TECHNISCHE UNIVERSITÄT MÜNCHEN
FAKULTÄT FÜR MATHEMATIK
LEHRSTUHL FÜR FINANZMATHEMATIK

Optimal Investment Strategies for Asset Liability Management

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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Die Dissertation wurde am 27.07.2020 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 05.10.2020 angenommen.

Abstract

Developments such as low interest rates, high volatility and risk-based regulatory regimes have increased the need for insurance companies and pension funds to manage their assets and liabilities integratively (Asset Liability Management). An important part is the choice of an adequate investment strategy which is favorable with respect to risk and return, but which also ensures that future liabilities can be fulfilled. Two major aspects which have to be considered in the portfolio optimization process are the inclusion of stochastic liabilities and the presence of constraints through regulatory regimes such as Solvency II in Europe. For the optimization with stochastic liabilities, we develop two portfolio optimization frameworks which take stochastic liabilities into account in different ways and derive optimal investment strategies in closed form. First, we maximize the terminal funding ratio and derive optimal investment strategies for initially well-funded and underfunded investors in a Cumulative Prospect Theory framework. Second, we establish a surplus optimization framework using a generalized martingale approach. The liabilities in this framework may be subject to index- or performance participation and may include unhedgeable risks. For the optimization with regulatory constraints, we consider risk constraints, in particular Solvency II-type constraints, which jointly depend on wealth and the investment strategy. We approximate the optimal constrained investment strategy through an iterative two-step approach for an investor with power utility. For general wealth-dependent risk constraints and general utility function, we show that the optimization problem can, under certain conditions, be reduced to an associate problem with a different utility function and constraints independent of wealth. The associate problem can be solved using known duality results. Following this approach, we derive optimal investment strategies in closed form for an investor with HARA utility. Within numerical studies, we illustrate the economic impact of various types of stochastic liabilities and the Solvency II constraints.

Zusammenfassung

Entwicklungen wie die Niedrigzinsphase, hohe Marktvolatilität und risikobasierte Regulierung haben dazu geführt, dass Versicherungen und Pensionsfonds ihre Aktiva und Verbindlichkeiten integriert steuern müssen (Asset Liability Management). Ein wichtiger Bestandteil dieser Steuerung ist die Wahl einer geeigneten Kapitalanlagestrategie, die einerseits im Hinblick auf das Risiko und die Rendite günstig ist, die aber andererseits sicherstellt, dass die Verbindlichkeiten erfüllt werden können. Zwei bedeutende Aspekte, die bei der Portfoliooptimierung beachtet werden müssen, sind die Berücksichtigung stochastischer Verbindlichkeiten und Nebenbedingungen durch regulatorische Vorschriften, wie Solvency II in Europa. Zur Portfoliooptimierung mit stochastischen Verbindlichkeiten entwickeln wir zwei Verfahren, in denen stochastische Verbindlichkeiten in unterschiedlicher Weise berücksichtigt werden und bei denen wir geschlossene Formeln für die optimalen Kapitalanlagestrategien erhalten. Zuerst maximieren wir die Finanzierungsquote und leiten optimale Strategien für ausreichend finanzierte und unterfinanzierte Investoren im Kontext der Cumulative Prospect Theory ab. Anschließend entwickeln wir ein Verfahren zur Optimierung des Überschusses durch Verwendung eines verallgemeinerten Martingalansatzes. Die Verbindlichkeiten können in diesem Verfahren an der Entwicklung eines Indexes oder den Kapitalanlagen selbst partizipieren und nicht replizierbare Risiken enthalten. Zur Optimierung mit regulatorischen Nebenbedingungen betrachten wir risikobasierte Nebenbedingungen, insbesondere wie bei Solvency II, die gleichzeitig vom Vermögen und von der Kapitalanlagestrategie abhängen. Wir approximieren die optimale Anlagestrategie durch ein iteratives Verfahren in zwei Schritten für einen Investor mit isoelastischer Nutzenfunktion. Für allgemeine risikobasierte Nebenbedingungen, die auch vom Vermögen abhängen und allgemeine Nutzenfunktion zeigen wir, dass das Optimierungsproblem unter geeigneten Voraussetzungen auf ein verwandtes Optimierungsproblem mit Nebenbedingungen, die nicht vom Vermögen abhängen und einer anderen Nutzenfunktion, zurückgeführt werden kann. Das verwandte Problem kann mit bekannten Dualitätsmethoden gelöst werden. Mit diesem Ansatz ermitteln wir geschlossene Lösungen für die optimale Kapitalanlagestrategien für einen Investor mit HARA-Nutzenfunktion. Mit numerischen Anwendungen veranschaulichen wir den ökonomischen Einfluss der stochastichen Verbindlichkeiten und Solvency II-Nebenbedingungen.

Acknowledgements

First of all, I like to express my gratitude to my supervisor Professor Rudi Zagst for the opportunity to work on my doctoral project at the Chair of Mathematical Finance. I am deeply thankful for his guidance and patience throughout my research studies and his perseverance in the discussion of problems for which a path forward was not at all visible. I am thankful to Professor Marcos Escobar for his dedication and especially for always being receptive to questions and ideas.

Moreover, I like to thank my coauthors Professor Sascha Desmettre, Ludwig Brummer and Paul Kriebel for the successful cooperation.

I am grateful for the support of Allianz Global Investors and for interesting and helpful discussions with Dr. Gerhard Scheuenstuhl.

Furthermore, I want to thank Professor Rudi Zagst, Professor Matthias Scherer and PD Dr. Aleksey Min for creating such a pleasant working atmosphere at M13 and for having an open ear on all questions on research, teaching and beyond. I am grateful to Dr. Oliver Schlick for his advice and for sharing his experience, and to my colleagues for many discussions on mathematics and much more.

Finally, I am thankful to my wife, my parents and my sister for their love, support and encouragement. Throughout my doctoral studies, they were the first ones to notice the *constraints* of being focused on an unsolved problem and the last ones to experience the *utility* of the joy emerging from a solved problem.

Contents

1 Introduction

1.1 Motivation and Literature Overview

Low interest rates, high market volatility and the shift towards risk-based regulatory regimes such as Solvency II in Europe represent major developments, which increase the need for insurance companies and pension funds to manage their assets and liabilities integratively (Asset Liability Management). Besides balancing the expected return and the risk according to their risk aversion and constructing portfolios such that diversification effects can be used best, insurance companies have to choose their investment strategy such that they have enough capital to meet their future liabilities, i.e. payments to the policy holders. The present value of the liabilities is stochastic as it may depend on, e.g., the level of interest rates, the value of other financial instruments to which the insurance policies may be linked, or the performance of the insurance company's asset portfolio. As a risk-based regulatory regime, the Solvency II framework for European insurance companies aims at protecting policy holders by obliging insurance companies to manage their risk in a way such that the yearly 99.5%-Value-at-Risk does not exceed the insurance company's own funds. As assets contribute to an important part to the risk of many insurance companies, these rules provide restrictions on the investment strategy. In this thesis, we present results for the following two areas of dynamic portfolio optimization in continuous time:

- Liability Driven Investment Strategies (LDI): portfolio optimization, which takes stochastic liabilities into account.
- Portfolio optimization with risk constraints which jointly depend on the investment strategy and wealth, especially Solvency II constraints.

Both areas, LDI and regulatory constraints, aim at ensuring that the insurance companies and pension funds can meet their liabilities. The LDI frameworks are based on the insurance company's own interest to stay solvent and on accounting and regulatory rules for determining the present value of future liabilities. The regulatory constraints represent externally imposed constraints by supervising authorities which force the management of insurance companies to limit their risks in the interest of the policy holder. With the recent international accounting standards IFRS (especially IAS 19) for pension funds and the European regulatory requirements Solvency II for insurance companies, the importance of investment strategies that are adapted to stochastic liabilities and regulatory constraints, has increased.

1.1.1 Literature on Liability Driven Investment Strategies

Whereas the optimization of the utility of terminal wealth and consumption over time, firstly presented in Merton (1969), is a widely accepted standard in the literature on intertemporal portfolio choice, liabilities were included in portfolio optimization frameworks in different ways. An extension of the mean-variance approach for a one-period setting with static investment strategies is discussed by Sharpe and Tint (1990), who consider the return of the (partial) terminal surplus (defined as the difference between assets and liabilities). Inspired by the approach in Sharpe and Tint (1990), Rudolf and Ziemba (2004) provide a continuous-time surplus optimization approach, but they consider a life-time surplus instead of the terminal surplus. Furthermore, Detemple and Rindisbacher (2008) directly aim at transforming the surplus optimization from Sharpe and Tint (1990) to a setting in continuous time with a maximization of the utility from the excess of liquid wealth over a minimum liability coverage. However, they focus on the inclusion of various stochastic factors and numerical results obtained from simulations. Ang et al. (2013) consider the downside risk inherent in the liabilities through an exchange option. In several generalizations of CPPI strategies, such as in Amenc et al. (2004), Kraus et al. (2011) and Bahaji (2014), the stochastic floor can be interpreted as stochastic liabilities as well. Martellini (2006) maximizes the expected utility of the terminal funding ratio (defined as the quotient of assets and liabilities). The author derives optimal investment strategies in a framework in which assets and liabilities are modeled as geometric Brownian motions. In spite of the increased need for LDI strategies, the literature overview illustrates that no scientific standard exists how liabilities should be included in the portfolio optimization. The literature on portfolio optimization with stochastic liabilities is intertwined with the literature on index-linked and performance-linked products, especially as insurance products which provide less guarantees and more performance participation to policy holders (for a detailed description of various types of products, see Korn and Wagner (2018)) become more common. The impact of different surplus distribution mechanisms on the risk exposure of insurance companies which sell performance-participating life insurance contracts is analyzed in Kling et al. (2009). As we further develop various existing techniques, our work is also related to the literature on these methods, which we adapt to LDI settings. In particular, these include the quantile approach for portfolio optimization in a Cumulative Prospect Theory (CPT) framework as intrduced in Jin and Zhou (2008) as well as the optimization with random utility functions and with random endowment. Random utility functions are used to solve portfolio optimization problems with a positive lower bound on the terminal wealth (see Korn (2005)), taking deferred capital gains taxes into account (see Seifried (2010)) or within an optimization, in which the portfolio may include a liquid and an illiquid risky asset as in Desmettre and Seifried (2016). In Hugonnier and Kramkov (2004) and Hugonnier et al. (2005), expected utility maximization problems with a random endowment at maturity are considered.

1.1.2 Literature on Constrained Portfolio Optimization

As introduced before, risk-based regulatory regimes impose constraints on the investment strategies. However, these constraints also depend on the own funds of insurance companies and therefore especially on the value of the asset portfolio itself. Consequently, insurance companies face the problem of how to optimize their investment strategy under these types of constraints. In the mathematical framework, such constraints can be represented by joint continuous-time restrictions on the investment strategy and on the wealth process. There is a significant literature devoted to continuous-time constraints solely on the investment strategy, see Cvitanić and Karatzas (1992), Cuoco (1997) and Lim and Choi (2009). Cvitanić and Karatzas (1992) present general results in form of a convex duality theory as well as applications with logarithmic utility and power utility. They work with constant constraint sets on the investment strategy and state briefly that the theory can be extended to random constraint sets in general. However, they do not cover the challenges of constraints depending on wealth which itself is a function of the investment strategy.
There are also many relevant papers that consider continuous-time constraints solely on wealth or funding ratio, see Korn and Trautmann (1995) and Korn (2005). Kraft and Steffensen (2013) study shortfall constraints on the terminal wealth. Examples of settings for insurance companies or pension funds include Detemple and Rindisbacher (2008) and Martellini and Milhau (2012). In Detemple and Rindisbacher (2008), a framework for the optimization of the excess coverage over a stochastic floor is developed. Martellini and Milhau (2012) consider the utility of the funding ratio and find optimal investment strategies with a lower bound on the terminal funding ratio as a constraint. More recently, Chen et al. (2019) solve a non-concave utility maximization problem with a fair pricing constraint, which is essentially a constraint on terminal wealth, too.
Literature on joint continuous-time constraints for the investment strategy and wealth is very rare. Portfolio optimization with joint constraints is considered in Zariphopoulou (1994) in a model with one risky asset. The author uses an approach based on viscosity solutions and obtains a solution for the optimal investment strategy in feedback form which depends on the value function. However, the value function is not stated explicitly. Moreno-Bromberg and Pirvu (2013) rely on BSDE and numerical calculations to tackle dynamic constraints of general type with no closed-form solutions. In Colwell et al. (2015), the authors deal with wealth-dependent constraints in the context of executive stock option pricing.

With respect to Solvency II, most of the literature focuses on the implementation and possible shortcomings of the framework (see, e.g., Eling et al. (2007) for an overview on the Solvency II framework, Gatzert and Wesker (2012) for a comparison to the Basel III framework for banks as well as Sandström (2007), Pfeifer and Strassburger (2008), Bauer et al. (2010) and Christiansen and Niemeyer (2014) for an analysis of the structure of capital requirements imposed by Solvency II). Closer to our objectives, Höring (2012) compares the market risk capital requirements of Solvency II with the Standard & Poor's rating model and analyzes the impact on asset allocation. Considering optimal investment strategies under Solvency II, Braun et al. (2015) investigate static efficient frontiers for the asset allocation under Solvency II-type constraints. A static optimization problem under Solvency II-type constraints is also considered in Kouwenberg (2017). The impact of the calibration of the equity risk module on the investment strategy is studied in Fischer and Schlütter (2015). Chen and Hieber (2016) study, in the context of constant strategies and continuous Value-at-Risk constraints on wealth, how negative effects of the regulation on the asset allocation can be overcome by proposing an alternative regulatory approach. In a continuous-time framework with discrete-time Value-at-Risk constraints, Shi and Werker (2012) explicitly refer to Solvency II in the context of short-term regulation for long-term investors.

1.2 Summary of the Results and Contributions to the Literature

This thesis is based on four research projects, which led to the following publications:

- Brummer, L., Wahl, M. and Zagst, R.: Liability Driven Investments with a Link to Behavioral Finance, Proceedings of the Innovations in Insurance, Risk- and Asset Management Conference, p. 275-311, World Scientific, 2018[1]
- Escobar, M., Kriebel, P., Wahl, M. and Zagst, R.: Portfolio Optimization under Solvency II, Annals of Operations Research, S.I.: Risk in Financial Economics, Issue 281, p. 193–227, 2019
- Escobar, M., Wahl, M. and Zagst, R.: Portfolio Optimization with Wealth-Dependent Risk Constraints, submitted to the European Journal of Operational Research, revise and resubmit, 2020[1]
- Desmettre, S., Wahl, M. and Zagst, R.: Dynamic Surplus Optimization with Performance- and Index-Linked Liabilities, submitted to Insurance: Mathematics and Economics, under review, https://ssrn.com/abstract=3592323, 2020[1]

Parts of this thesis are identical with or a reproduction with minor changes of these articles.

The following article was part of the doctoral research of the author, but is not part of this thesis:

- Engel, J., Wahl, M. and Zagst, R.: Forecasting Turbulence in the Asian and European Stock Market Using Regime-Switching Models. Quantitative Finance and Economics, Vol. 2, Issue 2, p. 388-406, 2018

[1]The author of this thesis is the leading author of the article.

1.3 Structure of the Thesis

The thesis is structured in the following way: Mathematical preliminaries, in particular the introduction of the market model, a presentation of some traditional utility functions and well-known portfolio optimization methods in continuous time are provided in Chapter 2.
In Chapter 3, we derive optimal investment strategies for funding ratio optimization in an expected utility and a CPT framework and compare them. Our results extend the expected utility funding ratio optimization approach from Martellini (2006) to a model with a CPT utility and distortion function. In detail, our contributions include:

- We embed funding ratio optimization in a CPT framework in a setting in continuous time.
- Within the CPT framework, we introduce an alternative distortion function, given by a modification of the Wang-distortion. This distortion is for certain parameter choices reverse S-shaped despite having only two parameters.
- For the CPT distortion functions, we provide an alternative interpretation, which makes them usable for modeling heavy-tailed returns.
- As the CPT utility function enhances the standard utility function with respect to risk-seeking behavior for funding levels below the reference point, we also contribute to the literature on underfunded pension plans.

Chapter 4 deals with the dynamic surplus optimization framework. We generalize Sharpe and Tint (1990) to a setting in continuous time. Furthermore, the consideration of liabilities generalizes parts of Desmettre and Seifried (2016) to a short position in the illiquid asset. We also provide closed-form solutions for the investment in the (liquid) asset, which extends Rudolf and Ziemba (2004) and Desmettre and Seifried (2016). In particular:

- We establish a general and flexible terminal surplus optimization framework in continuous time, which allows for dynamic investment strategies and stochastic liabilities.
- Within this framework, we derive closed-form solutions of the terminal wealth and the optimal investment strategy for various specific liability models, which may include unhedgeable risks and which may also be linked to the performance of an index or the wealth of the insurance company.
- For the different liability models, we study the impact on the optimal investment strategy and we derive implications for insurance product design. Hence, we also establish a link between the literature on portfolio optimization with stochastic liabilities and the literature on insurance product design.

