = \infty$.

(b) If $k \geqslant 1$, then $f'(x) \leqslant 0 \ \forall x \geqslant v_0$. The argument behind is the following:

$$\begin{aligned} f'(x) \leqslant 0 \ &\Leftrightarrow b_H\left[(x+\delta)^{b_H-1} - k\,(x-v_0)^{b_H-1}\right] \leqslant 0 \\ &\overset{0<b_H<1}{\Leftrightarrow} (x+\delta)^{b_H-1} \leqslant k\,(x-v_0)^{b_H-1} \overset{0<b_H<1}{\Leftrightarrow} x+\delta \geqslant k^{-\frac{1}{1-b_H}}\,(x-v_0) \\ &\Leftrightarrow x\left(1-k^{-\frac{1}{1-b_H}}\right) \geqslant -\left(\delta + k^{-\frac{1}{1-b_H}}\,v_0\right)\,. \end{aligned}$$

If $k > 1$, then $1 - k^{-\frac{1}{1-b_H}} > 0$ and therefore

$$f'(x) \leqslant 0 \ \Leftrightarrow x \geqslant -\frac{\delta + k^{-\frac{1}{1-b_H}}\,v_0}{1-k^{-\frac{1}{1-b_H}}}\,.$$

Since in this case $-\frac{\delta + k^{-\frac{1}{1-b_H}}\,v_0}{1-k^{-\frac{1}{1-b_H}}} < 0$ it is $f'(x) \leqslant 0 \ \forall x \geqslant v_0 \geqslant 0$.

If $k = 1$, then $f'(x) \leqslant 0 \ \Leftrightarrow x + \delta \geqslant x - v_0 \ \Leftrightarrow \ \delta \geqslant -v_0$. Since $\delta > 0$ and $-v_0 \leqslant 0$ it holds $\delta \geqslant -v_0$ and thus $f'(x) \leqslant 0 \ \forall x \geqslant v_0$. In summary, $x^\star = v_0$ is therefore optimal for the problem $\max_{x\geqslant v_0} f(x)$ with optimal value $f(x^\star) = f(v_0) = (v_0+\delta)^{b_H}$.

When $\inf_{c>0} k_H(c) \geqslant 1$ this corresponds to the latter case above where $k \geqslant 1$. Note $v_0^+ = v_0 \geqslant 0$. We conclude

$$\begin{aligned} \sup_{c\geqslant 0, v_+\geqslant v_0} v_H(c, v_+) &= \sup_{c\geqslant 0}\left\{ \theta(c)^{1-b_H} \underbrace{\sup_{v_+\geqslant v_0}\left\{(v_+ + \delta)^{b_H} - k_H(c)\,(v_+ - v_0)^{b_H}\right\}}_{=(v_0+\delta)^{b_H}\ \forall c} \right\} \\ &= (v_0+\delta)^{b_H}\,\theta(\infty)^{1-b_H} \overset{\text{Assumption 3.3.2}}{<} \infty\,. \end{aligned}$$

In total we have $\sup_{c\geqslant 0, v_+\geqslant v_0} v_H(c, v_+) < \infty$ and Lemma 3.9 tells that Problem (P) is well-posed.

2: Let $v_0 < 0$ and $\inf_{c>0} k_H(c) > 1$.

We consider the problem $\max_{x\geqslant 0} f(x)$ where $f(x) := (x+\delta)^{b_H} - k\,(x-v_0)^{b_H}$ and $k \geqslant 0$ fixed. Then the first derivative of f is $f'(x) = b_H\left[(x+\delta)^{b_H-1} - k\,(x-v_0)^{b_H-1}\right]$. Let $k > 1$, then there are the following two cases:

(a) $k < \left(-\frac{v_0}{\delta}\right)^{1-b_H}$:

This implies $-\frac{\delta + k^{-\frac{1}{1-b_H}}\,v_0}{1-k^{-\frac{1}{1-b_H}}} > 0$ because $k > 1$ and $\delta + k^{-\frac{1}{1-b_H}}\,v_0 < 0 \ \Leftrightarrow \ k < \left(-\frac{v_0}{\delta}\right)^{1-b_H}$. Analogically to the case $k > 1$ in the proof of Theorem 3.10.1 we have

$$f'(x) = \begin{cases} > 0, & x < -\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}, \\ = 0, & x = -\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}, \\ < 0, & x > -\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}. \end{cases}$$

Consequently, $x^\star = -\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}$ is the only maximimum point for the problem $\max_{x \geqslant 0} f(x)$ when $k < \left(-\frac{v_0}{\delta}\right)^{1-b_H}$ with optimal value

$$f(x^\star) = f\left(-\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}\right) = -\left(k^{\frac{1}{1-b_H}} - 1\right)^{1-b_H} \left(-\left(v_0 + \delta\right)\right)^{b_H} .$$

(b) $k \geqslant \left(-\frac{v_0}{\delta}\right)^{1-b_H}$:

Then analogically $-\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}} \leqslant 0$ which implies $f'(x) < 0 \; \forall x > 0$. Moreover, $f'(0) = 0$ when $k = \left(-\frac{v_0}{\delta}\right)^{1-b_H}$ and $f'(0) < 0$ when $k > \left(-\frac{v_0}{\delta}\right)^{1-b_H}$. Consequently, $x^\star = 0$ is the only maximizer of the problem $\max_{x \geqslant 0} f(x)$ when $k \geqslant \left(-\frac{v_0}{\delta}\right)^{1-b_H}$ with optimal value $f(x^\star) = f(0) = \delta^{b_H} - k\left(-v_0\right)^{b_H}$.

Bringing both cases together we conclude that the unique maximum point of problem $\max_{x \geqslant 0} f(x)$ is given by $x^\star = \max\left\{-\frac{\delta + k^{-\frac{1}{1-b_H}} v_0}{1 - k^{-\frac{1}{1-b_H}}}, 0\right\}$ with corresponding optimal value

$$\begin{aligned} f(x^\star) &= \left(x^\star + \delta\right)^{b_H} - k\left(x^\star - v_0\right)^{b_H} \\ &= \begin{cases} -\left(k^{\frac{1}{1-b_H}} - 1\right)^{1-b_H} \left(-\left(v_0 + \delta\right)\right)^{b_H}, & k < \left(-\frac{v_0}{\delta}\right)^{1-b_H}, \\ \delta^{b_H} - k\left(-v_0\right)^{b_H}, & k \geqslant \left(-\frac{v_0}{\delta}\right)^{1-b_H}. \end{cases} \end{aligned}$$

When $\inf_{c>0} k_H(c) > 1$, note $v_0^+ = 0$, we conclude for problem $\sup_{c \geqslant 0, v_+ \geqslant 0} v_H(c, v_+)$

$$\sup_{c \geqslant 0, v_+ \geqslant 0} v_H(c, v_+) = \sup_{c \geqslant 0}\left\{\theta(c)^{1-b_H} \sup_{v_+ \geqslant 0}\left\{\left(v_+ + \delta\right)^{b_H} - k_H(c)\left(v_+ - v_0\right)^{b_H}\right\}\right\}.$$

(a) $k_H(c) < \left(-\frac{v_0}{\delta}\right)^{1-b_H}$:

When $1 < k_H(c) < \left(-\frac{v_0}{\delta}\right)^{1-b_H}$ then it is $-v_0 > \delta$ and thus $-\left(\delta + v_0\right) > 0$. It follows

$$\sup_{v_+ \geqslant 0}\left\{\left(v_+ + \delta\right)^{b_H} - k_H(c)\left(v_+ - v_0\right)^{b_H}\right\} = -\left(k_H(c)^{\frac{1}{1-b_H}} - 1\right)^{1-b_H} \left(-\left(v_0 + \delta\right)\right)^{b_H} < 0 .$$

(b) $k_H(c) \geqslant \left(-\frac{v_0}{\delta}\right)^{1-b_H}$:

It follows

$$\sup_{v_+\geqslant 0}\left\{(v_+ + \delta)^{b_H} - k_H(c)\,(v_+ - v_0)^{b_H}\right\} = \delta^{b_H} - k_H(c)\,(-v_0)^{b_H} < \delta^{b_H}\,.$$

In summary, the above yields

$$\sup_{c\geqslant 0, v_+\geqslant 0} v_H(c,v_+) < \delta^{b_H}\sup_{c\geqslant 0}\left\{\theta(c)^{1-b_H}\right\} = \delta^{b_H}\theta(\infty)^{1-b_H} \overset{\text{Assumption 3.3.2}}{<} \infty\,.$$

In total we have $\sup_{c\geqslant 0, v_+\geqslant v_0} v_H(c,v_+) < \infty$ and Lemma 3.9 tells that Problem (P) is well-posed. □

Proof of Theorem 3.12. For clarity purposes, this proof consists of three parts, labeled: Lemma A.1, Lemma A.2 and Lemma A.3. The first lemma, A.1, is about monotonicity relations between λ, $\hat{z}$, c and v_+.

Lemma A.1.

1.

$$\lambda \nearrow \;\Leftrightarrow\; \hat{z} \nearrow\,,\ \textit{i.e. } \hat{z} \textit{ is increasing in } \lambda.$$

2. Let c be fixed.

$$v_+ \searrow \;\Rightarrow\; \lambda,\ \hat{z} \nearrow\,,\ \textit{i.e. } \lambda \textit{ and } \hat{z} \textit{ are decreasing in } v_+.$$

3. Let v_+ be fixed.

$$c \nearrow \;\Rightarrow\; \lambda,\ \hat{z} \nearrow\,,\ \textit{i.e. } \lambda \textit{ and } \hat{z} \textit{ are increasing in } c.$$

Proof of Lemma A.1. The important equations here are the budget equation and the relation between $\lambda > 0$ and $0 < \hat{z} < 1$:

$$\mathbb{E}\left[\tilde{Z}(U_+')^{-1}\left(\lambda\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right] = v_+,$$

$$\lambda\frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})} = U_+'(0+) > 0\,.$$

Moreover, recall that under Assumption 3.3.1 $F_{\tilde{Z}}^{-1}(y)/w_+'(y)$ is non-decreasing in y, thus the term $\frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})}$ is non-increasing in $\hat{z}$.

1. *$\hat{z}$ is increasing in λ:*

   Since $\lambda\frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})} = U_+'(0+)$ is a positive constant, when λ increases then the expression $\frac{F_{\tilde{Z}}^{-1}(1-\hat{z})}{w_+'(1-\hat{z})}$ has to decrease, thus $\hat{z}$ has to increase.

2. *Let c be fixed. λ and $\hat{z}$ are decreasing in v_+:*

   Note that $(U'_+)^{-1}(x)$ is decreasing in x. The left side of the first equation is therefore decreasing in λ and in $\hat{z}$, all other variables fixed. Therefore, when v_+ increases, then λ and $\hat{z}$, as both are interconnected, have to decrease.

3. *Let v_+ be fixed. λ and $\hat{z}$ are increasing in c:*

   Let v_+ be fixed. Note that the expectation on the left side of the first equation is increasing in c, all other variables fixed. Therefore, when c increases, then λ and $\hat{z}$ have to increase to compensate the increase in the expectation through c.

□

Before continuing with Lemmas A.2 and A.3, let us recall that in view of Theorem 3.8.1, the problem (P*) to be solved is

$$\begin{aligned}&\max_{c\geqslant 0, v_+\geqslant v_0} \mathcal{Y}(c,v_+)\\ &= \max_{c\geqslant 0, v_+\geqslant v_0}\left\{a_1(c,\hat{z})\left[(v_+ + a_2(c,\hat{z}))^{b_H} - \frac{a_3(c)}{a_1(c,\hat{z})}(v_+ - v_0 + a_4(c))^{b_H}\right] + a_5(c,\hat{z})\right\},\end{aligned}$$

where

$$\begin{aligned}
a_1(c,\hat{z}) &= (1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w'_+(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{1-b_H}\\
&= (1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\left(\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}}\left(w'_+(F_{\tilde{Z}}(\tilde{Z}))\right)^{\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{1-b_H},\\
a_2(c,\hat{z}) &= \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right],\\
a_3(c) &= (1-b_H)^{1-b_H}\frac{a_H^{b_H}}{b_H}\frac{k_{H-}w_-(1-F_{\tilde{Z}}(c))}{\left(\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}\right]\right)^{b_H}},\\
a_4(c) &= \frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}\right],\\
a_5(c,\hat{z}) &= -\frac{1-b_H}{b_H}d_H^{b_H}\mathbb{E}\left[w'_+(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] - \frac{1-b_H}{b_H}d_H^{b_H}k_{H-}w_-(1-F_{\tilde{Z}}(c))\\
&= -\frac{1-b_H}{b_H}d_H^{b_H}\left[\mathbb{E}\left[w'_+(F_{\tilde{Z}}(\tilde{Z}))\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right] + k_{H-}w_-(1-F_{\tilde{Z}}(c))\right].
\end{aligned}$$

Observe that:

1. $a_1(c,\hat{z}) > 0$ strictly increases in c while it strictly decreases in $\hat{z}$.
2. $a_2(c,\hat{z}) > 0$ strictly increases in c while it strictly decreases in $\hat{z}$.
3. $a_4(c) > 0$ strictly decreases in c.
4. $a_5(c,\hat{z}) < 0$ strictly increases in $\hat{z}$.

Lemma A.2. *The function*

$$\mathcal{Y}(c,v_+) = a_1(c,\hat{z})\left[(v_+ + a_2(c,\hat{z}))^{b_H} - \frac{a_3(c)}{a_1(c,\hat{z})}(v_+ - v_0 + a_4(c))^{b_H}\right] + a_5(c,\hat{z})$$

is strictly decreasing in v_+.

Proof of Lemma A.2. In more detail, we show

$$\frac{\partial \mathcal{Y}}{\partial v_+}(c,v_+,\hat{z}(c,v_+)) = \underbrace{\frac{\partial \mathcal{Y}}{\partial v_+}(c,v_+,\hat{z})}_{<0,(i)} + \underbrace{\frac{\partial \mathcal{Y}}{\partial \hat{z}}(c,v_+,\hat{z}(c,v_+))}_{\geqslant 0,(ii)}\underbrace{\frac{\partial \hat{z}}{\partial v_+}(c,v_+)}_{\leqslant 0,(iii)} < 0.$$

Then the problem $\max\limits_{c\geqslant 0, v_+\geqslant v_0} \mathcal{Y}(c,v_+)$ reduces to

$$\max_{c\geqslant 0, v_+\geqslant v_0} \mathcal{Y}(c,v_+) = \max_{c\geqslant 0} \mathcal{Y}(c,v_0)\,.$$

Before proving (i), (ii), (iii), let us rewrite the original problem as follows:

$$\begin{aligned}
\max_{c\geqslant 0, v_+\geqslant v_0} \mathcal{Y}(c,v_+) &= \max_{c\geqslant 0, v_+\geqslant v_0, \hat{z}=\hat{z}(z,c_+)} \mathcal{Y}(c,v_+,\hat{z}) \\
&= \max_{c\geqslant 0, v_+\geqslant v_0, \hat{z}=\hat{z}(c,v_+)} \Big\{a_1(c,\hat{z})\Big[(v_+ + a_2(c,\hat{z}))^{b_H} \\
&\qquad - \frac{a_3(c)}{a_1(c,\hat{z})}(v_+ - v_0 + a_4(c))^{b_H}\Big] + a_5(c,\hat{z})\Big\} \\
&= \max_{c\geqslant 0, v_+\geqslant v_0, \hat{z}=\hat{z}(c,v_+)} \Big\{h(c,v_+,\hat{z}) - a_3(c)(v_+ - v_0 + a_4(c))^{b_H}\Big\} \\
&= \max_{c\geqslant 0, v_+\geqslant v_0, \hat{z}=\hat{z}(c,v_+)} \{a_1(c,\hat{z})\, f(c,v_+,\hat{z}) + a_5(c,\hat{z})\}\,,
\end{aligned}$$

where

$$\begin{aligned}
h(c,v_+,\hat{z}) &= a_1(c,\hat{z})(v_+ + a_2(c,\hat{z}))^{b_H} + a_5(c,\hat{z})\,, \\
f(c,v_+,\hat{z}) &= (v_+ + a_2(c,\hat{z}))^{b_H} - \frac{a_3(c)}{a_1(c,\hat{z})}(v_+ - v_0 + a_4(c))^{b_H}\,.
\end{aligned}$$

(i) $\frac{\partial \mathcal{Y}}{\partial v_+}(c,v_+,\hat{z}) < 0$:

Differentiating $f(c,v_+,\hat{z})$ with respect to v_+ gives

$$\frac{\partial f}{\partial v_+}(c,v_+,\hat{z}) = b_H(v_+ + a_2(c,\hat{z}))^{b_H-1} - \frac{a_3(c)}{a_1(c,\hat{z})} b_H (v_+ - v_0 + a_4(c))^{b_H-1}\,.$$

$\frac{\partial f}{\partial v_+}(c,v_+,\hat{z}) < 0$ is equivalent to $\frac{a_3(c)}{a_1(c,\hat{z})}\left(\frac{v_+ + a_2(c,\hat{z})}{v_+ - v_0 + a_4(c)}\right)^{1-b_H} > 1$. First, $\left(\frac{v_+ + a_2(c,\hat{z})}{v_+ - v_0 + a_4(c)}\right)^{1-b_H} > 1$ is ensured by $v_0 > \sup_{c\geqslant 0, \hat{z}\in(0,1)}\{a_4(c) - a_2(c,\hat{z})\}$, with

$$a_4(c) - a_2(c,\hat{z}) = \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\left(\mathbb{1}_{\tilde{Z}>c} - \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right)\right]$$

$$\leqslant \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\right] = \frac{1-b_H}{a_H} d_H e^{-rT} .$$

Therefore, under Assumption 3.11.1, it is

$$v_0 > \frac{1-b_H}{a_H} d_H e^{-rT} \geqslant \sup_{c\geqslant 0, \hat{z}\in(0,1)} \left\{a_4(c) - a_2(c,\hat{z})\right\}$$

and consequently $\left(\frac{v_+ + a_2(c,\hat{z})}{v_+ - v_0 + a_4(c)}\right)^{1-b_H} > 1$.

Moreover, under Assumption 3.11.2 we have $\inf_{c\geqslant 0,\hat{z}\in(0,1)} \frac{a_3(c)}{a_1(c,\hat{z})} > 1$. The reason behind is the following inequality:

$$\begin{aligned}
\frac{a_3(c)}{a_1(c,\hat{z})} &= \frac{(1-b_H)^{1-b_H} \frac{a_H^{b_H}}{b_H} \frac{k_{H} - w_-(1-F_{\tilde{Z}}(c))}{\left(\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}\right]\right)^{b_H}}}{(1-b_H)^{1-b_H} \frac{a_H^{b_H}}{b_H} \left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))}\right]\right)^{1-b_H}} \\
&\overset{\text{minimum at } \hat{z}=0}{\geqslant} \frac{k_{H} - w_-(1-F_{\tilde{Z}}(c))}{\left(\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}>c}\right]\right)^{b_H} \left(\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant c}\right]\right)^{1-b_H}} = k_H(c).
\end{aligned}$$

Therefore $\inf_{c\geqslant 0,\hat{z}\in(0,1)} \frac{a_3(c)}{a_1(c,\hat{z})} \geqslant \inf_{c\geqslant 0} k_H(c) > 1$. In total we have shown $f(c, v_+, \hat{z})$ being strictly decreasing in v_+, and since $a_1(c,\hat{z}) > 0$ is independent from v_+, the value function $\mathcal{Y}(c, v_+, \hat{z})$ also strictly decreases in v_+.

(ii) $\frac{\partial \mathcal{Y}}{\partial \hat{z}}(c, v_+, \hat{z}(c, v_+)) \geqslant 0$:

For $\mathcal{Y}(c, v_+, \hat{z})$ to be increasing in $\hat{z}$ at $\hat{z}(c, v_+)$, we show $h(v_+, c, \hat{z}(c, v_+))$ increasing in $\hat{z}$ at $\hat{z}(c, v_+)$. At this point we use the budget condition to obtain an alternative representation for $\hat{z} = \hat{z}(c, v_+)$:

$$v_+ = \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\left[\left(\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})} \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} - 1\right] \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right].$$

This leads to the following relation at $\hat{z} = \hat{z}(c, v_+)$:

$$\begin{aligned}
&\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}} \left(w_+'(F_{\tilde{Z}}(\tilde{Z}))\right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right] \\
&= \left(\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\right)^{\frac{1}{1-b_H}} \left[\frac{v_+ a_H}{d_H(1-b_H)} + \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right]\right].
\end{aligned} \tag{A.6}$$

Then

$$a_1(c,\hat{z}) \overset{(A.6)}{=} \frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})} \frac{a_H}{b_H d_H^{1-b_H}} \left[v_+ + \frac{d_H(1-b_H)}{a_H} \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right]\right]^{1-b_H}.$$

We can then rewrite h by inserting the new representation for a_1:

$$h(c,v_+,\hat{z}(c,v_+)) = \frac{1-b_H}{b_H} d_H^{b_H} \mathbb{E}\left[\left(\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\tilde{Z} - w_+'(F_{\tilde{Z}}(\tilde{Z}))\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right] + \frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\frac{a_H}{b_H d_H^{1-b_H}} v_+ - \frac{1-b_H}{b_H} d_H^{b_H} k_{H-} w_-(1-F_{\tilde{Z}}(c)).$$

The last term $-\frac{1-b_H}{b_H} d_H^{b_H} k_{H-} w_-(1-F_{\tilde{Z}}(c))$ is independent of $\hat{z}$. Now notice that under Assumption 3.3, w_+ is to satisfy the monotonicity condition, therefore, the second piece $\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\frac{a_H}{b_H d_H^{1-b_H}} v_+$ increases in $\hat{z}$. To show that

$$\mathbb{E}\left[\left(\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\tilde{Z} - w_+'(F_{\tilde{Z}}(\tilde{Z}))\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right]$$

increases in $\hat{z}$, we calculate its partial derivative with respect to $\hat{z}$ independently of c and v_+, by applying the Leibniz integral rule (cf. Theorem 2.1),

$$\begin{aligned}
&\frac{d}{d\hat{z}}\mathbb{E}\left[\left(\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}\tilde{Z} - w_+'(F_{\tilde{Z}}(\tilde{Z}))\right)\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right] \\
&= \frac{d}{d\hat{z}}\int_0^{F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\left[\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}x - w_+'(F_{\tilde{Z}}(x))\right]dF_{\tilde{Z}}(x) \\
&= \left[\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c)) - w_+'((1-\hat{z})F_{\tilde{Z}}(c))\right]\underbrace{F_{\tilde{Z}}'(F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c)))}_{>0} \\
&\quad\times \underbrace{\frac{-F_{\tilde{Z}}(c)}{F_{\tilde{Z}}'(F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c)))}}_{<0} + \underbrace{\frac{d}{d\hat{z}}\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}}_{>0,\ \text{Assumption 3.3.1}}\underbrace{\int_0^{F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))} x dF_{\tilde{Z}}(x)}_{=\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}\right]>0} \\
&\geqslant 0
\end{aligned}$$

if

$$\frac{w_+'(1-\hat{z})}{F_{\tilde{Z}}^{-1}(1-\hat{z})}F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c)) - w_+'((1-\hat{z})F_{\tilde{Z}}(c)) \leqslant 0$$

for any $\hat{z}$, and c. Rewriting yields

$$\frac{F_{\tilde{Z}}^{-1}((1-\hat{z})F_{\tilde{Z}}(c))}{F_{\tilde{Z}}^{-1}(1-\hat{z})} \leqslant \frac{w_+'((1-\hat{z})F_{\tilde{Z}}(c))}{w_+'(1-\hat{z})}. \tag{A.7}$$

The monotonicity condition in Assumption 3.3.1 on w_+ implies

$$\frac{F_{\tilde{Z}}^{-1}(x)}{F_{\tilde{Z}}^{-1}(y)} \leqslant \frac{w_+'(x)}{w_+'(y)},\ 0 < x < y < 1$$

and thus, when setting $x := (1-\hat{z})F_{\tilde{Z}}(c)$ and $y := 1-\hat{z}$, Eq. (A.7) holds true.

In summary, h increases in $\hat{z}$ and therefore consequently $\mathcal{Y}(c, v_+)$ increases in $\hat{z}$, at $\hat{z} = \hat{z}(c, v_+)$.

(iii) $\frac{\partial \hat{z}}{\partial v_+}(c, v_+) \leqslant 0$:

This is already shown in Lemma A.1.

□

In consequence, $\mathcal{Y}(c, v_+)$ strictly decreases in v_+ for any $c > 0$. Since c is still a free variable, $v_+^\star = v_0$ is optimal. Then the problem of maximizing $\mathcal{Y}(c, v_+)$ with respect to c and v_+ reduces to

$$\max_{c \geqslant 0, v_+ \geqslant v_0} \mathcal{Y}(c, v_+) = \max_{c \geqslant 0} \left\{ \max_{v_+ \geqslant v_0} \mathcal{Y}(c, v_+) \right\} = \max_{c \geqslant 0} \mathcal{Y}(c, v_0),$$

with

$$\mathcal{Y}(c, v_0) = a_1 (c, \hat{z}) (v_0 + a_2 (c, \hat{z}))^{b_H} + \tilde{a}_5 (c, \hat{z}),$$

where

$$\tilde{a}_5 (c, \hat{z}) \quad = \quad -\frac{1 - b_H}{b_H} d_H^{b_H} \mathbb{E}\left[w'_+(F_{\tilde{Z}}(\tilde{Z})) \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right].$$

Notice that $\mathcal{Y}_-(c, v_0) = 0 \ \forall c > 0$, hence $\mathcal{Y}(c, v_0) = \mathcal{Y}_+(c, v_0)$.

Lemma A.3. *$\mathcal{Y}(c, v_0)$ increases in c.*

Proof of Lemma A.3. We prove

$$\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z}(c, v_0)) \quad = \quad \underbrace{\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z})}_{\geqslant 0, (i)} + \underbrace{\frac{\partial \mathcal{Y}}{\partial \hat{z}}(c, v_0, \hat{z}(c, v_0))}_{\geqslant 0, (ii)} \underbrace{\frac{\partial \hat{z}}{\partial c}(c, v_0)}_{\geqslant 0, (iii)} \geqslant 0.$$

(i) $\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z}) \geqslant 0$:

We calculate the partial derivative

$$\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z}) = \frac{\partial a_1}{\partial c} (c, \hat{z}) (v_0 + a_2 (c, \hat{z}))^{b_H} + a_1 (c, \hat{z}) b_H (v_0 + a_2 (c, \hat{z}))^{b_H - 1} \frac{\partial a_2}{\partial c} (c, \hat{z}) + \frac{\partial \tilde{a}_5}{\partial c} (c, \hat{z}) .$$

Define the constants $k_1 := (1 - b_H)^{1 - b_H} \frac{a_H^{b_H}}{b_H} > 0$, $k_2 := \frac{1 - b_H}{a_H} d_H > 0$, then $k_1 k_2^{b_H} = \frac{1 - b_H}{b_H} d_H^{b_H}$. By applying Leibniz integral rule we obtain the following expression for $\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z})$:

$$
\begin{aligned}
\frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z}) = k_1(1-b_H) & \left(\frac{v_0 + k_2 \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]}{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(\tilde{Z})) \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]} \right)^{b_H} \\
& \times \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right)^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(c)(1-\hat{z})) \right)^{\frac{1}{1-b_H}} \\
& \times F'_{\tilde{Z}} \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \left(\frac{\partial}{\partial c} F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \\
& + k_1 k_2 b_H \left(\frac{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(\tilde{Z})) \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]}{v_0 + k_2 \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]} \right)^{1-b_H} \\
& \times F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) F'_{\tilde{Z}} \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \left(\frac{\partial}{\partial c} F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \\
& - k_1 k_2^{b_H} w'_+(F_{\tilde{Z}}(c)(1-\hat{z})) F'_{\tilde{Z}} \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \left(\frac{\partial}{\partial c} F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right).
\end{aligned}
$$

Since

$$
F'_{\tilde{Z}} \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) \left(\frac{\partial}{\partial c} F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right) = F'_{\tilde{Z}}(c)(1-\hat{z}) > 0,
$$

it follows that

$$
\begin{aligned}
& \frac{\partial \mathcal{Y}}{\partial c}(c, v_0, \hat{z}) \geqslant 0 \\
\Leftrightarrow (1-b_H) & \left(\frac{v_0/k_2 + \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]}{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(\tilde{Z})) \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]} \right)^{b_H} \\
& \times \left(F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \right)^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(c)(1-\hat{z})) \right)^{\frac{1}{1-b_H}} \\
& + b_H \left(\frac{\mathbb{E}\left[\tilde{Z}^{-\frac{b_H}{1-b_H}} \left(w'_+(F_{\tilde{Z}}(\tilde{Z})) \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]}{v_0/k_2 + \mathbb{E}\left[\tilde{Z} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z}))} \right]} \right)^{1-b_H} F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) \\
& - w'_+(F_{\tilde{Z}}(c)(1-\hat{z})) \\
& \geqslant 0.
\end{aligned}
$$

With $x := F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c)(1-\hat{z})) > 0$ and $\Lambda(x) := \left(\frac{v_0/k_2 + \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z} \leqslant x}\right]}{\mathbb{E}\left[\tilde{Z} \left(\frac{w'_+(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}} \right)^{\frac{1}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant x} \right]} \right)^{1-b_H} \frac{w'_+(F_{\tilde{Z}}(x))}{x}$,

the above inequality reads

$$
(1-b_H)\left(\frac{v_0/k_2+\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant x}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{w_+'(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}}\right)^{\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant x}\right]}\right)^{b_H}\left(\frac{w_+'(F_{\tilde{Z}}(x))}{x}\right)^{\frac{b_H}{1-b_H}}
$$

$$
+\,b_H\left(\frac{\mathbb{E}\left[\tilde{Z}\left(\frac{w_+'(F_{\tilde{Z}}(\tilde{Z}))}{\tilde{Z}}\right)^{\frac{1}{1-b_H}}\mathbb{1}_{\tilde{Z}\leqslant x}\right]}{v_0/k_2+\mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant x}\right]}\right)^{1-b_H}\frac{x}{w_+'(F_{\tilde{Z}}(x))}-1\geqslant 0
$$

$$
\Leftrightarrow (1-b_H)\Lambda(x)^{\frac{b_H}{1-b_H}}+b_H\Lambda(x)^{-1}-1\geqslant 0
$$

$$
\overset{\Lambda(x)>0}{\Leftrightarrow} (1-b_H)\Lambda(x)^{\frac{1}{1-b_H}}-\Lambda(x)+b_H\geqslant 0. \tag{A.8}
$$

It remains to show (A.8) $\forall x>0$. The proof of this inequality uses Young's inequality, Young (1912), for $f:\mathbb{R}^+\to\mathbb{R}^+$ a continuous and strictly monotone increasing function with $f(0)=0$, which states that for any $z_1,z_2\geqslant 0$

$$
z_1z_2\leqslant\int_0^{z_1}f(x)dx+\int_0^{z_2}f^{-1}(y)dy.
$$

Taking $f(x)=x^p$, $p>0$, $z_2=1\geqslant 0$ and $p=\frac{1}{1-b_H}-1=\frac{b_H}{1-b_H}>0$. Then the inequality reads

$$
\forall z_1\geqslant 0:(1-b_H)z_1^{\frac{1}{1-b_H}}-z_1+b_H\geqslant 0\,.
$$

The minimum of the left side is 0 and appears at $z_1=1$. Therefore, since $\Lambda(x)=\Lambda(x,b_H)\geqslant 0$ $\forall x>0$, it holds

$$
\forall x\geqslant 0:(1-b_H)\Lambda(x)^{\frac{1}{1-b_H}}-\Lambda(x)+b_H\geqslant 0\,.
$$

Moreover, notice that strict inequality holds for almost every $x\geqslant 0$, since $(1-b_H)\Lambda(x)^{\frac{1}{1-b_H}}-\Lambda(x)+b_H=0$ only appears at $\Lambda(x)=1$. Therefore, one can conclude that $\frac{\partial\mathcal{Y}}{\partial c}(c,v_0,\hat{z})>0$ for almost every $c\geqslant 0$.

(ii) $\frac{\partial\mathcal{Y}}{\partial\hat{z}}(c,v_0,\hat{z}(c,v_0))\geqslant 0$:

As already mentioned in the proof of Lemma A.2, this holds because of the monotonicity condition on w_+ in Assumption 3.3.

(iii) $\frac{\partial\hat{z}}{\partial c}(c,v_0)\geqslant 0$:

This follows from Lemma A.1.

□

In summary, it follows that $\mathcal{Y}(c,v_+)$ strictly decreases in v_+ and $\mathcal{Y}(c,v_0)$ increases in c (and strictly increases for almost every c). Hence,

$$\max_{c\geqslant 0, v_+\geqslant v_0} \mathcal{Y}(c, v_+) = \max_{c\geqslant 0}\left\{\max_{v_+\geqslant v_0} \mathcal{Y}(c, v_+)\right\} = \max_{c\geqslant 0} \mathcal{Y}(c, v_0) = \mathcal{Y}(\infty, v_0),$$

and therefore $(\infty, v_0) = (c^\star, v_+^\star) = \operatorname{argmax}_{c\geqslant 0, v_+\geqslant v_0}\mathcal{Y}(c, v_+)$ is the unique optimal to Problem ($P^\star$). The corresponding $\hat{z}\left(c^\star, v_+^\star\right)$ is given by $\hat{z}\left(c^\star, v_+^\star\right) = \hat{z}\left(\infty, v_0\right)$.