- For the surplus and funding ratio optimization, we compare the optimal investment strategies.

In Chapter 5, we introduce the general constrained optimization framework, which includes the case of wealth-dependent constraints and the corresponding auxiliary markets and therefore extends the setting in Cvitanić and Karatzas (1992). For constraints independent of wealth, we state the optimal solution for an investor with logarithmic or power utility function and use these results to construct the iterative two-step approach and as an associate problem in Chapter 6. With the iterative two-step approach, we contribute to the literature in the following way:

- We transfer the limitations emerging from the Solvency II regulation to a convex constraint set for the investment strategy.
- In a continuous-time optimization framework which allows for dynamic investment strategies, we establish a two-step approach for an investor with power utility who faces Solvency II-type constraints. This approach can be applied iteratively to approximate the optimal constrained investment strategy. In a numerical study, we analyze the impact of the Solvency II constraints on the optimal allocation for several asset classes.
- For the iterative application of the two-step approach, we illustrate that the investment strategy converges numerically to the optimal constrained investment strategy obtained with a Bellman approach on a discrete grid if the intervals of the iterations converge to zero.

The results extend a portfolio optimization framework under Solvency II constraints in a one-period model in Braun et al. (2015) to a framework in continuous time. Whereas the two-step approach is an approximation for an investor with power utility, Chapter 6 deals with exact solutions:

- We develop a solution apporach for a rather general portfolio optimization problem in continuous time with wealth-dependent constraint set by showing that this problem can, under sufficient conditions, be reduced to an associate problem with a different utility function and a constraint set independent of wealth. This associate problem can then be solved by known convex duality methods in closed form.
- As an application of this solution approach, we provide closed-form solutions for the optimal investment strategy and optimal terminal wealth for an investor with Solvency II-type constraints and HARA utility.
- For the Solvency II constraint set and an investor with HARA utility, we analyze the impact of the constraints on the optimal investment strategy. We analyze the loss in utility and the reduction in risk caused by the constraints and we examine the trade-off between these two.

These results extend the applicability of the convex duality results from Cvitanić and Karatzas (1992) to the case of wealth-dependent constraint sets and Zariphopoulou (1994), where the optimal investment strategy under a simple wealth-dependent constraint set is expressed only in terms of the value function. We conclude in Chapter 7.

2 Mathematical Preliminaries

2.1 Market Model

To model the assets and liabilities, we use a probability space $(\Omega, \mathcal{H}, \mathbb{Q})$, with $\mathcal{H} = \mathcal{F} \vee \mathcal{G}$ and $\mathcal{F}$ and $\mathcal{G}$ are independent σ-algebras. We further assume that $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration in $\mathcal{F}$, generated by a d-dimensional standard Brownian motion $W = (W(t))_{t\in[0,T]}$, $W(t) = (W_1(t), ..., W_d(t))^T,\ t \in [0,T]$, where $T > 0$ denotes the time horizon for the investment. W is used to model the risky assets and hedgeable liability risks, whereas $\mathcal{G}$-measurable random variables are used to model the unhedgeable risks.

The introduction of the subsequent framework for the financial market, the wealth process and the investment strategy is adapted from Korn (1999), but we do not consider the possibility of consumption. By $\mathcal{M}$, we denote a financial market including one risk-free bond and d risky assets with price processes $(P_i(t))_{t\in[0,T]}$, $i = 0, ..., d$ given by

$$dP_0(t) = P_0(t)r(t)dt,$$

$$dP_i(t) = P_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right),\ P_i(0) = 1,\ i = 0, ..., d,$$

with deterministic market coefficients $r(t)$, $\mu(t)$, $\sigma(t)$, $t \in [0,T]$. We assume that $r(t)$ is non-negative, $\mu(t) \geq r(t)$ and that $r(t)$, $\mu(t)$ and $\sigma(t)$ are bounded on $[0,T]$. Furthermore, we assume that there exists a constant $\delta > 0$ such that for all $t \in [0,T]$

$$x^T\sigma(t)\sigma(t)^T x \geq \delta\|x\|^2.$$

In particular, this condition ensures that $\sigma(t)\sigma(t)^T$ is positive definite, $(\sigma(t)\sigma(t)^T)^{-1}$ is bounded and $\sigma(t)$ is invertible, so the financial market $\mathcal{M}$ is complete. The pricing kernel is given by

$$d\tilde{Z}(t) = -\tilde{Z}(t)\left(r(t)dt + \gamma(t)^T dW(t)\right),\ \tilde{Z}(0) = 1$$

and has the explicit representation

$$\tilde{Z}(t) = \exp\left(-\int_0^t r(s) + \frac{1}{2}\|\gamma(s)\|^2 ds - \int_0^t \gamma(s)^T dW(s)\right),$$

with the market price of risk

$$\gamma(t) := \sigma(t)^{-1}(\mu(t) - r(t)\mathbf{1}),$$

$\mathbf{1} := (1, ..., 1)^T \in \mathbb{R}^d$. With the above assumptions on the market coefficients, $\mathbb{E}[\tilde{Z}(T)] < \infty$. We have for the uniquely determined risk-neutral measure (see Bingham and Kiesel (2004), Chapter 6)

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\bigg|\mathcal{F}_t = P_0(t)\tilde{Z}(t)$$

and write

$$\tilde{Z}(t,T) := \frac{\tilde{Z}(T)}{\tilde{Z}(t)}.$$

We consider $\mathbb{F}$-adapted investment strategies $\pi = (\pi_1(t), ..., \pi_d(t))_{t\in[0,T]}$, with $\pi_i(t)$ denoting the fraction of wealth invested in asset $i \in \{1, ..., d\}$, $1 - \sum_{i=1}^d \pi_i(t)$ being the fraction of wealth invested in the risk-free asset and the corresponding wealth processes $(V^\pi(t))_{t\in[0,T]}$ with initial wealth by $v_0 > 0$. To simplify the notation, we sometimes write $V(t)$ for $V^\pi(t)$. The wealth process evolves according to

$$dV^\pi(t) = V^\pi(t)\left[\pi^T(t)\left(\mu(t)dt + \sigma(t)dW(t)\right) + (1 - \pi^T(t)\mathbf{1})r(t)dt\right]. \tag{2.1}$$

Definition 2.1.1 (Admissible Investment Strategies (see Korn (1999), p. 24-25))**.** *An investment strategy π such that* (2.1) *with initial wealth v_0 has a unique solution $V(t)$ satisfying*

$$\int_0^T \|\pi(t)V(t)\|^2 dt < \infty \ a.s.$$

and

$$V(t) \geq 0 \ a.s. \ for \ all \ t \in [0,T]$$

is called admissible. By $\Lambda(t,v)$, we denote the set of all admissible investment strategies on $[t,T]$ with $V(t) = v$ and we set $\Lambda(v_0) := \Lambda(0, v_0)$.

Remark 2.1.2. *The number of asset i in the portfolio is given by*

$$\varphi_0(t) := \frac{(1 - \pi(t)^T\mathbf{1})V(t)}{P_0(t)}, \ i = 0$$
$$and \ \varphi(t) := \left(\frac{\pi_i(t)V(t)}{P_i(t)}\right)_{i=1,...,d}.$$

With (2.1),

$$dV(t) = \sum_{i=0}^{d} \varphi_i(t) dP_i(t),$$

hence φ is self-financing.

In our applications, $\Lambda(v_0)$ might be further restricted to investment strategies, which guarantee that liabilities are hedged to a certain extent (Chapter 4 and Chapter 6) or which are subject to portfolio constraints (Chapter 5 and Chapter 6).

In cases in which we consider only one risky asset, we denote the Brownian motion, price process, corresponding parameters, market price of risk and allocation in the asset by W, P, μ, σ, γ and π instead of W_1, P_1, μ_1, σ_1, γ_1 and π_1 for simplicity.

2.2 Utility Functions

Definition 2.2.1 (Utility Function (adapted from Korn (1999), p. 38))**.** *Let $\mathbb{V} = (L, \infty)$, $L \geq 0$. A strictly concave, twice continuously differentiable function $U : \mathbb{V} \to \mathbb{R}$ satisfying*

$$\lim_{v \downarrow L} U'(v) = \infty \quad \text{and} \quad \lim_{v \to \infty} U'(v) = 0 \tag{2.2}$$

is called a utility function.

Besides the general form, we use the following special cases of utility functions. L represents the value of the liabilities.

Example 2.2.2 (Utility Functions)**.** *We consider the following examples for utility functions:*

- *Shifted logarithmic utility:*

$$U(v) = \log(v - L),\ \mathbb{V} = (L, \infty),\ L \geq 0, \tag{2.3}$$

  with the logarithmic utility

$$U(v) = \log(v),\ \mathbb{V} = (0, \infty) \tag{2.4}$$

  as a special case for $L = 0$.

- *HARA utility:*

$$U(v) = \frac{(v - L)^{\alpha}}{\alpha},\ \alpha < 1,\ \alpha \neq 0,\ \mathbb{V} = (L, \infty),\ L \geq 0, \tag{2.5}$$

with the power utility

$$U(v) = \frac{v^\alpha}{\alpha},\ \alpha < 1,\ \alpha \neq 0,\ \mathbb{V} = (0, \infty) \tag{2.6}$$

as a special case for $L = 0$.

We also interpret the shifted logarithmic utility as the logarithmic utility applied to the surplus and the HARA utility as the power utility applied to the surplus.

Definition 2.2.3 (Measures of Risk Aversion (see Korn (1999), p. 39)**.** *Let U be a utility function. We define the Arrow-Pratt measure of relative risk aversion as*

$$RRA(v) = -\frac{vU''(v)}{U'(v)}.$$

We also refer to $\frac{1}{RRA(v)}$ as the risk tolerance.

Example 2.2.4. *The relative risk aversion for the utility functions from Example 2.2.2 is given as follows:*

- *Shifted logarithmic utility* (2.3)*:*

  $$RRA(v) = \frac{v}{v - L},$$

  with $RRA(v) = 1$ for the logarithmic utility (2.4) *($L = 0$).*

- *HARA utility* (2.5)*:*

  $$RRA(v) = (1 - \alpha)\frac{v}{v - L},$$

  with $RRA(v) = 1 - \alpha$ for the power utility (2.6) *($L = 0$).*

The logarithmic utility and the power utility exhibit constant relative risk aversion. The shifted logarithmic (resp. logarithmic) utility can be interpreted as a limit of the HARA (resp. power) utility for $\alpha \to 0$.

To be able to maximize the expected utility later, we consider the subset $\Lambda'(v_0) \subset \Lambda(v_0)$ consisting of all $\pi \in \Lambda(v_0)$, satisfying

$$V^\pi(t) - L(t) \geq 0,\ \text{for}\ L(t) := e^{-\int_t^T r(s)ds} L,\ t \in [0, T]\ \text{and}\ \mathbb{E}\left[U^-(V^\pi(T))\right] < \infty,$$

with the negative part $U^-(v) := -\min\{U(v), 0\}$ of U. By $\Lambda'(t, v)$, we denote the restriction of $\Lambda'(v_0)$ to the interval $[t, T]$ with $V^\pi(t) = v$ and $\mathbb{E}\left[U^-(V^\pi(T))|\mathcal{F}_t\right] < \infty$.

Corresponding to U, we define the inverse marginal utility

$$I(y) := (U')^{-1}(y), \quad y > 0.$$

We always assume that

$$\hat{H}(y) := \mathbb{E}\left[\tilde{Z}(T)I(y\tilde{Z}(T))\right] < \infty \tag{2.7}$$

and

$$\mathbb{E}\left[U(I(y\tilde{Z}(T))\right] < \infty$$

for all $y > 0$. We denote the inverse of $\hat{H}$ by $\hat{\mathcal{Y}}$, which is well-defined (see Korn (1999), p. 65).

2.3 Methods from Dynamic Portfolio Optimization

The following two solution approaches from the literature provide the basis for most of the more advanced techniques, which we apply and develop further. At this point, we consider a basic optimization problem with utility from terminal wealth given by

$$\Phi(v_0) := \sup_{\pi \in \Lambda'(v_0)} \mathbb{E}\left[U(V^\pi(T))\right]. \tag{BP}$$

Our further optimization problems will be extensions of this problem.

2.3.1 Dynamic Programming Approach

This approach is based on a continuous-time version of the Bellman principle. The Bellman principle as introduced in Bellman (1957) states that, for an investment strategy π, which is optimal for the total problem (BP) starting at $t = 0$, the sub-strategy starting from a fixed point in time $t > 0$ onward must be optimal for the corresponding sub-problem too, independent of the initial states and initial decisions. The continuous-time version applied to portfolio optimization was introduced in Merton (1969) and Merton (1971). The following presentation of the approach is based on these publications and on Korn (1999). The value function is defined as

$$\Phi(t, v) := \sup_{\pi \in \Lambda'(t,v)} \mathbb{E}[U(V^\pi(T))|V^\pi(t) = v],\ t \in [0, T]. \tag{2.8}$$

As U is concave, $\Phi(t, v)$ is also concave in v. For (BP), the associated Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\sup_{\pi(t) \in \mathbb{R}^d} \left\{ v\pi(t)^T \left(\mu(t) - r(t)\mathbf{1}\right) \Phi_v(t, v) + \frac{1}{2} v^2 \|\pi^T(t)\sigma(t)\|^2 \Phi_{vv}(t, v) \right\} \tag{2.9}$$

$$+\Phi_t(t, v) + vr(t)\Phi_v(t, v) = 0$$

$$\Phi(T, v) = U(v). \tag{2.10}$$

The following theorem provides conditions under which the solution to the HJB equation is actually a solution to (2.8).

Theorem 2.3.1 (Verification Theorem (adapted from Korn (1999), p. 324))**.** *Let* $\Phi^* : [0,T] \times \mathbb{V} \to \mathbb{R}$ *be a continuous, polynomially bounded solution to* (2.9)-(2.10) *which satisfies* $\Phi^*(t,v) \in C^{1,2}([0,T] \times \mathbb{V})$ *. Furthermore, let* $\pi^* \in \Lambda(v_0)$ *satisfy*

$$\|\pi^*(t)\| \leq C,$$

$$\pi^*(t) \in \arg \sup_{\pi(t) \in \mathbb{R}^d} \left\{ v\pi(t)^T \left(\mu(t) - r(t)\mathbf{1}\right) \Phi^*_v(t,v) + \frac{1}{2} v^2 \|\pi^T(t)\sigma(t)\|^2 \Phi^*_{vv}(t,v) \right\}$$

for all $t \in [0,T]$ *and a constant* C*. Then,* $\Phi^*(t,v)$ *is optimal for* (2.8) *for all* $t \in [0,T]$*.*

Since, in our case, these conditions are always fulfilled for the power utility and HARA utility for $\alpha > 0$, we write Φ instead of Φ^* in the following chapters. The following results state the optimal investment strategy and value function, which serve as a reference point for our applications.

Corollary 2.3.2 (Optimal Investment Strategy, Logarithmic Utility)**.** *The optimal investment strategy for an investor with logarithmic utility* (2.4) *or shifted logarithmic utility* (2.3) *is given by*

$$\pi^*(t) = \pi^*(t, V^{\pi^*}(t)) = \left(\frac{V^{\pi^*}(t) - L(t)}{V^{\pi^*}(t)} \right) \left(\sigma(t)\sigma(t)^T\right)^{-1} \left(\mu(t) - r(t)\mathbf{1}\right)$$

and the value function is

$$\Phi(t,v) = \log(v - L(t))^\alpha + \varphi(t),$$

$$\varphi(t) = \int_t^T \frac{1}{2}\|\gamma(s)\|^2 + r(s) ds,$$

with $L = 0$ *for the logarithmic utility.*

Proof. This is a special case of Corollary 6.1.2 with $K(t,V(t)) = \mathbb{R}$, hence $X_{K(t,V(t))} = \{0 \in \mathbb{R}^d\}$ and $\lambda^*(t) \equiv 0$. □

Corollary 2.3.3 (Optimal Investment Strategy, HARA Utility)**.** *The optimal investment strategy for an investor with power utility* (2.6) *or HARA utility* (2.5) *is given by*

$$\pi^*(t) = \pi^*(t, V^{\pi^*}(t)) = \frac{1}{(1-\alpha)} \left(\frac{V^{\pi^*}(t) - L(t)}{V^{\pi^*}(t)} \right) \left(\sigma(t)\sigma(t)^T\right)^{-1} \left(\mu(t) - r(t)\mathbf{1}\right)$$

and the value function is

$$\Phi(t,v) = \frac{(v-L(t))^\alpha}{\alpha}\varphi(t),$$
$$\varphi(t) = e^{\alpha\int_t^T \frac{1}{2(1-\alpha)}\|\gamma(s)\|^2 + r(s)ds},$$

with $L = 0$ for the power utility.