According to Theorems 3.4 and 3.5, the optimal terminal wealth to Problem (P), when $(c^\star, v_+^\star)$ are the optimals to Problem ($P^\star$), is given by

$$X^\star = (U_+')^{-1}\left(\lambda^\star \frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right) \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(c^\star)(1-\hat{z}^\star))} - \frac{v_+^\star - v_0}{\mathbb{E}[\tilde{Z}\mathbb{1}_{\tilde{Z}>c^\star}]}\mathbb{1}_{\tilde{Z}>c^\star}\,.$$

As the optimal pair is $(c^\star, v_+^\star) = (\infty, v_0)$, it follows for $X^\star$

$$\begin{aligned} X^\star \overset{(3.9)}{=} & \left(\frac{v_0 + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}\right]}{\mathbb{E}\left[\tilde{Z}\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}\right]}\right)\left(\frac{\tilde{Z}}{w_+'(F_{\tilde{Z}}(\tilde{Z}))}\right)^{\frac{1}{b_H-1}} \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))} \\ & - \frac{1-b_H}{a_H} d_H \mathbb{1}_{\tilde{Z}\leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0))}, \end{aligned}$$

where $\lambda(\infty, v_0) > 0$, $\hat{z}(\infty, v_0) \in (0,1)$ satisfies $\mathbb{E}\left[\tilde{Z}X^\star\right] = v_0$ and the relation

$$\lambda(\infty, v_0)\frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0))}{w_+'(1-\hat{z}(\infty, v_0))} = U_+'(0+) = a_H d_H^{b_H-1}\,.$$

The optimal terminal wealth for a general reference wealth B follows similarly. Moreover, in view of Theorem 3.10.1 and Assumption 3.11.2, the original terminal wealth problem (P) is well-posed. □

A.2 Proofs: A Feasible Probability Distortion Function

We start with the verification of Eq. (3.12) and (3.13): First, recall the definition of $w(p)$ in (3.11):

$$w(p) := \left(\Phi\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)\right)^\alpha.$$

For the derivative we therefore infer

$$\begin{aligned} w'(p) &= \alpha \underbrace{\left(\Phi\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)\right)^{\alpha-1}}_{=(w(p))^{\frac{\alpha-1}{\alpha}}} \frac{d\Phi\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)}{dp} \\ &= \alpha\left(w(p)\right)^{\frac{\alpha-1}{\alpha}} \Phi'\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)\frac{d\Phi^{-1}(p)}{dp} \\ &= \alpha\left(w(p)\right)^{\frac{\alpha-1}{\alpha}} \frac{\Phi'\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)}{\Phi'\left(\Phi^{-1}(p)\right)} \overset{\Phi'=\psi}{=} \alpha\left(w(p)\right)^{\frac{\alpha-1}{\alpha}} \frac{\psi\left(\Phi^{-1}(p) - \delta\sigma_{\tilde{Z}}\right)}{\psi\left(\Phi^{-1}(p)\right)} \end{aligned}$$

$$
\begin{aligned}
&= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\Phi^{-1}(p)-\delta\sigma_{\tilde{Z}}\right)^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Phi^{-1}(p))^2}} \\
&= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} e^{-\frac{1}{2}\left(\Phi^{-1}(p)-\delta\sigma_{\tilde{Z}}\right)^2+\frac{1}{2}(\Phi^{-1}(p))^2} = \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} e^{-\frac{1}{2}\left(-2\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}+\delta^2\sigma_{\tilde{Z}}^2\right)} \\
&= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2}.
\end{aligned}
$$

As $F_{\tilde{Z}}^{-1}(p) = e^{\mu_{\tilde{Z}}+\sigma_{\tilde{Z}}\Phi^{-1}(p)}$ is the inverse function of $F_{\tilde{Z}}(x) = \Phi\left(\frac{\ln x-\mu_{\tilde{Z}}}{\sigma_{\tilde{Z}}}\right)$, where in the considered setup ($r = 0$) it holds $\mu_{\tilde{Z}} = -rT - \frac{1}{2}\sigma_{\tilde{Z}}^2 \overset{r=0}{=} -\frac{1}{2}\sigma_{\tilde{Z}}^2$, we further derive

$$
\begin{aligned}
w'(p) &= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2} = \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde{Z}}^{-1}(p) \frac{e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2}}{F_{\tilde{Z}}^{-1}(p)} \\
&= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde{Z}}^{-1}(p) \frac{e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2}}{e^{\mu_{\tilde{Z}}+\sigma_{\tilde{Z}}\Phi^{-1}(p)}} \\
&= \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde{Z}}^{-1}(p) e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2-\mu_{\tilde{Z}}-\sigma_{\tilde{Z}}\Phi^{-1}(p)} \\
&\overset{\mu_{\tilde{Z}}=-\frac{1}{2}\sigma_{\tilde{Z}}^2}{=} \alpha \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde{Z}}^{-1}(p) e^{\Phi^{-1}(p)\delta\sigma_{\tilde{Z}}-\frac{1}{2}\delta^2\sigma_{\tilde{Z}}^2+\frac{1}{2}\sigma_{\tilde{Z}}^2-\sigma_{\tilde{Z}}\Phi^{-1}(p)} \\
&= \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-(1-\delta)\sigma_{\tilde{Z}}\Phi^{-1}(p)} \left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde{Z}}^{-1}(p).
\end{aligned}
$$

This represents the first part of (3.12). The second part holds by inserting $F_{\tilde{Z}}(\tilde{Z})$ into $w'(p)$ and using $F_{\tilde{Z}}(x) \overset{r=0}{=} \Phi\left(\frac{\ln x+\frac{1}{2}\sigma_{\tilde{Z}}^2}{\sigma_{\tilde{Z}}}\right)$ with therefore

$$
\Phi^{-1}(F_{\tilde{Z}}(\tilde{Z})) = \Phi^{-1}\left(\Phi\left(\frac{\ln\tilde{Z}+\frac{1}{2}\sigma_{\tilde{Z}}^2}{\sigma_{\tilde{Z}}}\right)\right) = \frac{\ln\tilde{Z}+\frac{1}{2}\sigma_{\tilde{Z}}^2}{\sigma_{\tilde{Z}}}.
$$

We obtain:

$$
\begin{aligned}
w'(F_{\tilde{Z}}(\tilde{Z})) &= \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-(1-\delta)\sigma_{\tilde{Z}}\Phi^{-1}(F_{\tilde{Z}}(\tilde{Z}))} \left(w(F_{\tilde{Z}}(\tilde{Z}))\right)^{\frac{\alpha-1}{\alpha}} \underbrace{F_{\tilde{Z}}^{-1}(F_{\tilde{Z}}(\tilde{Z}))}_{=\tilde{Z}} \\
&\overset{(3.11)}{=} \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-(1-\delta)\sigma_{\tilde{Z}}\Phi^{-1}(F_{\tilde{Z}}(\tilde{Z}))} \left(\Phi\left(\Phi^{-1}(F_{\tilde{Z}}(\tilde{Z}))-\delta\sigma_{\tilde{Z}}\right)\right)^{\alpha-1} \tilde{Z} \\
&= \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-(1-\delta)\sigma_{\tilde{Z}}\frac{\ln\tilde{Z}+\frac{1}{2}\sigma_{\tilde{Z}}^2}{\sigma_{\tilde{Z}}}} \left(\Phi\left(\frac{\ln\tilde{Z}+\frac{1}{2}\sigma_{\tilde{Z}}^2}{\sigma_{\tilde{Z}}}-\delta\sigma_{\tilde{Z}}\right)\right)^{\alpha-1} \tilde{Z} \\
&= \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde{Z}}^2-\frac{1}{2}(1-\delta)\sigma_{\tilde{Z}}^2} \underbrace{e^{-(1-\delta)\ln\tilde{Z}}}_{=\tilde{Z}^{\delta-1}} \left(\Phi\left(\frac{\ln\tilde{Z}}{\sigma_{\tilde{Z}}}-\left(\delta-\frac{1}{2}\right)\sigma_{\tilde{Z}}\right)\right)^{\alpha-1} \tilde{Z} \\
&= \alpha e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde{Z}}^2} \left(\Phi\left(\frac{\ln\tilde{Z}}{\sigma_{\tilde{Z}}}-\left(\delta-\frac{1}{2}\right)\sigma_{\tilde{Z}}\right)\right)^{\alpha-1} \tilde{Z}^{\delta}.
\end{aligned}
$$

In view of $F_{\tilde{Z}}^{-1}(p) \overset{r=0}{=} e^{-\frac{1}{2}\sigma_{\tilde{Z}}^2+\sigma_{\tilde{Z}}\Phi^{-1}(p)}$, the formulas in (3.13) for $\alpha = 1$ are then a special case result of (3.12):

$$\begin{aligned} w'(p) &\overset{(3.12)}{=} \alpha e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde Z}^2-(1-\delta)\sigma_{\tilde Z}\Phi^{-1}(p)}\left(w(p)\right)^{\frac{\alpha-1}{\alpha}} F_{\tilde Z}^{-1}(p) \\ &\overset{\alpha=1}{=} e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde Z}^2-(1-\delta)\sigma_{\tilde Z}\Phi^{-1}(p)} F_{\tilde Z}^{-1}(p) = e^{\frac{1}{2}(1-\delta^2)\sigma_{\tilde Z}^2-\frac{1}{2}(1-\delta)\sigma_{\tilde Z}^2} F_{\tilde Z}^{-1}(p)^{\delta-1} F_{\tilde Z}^{-1}(p) \\ &= e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde Z}^2} F_{\tilde Z}^{-1}(p)^{\delta}, \\ w'(F_{\tilde Z}(\tilde Z)) &= e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde Z}^2} F_{\tilde Z}^{-1}(F_{\tilde Z}(\tilde Z))^{\delta} = e^{\frac{1}{2}(\delta-\delta^2)\sigma_{\tilde Z}^2}\tilde Z^{\delta}. \end{aligned}$$

Proof of Theorem 3.13. Assumption 3.11.1 is satisfied due to the condition on v_0. Let us turn to Assumption 3.11.2: $\inf_{c>0} k_H(c) > 1$. Since $\alpha_+ = 1$, $k_H(c)$ reduces to

$$\begin{aligned} k_H(c) &\overset{(3.10)}{=} k_{H-}\frac{w_-(1-F_{\tilde Z}(c))}{\left(\mathbb{E}\left[\tilde Z\left(\frac{w'_{A+}(F_{\tilde Z}(\tilde Z))}{\tilde Z}\right)^{\frac{1}{1-b_H}}\mathbb{1}_{\tilde Z\leqslant c}\right]\right)^{1-b_H}\left(\mathbb{E}\left[\tilde Z\mathbb{1}_{\tilde Z>c}\right]\right)^{b_H}} \\ &= \frac{k_{H-}}{e^{\frac{1}{2}\sigma_{\tilde Z}^2\left[\frac{b_H}{1-b_H}(1-\delta_+)^2\right]}}\frac{\left(\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}-(\delta_-+\frac{1}{2})\sigma_{\tilde Z}\right)\right)^{\alpha_-}}{\left(\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}+\frac{1}{2}\sigma_{\tilde Z}\right)\right)^{b_H}\left(\Phi\left(\frac{\ln c}{\sigma_{\tilde Z}}-\sigma_{\tilde Z}\left(\frac{\delta_+-b_H}{1-b_H}-\frac{1}{2}\right)\right)\right)^{1-b_H}}. \end{aligned}$$

Let $\alpha_- \leqslant b_H$. Moreover, let δ_+ such that $\frac{k_{H-}}{e^{\frac{1}{2}\sigma_{\tilde Z}^2\left[\frac{b_H}{1-b_H}(1-\delta_+)^2\right]}} > 1$ (for instance $\delta_+ = 1$) and $\delta_- \leqslant -1$. Then $\frac{1}{2}\sigma_{\tilde Z} \leqslant -(\delta_- + \frac{1}{2})\sigma_{\tilde Z}$ due to $\delta_- \leqslant -1$ and hence

$$k_H(c) > \frac{\left(\overbrace{\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}-(\delta_-+\frac{1}{2})\sigma_{\tilde Z}\right)}^{\geqslant\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}+\frac{1}{2}\sigma_{\tilde Z}\right)}\right)^{\alpha_-}}{\left(\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}+\frac{1}{2}\sigma_{\tilde Z}\right)\right)^{b_H}} \geqslant \left(\Phi\left(-\frac{\ln c}{\sigma_{\tilde Z}}+\frac{1}{2}\sigma_{\tilde Z}\right)\right)^{\alpha_--b_H} \geqslant 1\,.$$

Hence $\inf_{c>0} k_H(c) > 1$ and Assumption 3.11.2 is fulfilled. □

Proof of Theorem 3.14. 1: According to Theorem 3.12, if Assumption 3.11 is true, the optimal final payoff $X^\star$ is

$$\begin{aligned} &\left[\left(\frac{v_0-B+\frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde Z\mathbb{1}_{\tilde Z\leqslant F_{\tilde Z}^{-1}(1-\hat z(\infty,v_0-B))}\right]}{\mathbb{E}\left[\tilde Z\left(\frac{\tilde Z}{w'_+(F_{\tilde Z}(\tilde Z))}\right)^{\frac{1}{b_H-1}}\mathbb{1}_{\tilde Z\leqslant F_{\tilde Z}^{-1}(1-\hat z(\infty,v_0-B))}\right]}\right)\left(\frac{\tilde Z}{w'_+(F_{\tilde Z}(\tilde Z))}\right)^{-\frac{1}{1-b_H}}-\frac{1-b_H}{a_H}d_H\right] \\ &\times\mathbb{1}_{\tilde Z\leqslant F_{\tilde Z}^{-1}(1-\hat z(\infty,v_0-B))}+B\,. \end{aligned}$$

Define the constants $\chi_{A+} := \frac{v_0-B+\frac{1-b_H}{a_H}d_H\mathbb{E}\left[\tilde Z\mathbb{1}_{\tilde Z\leqslant F_{\tilde Z}^{-1}(1-\hat z(\infty,v_0-B))}\right]}{\mathbb{E}\left[\tilde Z\left(\frac{\tilde Z}{w'_+(F_{\tilde Z}(\tilde Z))}\right)^{\frac{1}{b_H-1}}\mathbb{1}_{\tilde Z\leqslant F_{\tilde Z}^{-1}(1-\hat z(\infty,v_0-B))}\right]}$ with $0 < \chi_{A+} < \infty$ and

$\tilde{\chi}_{A+} := \chi_{A+} e^{\frac{1}{2}\frac{\delta_+ - \delta_+^2}{1-b_H}\sigma_{\tilde{Z}}^2}$ with $0 < \tilde{\chi}_{A+} < \infty$. As $w'_+(F_{\tilde{Z}}(\tilde{Z})) \overset{(3.13)}{=} e^{\frac{1}{2}(\delta_+ - \delta_+^2)\sigma_{\tilde{Z}}^2} \tilde{Z}^{\delta_+}$ holds for $\alpha_+ = 1$, we have

$$X^\star = \left[\tilde{\chi}_{A+} \tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} - \frac{1-b_H}{a_H} d_H\right] \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))} + B\,.$$

The denominator of the constant term χ_{A+} can be expressed as

$$e^{\frac{1}{2}\frac{\delta_+ - \delta_+^2}{1-b_H}\sigma_{\tilde{Z}}^2} \mathbb{E}\left[\tilde{Z}^{\frac{\delta_+ - b_H}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right].$$

Then

$$\tilde{\chi}_{A+} = \chi_{A+} e^{\frac{1}{2}\frac{\delta_+ - \delta_+^2}{1-b_H}\sigma_{\tilde{Z}}^2} = \frac{v_0 - B + \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\tilde{Z}\mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right]}{\mathbb{E}\left[\tilde{Z}^{\frac{\delta_+ - b_H}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right]}.$$

2: First, notice that for any log-normally distributed variable Z with $\ln Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$ and any $x \geqslant 0$,

$$\frac{\ln\left(F_Z^{-1}(x)\right) - \mu_Z}{\sigma_Z} = \frac{\ln\left(e^{\mu_Z + \sigma_Z \Phi^{-1}(x)}\right) - \mu_Z}{\sigma_Z} = \Phi^{-1}(x).$$

Then, in view of part 1, the expected terminal payoff $\mathbb{E}\left[X^\star\right]$ is

$$\begin{aligned}
&\tilde{\chi}_{A+} \mathbb{E}\left[\tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right] - \frac{1-b_H}{a_H} d_H \mathbb{E}\left[\mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}\right] + B \\
&\overset{\text{Thm. 2.8}}{=} \tilde{\chi}_{A+} e^{\frac{1}{2}\sigma_{\tilde{Z}}^2\left[\left(\frac{1-\delta_+}{1-b_H}\right) + \left(\frac{1-\delta_+}{1-b_H}\right)^2\right]} \Phi\left(\Phi^{-1}\left(1 - \hat{z}(\infty, v_0 - B)\right) + \sigma_{\tilde{Z}}\left(\frac{1-\delta_+}{1-b_H}\right)\right) \\
&\quad - \frac{1-b_H}{a_H} d_H \Phi\left(\Phi^{-1}\left(1 - \hat{z}(\infty, v_0 - B)\right) - \sigma_{\tilde{Z}}\right) + B.
\end{aligned}$$

3: Now define $X_1^\star := \tilde{Z}^{-\frac{1-\delta_+}{1-b_H}} \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}$ and $X_2^\star := \mathbb{1}_{\tilde{Z} \leqslant F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}$, then

$$X^\star = \tilde{\chi}_{A+} X_1^\star - \frac{1-b_H}{a_H} d_H X_2^\star + B.$$

Since a complete market is assumed, there exists a replicating wealth process $V^\star(t)$, $t \in [0, T]$, such that

$$V^\star(T) = X^\star = \tilde{\chi}_{A+} X_1^\star - \frac{1-b_H}{a_H} d_H X_2^\star + B.$$

By the same argument, there exist replicating wealth processes $V_1^\star(t)$ and $V_2^\star(t)$ such that $V_1^\star(T) = X_1^\star$, $V_2^\star(T) = X_2^\star$. Hence,

$$V^\star(t) = \tilde{\chi}_{A+} V_1^\star(t) - \frac{1-b_H}{a_H} d_H V_2^\star(t) + B, \;\; \forall t \in [0, T].$$

In accordance with Theorem 2.9, the individual replicating wealth processes are

$$V_1^\star(t) = \tilde{Z}(t)^{-\frac{1-\delta_+}{1-b_H}} e^{\frac{1}{2}\sigma_{\tilde{Z}}^2(t)\left[\left(\frac{\delta_+-b_H}{1-b_H}\right)^2-\left(\frac{\delta_+-b_H}{1-b_H}\right)\right]} \times \Phi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))\right)-\ln\tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} - \sigma_{\tilde{Z}}(t)\left(\frac{\delta_+-b_H}{1-b_H}-\frac{1}{2}\right)\right)$$

and

$$V_2^\star(t) = \Phi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))\right)-\ln\tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} - \frac{1}{2}\sigma_{\tilde{Z}}(t)\right).$$

4: Theorem 2.9 states that the replicating risky relative portfolio process is given by

$$\hat{\pi}^\star(t)V^\star(t) = \tilde{\chi}_{A+}\hat{\pi}_1^\star(t)V_1^\star(t) - \frac{1-b_H}{a_H}d_H\hat{\pi}_2^\star(t)V_2^\star(t),\ \forall t\in[0,T],$$

where

$$\hat{\pi}_1^\star(t)V_1^\star(t) \overset{r=0}{=} \left\{\frac{\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))\right)^{\frac{\delta_+-b_H}{1-b_H}}}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)} \times \psi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))\right)-\ln\tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} + \frac{1}{2}\sigma_{\tilde{Z}}(t)\right) + \frac{1-\delta_+}{1-b_H}V_1^\star(t)\right\} \times \left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}),$$

$$\hat{\pi}_2^\star(t)V_2^\star(t) \overset{r=0}{=} \frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)} \times \psi\left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty,v_0-B))\right)-\ln\tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} + \frac{1}{2}\sigma_{\tilde{Z}}(t)\right)\left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}).$$

Then

$$\hat{\pi}^\star(t)V^\star(t) = \left(\tilde{\chi}_{A+}\hat{\pi}_1^\star(t)V_1^\star(t) - \frac{1-b_H}{a_H}d_H\hat{\pi}_2^\star(t)V_2^\star(t)\right)$$

$$= \Bigg[\tilde{\chi}_{A+} \frac{1-\delta_+}{1-b_H} V_1^\star(t) + \frac{F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))}{\sigma_{\tilde{Z}}(t)\tilde{Z}(t)}$$

$$\times \psi \left(\frac{\ln\left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))\right) - \ln \tilde{Z}(t)}{\sigma_{\tilde{Z}}(t)} + \frac{1}{2}\sigma_{\tilde{Z}}(t) \right)$$

$$\times \left\{ \tilde{\chi}_{A+} \left(F_{\tilde{Z}}^{-1}(1-\hat{z}(\infty, v_0 - B))\right)^{-\frac{1-\delta_+}{1-b_H}} - \frac{1-b_H}{a_H} d_H \right\} \Bigg]$$

$$\times \left(\sigma\sigma'\right)^{-1}(\mu - r\mathbf{1}).$$

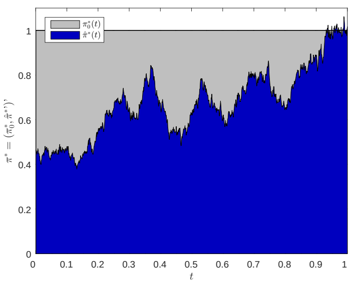
The relative investment in the bank account is

$$\pi_0^\star(t) = 1 - \hat{\pi}^\star(t)'\mathbb{1}.$$

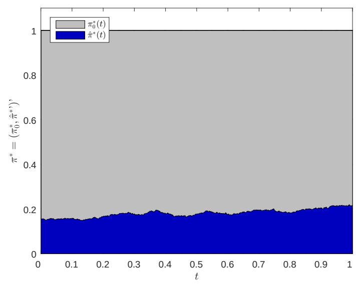
□

A.3 Numerical Case Study: Additional discussion

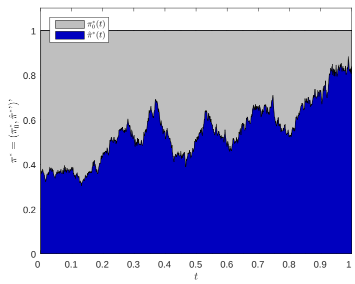
Here go some further illustrations, results and explanations coming from the numerical case study in Section 3.3. First, Figures A.1 and A.2 show area plots of the total investment strategies $\pi^\star(t)$ under each of the considered models, for the upward and downward moving underlying stock price from Figures 3.3 and 3.4. As $\hat{\pi}^\star(t) > 100\%$ for some $t \in [0, T]$ in Figure A.1, the investor takes leverage to participate in the increasing stock price. In these cases, $\pi_0^\star(t) < 0\%$ which is not indicated in the figures. Furthermore, the introduction of probability distortion functions can contribute in explaining mispricing of share prices or even the creation of so-called bubbles in financial markets. Bubbles can be characterized by stock prices, driven by market participants, significantly exceeding their fundamental value or fair price, potentially resulting in a market crash or bursting of the bubble. Youssefmir et al. (1998) for instance argue that investors, which do not fully behave rationally, can be responsible for the creation of a bubble. Smith et al. (1988) found out that bubbles are more likely to occur when people are less experienced. Let us assume that rationale expected utility with an objective weighting of probabilities leads to the true, fair stock prices. In particular Figures 3.3 and A.1 show that within a rising market, both CPT strategies overweight the risky asset when a concave w_+ is considered; the CRRA (CPT) and the HARA (EUT) utility models are equal, the two models only differ in the probability distortion. It can be observed that the CPT strategies with the concave w_+ lead to a much higher proportion of equity in the portfolio. This in turn leads to a higher demand of shares, which then drives the share price up and which finally leads to a higher portfolio value and thus to a higher total risky investment. The resulting increase in risky asset prices are not based on or connected to economic reasoning or value behind. Within CPT, it can be observed that HARA utility can even accelerate the creation of overpricing or even bubbles in the considered setting compared to CRRA utility. It is worth to mention that CPT normally assumes a small investor whose actions have no impact on the market. But when there is a sufficiently large fraction of actors in the market who behave according to CPT, then on average CPT can help explaining overpricing of share prices and the creation of bubbles. In contrast, a convex w_+ can help in explaining situations where investors significantly underweight risky assets in their portfolios.



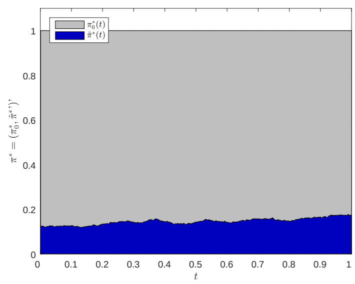
(a) HARA utility within CPT (w_+ concave).



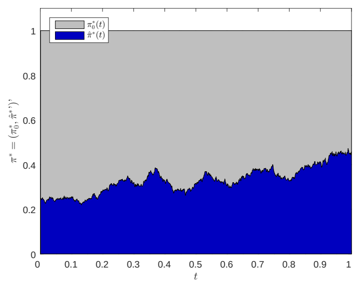
(b) HARA utility within CPT (w_+ convex).



(c) CRRA utility within CPT (w_+ concave).

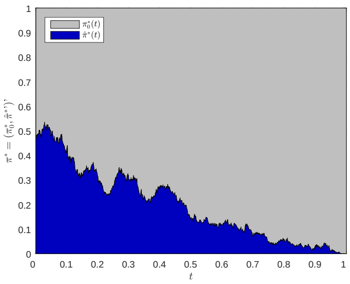


(d) CRRA utility within CPT (w_+ convex).

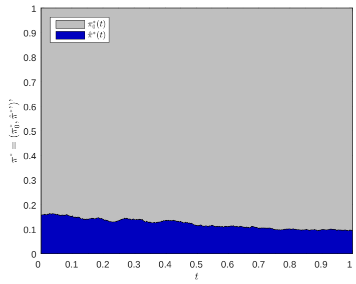


(e) HARA utility within EUT.

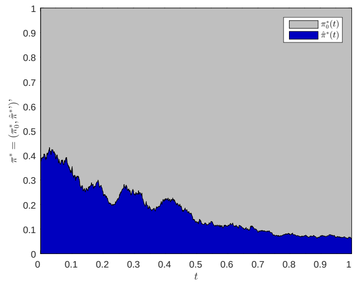
Figure A.1: Relative portfolio processes $\pi^\star(t) = (\pi_0^\star(t), \hat{\pi}^\star(t)')'$ under an upward movement of the stock price process $P_1(t)$ *(blue* $= \hat{\pi}^\star(t)$*, grey* $= \pi_0^\star(t)$*).*



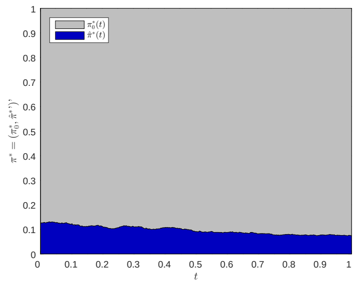
(a) HARA utility within CPT (w_+ concave).



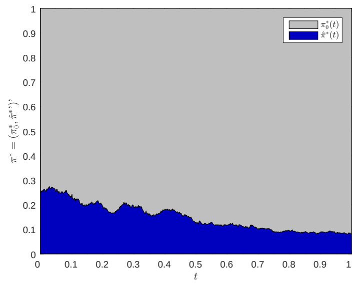
(b) HARA utility within CPT (w_+ convex).



(c) CRRA utility within CPT (w_+ concave).



(d) CRRA utility within CPT (w_+ convex).



(e) HARA utility within EUT.

Figure A.2: Relative portfolio processes $\pi^\star(t) = (\pi_0^\star(t), \hat{\pi}^\star(t)')'$ under a downward movement of the stock price process $P_1(t)$ *(blue* $= \hat{\pi}^\star(t)$, *grey* $= \pi_0^\star(t)$*).*

B Appendix to Chapter 4

B.1 Proofs: The consumption problem

Proof of Theorem 4.2. The Lagrangian of the Problem (4.8) subject to (4.9) is

$$\begin{aligned}\mathcal{L}(c,\lambda_1) &= \mathbb{E}\left[\int_0^T U_1(t,c(t))dt\right] - \lambda_1\left(\mathbb{E}\left[\int_0^T \tilde{Z}(t)\left(c(t)-y(t)\right)dt\right] - v_1\right)\\ &= \mathbb{E}\left[\int_0^T U_1(t,c(t)) - \lambda_1\left(\tilde{Z}(t)\left(c(t)-y(t)\right) - \frac{1}{T}v_1\right)dt\right].\end{aligned}$$

By the structure of the utility function, the optimal c_1 fulfills $c_1(t;v_1) > \bar{c}(t)$ and thus the first order conditions involve existence of a Lagrange multiplier $\lambda_1 = \lambda_1(v_1) > 0$ such that the optimal c_1 maximizes $\mathcal{L}(c,\lambda_1)$ and such that complementary slackness holds true. Hence it can be shown that the Karush-Kuhn-Tucker conditions besides the first derivative condition are satisfied.

Following Aase (2017), let $\nabla_h\mathcal{L}(c,\lambda_1;h)$ denote the directional derivative of $\mathcal{L}(c,\lambda_1)$ in the feasible direction h. The directional derivative of a function f in the direction h is generally defined by

$$\nabla_h f(x) = \lim_{y\to 0}\frac{f(x+hy)-f(x)}{y}.$$

If f is differentiable at x this results in

$$\nabla_h f(x) = f'(x)h.$$

In our case, for the inner function it holds

$$\begin{aligned}&\nabla_h\left(U_1(t,c(t)) - \lambda_1\left(\tilde{Z}(t)\left(c(t)-y(t)\right) - \frac{1}{T}v_1\right)\right)\\ &= \frac{\partial}{\partial c}\left(U_1(t,c(t)) - \lambda_1\left(\tilde{Z}(t)\left(c(t)-y(t)\right) - \frac{1}{T}v_1\right)\right)h(t)\\ &= \left(\frac{\partial}{\partial c}U_1(t,c(t)) - \lambda_1\tilde{Z}(t)\right)h(t).\end{aligned}$$

By the dominated convergence theorem, which allows interchanging expectation and differentiation, the first order condition gives

$$\begin{aligned}0 &= \mathbb{E}\left[\int_0^T\left(\frac{\partial}{\partial c}U_1(t,c(t)) - \lambda_1\tilde{Z}(t)\right)h(t)dt\right]\\ &= \mathbb{E}\left[\int_0^T\left(e^{-\beta t}a(t)\left(\frac{1}{1-b(t)}\left(c(t)-\bar{c}(t)\right)\right)^{b(t)-1} - \lambda_1\tilde{Z}(t)\right)h(t)dt\right]\end{aligned}$$

for all feasible h. In order to fulfill this condition for any h, the optimal consumption-rate process must be

$$c_1(t;v_1) = (1-b(t))\left(\lambda_1 \frac{e^{\beta t}}{a(t)}\tilde{Z}(t)\right)^{\frac{1}{b(t)-1}} + \bar{c}(t),\ t \in [0,T]. \tag{B.1}$$

Since $U_1(t,c)$ strictly increases in c, the budget constraint (4.9) for the optimal solution in (4.8) turns to equality, i.e.

$$\mathbb{E}\left[\int_0^T \tilde{Z}(t)\left(c_1(t;v_1) - y(t)\right)dt\right] = v_1.$$

When plugging in (B.1) and by Fubini, the budget condition turns into

$$\begin{aligned}
v_1 &= \mathbb{E}\left[\int_0^T \tilde{Z}(t)\left((1-b(t))\left(\lambda_1\frac{e^{\beta t}}{a(t)}\tilde{Z}(t)\right)^{\frac{1}{b(t)-1}} + \bar{c}(t) - y(t)\right)dt\right] \\
&= \int_0^T (1-b(t))\left(\lambda_1\frac{e^{\beta t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\mathbb{E}\left[\tilde{Z}(t)^{\frac{b(t)}{b(t)-1}}\right]dt + \int_0^T \mathbb{E}\left[\tilde{Z}(t)\right]\left(\bar{c}(t) - y(t)\right)dt \\
&= \int_0^T (1-b(t))\left(\lambda_1\frac{e^{\beta t}}{a(t)}\right)^{\frac{1}{b(t)-1}} e^{-\frac{b(t)}{b(t)-1}\left(r+\frac{1}{2}\|\gamma\|^2\right)t+\frac{1}{2}\left(\frac{b(t)}{b(t)-1}\right)^2\|\gamma\|^2 t}dt \\
&\quad + \int_0^T e^{-\left(r+\frac{1}{2}\|\gamma\|^2\right)t+\frac{1}{2}\|\gamma\|^2 t}\left(\bar{c}(t) - y(t)\right)dt \\
&= \int_0^T (1-b(t))\left(\frac{e^{\left[\beta - b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{1}{b(t)-1}}dt + \int_0^T e^{-rt}\left(\bar{c}(t) - y(t)\right)dt \\
&= \int_0^T (1-b(t))\left(\frac{e^{\left[\beta - b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{1}{b(t)-1}}dt + F_1(0).
\end{aligned}$$

Here we used that $\tilde{Z}(t)$ is a log-normal random variable and so is $\tilde{Z}(t)^{\frac{b(t)}{b(t)-1}}$. For any v_1, with $v_1 > F_1(0) = \int_0^T e^{-rt}\left(\bar{c}(t) - y(t)\right)dt$, the above equality determines $\lambda_1 > 0$ uniquely, since the integral in which λ_1 appears strictly decreases in λ_1 and has the limits 0 and ∞ as λ_1 approaches ∞ and 0. It follows immediately that the condition $v_1 > \int_0^T e^{-rt}\left(\bar{c}(t) - y(t)\right)dt$ in (4.10) is inevitable. The optimal wealth process $V_1(t;v_1)$ which arises by applying $c_1(t;v_1)$ is

$$\begin{aligned}
V_1(t;v_1) &= \mathbb{E}\left[\int_t^T \frac{\tilde{Z}(s)}{\tilde{Z}(t)}\left(c_1(s;v_1) - y(s)\right)ds\middle|\mathcal{F}_t\right] \\
&= \frac{1}{\tilde{Z}(t)}\mathbb{E}\left[\int_t^T \tilde{Z}(s)\left\{(1-b(s))\left(\lambda_1\frac{e^{\beta s}}{a(s)}\tilde{Z}(s)\right)^{\frac{1}{b(s)-1}} + \bar{c}(s) - y(s)\right\}ds\middle|\mathcal{F}_t\right] \\
&= \frac{1}{\tilde{Z}(t)}\left\{\mathbb{E}\left[\int_t^T (1-b(s))\left(\lambda_1\frac{e^{\beta s}}{a(s)}\right)^{\frac{1}{b(s)-1}}\tilde{Z}(s)^{\frac{b(s)}{b(s)-1}}ds\middle|\mathcal{F}_t\right]\right. \\
&\qquad\left. + \mathbb{E}\left[\int_t^T \tilde{Z}(s)\left(\bar{c}(s) - y(s)\right)ds\middle|\mathcal{F}_t\right]\right\}
\end{aligned}$$

$$= \frac{1}{\tilde{Z}(t)} \Bigg\{ \int_t^T (1-b(s)) \left(\lambda_1 \frac{e^{\beta s}}{a(s)}\right)^{\frac{1}{b(s)-1}} \mathbb{E}\left[\tilde{Z}(s)^{\frac{b(s)}{b(s)-1}} \Big| \mathcal{F}_t\right] ds + \int_t^T (\bar{c}(s)-y(s))\, \mathbb{E}\left[\tilde{Z}(s)\Big|\mathcal{F}_t\right] ds \Bigg\}.$$

$\tilde{Z}(s)$ can be written as $\frac{\tilde{Z}(s)}{\tilde{Z}(t)}\tilde{Z}(t)$ where $\frac{\tilde{Z}(s)}{\tilde{Z}(t)}$ is independent of $\mathcal{F}_t$ and $\tilde{Z}(t)$ is $\mathcal{F}_t$-measurable. Therefore it follows

$$\begin{aligned}
\mathbb{E}\left[\tilde{Z}(s)\Big|\mathcal{F}_t\right] &= \tilde{Z}(t)\mathbb{E}\left[\frac{\tilde{Z}(s)}{\tilde{Z}(t)}\right] = \tilde{Z}(t)e^{-\left(r+\frac{1}{2}\|\gamma\|^2\right)(s-t)+\frac{1}{2}\|\gamma\|^2(s-t)} = \tilde{Z}(t)e^{-r(s-t)}, \\
\mathbb{E}\left[\tilde{Z}(s)^\eta\Big|\mathcal{F}_t\right] &= \tilde{Z}(t)^\eta\mathbb{E}\left[\left(\frac{\tilde{Z}(s)}{\tilde{Z}(t)}\right)^\eta\right] = \tilde{Z}(t)^\eta e^{-\eta\left(r+\frac{1}{2}\|\gamma\|^2\right)(s-t)+\frac{1}{2}\eta^2\|\gamma\|^2(s-t)} \\
&= \tilde{Z}(t)^\eta e^{-\eta\left(r-\frac{1}{2}(\eta-1)\|\gamma\|^2\right)(s-t)}
\end{aligned}$$

for any $\eta \in \mathbb{R}$, where we used that $\frac{\tilde{Z}(s)}{\tilde{Z}(t)}$ and thus $\left(\frac{\tilde{Z}(s)}{\tilde{Z}(t)}\right)^\eta$ are log-normally distributed. Define the function g by

$$g(s,t;v_1) = (1-b(s)) \left(\frac{e^{\beta s - b(s)\left(r-\frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(s-t)}}{a(s)}\right)^{\frac{1}{b(s)-1}} \lambda_1^{\frac{1}{b(s)-1}},$$

then the optimal wealth process is given by

$$V_1(t;v_1) = \int_t^T g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds + F_1(t) \tag{B.2}$$

with $F_1(t)$ defined in (4.10). The dynamics can be calculated as

$$\begin{aligned}
dV_1(t;v_1) = & \left(-g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}}dt + \int_t^T d_t\left(g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right)ds\right) \\
& + \left(-(\bar{c}(t)-y(t))\,dt + \int_t^T d_t\left(e^{-r(s-t)}(\bar{c}(s)-y(s))\right)ds\right) \\
= & -g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}}dt + \int_t^T d_t\left(g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right)ds - (\bar{c}(t)-y(t))\,dt \\
& + \left(\int_t^T re^{-r(s-t)}(\bar{c}(s)-y(s))\,ds\right)dt \\
= & \left(-g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - (\bar{c}(t)-y(t)) + \int_t^T re^{-r(s-t)}(\bar{c}(s)-y(s))\,ds\right)dt \\
& + \int_t^T d_t\left(g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right)ds.
\end{aligned}$$

Notice that by Itô's formula,

$$d\left(\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right) = \frac{1}{b(s)-1}\tilde{Z}(t)^{\frac{1}{b(s)-1}-1}d\tilde{Z}(t) + \frac{1}{2}\frac{1}{b(s)-1}\left(\frac{1}{b(s)-1}-1\right)\tilde{Z}(t)^{\frac{1}{b(s)-1}-2}\tilde{Z}(t)^2\|\gamma\|^2dt$$

$$= \tilde{Z}(t)^{\frac{1}{b(s)-1}} \left\{ \left[-\frac{1}{b(s)-1} r + \frac{1}{2}\frac{1}{b(s)-1}\left(\frac{1}{b(s)-1} - 1\right)\|\gamma\|^2 \right] dt - \frac{1}{b(s)-1}\gamma' dW(t) \right\}.$$

Moreover, it holds

$$\begin{aligned} d_t g(s,t;v_1) &= (1-b(s)) \left(\frac{e^{\beta s - b(s)\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)s}}{a(s)} \right)^{\frac{1}{b(s)-1}} \lambda_1^{\frac{1}{b(s)-1}} d_t \left(e^{\frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)t} \right) \\ &= (1-b(s)) \left(\frac{e^{\beta s - b(s)\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)s}}{a(s)} \right)^{\frac{1}{b(s)-1}} \lambda_1^{\frac{1}{b(s)-1}} \\ &\quad \times \frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right) e^{\frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)t} dt \\ &= \frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(1-b(s)) \left(\frac{e^{\beta s - b(s)\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(s-t)}}{a(s)} \right)^{\frac{1}{b(s)-1}} \lambda_1^{\frac{1}{b(s)-1}} dt \\ &= \frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right) g(s,t;v_1) dt. \end{aligned}$$

With this we obtain

$$\begin{aligned} & d_t\left(g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right) = g(s,t;v_1) d\left(\tilde{Z}(t)^{\frac{1}{b(s)-1}}\right) + \tilde{Z}(t)^{\frac{1}{b(s)-1}} d_t g(s,t;v_1) + 0 \\ &= g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} \\ &\quad \times \left\{ \left[-\frac{1}{b(s)-1} r + \frac{1}{2}\frac{1}{b(s)-1}\left(\frac{1}{b(s)-1} - 1\right)\|\gamma\|^2 \right] dt - \frac{1}{b(s)-1}\gamma' dW(t) \right\} \\ &\quad + \tilde{Z}(t)^{\frac{1}{b(s)-1}} \frac{b(s)}{b(s)-1}\left(r - \frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right) g(s,t;v_1) dt \\ &= g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} \left\{ \left(r - \frac{1}{b(s)-1}\|\gamma\|^2\right) dt - \frac{1}{b(s)-1}\gamma' dW(t) \right\}. \end{aligned}$$

Define

$$Y(t) = \int_t^T \frac{1}{b(s)-1} g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds.$$

In summary, the dynamics of the optimal wealth process is then given by

$$\begin{aligned} dV_1(t;v_1) = & \left(-g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - (\bar{c}(t) - y(t)) + \int_t^T r e^{-r(s-t)} (\bar{c}(s) - y(s))\, ds \right) dt \\ & + \int_t^T g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} \left\{ \left(r - \frac{1}{b(s)-1}\|\gamma\|^2\right) dt - \frac{1}{b(s)-1}\gamma' dW(t) \right\} ds \end{aligned}$$

$$
\begin{aligned}
&= \Bigg(- g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - (\bar{c}(t) - y(t)) + \int_t^T re^{-r(s-t)} \left(\bar{c}(s) - y(s)\right) ds \\
&\quad + \int_t^T \left(r - \frac{1}{b(s)-1}\|\gamma\|^2 \right) g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds \Bigg) dt \\
&\quad - \underbrace{\left(\int_t^T \frac{1}{b(s)-1} g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds \right)}_{=Y(t)} \gamma' dW(t) \\
&= \Bigg\{ r \underbrace{\left(\int_t^T e^{-r(s-t)} \left(\bar{c}(s) - y(s)\right) ds + \int_t^T g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds \right)}_{=V_1(t;v_1)} \\
&\quad - g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - (\bar{c}(t) - y(t)) - \|\gamma\|^2 \underbrace{\int_t^T \frac{1}{b(s)-1} g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}} ds}_{=Y(t)} \Bigg\} dt \\
&\quad - Y(t)\gamma' dW(t) \\
&= \left(rV_1(t;v_1) - g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - (\bar{c}(t) - y(t)) - \|\gamma\|^2 Y(t) \right) dt - Y(t)\gamma' dW(t) \\
&= \mu_{V_1}(t)dt - Y(t)\gamma' dW(t) \qquad \text{(B.3)}
\end{aligned}
$$

with drift

$$
\mu_{V_1}(t) = rV_1(t;v_1) - g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} - \bar{c}(t) + y(t) - \|\gamma\|^2 Y(t).
$$

By (B.1) it follows

$$
c_1(t;v_1) = (1 - b(t)) \left(\lambda_1 \frac{e^{\beta t}}{a(t)} \tilde{Z}(t) \right)^{\frac{1}{b(t)-1}} + \bar{c}(t) = g(t,t;v_1)\tilde{Z}(t)^{\frac{1}{b(t)-1}} + \bar{c}(t).
$$

Hence

$$
\mu_{V_1}(t) = rV_1(t;v_1) - c_1(t;v_1) + y(t) - \|\gamma\|^2 Y(t).
$$

In order to determine the optimal investment strategy $\pi_1(t;v_1)$ to Problem (4.8) we compare the optimal wealth dynamics in (4.1) and (B.3):

$$
\begin{aligned}
dV_1(t;v_1) &= V_1(t;v_1) \left[\left(r + \hat{\pi}_1(t;v_1)' \left(\mu - r\mathbf{1}\right) \right) dt + \hat{\pi}_1(t;v_1)'\sigma dW(t) \right] - c_1(t;v_1)dt + y(t)dt, \\
dV_1(t;v_1) &= \left(rV_1(t;v_1) - c_1(t;v_1) + y(t) - \|\gamma\|^2 Y(t) \right) dt - Y(t)\gamma' dW(t).
\end{aligned}
$$

Matching the diffusion terms yields the equality

$$
\hat{\pi}_1(t;v_1) = -\frac{Y(t)}{V_1(t;v_1)} \Sigma^{-1}(\mu - r\mathbf{1})
$$

which simultaneously matches the drift terms. Therefore, if we insert the formula for $Y(t)$, we obtain

$$\hat{\pi}_1(t;v_1) = -\frac{\int_t^T \frac{1}{b(s)-1} g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds}{V_1(t;v_1)}\Sigma^{-1}(\mu - r\mathbf{1}) \tag{B.4}$$

which holds true for a general function $b(t)$ that does not have to be continuous. If $b(t)$ is a continuous function, by the first mean value theorem for integrals (cf. Theorem 2.2) it furthermore follows that there exists $\tilde{t}_1 \in (t,T)$ such that

$$\begin{aligned} Y(t) &= \int_t^T \frac{1}{b(s)-1} g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds = \frac{1}{b(\tilde{t}_1)-1}\int_t^T g(s,t;v_1)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds \\ &\overset{(B.2)}{=} \frac{1}{b(\tilde{t}_1)-1}\left(V_1(t;v_1) - F_1(t)\right). \end{aligned}$$

This determines the optimal investment strategy to be

$$\hat{\pi}_1(t;v_1) = \frac{1}{1-b(\tilde{t}_1)}\Sigma^{-1}(\mu - r\mathbf{1})\frac{V_1(t;v_1) - F_1(t)}{V_1(t;v_1)}. \tag{B.5}$$

□

Proof of Theorem 4.4. First, the value function of this problem is

$$\begin{aligned} \mathcal{V}_1(v_1) &= \mathbb{E}\left[\int_0^T U_1(t, c_1(t;v_1))dt\right] = \mathbb{E}\left[\int_0^T e^{-\beta t}\frac{1-b(t)}{b(t)}a(t)\left(\frac{1}{1-b(t)}\left(c_1(t;v_1) - \bar{c}(t)\right)\right)^{b(t)}dt\right] \\ &\overset{\text{Thm. 4.2}}{=} \mathbb{E}\left[\int_0^T e^{-\beta t}\frac{1-b(t)}{b(t)}a(t)\left(\lambda_1\frac{e^{\beta t}}{a(t)}\tilde{Z}(t)\right)^{\frac{b(t)}{b(t)-1}}dt\right] \\ &= \int_0^T e^{-\beta t}\frac{1-b(t)}{b(t)}a(t)\left(\lambda_1\frac{e^{\beta t}}{a(t)}\right)^{\frac{b(t)}{b(t)-1}}\mathbb{E}\left[\tilde{Z}(t)^{\frac{b(t)}{b(t)-1}}\right]dt \\ &= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\beta t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{b(t)}{b(t)-1}}\mathbb{E}\left[\tilde{Z}(t)^{\frac{b(t)}{b(t)-1}}\right]dt \\ &= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\beta t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{b(t)}{b(t)-1}}e^{-\frac{b(t)}{b(t)-1}\left(r+\frac{1}{2}\|\gamma\|^2\right)t+\frac{1}{2}\left(\frac{b(t)}{b(t)-1}\right)^2\|\gamma\|^2 t}dt \\ &= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{b(t)}{b(t)-1}}dt, \end{aligned}$$

where λ_1 is subject to (4.11). From differentiating both sides of Eq. (4.11) with respect to v_1 we derive

$$\begin{aligned} 1 &= \frac{\partial}{\partial v_1}\int_0^T (1-b(t))\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\lambda_1^{\frac{1}{b(t)-1}}dt \\ &= \int_0^T (1-b(t))\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}}\frac{\partial}{\partial v_1}\left(\lambda_1^{\frac{1}{b(t)-1}}\right)dt. \end{aligned} \tag{B.6}$$

This helps to identify $\mathcal{V}_1'(v_1)$ to be

$$\begin{aligned}
\mathcal{V}_1'(v_1) &= \frac{\partial}{\partial v_1}\int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \lambda_1^{\frac{b(t)}{b(t)-1}}dt \\
&= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \frac{\partial}{\partial v_1}\left(\lambda_1^{\frac{b(t)}{b(t)-1}}\right)dt \\
&= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \frac{\partial}{\partial v_1}\left(\left(\lambda_1^{\frac{1}{b(t)-1}}\right)^{b(t)}\right)dt \\
&= \int_0^T \frac{1-b(t)}{b(t)}\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} b(t)\left(\lambda_1^{\frac{1}{b(t)-1}}\right)^{b(t)-1}\frac{\partial}{\partial v_1}\left(\lambda_1^{\frac{1}{b(t)-1}}\right)dt \\
&= \lambda_1\int_0^T (1-b(t))\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \frac{\partial}{\partial v_1}\left(\lambda_1^{\frac{1}{b(t)-1}}\right)dt \overset{(B.6)}{=} \lambda_1.
\end{aligned}$$

(B.6) further implies concavity of $\mathcal{V}_1(v_1)$ as

$$\begin{aligned}
1 &= \int_0^T (1-b(t))\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \frac{\partial}{\partial v_1}\left(\lambda_1^{\frac{1}{b(t)-1}}\right)dt \\
&= \int_0^T (1-b(t))\left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \frac{1}{b(t)-1}\lambda_1^{\frac{1}{b(t)-1}-1}\left(\frac{\partial}{\partial v_1}\lambda_1\right)dt \\
&= -\left(\frac{\partial}{\partial v_1}\lambda_1\right)\int_0^T \left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \lambda_1^{-\frac{b(t)-2}{b(t)-1}}dt
\end{aligned}$$

and thus

$$\mathcal{V}_1''(v_1) = \frac{\partial}{\partial v_1}\lambda_1 = -\left(\int_0^T \left(\frac{e^{\left[\beta-b(t)\left(r-\frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{a(t)}\right)^{\frac{1}{b(t)-1}} \lambda_1^{-\frac{b(t)-2}{b(t)-1}}dt\right)^{-1} < 0.$$

□

B.2 Proofs: The terminal wealth problem

Proof of Theorem 4.5. The Lagrangian of the Problem (4.13) subject to (4.14) is

$$\mathcal{L}(V,\lambda_2) = \mathbb{E}\left[U_2(V)\right] - \lambda_2\left(\mathbb{E}\left[\tilde{Z}(T)V\right] - v_2\right) = \mathbb{E}\left[U_2(V) - \lambda_2\left(\tilde{Z}(T)V - v_2\right)\right].$$

First of all, it is clear that $c_2(t;v_2)\equiv 0$. By the structure of the utility function, the optimal V_2 fulfills $V_2(T;v_2)>F$ and thus the first order conditions involve existence of a Lagrange multiplier $\lambda_2=\lambda_2(v_2)>0$ such that the optimal V_2 maximizes $\mathcal{L}(V,\lambda_2)$ and such that complementary slackness holds true. Hence it can be shown that the Karush-Kuhn-Tucker conditions besides the first derivative condition are satisfied. By the dominated convergence theorem, the first order condition with respect to the directional derivative gives

$$0=\mathbb{E}\left[\left(\frac{\partial}{\partial V}U_2(V)-\lambda_2\tilde{Z}(T)\right)h\right]=\mathbb{E}\left[\left(e^{-\beta T}\hat{a}\left(\frac{1}{1-\hat{b}}(V-F)\right)^{\hat{b}-1}-\lambda_2\tilde{Z}(T)\right)h\right],$$

which has to be satisfied for all suitable h; hence the optimal terminal wealth has to fulfill

$$V_2(T;v_2)=(1-\hat{b})\left(\lambda_2\frac{e^{\beta T}}{\hat{a}}\tilde{Z}(T)\right)^{\frac{1}{\hat{b}-1}}+F. \tag{B.7}$$

Since $U_2(V)$ strictly increases in V, complementary slackness implies equality for the budget constraint

$$\mathbb{E}\left[\tilde{Z}(T)V_2(T;v_2)\right]=v_2.$$

Using (B.7), this gives

$$\begin{aligned}
v_2&=\mathbb{E}\left[\tilde{Z}(T)\left((1-\hat{b})\left(\lambda_2\frac{e^{\beta T}}{\hat{a}}\tilde{Z}(T)\right)^{\frac{1}{\hat{b}-1}}+F\right)\right]=(1-\hat{b})\left(\lambda_2\frac{e^{\beta T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}\mathbb{E}\left[\tilde{Z}(T)^{\frac{\hat{b}}{\hat{b}-1}}\right]+F\mathbb{E}\left[\tilde{Z}(T)\right]\\
&=(1-\hat{b})\left(\lambda_2\frac{e^{\beta T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}e^{-\frac{\hat{b}}{\hat{b}-1}(r+\frac{1}{2}\|\gamma\|^2)T+\frac{1}{2}\left(\frac{\hat{b}}{\hat{b}-1}\right)^2\|\gamma\|^2T}+Fe^{-(r+\frac{1}{2}\|\gamma\|^2)T+\frac{1}{2}\|\gamma\|^2T}\\
&=(1-\hat{b})\left(\frac{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}\lambda_2^{\frac{1}{\hat{b}-1}}+e^{-rT}F\\
&=(1-\hat{b})\left(\frac{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}\lambda_2^{\frac{1}{\hat{b}-1}}+F_2(0).
\end{aligned}$$

Solving for λ_2 yields

$$\lambda_2=\left(\frac{v_2-F_2(0)}{(1-\hat{b})\left(\frac{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}}\right)^{\hat{b}-1}=e^{-\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\left(1-\hat{b}\right)^{1-\hat{b}}\hat{a}\left(v_2-F_2(0)\right)^{\hat{b}-1} \tag{B.8}$$

where $v_2>F_2(0)=e^{-rT}F$ in (4.15) is required. Plugging this back into (B.7), the optimal terminal wealth is

$$\begin{aligned}
V_2(T;v_2) &= (1-\hat{b})\left(\frac{e^{\beta T}}{\hat{a}}\tilde{Z}(T)\right)^{\frac{1}{\hat{b}-1}}\lambda_2^{\frac{1}{\hat{b}-1}} + F \\
&= (1-\hat{b})\left(\frac{e^{\beta T}}{\hat{a}}\tilde{Z}(T)\right)^{\frac{1}{\hat{b}-1}}\left(\frac{v_2 - F_2(0)}{(1-\hat{b})\left(\frac{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}{\hat{a}}\right)^{\frac{1}{\hat{b}-1}}}\right) + F \\
&= (v_2 - F_2(0))\left(e^{\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)\right)^{\frac{1}{\hat{b}-1}} + F \\
&= (v_2 - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)^{\frac{1}{\hat{b}-1}} + F. \qquad \text{(B.9)}
\end{aligned}$$

The optimal wealth process replicates $V_2(T;v_2)$ and is uniquely given by

$$\begin{aligned}
V_2(t;v_2) &= \mathbb{E}\left[\frac{\tilde{Z}(T)}{\tilde{Z}(t)}V_2(T;v_2)\Big|\mathcal{F}_t\right] \\
&= \frac{1}{\tilde{Z}(t)}\mathbb{E}\left[\tilde{Z}(T)\left\{(v_2 - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)^{\frac{1}{\hat{b}-1}} + F\right\}\Big|\mathcal{F}_t\right] \\
&= \frac{1}{\tilde{Z}(t)}\left\{(v_2 - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\mathbb{E}\left[\tilde{Z}(T)^{\frac{\hat{b}}{\hat{b}-1}}\Big|\mathcal{F}_t\right] + F\mathbb{E}\left[\tilde{Z}(T)\Big|\mathcal{F}_t\right]\right\} \\
&= \frac{1}{\tilde{Z}(t)}\left\{(v_2 - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(t)^{\frac{\hat{b}}{\hat{b}-1}}e^{-\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(T-t)} + F\tilde{Z}(t)e^{-r(T-t)}\right\}.
\end{aligned}$$

This finally gives

$$V_2(t;v_2) = (v_2 - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t) \qquad \text{(B.10)}$$

with $F_2(t)$ defined in (4.15). Recall that

$$d\left(\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\right) = \tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\left\{\left[-\frac{1}{\hat{b}-1}r + \frac{1}{2}\frac{1}{\hat{b}-1}\left(\frac{1}{\hat{b}-1}-1\right)\|\gamma\|^2\right]dt - \frac{1}{\hat{b}-1}\gamma' dW(t)\right\}.$$

It follows by Itô

$$\begin{aligned}
&d\left(e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\right) \\
&= e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}d\left(\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\right) + \tilde{Z}(t)^{\frac{1}{\hat{b}-1}}d\left(e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\right) + 0 \\
&= e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\left\{\left[-\frac{1}{\hat{b}-1}r + \frac{1}{2}\frac{1}{\hat{b}-1}\left(\frac{1}{\hat{b}-1}-1\right)\|\gamma\|^2\right]dt - \frac{1}{\hat{b}-1}\gamma' dW(t)\right\} \\
&\quad + \tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}dt
\end{aligned}$$

$$= e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}$$
$$\times\left\{\left[-\frac{1}{\hat{b}-1}r+\frac{1}{2}\frac{1}{\hat{b}-1}\left(\frac{1}{\hat{b}-1}-1\right)\|\gamma\|^2\right]dt-\frac{1}{\hat{b}-1}\gamma' dW(t)+\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)dt\right\}$$
$$= e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\left\{\frac{1}{\hat{b}-1}\left[(\hat{b}-1)r+\frac{1}{2}\left(\frac{1}{\hat{b}-1}-1-\frac{\hat{b}}{\hat{b}-1}\right)\|\gamma\|^2\right]dt-\frac{1}{\hat{b}-1}\gamma' dW(t)\right\}$$
$$= e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\left\{\left(r-\frac{1}{\hat{b}-1}\|\gamma\|^2\right)dt-\frac{1}{\hat{b}-1}\gamma' dW(t)\right\}.$$

Then the optimal wealth dynamics can be calculated as

$$\begin{aligned}
dV_2(t;v_2) =&\ (v_2-F_2(0))\, d\left(e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}\right)+rF_2(t)dt \\
=&\ \underbrace{(v_2-F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}}}_{\overset{(B.10)}{=}V_2(t;v_2)-F_2(t)}\left\{\left(r-\frac{1}{\hat{b}-1}\|\gamma\|^2\right)dt-\frac{1}{\hat{b}-1}\gamma' dW(t)\right\} \\
&+rF_2(t)dt \\
=&\ rV_2(t;v_2)dt+(V_2(t;v_2)-F_2(t))\left\{-\frac{1}{\hat{b}-1}\|\gamma\|^2 dt-\frac{1}{\hat{b}-1}\gamma' dW(t)\right\}.
\end{aligned}$$

Comparing the diffusion term with the one from (4.1) for $y(t)\equiv 0$ implies

$$\hat{\pi}_2(t;v_2)=\frac{1}{1-\hat{b}}\Sigma^{-1}(\mu-r\mathbf{1})\frac{V_2(t;v_2)-F_2(t)}{V_2(t;v_2)} \tag{B.11}$$

which automatically matches the drifts iff $c_2(t;v_2)\equiv 0$. □

Proof of Theorem 4.7. The value function of this problem is given by

$$\begin{aligned}
\mathcal{V}_2(v_2) =&\ \mathbb{E}\left[U_2(V_2(T;v_2))\right]=\mathbb{E}\left[U_2\left((v_2-F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)^{\frac{1}{\hat{b}-1}}+F\right)\right] \\
=&\ e^{-\beta T}\frac{1-\hat{b}}{\hat{b}}\hat{a}\left(\frac{1}{1-\hat{b}}\right)^{\hat{b}}(v_2-F_2(0))^{\hat{b}}\, e^{\frac{\hat{b}^2}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\mathbb{E}\left[\tilde{Z}(T)^{\frac{\hat{b}}{\hat{b}-1}}\right] \\
=&\ e^{-\beta T}\frac{\left(1-\hat{b}\right)^{1-\hat{b}}}{\hat{b}}\hat{a}\,(v_2-F_2(0))^{\hat{b}}\, e^{\frac{\hat{b}^2}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}e^{-\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T} \\
=&\ e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\frac{\left(1-\hat{b}\right)^{1-\hat{b}}}{\hat{b}}\hat{a}\,(v_2-F_2(0))^{\hat{b}}.
\end{aligned}$$

This implies

$$\begin{aligned}
\mathcal{V}_2'(v_2) =&\ e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\frac{\left(1-\hat{b}\right)^{1-\hat{b}}}{\hat{b}}\hat{a}\hat{b}\,(v_2-F_2(0))^{\hat{b}-1} \\
=&\ e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\left(1-\hat{b}\right)^{1-\hat{b}}\hat{a}\,(v_2-F_2(0))^{\hat{b}-1}
\end{aligned}$$

$$\overset{\text{(B.8)}}{=} \lambda_2 > 0.$$

Due to the assumption $v_2 - F_2(0) > 0$ in (4.15), it is straightforward that $\mathcal{V}_2''(v_2) = \frac{\partial}{\partial v_2}\lambda_2 < 0$:

$$\mathcal{V}_2''(v_2) = -e^{\left[-\beta+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\left(1-\hat{b}\right)^{2-\hat{b}}\hat{a}\left(v_2 - F_2(0)\right)^{\hat{b}-2} < 0.$$

□

B.3 Proofs: Optimal merging of the individual solutions

Proof of Theorem 4.8.