Proof. This is a special case of Corollary 6.1.3 with $K(t,V(t)) = \mathbb{R}$, hence $X_{K(t,V(t))} = \{0 \in \mathbb{R}^d\}$ and $\lambda^*(t) \equiv 0$. □

Remark 2.3.4. *Note that $\frac{V^{\pi^*}(t)-L(t)}{V^{\pi^*}(t)} = 1 - \frac{L(t)}{V^{\pi^*}(t)}$ is bounded for $\pi \in \Lambda'(v_0)$. Since the market coefficients and $(\sigma(t)\sigma(t)^T)^{-1}$ are also bounded, there exists a constant $C > 0$ with $\|\pi^*(t)\| \leq C$ and the conditions of Theorem 2.3.1 are satisfied.*

The optimal investment strategy can be interpreted as a two-fund separation, with the performance seeking portfolio (growth optimal portfolio) being defined as

$$\pi^{PS} := (\sigma(t)\sigma(t)^T)^{-1}(\mu(t) - r(t)\mathbf{1}). \tag{2.11}$$

The allocation in this portfolio is scaled by the relative risk tolerance of the utility. The remaining wealth $1 - \pi(t)^T\mathbf{1}$ is allocated in the risk-free asset.

2.3.2 Martingale Approach

Using the martingale approach, the optimization problem (BP) is in solved two steps. In the first step, the optimal terminal wealth is found. In the second step, the optimal investment strategy is determined as a strategy which replicates the optimal terminal wealth. As we also use this decomposition, we present the foundation for this approach in Theorem 2.3.5 and the optimal terminal wealth in Theorem 2.3.6. In our applications, the procedure for obtaining the investment strategy is very specific in each case, so we omit this step here. The following results are adapted from Korn (1999).

Theorem 2.3.5 (Completeness of the Market Model, see Korn (1999), p. 25-26)**.** *For every $\pi \in \Lambda(v_0)$ and corresponding wealth process $V^\pi(t)$, we have*

$$\mathbb{E}\left[\tilde{Z}(t)V^\pi(t)\right] \leq v_0.$$

Furthermore, for every non-negative, $\mathcal{F}_T$-measurable random variable V with

$$v_V := \mathbb{E}\left[\tilde{Z}(T)V\right] < \infty,$$

there exists an investment strategy $\pi \in \Lambda(v_V)$ *such that the corresponding wealth process* V^π *satisfies*

$$V^\pi(T) = V.$$

We define a set

$$\mathcal{V} := \left\{ V\, \mathcal{F}_T\text{-measurable: } V \geq 0,\ \mathbb{E}\left[U(V)^-\right] < \infty,\ \mathbb{E}\left[\tilde{Z}(T)V\right] \leq v_0 \right\},$$

on which we search for an optimal terminal wealth. Theorem 2.3.5 guarantees that there is a corresponding investment strategy, which is then optimal to (BP).

Theorem 2.3.6 (Optimal Terminal Wealth, see Korn (1999), p. 68)**.** *The optimal terminal wealth for* (BP) *is given by*

$$V^*(T) = I(\hat{\mathcal{Y}}(v_0)\tilde{Z}(T)).$$

3 Funding Ratio Optimization

In this chapter, we consider the optimization of the terminal funding ratio. In Section 3.1, we present a funding ratio optimization framework adapted from Martellini (2006). As a generalization of this framework, we apply the quantile optimization approach from Jin and Zhou (2008) to solve a funding ratio optimization problem in a CPT context in Section 3.2. The results are analyzed and compared in Section 3.3. Large parts of this chapter coincide with Brummer et al. (2018).

For the market model, we assume constant coefficients μ, σ and r throughout the whole chapter. We model the liability process L_ϵ, as in Fombellida (2004) and Martellini (2006), by a geometric Brownian motion following

$$dL_\epsilon(t) = L_\epsilon(t)\left(\mu_L dt + \sigma_L dW(t) + \sigma_\epsilon dW_\epsilon(t)\right),$$

with constant $L_\epsilon(0)$, constant drift μ_L, hedgeable risks related to W and non-hedgeable risks related to a Brownian motion W_ϵ. The liabilities have the explicit representation

$$L_\epsilon(t) = L_\epsilon(0)\exp\left(\left(\mu_L - \frac{\|\sigma_L\|^2 + \sigma_\epsilon^2}{2}\right)t + \sigma_L W(t) + \sigma_\epsilon W(t)\right). \tag{3.1}$$

We assume that the filtration generated by W_ϵ is a filtration in $\mathcal{G}$. In the context of an insurance company or pension fund, the non-hedgeable risks could represent, e.g., actuarial risks like mortality/longevity risk or underwriting risk. If we have $\sigma_\epsilon = 0$, all the liability risks emerge from W and the liabilities can be hedged. The funding ratio is defined as

$$F^\pi(t) := \frac{V^\pi(t)}{L_\epsilon(t)},\ t \in [0, T]$$

for a corresponding portfolio process π. For ease of notation, we will also write $F(t)$ instead of $F^\pi(t)$. Applying Itô's formula,

$$\begin{aligned} d\left(\frac{1}{L_\epsilon(t)}\right) &= -\frac{1}{L_\epsilon(t)}\left(\mu_L dt + \sigma_L dW(t) + \sigma_\epsilon dW_\epsilon(t)\right) + \frac{1}{L_\epsilon(t)}\left(\sigma_L^T\sigma_L + \sigma_\epsilon^2\right)dt \\ &= \frac{1}{L_\epsilon(t)}\left[(\sigma_L^T\sigma_L + \sigma_\epsilon^2 - \mu_L)dt - \sigma_L dW(t) - \sigma_\epsilon dW_\epsilon(t)\right] \end{aligned}$$

and applying it again, the SDE of the funding ratio can be written with (2.1) as

$$\begin{aligned}
dF^\pi(t) =& d\left(\frac{V^\pi(t)}{L_\epsilon(t)}\right)\\
=& \frac{1}{L_\epsilon(t)} dV^\pi(t) + V^\pi(t) d\left(\frac{1}{L_\epsilon(t)}\right) - \frac{V^\pi(t)}{L_\epsilon(t)} \sigma_L \sigma^T \pi(t) dt\\
=& \frac{V^\pi(t)}{L_\epsilon(t)} \left[\pi^T(t)\left(\mu dt + \sigma dW(t)\right) + (1 - \pi^T(t)\mathbf{1}) r dt\right]\\
&+ \frac{V^\pi(t)}{L_\epsilon(t)} \left[(\sigma_L^T \sigma_L + \sigma_\epsilon^2 - \mu_L) dt - \sigma_L dW(t) - \sigma_\epsilon dW_\epsilon(t)\right] - \frac{V^\pi(t)}{L_\epsilon(t)} \sigma_L \sigma^T \pi(t) dt\\
=& F^\pi(t) \left[\mu_F^\pi(t) dt + (\pi(t)^T \sigma - \sigma_L) dW(t) - \sigma_\epsilon dW_\epsilon(t)\right], \qquad (3.2)
\end{aligned}$$

with

$$\mu_F^\pi(t) := r + \pi(t)^T(\mu - r\mathbf{1}) + \sigma_L \sigma_L^T + \sigma_\epsilon^2 - \mu_L - \sigma_L \sigma^T \pi(t).$$

We define

$$\sigma_F^\pi(t) := \left(\|\pi(t)^T\sigma - \sigma_L\|^2 + \sigma_\epsilon^2\right)^{\frac{1}{2}}$$

as well as a Brownian motion $\bar{W}$ by

$$\sigma_F^\pi(t)\bar{W}(t) = \left(\pi(t)^T\sigma - \sigma_L\right) W(t) - \sigma_\epsilon W_\epsilon(t).$$

Then, (3.2) can be written as

$$dF^\pi(t) = F^\pi(t)\left[\mu_F^\pi(t) dt + \sigma_F^\pi(t) d\bar{W}(t)\right],$$

We see that the liability hedging portfolio, as introduced in Martellini (2006)

$$\pi^{LH} := (\sigma^T)^{-1}\sigma_L^T \qquad (3.3)$$

minimizes the volatility of the funding ratio by eliminating all hedgeable risks.

3.1 Funding Ratio Optimization in an Expected Utility Framework

In this section, we consider a standard expected utility framework, in which we want to maximize the expected utility of the terminal funding ratio instead of the terminal wealth. We solve the expected utility funding ratio optimization problem

$$\sup_{\pi\in\Lambda'(v_0)} \mathbb{E}\left[U\left(F^\pi(T)\right)\right] \qquad \text{(EUFP)}$$

for a general utility function U and $\Lambda'(v_0)$ consisting of all $\pi \in \Lambda(v_0)$ satisfying $\mathbb{E}\left[U^-(F^\pi(T))\right] < \infty$. For the application of the HJB approach, we define the value function in terms of the funding ratio here, i.e.

$$\Phi(t,v) := \sup_{\pi \in \Lambda'(t,v)} \mathbb{E}[U(F^\pi(T))|F^\pi(t) = v]. \tag{3.4}$$

The solution to this problem is presented in the following theorem, which can be found in Martellini (2006).

Theorem 3.1.1 (Three-Fund Separation, Expected Utility Theory)**.** *The optimal investment strategy π^* to* (EUFP) *is given by*

$$\pi^*(t,F(t)) = \left(1 - \lambda^{EU}(t,F(t))\right)\pi^{LH} + \lambda^{EU}(t,F(t))\pi^{PS}$$

with $\lambda^{EU}(t,F(t))$ given by the relative risk tolerance of $\Phi(t,F(t))$

$$\lambda^{EU}(t,F(t)) := -\frac{\Phi_v(t,F(t))}{F(t)\Phi_{vv}(t,F(t))},$$

the liability hedging portfolio as in (3.3) and the performance seeking portfolio from (2.11)*. The remaining fraction of wealth $1 - \pi^*(t,F(t))^T\mathbf{1}$ is invested in the risk-free asset.*

Proof. See Appendix A.1. ☐

As in Martellini (2006), the optimal investment strategy can be interpreted as a three-fund separation, with the funds being the performance seeking portfolio, liability hedging portfolio and risk-free asset. In the following corollary, we specify the utility function to get an explicit expression for the value function and consequently the optimal investment strategy. This result can also be found in Martellini (2006).

Corollary 3.1.2 (Three-Fund Separation, Power Utility)**.** *For the power utility, we get*

$$\lambda^{EU}(t,F(t)) = -\frac{\Phi_v(t,F(t))}{F(t)\Phi_{vv}(t,F(t))} = \frac{1}{1-\alpha}.$$

Proof. The proof can be found in Appendix A.1. ☐

Remark 3.1.3. *Although we consider unhedgeable risks associated with σ_ϵ, they do not have an impact on the optimal investment strategy for the example with power utility. To cover these risks, an additional capital buffer would have to be used. Therefore, we assume in the following that a certain part of the wealth is used for this purpose and we only deal with the hedgeable risks, i.e. $\sigma_\epsilon = 0$ in the following section.*

3.2 Funding Ratio Optimization in a CPT Framework

In this section, we generalize the funding ratio optimization to Cumulative Prospect Theory (CPT) by applying the approach from Jin and Zhou (2008) to the terminal funding ratio. Similar to the martingale approach, the optimal terminal funding ratio is determined first in this approach and the replicating strategy is calculated later. Thus, a complete market is required. So we choose, supported by the reasoning in Remark 3.1.3, $\sigma_\epsilon = 0$ in the whole section and denote the corresponding liability process by $L := L_\epsilon$. We further assume that the discounted liability process $\frac{L}{P_0}$ is a $\tilde{\mathbb{Q}}$-martingale. Then, we can change the numéraire from the risk-free asset to the liability process L. By $\mathbb{Q}_L$, we denote the risk-neutral measure under the numéraire L. Then, $F^\pi(t) = \frac{V^\pi(t)}{L(t)}$ is a $\mathbb{Q}_L$-martingale and the pricing kernel with respect to the new numéraire is given by (see Bingham and Kiesel (2004), p. 239)

$$\begin{aligned} Z_L(t) := & \frac{d\mathbb{Q}_L}{d\mathbb{Q}}\bigg|\mathcal{F}_t = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\frac{d\mathbb{Q}_L}{d\tilde{\mathbb{Q}}}\bigg|\mathcal{F}_t = \frac{L(t)}{L(0)P_0(t)}\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\bigg|\mathcal{F}_t = \tilde{Z}(t)\frac{L(t)}{L(0)} \\ = & \exp\left(\left(\mu_L - \frac{1}{2}\|\sigma_L\|^2\right)t + \sigma_L W(t)\right)\exp\left(-\left(r + \frac{1}{2}\|\gamma\|^2\right)t - \gamma^T W(t)\right) \\ = & \exp\left(\left(\mu_L - r - \frac{1}{2}\left(\|\sigma_L\|^2 + \|\gamma\|^2\right)\right)t + (\sigma_L - \gamma^T)W(t)\right). \end{aligned}$$

Since $\tilde{Z}(t)\frac{L(t)}{L(0)}$ is a $\mathbb{Q}$-martingale, we have

$$Z_L(t) = \exp\left(-\frac{1}{2}\|\sigma_L - \gamma^T\|^2 t + (\sigma_L - \gamma^T)W(t)\right). \tag{3.5}$$

For $z \in (0, \infty)$,

$$\begin{aligned} & Z_L(t) \le z \\ \Leftrightarrow\ & \exp\left(-\frac{1}{2}\|\sigma_L - \gamma^T\|^2 t + (\sigma_L - \gamma^T)W(t)\right) \le z \\ \Leftrightarrow\ & (\sigma_L - \gamma^T)W(t) \le \log(z) + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 t. \end{aligned}$$

Hence, the distribution function of $Z_L(t)$ is given by

$$\mathbb{Q}_{Z_L(t)}(z) := \mathbb{Q}(Z_L(t) \le z) = \Phi\left(\frac{\log(z) + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 t}{\|\sigma_L - \gamma^T\|\sqrt{t}}\right) \tag{3.6}$$

and the quantile function by

$$q_{Z_L(t)}(p) = \exp\left(-\frac{1}{2}\|\sigma_L - \gamma^T\|^2 t + \|\sigma_L - \gamma^T\|\sqrt{t}\Phi^{-1}(p)\right),$$

with Φ and Φ^{-1} denoting the distribution function and the quantile of the standard normal distribution. For the rest of the chapter, we only work with this pricing kernel. For ease of notation, we write $Z := Z_L(T)$ as well as $\mathbb{Q}_Z$ for $\mathbb{Q}_{Z_L(T)}$ and q_Z for $q_{Z_L(T)}$.

3.2.1 Introduction to CPT

Introduced in Kahneman and Tversky (1979) and Tversky and Kahneman (1992), CPT extends expected utility theory with respect to the following aspects:

- gains and losses are treated differently with respect to the utility and
- probabilities are distorted.

Motivated by experiments in Kahneman and Tversky (1979), which exhibit that people are risk-averse with respect to gains, but risk-seeking with respect to losses, different utility functions are used for gains and losses. They are separated by a reference point. To model this behavior, a concave utility function is applied to gains and a convex utility function is applied to losses. Applied to our funding ratio setting, which is a perspective predominantly taken by institutional investors such as insurance companies or pension funds, an alternative interpretation is also possible: In the case of underfunding, the management of the pension plan might be willing to take more risk than in the case of well-funding to achieve a better funding status on the long run. Especially within low interest-rate environments, pension funds might have to take more risk in trying to generate an adequate return. For the probability distortion, which was originally introduced to model the observed behavior that people tend to overestimate small probabilities and underestimate large probabilities (which is an irrational bias), an alternative interpretation is also possible. Through the distortion of the probabilities, heavier tails in the distribution of asset returns, which are not captured in the Black-Scholes market model, can be included. In this case, the probability distortion could even be fitted using market data. Since typical distortion functions (see Tversky and Kahneman (1992) and Prelec (1998)) cannot be used for the quantile optimization approach presented in Jin and Zhou (2008), Jin and Zhou (2008) introduce their own probability function. As this probability function has many parameters which would have to be fitted, we introduce a modification of the distortion function presented in Wang (2000). Our distortion function can be reverse-S shaped, has only two parameters and is suitable for the quantile optimization approach. In the following sections, we present the CPT optimization problem and apply the approach from Jin and Zhou (2008) to a funding ratio optimization.

3.2.2 CPT Funding Ratio Optimization Problem

We consider the funding ratio compared to a constant reference point B and introduce the CPT utility and probability distortion functions.

Definition 3.2.1 (CPT Utility Function)**.** *A CPT utility function $U : \mathbb{R} \to \mathbb{R}$ is a function of the form*

$$U(v) := U_+((v-B)^+)\mathbb{1}_{v \geq B}(v) - U_-((v-B)^-)\mathbb{1}_{v<B}(v), \tag{3.7}$$

with the positive and negative parts

$$(v)^+ := \max\{v, 0\}, \quad (v)^- := -\min\{v, 0\}$$

and $U_+, U_- : \mathbb{V} = [0, \infty) \to [0, \infty)$ are functions which satisfy the conditions of standard utility functions and additionally

$$U_+(0) = U_-(0) = 0 \text{ and } U_-'(v) > U_+'(v), \ v \in \mathbb{V}.$$

The reference point B represents the funding ratio, for which the pension plan is considered to be adequately funded. A natural example would be $B = 1$, representing a fully funded status.

Definition 3.2.2 (Probability Distortion Function)**.** *A probability distortion function is a strictly increasing, twice differentiable function $w : [0, 1] \to [0, 1]$ with*

$$w(0) = 0, \quad w(1) = 1, \quad w' > 0.$$

By w_+ (resp. w_-), we denote distortion functions applied to gains (resp. losses).