1. $\mathcal{V}(v_0) \geqslant \sup_{v_1 \geqslant F_1(0),\ v_2 \geqslant F_2(0),\ v_1+v_2=v_0} \{\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2)\}$:

   Let $(\pi_1(t; v_1), c_1(t; v_1))$ and $(\pi_2(t; v_2), c_2(t; v_2))$ denote the optimal controls to Problems (4.8) and (4.13) with optimal wealth processes $V_1(t; v_1)$ and $V_2(t; v_2)$ to the initial wealths $v_1 \geqslant F_1(0)$ and $v_2 \geqslant F_2(0)$. Then, as the budget constraints for the optimal solutions to all three problems hold with equality,

$$\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2) = \mathbb{E}\left[\int_0^T U_1(t, c_1(t; v_1))dt + U_2(V_2(T; v_2))\right] \leqslant \sup_{(\pi,c)\in\Lambda} \mathcal{J}(\pi, c; v_0) = \mathcal{V}(v_0)$$

   for all v_1, v_2 with $v_1 + v_2 = v_0$. Thus

$$\mathcal{V}(v_0) \geqslant \sup_{v_1 \geqslant F_1(0),\ v_2 \geqslant F_2(0),\ v_1+v_2=v_0} \{\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2)\}.$$

2. $\mathcal{V}(v_0) \leqslant \sup_{v_1 \geqslant F_1(0),\ v_2 \geqslant F_2(0),\ v_1+v_2=v_0} \{\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2)\}$:

   Let $(\pi^\star, c^\star)$ denote the optimal controls which maximize $\mathcal{V}(v_0)$ with optimal wealth process $V^\star$ to the initial wealth $v_0 > 0$. Define

$$v_1 = \mathbb{E}\left[\int_0^T \tilde{Z}(t)\left(c^\star(t) - y(t)\right)dt\right],\ v_2 = \mathbb{E}\left[\tilde{Z}(T)V^\star(T)\right].$$

   Then, $v_1 + v_2 = v_0$ and

$$\mathcal{V}(v_0) = \mathbb{E}\left[\int_0^T U_1(t, c^\star(t))dt\right] + \mathbb{E}\left[U_2(V^\star(T))\right] \leqslant \mathcal{V}_1(v_1) + \mathcal{V}_2(v_2).$$

   Hence

$$\mathcal{V}(v_0) \leqslant \sup_{v_1 \geqslant F_1(0),\ v_2 \geqslant F_2(0),\ v_1+v_2=v_0} \{\mathcal{V}_1(v_1) + \mathcal{V}_2(v_2)\}.$$

□

Proof of Lemma 4.9. In accordance with Theorem 4.8 and by expressing $v_2 = v_0 - v_1$, the candidate for the optimal $v_1^\star$ is the one that satisfies the first order derivative condition on the budget

$$0 = \frac{\partial}{\partial v_1}\left(\mathcal{V}_1(v_1) + \mathcal{V}_2(v_0 - v_1)\right) = \mathcal{V}_1'(v_1) - \mathcal{V}_2'(v_0 - v_1)$$

such that $v_1^\star \geqslant F_1(0)$, $v_2^\star = v_0 - v_1^\star$ with $v_2^\star \geqslant F_2(0)$; thus $F_1(0) \leqslant v_1^\star \leqslant v_0 - F_2(0)$. Theorems 4.4 and 4.7 tell that $\mathcal{V}_1(v_1)$ and $\mathcal{V}_2(v_2)$ are strictly concave functions in v_1 respectively v_2. Therefore, it follows

$$0 = \frac{\partial^2}{\partial v_1^2}\left(\mathcal{V}_1(v_1) + \mathcal{V}_2(v_0 - v_1)\right) = \mathcal{V}_1''(v_1) + \mathcal{V}_2''(v_0 - v_1) < 0.$$

This implies that the candidate $v_1^\star$ that solves Eq. (4.16), together with $v_2^\star = v_0 - v_1^\star$, is the solution when the constraint $F_1(0) \leqslant v_1^\star \leqslant v_0 - F_2(0)$ applies. Furthermore, in accordance with Theorems 4.4 and 4.7 we have

$$\begin{aligned}
\mathcal{V}_1'(v_1) &= \lambda_1,\\
\mathcal{V}_2'(v_2) &= \lambda_2 = e^{-\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\left(1-\hat{b}\right)^{1-\hat{b}}\hat{a}\left(v_2 - F_2(0)\right)^{\hat{b}-1}.
\end{aligned}$$

By equating $\mathcal{V}_1'(v_1)$ and $\mathcal{V}_2'(v_0 - v_1)$ we obtain

$$\begin{aligned}
(4.16) \;&\Leftrightarrow \mathcal{V}_1'(v_1) = \mathcal{V}_2'(v_0 - v_1)\\
&\Leftrightarrow \lambda_1 = \lambda_2 = e^{-\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}\left(1-\hat{b}\right)^{1-\hat{b}}\hat{a}\left(v_0 - v_1 - F_2(0)\right)^{\hat{b}-1}.
\end{aligned}$$

Inserting λ_1 in Eq. (4.11), the optimal $v_1^\star$ is the solution to

$$v_1 - \int_0^T \chi(t)\left(v_0 - v_1 - F_2(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt = F_1(0),$$

where the continuous function $\chi(t)$ is defined by

$$\chi(t) = (1 - b(t))\left(1-\hat{b}\right)^{\frac{1-\hat{b}}{b(t)-1}}\left(\frac{\hat{a}}{a(t)}\right)^{\frac{1}{b(t)-1}}\left(\frac{e^{\left[\beta - b(t)\left(r - \frac{1}{2}\frac{1}{b(t)-1}\|\gamma\|^2\right)\right]t}}{e^{\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}\right)^{\frac{1}{b(t)-1}} > 0.$$

It remains to verify $F_1(0) \leqslant v_1^\star \leqslant v_0 - F_2(0)$ and uniqueness of $v_1^\star$. For this sake, define the function f by

$$f : (-\infty, v_0 - F_2(0)],\; f(x) = x - \int_0^T \chi(t)\left(v_0 - x - F_2(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt - F_1(0).$$

$v_1^\star$ is the root of the function f, i.e. $f(v_1^\star) = 0$, if it holds $v_1^\star \geqslant F_1(0)$. f is continuous in x, the exponent $\frac{\hat{b}-1}{b(t)-1}$ within the first integral is positive. Furthermore, due to $v_0 > F(0)$ claimed in (4.6) and $F(t) = F_1(t) + F_2(t)$, we have for the limits

$$\begin{aligned}
\lim_{x\searrow F_1(0)} f(x) &= -\int_0^T \chi(t)\left(v_0 - F_1(0) - F_2(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt = -\int_0^T \chi(t)\left(v_0 - F(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt < 0,\\
\lim_{x\nearrow v_0 - F_2(0)} f(x) &= v_0 - F_2(0) - \int_0^T \chi(t)\left(v_0 - (v_0 - F_2(0)) - F_2(0)\right)^{\frac{\hat{b}-1}{b(t)-1}} dt - F_1(0)
\end{aligned}$$

$$= v_0 - F_2(0) - F_1(0) = v_0 - F(0) > 0.$$

Note, $F_1(0) \leqslant v_1 = v_0 - v_2 \leqslant v_0 - F_2(0)$ for general v_1 and v_2. Additionally, f is strictly monotone increasing in x since

$$f'(x) = 1 + \int_0^T \chi(t) \frac{\hat{b}-1}{b(t)-1} \left(v_0 - x - F_2(0)\right)^{\frac{\hat{b}-b(t)}{b(t)-1}} dt > 0, \ \forall x \leqslant v_0 - F_2(0).$$

We conclude that there exists a unique root $x \in [F_1(0), v_0 - F_2(0)]$ such that $f(x) = 0$. Therefore, we conclude that the optimal v_1^* and $v_2^* = v_0 - v_1^*$ exist and are unique. v_1^* is the solution to Eq. (4.17). The optimal Lagrange multiplier $\lambda_1^\star = \lambda_1(v_1^\star)$ is then given by

$$\lambda_1^\star = \left(1-\hat{b}\right)^{1-\hat{b}} \hat{a} e^{-\left[\beta - \hat{b}\left(r - \frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T} \left(v_0 - v_1^\star - F_2(0)\right)^{\hat{b}-1}.$$

□

Proof of Theorem 4.10. Starting with $V^\star(t;v_0) = V_1(t;v_1^\star) + V_2(t;v_2^\star)$ we compare the dynamics of both sides of the equation:

$$dV^\star(t;v_0) = dV_1(t;v_1^\star) + dV_2(t;v_2^\star). \tag{B.12}$$

Eq. (4.1) for $V^\star(t;v_0)$, $V_1(t;v_1^\star)$ and $V_2(t;v_2^\star)$, with $y(t) \equiv 0$ for $V_2(t;v_2^\star)$, provides

$$\begin{aligned} dV^\star(t;v_0) &= V^\star(t;v_0)\left[\left(r + \hat{\pi}^\star(t;v_0)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}^\star(t;v_0)'\sigma dW(t)\right] - c^\star(t;v_0)dt + y(t)dt, \\ dV_1(t;v_1^\star) &= V_1(t;v_1^\star)\left[\left(r + \hat{\pi}_1(t;v_1^\star)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}_1(t;v_1^\star)'\sigma dW(t)\right] - c_1(t;v_1^\star)dt + y(t)dt, \\ dV_2(t;v_2^\star) &= V_2(t;v_2^\star)\left[\left(r + \hat{\pi}_2(t;v_2^\star)'\left(\mu - r\mathbf{1}\right)\right)dt + \hat{\pi}_2(t;v_2^\star)'\sigma dW(t)\right]. \end{aligned}$$

Comparing the diffusion terms in (B.12) gives

$$\hat{\pi}^\star(t;v_0) = \frac{\hat{\pi}_1(t;v_1^\star)V_1(t;v_1^\star) + \hat{\pi}_2(t;v_2^\star)V_2(t;v_2^\star)}{V^\star(t;v_0)}.$$

Inserting this back and comparing the drift terms finally leads to

$$c^\star(t;v_0) = c_1(t;v_1^\star).$$

Notice that the pair $(\hat{\pi}^\star, c^\star)$ is admissible, i.e. $(\hat{\pi}^\star, c^\star) \in \Lambda$ because $(\hat{\pi}_1, c_1) \in \Lambda_1$ and $(\hat{\pi}_2, 0) \in \Lambda_2$ which implies

$$V^\star(t;v_0) = \underbrace{V_1(t;v_1^\star)}_{\geqslant -\int_t^T e^{-r(s-t)}y(s)ds} + \underbrace{V_2(t;v_2^\star)}_{\geqslant 0} \geqslant -\int_t^T e^{-r(s-t)}y(s)ds, \ \mathbb{P}-a.s., \ \forall t \in [0,T].$$

Using the solutions in Theorems 4.2 and 4.5 we derive the following for the utility setup in (4.5):

$$\begin{aligned} \hat{\pi}^\star(t;v_0) &= \Sigma^{-1}(\mu - r\mathbf{1}) \frac{\frac{1}{1-b(\tilde{t}_1^\star)}\left(V_1(t;v_1^\star) - F_1(t)\right) + \frac{1}{1-\hat{b}}\left(V_2(t;v_2^\star) - F_2(t)\right)}{V^\star(t;v_0)}, \\ c^\star(t;v_0) &= c_1(t;v_1^\star) = g(t,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(t)-1}} + \bar{c}(t) = (1-b(t))\left(\lambda_1^\star \frac{e^{\beta t}}{a(t)}\tilde{Z}(t)\right)^{\frac{1}{b(t)-1}} + \bar{c}(t), \end{aligned}$$

$$
\begin{aligned}
V^\star(t;v_0) &= V_1(t;v_1^\star) + V_2(t;v_2^\star) \\
&= \int_t^T g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds + F_1(t) + (v_2^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t) \\
&= \int_t^T g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds + (v_2^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F(t), \\
V^\star(T;v_0) &= (v_2^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T}\tilde{Z}(T)^{\frac{1}{\hat{b}-1}} + F, \\
V_1(t;v_1^\star) &= \int_t^T g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds + F_1(t), \\
V_2(t;v_2^\star) &= (v_2^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}\tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t),
\end{aligned}
$$

for all $t \in [0,T]$, with

$$
\begin{aligned}
g(s,t;v_1^\star) &= (1-b(s))\left(\frac{e^{\beta s - b(s)\left(r-\frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(s-t)}}{a(s)}\right)^{\frac{1}{b(s)-1}}(\lambda_1^\star)^{\frac{1}{b(s)-1}} \\
&= (1-b(s))\left(1-\hat{b}\right)^{\frac{1-\hat{b}}{b(s)-1}}\left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{b(s)-1}}\left(\frac{e^{\beta s - b(s)\left(r-\frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)(s-t)}}{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}}\right)^{\frac{1}{b(s)-1}} \\
&\quad \times (v_0 - v_1^\star - F_2(0))^{\frac{\hat{b}-1}{b(s)-1}} \\
&\overset{(4.18)}{=} \chi(s) e^{\frac{b(s)}{b(s)-1}\left(r-\frac{1}{2}\frac{1}{b(s)-1}\|\gamma\|^2\right)t}(v_0 - v_1^\star - F_2(0))^{\frac{\hat{b}-1}{b(s)-1}}.
\end{aligned}
$$

Furthermore, $\tilde{t}_1^\star = \tilde{t}_1(v_1^\star) \in (t,T)$ solves (4.12):

$$
b(\tilde{t}_1^\star) = 1 + \frac{\int_t^T g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds}{\int_t^T \frac{1}{b(s)-1} g(s,t;v_1^\star)\tilde{Z}(t)^{\frac{1}{b(s)-1}}ds}.
$$

□

Proof of Remark 4.11. The formula for the optimal investment strategy is straightforward from Theorem 4.10 as $b(\tilde{t}_1^\star) \equiv \hat{b}$ and $V^\star(t;v_0) = V_1(t;v_1^\star) + V_2(t;v_2^\star)$ for any $t \in [0,T]$. The optimal $v_1^\star$ can be determined by Lemma 4.9 as the solution to Eq. (4.17):

$$
v_1^\star - (v_0 - v_1^\star - F_2(0))\int_0^T \chi(t)dt = F_1(0),
$$

where

$$
\chi(t) = \left(\frac{\hat{a}}{a(t)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right](T-t)}.
$$

Therefore,

$$
v_1^\star = \frac{(v_0 - F_2(0))\int_0^T \chi(t)dt + F_1(0)}{\int_0^T \chi(t)dt + 1}
$$

is the optimal budget to the consumption problem, $v_2^\star = v_0 - v_1^\star$ is the optimal budget to the terminal wealth problem. Furthermore, by Lemma 4.9 one knows

$$(\lambda_1^\star)^{\frac{1}{\hat{b}-1}} = \frac{1}{1-\hat{b}} \left(\frac{\hat{a}}{e^{\left[\beta-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}} \right)^{\frac{1}{\hat{b}-1}} (v_0 - v_1^\star - F_2(0)) .$$

This enables us to calculate $g(s,t;v_1^\star)$ to be

$$\begin{aligned}
g(s,t;v_1^\star) &= (1-\hat{b}) \left(\frac{e^{\beta s-\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t)}}{a(s)} \right)^{\frac{1}{\hat{b}-1}} (\lambda_1^\star)^{\frac{1}{\hat{b}-1}} \\
&= \left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta(T-s)+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t-T)\right]} (v_0 - v_1^\star - F_2(0)) \\
&= \left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta(T-s)+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t-T)\right]} \\
&\quad \times \left(v_0 - \frac{(v_0 - F_2(0)) \int_0^T \chi(t)dt + F_1(0)}{\int_0^T \chi(t)dt + 1} - F_2(0) \right) \\
&= \left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta(T-s)+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t-T)\right]} \left(\frac{v_0 - F_2(0) - F_1(0)}{\int_0^T \chi(t)dt + 1} \right) \\
&= \left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta(T-s)+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t-T)\right]} \left(\frac{v_0 - F(0)}{\int_0^T \chi(t)dt + 1} \right)
\end{aligned}$$

with $F(0) = \int_0^T e^{-rs} (\bar{c}(s) - y(s))\, ds + e^{-rT} F$ defined in (4.7), and thus using Theorem 4.10:

$$\begin{aligned}
V_1(t;v_1^\star) &= \int_t^T g(s,t;v_1^\star) \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} ds + F_1(t) \\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(\frac{v_0 - F(0)}{\int_0^T \chi(t)dt + 1} \right) \int_t^T \left(\frac{\hat{a}}{a(s)}\right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\left[\beta(T-s)+\hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)(s-t-T)\right]} ds + F_1(t) \\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} (v_0 - F(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\int_t^T \chi(s)ds}{\int_0^T \chi(t)dt + 1} + F_1(t).
\end{aligned}$$

With, again from Theorem 4.10,

$$\begin{aligned}
V_2(t;v_2^\star) &= (v_2^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t) \\
&= (v_0 - v_1^\star - F_2(0))\, e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t) \\
&= \left(\frac{v_0 - F_2(0) - F_1(0)}{\int_0^T \chi(t)dt + 1} \right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} + F_2(t) \\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} (v_0 - F(0)) \frac{e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t}}{\int_0^T \chi(t)dt + 1} + F_2(t)
\end{aligned}$$

because $F(t) = F_1(t) + F_2(t)\ \forall t \in [0,T]$, it follows

$$\begin{aligned}
V^\star(t;v_0) &= V_1(t;v_1^\star) + V_2(t;v_2^\star)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) \frac{1}{\int_0^T \chi(t)dt + 1} \left\{ e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \int_t^T \chi(s)ds + e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \right\}\\
&\quad + F_1(t) + F_2(t)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\int_t^T \chi(s)ds + 1}{\int_0^T \chi(t)dt + 1} + F(t).
\end{aligned}$$

Finally, the optimal consumption-rate process can then be determined from Theorem 4.10 as

$$\begin{aligned}
c^\star(t;v_0) &= (1-\hat{b}) \left(\lambda_1^\star \frac{e^{\beta t}}{a(t)} \tilde{Z}(t) \right)^{\frac{1}{\hat{b}-1}} + \bar{c}(t)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(\frac{e^{\beta t}}{a(t)} \right)^{\frac{1}{\hat{b}-1}} \left(\frac{\hat{a}}{e^{\left[\beta - \hat{b}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)\right]T}} \right)^{\frac{1}{\hat{b}-1}} \left(\frac{v_0 - F(0)}{\int_0^T \chi(t)dt + 1} \right) + \bar{c}(t)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) \left(\frac{\hat{a}}{a(t)} \right)^{\frac{1}{\hat{b}-1}} e^{-\frac{1}{\hat{b}-1}\beta(T-t) + \frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)T} \left(\frac{1}{\int_0^T \chi(t)dt + 1} \right) + \bar{c}(t)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\chi(t)}{\int_0^T \chi(t)dt + 1} + \bar{c}(t)\\
&= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\int_t^T \chi(s)ds + 1}{\int_0^T \chi(t)dt + 1} \frac{\chi(t)}{\int_t^T \chi(s)ds + 1} + \bar{c}(t)\\
&= \frac{\chi(t)}{\int_t^T \chi(s)ds + 1} \left(V^\star(t;v_0) - F(t) \right) + \bar{c}(t).
\end{aligned}$$

By defining

$$\zeta(t) = \frac{\chi(t)}{\int_t^T \chi(s)ds + 1} > 0$$

we obtain

$$c^\star(t;v_0) = \zeta(t) \left(V^\star(t;v_0) - F(t) \right) + \bar{c}(t).$$

With the definition of $\zeta(t)$, the optimal wealth process finally can be written as

$$\begin{aligned}
V^\star(t;v_0) &= \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\int_t^T \chi(s)ds + 1}{\int_0^T \chi(t)dt + 1} + F(t)\\
&= \frac{1}{\zeta(t)} \tilde{Z}(t)^{\frac{1}{\hat{b}-1}} \left(v_0 - F(0)\right) e^{\frac{\hat{b}}{\hat{b}-1}\left(r-\frac{1}{2}\frac{1}{\hat{b}-1}\|\gamma\|^2\right)t} \frac{\chi(t)}{\int_0^T \chi(t)dt + 1} + F(t).
\end{aligned}$$

□

C Appendix to Chapter 5

C.1 Additional results to Section 5.1.2

C.1.1 Special cases

The upcoming corollary states the results if the wealth benchmark $B(t)$ for the buffer mechanism $c(t)$ is constant (thus $dB(t) = 0dt$: zero benchmark return).

Corollary C.1 (Constant buffer benchmark: $B(t) \equiv B$)**.** *In the case where $B(t) \equiv B$ (i.e. constant buffer wealth benchmark) we have $dB(t) = B'(t)dt = 0$ and thus the formula for $\hat{\pi}^{\star}_{\alpha,B}(t)$ reduces to*

$$\hat{\pi}^{\star}_{\alpha,B}(t) = \frac{1+\alpha(t)}{1-\hat{b}} \frac{V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^{\star}_{\alpha,B}(t)} \Sigma^{-1} \left(\mu - r\mathbf{1}\right),$$

which is of a PPI type with cohort-age dependent but state- or market-independent multiple, and where

$$\tilde{F}_{\alpha,B}(t) = Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T \frac{y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds,$$

with $\tilde{F}_{\alpha,B}(T) = F$ and $\tilde{F}'_{\alpha,B}(t) = \frac{1}{1+\alpha(t)}\left(r\tilde{F}_{\alpha,B}(t) + y(t)\right)$. For the wealth process of the pension fund we obtain

$$\begin{aligned} dV^{\star}_{\alpha,B}(t) &= (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] \\ &\quad + \tilde{F}'_{\alpha,B}(t)dt, \\ d\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) &= (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right], \end{aligned}$$

and

$$\begin{aligned} V^{\star}_{\alpha,B}(t) &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\left[r\int_0^t \frac{1}{1+\alpha(s)}ds + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t\right] + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}. \end{aligned}$$

Additionally, the formula for the accumulated buffer account reads

$$C^{\star}_{\alpha,B}(t) = \alpha(t)\left(V^{\star}_{\alpha,B}(t) - B\right) - e^{rt}\alpha(0)(v_0 - B) + \int_0^t e^{r(t-s)} \left(r - \frac{\alpha'(s)}{\alpha(s)}\right) \alpha(s) \left(V^{\star}_{\alpha,B}(s) - B\right) ds.$$

$B(t)$ can be regarded as a wealth benchmark for the buffer mechanism $c(t)$ and $\tilde{F}_{\alpha,B}(t)$ as a wealth benchmark for the investment strategy $\hat{\pi}^{\star}_{\alpha,B}(t)$. The following corollary shows the solution if both wealth benchmarks coincide.

Corollary C.2 (Equal benchmarks for buffer rate and investment strategy: $B(t) = \tilde{F}_{\alpha,B}(t)$)**.** *If the wealth benchmark $B(t)$ of the buffer rate process coincides with the wealth benchmark $\tilde{F}_{\alpha,B}(t)$ of the optimal investment strategy, then*

$$\hat{\pi}^{\star}_{\alpha,B}(t) = \frac{1+\alpha(t)}{1-\hat{b}} \frac{V^{\star}_{\alpha,B}(t) - B(t)}{V^{\star}_{\alpha,B}(t)} \Sigma^{-1} (\mu - r\mathbf{1}),$$

which is a PPI strategy with time-t floor $\tilde{F}_{\alpha,B}(t) = B(t)$ being equal to the wealth benchmark $B(t)$ of the buffer rate process. Furthermore, we obtain

$$dV^{\star}_{\alpha,B}(t) = (V^{\star}_{\alpha,B}(t) - B(t)) \left[\left(\frac{1}{1+\alpha(t)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right] + B'(t)dt,$$

$$d\left(V^{\star}_{\alpha,B}(t) - B(t)\right) = (V^{\star}_{\alpha,B}(t) - B(t)) \left[\left(\frac{1}{1+\alpha(t)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right],$$

and

$$\begin{aligned} V^{\star}_{\alpha,B}(t) &= B(t) + (v_0 - B(0)) e^{\left[r \int_0^t \frac{1}{1+\alpha(s)} ds + \left(\frac{1}{1-\hat{b}} \|\gamma\|^2 - \frac{1}{2} \left(\frac{1}{1-\hat{b}} \right)^2 \|\gamma\|^2 \right) t \right] + \frac{1}{1-\hat{b}} \gamma' W(t)} \\ &= B(t) + (v_0 - B(0)) e^{\frac{1}{1-\hat{b}} \left[r \left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds \right) - \left(r + \frac{1}{2} \frac{\hat{b}}{1-\hat{b}} \|\gamma\|^2 \right) t \right]} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}. \end{aligned}$$

In this case ($B(t) = \tilde{F}_{\alpha,B}(t)$), $B(t)$ needs to fulfill

$$B'(t) = \frac{1}{1+\alpha(t)} \left(rB(t) + \alpha(t)B'(t) + y(t) \right) \Leftrightarrow B'(t) = rB(t) + y(t)$$

with terminal condition $B(T) = \tilde{F}_{\alpha,B}(T) = F$. The solution to this ODE is given by

$$B(t) = e^{-r(T-t)} F - \int_t^T e^{-r(s-t)} y(s) ds, \;\; B(T) = F.$$

Hence, $B(t)$ grows with inflows $y(t)$ and interest rate r which means that an investment return benchmark of r is considered. The accumulated buffer account formula becomes

$$\begin{aligned} C^{\star}_{\alpha,B}(t) &= \alpha(t) \left(V^{\star}_{\alpha,B}(t) - B(t) \right) - e^{rt} \alpha(0)(v_0 - B(0)) \\ &\quad + \int_0^t e^{r(t-s)} \left(r - \frac{\alpha'(s)}{\alpha(s)} \right) \alpha(s) \left(V^{\star}_{\alpha,B}(s) - B(s) \right) ds. \end{aligned}$$

Finally, we are interested in special settings where the optimal allocation coincides with a constant-mix strategy. One can show that this is the case if the buffer rule parameter is constant, $\alpha(t) \equiv \alpha$, and if $\tilde{F}_{\alpha,B}(t) = 0$.

Corollary C.3 (Optimal constant-mix strategy)**.** *Let $\alpha(t) \equiv \alpha$ and let $\tilde{F}_{\alpha,B}(t) = 0 \;\forall t \in [0,T]$. The latter holds if and only if $F = 0$ (i.e. CRRA utility function), since $\tilde{F}_{\alpha,B}(T) = F$ and the* *remainder of* $\tilde{F}_{\alpha,B}(t)$*, which is* $\int_t^T \frac{\alpha(s)B'(s)+y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$ *equates to zero* $\forall t \in [0,T]$ *because* $0 = \tilde{F}'_{\alpha,B}(t) \overset{(C.15)}{=} \frac{r\tilde{F}_{\alpha,B}(t)+\alpha(t)B'(t)+y(t)}{1+\alpha(t)} = \frac{\alpha(t)B'(t)+y(t)}{1+\alpha(t)}$*. In this case, i.e.* $\alpha(t)B'(t) + y(t) = 0 \Leftrightarrow B'(t) = -\frac{1}{\alpha(t)}y(t)$ *(*$B(t)$ *decreases in* $y(t) > 0$*), the wealth benchmark* $B(t)$ *is forced to take the form*

$$B(t) = B_T + \int_t^T \frac{1}{\alpha(s)} y(s)ds, \ B(T) = B_T.$$

The optimal constant-mix strategy then turns out to be

$$\hat{\pi}^\star_{\alpha,B}(t) \equiv \frac{1+\alpha}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1}).$$

Moreover, it holds

$$dV^\star_{\alpha,B}(t) = V^\star_{\alpha,B}(t)\left[\left(\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right],$$

and

$$V^\star_{\alpha,B}(t) = v_0 e^{\left[\frac{1}{1+\alpha}r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right]t+\frac{1}{1-\hat{b}}\gamma' W(t)} = v_0 e^{-\frac{1}{1-\hat{b}}\left[\frac{\alpha+\hat{b}}{1+\alpha}r+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right]t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.$$

If there is only one risky asset in the financial market ($N = 1$*), i.e. if a risky fund is considered, then the optimal constant-mix strategy is*

$$\hat{\pi}^\star_{\alpha,B}(t) \equiv \frac{1+\alpha}{1-\hat{b}}\frac{\mu-r}{\sigma^2} = (1+\alpha)MP = \frac{1+\alpha}{1-\hat{b}}\frac{1}{\sigma}SR,$$

where $MP := \frac{1}{1-\hat{b}}\frac{\mu-r}{\sigma^2}$ *is the Merton portfolio (cf. Merton (1969), Merton (1971)) and* $SR := \frac{\mu-r}{\sigma}$ *denotes the Sharpe Ratio (cf. Sharpe (1966), Sharpe (1994)) of the risky asset.*

Finally, the formula for the accumulated buffer becomes

$$C^\star_{\alpha,B}(t) = \alpha\left[\left(V^\star_{\alpha,B}(t) - B(t)\right) - e^{rt}(v_0 - B(0)) + r\int_0^t e^{r(t-s)}\left(V^\star_{\alpha,B}(s) - B(s)\right)ds\right].$$

C.1.2 Purely accumulated buffer

Let $\alpha(t) \equiv \alpha$ and $C^\star_{\alpha,B}(t) = \int_0^t c(s)ds$, $dC^\star_{\alpha,B}(t) = c(t)dt = \alpha d(V^\star_{\alpha,B}(t) - B(t))$. Under Assumption 5.6, we further aim to limit the downside risk, quantified by the risk measure Value-at-Risk (quantile). Thus, we now examine the Value-at-Risk constraint $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for some level $\beta \in (0,1]$. Note that for $\beta = 0$, $VaR_0\left(C^\star_{\alpha,B}(t)\right) \geqslant 0 \Leftrightarrow C^\star_{\alpha,B}(t) \geqslant 0$ which is considered in Theorem 5.7. Here go the Value-at-Risk closed-form results:

Theorem C.4 (Non-negativity of the Value-at-Risk $VaR_\beta\left(C^\star_{\alpha,B}(t)\right)$)**.** *Let* $\beta \in (0,1]$ *and let Assumption 5.6 hold, then the formula for* $VaR_\beta\left(C^\star_{\alpha,B}(t)\right)$ *is given by*

$$VaR_\beta\left(C^\star_{\alpha,B}(t)\right) = \alpha\left[f_{VaR}(t) + \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha}\right)Y(T) + \frac{1-\delta}{1+\alpha}Y(t)\right]$$

with

$$f_{VaR}(t) := e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} \geqslant 0.$$

Moreover, the following claims hold:

1. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ *for some* $t \in [0,T]$*:*

$$\begin{aligned}
& VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0 \\
\Leftrightarrow\ & \mathbb{P}\left(C^\star_{\alpha,B}(t) < 0\right) = \beta \\
\Leftrightarrow\ & \delta \geqslant 0,\ \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right] :\ f_{VaR}(t) + \frac{1-\delta}{1+\alpha}Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T) \\
\Leftrightarrow\ & \begin{cases}
\delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right],\ \textit{or} \\
\delta \in [0,1],\ \tilde{\delta} \in \left[0, \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}\right), \\
\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)-\frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}} - 1\right]\right),\ \textit{or} \\
\delta > 1,\ \tilde{\delta} \in [0,1],\ \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)-\frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}} - 1\right]\right).
\end{cases}
\end{aligned}$$

2. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ *for all* $t \in [0,T]$*:*

   a) *Let* $\beta \leqslant 1 - \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}}\right) \in (0, 0.5]$ *and* $\delta \leqslant 1$*:*

$$\begin{aligned}
& VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0\ \forall t \in [0,T] \\
\textit{if}\ & \delta \in [0,1],\ \tilde{\delta} \in \left[0, \frac{1+\alpha\delta}{1+\alpha}\right] :\ f_{VaR}(T) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T) \\
\textit{i.e. if}\ & \delta \in [0,1],\ \tilde{\delta} \in \left[0, \frac{1+\alpha\delta}{1+\alpha}\right],
\end{aligned}$$

$$\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{\delta}}\right)^2\|\gamma\|^2 T}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{\delta}}\|\gamma\|^2 T}}-1\right]\right).$$

b) *Let* $\beta \leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{\delta}}\right)^2\|\gamma\|^2 T}-1\right)}}\right) \in (0, 0.5]$ *and* $\delta > 1$*:*

$$\begin{aligned}
&VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0 \ \forall t \in [0,T] \\
\Leftrightarrow\ & \delta > 1,\ \tilde{\delta} \in [0,1]:\ f_{VaR}(T) \geqslant \left(\delta-\tilde{\delta}\right)Y(T) \\
\Leftrightarrow\ & \delta > 1,\ \tilde{\delta} \in [0,1],\ \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{\delta}}\right)^2\|\gamma\|^2 T}-1\right)}}\left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{\delta}}\|\gamma\|^2 T}}-1\right]\right).
\end{aligned}$$

3. $VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0$*:*

$$\begin{aligned}
&VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0 \\
\Leftrightarrow\ & \delta \geqslant 0,\ \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right]:\ f_{VaR}(T) \geqslant \left(\delta-\tilde{\delta}\right)Y(T) \\
\Leftrightarrow\ & \begin{cases} \delta \in [0,1],\ \tilde{\delta} \in \left[\delta, \frac{1+\alpha\delta}{1+\alpha}\right],\ or \\ \delta \geqslant 0,\ \tilde{\delta} \in [0, \min\{\delta,1\}],\ \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{\delta}}\right)^2\|\gamma\|^2 T}-1\right)}}\left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{\delta}}\|\gamma\|^2 T}}-1\right]\right). \end{cases}
\end{aligned}$$

Proof. The proof is given in Appendix C.2. □

Theorem C.4 provides possible parameter choices such that there is a lower bound on the quantile of the accumulated buffer $C^\star_{\alpha,B}(t)$. Hence, some worst-case loss probability can be guaranteed, while a possibly negative buffer rate $c(t)$ can be used for smoothing purposes. Notice that we are interested in the "left tail" of the buffer distribution, hence it is suitable to look at small β values. In Theorem C.4 one can observe that the selection criteria on δ and $\tilde{\delta}$ can be reformulated as a boundary on β to some extent. Notice that the conditions in Theorem C.4 without any β involved coincide with the respective conditions for $C^\star_{\alpha,B}(t) \geqslant 0$ in Theorem 5.7 because if $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for all $\beta \geqslant 0$, then in particular for $\beta = 0$. Also note that the conditions for 2. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for all $t \in [0,T]$ and 3. $VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0$ are logically independent of t. Together with

$$\tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right] \Leftrightarrow \frac{1-\delta}{1+\alpha}Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T)$$

from Theorem 5.7, it further directly shows the straightforward relation

$$C^{\star}_{\alpha,B}(t) \geqslant 0 \Rightarrow VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0.$$

This nicely shows that $f_{VaR}(t)$ plays the difference between $C^{\star}_{\alpha,B}(t) \geqslant 0$ and $VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0$ and relaxes the stricter $C^{\star}_{\alpha,B}(t) \geqslant 0$ condition.

Analogically to Remark 5.8 for $C^{\star}_{\alpha,B}(t) \geqslant 0$, we provide some interesting features to the Value-at-Risk constraint $VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0$ that arise from Theorem C.4.

Remark C.5 (Comments on the non-negativity of the Value-at-Risk $VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right)$)**.** *Let Assumption 5.6 hold true.*

1. *The f_{VaR}-inequality conditions on the choice of δ and $\tilde{\delta}$ in Theorem C.4 become less restrictive for larger β values, i.e. if an inequality condition is fulfilled for β, then it is automatically fulfilled for all $\tilde{\beta} \geqslant \beta$. That should clearly hold true since by definition it generally holds*

$$VaR_{\tilde{\beta}}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \;\; \forall \tilde{\beta} \geqslant \beta.$$

2. *From Theorem C.4 we know that*

$$\begin{aligned} & VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0 \\ \Leftrightarrow\; & \delta \geqslant 0,\; \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right]:\; f_{VaR}(t) + \frac{1-\delta}{1+\alpha}Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T). \end{aligned}$$

Let $\beta \leqslant 1 - \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\delta}\right)^2 \|\gamma\|^2 T} - 1\right)}}\right)$ *and $\delta > 1$. From (the proof of) Theorem C.4 we know that in this situation*

$$f_{VaR}(t) + \frac{1-\delta}{1+\alpha}Y(t)$$

is monotone decreasing in t. Thus,

$$\begin{aligned} & VaR_{\beta}\left(C^{\star}_{\alpha,B}(s)\right) \geqslant 0 \;\forall s \leqslant t \in [0,T] \\ \Leftrightarrow\; & \delta > 1,\; \tilde{\delta} \in [0,1]:\; f_{VaR}(s) + \frac{1-\delta}{1+\alpha}Y(s) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) \;\forall s \leqslant t \in [0,T] \\ \Leftrightarrow\; & \delta > 1,\; \tilde{\delta} \in [0,1]:\; \min_{s\in[0,t]}\left\{f_{VaR}(s) + \frac{1-\delta}{1+\alpha}Y(s)\right\} \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) \\ \overset{monotonicity}{\Leftrightarrow}\; & \delta > 1,\; \tilde{\delta} \in [0,1]:\; f_{VaR}\left(\max_{s\in[0,t]}\{s\}\right) + \frac{1-\delta}{1+\alpha}Y\left(\max_{s\in[0,t]}\{s\}\right) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) \end{aligned}$$

$$\Leftrightarrow \delta > 1,\ \tilde{\delta} \in [0,1]:\ f_{VaR}(t) + \frac{1-\delta}{1+\alpha} Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T)$$
$$\Leftrightarrow \delta > 1,\ \tilde{\delta} \in [0,1]:\ VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$$
$$\Leftrightarrow \delta > 1,\ \tilde{\delta} \in [0,1]:\ VaR_\beta\left(C^\star_{\alpha,B}\left(\max_{s\in[0,t]}\{s\}\right)\right) \geqslant 0.$$

Thus, "the largest time wins", i.e. is essential. Under $\delta > 1$, $\tilde{\delta} \in [0,1]$, it generally follows that if the parameters are selected such that $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ $\mathbb{P}$-a.s., then it also is $VaR_\beta\left(C^\star_{\alpha,B}(s)\right) \geqslant 0$ for all times s prior to time t, $s < t$:

$$VaR_\beta\left(C^\star_{\alpha,B}(s)\right) \geqslant 0\ \forall s \leqslant t \in [0,T] \Leftrightarrow VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0. \tag{C.1}$$

In particular, for $\delta > 1$, $\tilde{\delta} \in [0,1]$ ($\delta > \tilde{\delta}$), it holds $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0\ \forall t \in [0,T] \Leftrightarrow VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0$, which is also reflected in Theorem C.4 part 3..