We assume that the following monotonicity condition holds:

$$\frac{q_Z(y)}{w_+'(y)} \text{ is non-decreasing for } y \in (0, 1]. \tag{M}$$

The distortion function we use later satisfies this condition, as shown in Lemma (3.2.5). As we maximize the expected distorted utility, we define a value function with the CPT utility function from Definition 3.2.1 and the distortion function from Definition 3.2.2 as

$$\mathbb{U}(\bar{F}) := \mathbb{U}_+((\bar{F})^+) - \mathbb{U}_-((\bar{F})^-),$$

with $\bar{F} := F(T) - B$,

$$\begin{aligned}
\mathbb{U}_+(\bar{F}) &:= \int_0^\infty w_+(\mathbb{Q}(U_+(\bar{F}) > x))dx = \int_0^\infty w_+(1 - \mathbb{Q}_{\bar{F}}(U_+^{-1}(x)))dx \\
&= \int_0^\infty \int_{U_+^{-1}(x)}^\infty d(-w_+(1 - \mathbb{Q}_{\bar{F}}(y)))dx \\
&= \int_0^\infty U_+(y)d(-w_+(1 - \mathbb{Q}_{\bar{F}}(y))) = \mathbb{E}\left[U_+(\bar{F})w_+'(1 - \mathbb{Q}_{\bar{F}}(\bar{F}))\right],
\end{aligned}$$

where $\mathbb{Q}_{\bar{F}}$ denotes the distribution function of $\bar{F}$ and

$$\mathbb{U}_-(\bar{F}) := \int_0^\infty w_-(\mathbb{Q}(U_-(\bar{F}) > x))dx = \mathbb{E}\left[U_-(\bar{F})w'_-(1-\mathbb{Q}_{\bar{F}}(\bar{F}))\right].$$

The CPT optimization problem is defined as

$$\begin{aligned} &\sup_{\bar{F}}\ \mathbb{U}(\bar{F}) \\ &\text{s.t. } \mathbb{E}[Z\bar{F}] = \frac{V(0)}{L(0)} - B \\ &\bar{F} \text{ is } \mathcal{F}_T\text{-measurable and bounded from below.} \end{aligned} \tag{CPTOP}$$

With the utility function being concave only above the reference point, usual portfolio optimization methods such as the martingale approach are not applicable anymore. To overcome this problem, Jin and Zhou (2008) introduce a CPT optimization approach, which we apply to the funding ratio optimization in the following section.

3.2.3 CPT Optimization Method

Jin and Zhou (2008) propose a solution approach by splitting the problem into a Gains Problem and a Loss Problem. Both problems can, under sufficient conditions, be solved separately. The solutions depend on parameters c and v_+. In a so-called Gluing Problem, the optimal values for c and v_+ are determined. Theorem 3.2.3 ensures that this solution approach is equivalent to solving the original problem.

We consider the part of the terminal funding ratio which exceeds or falls short of the reference point B, i.e. $\bar{F} = (\bar{F})^+ - (\bar{F})^-$. $(\bar{F})^+$ only influences $\mathbb{U}_+$ and $(\bar{F})^-$ only influences $\mathbb{U}_-$. We define the set where $\bar{F}$ exceeds the reference point as

$$A := \{\bar{F} \geq 0\}.$$

Further, we denote the initial funding ratio needed to replicate $(\bar{F})^+$ by v_+, i.e.

$$v_+ := \mathbb{E}[Z(\bar{F})^+].$$

With $F(T) = (\bar{F})_+ - (\bar{F})_- + B$ and $\mathbb{E}\left[ZF(T)\right] = \frac{V(0)}{L(0)}$, the budget in terms of the funding ratio for $(\bar{F})_-$ is given by $\mathbb{E}[Z(\bar{F})^-] = v_+ - \left(\frac{V(0)}{L(0)} - B\right)$.

In the *Gains Problem* (GP), we optimize $(\bar{F})^+$ with given initial funding ratio v_+, i.e.

we solve

$$\begin{aligned}
&\sup_{\bar{F}} \quad \mathbb{U}_+((\bar{F})^+) \\
&\text{s.t.} \quad \mathbb{E}[Z(\bar{F})^+] = v_+, \\
&\bar{F} \text{ is } \mathcal{F}_T\text{-measurable and lower-bounded.}
\end{aligned} \tag{GP}$$

The shortfall of the terminal funding ratio $(\bar{F})^-$ is different from zero on $A^c = \{\bar{F} < 0\}$. We optimize $(\bar{F})^-$ in the *Loss Problem* (LP) given by

$$\begin{aligned}
&\inf_{\bar{F}} \quad \mathbb{U}_-((\bar{F})^-) \\
&\text{s.t.} \quad \mathbb{E}[Z(\bar{F})^-] = v_+ - \left(\frac{V(0)}{L(0)} - B\right) \\
&\bar{F} \text{ is } \mathcal{F}_T\text{-measurable and upper-bounded.}
\end{aligned} \tag{LP}$$

Both problems, (GP) and (LP), depend on the parameters A and v_+. Therefore, we denote the optimal solutions to these problems by $\Phi_+(v_+, A)$ for (GP) and $\Phi_-(v_+, A)$ for (LP). The connection between (GP) and (LP) is established by finding a pair (v_+^*, A^*) which maximizes $\Phi_+(v_+, A) - \Phi_-(v_+, A)$. This problem is called the *Gluing Problem*. To simplify the parametrization with respect to the set A, Jin and Zhou (2008) (Theorem 5.1) show that an optimal set A^* is always of the form $A^* = \{Z \leq c^*\}, \quad c^* \geq 0$. This simplifies the Gluing Problem to finding solutions of the form $\Phi_\pm(v_+, c) := \Phi_\pm(v_+, \{Z \leq c\})$, i.e. we solve an optimization problem in only two real variables. In the following, we give an intuition why this simplification is possible. First, we note that $\mathbb{Q}_Z(Z)$ and subsequently $1 - \mathbb{Q}_Z(Z)$ are uniformly distributed and $A = \{\bar{F} \geq 0\}$. Let

$$c := q_Z(\mathbb{Q}(A)) = q_Z(\mathbb{Q}(\bar{F} \geq 0)) = q_Z(\mathbb{Q}(F \geq B)).$$

Then,

$$\mathbb{Q}(Z \leq c) = \mathbb{Q}(Z \leq q_Z(\mathbb{Q}(A))) = \mathbb{Q}(A) = \mathbb{Q}(\bar{F} \geq 0)$$

for all relevant $\bar{F}$. Hence, $Z \leq c \Leftrightarrow \bar{F} \geq 0$ as F will be a monotonic function of Z.
The Gluing Problem (GLUE) is defined as

$$\begin{aligned}
&\sup_{(v_+,c)} \quad \Phi_+(v_+, c) - \Phi_-(v_+, c) \\
&\quad s.t.\ 0 \leq c \leq \infty.
\end{aligned} \tag{GLUE}$$

By solving (GLUE), we obtain the optimal parameters c^* and v_+^*. We showed $Z \leq c \Leftrightarrow \bar{F} \geq 0$ in all relevant cases, so for the optimal solutions, it holds

$$(\bar{F}^*)^+ = \bar{F}^* \mathbb{1}_{\bar{F}^* \geq 0} = \bar{F}^* \mathbb{1}_{Z \leq c^*}$$

as well as

$$(\bar{F}^*)^- = -\bar{F}^* \mathbb{1}_{\bar{F}^*<0} = -\bar{F}^* \mathbb{1}_{Z>c^*}.$$

Consequently, we can write (GP) and (LP) equivalently as

$$\begin{aligned} \sup_{\bar{F}} \quad & \mathbb{U}_+(\bar{F}\mathbb{1}_{Z\le c}) \\ \text{s.t.} \quad & \mathbb{E}[Z\bar{F}\mathbb{1}_{Z\le c}] = v_+, \\ & \bar{F}\mathbb{1}_{Z\le c} \ge 0 \end{aligned} \tag{GP'}$$

and

$$\begin{aligned} \inf_{\bar{F}} \quad & \mathbb{U}_-(-\bar{F}\mathbb{1}_{Z>c}) \\ \text{s.t. } & \mathbb{E}[-Z\bar{F}\mathbb{1}_{Z>c}] = v_+ - \left(\frac{V(0)}{L(0)} - B\right) \\ & -\bar{F}\mathbb{1}_{Z>c} \ge 0. \end{aligned} \tag{LP'}$$

3.2.4 Optimal CPT Funding Ratio

Jin and Zhou (2008) provide an explicit solution for power utility, which we adapt in this section. We use

$$U_+(v) = v^\alpha, \quad U_-(v) = \beta v^\alpha, \quad \alpha \in (0,1), \quad \beta > 1$$

and define

$$\begin{aligned} G(c) &:= \mathbb{E}\left[Z^{\frac{\alpha}{\alpha-1}} w_+'(\mathbb{Q}_Z(Z))^{\frac{1}{1-\alpha}} \mathbb{1}_{Z\le c}\right], \\ \text{and } k(c) &:= \frac{\beta w_-(1-\mathbb{Q}_Z(c))}{G(c)^{1-\alpha}\mathbb{E}[Z\mathbb{1}_{Z>c}]^\alpha}. \end{aligned}$$

We always assume

$$\inf_{c>0} k(c) \ge 1. \tag{K}$$

In the following result, adapted from Jin and Zhou (2008), we provide explicit solutions for the optimal terminal funding ratio depending on the initial funding ratio.

Theorem 3.2.3 (Optimal Terminal Funding Ratio, CPT)**.** *Let (M) and (K) be satisfied. If $\frac{V(0)}{L(0)} \ge B$, the optimal solution $\bar{F}^*$ to* (CPTOP) *is given by*

$$\bar{F}^* = (\bar{F}^*)^+ = \frac{\frac{V(0)}{L(0)} - B}{G(\infty)} \left(\frac{Z}{w_+'(\mathbb{Q}_Z(Z))}\right)^{\frac{1}{\alpha-1}}$$

and the optimal terminal funding ratio is $F^*(T) = \bar{F}^* + B$.

If $\frac{V(0)}{L(0)} < B$, *then the following holds:*

- *If* $\inf_{c>0} k(c) = 1$, *the supremum value of* (CPTOP) *is* 0 *but not attainable.*
- *If* $\inf_{c>0} k(c) > 1$, *the Problem* (CPTOP) *admits an optimal solution if and only if the problem*

$$\inf_{0 \leq c < \infty} \left(\frac{\beta w_-(1 - \mathbb{Q}_Z(c))}{\mathbb{E}[Z \mathbb{1}_{Z>c}]^\alpha} \right)^{\frac{1}{1-\alpha}} - G(c) \tag{C}$$

  admits an optimal solution c^*.

  - *If* $c^* = 0$ *is the only solution to* (C), *then*

$$\bar{F}^* = \frac{V(0)}{L(0)} - B$$

    and the optimal terminal funding ratio is $F^* = \bar{F}^* + B = \frac{V(0)}{L(0)}$.

  - *If* $c^* > 0$, *the optimal solution to* (CPTOP) *is given by*

$$\begin{aligned}\bar{F}^* =& (\bar{F}^*)^+ - (\bar{F}^*)^- \\ =& \frac{v_+^*}{G(c^*)} \left(\frac{Z}{w_+'(\mathbb{Q}_Z(Z))} \right)^{\frac{1}{\alpha-1}} \mathbb{1}_{Z \leq c^*} - \frac{v_+^* - \frac{V(0)}{L(0)} + B}{\mathbb{E}[Z \mathbb{1}_{Z>c^*}]} \mathbb{1}_{Z>c^*}\end{aligned}$$

    with

$$v_+^* = \frac{B - \frac{V(0)}{L(0)}}{k(c^*)^{\frac{1}{1-\alpha}} - 1}$$

    and the optimal terminal funding ratio is $F^* = \bar{F}^* + B$.

Proof. The proof works along Jin and Zhou (2008) with the wealth being replaced by the funding ratio and the change of numéraire as described in Section 2. □

Remark 3.2.4. *With* $I(y) = y^{\frac{1}{\alpha-1}}$ *for the power utility, the parts* $\frac{\frac{V(0)}{L(0)} - B}{G(\infty)} \left(\frac{Z}{w_+'(\mathbb{Q}_Z(Z))} \right)^{\frac{1}{\alpha-1}}$ *and* $\frac{v_+^*}{G(c^*)} \left(\frac{Z}{w_+'(\mathbb{Q}_Z(Z))} \right)^{\frac{1}{\alpha-1}}$ *can with Theorem 2.3.6 be interpreted as an optimal terminal wealth of a standard optimization problem which is adjusted by a distorting factor and scaled to the available budget for* $(\bar{F}^*)^+$.

In the following section, we specify the distortion in order to achieve more explicit results for the optimal terminal funding ratio and the corresponding investment strategies.

3.2.5 Modified Wang-Distortion Function

We consider the distortion from Wang (2000), which can be written as[2]

$$\begin{aligned}\bar{w}(p) :=& \int_0^{q_Z(p)} r f_Z(r) dr = \mathbb{E}\left[Z \mathbb{1}_{Z \leq q_Z(p)}\right] \\ =&\Phi\left(\frac{\log(q_Z(p)) - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) = \Phi\left(\Phi^{-1}(p) - \|\sigma_L - \gamma^T\|\sqrt{T}\right),\end{aligned}$$

with f_Z denoting the density function of Z. While $\bar{w}$ is a probability distortion function according to Definition 3.2.2, it is convex instead of reverse-S-shaped. A proof is provided for a slightly more general statement in Appendix A.2, Lemma A.2.1.

With the following generalization to this distortion function, we get a distortion function $w(p)$, which can be reverse-S-shaped. It is defined by

$$w(p) := \Phi^\eta\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right),$$

with parameters $\eta \in (0, 1]$, $\delta \in (0, 1]$ and $\Phi^\eta(\cdot) := (\Phi(\cdot))^\eta$. Moreover, we define

$$Z_\delta := \exp\left(-\frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T + \delta(\sigma_L - \gamma^T)W(T)\right),$$

which has, by the same derivation as for $\mathbb{Q}_Z$, the cumulative distribution function

$$\mathbb{Q}_{Z_\delta}(z) = \Phi\left(\frac{\log z + \frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T}{\delta\|\sigma_L - \gamma^T\|\sqrt{T}}\right).$$

The corresponding density is denoted by f_{Z_δ} and the quantile function is given by

$$q_{Z_\delta}(p) = \exp\left(-\frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T + \delta\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right).$$

Further, we define

$$\begin{aligned}\bar{w}_\delta(p) :=& \int_0^{q_{Z_\delta}(p)} r f_{Z_\delta}(r) dr = \mathbb{E}\left[Z_\delta \mathbb{1}_{Z_\delta \leq q_{Z_\delta}(p)}\right] \\ =&\Phi\left(\frac{\log(q_{Z_\delta}(p)) - \frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T}{\delta\|\sigma_L - \gamma^T\|\sqrt{T}}\right) = \Phi\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right),\end{aligned}$$

[2] For $\nu \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}^+$ and $Y \sim \mathcal{LN}(\mu, \sigma^2)$

$$\mathbb{E}[Y^\nu \mathbb{1}_{Y \in (c_1, c_2)}] = \exp\left(\nu\mu + \frac{1}{2}\nu^2\sigma^2\right)\left(\Phi\left(\frac{\log c_2 - \mu - \nu\sigma^2}{\sigma}\right) - \Phi\left(\frac{\log c_1 - \mu - \nu\sigma^2}{\sigma}\right)\right).$$

with Footnote 2 as above. Hence,

$$w(p) = (\bar{w}_\delta(p))^\eta .$$

Lemma 3.2.5. *Let $\delta \in (0,1]$ and $\eta \in (0,1]$. Then, w satisfies* (M) *and its first derivative is given by*

$$\begin{aligned} w'(p) =& \eta \Phi^{\eta-1}\left(\Phi^{-1}(p) - \delta \|\sigma_L - \gamma^T\|\sqrt{T}\right) q_Z(p) \\ & \cdot \exp\left(\frac{1}{2}(1-\delta^2)\|\sigma_L - \gamma^T\|^2 T + (\delta - 1)\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right). \end{aligned}$$

Furthermore, w is reverse S-shaped for $\eta \in (0,1)$.