Moreover, we observe the reversed relationship compared to

$$C^\star_{\alpha,B}(s) \geqslant 0\ \forall s \geqslant t \in (0,T] \Leftrightarrow C^\star_{\alpha,B}(t) \geqslant 0.$$

But remember that here we forced $\delta \leqslant 1$ and for the Value-at-Risk equivalence above $\delta > 1$. For the Value-at-Risk, we have no similar conclusion under $\delta \leqslant 1$ because in (the proof of) Theorem C.4 we only have a lower bound on $f_{VaR}(t) + \frac{1-\delta}{1+\alpha}Y(t)$ with $\min_t\{f_{VaR}(t)\} = f_{VaR}(\max_t\{t\})$ but $\min_t\left\{\frac{1-\delta}{1+\alpha}Y(t)\right\} = \frac{1-\delta}{1+\alpha}Y(\min_t\{t\})$, and hence no equivalence.

3. *Theorem C.4 part 2. shows that $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for all $t \in [0,T]$ only holds for specific β values. One can show that the feasibility region for β is non-empty iff the entire accumulated time-T human capital $Y(T)$ is upper-bounded. We start with part 2. b) as this represents an equivalent condition, whereas part 2. a) only represents an "if" condition:*

Part 2. b) Case $\delta > 1$:

Let $\delta > 1$ and $\tilde{\delta} \in [0,1]$. The feasible region for β is

$$\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T} - 1\right)}}\left[\frac{\ln\left(\left(\delta - \tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 T}} - 1\right]\right),$$

$$\beta \leqslant 1 - \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T} - 1\right)}}\right)$$

which is non-empty iff

$$\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}\left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2T}}-1\right]\right)$$

$$\leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}\right)$$

$$\Leftrightarrow \frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}\left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2T}}-1\right]\leqslant -\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}$$

$$\Leftrightarrow \frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2T}}\leqslant 0 \Leftrightarrow \ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)\leqslant 0 \Leftrightarrow \left(\delta-\tilde{\delta}\right)Y(T)\leqslant 1$$

$$\Leftrightarrow Y(T)\leqslant \frac{1}{\delta-\tilde{\delta}}.$$

In summary, if $\delta > 1$ and $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ should be satisfied throughout the entire investment period, this leads to a necessary upper bound on $Y(T)$. The economic explanation behind is as follows: If $\delta > 1$ and $Y(T)$ is very large (and so is $y(t)$), then $dB(t) = \delta y(t)dt$ gets very large and the deterministic part of the change in the buffer account $dC^\star_{\alpha,B}(t) = \alpha d(V^\star_{\alpha,B}(t) - B(t))$ in Eq. (5.10), which is $\alpha\frac{1-\delta}{1+\alpha}y(t)dt < 0$, becomes strongly negative. Therefore, even if the investment return is positive, i.e. $dV^\star_{\alpha,B}(t) > 0$, the dynamics of the buffer can be negative, i.e. $dC^\star_{\alpha,B}(t) < 0$. Only a strongly positive investment return can outweigh the negative deterministic drift part. This is because $y(t)dt$ is inside the stochastic differential equation for $V^\star_{\alpha,B}(t)$, but $\delta y(t)dt > y(t)dt$ in the dynamics of the buffer wealth benchmark $B(t)$. Hence, the larger $y(t)$, the larger the probability of $C^\star_{\alpha,B}(t)$ to fall below 0.

Part 2. a) Case $\delta \leqslant 1$:

Let $\delta \in [0,1]$ and $\tilde{\delta} \in \left[0, \frac{1+\alpha\delta}{1+\alpha}\right]$. The potential region for β is

$$\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2T}}-1\right]\right),$$

$$\beta \leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2T}-1\right)}}\right).$$

For this to be non-empty, we must have

$$Y(T) \leqslant \frac{1}{\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}}.$$

Thus, $Y(T)$ needs again to be upper-bounded. An economic reasoning could be the following: $C^\star_{\alpha,B}(t)$ changes with the rate $dC^\star_{\alpha,B}(t) = c(t)dt = \alpha d(V^\star_{\alpha,B}(t) - B(t))$. If $Y(T)$ and thus also $y(t)$ is large, then the deterministic drift part of $dC^\star_{\alpha,B}(t)$ in Eq. (5.10), which is $\alpha\frac{1-\delta}{1+\alpha}y(t)dt > 0$, can become strongly positive. This has the effect that $V^\star_{\alpha,B}(t)$ gets quite low compared to the time-t present value of the future human capital $HC(t) = Y(T) - Y(t)$. This dis-balance leads to a much higher relative risky investment $\hat{\pi}^\star_{\alpha,B}(t)$ due to "low current wealth, but high promised future inflows". Due to this much higher investment downside risk, the probability of a high potential loss in $V^\star_{\alpha,B}(t)$ (i.e. $dV^\star_{\alpha,B}(t)$ very small), and therefore also in $C^\star_{\alpha,B}(t)$ due to the relation $dC^\star_{\alpha,B}(t) = \alpha\left(dV^\star_{\alpha,B}(t) - dB(t)\right)$, becomes too large and therefore a non-negative Value-at-Risk level cannot be guaranteed. Finally, as already mentioned, in the situation where $\delta \leqslant 1$, Theorem C.4 only provides an "if" condition, but no equivalent condition as for the case $\delta > 1$. Thus, the "if" condition does not rule out a high $Y(T)$ in general.

C.2 Proofs to Section 5.1.2 and Appendix C.1

Proof of Theorem 5.1. The corresponding Hamilton-Jacobi-Bellman (HJB) equation for the value function $\Phi = \Phi(t, V) = \mathcal{V}$ is given by

$$\begin{aligned}\Phi_t(t,V) + \max_\pi \Bigg\{ &\Phi_V(t,V)\left[V\frac{1}{1+\alpha(t)}\left(r + \hat{\pi}(t)'\left(\mu - r\mathbf{1}\right)\right) + \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t) + y(t)\right)\right] \\ &+ \frac{1}{2}\Phi_{VV}(t,V)V^2\left(\frac{1}{1+\alpha(t)}\right)^2 \hat{\pi}(t)'\Sigma\hat{\pi}(t)\Bigg\} = 0.\end{aligned} \tag{C.2}$$

The terminal boundary condition is $\Phi(T,V) = U(V)$. The first order condition of the maximization, i.e. equating the first derivative of the maximum with respect to π to zero, leads to

$$\hat{\pi}^\star(t) = -(1+\alpha(t))\frac{\Phi_V(t,V(t))}{\Phi_{VV}(t,V(t))V(t)}\Sigma^{-1}\left(\mu - r\mathbf{1}\right). \tag{C.3}$$

Inserting this back in the HJB gives

$$\begin{aligned}\Phi_t(t,V) + \Bigg\{ &\Phi_V(t,V)\Bigg[V\frac{1}{1+\alpha(t)}\Bigg(r - (1+\alpha(t))\frac{\Phi_V(t,V)}{\Phi_{VV}(t,V)V}\underbrace{\left(\mu - r\mathbf{1}\right)'\Sigma^{-1}\left(\mu - r\mathbf{1}\right)}_{=\|\gamma\|^2}\Bigg) \\ &+ \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t) + y(t)\right)\Bigg] \\ &+ \frac{1}{2}\Phi_{VV}(t,V)V^2\left(\frac{\Phi_V(t,V)}{\Phi_{VV}(t,V)V}\right)^2\underbrace{\left(\mu - r\mathbf{1}\right)'\Sigma^{-1}\left(\mu - r\mathbf{1}\right)}_{=\|\gamma\|^2}\Bigg\} = 0\end{aligned}$$

and thus we obtain

$$\Phi_t(t,V) + \left[\frac{1}{1+\alpha(t)} rV + \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t) + y(t)\right)\right]\Phi_V(t,V) - \frac{1}{2}\|\gamma\|^2 \frac{\Phi_V(t,V)^2}{\Phi_{VV}(t,V)} = 0. \tag{C.4}$$

Now recall the utility function U from (5.5):

$$U(v) = \hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(v-F)\right)^{\hat{b}}$$

The ansatz for the value function is

$$\Phi(t,V) = h(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}, \tag{C.5}$$

with deterministic, differentiable functions $h(t)$, $f(t)$ such that $h(t) \neq 0 \ \forall t \in [0,T]$. The partial derivatives are

$$\begin{aligned}
\Phi_t(t,V) &= h'(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}} - h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} f'(t),\\
\Phi_V(t,V) &= h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1},\\
\Phi_{VV}(t,V) &= -h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-2}.
\end{aligned}$$

With this, in case the value function fulfills the HJB equation, the optimal portfolio composition would be

$$\hat{\pi}^\star(t) = \frac{1+\alpha(t)}{1-\hat{b}}\frac{V(t)-f(t)}{V(t)}\Sigma^{-1}(\mu - r\mathbf{1}) \tag{C.6}$$

which is of a Proportional Portfolio Insurance (PPI) type.

Notice that the boundary condition $\Phi(T,V) = U(V)$ is satisfied for all feasible V iff

$$h(T)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(T))\right)^{\hat{b}} = \hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-F)\right)^{\hat{b}},$$

hence iff

$$h(T) = 1, \ f(T) = F. \tag{C.7}$$

When we insert the ansatz for $\Phi(t,V)$ and its partial derivatives into the HJB equation, then it boils down to

$$
\begin{aligned}
0 \overset{!}{=}\ & h'(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}} - h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} f'(t) \\
&+ \left[\frac{1}{1+\alpha(t)}rV + \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t)+y(t)\right)\right] h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} \\
&- \frac{1}{2}\|\gamma\|^2 \frac{\left(h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}\right)^2}{-h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-2}} \\
=\ & h'(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}} - f'(t)h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} \\
&+ \left[\underbrace{r(1-\hat{b})\left(\frac{1}{1-\hat{b}}(V-f(t)+f(t))\right)}_{=rV} + \left(\alpha(t)B'(t)+y(t)\right)\right]\frac{1}{1+\alpha(t)}h(t)\hat{a} \\
&\times \left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} + \frac{1}{2}\|\gamma\|^2 h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}} \\
=\ & \left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}\hat{a}\frac{1-\hat{b}}{\hat{b}}\left\{h'(t)+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2 h(t) + \frac{1}{1+\alpha(t)}r\hat{b}h(t)\right\} \\
&- \left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1} h(t)\hat{a}\left\{f'(t) - \frac{1}{1+\alpha(t)}rf(t) - \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t)+y(t)\right)\right\}.
\end{aligned}
$$

As long as the HJB equation has to hold for any V, it must be

$$
h'(t) = -\hat{b}\left(\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2 + \frac{1}{1+\alpha(t)}r\right)h(t),\ \ h(T)=1, \tag{C.8}
$$

$$
f'(t) = \frac{1}{1+\alpha(t)}rf(t) + \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t)+y(t)\right),\ \ f(T)=F, \tag{C.9}
$$

$\forall t \in [0,T]$. The solution to the ordinary differential equation (ODE) on h is given by

$$
h(t) = e^{\int_t^T \hat{b}\left(\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2 + \frac{1}{1+\alpha(s)}r\right)ds} = e^{\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2(T-t) + \hat{b}r\int_t^T \frac{1}{1+\alpha(s)}ds}. \tag{C.10}
$$

The solution to the ODE on f is

$$
f(t) = Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T \frac{\alpha(s)B'(s)+y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds. \tag{C.11}
$$

This results from applying Theorem C.6 in Appendix C.3. For the optimal portfolio allocation it follows

$$
\begin{aligned}
\hat{\pi}^{\star}_{\alpha,B}(t) &= \frac{1+\alpha(t)}{1-\hat{b}} \frac{V(t) - Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} + \int_t^T \frac{\alpha(s)B'(s)+y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds}{V(t)} \Sigma^{-1}(\mu - r\mathbf{1}) \\
&= \frac{1+\alpha(t)}{1-\hat{b}} \frac{V(t) - \tilde{F}_{\alpha,B}(t)}{V(t)} \Sigma^{-1}(\mu - r\mathbf{1}),
\end{aligned} \tag{C.12}
$$

where we defined the PPI floor $\tilde{F}_{\alpha,B}(t)$ to be

$$\tilde{F}_{\alpha,B}(t) := Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T \frac{\alpha(s)B'(s)+y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds,\ \tilde{F}_{\alpha,B}(T) = F. \tag{C.13}$$

We now substitute $\hat{\pi}^\star_{\alpha,B}(t)$ into the dynamics of $V(t)$ above that arises from Eq. (5.1) with the specific buffer rule $c(t)$:

$$\begin{aligned}
dV^\star_{\alpha,B}(t) &= \frac{1}{1+\alpha(t)}\left(rV^\star_{\alpha,B}(t) + \left\{\frac{1+\alpha(t)}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\Sigma^{-1}(\mu - r\mathbf{1})\right\}'(\mu - r\mathbf{1})\right)dt \\
&\quad + \frac{1}{1+\alpha(t)}\left\{\frac{1+\alpha(t)}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\Sigma^{-1}(\mu - r\mathbf{1})\right\}'\sigma dW(t) \\
&\quad + \frac{1}{1+\alpha(t)}\big(\alpha(t)B'(t)+y(t)\big)dt \\
&= \Bigg(\frac{1}{1+\alpha(t)}rV^\star_{\alpha,B}(t) + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\underbrace{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})}_{=\|\gamma\|^2} \\
&\qquad + \frac{1}{1+\alpha(t)}\big(\alpha(t)B'(t)+y(t)\big)\Bigg)dt + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\underbrace{(\mu - r\mathbf{1})'\Sigma^{-1}\sigma}_{=\gamma'} dW(t) \\
&= \left(\frac{1}{1+\alpha(t)}rV^\star_{\alpha,B}(t) + \frac{1}{1-\hat{b}}\|\gamma\|^2(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)) + \frac{1}{1+\alpha(t)}\big(\alpha(t)B'(t)+y(t)\big)\right)dt \\
&\quad + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\gamma' dW(t) \\
&= \Bigg[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)) \\
&\qquad + \frac{1}{1+\alpha(t)}\big(\alpha(t)B'(t)+y(t)+r\tilde{F}_{\alpha,B}(t)\big)\Bigg]dt + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\gamma' dW(t) \\
&= (V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] \\
&\quad + \frac{1}{1+\alpha(t)}\big(\alpha(t)B'(t)+y(t)+r\tilde{F}_{\alpha,B}(t)\big)dt.
\end{aligned} \tag{C.14}$$

The SDE for the cushion $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$ is given by

$$d\left(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = dV^\star_{\alpha,B}(t) - d\tilde{F}_{\alpha,B}(t) = dV^\star_{\alpha,B}(t) - \tilde{F}'_{\alpha,B}(t)dt.$$

By using the Leibniz integral rule we obtain

$$\tilde{F}'_{\alpha,B}(t) = r\frac{1}{1+\alpha(t)}\tilde{F}_{\alpha,B}(t) + \frac{\alpha(t)B'(t)+y(t)}{1+\alpha(t)} = \frac{1}{1+\alpha(t)}\left(r\tilde{F}_{\alpha,B}(t) + \alpha(t)B'(t) + y(t)\right). \tag{C.15}$$

Hence,

$$dV^\star_{\alpha,B}(t) = (V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] + \tilde{F}'_{\alpha,B}(t)dt \tag{C.16}$$

and

$$d\left(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right]$$
$$+ \frac{1}{1+\alpha(t)}\left(\alpha(t)B'(t) + y(t) + r\tilde{F}_{\alpha,B}(t)\right)dt$$
$$- \frac{1}{1+\alpha(t)}\left(r\tilde{F}_{\alpha,B}(t) + \alpha(t)B'(t) + y(t)\right)dt$$
$$= (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(\frac{1}{1+\alpha(t)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right]. \tag{C.17}$$

The formula shows that $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$ follows a geometric Brownian motion with

$$V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) = (v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^t \frac{1}{1+\alpha(s)}ds + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t\right] + \frac{1}{1-\hat{b}}\gamma' W(t)}$$

under $\mathbb{P}$, consequently

$$V^\star_{\alpha,B}(t) = \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^t \frac{1}{1+\alpha(s)}ds + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t\right] + \frac{1}{1-\hat{b}}\gamma' W(t)}$$
$$= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}. \tag{C.18}$$

By partial integration and the Leibniz integral rule, the formula for $\tilde{F}_{\alpha,B}(t)$ can furthermore be transformed to

$$\tilde{F}_{\alpha,B}(t) = Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T \frac{\alpha(s)B'(s) + y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$$
$$= Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \int_t^T B'(s)\frac{\alpha(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds - \int_t^T \frac{y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$$
$$\overset{\text{partial integration}}{=} Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \left\{ B(s)\frac{\alpha(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du}\Big|_{s=t}^{s=T}\right.$$
$$- \int_t^T B(s)\left(\frac{\alpha'(s)(1+\alpha(s)) - \alpha(s)\alpha'(s)}{(1+\alpha(s))^2} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du}\right.$$
$$\left.\left. + \frac{\alpha(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du}(-r\frac{1}{1+\alpha(s)})\right) ds \right\}$$
$$- \int_t^T \frac{y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$$
$$= Fe^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} - \left(B(T)\frac{\alpha(T)}{1+\alpha(T)} e^{-r\int_t^T \frac{1}{1+\alpha(u)}du} - B(t)\frac{\alpha(t)}{1+\alpha(t)}\right)$$
$$- \int_t^T \frac{\alpha(s)}{(1+\alpha(s))^2}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right) B(s) e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$$
$$- \int_t^T \frac{y(s)}{1+\alpha(s)} e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds$$

$$= \left(F - \frac{\alpha(T)}{1+\alpha(T)}B(T)\right) e^{-r\int_t^T \frac{1}{1+\alpha(s)}ds} + \frac{\alpha(t)}{1+\alpha(t)}B(t)$$
$$- \int_t^T \frac{1}{1+\alpha(s)} \left(\frac{\alpha(s)}{1+\alpha(s)} \left(r - \frac{\alpha'(s)}{\alpha(s)}\right) B(s) + y(s)\right) e^{-r\int_t^s \frac{1}{1+\alpha(u)}du} ds.$$

Finally, using above results we can calculate

$$\begin{aligned} C^\star_{\alpha,B}(t) &= \int_0^t c(s)e^{r(t-s)}ds \overset{(5.7)}{=} \int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - B(s)) \\ &= \int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s) + \tilde{F}_{\alpha,B}(s) - B(s)) \\ &= \int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s)) + \int_0^t e^{r(t-s)}\alpha(s)d(\tilde{F}_{\alpha,B}(s) - B(s)). \end{aligned}$$

The second part becomes

$$\begin{aligned} \int_0^t e^{r(t-s)}\alpha(s)d(\tilde{F}_{\alpha,B}(s) - B(s)) &= \int_0^t e^{r(t-s)}\alpha(s)(\tilde{F}'_{\alpha,B}(s) - B'(s))ds \\ &= \int_0^t e^{r(t-s)}\alpha(s) \\ &\quad \times \left(\frac{1}{1+\alpha(s)}\left(r\tilde{F}_{\alpha,B}(s) + \alpha(s)B'(s) + y(s)\right) - B'(s)\right) ds \\ &= \int_0^t e^{r(t-s)}\frac{\alpha(s)}{1+\alpha(s)}\left(r\tilde{F}_{\alpha,B}(s) - B'(s) + y(s)\right) ds. \end{aligned}$$

The first part can be traced down to

$$\begin{aligned} &\int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s)) \\ &\overset{(C.17)}{=} \int_0^t e^{r(t-s)}\alpha(s)(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s)) \left[\left(\frac{1}{1+\alpha(s)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) ds + \frac{1}{1-\hat{b}}d\left(\gamma' W(s)\right)\right] \\ &\overset{(C.18)}{=} \int_0^t e^{r(t-s)}\alpha(s)(v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} \\ &\qquad \times \left[\left(\frac{1}{1+\alpha(s)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) ds + \frac{1}{1-\hat{b}}d\left(\gamma' W(s)\right)\right] \\ &\overset{(C.15)}{=} \int_0^t e^{r(t-s)}\alpha(s)(v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} \frac{1}{1-\hat{b}}d\left(\gamma' W(s)\right) \\ &\qquad + \int_0^t e^{r(t-s)}\alpha(s)(v_0 - \tilde{F}_{\alpha,B}(0))e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} \\ &\qquad \times \left(\frac{1}{1+\alpha(s)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{v_0 - \tilde{F}_{\alpha,B}(0)}{1-\hat{b}} \int_0^t e^{r(t-s)} \alpha(s) e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} d\left(\gamma' W(s)\right) \\
&\quad + \left(v_0 - \tilde{F}_{\alpha,B}(0)\right) \int_0^t e^{r(t-s)} \alpha(s) \left(\frac{1}{1+\alpha(s)} r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right) \\
&\quad \times e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} ds.
\end{aligned}
$$

Let us define the stochastic process X via $X(t) := \gamma' W(t) = \sum_{i=1}^N \gamma_i W_i(t) \sim \mathcal{N}\left(0, \|\gamma\|^2 t\right)$. Moreover, define

$$
f(t, X) := g(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}} X}
$$

with $g(t) := e^{-rt}\alpha(t) e^{r\int_0^t \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t}$. f is selected such that

$$
\int_0^t f_X(s, X(s)) dX(s) = \int_0^t e^{-rs}\alpha(s) e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} d\left(\gamma' W(s)\right),
$$

which corresponds to the very first integral in the formula above. Due to f being twice continuously differentiable in each argument, we can apply the Itô formula

$$
df(t, X) = f_t(t, X)dt + f_X(t, X)dX + \frac{1}{2} f_{XX}(t, X) d\langle X\rangle
$$

with $d\langle X\rangle_t = \|\gamma\|^2 dt$. The needed derivatives are

$$
f_t(t, X) = g'(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}X}, \;\; f_X(t, X) = g(t) e^{\frac{1}{1-\hat{b}}X}, \;\; f_{XX}(t, X) = g(t) \frac{1}{1-\hat{b}} e^{\frac{1}{1-\hat{b}}X}.
$$

This gives

$$
f(t, X(t)) - f(0, X(0)) = \int_0^t f_t(s, X(s)) ds + \int_0^t f_X(s, X(s)) dX(s) + \frac{1}{2}\int_0^t f_{XX}(s, X(s)) d\langle X\rangle_s,
$$

where the left-hand side of the equation reduces to

$$
\begin{aligned}
f(t, X(t)) - f(0, X(0)) &= g(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}X(t)} - \underbrace{g(0)}_{=\alpha(0)} (1-\hat{b}) e^{\frac{1}{1-\hat{b}} \overbrace{X(0)}^{=0}} \\
&= g(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}X(t)} - \alpha(0)(1-\hat{b}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& g(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}X(t)} - \alpha(0)(1-\hat{b}) \\
&= \int_0^t g'(s)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}X(s)} ds + \int_0^t f_X(s, X(s)) dX(s) + \frac{1}{2}\int_0^t g(s)\frac{1}{1-\hat{b}} e^{\frac{1}{1-\hat{b}}X(s)} \|\gamma\|^2 ds \\
&= \int_0^t f_X(s, X(s)) dX(s) + \int_0^t e^{\frac{1}{1-\hat{b}}X(s)} \left(g'(s)(1-\hat{b}) + \frac{1}{2} g(s) \frac{1}{1-\hat{b}}\|\gamma\|^2\right) ds,
\end{aligned}
$$

which finally leads to

$$\begin{aligned}
&\int_0^t f_X(s,X(s))dX(s)\\
&= g(t)(1-\hat{b})e^{\frac{1}{1-\hat{b}}X(t)} - \alpha(0)(1-\hat{b}) - \int_0^t e^{\frac{1}{1-\hat{b}}X(s)}\left(g'(s)(1-\hat{b}) + \frac{1}{2}g(s)\frac{1}{1-\hat{b}}\|\gamma\|^2\right)ds\\
&= g(t)(1-\hat{b})e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0)(1-\hat{b}) - \int_0^t e^{\frac{1}{1-\hat{b}}\gamma' W(s)}\left(g'(s)(1-\hat{b}) + \frac{1}{2}g(s)\frac{1}{1-\hat{b}}\|\gamma\|^2\right)ds.
\end{aligned}$$

g' can be expressed in terms of g:

$$\begin{aligned}
g'(t) &= \alpha'(t)e^{-rt+r\int_0^t \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t} + \alpha(t)e^{-rt+r\int_0^t \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t}\\
&\quad \times \left[-r + r\frac{1}{1+\alpha(t)} + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)\right]\\
&= \left[\frac{\alpha'(t)}{\alpha(t)} - r + r\frac{1}{1+\alpha(t)} + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)\right]g(t)\\
&= \left[\frac{\alpha'(t)}{\alpha(t)} - r\frac{\alpha(t)}{1+\alpha(t)} + \left(1 - \frac{1}{2}\frac{1}{1-\hat{b}}\right)\frac{1}{1-\hat{b}}\|\gamma\|^2\right]g(t).
\end{aligned}$$

Then it follows

$$\begin{aligned}
&g'(t)(1-\hat{b}) + \frac{1}{2}g(t)\frac{1}{1-\hat{b}}\|\gamma\|^2\\
&= \left[\frac{\alpha'(t)}{\alpha(t)} - r\frac{\alpha(t)}{1+\alpha(t)} + \left(1 - \frac{1}{2}\frac{1}{1-\hat{b}}\right)\frac{1}{1-\hat{b}}\|\gamma\|^2\right]g(t)(1-\hat{b}) + \frac{1}{2}g(t)\frac{1}{1-\hat{b}}\|\gamma\|^2\\
&= \left[(1-\hat{b})\left(\frac{\alpha'(t)}{\alpha(t)} - r\frac{\alpha(t)}{1+\alpha(t)}\right) + \left(1 - \frac{1}{2}\frac{1}{1-\hat{b}}\right)\|\gamma\|^2 + \frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right]g(t)\\
&= \left[(1-\hat{b})\left(\frac{\alpha'(t)}{\alpha(t)} - r\frac{\alpha(t)}{1+\alpha(t)}\right) + \|\gamma\|^2\right]g(t).
\end{aligned}$$

Inserting this result into the formula for $\int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s))$ leads to

$$\begin{aligned}
&\int_0^t e^{r(t-s)}\alpha(s)d(V^\star_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s))\\
&= \frac{v_0 - \tilde{F}_{\alpha,B}(0)}{1-\hat{b}}\int_0^t e^{r(t-s)}\alpha(s)e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} d\left(\gamma' W(s)\right)\\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0))\int_0^t e^{r(t-s)}\alpha(s)\left(\frac{1}{1+\alpha(s)}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)\\
&\quad \times e^{\left[r\int_0^s \frac{1}{1+\alpha(u)}du + \left(\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s\right] + \frac{1}{1-\hat{b}}\gamma' W(s)} ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{v_0 - \tilde{F}_{\alpha,B}(0)}{1-\hat{b}} e^{rt} \\
&\quad \times \left[g(t)(1-\hat{b}) e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0)(1-\hat{b}) - \int_0^t e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \left(g'(s)(1-\hat{b}) + \frac{1}{2} g(s) \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) ds \right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t g(s) e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \left(\frac{1}{1+\alpha(s)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) ds \\
&= (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g(t) e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0) - \int_0^t e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \left(g'(s) + \frac{1}{2} g(s) \left(\frac{1}{1-\hat{b}} \right)^2 \|\gamma\|^2 \right) ds \right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t g(s) e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \left(\frac{1}{1+\alpha(s)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) ds \\
&= (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g(t) e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0) \right] + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \\
&\quad \times \int_0^t e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \left[g(s) \left(\frac{1}{1+\alpha(s)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) - \left(g'(s) + \frac{1}{2} g(s) \left(\frac{1}{1-\hat{b}} \right)^2 \|\gamma\|^2 \right) \right] ds \\
&= (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g(t) e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0) \right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t g(s) e^{\frac{1}{1-\hat{b}}\gamma' W(s)} \\
&\quad \times \left[\left(\frac{1}{1+\alpha(s)} r + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) - \left\{ \left(\frac{\alpha'(s)}{\alpha(s)} - r \frac{\alpha(s)}{1+\alpha(s)} \right) + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right\} \right] ds \\
&= (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g(t) e^{\frac{1}{1-\hat{b}}\gamma' W(t)} - \alpha(0) \right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)} \right) g(s) e^{\frac{1}{1-\hat{b}}\gamma' W(s)} ds \\
&\quad + \int_0^t e^{r(t-s)} \frac{\alpha(s)}{1+\alpha(s)} \left(r\tilde{F}_{\alpha,B}(s) - B'(s) + y(s) \right) ds \\
&\overset{(2.3)}{=} (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g(t) e^{-\frac{1}{1-\hat{b}}\left(r + \frac{1}{2}\|\gamma\|^2\right)t} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - \alpha(0) \right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)} \right) g(s) e^{-\frac{1}{1-\hat{b}}\left(r + \frac{1}{2}\|\gamma\|^2\right)s} \tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds \\
&= (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \left[g_C(t) \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - \alpha(0) \right] + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt} \int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)} \right) g_C(s) \tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds,
\end{aligned}
$$

with

$$
\begin{aligned}
g_C(t) &:= g(t) e^{-\frac{1}{1-\hat{b}}\left(r + \frac{1}{2}\|\gamma\|^2\right)t} \\
&= e^{-rt} \alpha(t) e^{r \int_0^t \frac{1}{1+\alpha(u)} du + \left(\frac{1}{1-\hat{b}} \|\gamma\|^2 - \frac{1}{2} \left(\frac{1}{1-\hat{b}} \right)^2 \|\gamma\|^2 \right) t} e^{-\frac{1}{1-\hat{b}}\left(r + \frac{1}{2}\|\gamma\|^2\right)t} \\
&= \alpha(t) e^{-r\left(t + \frac{1}{1-\hat{b}} t - \int_0^t \frac{1}{1+\alpha(u)} du \right) + \frac{1}{2} \frac{1}{1-\hat{b}} \|\gamma\|^2 \left(1 - \frac{1}{1-\hat{b}} \right) t} \\
&= \alpha(t) e^{\frac{1}{1-\hat{b}} \left[-r\left((2-\hat{b})t - \int_0^t \frac{1-\hat{b}}{1+\alpha(u)} du \right) - \frac{1}{2} \frac{\hat{b}}{1-\hat{b}} \|\gamma\|^2 t \right]}.
\end{aligned}
$$

Bringing everything together, we obtain for the buffer account:

$$
\begin{aligned}
C^\star_{\alpha,B}(t) &= (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\left[g_C(t)\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - \alpha(0)\right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)}\right) g_C(s)\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds \\
&\quad + \int_0^t e^{r(t-s)}\frac{\alpha(s)}{1+\alpha(s)}\left(r\tilde{F}_{\alpha,B}(s) - B'(s) + y(s)\right) ds.
\end{aligned}
$$

We further obtain by partial integration

$$
\begin{aligned}
e^{rt} & \underbrace{\int_0^t e^{-rs}\alpha(s)\left(\tilde{F}'_{\alpha,B}(s) - B'(s)\right) ds}_{= e^{-rs}\alpha(s)\left(\tilde{F}_{\alpha,B}(s)-B(s)\right)\Big|_{s=0}^{s=t} - \int_0^t (-re^{-rs}\alpha(s)+e^{-rs}\alpha'(s))\left(\tilde{F}_{\alpha,B}(s)-B(s)\right)ds} \\
&= e^{rt}\left[e^{-rt}\alpha(t)\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - \alpha(0)\left(\tilde{F}_{\alpha,B}(0) - B(0)\right)\right] \\
&\quad + e^{rt}\int_0^t e^{-rs}\alpha(s)\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds.
\end{aligned}
$$

The formula for $C^\star_{\alpha,B}(t)$ then becomes

$$
\begin{aligned}
C^\star_{\alpha,B}(t) &= (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\left[g_C(t)\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - \alpha(0)\right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)}\right) g_C(s)\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds \\
&\quad + e^{rt}\left[e^{-rt}\alpha(t)\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - \alpha(0)\left(\tilde{F}_{\alpha,B}(0) - B(0)\right)\right] \\
&\quad + e^{rt}\int_0^t e^{-rs}\alpha(s)\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds \\
&\overset{\text{def. of } g_C}{=} (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\left[\alpha(t)e^{\frac{1}{1-\hat{b}}\left[-r\left((2-\hat{b})t - \int_0^t \frac{1-\hat{b}}{1+\alpha(u)}du\right) - \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2 t\right]}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} - \alpha(0)\right] \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt}\int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)e^{\frac{1}{1-\hat{b}}\left[-r\left((2-\hat{b})s - \int_0^s \frac{1-\hat{b}}{1+\alpha(u)}du\right) - \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2 s\right]}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds \\
&\quad + e^{rt}\left[e^{-rt}\alpha(t)\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - \alpha(0)\left(\tilde{F}_{\alpha,B}(0) - B(0)\right)\right] \\
&\quad + e^{rt}\int_0^t e^{-rs}\alpha(s)\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds.
\end{aligned}
$$

From (C.18) we know

$$
V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) = (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.
$$

It follows

$$
\begin{aligned}
C^{\star}_{\alpha,B}(t) &= \alpha(t)\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) e^{rt} e^{-\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} \\
&\quad \times e^{\frac{1}{1-\hat{b}}\left[-r\left((2-\hat{b})t - \int_0^t \frac{1-\hat{b}}{1+\alpha(u)} du\right) - \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2 t\right]} - \alpha(0)(v_0 - \tilde{F}_{\alpha,B}(0))e^{rt} \\
&\quad + e^{rt}\int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(V^{\star}_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s)\right) \\
&\quad \times e^{-\frac{1}{1-\hat{b}}\left[r\left(\int_0^s \frac{1-\hat{b}}{1+\alpha(u)} du\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)s\right]} e^{\frac{1}{1-\hat{b}}\left[-r\left((2-\hat{b})s - \int_0^s \frac{1-\hat{b}}{1+\alpha(u)} du\right) - \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2 s\right]} ds \\
&\quad + e^{rt}\left[e^{-rt}\alpha(t)\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - \alpha(0)\left(\tilde{F}_{\alpha,B}(0) - B(0)\right)\right] \\
&\quad + e^{rt}\int_0^t e^{-rs}\alpha(s)\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds \\
&= \alpha(t)\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) - \alpha(0)(v_0 - \tilde{F}_{\alpha,B}(0))e^{rt} \\
&\quad + e^{rt}\int_0^t \left(r - \frac{\alpha'(s)}{\alpha(s)}\right) e^{-rs}\alpha(s)\left(V^{\star}_{\alpha,B}(s) - \tilde{F}_{\alpha,B}(s)\right) ds \\
&\quad + e^{rt}\left[e^{-rt}\alpha(t)\left(\tilde{F}_{\alpha,B}(t) - B(t)\right) - \alpha(0)\left(\tilde{F}_{\alpha,B}(0) - B(0)\right)\right] \\
&\quad + e^{rt}\int_0^t e^{-rs}\alpha(s)\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds \\
&= \alpha(t)\left(V^{\star}_{\alpha,B}(t) - B(t)\right) - e^{rt}\alpha(0)(v_0 - B(0)) \\
&\quad + \int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(V^{\star}_{\alpha,B}(s) - B(s)\right) ds.
\end{aligned}
$$

□

Proof of Theorem 5.2.