Proof. See Appendix A.2. □

The shape of w is further illustrated for various parameters in Figure 3.1, where we can also observe that w is reverse-S-shaped for $\eta = 0.5$, which is particularly visible for the case with $\delta = 1$ at the same time. To get a better understanding of the effect of the distortion, we apply it to an example for illustrative purposes. For a random variable Y with distribution function $\mathbb{Q}_Y$ and density function f_Y, we consider

$$\mathbb{U}_+(Y) = \mathbb{E}\left[U_+(Y) w'_+(1 - \mathbb{Q}_Y(Y))\right] = \int_{-\infty}^{\infty} U_+(y) w'_+(1 - \mathbb{Q}_Y(y)) f_Y(y) dy,$$

which we interpret as an expectation of $U_+(Y)$ under a distorted probability measure. The distorted density function of Y is given by

$$f_Y^w(y) := w'_+(1 - \mathbb{Q}_Y(y)) f_Y(y)$$

and the corresponding distribution function by

$$\mathbb{Q}_Y^w(y) := \int_{-\infty}^{y} f_Y^w(s) ds = 1 - w_+(1 - \mathbb{Q}_Y(y)).$$

For Y being standard normally distributed, Figure 3.2 illustrates the distorted density function f_Y^w for various parameter choices. For a fixed value of η, a variation in δ results in a shift of the distribution. If Y represents the return of the wealth or the funding ratio, an increase in δ results in an increase in the probability for very low returns, which might be a desirable property to adjust the Black-Scholes market model for a higher downside risk. For $\eta = 1$, the whole density function is just shifted to the left, so the expected return also decreases. With a decrease in η, the upper tail of the distribution is emphasized. Combining both effects, the probability for events from both tails is increased (see the graph for $\eta = 0.5,\ \delta = 1$ in Figure 3.2).

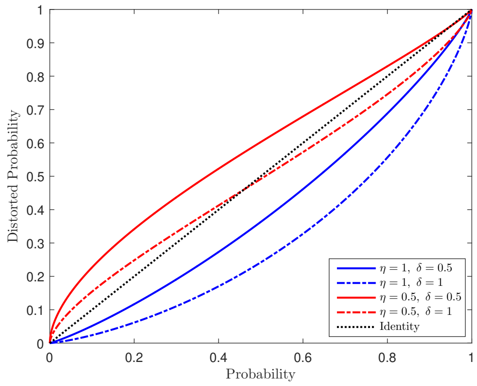

Figure 3.1: Distortion function w.

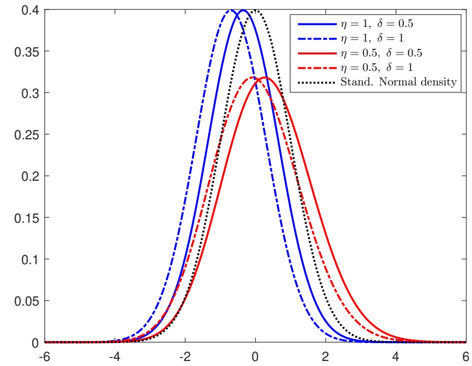

Figure 3.2: Distorted density of a standard normally distributed random variable.

3.2.6 Optimal Investment Strategy for the Modified Wang-Distortion

For the modified Wang-distortion function w, we derive the optimal terminal funding ratio and investment strategy for the well-funded case and the underfunded case separately in this section. We apply w to both gains and losses, denote the corresponding distortions by w_+ and w_- and assume for the corresponding parameters $\delta_\pm \in (0,1]$ and $\eta_\pm \in (0,1]$. Therefore, with Lemma 3.2.5, (M) holds. Note that the distortion on the losses is needed to ensure that the problem is well-posed.

Optimal Investment Strategy in the Well-Funded Case

Theorem 3.2.6 (Three-Fund Separation, CPT, $\frac{V(0)}{L(0)} \geq B$)**.** *Let (K) be satisfied and* $\frac{V(0)}{L(0)} \geq B$*. The optimal terminal funding ratio is given by*

$$F^* - B = \bar{F}^* = \frac{\frac{V(0)}{L(0)} - B}{G(\infty)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)}{\eta_+ \Phi^{\eta_+ - 1}\left(\frac{\log Z + (\frac{1}{2} - \delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \right)^{\frac{1}{\alpha - 1}}$$

and the optimal investment strategy is given by

$$\pi^*(t) = \lambda^{CPT}((\bar{F}^*)^+(t, Z_L(t)), B)\pi^{PS} + \left(1 - \lambda^{CPT}((\bar{F}^*)^+(t, Z_L(t)), B)\right)\pi^{LH},$$

with

$$\lambda^{CPT}((\bar{F}^*)^+(t, Z_L(t)), B) = \frac{-Z_L(t)\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))}{(\bar{F}^*)^+(t, Z_L(t)) + B},$$

where the funding ratio at time t corresponding to $(\bar{F}^*)^+$ is denoted by

$$(\bar{F}^*)^+(t, Z_L(t)) := \mathbb{E}_{\mathbb{Q}_L}\left[(\bar{F}^*)^+|\mathcal{F}_t\right]$$

and $\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))$ is the derivative of $(\bar{F}^*)^+(t, Z_L(t))$ with respect to the second component.

Proof. With Lemma 3.2.5,

$$\begin{aligned}
w'_+(\mathbb{Q}_Z(Z)) =& \eta_+ \Phi^{\eta_+-1}\left(\frac{\log Z + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}} - \delta_+\|\sigma_L - \gamma^T\|\sqrt{T}\right) \cdot Z \\
& \cdot \exp\left(\frac{1}{2}(1-\delta_+^2)\|\sigma_L - \gamma^T\|^2 T + (\delta_+ - 1)\left(\log Z + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T\right)\right) \\
=& \eta_+ \Phi^{\eta_+-1}\left(\frac{\log Z + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}} - \delta_+\|\sigma_L - \gamma^T\|\sqrt{T}\right) \\
& \cdot \exp\left(-\frac{1}{2}\delta_+^2\|\sigma_L - \gamma^T\|^2 T + \delta_+\|\sigma_L - \gamma^T\|\sqrt{T}\frac{\log Z + \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) \\
=& \eta_+ \Phi^{\eta_+-1}\left(\frac{\log Z + (\frac{1}{2} - \delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) \\
& \cdot \exp\left(\delta_+ \log Z - \frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right). \\
=& \eta_+ \Phi^{\eta_+-1}\left(\frac{\log Z + (\frac{1}{2} - \delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) \cdot \\
& Z^{\delta_+} \exp\left(-\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right).
\end{aligned}$$

Applying Theorem 3.2.3 for the well-funded case, we obtain $F^* = (\bar{F}^*)^+ + B$, with

$$\begin{aligned}
(\bar{F}^*)^+ =& \frac{\frac{V(0)}{L(0)} - B}{G(\infty)}\left(\frac{Z}{w'_+(\mathbb{Q}_Z(Z))}\right)^{\frac{1}{\alpha-1}} \\
=& \frac{\frac{V(0)}{L(0)} - B}{G(\infty)}\left(\frac{Z^{1-\delta_+}\exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)}{\eta_+\Phi^{\eta_+-1}\left(\frac{\log Z + (\frac{1}{2}-\delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)}\right)^{\frac{1}{\alpha-1}}.
\end{aligned}$$

We use Appendix A.3 with $(\bar{F}^*)^-(t, Z_L(t)) = 0$ to receive the optimal investment strategy for $F^*(t, Z_L(t)) = (\bar{F}^*)^+(t, Z_L(t)) + B$:

$$\begin{aligned}\pi^*(t) =&\frac{1}{F^*(t, Z_L(t))}\left(Z_L(t)\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))(\pi^{LH} - \pi^{PS}) + \left((\bar{F}^*)^+(t, Z_L(t)) + B\right)\pi^{LH}\right)\\ =&\frac{-Z_L(t)\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))}{(\bar{F}^*)^+(t, Z_L(t)) + B}\pi^{PS} + \left(1 - \frac{-Z_L(t)\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))}{(\bar{F}^*)^+(t, Z_L(t)) + B}\right)\pi^{LH}\end{aligned}$$

□

In the preceding theorem, we received again a three-fund separation. Since $(\bar{F}^*)^- = 0$, the funding ratio never falls below the reference point B. In the following corollary, we consider the case $\eta_+ = 1$ to get a more explicit result for λ^{CPT}.

Corollary 3.2.7 (Three-Fund Separation, CPT, $\frac{V(0)}{L(0)} \geq B,\ \eta_+ = 1$)**.** *Let (K) be satisfied, $\frac{V(0)}{L(0)} \geq B$ and $\eta_+ = 1$. The optimal terminal funding ratio is then given by*

$$F^* - B = \bar{F}^* = (\bar{F}^*)^+ = \left(\frac{V(0)}{L(0)} - B\right)\exp\left(\frac{1}{2}\frac{(1-\delta_+)(\delta_+ - \alpha)}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right) Z^{\frac{1-\delta_+}{\alpha - 1}}$$

and the optimal investment strategy is given by

$$\pi^*(t) = \lambda^{CPT}(F^{\pi^*}(t), B)\pi^{PS} + (1 - \lambda^{CPT}(F^{\pi^*}(t), B))\pi^{LH},$$

with

$$\lambda^{CPT}(F^{\pi^*}(t), B) = \frac{F^{\pi^*}(t) - B}{F^{\pi^*}(t)} \cdot \frac{1-\delta_+}{1-\alpha}.$$

Proof. For $\eta_+ = 1$, we have with the proof of Theorem 3.2.6

$$w'_+(\mathbb{Q}_Z(Z)) = Z^{\delta_+}\exp\left(-\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right).$$

Thus, we have with Footnote 2 and with $\nu = \frac{\delta_+ - \alpha}{1-\alpha}$, $c_2 = c$, and $c_1 = 0$

$$\begin{aligned}
G(c) =& \mathbb{E}\left[Z^{\frac{\alpha}{\alpha-1}} w'_+(\mathbb{Q}_Z(Z))^{\frac{1}{1-\alpha}} \mathbb{1}_{Z\leq c}\right] \\
=& \exp\left(-\frac{1}{2}\frac{\delta_+^2 - \delta_+}{1-\alpha} \|\sigma_L - \gamma^T\|^2 T\right) \mathbb{E}\left[Z^{\frac{\delta_+ - \alpha}{1-\alpha}} \mathbb{1}_{Z\leq c}\right] \\
=& \exp\left(\frac{1}{2}\left(\left(\frac{\delta_+ - \alpha}{1-\alpha}\right)^2 - \frac{\delta_+ - \alpha}{1-\alpha} - \frac{\delta_+^2 - \delta_+}{1-\alpha}\right) \|\sigma_L - \gamma^T\|^2 T\right) \\
& \cdot \Phi\left(\frac{\log c + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right) \|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) \\
=& \exp\left(\frac{1}{2}\frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2} \|\sigma_L - \gamma^T\|^2 T\right) \Phi\left(\frac{\log c + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right) \|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right),
\end{aligned}$$

since

$$\begin{aligned}
\left(\frac{\delta_+^2 - \alpha}{1-\alpha}\right)^2 - \frac{\delta_+ - \alpha}{1-\alpha} - \frac{\delta_+^2 - \delta_+}{1-\alpha} &= \frac{(\delta_+ - \alpha)^2 + (1-\alpha)(\alpha - \delta_+^2)}{(1-\alpha)^2} \\
&= \frac{-2\delta_+\alpha + \alpha + \alpha\delta_+^2}{(1-\alpha)^2} = \frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2}.
\end{aligned}$$

Applying Theorem 3.2.6, the optimal funding ratio reads

$$\begin{aligned}
(\bar{F}^*)^+ =& \frac{\frac{V(0)}{L(0)} - B}{G(\infty)} \left(Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)\right)^{\frac{1}{\alpha-1}} \\
=& \left(\frac{V(0)}{L(0)} - B\right) \exp\left(-\frac{1}{2}\frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right) \\
& \cdot Z^{\frac{1-\delta_+}{\alpha-1}} \exp\left(\frac{1}{2}\frac{\delta_+^2 - \delta_+}{\alpha - 1}\|\sigma_L - \gamma^T\|^2 T\right) \\
=& \left(\frac{V(0)}{L(0)} - B\right) \exp\left(\frac{1}{2}\frac{(1-\delta_+)(\delta_+ - \alpha)}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right) Z^{\frac{1-\delta_+}{\alpha-1}},
\end{aligned}$$

as

$$\begin{aligned}
-\frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2} + \frac{\delta_+^2 - \delta_+}{\alpha - 1} =& \frac{-(\delta_+ - 1)^2\alpha + \delta_+(\delta_+ - 1)(\alpha - 1)}{(1-\alpha)^2} \\
=& \frac{(1-\delta_+)(\delta_+\alpha - \alpha - \delta_+\alpha + \delta_+)}{(1-\alpha)^2} = \frac{(\delta_+ - 1)(\delta_+ - \alpha)}{(1-\alpha)^2}.
\end{aligned}$$

The value of $(\bar{F}^*)^+$ at time t, denoted by $(\bar{F}^*)^+(t, Z_L(t))$, can be calculated using (A.12) in Appendix A.3 with $\nu = \frac{1-\delta_+}{\alpha-1}$, $c_2 = \infty$, and $c_1 = 0$ and is given by

$$
\begin{aligned}
(\bar{F}^*)^+(t, Z_L(t)) = &\left(\frac{V(0)}{L(0)} - B\right) \exp\left(\frac{1}{2}\frac{(1-\delta_+)(\delta_+-\alpha)}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right) Z_L(t)^{\frac{1-\delta_+}{\alpha-1}} \cdot \\
&\exp\left(\frac{1}{2}\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 (T-t)\right) \\
= &\left(\frac{V(0)}{L(0)} - B\right) Z_L(t)^{\frac{1-\delta_+}{\alpha-1}} \exp\left(-\frac{1}{2}\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 t\right),
\end{aligned}
$$

due to

$$
\begin{aligned}
\frac{(1-\delta_+)(\delta_+-\alpha)}{(1-\alpha)^2} &= \frac{(1-\delta_+)(\delta_+-1+1-\alpha)}{(1-\alpha)^2} \\
&= -\frac{(1-\delta_+)^2 + (1-\delta_+)(\alpha-1)}{(1-\alpha)^2} \\
&= -\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}.
\end{aligned}
$$

With (A.13) in Appendix A.3, the corresponding replicating strategy can be written as

$$
\pi_+(t) = \pi^{LH} + \frac{1-\delta_+}{1-\alpha}(\pi^{PS} - \pi^{LH}).
$$

As $(\bar{F}^*)^-(t, Z_L(t)) = 0$, the replicating strategy for $F^*(t, Z_L(t)) = (\bar{F}^*)^+(t, Z_L(t)) + B$ is with (A.11) in Appendix A.3 given by

$$
\begin{aligned}
\pi^*(t) =& \frac{1}{(\bar{F}^*)^+(t, Z_L(t)) + B}\left((\bar{F}^*)^+(t, Z_L(t))\pi_+(t) + B\pi^{LH}\right) \\
=& \pi^{LH} + \frac{(\bar{F}^*)^+(t, Z_L(t))}{(\bar{F}^*)^+(t, Z_L(t)) + B} \cdot \frac{1-\delta_+}{1-\alpha}(\pi^{PS} - \pi^{LH}) \\
=& \pi^{LH} + \frac{F^*(t, Z_L(t)) - B}{F^*(t, Z_L(t))} \cdot \frac{1-\delta_+}{1-\alpha}(\pi^{PS} - \pi^{LH}).
\end{aligned}
$$

□

Again, π^* can be represented as a three-fund separation with the liability hedging portfolio, the performance seeking portfolio and the risk-free asset. With $1-\alpha > 0$, we interpret $\frac{1-\delta_+}{1-\alpha}$ as a CPT-distorted factor of the risk appetite of the investor. As observed before, a higher δ_+ emphasizes the lower asset returns. This leads to a lower allocation in the performance seeking portfolio, a higher allocation in the liability hedging portfolio and therefore a more cautious investment strategy. We also observe that the investment in the performance seeking portfolio corresponds to a CPPI-strategy with multiplier $m := \frac{1-\delta_+}{1-\alpha}$. This component ensures, together with the allocation in the liability hedging portfolio, that the funding ratio never falls below the reference point B.

Thus, the funding ratio is always in the area of the CPT utility, in which the investor is risk-averse and the risk-seeking part, i.e. U_-, does not have an impact on the strategy.

Optimal Investment Strategy with Initial Underfunding

In case $\frac{V(0)}{L(0)} < B$ and $c^* = 0$ being the only solution to (C), the optimal terminal funding ratio is with Theorem 3.2.3 given by

$$F^* = \frac{V(0)}{L(0)}.$$

In this case the optimal strategy is to hedge the liabilities perfectly with

$$\pi^{LH} = (\sigma^T)^{-1}\sigma_L^T.$$

With this investment strategy, the funding ratio is kept constant. For the investor, the risk of an even lower funding ratio outweighs the potential profit generated by an investment in the performance seeking portfolio. Since $Z \leq c^* \Leftrightarrow F^* \geq B$,

$$\mathbb{Q}(F^* \geq B) = \mathbb{Q}(Z \leq 0) = 0,$$

for $c^* = 0$. This means that there is zero probability that funded or well-funded status can be achieved and thus no reason to risk anything and try. Therefore, we only consider the case $c^* > 0$ in the following.