- Expected fund wealth:

  From Theorem 5.1 it follows

$$
\begin{aligned}
\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] &= \mathbb{E}\left[\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right] \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} \mathbb{E}\left[\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right] \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds\right) - \left(r + \frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} \\
&\quad \times e^{\frac{1}{1-\hat{b}}\left(r + \frac{1}{2}\|\gamma\|^2\right)t + \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)} ds\right) + \|\gamma\|^2 t\right]},
\end{aligned}
$$

  where we used that $\tilde{Z}(t)^{\eta} = e^{-\eta\left(r + \frac{1}{2}\|\gamma\|^2\right)t - \eta\gamma' W(t)}$, $\eta \in \mathbb{R}$, is log-normally distributed with mean $\mu_{\tilde{Z}(t)^{\eta}} = -\eta\left(r + \frac{1}{2}\|\gamma\|^2\right)t$ and variance $\sigma^2_{\tilde{Z}(t)^{\eta}} = \eta^2\|\gamma\|^2 t$, and that $\mathbb{E}\left[e^{Z}\right] = e^{\mu_Z + \frac{1}{2}\sigma_Z^2}$ for a normally distributed random variable $Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$.

- Variance of the fund wealth:

  Theorem 5.1 implies

$$\begin{aligned}
Var\left(V^{\star}_{\alpha,B}(t)\right) &= Var\left(\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)-\left(r+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{\frac{2}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)-\left(r+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} Var\left(\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{\frac{2}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)-\left(r+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right)t\right]} e^{\frac{2}{1-\hat{b}}\left(r+\frac{1}{2}\|\gamma\|^2\right)t+\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} \\
&\quad \times \left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{\frac{2}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)+\|\gamma\|^2 t\right]} \left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right).
\end{aligned}$$

Here we used that $Var\left(e^Z\right) = e^{2\mu_Z+\sigma_Z^2}\left(e^{\sigma_Z^2}-1\right)$ for a normally distributed random variable $Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$.

- Value-at-Risk/Quantiles of the fund wealth distribution with level $\beta \in [0,1]$:

The Value-at-Risk is defined as

$$VaR_\beta\left(V^{\star}_{\alpha,B}(t)\right) := \inf\left\{x \in \mathbb{R} : F_{V^{\star}_{\alpha,B}(t)}(x) \geqslant \beta\right\},$$

where $F_{V^{\star}_{\alpha,B}(t)}$ denotes the cumulative distribution function of the fund wealth $V^{\star}_{\alpha,B}(t)$, i.e. $F_{V^{\star}_{\alpha,B}(t)}(x) = \mathbb{P}(V^{\star}_{\alpha,B}(t) \leqslant x)$. From Theorem 5.1 we know that $V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) > 0$ is log-normally distributed with mean $\mathbb{E}\left[V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right] = \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t)$ and variance $Var\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = Var\left(V^{\star}_{\alpha,B}(t)\right)$; for $\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]$ and $Var\left(V^{\star}_{\alpha,B}(t)\right)$ see the formulas above. Hence,

$$\begin{aligned}
F_{V^{\star}_{\alpha,B}(t)}(x) &= \mathbb{P}(V^{\star}_{\alpha,B}(t) \leqslant x) \\
&= \mathbb{P}(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) \leqslant x - \tilde{F}_{\alpha,B}(t)) = F_{V^{\star}_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)}(x - \tilde{F}_{\alpha,B}(t)) \\
&= \Phi\left(\frac{\ln(x - \tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right]}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right)}}\right) \\
&= \Phi\left(\frac{\ln(x - \tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}\right)
\end{aligned}$$

for any $x > \tilde{F}_{\alpha,B}(t)$. $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. Let $F^{-1}_{V^{\star}_{\alpha,B}(t)}$ denote the continuous inverse distribution function or quantile function of $F_{V^{\star}_{\alpha,B}(t)}$. Then

$$VaR_\beta\left(V^{\star}_{\alpha,B}(t)\right) = F^{-1}_{V^{\star}_{\alpha,B}(t)}(\beta).$$

We determine $F^{-1}_{V^{\star}_{\alpha,B}(t)}$:

$$
\begin{aligned}
F_{V^{\star}_{\alpha,B}(t)}(x) &= \Phi\left(\frac{\ln(x-\tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}\right) = \beta \\
\Leftrightarrow \frac{\ln(x-\tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}} &= \Phi^{-1}(\beta) \\
\Leftrightarrow \ln(x-\tilde{F}_{\alpha,B}(t)) &= \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t) + \Phi^{-1}(\beta)\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)} \\
\overset{\text{monotonicity of ln}}{\Leftrightarrow} x &= \tilde{F}_{\alpha,B}(t) + e^{\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t) + \Phi^{-1}(\beta)\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}.
\end{aligned}
$$

Thus it follows

$$
VaR_{\beta}\left(V^{\star}_{\alpha,B}(t)\right) = F^{-1}_{V^{\star}_{\alpha,B}(t)}(\beta) = \tilde{F}_{\alpha,B}(t) + e^{\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t) + \Phi^{-1}(\beta)\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}},
$$

with $\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]$ and $Var\left(V^{\star}_{\alpha,B}(t)\right)$ given in the formulas above.

- Shortfall probability of the fund wealth with threshold $s > \tilde{F}_{\alpha,B}(t)$:

  The shortfall probability was already calculated in the proof of the Value-at-Risk formula:

$$
\mathbb{P}\left(V^{\star}_{\alpha,B}(t) \leqslant s\right) = F_{V^{\star}_{\alpha,B}(t)}(s) = \Phi\left(\frac{\ln(s-\tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}\right)
$$

  for the threshold $s > \tilde{F}_{\alpha,B}(t)$.

- Expected accumulated buffer account:

  In view of Theorem 5.1 and the first part on the expected fund wealth, the expected value of the collected buffer equals

$$
\begin{aligned}
\mathbb{E}\left[C^{\star}_{\alpha,B}(t)\right] &= \alpha(t)\left(\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - B(t)\right) - e^{rt}\alpha(0)(v_0 - B(0)) \\
&\quad + \mathbb{E}\left[\int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(V^{\star}_{\alpha,B}(s) - B(s)\right)ds\right] \\
&= \alpha(t)\left(\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] - B(t)\right) - e^{rt}\alpha(0)(v_0 - B(0)) \\
&\quad + \int_0^t e^{r(t-s)}\left(r - \frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\left(\mathbb{E}\left[V^{\star}_{\alpha,B}(s)\right] - B(s)\right)ds
\end{aligned}
$$

$$\begin{aligned}&\overset{\text{part on }\mathbb{E}\left[V^{\star}_{\alpha,B}(s)\right]}{=} \alpha(t)\left(\tilde{F}_{\alpha,B}(t)+(v_0-\tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^t \frac{1-\hat{b}}{1+\alpha(s)}ds\right)+\|\gamma\|^2 t\right]}-B(t)\right)\\ &\quad - e^{rt}\alpha(0)(v_0-B(0))+\int_0^t e^{r(t-s)}\left(r-\frac{\alpha'(s)}{\alpha(s)}\right)\alpha(s)\\ &\quad \times\left(\tilde{F}_{\alpha,B}(s)+(v_0-\tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\left[r\left(\int_0^s \frac{1-\hat{b}}{1+\alpha(u)}du\right)+\|\gamma\|^2 s\right]}-B(s)\right)ds.\end{aligned}$$

□

Calculations below Assumption 5.6. By applying the special form of $Y(t)$ and $B(t)$ we obtain

$$\begin{aligned}\tilde{F}_{\alpha,B}(t)&=F-\int_t^T\frac{\alpha B'(s)+y(s)}{1+\alpha}ds=F-\int_t^T\frac{\alpha\delta y(s)+y(s)}{1+\alpha}ds=F-\frac{1+\alpha\delta}{1+\alpha}\int_t^T y(s)ds\\ &=F-\frac{1+\alpha\delta}{1+\alpha}\left(Y(T)-Y(t)\right),\end{aligned}$$

with

$$\tilde{F}_{\alpha,B}(0)=F-\frac{1+\alpha\delta}{1+\alpha}Y(T).$$

With the definition of F this becomes

$$\tilde{F}_{\alpha,B}(t)=v_0+\frac{1+\alpha\delta}{1+\alpha}Y(t)+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T),$$

with

$$\tilde{F}_{\alpha,B}(0)=v_0+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T).$$

Furthermore,

$$\begin{aligned}\tilde{F}_{\alpha,B}(t)-B(t)&=F-B(0)-\frac{1+\alpha\delta}{1+\alpha}Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\\ &=v_0-B(0)+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\end{aligned}$$

and

$$d(\tilde{F}_{\alpha,B}(t)-B(t))=\frac{1-\delta}{1+\alpha}y(t)dt.$$

From this it follows

$$\begin{aligned}c(t)dt&\overset{(5.7)}{=}\alpha d(V^{\star}_{\alpha,B}(t)-B(t))=\alpha\left(d(V^{\star}_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))+d(\tilde{F}_{\alpha,B}(t)-B(t))\right)\\ &\overset{\text{Cor. 5.3, }r=0}{=}\alpha\left((V^{\star}_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\frac{1}{1-\hat{b}}\|\gamma\|^2dt+\frac{1}{1-\hat{b}}\gamma' dW(t)\right]+\frac{1-\delta}{1+\alpha}y(t)dt\right)\end{aligned}$$

and

$$\begin{aligned}
C^\star_{\alpha,B}(t) &\overset{(5.9)}{=} \alpha\left[(V^\star_{\alpha,B}(t)-B(t))-(v_0-B(0))\right]\\
&\overset{\text{Cor. 5.3}}{=} \alpha\left[(v_0-\tilde{F}_{\alpha,B}(0))e^{-\frac{1}{1-\hat{b}}\left[\frac{\alpha+\hat{b}}{1+\alpha}r+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2\right]t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}+\tilde{F}_{\alpha,B}(t)-B(t)-v_0+B(0)\right]\\
&\overset{r=0}{=} \alpha\left[(v_0-\tilde{F}_{\alpha,B}(0))e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}+\tilde{F}_{\alpha,B}(t)-B(t)-v_0+B(0)\right]\\
&= \alpha\Bigg[\left(v_0-F+\frac{1+\alpha\delta}{1+\alpha}Y(T)\right)e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}-v_0+F-\frac{1+\alpha\delta}{1+\alpha}Y(T)\\
&\qquad+\frac{1-\delta}{1+\alpha}Y(t)\Bigg]\\
&= \alpha\left[\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\right)e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right].
\end{aligned}$$

□

Proof of Theorem 5.7. From Eq. (5.11) it follows that

$$C^\star_{\alpha,B}(t)=\alpha\left[\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\right)e^{-\frac{1}{2}\frac{\hat{b}}{(1-\hat{b})^2}\|\gamma\|^2t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right]$$

with

$$C^\star_{\alpha,B}(t)\geqslant 0\ \mathbb{P}\text{-a.s.}\Leftrightarrow C^\star_{\alpha,B}(t)\geqslant 0\ \forall\tilde{Z}(t)\in(0,\infty)\Leftrightarrow C^\star_{\alpha,B}(t)\geqslant 0\ \forall P_i(t)\in(0,\infty).$$

Furthermore, due to Eq. (5.12) we have

$$\tilde{\delta}\leqslant\min\left\{\frac{1+\alpha\delta}{1+\alpha},1\right\}\leqslant\frac{1+\alpha\delta}{1+\alpha}.$$

Therefore,

$$C^\star_{\alpha,B}(t)\overset{\tilde{Z}(t)\in(0,\infty)}{\geqslant}\alpha\left[\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right]$$

with

$$C^\star_{\alpha,B}(t)\geqslant 0\Leftrightarrow\alpha\left[\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right]\geqslant 0\Leftrightarrow\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\geqslant 0.$$

1. $C^\star_{\alpha,B}(t)\geqslant 0$ for some $t\in[0,T]$:

   We have

   $$C^\star_{\alpha,B}(t)\geqslant 0\Leftrightarrow\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\geqslant 0\overset{Y(T)>0}{\Leftrightarrow}\tilde{\delta}\geqslant\delta+\frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}$$

   and

   $$\tilde{\delta}\in\left[0,\min\left\{\frac{1+\alpha\delta}{1+\alpha},1\right\}\right]$$

which follows from Assumption 5.6. In summary,

$$C^{\star}_{\alpha,B}(t) \geqslant 0 \Leftrightarrow \delta \geqslant 0,\ \tilde{\delta} \in \left[\max\left\{\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, 0\right\}, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right].$$

We now elaborate on conditions for a non-empty feasibility region for $\tilde{\delta}$, i.e. on conditions such that $\max\left\{\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, 0\right\} < \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}$. First notice that

$$\max\left\{\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, 0\right\} = \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}$$

because for $0 \leqslant \delta \leqslant 1$:

$$\underbrace{\delta}_{\geqslant 0} + \underbrace{\frac{1-\delta}{1+\alpha}}_{\geqslant 0}\underbrace{\frac{Y(T)-Y(t)}{Y(T)}}_{\geqslant 0} \geqslant 0$$

and for $\delta > 1$:

$$\delta + \underbrace{\frac{1-\delta}{1+\alpha}}_{<0}\underbrace{\frac{Y(T)-Y(t)}{Y(T)}}_{\in\left[0, \frac{Y(T)-Y(0)}{Y(T)}\right]} \geqslant \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(0)}{Y(T)} = \delta + \frac{1-\delta}{1+\alpha} = \frac{1+\alpha\delta}{1+\alpha} \geqslant 0.$$

In addition, we have

$$\min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\} = \begin{cases} \frac{1+\alpha\delta}{1+\alpha}, & \text{if } \delta \leqslant 1 \\ 1, & \text{if } \delta > 1. \end{cases}$$

From this it follows for the case $\delta \leqslant 1$:

$$\delta + \underbrace{\frac{1-\delta}{1+\alpha}}_{\geqslant 0}\underbrace{\frac{Y(T)-Y(t)}{Y(T)}}_{\leqslant 1} \leqslant \delta + \frac{1-\delta}{1+\alpha} = \frac{1+\alpha\delta}{1+\alpha} = \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}$$

but for the case $\delta > 1$:

$$\begin{aligned}\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)} &\geqslant \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(0)}{Y(T)} = \delta + \frac{1-\delta}{1+\alpha} \\ &= 1 + \frac{\alpha(\delta-1)}{1+\alpha} > 1 = \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}.\end{aligned}$$

Thus, only the case $\delta \leqslant 1$ where $\min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\} = \frac{1+\alpha\delta}{1+\alpha}$ leads to a non-empty region for $\tilde{\delta}$. In total we obtain

$$\begin{aligned}C^{\star}_{\alpha,B}(t) \geqslant 0 &\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right] \\ &\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right].\end{aligned}$$

2. $C^{\star}_{\alpha,B}(t) \geqslant 0$ for all $t \in [0,T]$:

   If $C^{\star}_{\alpha,B}(t) \geqslant 0$ should now hold for all $t \in [0,T]$, it has to hold for the infimum:

$$C^{\star}_{\alpha,B}(t) \geqslant 0 \ \forall t \in [0,T] \Leftrightarrow \inf_{t \in [0,T]} \left\{ C^{\star}_{\alpha,B}(t) \right\} \geqslant 0.$$

   It is

$$\begin{aligned} \inf_{t \in [0,T]} \left\{ C^{\star}_{\alpha,B}(t) \right\} \geqslant 0 &\Leftrightarrow \inf_{t \in [0,T]} \left\{ \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) Y(T) + \frac{1-\delta}{1+\alpha} Y(t) \right\} \geqslant 0 \\ &\Leftrightarrow \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) Y(T) + \inf_{t \in [0,T]} \left\{ \frac{1-\delta}{1+\alpha} Y(t) \right\} \geqslant 0 \end{aligned}$$

   with

$$\inf_{t \in [0,T]} \left\{ \frac{1-\delta}{1+\alpha} Y(t) \right\} = \begin{cases} \frac{1-\delta}{1+\alpha} Y(T) & , \text{ if } \delta > 1 \\ \frac{1-\delta}{1+\alpha} Y(0) = 0, & \text{if } \delta \leqslant 1. \end{cases}$$

   Thus,

$$\inf_{t \in [0,T]} \left\{ C^{\star}_{\alpha,B}(t) \right\} \geqslant 0 \Leftrightarrow 0 \leqslant \begin{cases} \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) Y(T) + \frac{1-\delta}{1+\alpha} Y(T) = \left(\tilde{\delta} - \delta \right) Y(T), & \text{if } \delta > 1 \\ \left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) Y(T) & , \text{ if } \delta \leqslant 1. \end{cases}$$

   In the first case, $\delta > 1$, it must be

$$\left(\tilde{\delta} - \delta \right) \underbrace{Y(T)}_{\geqslant 0} \geqslant 0 \Leftrightarrow \tilde{\delta} \geqslant \delta > 1.$$

   This is a contradiction to $\tilde{\delta} \leqslant 1$ in

$$\tilde{\delta} \in \left[0, \min \left\{ \frac{1+\alpha\delta}{1+\alpha}, 1 \right\} \right]$$

   that follows from Assumption 5.6.

   In the second case, $\delta \leqslant 1$, we must have

$$\left(\tilde{\delta} - \frac{1+\alpha\delta}{1+\alpha} \right) \underbrace{Y(T)}_{\geqslant 0} \geqslant 0 \Leftrightarrow \tilde{\delta} \geqslant \frac{1+\alpha\delta}{1+\alpha}$$

   which only holds for the limiting case $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$ in view of

$$\tilde{\delta} \in \left[0, \min \left\{ \frac{1+\alpha\delta}{1+\alpha}, 1 \right\} \right],$$

   again following from Assumption 5.6.

   In summary, $C^{\star}_{\alpha,B}(t) \geqslant 0$ for all $t \in [0,T]$ is only possible for the limiting case where $\delta \in [0,1]$ and $\tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}$.

An alternative proof based on part 1.:

If $C^\star_{\alpha,B}(t) \geqslant 0$ should now hold for all $t \in [0,T]$, then part 1. gives

$$\begin{aligned}
C^\star_{\alpha,B}(t) \geqslant 0 \ \forall t \in [0,T] &\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right] \ \forall t \in [0,T] \\
&\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\sup_{t\in[0,T]}\left\{\delta + \underbrace{\frac{1-\delta}{1+\alpha}}_{\geqslant 0}\underbrace{\frac{Y(T)-Y(t)}{Y(T)}}_{\geqslant 0}\right\}, \frac{1+\alpha\delta}{1+\alpha}\right] \\
&\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(0)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right] \\
&\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\frac{1+\alpha\delta}{1+\alpha}, \frac{1+\alpha\delta}{1+\alpha}\right] \\
&\Leftrightarrow \delta \in [0,1],\ \tilde{\delta} = \frac{1+\alpha\delta}{1+\alpha}.
\end{aligned}$$

3. $C^\star_{\alpha,B}(T) \geqslant 0$:

   From part 1. we know

$$C^\star_{\alpha,B}(t) \geqslant 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right].$$

   If $t = T$, the equivalency becomes

$$C^\star_{\alpha,B}(T) \geqslant 0 \Leftrightarrow \delta \in [0,1],\ \tilde{\delta} \in \left[\delta, \frac{1+\alpha\delta}{1+\alpha}\right].$$

□

Proof of Theorem C.4. Let $\beta \in (0,1]$ be given. We want to have a positive Value-at-Risk for this level β, i.e. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$. Note that the distribution of $C^\star_{\alpha,B}(t)$ is continuous. From Theorem 5.5 we have

$$\begin{aligned}
VaR_\beta\left(C^\star_{\alpha,B}(t)\right) &= \alpha VaR_\beta\left(V^\star_{\alpha,B}(t)\right) - \alpha\left[B(t) + (v_0 - B(0))\right] \\
&\overset{\text{Thm. 5.2}}{=} \alpha\left[\tilde{F}_{\alpha,B}(t) + e^{\mathbb{E}\left[V^\star_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t) + \Phi^{-1}(\beta)\sqrt{Var\left(V^\star_{\alpha,B}(t)\right)}}\right] - \alpha\left[B(t) + (v_0 - B(0))\right] \\
&= \alpha e^{\mathbb{E}\left[V^\star_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t) + \Phi^{-1}(\beta)\sqrt{Var\left(V^\star_{\alpha,B}(t)\right)}} - \alpha\left[B(t) - \tilde{F}_{\alpha,B}(t) + (v_0 - B(0))\right] \\
&\overset{\text{Thm. 5.2}}{=} \alpha e^{\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\left(\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)t} - \tilde{F}_{\alpha,B}(t)} \\
&\quad \times e^{\Phi^{-1}(\beta)\sqrt{(v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{2\left(\frac{1}{1+\alpha}r + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)t}\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)}} \\
&\quad - \alpha\left[B(t) - \tilde{F}_{\alpha,B}(t) + (v_0 - B(0))\right]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha e^{(v_0-\tilde{F}_{\alpha,B}(0))e^{\left(\frac{1}{1+\alpha}r+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t}+\Phi^{-1}(\beta)(v_0-\tilde{F}_{\alpha,B}(0))e^{\left(\frac{1}{1+\alpha}r+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t}\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}} \\
&\quad - \alpha\left[B(t)-\tilde{F}_{\alpha,B}(t)+(v_0-B(0))\right] \\
&= \alpha\Bigg[e^{(v_0-\tilde{F}_{\alpha,B}(0))e^{\left(\frac{1}{1+\alpha}r+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} - B(t)+\tilde{F}_{\alpha,B}(t) \\
&\qquad - (v_0-B(0))\Bigg].
\end{aligned}
$$

If we insert the setting and parameterization from Assumption 5.6 this becomes

$$
\begin{aligned}
&VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \\
&\overset{r=0}{=} \alpha\left[e^{(v_0-\tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} - B(t)+\tilde{F}_{\alpha,B}(t)-(v_0-B(0))\right] \\
&= \alpha\left[e^{(v_0-\tilde{F}_{\alpha,B}(0))e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} + \left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right] \\
&= \alpha\left[e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} + \left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right] \\
&= \alpha\left[f_{VaR}(t)+\left(\tilde{\delta}-\frac{1+\alpha\delta}{1+\alpha}\right)Y(T)+\frac{1-\delta}{1+\alpha}Y(t)\right],
\end{aligned}
$$

where we define for convenience

$$
f_{VaR}(t) := e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} \geqslant 0. \tag{C.19}
$$

1. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for some $t \in [0,T]$:

   It immediately follows

   $$VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0 \Leftrightarrow f_{VaR}(t)+\frac{1-\delta}{1+\alpha}Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T).$$

   If $\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)-\frac{1-\delta}{1+\alpha}Y(t) > 0$, then the inequality condition can be reformulated as a lower bound condition on β:

$$
\begin{aligned}
& f_{VaR}(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t) \\
\Leftrightarrow\ & e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t) \\
\Leftrightarrow\ & \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t} \left(1 + \Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)}\right) \\
& \geqslant \ln\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t)\right) \\
\Leftrightarrow\ & \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)}} \left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}} - 1\right]\right).
\end{aligned}
$$

We now look at the sign of $\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t)$:
First, $\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t) > 0$ if $\delta > 1$. Second, if $\delta \leqslant 1$, we have the following:

$$
\begin{aligned}
& \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) - \frac{1-\delta}{1+\alpha} Y(t) = \left(\delta - \tilde{\delta}\right) Y(T) + \frac{1-\delta}{1+\alpha}\left(Y(T) - Y(t)\right) \geqslant 0 \\
\overset{Y(T)>0}{\Leftrightarrow}\ & \tilde{\delta} \leqslant \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T) - Y(t)}{Y(T)}.
\end{aligned}
$$

In summary, we obtain

$$
\begin{aligned}
& VaR_\beta\left(C^{\star}_{\alpha,B}(t)\right) \geqslant 0 \\
\Leftrightarrow\ & \delta \geqslant 0,\ \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right]:\ f_{VaR}(t) + \frac{1-\delta}{1+\alpha} Y(t) \geqslant \left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) \\
\Leftrightarrow\ & \begin{cases}
\delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right], \text{ or} \\
\delta \in [0,1],\ \tilde{\delta} \in \left[0, \min\left\{\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right], \\
\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T) - \frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}} - 1\right]\right), \text{ or} \\
\delta > 1,\ \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right], \\
\beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T) - \frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 t}} - 1\right]\right)
\end{cases}
\end{aligned}
$$

$$\Leftrightarrow \begin{cases} \delta \in [0,1],\ \tilde{\delta} \in \left[\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}, \frac{1+\alpha\delta}{1+\alpha}\right], \text{ or} \\ \delta \in [0,1],\ \tilde{\delta} \in \left[0, \delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)}\right), \\ \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 t}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)-\frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t}}-1\right]\right), \text{ or} \\ \delta > 1,\ \tilde{\delta} \in [0,1],\ \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 t}-1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)-\frac{1-\delta}{1+\alpha}Y(t)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t}}-1\right]\right), \end{cases}$$

where the last equivalence holds due to $\delta + \frac{1-\delta}{1+\alpha}\frac{Y(T)-Y(t)}{Y(T)} \leqslant \frac{1+\alpha\delta}{1+\alpha} \leqslant 1$ under $\delta \leqslant 1$, and $\frac{1+\alpha\delta}{1+\alpha} > 1$ under $\delta > 1$.

2. $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for all $t \in [0,T]$:

We now elaborate on $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ for all $t \in (0,T]$. Recall that in view of Eq. (5.12) it is

$$\tilde{\delta} \leqslant \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\} \leqslant \frac{1+\alpha\delta}{1+\alpha},$$

which leads to

$$\begin{aligned} &VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0\ \forall t \in [0,T] \\ \Leftrightarrow\ &\underbrace{e^{\overbrace{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)}^{\geqslant 0} Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)}}_{\geqslant 0} + \frac{1-\delta}{1+\alpha}\underbrace{Y(t)}_{\geqslant 0} \\ \geqslant\ &\underbrace{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)}_{\geqslant 0}\underbrace{Y(T)}_{\geqslant 0}\ \ \forall t \in [0,T]. \end{aligned}$$

Thus it is unclear if the inequality is fulfilled.