Proposition 3.2.8 (Optimal Terminal Funding Ratio, CPT, $\frac{V(0)}{L(0)} < B$)**.** *Let* $\inf_{c>0} k(c) > 1$ *be satisfied and* $\frac{V(0)}{L(0)} < B$*. If* (C) *has an optimal solution* $c^* > 0$*, the optimal terminal funding ratio is given by*

$$F^* - B = \bar{F}^* = (\bar{F}^*)^+ - (\bar{F}^*)^-,$$

with

$$(\bar{F}^*)^+ = \frac{v_+^*}{G(c^*)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)}{\eta_+ \Phi^{\eta_+ - 1}\left(\frac{\log Z + (\frac{1}{2} - \delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \right)^{\frac{1}{\alpha - 1}} \mathbb{1}_{Z \leq c^*}$$

$$(\bar{F}^*)^- = \frac{v_+^* - \frac{V(0)}{L(0)} + B}{1 - \Phi\left(\frac{\log c^* - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \mathbb{1}_{Z > c^*}$$

and

$$v_+^* = \frac{B - \frac{V(0)}{L(0)}}{k(c^*)^{\frac{1}{1-\alpha}} - 1}.$$

Proof. We proceed as in the proof of Theorem 3.2.6 and receive by an application of Theorem 3.2.3

$$\begin{aligned}(\bar{F}^*)^+ =& \frac{v_+^*}{G(c^*)} \left(\frac{Z}{w'_+(\mathbb{Q}_Z(Z))} \right)^{\frac{1}{\alpha-1}} \mathbb{1}_{Z \le c^*} \\ =& \frac{v_+^*}{G(c^*)} \left(\frac{Z^{1-\delta_+} \exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)}{\eta_+ \Phi^{\eta_+ - 1}\left(\frac{\log Z + (\frac{1}{2} - \delta_+)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \right)^{\frac{1}{\alpha-1}} \mathbb{1}_{Z \le c^*}.\end{aligned}$$

The other part is with Footnote (2) given by

$$(\bar{F}^*)^- = \frac{v_+^* - \frac{V(0)}{L(0)} + B}{\mathbb{E}[Z \mathbb{1}_{Z > c^*}]} \mathbb{1}_{Z > c^*} = \frac{v_+^* - \frac{V(0)}{L(0)} + B}{1 - \Phi\left(\frac{\log c^* - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \mathbb{1}_{Z > c^*}.$$

□

While the CPPI-part for the performance seeking portfolio in the well-funded case ensures that the funding ratio never falls below the reference point B (and therefore never falls into the risk seeking area U_- of the utility function in T), the optimal investment strategy in the underfunded case corresponds to a leveraged strategy. Starting in the risk-seeking area U_- of the CPT utility, the investor tries to achieve a terminal funding ratio above the reference point B. In case the financial market evolves in a favorable way, more precise, if $Z \le c^*$, then the investor receives the terminal funding ratio $(\bar{F}^*)^+ + B$. Otherwise, i.e. $Z > c^*$, the investor suffers a loss and receives the constant funding ratio $B - (\bar{F}^*)^-$. Again, we consider the case $\eta = 1$ for a more explicit result.

Theorem 3.2.9 (Three-Fund Separation, CPT, $\frac{V(0)}{L(0)} < B$, $\eta_+ = 1$)**.** *Let* $\inf_{c>0} k(c) > 1$, $\frac{V(0)}{L(0)} < 0$ *and* $\eta_+ = 1$. *If* (C) *has an optimal solution* $c^* > 0$, *the optimal terminal funding ratio is given by*

$$F^* - B = \bar{F}^* = (\bar{F}^*)^+ - (\bar{F}^*)^-,$$

with

$$
\begin{aligned}
(\bar{F}^*)^+ =& \frac{v_+^*}{\Phi\left(\frac{\log c^* + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \cdot \\
& \exp\left(\frac{1}{2}\frac{(1-\delta_+)(\delta_+ - \alpha)}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right) Z^{\frac{1-\delta_+}{\alpha - 1}} \mathbb{1}_{Z \le c^*}, \\
(\bar{F}^*)^- =& \frac{v_+^* - \frac{V(0)}{L(0)} + 1}{1 - \Phi\left(\frac{\log c^* - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \mathbb{1}_{Z > c^*}
\end{aligned}
$$

and

$$
v_+^* = \frac{B - \frac{V(0)}{L(0)}}{k(c^*)^{\frac{1}{1-\alpha}} - 1}.
$$

The optimal investment strategy is given by

$$
\pi^*(t) = \lambda^{CPT} \cdot \pi^{PS} + (1 - \lambda^{CPT}) \cdot \pi^{LH},
$$

with

$$
\begin{aligned}
\lambda^{CPT} =& \frac{1}{F(t)^\pi} \cdot \left(F_+^\pi(t) \cdot \lambda_+ - F_-^\pi(t) \cdot \lambda_-\right) \\
\lambda_+ =& \left(\frac{1-\delta_+}{1-\alpha} + \frac{1}{\|\sigma_L - \gamma^T\|\sqrt{T-t}} \cdot \frac{\phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right)}{\Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right)}\right), \\
\lambda_- =& \frac{\phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)}{1 - \Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)} \cdot \frac{1}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}, \\
F_+^\pi(t) =& \frac{v_+^* Z_L(t)^{\frac{1-\delta_+}{\alpha-1}} \cdot \Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right)}{\Phi\left(\frac{\log c^* - \frac{1}{2}\cdot\frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \cdot \exp\left(-\frac{1}{2}\left(\frac{1-\delta_+}{\alpha-1} + 1\right) \cdot \frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 t\right), \\
F_-^\pi(t) =& \frac{v_+^* - \frac{V(0)}{L(0)} + B}{1 - \Phi\left(\frac{\log c^* - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \cdot \left(1 - \Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)\right),
\end{aligned}
$$

ϕ denoting the density function of a standard normally distributed random variable and $d(c,s,v) := \frac{\log(c) - \left(v + \frac{1}{2}\right)\|\sigma_L - \gamma^T\|^2 s}{\|\sigma_L - \gamma^T\|\sqrt{s}}$.

Proof. From the proof of Corollary 3.2.7, we know that

$$G(c^*) = \exp\left(\frac{1}{2}\frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2}\|\sigma_L - \gamma^T\|^2 T\right)\Phi\left(\frac{\log c^* + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right).$$

Applying Proposition 3.2.8 and inserting $G(c^*)$ leads to

$$\begin{aligned}(\bar{F}^*)^+ =& \frac{v_+^*}{G(c^*)}\left(Z^{1-\delta_+}\exp\left(\frac{1}{2}(\delta_+^2 - \delta_+)\|\sigma_L - \gamma^T\|^2 T\right)\right)^{\frac{1}{\alpha-1}} \mathbb{1}_{Z \le c^*} \\ =& \frac{v_+^*}{\Phi\left(\frac{\log c^* + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)} \cdot \\ & \exp\left(-\frac{1}{2}\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 T\right) Z^{\frac{1-\delta_+}{\alpha-1}}\mathbb{1}_{Z \le c^*},\end{aligned}$$

since (see the proof of Corollary 3.2.7)

$$-\frac{(\delta_+ - 1)^2\alpha}{(1-\alpha)^2} + \frac{\delta_+^2 - \delta_+}{\alpha - 1} = \frac{(1-\delta_+)(\delta_+ - \alpha)}{(1-\alpha)^2} = -\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}.$$

With (A.12) from Appendix A.3,

$$\begin{aligned}(\bar{F}^*)^+(t, Z_L(t)) =& \frac{v_+^* Z_L(t)^{\frac{1-\delta_+}{\alpha-1}}}{\Phi\left(\frac{\log c^* + \left(\frac{1}{2} - \frac{\delta_+ - \alpha}{1-\alpha}\right)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)}\Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right) \\ & \cdot \exp\left(-\frac{1}{2}\left(\frac{1-\delta_+}{\alpha-1}+1\right)\frac{1-\delta_+}{\alpha-1}\|\sigma_L - \gamma^T\|^2 t\right)\end{aligned}$$

and the corresponding replicating strategy is with (A.13) from Appendix A.3 given by

$$\pi_+(t) = \pi^{LH} + \left(\frac{1-\delta_+}{1-\alpha} + \frac{1}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}\frac{\phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right)}{\Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, \frac{1-\delta_+}{\alpha-1}\right)\right)}\right)(\pi^{PS} - \pi^{LH}).$$

The second part of the terminal funding ratio is also given by Proposition 3.2.8. With (A.12) from Appendix A.3, $c_2 = \infty, c_1 = c^*$ and $\nu = 0$, we have

$$(\bar{F}^*)^-(t, Z_L(t)) = \frac{v_+^* - \frac{V(0)}{L(0)} + B}{1 - \Phi\left(\frac{\log c^* - \frac{1}{2}\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)}\left(1 - \Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)\right).$$

Moreover, the replicating strategy π_- of $(\bar{F}^*)^-$ reads

$$\pi_- = \pi^{LH} - \frac{\phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)}{1-\Phi\left(d\left(\frac{c^*}{Z_L(t)}, T-t, 0\right)\right)} \frac{1}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}(\pi^{PS} - \pi^{LH}).$$

Finally, the total optimal investment strategy consisting of both parts can be written as (see (A.11) in Appendix A.3)

$$\pi^*(t) = \frac{1}{F^*(t, Z_L(t))}\left((\bar{F}^*)^+(t, Z_L(t))\pi_+(t) - (\bar{F}^*)^-(t, Z_L(t))\pi_-(t) + B\pi^{LH}\right),$$

which corresponds to the representation from the statement of the theorem as $F^*(t, Z_L(t)) = (\bar{F}^*)^+(t, Z_L(t)) - (\bar{F}^*)^-(t, Z_L(t)) + B$.

□

3.3 Connection Between Expected Utility and CPT Funding Ratio Optimization

In the previous section, we introduced the CPT funding ratio optimization as a generalization of the expected utility framework. We illustrate the connections between the results from Corollary 3.1.2 and Corollary 3.2.7 in this section. While the optimal investment strategy in the expected utility is a constant mix strategy, so the allocations in the performance seeking portfolio and the liability hedging portfolio are independent of $F(t)$, the optimal investment strategy for the CPT approach with $\frac{V(0)}{L(0)} \geq B$ dynamically depends on $F(t)$. In Figure 3.3 and Figure 3.4, the weights of the performance seeking portfolio and the liability hedging portfolio in $t = 0$ are illustrated. We choose $\alpha = -1$ for both, the expected utility approach with power utility, and the CPT approach and consider only the downside risk component of the distortion by setting $\eta = 1$ in the CPT example. The optimal investment strategy in the CPT example differs from the optimal strategy in the expected utility example by the consideration of the reference point B and the factor $1-\delta_+$ caused by the distortion. Figures 3.3 and 3.4 show the convergence of the allocations in the CPT approach to the allocations of the expected utility approach as the distortion is reduced ($\delta_+ \to 0$) and as the reference point is decreased ($B \to 0$). Hence, the CPT approach can be considered as a generalization of the expected utility approach with the expected utility approach representing the special case of a reference point $B = 0$ and without distortion ($\delta_+ = 0$, $\eta = 1$).

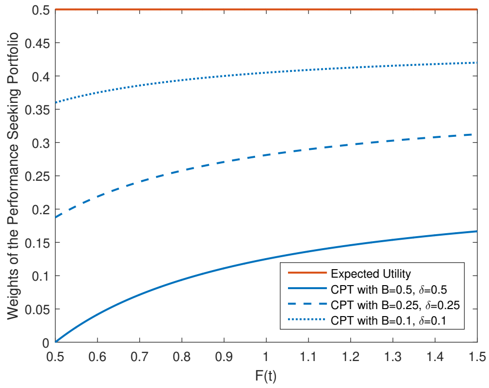

Figure 3.3: Weights of the performance seeking portfolio in the different settings.

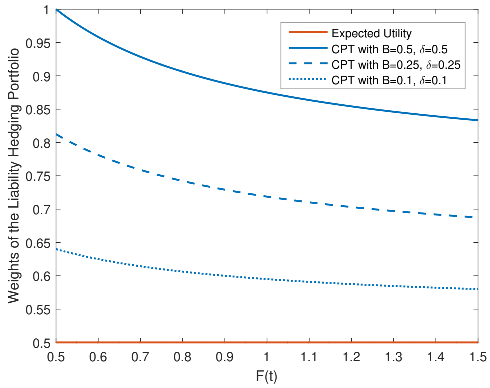

Figure 3.4: Weights of the liability hedging portfolio in the different settings.

4 Surplus Optimization

In this chapter, we maximize the utility of the terminal surplus. In Section 4.1, we introduce the liability models and present the surplus optimization problem. The generalized martingale approach is introduced in Section 4.2 and applied to receive the optimal terminal wealth. Furthermore, the corresponding investment strategy is derived. For several specific liability models, more explicit results are presented in Sections 4.3 and 4.4 and compared in Section 4.5. Section 4.6 contains a comparison to results from Chapter 3. Large parts of this chapter coincide with Desmettre et al. (2020) and minor parts coincide with Brummer et al. (2018). Throughout the whole chapter, we assume $d = 1$, i.e. we consider only one risky asset.

4.1 Surplus Optimization Framework

For the maximization of the expected utility of the surplus of the assets over the liabilities at time T, we define the surplus as introduced in Sharpe and Tint (1990) by

$$S(T) := V(T) - \psi_L L(T, V(T)).$$

The factor $\psi_L \in (0, 1]$ is constant as in Sharpe and Tint (1990) and allows for a flexible portion of consideration of the liabilities. The liabilities $L(T, V(T))$ are further specified in the following sections.

4.1.1 General Liability Model

To set up the portfolio optimization problem, we introduce the liabilities and the generalized martingale approach in a way which is inspired by Desmettre and Seifried (2016). They develop a two-step method for a setting that includes an illiquid asset which can only be traded at the beginning of the time horizon. In the first step, the authors find an optimal liquid portfolio given an arbitrary, but fixed illiquid investment and in the second step, they optimize the amount invested in the illiquid asset. We extend the methods used in the first step to our setting by interpreting the liability position as a short position in an illiquid asset and allowing for such a short position. Furthermore, we derive closed-form solutions for the optimal trading strategies in the setting of replicable liabilities and for various types of liabilities with performance participation. We assume that the value of the liabilities at time T can be modeled by random variables $L_1(T)$

and $L_2(T)$ and that we cannot invest in the liabilities directly. $L_1(T)$ will be used to model a performance-participating part and $L_2(T)$ will be used to model a part which is not directly depending on wealth. $L_2(T)$ can be interpreted as an index-linked part. For both, L_1 and L_2, we want to allow for hedgeable and non-hedgeable components. The hedgeable risks can be interpreted as, e.g., interest-rate risks and the non-hedgeable risks as, e.g., inflation risks (income growth of the policy holder), mortality risk or operational risk inherent in the liabilities. The hedgeable components are modeled by a stochastic process X which satisfies the following assumption.

Assumption (LX). *The process X follows*

$$dX(t) = \mu_X(t, X(t))dt + \sigma_X(t, X(t))dW(t).$$

For $t \in [0, T]$, $X(T)$ can be written as a function of $X(t)$ and an increment $\xi(t, T)$, i.e.

$$X(T) = g(X(t), \xi(t, T)),$$

where $\xi(t, T)$ is independent of $X(t)$ and g is twice differentiable with respect to the first component.

Note that $X(T)$ is replicable, since X is driven by the same Brownian motion W as the risky asset. To model the unhedgeable components of the liabilities, we use $\mathcal{G}$-measurable random variables $\mathcal{U}_1$ and $\mathcal{U}_2$. We consider liabilities $L(T, v)$ and always impose the following assumptions (L1) and (L2). (L1) states the general structure of the liabilities. (L2) contains conditions to exclude the possibility of unavoidable bankruptcy.

Assumption (L1). *The liabilities are of the form*

$$L(T, v) = L(T, v, X(T), \mathcal{U}_1, \mathcal{U}_2) = vL_1(T, X(T), \mathcal{U}_1) + L_2(T, X(T), \mathcal{U}_2),\ v > 0, \quad \text{(L1)}$$

with non-negative, $\mathcal{H}$-measurable functions $L_i(T, X(T), \mathcal{U}_i)$, $i = 1, 2$.

We call the left part of the sum $vL_1(T, X(T), \mathcal{U}_1)$ *performance-linked* and the right part $L_2(T, X(T), \mathcal{U}_2)$ *index-linked*. To simplify the notation, we write $L(T)$ or $L(T, v)$ instead of $L(T, v, X(T), \mathcal{U}_1, \mathcal{U}_2)$ and $L_i(T)$ instead of $L_i(T, X(T), \mathcal{U}_i)$, $i = 1, 2$ sometimes. For each $\omega \in \Omega$, we further assume the existence of the worst case scenario with respect to the unhedgeable risks defined by

$$\hat{\omega}_i := \arg \sup_{\hat{\omega} \in \Omega} L_i(T, X(T, \omega), \mathcal{U}_i(\hat{\omega})),\ i = 1, 2.$$

In T, the liabilities are covered even for the worst outcomes of the unhedgeable risk components, if

$$V(T, \omega) \geq \psi_L V(T, \omega) L_1(T, X(T, \omega), \mathcal{U}_1(\hat{\omega}_1)) + \psi_L L_2(T, X(T, \omega), \mathcal{U}_2(\hat{\omega_2})), \quad (4.1)$$

i.e. if $V(T,\omega) \geq \hat{v}_0(\omega)$ with

$$\hat{v}_0(\omega) := \frac{\psi_L L_2(T, X(T,\omega), \mathcal{U}_2(\hat{\omega}_2))}{1 - \psi_L L_1(T, X(T,\omega), \mathcal{U}_1(\hat{\omega}_1))}. \tag{4.2}$$

Assumption (L2). *There exists a constant* $k_1 \in \left[0, \frac{1}{\psi_L}\right)$*, and a* $\tilde{\mathbb{Q}}$*-integrable random variable* $k_2(\omega)$ *such that* $\mathbb{Q}$*-a.s.*

$$k_1 \geq L_1(T, X(T), \mathcal{U}_1) \tag{L2.1}$$

$$\text{and } k_2(\omega) \geq \sup_{\hat{\omega}\in\Omega} L_2(T, X(T,\omega), \mathcal{U}_2(\hat{\omega})). \tag{L2.2}$$

Furthermore,

$$v_0 \geq \mathbb{E}\left[\tilde{Z}(T)\hat{v}_0(\omega)\right]. \tag{L2.3}$$

In particular, this means that the investor has enough initial capital to hedge the worst outcomes of the unhedgeable risk components associated with $\mathcal{U}_i$, $i = 1, 2$. For a pension plan or insurance company with the liabilities consisting of the discounted cash flows of the future payments, an upper bound could be, e.g., the sum of all the (non-discounted) payments or a cap in benefits to the policy holders (see Ekern (1996)).