If $VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0$ should hold for all $t \in [0,T]$, the equivalent inequality condition has to hold also for all $t \in [0,T]$ or its infimum:

$$\begin{aligned} &VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0\ \forall t \in [0,T] \\ \Leftrightarrow\ &\inf_{t\in[0,T]}\left\{f_{VaR}(t) + \frac{1-\delta}{1+\alpha}Y(t)\right\} \geqslant \left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T). \end{aligned}$$

We examine the infimum and its lower bound in what follows. Since the expression $\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t} \geqslant 0$ in $f_{VaR}(t)$ in any case, we first look at the second part

$1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}$ in $f_{VaR}(t)$ which is to be minimized. Note that we would like to examine the "left tail" of the distribution, hence it is suitable to look at

$$\beta \leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}}\right), \tag{C.20}$$

where

$$\underbrace{1-\Phi\left(\underbrace{\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}}}_{\in(0,\infty)}\right)}_{\in[0.5,1)} \in (0,0.5]$$

implies

$$\beta \leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}}\right) \Rightarrow \beta \leqslant 0.5$$

with

$$\beta \leqslant 0.5 \Leftrightarrow \Phi^{-1}(\beta) \leqslant 0.$$

Then

$$\begin{aligned}\inf_{t\in[0,T]}\left\{1+\underbrace{\Phi^{-1}(\beta)}_{\leqslant 0}\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right\} &= 1+\Phi^{-1}(\beta)\sup_{t\in[0,T]}\left\{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right\}\\ &= 1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}.\end{aligned}$$

Notice that the above restriction on β was selected such that

$$1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)} \leqslant 0 \Leftrightarrow \beta \leqslant 1-\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 T}-1\right)}}\right).$$

With these thoughts we arrive at

$$\begin{aligned}
\inf_{t\in[0,T]}\{f_{VaR}(t)\} &= \inf_{t\in[0,T]}\left\{ e^{\overbrace{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t}}^{\geqslant 0}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right)} \right\} \\
&= e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\sup_{t\in[0,T]}\left\{e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 t}\right\}\overbrace{\inf_{t\in[0,T]}\left\{1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 t}-1\right)}\right\}}^{\leqslant 0}} \\
&= e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 T}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 T}-1\right)}\right)} \\
&= f_{VaR}(T)
\end{aligned}$$

with $f_{VaR}(T)\in[0,1]$. Hence, the infimum of $f_{VaR}(t)$ is attained at $t=T$:

$$\inf_{t\in[0,T]}\{f_{VaR}(t)\} = f_{VaR}(T) = e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\tilde{b}}\|\gamma\|^2 T}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\tilde{b}}\right)^2\|\gamma\|^2 T}-1\right)}\right)}.$$

Now, we split our analysis into two cases:

a) Let $\delta\leqslant 1$:

Then

$$\inf_{t\in[0,T]}\left\{\underbrace{\frac{1-\delta}{1+\alpha}}_{\geqslant 0}Y(t)\right\} = \frac{1-\delta}{1+\alpha}\underbrace{\inf_{t\in[0,T]}\{Y(t)\}}_{=Y(0)} = \frac{1-\delta}{1+\alpha}Y(0)=0$$

and therefore

$$\begin{aligned}
\inf_{t\in[0,T]}\left\{f_{VaR}(t)+\frac{1-\delta}{1+\alpha}Y(t)\right\} &\geqslant \inf_{t\in[0,T]}\{f_{VaR}(t)\} + \inf_{t\in[0,T]}\left\{\frac{1-\delta}{1+\alpha}Y(t)\right\} \\
&= \inf_{t\in[0,T]}\{f_{VaR}(t)\} = f_{VaR}(T).
\end{aligned}$$

In total, coming back to the Value-at-Risk condition, we obtain

$$\begin{aligned}
& VaR_\beta\left(C^\star_{\alpha,B}(t)\right)\geqslant 0\ \forall t\in[0,T] \\
\Leftrightarrow\ & \inf_{t\in[0,T]}\left\{f_{VaR}(t)+\frac{1-\delta}{1+\alpha}Y(t)\right\}\geqslant\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T) \\
\Leftarrow\ & f_{VaR}(T)\geqslant\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T),
\end{aligned}$$

where the last sufficient condition can be transferred to

$$f_{VaR}(T) \geqslant \underbrace{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T)}_{\geqslant 0}$$

$$\Leftrightarrow \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2 T} - 1\right)}}\left[\frac{\ln\left(\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 T}} - 1\right]\right),$$

that is a lower bound condition on β. We observe

$$VaR_\beta\left(C^\star_{\alpha,B}(t)\right) \geqslant 0 \; \forall t \in [0,T]$$

$$\text{if } \underbrace{e^{\overbrace{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 T}}^{\geqslant 0} \overbrace{\left(1 + \Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2 T} - 1\right)}\right)}^{\leqslant 0}}}_{\in [0,1]}$$

$$\geqslant \underbrace{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right)}_{\geqslant 0} Y(T)$$

with

$$\tilde{\delta} \in \left[0, \underbrace{\min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}}_{=\frac{1+\alpha\delta}{1+\alpha},\text{ for } \delta \leqslant 1}\right] = \left[0, \frac{1+\alpha\delta}{1+\alpha}\right].$$

b) Let $\delta > 1$:

We show above that, under the assumption $\beta \leqslant 1 - \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2 T} - 1\right)}}\right)$, the infimum of $f_{VaR}(t)$ is attained at $t = T$:

$$\inf_{t \in [0,T]} \{f_{VaR}(t)\} = f_{VaR}(T) = e^{\left(\frac{1+\alpha\delta}{1+\alpha} - \tilde{\delta}\right) Y(T) e^{\frac{1}{1-\hat{b}}\|\gamma\|^2 T}\left(1 + \Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2 T} - 1\right)}\right)}.$$

Moreover,

$$\inf_{t\in[0,T]}\left\{\underbrace{\frac{1-\delta}{1+\alpha}}_{<0}Y(t)\right\}=\frac{1-\delta}{1+\alpha}\underbrace{\sup_{t\in[0,T]}\{Y(t)\}}_{=Y(T)}=\frac{1-\delta}{1+\alpha}Y(T),$$

i.e. the infimum of $\frac{1-\delta}{1+\alpha}Y(t)$ is also attained at $t=T$. Altogether, this gives equality:

$$\inf_{t\in[0,T]}\left\{f_{VaR}(t)+\frac{1-\delta}{1+\alpha}Y(t)\right\}=f_{VaR}(T)+\frac{1-\delta}{1+\alpha}Y(T).$$

Finally, coming back to the Value-at-Risk condition, we get

$$\begin{aligned}
&VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right)\geqslant 0\ \forall t\in[0,T]\\
\Leftrightarrow\ &\inf_{t\in[0,T]}\left\{f_{VaR}(t)+\frac{1-\delta}{1+\alpha}Y(t)\right\}\geqslant\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\\
\Leftrightarrow\ &f_{VaR}(T)+\frac{1-\delta}{1+\alpha}Y(T)\geqslant\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)\\
\Leftrightarrow\ &f_{VaR}(T)\geqslant\left(\delta-\tilde{\delta}\right)Y(T).
\end{aligned}$$

Since $\delta>1$ in this case and $\tilde{\delta}\leqslant 1$ is required, it follows that $\tilde{\delta}\leqslant 1<\delta$ and the term $\left(\delta-\tilde{\delta}\right)Y(T)\geqslant 0$ becomes always non-negative. Therefore, we can rewrite the condition as follows:

$$\begin{aligned}
&VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right)\geqslant 0\ \forall t\in[0,T]\\
\Leftrightarrow\ &f_{VaR}(T)\geqslant\left(\delta-\tilde{\delta}\right)Y(T)\\
\Leftrightarrow\ &\beta\geqslant\Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-\hat{\delta}}\right)^2\|\gamma\|^2T}-1\right)}}\left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{\delta}}\|\gamma\|^2T}}-1\right]\right).
\end{aligned}$$

We observe

$$\begin{aligned}
&VaR_{\beta}\left(C^{\star}_{\alpha,B}(t)\right)\geqslant 0\ \forall t\in[0,T]\\
\Leftrightarrow\ &\underbrace{e^{\overbrace{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-\hat{\delta}}\|\gamma\|^2T}}^{\geqslant 0}\overbrace{\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-\hat{\delta}}\right)^2\|\gamma\|^2T}-1\right)}\right)}^{\leqslant 0}}}_{\in[0,1]}\geqslant\left(\delta-\tilde{\delta}\right)Y(T)
\end{aligned}$$

with

$$\tilde{\delta} \in \left[0, \underbrace{\min\left\{ \frac{1+\alpha\delta}{1+\alpha}, 1 \right\}}_{=1,\text{ for } \delta>1} \right] = [0,1].$$

3. $VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0$:

   In particular we have at terminal time T:

$$\begin{aligned} VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0 &\Leftrightarrow f_{VaR}(T) \geqslant \left(\delta - \tilde{\delta}\right) Y(T) \\ &\Leftrightarrow \underbrace{e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-b}\|\gamma\|^2 T}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-b}\right)^2\|\gamma\|^2 T}-1\right)}\right)}}_{\geqslant 0} \geqslant \left(\delta - \tilde{\delta}\right) \underbrace{Y(T)}_{\geqslant 0}. \end{aligned}$$

   We observe the two cases $\tilde{\delta} \geqslant \delta$ and $\tilde{\delta} < \delta$. First, for any $\tilde{\delta} \geqslant \delta$:

$$\delta \geqslant 0,\ \tilde{\delta} \in \left[\delta, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right].$$

   This region for $\tilde{\delta}$ is non-empty iff $\delta \leqslant 1$ with then $\min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\} = \frac{1+\alpha\delta}{1+\alpha}$ and we have to restrict

$$\delta \in [0,1],\ \tilde{\delta} \in \left[\delta, \frac{1+\alpha\delta}{1+\alpha}\right].$$

   In this case, Theorem 5.7 shows that

$$\delta \in [0,1],\ \tilde{\delta} \in \left[\delta, \frac{1+\alpha\delta}{1+\alpha}\right] \Leftrightarrow C^\star_{\alpha,B}(T) \geqslant 0.$$

   Second, for any $\tilde{\delta} < \delta$, the condition on δ and $\tilde{\delta}$ can be rewritten as a condition on β:

$$\begin{aligned} VaR_\beta\left(C^\star_{\alpha,B}(T)\right) \geqslant 0 &\Leftrightarrow f_{VaR}(T) \geqslant \left(\delta - \tilde{\delta}\right) Y(T) \\ &\Leftrightarrow \underbrace{e^{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-b}\|\gamma\|^2 T}\left(1+\Phi^{-1}(\beta)\sqrt{\left(e^{\left(\frac{1}{1-b}\right)^2\|\gamma\|^2 T}-1\right)}\right)}}_{\geqslant 0} \geqslant \underbrace{\left(\delta - \tilde{\delta}\right) Y(T)}_{\geqslant 0} \\ &\Leftrightarrow \beta \geqslant \Phi\left(\frac{1}{\sqrt{\left(e^{\left(\frac{1}{1-b}\right)^2\|\gamma\|^2 T}-1\right)}} \left[\frac{\ln\left(\left(\delta-\tilde{\delta}\right)Y(T)\right)}{\left(\frac{1+\alpha\delta}{1+\alpha}-\tilde{\delta}\right)Y(T)e^{\frac{1}{1-b}\|\gamma\|^2 T}} - 1 \right] \right). \end{aligned}$$

   Moreover, for $\tilde{\delta} < \delta$ the restriction

$$\delta \geqslant 0,\ \tilde{\delta} \in \left[0, \min\left\{\frac{1+\alpha\delta}{1+\alpha}, 1\right\}\right],\ \tilde{\delta} < \delta$$

becomes

$$\delta \geqslant 0, \;\; \tilde{\delta} \in [0, \min\{\delta, 1\}]$$

due to

$$\min\left\{\delta, \frac{1+\alpha\delta}{1+\alpha}, 1\right\} = \begin{cases} \delta, & \text{if } \delta \leqslant 1 \\ 1, & \text{if } \delta > 1 \end{cases} = \min\{\delta, 1\},$$

where we added the case $\tilde{\delta} = \delta$ for convenience, which does not falsify any result.

□

C.3 A solution to an ordinary differential equation

Let the following ordinary differential equation (ODE) on a function $f(t)$ with $\delta(t) \in \mathbb{R}\backslash\{0\}$, $\epsilon(t) \in \mathbb{R}$ be given:

$$\begin{aligned} f'(t) &= \delta(t)f(t) + \epsilon(t), \\ f(T) &= f_T. \end{aligned} \tag{C.21}$$

Theorem C.6. *The solution to the ODE* (C.21) *is*

$$f(t) = e^{-\int_t^T \delta(s)ds} f_T - \int_t^T e^{-\int_t^s \delta(u)du} \epsilon(s)ds.$$

Proof. Define the function $g(t) := e^{-\int_0^t \delta(s)ds}$ such that $g'(t) = -\delta(t)g(t)$. Now define $z(t) := f(t)g(t)$. The product rule gives

$$z'(t) = f(t)g'(t) + g(t)f'(t) = -\delta(t)g(t)f(t) + g(t)\left(\delta(t)f(t) + \epsilon(t)\right) = g(t)\epsilon(t).$$

Therefore, the ODE in z that is to be solved is given by

$$\begin{aligned} z'(t) &= e^{-\int_0^t \delta(s)ds} \epsilon(t), \\ z(T) &= e^{-\int_0^T \delta(s)ds} f_T. \end{aligned}$$

To solve this ODE we can apply the technique of separation of variables. It follows

$$dz = e^{-\int_0^t \delta(s)ds} \epsilon(t)dt,$$

and by integrating both sides we obtain

$$z(t) = \int_{t_0}^t e^{-\int_0^s \delta(u)du} \epsilon(s)ds + K$$

for some $t_0 \leqslant t$ and some constant $K \in \mathbb{R}$ that is such that the terminal condition $z(T) = e^{-\int_0^T \delta(s)ds} f_T$ holds true, i.e.

$$e^{-\int_0^T \delta(s)ds} f_T = \int_{t_0}^T e^{-\int_0^s \delta(u)du} \epsilon(s)ds + K$$

which leads to

$$K = e^{-\int_0^T \delta(s)ds} f_T - \int_{t_0}^T e^{-\int_0^s \delta(u)du} \epsilon(s)ds$$

and thus

$$z(t) = e^{-\int_0^T \delta(s)ds} f_T - \int_t^T e^{-\int_0^s \delta(u)du} \epsilon(s)ds.$$

Finally, we arrive at the solution to the ODE (C.21):

$$f(t) = \frac{z(t)}{g(t)} = e^{\int_0^t \delta(s)ds} \left(e^{-\int_0^T \delta(s)ds} f_T - \int_t^T e^{-\int_0^s \delta(u)du} \epsilon(s)ds \right) = e^{-\int_t^T \delta(s)ds} f_T - \int_t^T e^{-\int_t^s \delta(u)du} \epsilon(s)ds.$$

□

C.4 An alternative buffer rate process as a proportion of the fund surplus

The second proposal for a buffer scheme, as an alternative to the one in Section 5.1.2, smooths the portfolio wealth around the benchmark wealth $B(t)$. We consider the alternative buffer rule $c(t) = c(t, V)$ defined by

$$c(t)dt = \alpha(t)(V(t) - B(t))dt, \ \alpha(t) \in [0, 1] \tag{C.22}$$

for some deterministic wealth benchmark $B(t)$. Hence, the buffer rate equals the proportion $\alpha(t)$ of the surplus $V(t) - B(t)$. Whenever the portfolio wealth exceeds the benchmark wealth, i.e. $V(t) > B(t)$, then some fraction of this surplus is taken out of the portfolio and put into the buffer account. Similar to Section 5.1.2, the proposed buffer rule defined in (C.22) and also the accumulated buffer account can turn negative in order to smooth the investment portfolio in bad market times. In good times, when $V(t) > B(t)$, then some fraction of this surplus is used to build a buffer that can be used in bad times. In those bad times, when $V(t) < B(t)$, then a fraction of the absolute value of this negative surplus is taken from the buffer account to increase the portfolio value and to bring it back closer to the benchmark $B(t)$. Notice that $c(t) = \alpha(t)(V(t) - B(t))$ is to be interpreted as a rate and not a total amount.

When we insert $c(t)$ from (C.22) into the formula for the wealth dynamics, then (5.1) becomes

$$\begin{aligned} dV(t) &= V(t) \left[\left(r + \hat{\pi}(t)' (\mu - r\mathbf{1}) \right) dt + \hat{\pi}(t)' \sigma dW(t) \right] - \alpha(t)(V(t) - B(t))dt + y(t)dt \\ &= V(t) \left[\left(r - \alpha(t) + \hat{\pi}(t)' (\mu - r\mathbf{1}) \right) dt + \hat{\pi}(t)' \sigma dW(t) \right] + \alpha(t)B(t)dt + y(t)dt. \end{aligned}$$

The formula shows that $\alpha(t)$ can to some part be interpreted as a reduction in the interest rate r.

The reasoning why this buffer rule is in the appendix is because we prefer the first buffer rule introduced in Section 5.1.2. The explanation is the following:

Let us look at the situation where $V(t) > B(t)$. Moreover, let $V(t+\Delta) < V(t)$, i.e. the investment return in the period $[t, t+\Delta]$ is negative although there are some positive inflows in terms of $y(t)$. Let nevertheless $V(t+\Delta) > B(t+\Delta)$ hold with $B(t+\Delta) > B(t)$, due to a strong outperformance of $V(s)$ over $B(s)$ prior to time t, i.e. for $s \leqslant t$. This means, the difference in $V - B$ shrinks from time t to $t+\Delta$, but is still positive. Then Eq. (C.22) supposes that some amount is taken out of the portfolio and put into the buffer account even if there is a loss in the wealth process $V(t)$ while having an increase in the wealth benchmark $B(t)$. This is counterintuitive and does not happen for the buffer proposed in Eq. (5.7) in Section 5.1.2.

C.4.1 General solution

The next theorem summarizes the closed-form results for the optimal control $\hat{\pi}^{\star}_{\alpha,B}(t)$ and the corresponding optimal fund wealth process $V^{\star}_{\alpha,B}(t) = V(t, \hat{\pi}^{\star}_{\alpha,B})$. Let $v_0 > \tilde{F}_{\alpha,B}(0)$ be assumed from now on[1].

Theorem C.7 (General solution)**.** *The optimal portfolio process is given by*

$$\hat{\pi}^{\star}_{\alpha,B}(t) = \frac{1}{1-\hat{b}} \frac{V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^{\star}_{\alpha,B}(t)} \Sigma^{-1} (\mu - r\mathbf{1}),$$

which is of a Constant Proportion Portfolio Insurance (CPPI) type with constant multiple and where we define

$$\tilde{F}_{\alpha,B}(t) := F e^{-r(T-t)+\int_t^T \alpha(s)ds} - \int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du} (\alpha(s)B(s) + y(s))ds, \;\; \tilde{F}_{\alpha,B}(T) = F,$$

with $\tilde{F}'_{\alpha,B}(t) = (r - \alpha(t))\tilde{F}_{\alpha,B}(t) + \alpha(t)B(t) + y(t)$. The associated optimal wealth process $V^{\star}_{\alpha,B}(t)$ of the pension fund follows the SDE

$$dV^{\star}_{\alpha,B}(t) = (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right] + \tilde{F}'_{\alpha,B}(t)dt,$$

the fund surplus $V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)$ is log-normally distributed and follows the SDE

$$d\left(V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = (V^{\star}_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)) \left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right],$$

hence

$$\begin{aligned} V^{\star}_{\alpha,B}(t) &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{\left(r + \frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right)t - \int_0^t \alpha(s)ds + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r + \frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}} \\ &> \tilde{F}_{\alpha,B}(t). \end{aligned}$$

The optimal accumulated buffer account equals

[1] $\tilde{F}_{\alpha,B}(0)$ is defined in the upcoming Theorem C.7. The reasoning why this condition is supposed was already explained in Section 5.1.2.

$$\begin{aligned}C^{\star}_{\alpha,B}(t) &= \int_0^t c(s)e^{r(t-s)}ds = \int_0^t e^{r(t-s)}\alpha(s)(V^{\star}_{\alpha,B}(s) - B(s))ds \\ &= (v_0 - \tilde{F}_{\alpha,B}(0)) \int_0^t e^{r(t-s)}\alpha(s)e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s-\int_0^s \alpha(u)du}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds \\ &\quad + \int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right)ds.\end{aligned}$$

Proof. The proof is provided in Appendix C.5. □

The formula for the investment strategy $\hat{\pi}^{\star}_{\alpha,B}(t)$ shows that the buffer rule has no impact on the constant CPPI multiplier, but influences the optimal portfolio allocation strategy through the CPPI floor $\tilde{F}_{\alpha,B}(t)$. Further the formulas show that under $\tilde{F}_{\alpha,B}(t) \geqslant B(t)$ (for instance see Corollary C.11), it is $c(t) \geqslant 0$ and therefore $C^{\star}_{\alpha,B}(t) \geqslant 0$.

We now have a look at the expectation, variance, Value-at-Risk and shortfall probability of the fund wealth distribution and the expectation of the buffer account to assess its risk.

Theorem C.8 (Fund wealth characteristics)**.** *Let $\Phi(\cdot)$ denote the cumulative distribution function of a standard normal random variable.*

- *Expected fund wealth:*

$$\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] = \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{rt+\frac{1}{1-\hat{b}}\|\gamma\|^2 t-\int_0^t \alpha(s)ds}$$

- *Variance of the fund wealth:*

$$Var\left(V^{\star}_{\alpha,B}(t)\right) = (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{2\left(rt+\frac{1}{1-\hat{b}}\|\gamma\|^2 t-\int_0^t \alpha(s)ds\right)}\left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right)$$

- *Value-at-Risk/Quantiles of the fund wealth distribution with level $\beta \in [0,1]$:*

$$VaR_{\beta}\left(V^{\star}_{\alpha,B}(t)\right) = F^{-1}_{V^{\star}_{\alpha,B}(t)}(\beta) = \tilde{F}_{\alpha,B}(t) + e^{\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]-\tilde{F}_{\alpha,B}(t)+\Phi^{-1}(\beta)\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}$$

- *Shortfall probability of the fund wealth with threshold $s > \tilde{F}_{\alpha,B}(t)$:*

$$\mathbb{P}\left(V^{\star}_{\alpha,B}(t) \leqslant s\right) = F_{V^{\star}_{\alpha,B}(t)}(s) = \Phi\left(\frac{\ln(s - \tilde{F}_{\alpha,B}(t)) - \mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right] + \tilde{F}_{\alpha,B}(t)}{\sqrt{Var\left(V^{\star}_{\alpha,B}(t)\right)}}\right)$$

- *Expected accumulated buffer account:*

$$\mathbb{E}\left[C^{\star}_{\alpha,B}(t)\right] = \int_0^t e^{r(t-s)}\alpha(s)(\mathbb{E}\left[V^{\star}_{\alpha,B}(s)\right] - B(s))ds$$

$$= \int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds$$
$$+ (v_0 - \tilde{F}_{\alpha,B}(0)) \int_0^t e^{r(t-s)}\alpha(s)e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} ds$$

Proof. The proof is given in Appendix C.5. □

The numbers in Theorem C.8 can be applied to assess and manage the portfolio's risks or for estimating or selecting the model parameters.

C.4.2 Special cases

We consider some special cases that arise from Theorem C.7. First of all, if the buffer rule parameter is constant, $\alpha(t) \equiv \alpha$, then the formulas can be reduced to:

Corollary C.9 (Constant buffer rule parameter: $\alpha(t) \equiv \alpha$)**.** *If $\alpha(t) \equiv \alpha$, then the optimal portfolio process is given by*

$$\hat{\pi}^\star_{\alpha,B}(t) = \frac{1}{1-\hat{b}} \frac{V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^\star_{\alpha,B}(t)} \Sigma^{-1}(\mu - r\mathbf{1}),$$

which is a CPPI strategy with constant multiple and cohort-age dependent but state- or market-independent cushion floor

$$\tilde{F}_{\alpha,B}(t) = Fe^{-(r-\alpha)(T-t)} - \int_t^T e^{-(r-\alpha)(s-t)}(\alpha B(s) + y(s))ds, \ \tilde{F}_{\alpha,B}(T) = F,$$

with $\tilde{F}'_{\alpha,B}(t) = (r-\alpha)\tilde{F}_{\alpha,B}(t) + \alpha B(t) + y(t)$. Moreover, it holds

$$dV^\star_{\alpha,B}(t) = (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(r - \alpha + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] + \tilde{F}'_{\alpha,B}(t)dt,$$
$$d\left(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(r - \alpha + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right],$$

and

$$V^\star_{\alpha,B}(t) = \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{\left(r-\alpha+\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t + \frac{1}{1-\hat{b}}\gamma' W(t)}$$
$$= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0))e^{-\left[\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)+\alpha\right]t}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.$$

Finally, the formula for the accumulated buffer account simplifies to

$$C^\star_{\alpha,B}(t) = (v_0 - \tilde{F}_{\alpha,B}(0))\alpha \int_0^t e^{r(t-s)}e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \alpha s}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds$$
$$+ \alpha \int_0^t e^{r(t-s)}\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds.$$

If the buffer mechanism is absent, i.e. if $\alpha(t)$ is forced to zero, then Corollary 5.4 already provides the respective objects of interest.

The upcoming corollary states the results if the wealth benchmark $B(t) \equiv 0$ for the buffer mechanism $c(t)$ is set to zero.

Corollary C.10 (No buffer benchmark: $B(t) \equiv 0$)**.** *In the case where $B(t) \equiv 0$ (i.e. no buffer wealth benchmark) the formula for $\hat{\pi}^\star_{\alpha,0}(t)$ reduces to*

$$\hat{\pi}^\star_{\alpha,0}(t) = \frac{1}{1-\hat{b}} \frac{V^\star_{\alpha,0}(t) - \tilde{F}_{\alpha,0}(t)}{V^\star_{\alpha,0}(t)} \Sigma^{-1} (\mu - r\mathbf{1}),$$

which is of a CPPI type with cohort-age dependent floor

$$\tilde{F}_{\alpha,0}(t) = Fe^{-r(T-t)+\int_t^T \alpha(s)ds} - \int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du} y(s)ds, \ \ \tilde{F}_{\alpha,B}(T) = F,$$

with $\tilde{F}'_{\alpha,0}(t) = (r - \alpha(t))\, \tilde{F}_{\alpha,0}(t) + y(t)$. For the wealth process of the pension fund we obtain

$$dV^\star_{\alpha,0}(t) = (V^\star_{\alpha,0}(t) - \tilde{F}_{\alpha,0}(t)) \left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right] + \tilde{F}'_{\alpha,0}(t)dt,$$

$$d\left(V^\star_{\alpha,0}(t) - \tilde{F}_{\alpha,0}(t)\right) = (V^\star_{\alpha,0}(t) - \tilde{F}_{\alpha,0}(t)) \left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}} \|\gamma\|^2 \right) dt + \frac{1}{1-\hat{b}} \gamma' dW(t) \right],$$

and

$$\begin{aligned} V^\star_{\alpha,0}(t) &= \tilde{F}_{\alpha,0}(t) + (v_0 - \tilde{F}_{\alpha,0}(0)) e^{\left(r + \frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2 \|\gamma\|^2\right)t - \int_0^t \alpha(s)ds + \frac{1}{1-\hat{b}}\gamma' W(t)} \\ &= \tilde{F}_{\alpha,0}(t) + (v_0 - \tilde{F}_{\alpha,0}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r + \frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}. \end{aligned}$$

Additionally, the formula for the accumulated buffer account reads

$$\begin{aligned} C^\star_{\alpha,0}(t) = (v_0 - \tilde{F}_{\alpha,0}(0)) &\int_0^t e^{r(t-s)} \alpha(s) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r + \frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} \tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds \\ &+ \int_0^t e^{r(t-s)} \alpha(s) \tilde{F}_{\alpha,0}(s) ds. \end{aligned}$$

Moreover, $B(t)$ can be regarded as a wealth benchmark for the buffer rule $c(t)$ and $\tilde{F}_{\alpha,B}(t)$ as a wealth benchmark for the investment strategy $\hat{\pi}^\star_{\alpha,B}(t)$. The following corollary shows the solution if both wealth benchmarks coincide.

Corollary C.11 (Equal benchmarks for buffer rate and investment strategy: $B(t) = \tilde{F}_{\alpha,B}(t)$)**.** *If the wealth benchmark $B(t)$ of the buffer rate process coincides with the wealth benchmark $\tilde{F}_{\alpha,B}(t)$ of the optimal investment strategy, then*

$$\hat{\pi}^\star_{\alpha,B}(t) = \frac{1}{1-\hat{b}} \frac{V^\star_{\alpha,B}(t) - B(t)}{V^\star_{\alpha,B}(t)} \Sigma^{-1} (\mu - r\mathbf{1}),$$

which is a CPPI strategy with time-t floor $\tilde{F}_{\alpha,B}(t) = B(t)$ being equal to the wealth benchmark $B(t)$ of the buffer rate process. Furthermore, we obtain

$$
\begin{aligned}
dV^\star_{\alpha,B}(t) &= (V^\star_{\alpha,B}(t) - B(t))\left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] + B'(t)dt,\\
d\left(V^\star_{\alpha,B}(t) - B(t)\right) &= (V^\star_{\alpha,B}(t) - B(t))\left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
V^\star_{\alpha,B}(t) &= B(t) + (v_0 - B(0))e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds + \frac{1}{1-\hat{b}}\gamma' W(t)}\\
&= B(t) + (v_0 - B(0))e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.
\end{aligned}
$$

In this case ($B(t) = \tilde{F}_{\alpha,B}(t)$), $B(t)$ needs to fulfill

$$B'(t) = (r - \alpha(t))\,B(t) + \alpha(t)B(t) + y(t) \Leftrightarrow B'(t) = rB(t) + y(t)$$

with terminal condition $B(T) = \tilde{F}_{\alpha,B}(T) = F$.

$$B(t) = e^{-r(T-t)}F - \int_t^T e^{-r(s-t)}y(s)ds,\;\; B(T) = F.$$

Hence, $B(t)$ grows with inflows $y(t)$ and interest rate r which means that an investment return benchmark of r is considered. The accumulated buffer account formula becomes

$$C^\star_{\alpha,B}(t) = (v_0 - B(0))\int_0^t e^{r(t-s)}\alpha(s)e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \int_0^s \alpha(u)du}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds.$$

Finally, we are interested in those settings in which the optimal portfolio rule turns to a constant-mix strategy. It can be shown that this is the case iff $\tilde{F}_{\alpha,B}(t) = 0$.

Corollary C.12 (Optimal constant-mix strategy)**.** *Let $\tilde{F}_{\alpha,B}(t) = 0$ $\forall t \in [0,T]$ which holds if and only if $F = 0$ (i.e. CRRA utility function), since $\tilde{F}_{\alpha,B}(T) = F$ and the remainder of $\tilde{F}_{\alpha,B}(t)$, which is $\int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du}(\alpha(s)B(s) + y(s))ds$ equates to zero $\forall t \in [0,T]$ because $0 = \tilde{F}'_{\alpha,B}(t) \overset{(C.34)}{=} (r-\alpha(t))\,\tilde{F}_{\alpha,B}(t) + \alpha(t)B(t) + y(t) = \alpha(t)B(t) + y(t)$. In this case, i.e. $\alpha(t)B(t) + y(t) = 0$, the wealth benchmark $B(t)$ is forced to take the form*

$$B(t) = -\frac{1}{\alpha(t)}y(t).$$

Therefore, $B(t) < 0$ if $y(t) > 0$. The optimal constant-mix strategy then turns out to be

$$\hat{\pi}^\star_{\alpha,B}(t) \equiv \frac{1}{1-\hat{b}}\Sigma^{-1}(\mu - r\mathbf{1})$$

which coincides with the Merton portfolio. Moreover, it holds

$$dV^\star_{\alpha,B}(t) = V^\star_{\alpha,B}(t)\left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right],$$

and

$$V^{\star}_{\alpha,B}(t) = v_0 e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t-\int_0^t \alpha(s)ds+\frac{1}{1-\hat{b}}\gamma' W(t)}$$
$$= v_0 e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t-\int_0^t \alpha(s)ds}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.$$

If there is only one risky asset in the financial market ($N = 1$), i.e. if a risky fund is considered, then the optimal constant-mix strategy is

$$\hat{\pi}^{\star}_{\alpha,B}(t) \equiv \frac{1}{1-\hat{b}}\frac{\mu - r}{\sigma^2} = MP = \frac{1}{1-\hat{b}}\frac{1}{\sigma}SR,$$

where $MP := \frac{1}{1-\hat{b}}\frac{\mu-r}{\sigma^2}$ is the Merton portfolio and $SR := \frac{\mu-r}{\sigma}$ denotes the Sharpe Ratio of the risky asset.

Finally, the formula for the accumulated buffer becomes

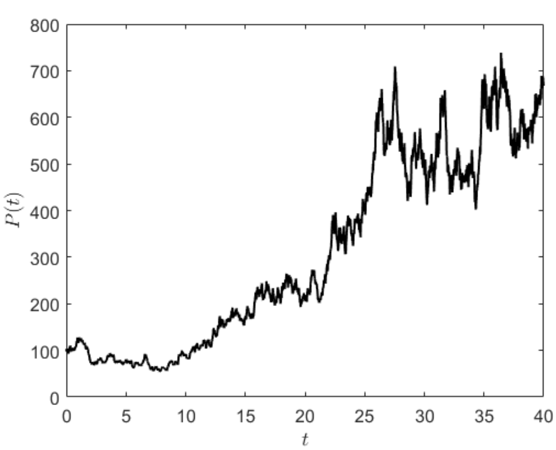
$$C^{\star}_{\alpha,B}(t) = \int_0^t e^{r(t-s)}\alpha(s)(V^{\star}_{\alpha,B}(s) - B(s))ds$$
$$= v_0\int_0^t e^{r(t-s)}\alpha(s)e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s-\int_0^s \alpha(u)du}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds - \int_0^t e^{r(t-s)}\alpha(s)B(s)ds$$
$$= v_0\int_0^t e^{r(t-s)}\alpha(s)e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s-\int_0^s \alpha(u)du}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds + \int_0^t e^{r(t-s)}y(s)ds.$$

C.4.3 Scenario generation and numerical analysis of the optimal pension fund strategy

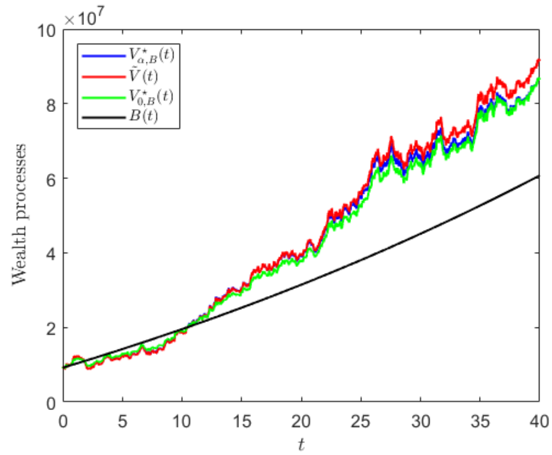
Setting. We assume the very same setting as in the numerical case study in Section 5.1.2.3; the single difference is that the smoothing parameter α is chosen to be $\alpha = 1\%$ instead of $\alpha = 40\%$. This different selection is due to the different impact and meaning of the α parameter within both buffer rules[2].

Simulation results. For completeness we provide some comparable pictures as in Section 5.1.2.3. In Figure C.7 the Value-at-Risk curves for $V^{\star}_{\alpha,B}(T)$ and $V^{\star}_{\alpha,0}(T)$, as well as $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$ and $\tilde{V}(T)$, almost coincide for $\beta \in (0, 50\%]$, but diverge for larger β values.

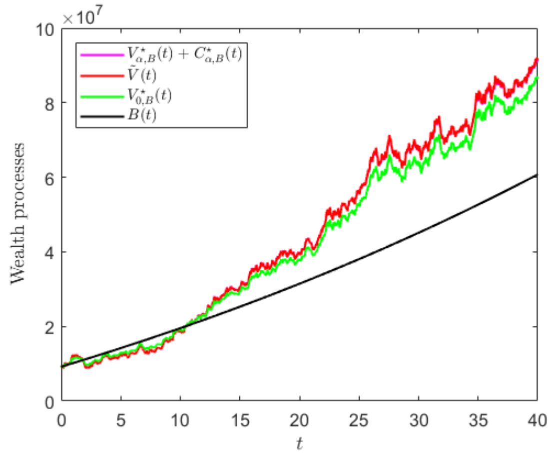
[2] With $\alpha = 1\%$ we obtain equal initial portfolio weights $\pi^{\star}_{\alpha,B}(0)$ ($\hat{\pi}^{\star}_{\alpha,B}(0) \approx 90\%$). For higher α values, the smoothing feature of the buffer might look more pronounced, but then the setting leads to a highly leveraged investment strategy particularly at the beginning of the investment period.



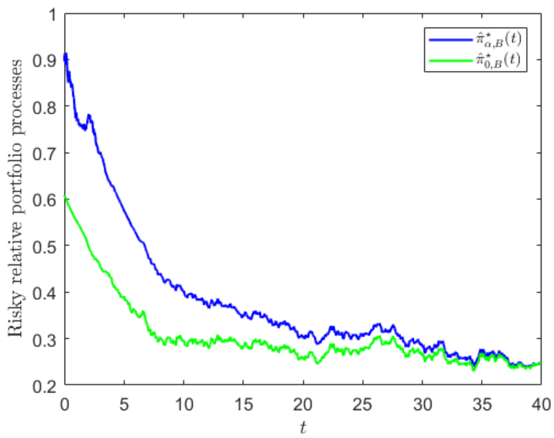
(a) Stock price $P(t)$.



(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

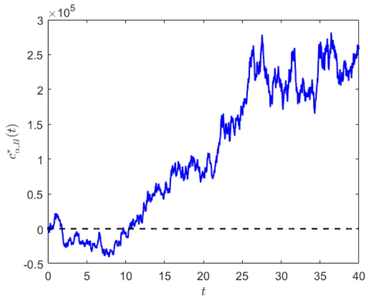


(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

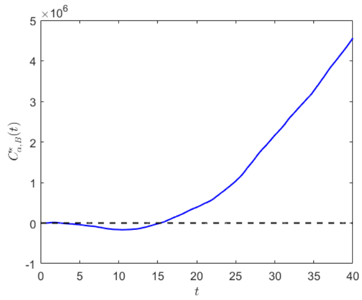


(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

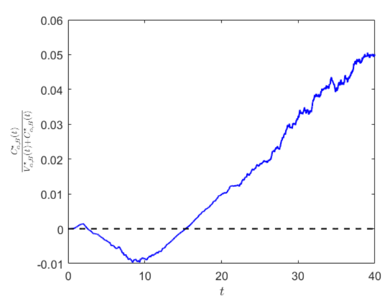
Figure C.1: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 1\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bull market.



(a) Buffer rate $c^\star_{\alpha,B}(t)$.

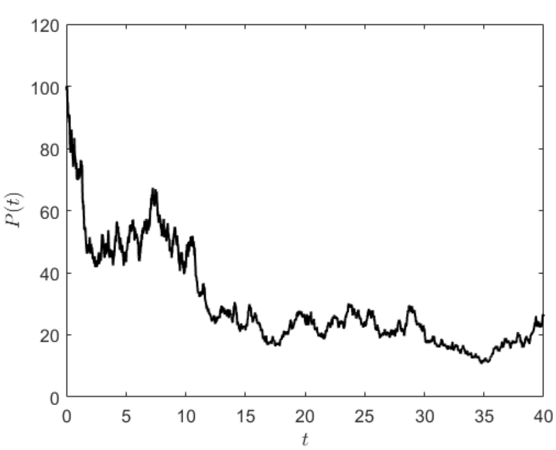


(b) Buffer account $C^\star_{\alpha,B}(t)$.

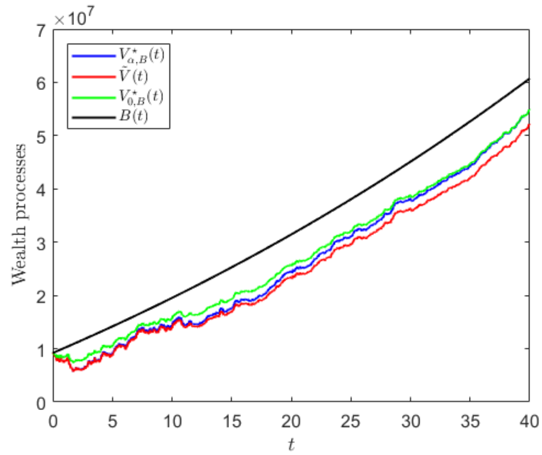


(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t)+C^\star_{\alpha,B}(t)$.

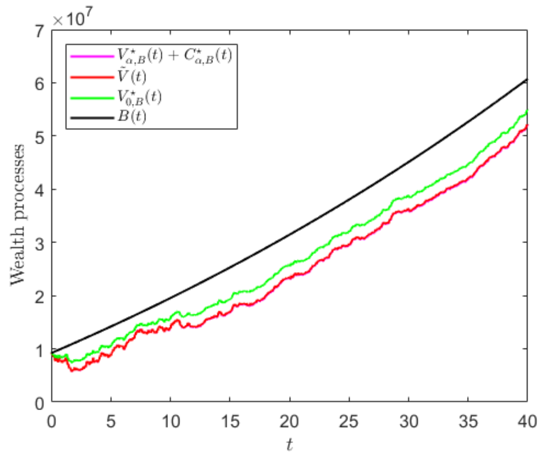
Figure C.2: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bull market.



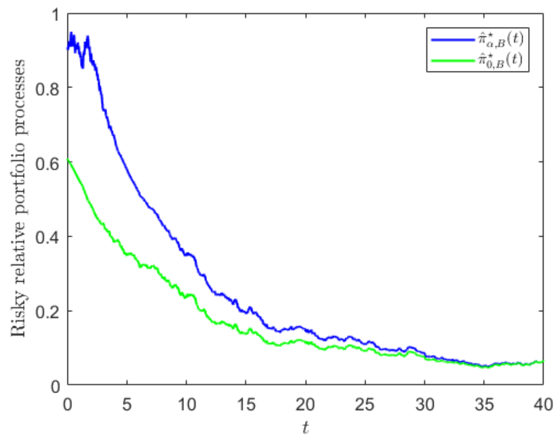
(a) Stock price $P(t)$.



(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

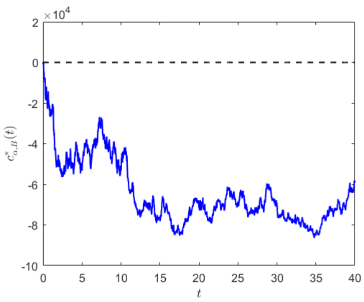


(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

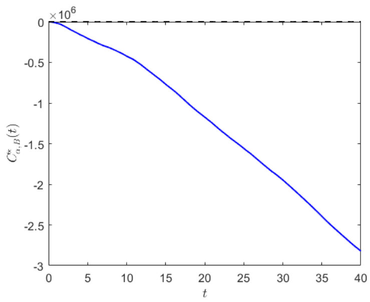


(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

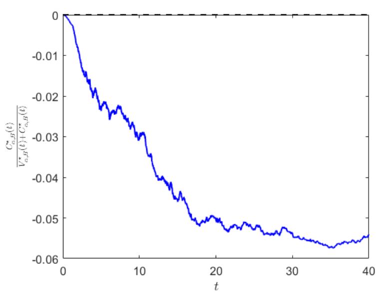
Figure C.3: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 1\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a bear market.



(a) Buffer rate $c^\star_{\alpha,B}(t)$.

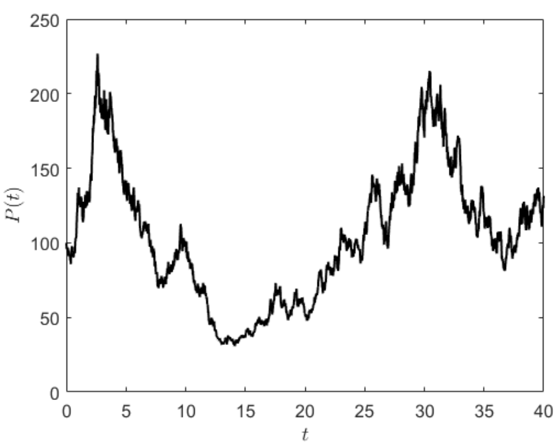


(b) Buffer account $C^\star_{\alpha,B}(t)$.

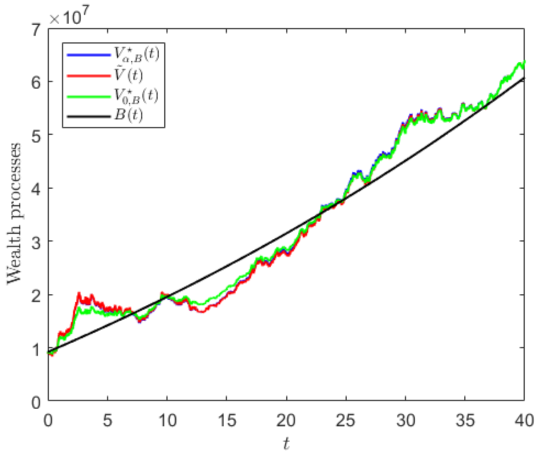


(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t)+C^\star_{\alpha,B}(t)$.

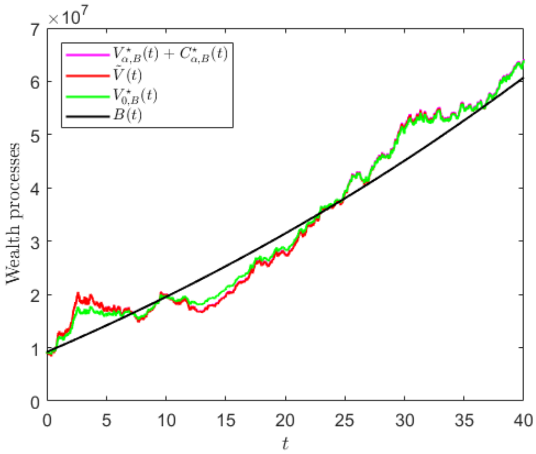
Figure C.4: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a bear market.



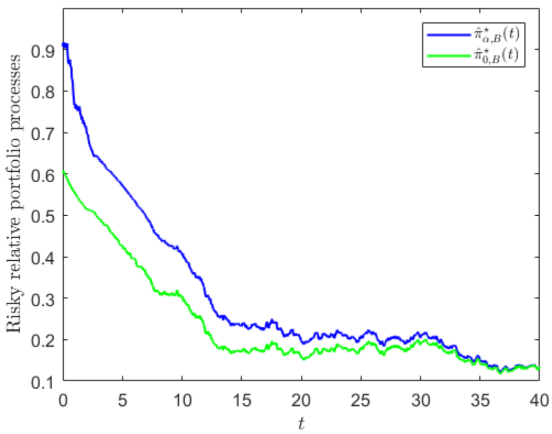
(a) Stock price $P(t)$.



(b) Reported wealths $V^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

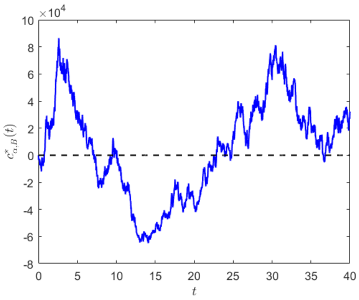


(c) Total wealths $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$, $V^\star_{0,B}(t)$, $\tilde{V}(t)$.

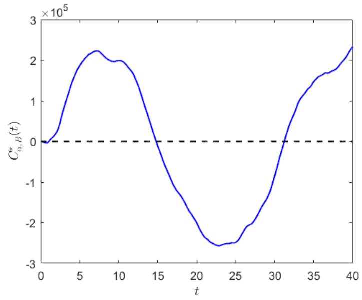


(d) Relative risky portfolio $\hat{\pi}^\star_{\alpha,B}(t)$, $\hat{\pi}^\star_{0,B}(t)$.

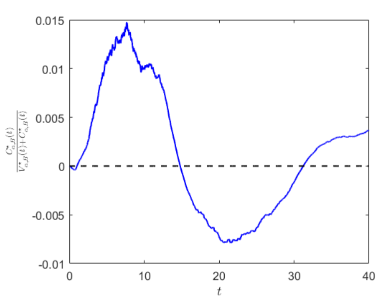
Figure C.5: Stock price process, wealth and risky relative portfolio processes for the smoothed ($\alpha = 1\%$) and unsmoothed ($\alpha = 0\%$) portfolio in a non-directional market.



(a) Buffer rate $c^\star_{\alpha,B}(t)$.

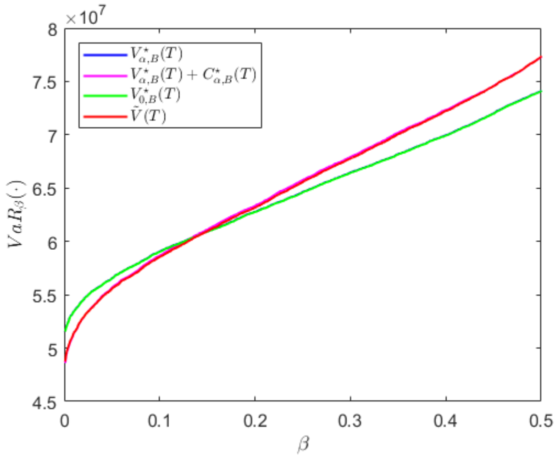


(b) Buffer account $C^\star_{\alpha,B}(t)$.



(c) Ratio between the buffer account $C^\star_{\alpha,B}(t)$ and the total wealth $V^\star_{\alpha,B}(t) + C^\star_{\alpha,B}(t)$.

Figure C.6: Buffer rate, buffer account and buffer account-to-total wealth ratio evolution in a non-directional market.



(a) $VaR_{\beta}(\cdot)$.

Figure C.7: $VaR_{\beta}(\cdot)$ vs. β for the terminal portfolio values $V^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,B}(T) + C^{\star}_{\alpha,B}(T)$, $V^{\star}_{\alpha,0}(T)$ and $\tilde{V}(T)$.

	$\mathbb{E}[\cdot]$	$Sd(\cdot)$	$SR(\cdot)$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$
$V^{\star}_{\alpha,B}(T)$	8.1182	2.5272	3.2123	5.6551	5.3458
$V^{\star}_{\alpha,B}(T)+C^{\star}_{\alpha,B}(T)$	8.5151	2.9373	2.8990	5.5655	5.1361
$V^{\star}_{0,B}(T)$	8.1160	2.5283	3.2101	5.6530	5.3440
$\tilde{V}(T)$	8.6042	3.1554	2.7268	5.5564	5.1426
	$\mathbb{P}(\cdot < 0)$	$\mathbb{E}[\cdot]$	$VaR_{0.05}(\cdot)$	$VaR_{0.01}(\cdot)$	
$C^{\star}_{\alpha,B}(T)$	16.34%	0.39693	−0.14382	−0.24864	

Table C.1: Terminal performance numbers (values $\cdot 10^7$ except for $\mathbb{P}(C^{\star}_{\alpha,B}(T) < 0)$ and $SR(\cdot)$) under the optimal and the comparative investment strategies under 10,000 simulations and annual rebalancing.

C.5 Proofs to Appendix C.4

Proof of Theorem C.7. The HJB equation for the value function $\Phi = \Phi(t, V) = \mathcal{V}$ reads

$$\begin{aligned} \Phi_t(t,V) + \max_{\pi} \Big\{ \Phi_V(t,V) \left[V \left(r - \alpha(t) + \hat{\pi}(t)' \left(\mu - r\mathbf{1} \right) \right) + \alpha(t)B(t) + y(t) \right] \\ + \frac{1}{2} \Phi_{VV}(t,V) V^2 \hat{\pi}(t)' \Sigma \hat{\pi}(t) \Big\} = 0. \end{aligned} \tag{C.23}$$

The terminal boundary condition is $\Phi(T, V) = U(V)$. The first order condition of the maximization, i.e. equating the first derivative of the maximum with respect to π to zero, leads to

$$\hat{\pi}^\star(t) = -\frac{\Phi_V(t, V(t))}{\Phi_{VV}(t, V(t))V(t)} \Sigma^{-1} \left(\mu - r\mathbf{1} \right). \tag{C.24}$$

Hence, $\alpha(t)$ has no direct impact on $\hat{\pi}^\star(t)$ but might have an indirect influence through $\Phi(t, V)$. Inserting this back in the HJB gives

$$\Phi_t(t,V) + \left[V \left(r - \alpha(t) \right) + \alpha(t)B(t) + y(t) \right] \Phi_V(t,V) - \frac{1}{2} \|\gamma\|^2 \frac{\Phi_V(t,V)^2}{\Phi_{VV}(t,V)} = 0.$$

Now recall the utility function U from (5.5):

$$U(v) = \hat{a} \frac{1 - \hat{b}}{\hat{b}} \left(\frac{1}{1 - \hat{b}} (v - F) \right)^{\hat{b}}$$

The ansatz for the value function is

$$\Phi(t,V) = h(t) \hat{a} \frac{1 - \hat{b}}{\hat{b}} \left(\frac{1}{1 - \hat{b}} (V - f(t)) \right)^{\hat{b}}, \tag{C.25}$$

with deterministic, differentiable functions $h(t)$, $f(t)$ such that $h(t) \neq 0 \ \forall t \in [0, T]$. The partial derivatives are

$$\begin{aligned} \Phi_t(t,V) &= h'(t) \hat{a} \frac{1 - \hat{b}}{\hat{b}} \left(\frac{1}{1 - \hat{b}} (V - f(t)) \right)^{\hat{b}} - h(t) \hat{a} \left(\frac{1}{1 - \hat{b}} (V - f(t)) \right)^{\hat{b}-1} f'(t), \\ \Phi_V(t,V) &= h(t) \hat{a} \left(\frac{1}{1 - \hat{b}} (V - f(t)) \right)^{\hat{b}-1}, \\ \Phi_{VV}(t,V) &= -h(t) \hat{a} \left(\frac{1}{1 - \hat{b}} (V - f(t)) \right)^{\hat{b}-2}. \end{aligned}$$

With this, in case the value function fulfills the HJB equation, the optimal portfolio composition would be

$$\hat{\pi}^\star(t) = \frac{1}{1 - \hat{b}} \frac{V(t) - f(t)}{V(t)} \Sigma^{-1} \left(\mu - r\mathbf{1} \right)$$

which is of a Constant Proportion Portfolio Insurance (CPPI) type. Notice that the boundary condition $\Phi(T, V) = U(V)$ is satisfied for all feasible V iff

$$h(T)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(T))\right)^{\hat{b}}=\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-F)\right)^{\hat{b}},$$

hence iff

$$h(T)=1,\ f(T)=F. \tag{C.26}$$

When we insert the ansatz for $\Phi(t,V)$ and its partial derivatives into the HJB equation, then it boils down to

$$\begin{aligned}
0 &\stackrel{!}{=} h'(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}-h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}f'(t)\\
&\quad+\left[V\left(r-\alpha(t)\right)+\alpha(t)B(t)+y(t)\right]h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}\\
&\quad-\frac{1}{2}\|\gamma\|^2\frac{\left(h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}\right)^2}{-h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-2}}\\
&= h'(t)\hat{a}\frac{1-\hat{b}}{\hat{b}}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}-f'(t)h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}\\
&\quad+\left[\underbrace{(r-\alpha(t))\,(1-\hat{b})\left(\frac{1}{1-\hat{b}}(V-f(t)+f(t))\right)}_{=V(r-\alpha(t))}+\alpha(t)B(t)+y(t)\right]h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}\\
&\quad+\frac{1}{2}\|\gamma\|^2h(t)\hat{a}\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}\\
&= \left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}}\hat{a}\frac{1-\hat{b}}{\hat{b}}\left\{h'(t)+\frac{1}{2}\frac{\hat{b}}{1-\hat{b}}\|\gamma\|^2h(t)+\hat{b}\left(r-\alpha(t)\right)h(t)\right\}\\
&\quad-\left(\frac{1}{1-\hat{b}}(V-f(t))\right)^{\hat{b}-1}h(t)\hat{a}\left\{f'(t)-\left(r-\alpha(t)\right)f(t)-\left(\alpha(t)B(t)+y(t)\right)\right\}.
\end{aligned}$$

As long as the HJB equation has to hold for any V, it must be

$$h'(t)=-\hat{b}\left(\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2+r-\alpha(t)\right)h(t),\ h(T)=1, \tag{C.27}$$

$$f'(t)=\left(r-\alpha(t)\right)f(t)+\left(\alpha(t)B(t)+y(t)\right),\ f(T)=F, \tag{C.28}$$

$\forall t\in[0,T]$. The solution to the ordinary differential equation (ODE) on h is given by

$$h(t)=e^{\int_t^T\hat{b}\left(\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2+r-\alpha(s)\right)ds}=e^{\hat{b}\left(\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2+r\right)(T-t)-\hat{b}\int_t^T\alpha(s)ds}. \tag{C.29}$$

The solution to the ODE on f is

$$
\begin{aligned}
f(t) &= Fe^{-\int_t^T (r-\alpha(s))ds} - \int_t^T e^{-\int_t^s (r-\alpha(u))du}(\alpha(s)B(s)+y(s))ds \\
&= Fe^{-r(T-t)+\int_t^T \alpha(s)ds} - \int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du}(\alpha(s)B(s)+y(s))ds. \qquad \text{(C.30)}
\end{aligned}
$$

This results from applying Theorem C.6 in Appendix C.3. For the optimal portfolio allocation it follows

$$
\begin{aligned}
\hat{\pi}^\star_{\alpha,B}(t) &= \frac{1}{1-\hat{b}} \frac{V(t) - \left(Fe^{-r(T-t)+\int_t^T \alpha(s)ds} - \int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du}(\alpha(s)B(s)+y(s))ds\right)}{V(t)} \Sigma^{-1}(\mu - r\mathbf{1}) \\
&= \frac{1}{1-\hat{b}} \frac{V(t) - \tilde{F}_{\alpha,B}(t)}{V(t)} \Sigma^{-1}(\mu - r\mathbf{1}), \qquad \text{(C.31)}
\end{aligned}
$$

which now is a CPPI with constant multiple and floor

$$
\tilde{F}_{\alpha,B}(t) := Fe^{-r(T-t)+\int_t^T \alpha(s)ds} - \int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du}(\alpha(s)B(s)+y(s))ds,\ \tilde{F}_{\alpha,B}(T) = F. \qquad \text{(C.32)}
$$

We now substitute $\hat{\pi}^\star_{\alpha,B}(t)$ into the dynamics of $V(t)$ above that arises from Eq. (5.1) with the specific buffer rule $c(t)$:

$$
\begin{aligned}
dV^\star_{\alpha,B}(t) &= V^\star_{\alpha,B}(t)\left[\left(r - \alpha(t) + \hat{\pi}(t)'(\mu - r\mathbf{1})\right)dt + \hat{\pi}(t)'\sigma dW(t)\right] + \alpha(t)B(t)dt + y(t)dt \\
&= V^\star_{\alpha,B}(t)\left(r - \alpha(t) + \left\{\frac{1}{1-\hat{b}} \frac{V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^\star_{\alpha,B}(t)} \Sigma^{-1}(\mu - r\mathbf{1})\right\}'(\mu - r\mathbf{1})\right)dt \\
&\quad + V^\star_{\alpha,B}(t)\left\{\frac{1}{1-\hat{b}} \frac{V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)}{V^\star_{\alpha,B}(t)} \Sigma^{-1}(\mu - r\mathbf{1})\right\}'\sigma dW(t) + \alpha(t)B(t)dt + y(t)dt \\
&= V^\star_{\alpha,B}(t)(r-\alpha(t))dt + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\underbrace{(\mu - r\mathbf{1})'\Sigma^{-1}(\mu - r\mathbf{1})}_{=\|\gamma\|^2}dt \\
&\quad + \frac{1}{1-\hat{b}}(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\underbrace{(\mu - r\mathbf{1})'\Sigma^{-1}\sigma}_{=\gamma'} dW(t) + \alpha(t)B(t)dt + y(t)dt \\
&= (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))(r-\alpha(t))dt + (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\frac{1}{1-\hat{b}}\|\gamma\|^2 dt \\
&\quad + (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\frac{1}{1-\hat{b}}\gamma' dW(t) + \alpha(t)B(t)dt + y(t)dt + \tilde{F}_{\alpha,B}(t)(r-\alpha(t))dt \\
&= (V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t))\left[\left(r - \alpha(t) + \frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt + \frac{1}{1-\hat{b}}\gamma' dW(t)\right] \\
&\quad + \left(\alpha(t)B(t) + y(t) + \tilde{F}_{\alpha,B}(t)(r-\alpha(t))\right)dt. \qquad \text{(C.33)}
\end{aligned}
$$

By applying the Leibniz integral rule we obtain

$$
\begin{aligned}
\tilde{F}'_{\alpha,B}(t) &= Fe^{-r(T-t)+\int_t^T \alpha(s)ds}(r-\alpha(t)) \\
&\qquad - (r-\alpha(t))\int_t^T e^{-r(s-t)+\int_t^s \alpha(u)du}(\alpha(s)B(s)+y(s))ds + (\alpha(t)B(t)+y(t))
\end{aligned}
$$

$$= \left(r-\alpha(t)\right)\tilde{F}_{\alpha,B}(t)+\alpha(t)B(t)+y(t). \tag{C.34}$$

Hence,

$$dV^\star_{\alpha,B}(t) = (V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\left(r-\alpha(t)+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt+\frac{1}{1-\hat{b}}\gamma' dW(t)\right]+\tilde{F}'_{\alpha,B}(t)dt. \tag{C.35}$$

Furthermore, the SDE for the cushion $V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)$ is given by

$$\begin{aligned}
d\left(V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)\right) &= dV^\star_{\alpha,B}(t)-d\tilde{F}_{\alpha,B}(t) = dV^\star_{\alpha,B}(t)-\tilde{F}'_{\alpha,B}(t)dt\\
&= (V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\left(r-\alpha(t)+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt+\frac{1}{1-\hat{b}}\gamma' dW(t)\right]\\
&\quad+\left(\alpha(t)B(t)+y(t)+\tilde{F}_{\alpha,B}(t)\left(r-\alpha(t)\right)\right)dt\\
&\quad-\left[\left(r-\alpha(t)\right)\tilde{F}_{\alpha,B}(t)+\alpha(t)B(t)+y(t)\right]dt\\
&= (V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t))\left[\left(r-\alpha(t)+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)dt+\frac{1}{1-\hat{b}}\gamma' dW(t)\right].
\end{aligned} \tag{C.36}$$

The formula shows that $V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t)$ follows a geometric Brownian motion with

$$V^\star_{\alpha,B}(t)-\tilde{F}_{\alpha,B}(t) = (v_0-\tilde{F}_{\alpha,B}(0))e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t-\int_0^t\alpha(s)ds+\frac{1}{1-\hat{b}}\gamma' W(t)}$$

under $\mathbb{P}$, consequently

$$\begin{aligned}
V^\star_{\alpha,B}(t) &= \tilde{F}_{\alpha,B}(t)+(v_0-\tilde{F}_{\alpha,B}(0))e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)t-\int_0^t\alpha(s)ds+\frac{1}{1-\hat{b}}\gamma' W(t)}\\
&= \tilde{F}_{\alpha,B}(t)+(v_0-\tilde{F}_{\alpha,B}(0))e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t-\int_0^t\alpha(s)ds}\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}.
\end{aligned} \tag{C.37}$$

Finally, using above results we can calculate

$$\begin{aligned}
C^\star_{\alpha,B}(t) &= \int_0^t c(s)e^{r(t-s)}ds \overset{(C.22)}{=} \int_0^t e^{r(t-s)}\alpha(s)(V^\star_{\alpha,B}(s)-B(s))ds\\
&\overset{(C.37)}{=} \int_0^t e^{r(t-s)}\alpha(s)\\
&\qquad\times\left((v_0-\tilde{F}_{\alpha,B}(0))e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s-\int_0^s\alpha(u)du+\frac{1}{1-\hat{b}}\gamma' W(s)}+\tilde{F}_{\alpha,B}(s)-B(s)\right)ds\\
&= (v_0-\tilde{F}_{\alpha,B}(0))\int_0^t e^{r(t-s)}\alpha(s)e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2-\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s-\int_0^s\alpha(u)du+\frac{1}{1-\hat{b}}\gamma' W(s)}ds\\
&\quad+\int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s)-B(s)\right)ds\\
&= (v_0-\tilde{F}_{\alpha,B}(0))\int_0^t e^{r(t-s)}\alpha(s)e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s-\int_0^s\alpha(u)du}\tilde{Z}(s)^{-\frac{1}{1-\hat{b}}}ds\\
&\quad+\int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s)-B(s)\right)ds
\end{aligned}$$

$$= (v_0 - \tilde{F}_{\alpha,B}(0)) \int_0^t e^{r(t-s)} \alpha(s) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} \tilde{Z}(s)^{-\frac{1}{1-\hat{b}}} ds + \int_0^t e^{r(t-s)} \alpha(s)$$
$$\times \left(F e^{-r(T-s)+\int_s^T \alpha(u)du} - \int_s^T e^{-r(u-s)+\int_s^u \alpha(v)dv} (\alpha(u)B(u) + y(u))du - B(s) \right) ds.$$

□

Proof of Theorem C.8.

- Expected fund wealth:

  From Theorem C.7 it follows

$$\begin{aligned}
\mathbb{E}\left[V^\star_{\alpha,B}(t)\right] &= \mathbb{E}\left[\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right] \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} \mathbb{E}\left[\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right] \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} e^{\frac{1}{1-\hat{b}}\left(r+\frac{1}{2}\|\gamma\|^2\right)t + \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} \\
&= \tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{rt + \frac{1}{1-\hat{b}}\|\gamma\|^2 t - \int_0^t \alpha(s)ds}.
\end{aligned}$$

  Here we used that $\tilde{Z}(t)^\eta = e^{-\eta\left(r+\frac{1}{2}\|\gamma\|^2\right)t - \eta\gamma' W(t)}$, $\eta \in \mathbb{R}$, is log-normally distributed with mean $-\eta\left(r + \frac{1}{2}\|\gamma\|^2\right)t$ and variance $\eta^2\|\gamma\|^2 t$, and that $\mathbb{E}\left[e^Z\right] = e^{\mu_Z + \frac{1}{2}\sigma_Z^2}$ for a normally distributed random variable $Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$.

- Variance of the fund wealth:

  Theorem C.7 implies

$$\begin{aligned}
Var\left(V^\star_{\alpha,B}(t)\right) &= Var\left(\tilde{F}_{\alpha,B}(t) + (v_0 - \tilde{F}_{\alpha,B}(0)) e^{-\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - \int_0^t \alpha(s)ds} \tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{-2\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - 2\int_0^t \alpha(s)ds} Var\left(\tilde{Z}(t)^{-\frac{1}{1-\hat{b}}}\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{-2\frac{\hat{b}}{1-\hat{b}}\left(r+\frac{1}{2}\frac{1}{1-\hat{b}}\|\gamma\|^2\right)t - 2\int_0^t \alpha(s)ds} e^{\frac{2}{1-\hat{b}}\left(r+\frac{1}{2}\|\gamma\|^2\right)t + \left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} \\
&\quad \times \left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right) \\
&= (v_0 - \tilde{F}_{\alpha,B}(0))^2 e^{2\left(rt + \frac{1}{1-\hat{b}}\|\gamma\|^2 t - \int_0^t \alpha(s)ds\right)} \left(e^{\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 t} - 1\right),
\end{aligned}$$

  where we used that $Var\left(e^Z\right) = e^{2\mu_Z + \sigma_Z^2}\left(e^{\sigma_Z^2} - 1\right)$ for a normally distributed random variable $Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$.

- Value-at-Risk/Quantiles of the fund wealth distribution with level $\beta \in [0, 1]$:

  From Theorem C.7 we know that $V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t) > 0$ is log-normally distributed with mean $\mathbb{E}\left[V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right] = \mathbb{E}\left[V^\star_{\alpha,B}(t)\right] - \tilde{F}_{\alpha,B}(t)$ and variance $Var\left(V^\star_{\alpha,B}(t) - \tilde{F}_{\alpha,B}(t)\right) = Var\left(V^\star_{\alpha,B}(t)\right)$; for $\mathbb{E}\left[V^\star_{\alpha,B}(t)\right]$ and $Var\left(V^\star_{\alpha,B}(t)\right)$ see the formulas above. The rest follows from the Value-at-Risk proof in Theorem 5.2. The difference lies in the different values for $\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]$, $Var\left(V^{\star}_{\alpha,B}(t)\right)$ and $\tilde{F}_{\alpha,B}(t)$.

- Shortfall probability of the fund wealth with threshold $s > \tilde{F}_{\alpha,B}(t)$:

  The shortfall probability was already calculated in the proof of the Value-at-Risk formula, see proof of Theorem 5.2, but with different $\mathbb{E}\left[V^{\star}_{\alpha,B}(t)\right]$ and $Var\left(V^{\star}_{\alpha,B}(t)\right)$ values to be inserted into the formula.

- Expected accumulated buffer account:

  In view of Theorem C.7, the expected value of the collected buffer equals

$$
\begin{aligned}
\mathbb{E}\left[C^{\star}_{\alpha,B}(t)\right] &= \mathbb{E}\left[(v_0 - \tilde{F}_{\alpha,B}(0)) \int_0^t e^{r(t-s)}\alpha(s) e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s - \int_0^s \alpha(u)du + \frac{1}{1-\hat{b}}\gamma' W(s)} ds\right] \\
&\quad + \mathbb{E}\left[\int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds\right] \\
&= \int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds + (v_0 - \tilde{F}_{\alpha,B}(0)) \\
&\quad \times \int_0^t e^{r(t-s)}\alpha(s) e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} \mathbb{E}\left[e^{\frac{1}{1-\hat{b}}\gamma' W(s)}\right] ds \\
&= \int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds + (v_0 - \tilde{F}_{\alpha,B}(0)) \\
&\quad \times \int_0^t e^{r(t-s)}\alpha(s) e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2 - \frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} e^{\frac{1}{2}\left(\frac{1}{1-\hat{b}}\right)^2\|\gamma\|^2 s} ds \\
&= \int_0^t e^{r(t-s)}\alpha(s)\left(\tilde{F}_{\alpha,B}(s) - B(s)\right) ds \\
&\quad + (v_0 - \tilde{F}_{\alpha,B}(0)) \int_0^t e^{r(t-s)}\alpha(s) e^{\left(r+\frac{1}{1-\hat{b}}\|\gamma\|^2\right)s - \int_0^s \alpha(u)du} ds,
\end{aligned}
$$

  where we used that $\gamma' W(s) \sim \mathcal{N}\left(0, \|\gamma\|^2 s\right)$.

□

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