4.1.2 Specific Liability Models

In this section, we present some specific liability models, for which we obtain more explicit results later. We also discuss Assumptions (LX), (L1) and (L2), where (L2) is always evaluated assuming that the investor has enough initial capital, i.e. (L2.3) holds.

In some examples, we assume that X follows a geometric Brownian motion

$$dX(t) = X(t)\Big(\hat{\mu}_X dt + \hat{\sigma}_X dW(t)\Big),\ X(0) = 1, \tag{4.3}$$

with constant coefficients $\hat{\mu}_X$ and $\hat{\sigma}_X$ and W being the Brownian motion which also drives the risky asset. Since

$$X(T) = X(t) e^{\left(\hat{\mu}_X - \frac{1}{2}\hat{\sigma}_X^2\right)(T-t) + \hat{\sigma}_X (W(T)-W(t))},$$

X satisfies Assumption (LX) with $\xi(t,T) = e^{\left(\hat{\mu}_X - \frac{1}{2}\hat{\sigma}_X^2\right)(T-t) + \hat{\sigma}_X (W(T)-W(t))}$ and $g(x,\xi) = x\xi$. We assume that the investor cannot trade in X, only in the risky asset P and the risk-free asset P_0. Since the Brownian motions driving P and X are the same, $X(T)$ is still replicable with P. However, since trading in X is not allowed, the model is also free of arbitrage.

Example 4.1.1 (Geometric Brownian Motion)**.** *For the liabilities from Chapter 3 as in* (3.1)*,* (L2.2) *only holds if there are no unhedgeable risks, i.e.* $\sigma_\epsilon = 0$*. In this case, a version of the liabilities in the market with* $d = 1$ *of the liabilities from Chapter 3 is given with*

$$L_1(T) = 0,\ L_2(T, X(T)) = X(T),$$

X *as in* (4.3)*, with* $\hat{\mu}_X = \mu_L$*,* $\hat{\sigma}_X = \|\sigma_L\|$ *and* $k_2(\omega) = X(T, \omega)$*. These liabilities satisfy Assumptions (LX), (L1) and (L2) with* $k_1 = 0$*.*

Example 4.1.2 (Replicable Liabilities)**.** *If we do not consider unhedgeable risks associated with* $\mathcal{U}_i, i = 1, 2$*, but a general process* X *satisfying Assumption (LX) as well as general* $L_1(T)$ *and* $L_2(T)$ *satisfying Assumption (L2), we have*

$$L(T, v) = vL_1(T) + L_2(T) = vL_1(T, X(T)) + L_2(T, X(T)). \tag{RL}$$

In this case, we can set $k_2(\omega) := L_2(T, X(T, \omega))$*.*

Example 4.1.3 (Index-Linked Liabilities with Capped Maximum Benefits (ILCB))**.** *Motivated by Ekern (1996), we consider liabilities of the form*

$$L_1(T) = 0,\ \ L(T, X(t)) = L_2(T, X(T)) = L(0)f(X(T)),\ L(0) > 0,\ t \in [0, T], \tag{ILCB}$$

with capped maximum benefits, i.e.

$$f(x) = \min\{x, K\} \tag{4.4}$$

and X *as in* (4.3)*. This type of liabilities satisfies Assumptions (LX), (L1) and (L2) with* $L_1(T) = 0$*,* $L_2(T, X(t)) = L(0)f(X(T))$*,* $k_1 = 0$ *and* $k_2(\omega) \equiv L(0)K$*. The cap can be interpreted, e.g., as a special product feature to limit the insurance company's risk (see Ekern (1996)), as an implicit guarantee by the supervising authority to change rules in case of industry-wide underfunding, or as some form of natural upper-bound as described in Section 4.1.1.*

In the following examples, we assume that the payment to the policy holder is depending on the performance of the asset portfolio of the insurance company. Such mechanisms can be found in various types of insurance contracts, see e.g. Korn and Wagner (2018) or Kling et al. (2009).

Example 4.1.4 (Performance-Linked Liabilities (PLU))**.** *We consider performance-linked liabilities with unhedgeable risks of the form*

$$L_2(T) = 0,\ \ L(T, v) = vL_1(T, X(T), \mathcal{U}_1), \tag{PLU}$$

with general, $L_1(T, X(T), \mathcal{U}_1)$ *satisfying Assumptions (LX) and (L2). Since the future payments to the clients, and subsequently the value of the liabilities, depend on the performance of the assets for various insurance products, the value of the liabilities is assumed* *to be proportional to v here. The term $L_1(T)$ can be used to model a component of the liabilities that is not directly connected to the wealth. This may include both unhedgeable risks, e.g. mortality risk and hedgeable risks such as interest-rate risk.*

Example 4.1.5 (Performance-Linked Liabilities with Capped Benefits and Unhedgeable Risks (PLCBU))**.** *We specify the model from the previous example further using an affine model. The use of an affine model in this context can also be found in Höcht et al. (2008). In this example, we assume that $\mathcal{U}_1$ is uniformly distributed on $[c_1, c_2]$, with $c_1, c_2 \geq 0$, $c_1 < c_2$. Furthermore, we consider liabilities of the form*

$$L_2(T) = 0,\;\; L(T, v) = vL_1(T, X(T), \mathcal{U}_1) = vL(0)\left(\beta_1 f(X(T)) + \beta_2 \mathcal{U}_1\right),\;\; L(0) > 0, \tag{PLCBU}$$

with X as in (4.3)*, f being a strictly positive function, which is bounded from above by a constant K with $0 < K < \frac{1}{\beta_1}\left(\frac{1}{\psi_L L(0)} - \beta_2 c_2\right)$, $\beta_1, \beta_2 \geq 0$, and almost everywhere twice continuously differentiable. These liabilities are a special case of* (PLU) *and satisfy Assumptions (LX) (see above) and (L2) with $L_1(T) = L(0)(\beta_1 f(X(T)) + \beta_2 \mathcal{U}_1)$, $L_2(T) = 0$, $k_1 = L(0)(\beta_1 K + \beta_2 c_2)$ and $k_2(\omega) = 0$ (see also the proof of Corollary 4.4.3 for* (L2.1)*). In addition to the consideration of a general f, we also deal with special choices of f:*

Choosing f as in (4.4) *introduces a positive correlation between the risky asset and $L_1(T)$. In the context of performance-linked liabilities, this leads to liabilities, which are more sensitive to market changes than the wealth process. If we use the function*

$$f(x) = \min\left\{\frac{1}{x}, K\right\}, \tag{4.5}$$

this leads to a framework in which the policy holder participates only partially in the performance of the assets. In particular, the liabilities can be written as

$$L(T, v) = vL_1(T, X(T), \mathcal{U}_1) = L(0)\left(\beta_1 \min\left\{\frac{v}{X(T)}, vK\right\} + \beta_2 \mathcal{U}_1 v\right)$$

and can therefore be interpreted as a capped relative performance of the insurance company's wealth compared to an index. In addition, there is an unhedgeable component which can be nicely interpreted in the context of mortality risk: while the first term $\beta_1 \min\left\{\frac{v}{X(T)}, vK\right\}$ includes current estimates of the mortality, additional capital $\beta_2 \mathcal{U}_1 v$ must be provided to cover the risk that the mortality changes more than expected in an unfavorable way. In this context, the exact amount of additional capital required is unknown in $t = 0$.

Example 4.1.6 (Performance Linked Liabilities with Capped Benefits (PLCBU*))**.** *In Chapter 3, we considered, inspired by Martellini (2006), liabilities modeled as geometric Brownian motions which may also include unhedgeable risks. We adapt this model for* *the liabilities to the context of performance-linked liabilities in this section. We consider more general liabilities of the form*

$$L_2(T) = 0, \;\; L(T, v) := vL_1(T, X(T), \mathcal{U}_1),$$

with

$$L_1(T, X(T), \mathcal{U}_1) := L(0)f(X(T)\mathcal{U}_1(T)), \;\; L(0) > 0, \tag{PLCBU*}$$

with X as in (4.3)*, f being a strictly positive function, which is bounded from above by a constant K with $0 < K < \frac{1}{\psi_L L(0)}$, almost everywhere twice continuously differentiable and $\mathcal{U}_1(t)$ given by the SDE*

$$d\mathcal{U}_1(t) = \mathcal{U}_1(t)\hat{\sigma}_\epsilon dW_\epsilon(t), \;\; \mathcal{U}_1(0) = 1,$$

with constant $\hat{\sigma}_\epsilon$ and W_ϵ being a Brownian motion which is independent of W. As in the previous example, these liabilities are a special case of (PLU) *and satisfy Assumptions (LX) (see above) and (L2) with $L_1(T) = L(0)f(X(T)\mathcal{U}_1)$, $L_2(T) = 0$, $k_1 = L(0)K$ and $k_2(\omega) = 0$ (see also the proof of Corollary 4.4.7 for* (L2.1)*). In addition to the consideration of a general f, we also deal with special choices of f:*

Scheuenstuhl and Zagst (2008) use geometric Brownian motions to model stock prices with a market risk component and an idiosyncratic component. We proceed similarly to model an index and the risk that the actual portfolio of the insurance company deviates from this index. For X as in (4.3) *representing an index, we interpret $X\mathcal{U}_1$ as a fund which uses the index X as a benchmark. The SDE of the fund is with Itô's formula given by*

$$\begin{aligned} d(X(t)\mathcal{U}_1(t)) &= X(t)\mathcal{U}_1(t)\hat{\sigma}_\epsilon dW_\epsilon(t) + \mathcal{U}_1(t)X(t)\left[\hat{\mu}_X dt + \hat{\sigma}_X dW(t)\right] \\ &= X(t)\mathcal{U}_1(t)\left[\hat{\mu}_X dt + \hat{\sigma}_X dW(t) + \hat{\sigma}_\epsilon dW_\epsilon(t)\right], \end{aligned}$$

with $\hat{\sigma}_\epsilon$ representing the risk that the portfolio deviates from the index. For f as in (4.5)*, the value of the liabilities can be written as*

$$L(T, v) = vL_1(T, X(T)) = L(0)\min\left\{\frac{v}{X(T)\mathcal{U}_1(T)}, vK\right\}.$$

As in the previous example, the liabilities can be interpreted as a capped relative performance of an asset portfolio, which is, in this case, compared to a fund $X\mathcal{U}_1$.

Remark 4.1.7. *This liability model can be interpreted as a performance-linked version of the model in Chapter 3 for the special case of X being a Brownian motion. In contrast to Chapter 3 and Example 4.1.1, the cap is necessary here to avoid the possibility of bankruptcy caused by unlimited unhedgeable risks. In a pure funding ratio optimization, such as in the expected utility setting in Chapter 3 or in a setting with $L_2(T, X(T))$ instead of $L_1(T, X(T))$ as in Example 4.1.1, no cap is necessary.*

4.1.3 Surplus Optimization Problem

We now introduce the portfolio optimization problem for a general utility function as in Definition 2.2.1 with $\mathbb{V} = (0, \infty)$. In contrast to Chapter 3, where the CPT utility function was also defined for a funding level below the reference point, we exclude a negative surplus here due to the use of a traditional utility function. The set of admissible strategies $\Lambda'(v_0)$ corresponding to the initial wealth v_0 contains all admissible strategies which satisfy

$$V(t) - \mathbb{E}\left[\tilde{Z}(t,T)\hat{v}_0(\omega)|\mathcal{F}_t\right] \geq 0 \;\; \mathbb{Q}\text{-a.s. for all } t \in [0,T] \tag{4.6}$$

and $\mathbb{E}\left[U^-(S(T))\right] < \infty$.

Remark 4.1.8. *For $V(T,\omega) \geq \hat{v}_0(\omega)$, the investor has enough capital to cover the worst outcomes of the unhedgebale risks in T (see* (4.1) *and* (4.2)*). Consequently, if* (4.6) *holds, the investor has enough capital to cover the present value of the liabilities with respect to the worst outcomes of the unhedgeable risk components in $t \in [0,T]$. Note that we assume that the investor has enough initial capital (see Assumption (L2.3)).*

We aim at finding the optimal allocation in the risky asset and the risk-free asset such that the *expected utility of the terminal surplus is maximized*:

$$\max_{\pi \in \Lambda'(v_0)} \mathbb{E}\left[U(S(T))\right] = \max_{\pi \in \Lambda'(v_0)} \mathbb{E}\left[U(V^\pi(T) - \psi_L L(T, V(T)))\right] \tag{P$_\text{S}$}$$

4.2 Generalized Martingale Approach for Surplus Optimization

In this section, we present the generalized martingale approach, which we adapt from Desmettre and Seifried (2016).

Remark 4.2.1. *We wish to stress again that we focus on the maximization of the expected terminal surplus*

$$\max_{\pi \in \Lambda'(v_0)} \mathbb{E}\left[U(V^\pi(T) - \psi_L L(T, V^\pi(T)))\right],$$

in the presence of liabilities. In contrast, Desmettre and Seifried (2016) have focused on the optimization with fixed-term securities

$$\max_{(\psi,\pi) \in \Lambda'(v_0)} \mathbb{E}\left[U(V^{(\psi,\pi)}(T) + \psi F(T))\right],$$

where ψ denotes the units invested into the fixed-term security $F(T)$, and Seifried (2010) has focused on the optimal portfolio problem with deferred capital gains taxes

$$\max_{\pi \in \Lambda'(v_0)} \mathbb{E}\left[U\left(v_0 \mathbb{1}_{\{V^\pi(T) \le v_0\}} + (1-\alpha)\left(V^\pi(T) - v_0\right)\mathbb{1}_{\{V^\pi(T) > v_0\}}\right)\right],$$

where $\alpha \in [0,1)$ is the investor's personal tax rate.

Different from Desmettre and Seifried (2016), where an optimization over ψ leads to the investment in the fixed-term asset, we assume ψ_L to be constant since it is in our setting rather a property of the preferences of the investor than a control variable. This assumption is in line with Sharpe and Tint (1990) and Detemple and Rindisbacher (2008). To utilize the approach from Desmettre and Seifried (2016), we introduce the random utility functions

$$\begin{aligned}\bar{U}_\omega(v) :=& U\left(v - \psi_L L(T,v)\right) \\ =& U(v - \psi_L L(T, v, X(T,\omega), \mathcal{U}_1(\omega), \mathcal{U}_2(\omega))),\ v \in (\hat{v}_0(\omega), \infty),\ \omega \in \Omega,\end{aligned}$$

and $\hat{U}_\omega : (\hat{v}_0(\omega), \infty) \to \mathbb{R}$, $\omega \in \Omega$, given by

$$\hat{U}_\omega(v) := \mathbb{E}\left[\bar{U}_\omega(v) | \mathcal{F}_T\right] = \mathbb{E}\left[U\left(v - \psi_L L(T,v)\right) | \mathcal{F}_T\right].$$

Note that $\hat{v}_0(\omega) \in \left(0, \frac{\psi_L k_2(\omega)}{1 - \psi_L k_1}\right]$. Also note that for the case of replicable liabilities, i.e. there are no unhedgeable components $\mathcal{U}_1$ and $\mathcal{U}_2$, we obviously have

$$\hat{U}_\omega(v) = \bar{U}_\omega(v), \quad \omega \in \Omega.$$

In our surplus optimization framework, in contrast to Desmettre and Seifried (2016), we replace the illiquid asset with a corresponding positive payoff by liabilities that generate a negative value

$$-\psi_L L(T,v) < 0.$$

The subsequent results are needed for further derivations.

Lemma 4.2.2. *The random utility function $\hat{U}_\omega$ is differentiable for almost every $\omega \in \Omega$ with*

$$\hat{U}'_\omega(v) = \mathbb{E}\left[(1 - \psi_L L_1(T)) U'\left(v - \psi_L L(T,v)\right) \middle| \mathcal{F}_T\right].$$

Furthermore, $\hat{U}'_\omega$ is strictly monotonically decreasing, $\hat{U}'_\omega(\hat{v}_0(\omega)) \in (0, \infty]$, $\hat{U}'_\omega(v) > 0$ for all $v > \hat{v}_0(\omega)$ and $\hat{U}'_\omega(v) \to 0$ as $v \to \infty$.

Proof. For $\omega \in \Omega$, $\hat{v} > \hat{v}_0(\omega)$ arbitrary but fixed and ϵ small enough, we have

$$\begin{aligned}
0 &< \frac{\partial}{\partial v} U(v - \psi_L L(T,v)) \\
&= \frac{\partial}{\partial v} U((1-\psi_L L_1(T))v - \psi_L L_2(T))) \\
&= (1-\psi_L L_1(T))U'((1-\psi_L L_1(T))v - \psi_L L_2(T)) \\
&\leq U'((1-\psi_L L_1(X(T,\omega),\mathcal{U}_1(\hat{\omega}_1))(\hat{v}-\epsilon) - \psi_L L_2(X(T,\omega),\mathcal{U}(\hat{\omega}_2)))
\end{aligned}$$

for all $v \in (\hat{v}-\epsilon, \hat{v}+\epsilon)$ since $1-\psi_L L_1(T) > 0$, U is increasing and U' decreasing. Thus, given $\mathcal{F}_T$, there is a constant upper bound for $\frac{\partial}{\partial v}U(v-\psi_L L(T,v))$ and differentiation and integration at $\hat{v}$ can be interchanged by dominated convergence. The other properties follow by the characteristics of U' as well as $\hat{v}_0$. □

Since $\hat{U}_\omega$ is differentiable with $\hat{U}'_\omega : (\hat{v}_0(\omega), \infty) \to (0, \hat{U}'_\omega(\hat{v}_0(\omega)))$ and due to its monotonicity, we can define the inverse marginal utility corresponding to $\hat{U}_\omega$ and we denote it by $\hat{I}_\omega : (0,\infty) \to (\hat{v}_0(\omega), \infty)$, where we set $\hat{I}_\omega(y) := \hat{v}_0(\omega)$ for $y \geq \hat{U}'_\omega(\hat{v}_0(\omega))$. Note that, due to the structure of the liabilities, $\hat{I}_\omega$ is a deterministic function of y and $X(T)$. Therefore, we also write

$$\hat{I}_\omega(y) = \mathcal{I}(y, X(T)) = \mathcal{I}(y, g(X(t), \xi(t,T))).$$

The following assumption will be used for the derivation of optimal investment strategies.

Assumption (LS). *$\mathcal{I}(y\tilde{Z}(t,T), g(x,\xi(t,T)))$ is almost everywhere twice continuously differentiable with respect to both, x and y. $L_i(T, g(x,\xi(t,T))))$, $i = 1,2$ are almost everywhere twice continuously differentiable with respect to x.*

Furthermore, we define

$$\begin{aligned}
H(t,y,x) :=& \mathbb{E}\left[\tilde{Z}(t,T)\hat{I}_\omega(y\tilde{Z}(t,T))|\mathcal{F}_t\right] \\
=& \mathbb{E}\left[\tilde{Z}(t,T)\mathcal{I}\left(y\tilde{Z}(t,T), g(x,\xi(t,T))\right)|\mathcal{F}_t\right] \; x,y>0,\; t\in[0,T],
\end{aligned}$$

The following Lemma provides a necessary condition required for the later application of the generalized martingale approach.

Lemma 4.2.3. *It holds that*

$$H(0,y,X(0)) = \mathbb{E}[\tilde{Z}(T)\hat{I}_\omega(y\tilde{Z}(T))] < \infty, \quad \textit{for all } y > 0.$$

Furthermore, $H(0,y,X(0))$ is continuous and strictly monotonically decreasing in y.

Proof. We have for $v > \hat{v}_0(\omega)$ by dominated convergence as in the proof of Lemma 4.2.2

and as U' is strictly monotonically decreasing

$$\begin{aligned}\hat{U}'_\omega(v) =&\mathbb{E}\left[\frac{\partial}{\partial v}U\left(v-\psi_L L(T,v)\right)\Big|\mathcal{F}_T\right]\\ =&\mathbb{E}\left[(1-\psi_L L_1(T))U'((1-\psi_L L_1(T))v-\psi_L L_2(T))\big|\mathcal{F}_T\right]\\ \leq& U'\left((1-\psi_L L_1(X(T,\omega),\mathcal{U}_1(\hat{\omega}_1)))v-\psi_L L_2(X(T,\omega),\mathcal{U}_2(\hat{\omega}_2))\right)\\ =:&M_\omega(v)\end{aligned}$$

for a.e. $\omega\in\Omega$. Note that $M_\omega(v)$ is continuous and strictly monotonically decreasing and therefore its inverse

$$I_\omega^M(y)=\begin{cases}(M_\omega)^{-1}(y), & 0<y<\hat{U}'(\hat{v}_0(\omega))\\ \hat{v}_0(\omega), & \text{else}\end{cases}$$

is monotonically decreasing as well. Then, it follows for all $0<y<\hat{U}'_\omega(\hat{v}_0(\omega))$

$$y=\hat{U}'_\omega\left(\hat{I}_\omega(y)\right)\leq M_\omega\left(\hat{I}_\omega(y)\right).$$

By the monotonicity of $I_\omega^M(y)$, we have

$$I_\omega^M(y)\geq I_\omega^M\left(M_\omega\left(\hat{I}_\omega(y)\right)\right)=\hat{I}_\omega(y).$$

Since both sides are equal for $y>\hat{U}'_\omega(\hat{v}_0(\omega))$, we learn that $\hat{I}_\omega(y)\leq I_\omega^M(y)$ for almost every $\omega\in\Omega$. As $L_i(X(T,\omega),\mathcal{U}_i(\hat{\omega}_i))$ are $\mathcal{F}_T$-measurable for $i=1,2$, and due to the boundedness conditions (L2.1) and (L2.2),

$$\begin{aligned}&I_\omega^M(y)\\ &\leq\max\left\{\frac{1}{1-\psi_L L_1(X(T,\omega),\mathcal{U}_1(\hat{\omega}_1))}\left(I\left(y\right)+\psi_L L_2(X(T,\omega),\mathcal{U}_2(\hat{\omega}_2))\right);\hat{v}_0(\omega)\right\}\\ &\leq\max\left\{\frac{1}{1-\psi_L k_1}\left(I(y)+\psi_L k_2(\omega)\right);\hat{v}_0(\omega)\right\}.\end{aligned}$$

The statement follows with (2.7) as $k_2(\omega)$ is $\tilde{\mathbb{Q}}$-integrable by Assumption (L2). The continuity and monotonicity of $H(0,y,X(0))$ follow with Lemma 4.2.2. □

Optimal Terminal Wealth and Investment Strategy

The following result states that the well-known calculation of the optimal terminal wealth using the martingale approach (see, e.g., Theorem 7.6 (p. 114) in Karatzas and Shreve (1998)) which can also be transferred to the case with a random utility function, which is based on the terminal surplus $S(T)=V(T)-\psi_L L(T,V)$.

Theorem 4.2.4 (Optimal Terminal Wealth)**.** *The optimal terminal wealth for* ($\mathrm{P_S}$) *is given by*

$$V^*(T) = \hat{I}_\omega(Y(v_0)\tilde{Z}(T)),$$

with $Y(\cdot)$ *being the inverse of* $H(0,\cdot,X(0))$*. The optimal terminal surplus is then given by*

$$S^*(T) := V^*(T) - \psi_L L(T, V^*(T)).$$

Proof. The proof is provided in Appendix B. □

In case of a surplus optimization without performance-linked liabilities, i.e. $L_1(T) = 0$, we have for very unfavorable market developments, i.e. high values of $\tilde{Z}(T)$, $V^*(T) = \hat{I}_\omega(Y(v_0)\tilde{Z}(T)) = \hat{v}_0(\omega) = \sup_{\hat{\omega}\in\Omega} \psi_L L_2(T, X(T,\omega), \mathcal{U}_2(\hat{\omega}))$. Thus, the surplus $S^*(T)$ consists only of the difference between the worst case scenario of the unhedgeable risks $\hat{v}_0(\omega)$ and the actual realization $\psi_L L_2(T, X(T,\omega), \mathcal{U}_2(\omega))$.

For the optimal terminal wealth, we now deduce the corresponding replicating strategy.

Theorem 4.2.5 (Optimal Investment Strategy)**.** *Let Assumption (LS) be satisfied. The investment strategy corresponding to the optimal terminal wealth from Theorem 4.2.4 is given by*

$$\pi^*(t) = \frac{1}{\sigma(t)V^*(t)} \left[-H_y\left(t, \mathcal{Y}(t), X(t)\right)\mathcal{Y}(t)\gamma(t) + H_x\left(t, \mathcal{Y}(t), X(t)\right)\sigma_X(t, X(t))\right],$$

with

$$\mathcal{Y}(t) := Y(v_0)\tilde{Z}(t) = Y(v_0)e^{-\int_0^t r(s)+\frac{1}{2}\gamma(s)^2 ds - \int_0^t \gamma(s)dW(s)}.$$

Furthermore,

$$V^*(t) = H(t, \mathcal{Y}(t), X(t)).$$

Proof. The SDE of $\mathcal{Y}$ can be written as

$$d\mathcal{Y}(t) = \mathcal{Y}(t)\left[-r(t)ds - \gamma(t)dW(t)\right].$$

By Theorem 4.2.4,

$$V^*(T) = \hat{I}_\omega(Y(v_0)\tilde{Z}(T)) = \hat{I}_\omega(\mathcal{Y}(T)).$$

We thus have with Assumption (LX)

$$\begin{aligned}V^*(t) =&\mathbb{E}\left[\tilde{Z}(t,T)\hat{I}_\omega\left(\mathcal{Y}(T)\right)|\mathcal{F}_t\right]\\ =&\mathbb{E}\left[\tilde{Z}(t,T)\mathcal{I}\left(\mathcal{Y}(T),X(T)\right)|\mathcal{F}_t\right]\\ =&\mathbb{E}\left[\tilde{Z}(t,T)\mathcal{I}\left(\mathcal{Y}(t)\tilde{Z}(t,T),g(X(t),\xi(t,T))\right)|\mathcal{F}_t\right] = H\left(t,\mathcal{Y}(t),X(t)\right).\end{aligned}$$

As Assumption (LS) holds, we can apply Itô's formula and obtain

$$\begin{aligned}dV^*(t) =&dH\left(t,\mathcal{Y}(t),X(t)\right)\\ =&H_t\left(t,\mathcal{Y}(t),X(t)\right)dt + H_y\left(t,\mathcal{Y}(t),X(t)\right)d\mathcal{Y}(t) + H_x\left(t,\mathcal{Y}(t),X(t)\right)dX(t)\\ &+\left[-H_{yx}\left(t,\mathcal{Y}(t),X(t)\right)\mathcal{Y}(t)\gamma(t)\sigma_X(t,X(t)) + \frac{1}{2}\mathcal{Y}(t)^2\gamma(t)^2H_{yy}\left(t,\mathcal{Y}(t),X(t)\right)\right.\\ &\left.+\frac{1}{2}\sigma_X(t,X(t))^2H_{xx}\left(t,\mathcal{Y}(t),X(t)\right)\right]dt\\ =&\left[H_t\left(t,\mathcal{Y}(t),X(t)\right) - r(t)\mathcal{Y}(t)H_y\left(t,\mathcal{Y}(t),X(t)\right) + \mu_X(t,X(t))H_x\left(t,\mathcal{Y}(t),X(t)\right)\right.\\ &-H_{yx}\left(t,\mathcal{Y}(t),X(t)\right)\mathcal{Y}(t)\gamma(t)\sigma_X(t,X(t)) + \frac{1}{2}\mathcal{Y}(t)^2\gamma(t)^2H_{yy}\left(t,\mathcal{Y}(t),X(t)\right)\\ &\left.+\frac{1}{2}\sigma_X(t,X(t))^2H_{xx}\left(t,\mathcal{Y}(t),X(t)\right)\right]dt\\ &+\left[-H_y(t,\mathcal{Y}(t))\mathcal{Y}(t)\gamma(t) + H_x(t,\mathcal{Y}(t))\sigma_X(t,X(t))\right]dW(t).\end{aligned}$$

Comparing the coefficients of the diffusion terms of this SDE and (2.1), we receive

$$V^*(t)\pi(t)\sigma(t) = -H_y\left(t,\mathcal{Y}(t),X(t)\right)\mathcal{Y}(t)\gamma(t) + H_x\left(t,\mathcal{Y}(t),X(t)\right)\sigma_X(t,X(t)),$$

which completes the proof. □

4.3 Performance- and Index-Linked Liabilities

In this section, we consider an application with replicable liabilities, i.e. we assume liabilities of the form (RL). Throughout the whole section, we consider a power utility function of the form (2.6) to model the preferences of the investor.

4.3.1 Optimal Investment Strategy with Replicable Liabilities

As a direct consequence of Theorem 4.2.5 we obtain:

Corollary 4.3.1 (Power Utility and Replicable Liabilities (RL))**.** *Let Assumption (LS) be satisfied. The optimal terminal wealth for an investor with power utility function* (2.6) *and liabilities as in* (RL) *is given by*

$$V^*(T) = \bar{I}_\omega(Y(v_0)\tilde{Z}(T)) = \frac{1}{1-\psi_L L_1(T)}\left(\left(\frac{Y(v_0)\tilde{Z}(T)}{1-\psi_L L_1(T)}\right)^{\frac{1}{\alpha-1}} + \psi_L L_2(T)\right).$$

and

$$Y(v_0) = \left(\frac{v_0 - \mathbb{E}\left[\tilde{Z}(T)\frac{\psi_L L_2(T)}{1-\psi_L L_1(T)}\right]}{\mathbb{E}\left[\left(\frac{\tilde{Z}(T)}{1-\psi_L L_1(T)}\right)^{\frac{\alpha}{\alpha-1}}\right]}\right)^{\alpha-1}.$$

Furthermore,

$$\begin{aligned} V^*(t) =& H(t,\mathcal{Y}(t),X(t)) \\ =& \mathcal{Y}(t)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\left(\frac{\tilde{Z}(t,T)}{1-\psi_L L_1(T,g(X(t),\xi(t,T)))}\right)^{\frac{\alpha}{\alpha-1}}\middle|\mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\tilde{Z}(t,T)\frac{\psi_L L_2(T,g(X(t),\xi(t,T)))}{1-\psi_L L_1(T,g(X(t),\xi(t,T)))}\middle|\mathcal{F}_t\right] \end{aligned}$$

and the optimal investment strategy is given by

$$\pi^*(t) = \pi_M(t) + \pi_{PL}(t) + \pi_{IL}(t) + \pi_{mi}(t),$$

with the Merton portfolio

$$\pi_M(t) = \frac{\gamma(t)}{(1-\alpha)\sigma(t)},$$

performance-linked part

$$\begin{aligned} \pi_{PL}(t) =& \frac{\sigma_X(t,X(t))}{\sigma(t)V^*(t)}\mathbb{E}\left[\tilde{Z}(t,T)\left(\frac{\alpha}{1-\alpha}\right)(-\psi_L L_{1,x}(T,X(T)))g_x(X(t),\xi(t,T))\right. \\ &\left.\cdot\left[1-\psi_L L_1(T,g(X(t),\xi(t,T)))\right]^{\frac{1-2\alpha}{\alpha-1}}\left(\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}}\middle|\mathcal{F}_t\right], \end{aligned}$$

index-linked part

$$\pi_{IL}(t) = \frac{\psi_L}{\sigma(t)V^*(t)}\mathbb{E}\left[\tilde{Z}(t,T)\frac{(1-\alpha)\sigma_X(t,X(t))L_{2,x}(T,X(T)g_x(X(t),\xi(t,T)) - \gamma(t)L_2(T,X(T))}{(1-\alpha)(1-\psi_L L_1(T,X(T)))}\middle|\mathcal{F}_t\right]$$

and a mixed part

$$\pi_{mi}(t) = \frac{\sigma_X(t,X(t))}{\sigma(t)V^*(t)}\mathbb{E}\left[\tilde{Z}(t,T)\frac{\psi_L L_2(T,X(T))\psi_L L_{1,x}(T,X(T))g_x(X(t),\xi(t,T)))}{(1-\psi_L L_1(T,X(T)))^2}\middle|\mathcal{F}_t\right],$$

where $L_{i,x}(T, X(T))$, $i = 1, 2$ denote the derivatives of $L_i(T, X(T))$ with respect to the second component.

Proof. We have $\hat{U}_\omega = \bar{U}_\omega$ since the liabilities are replicable with

$$\bar{U}_\omega(v) = U(v - \psi_L L(T, v)) = \frac{1}{\alpha}(v - \psi_L (vL_1(T) + L_2(T)))^\alpha, \; v > \hat{v}_0(\omega) = \frac{\psi_L L_2(T)}{1 - \psi_L L_1(T)}$$

and

$$\bar{U}'_\omega(v) = (1 - \psi_L L_1(T))(v - \psi_L(vL_1(T) + L_2(T)))^{\alpha-1}, \; v > \frac{\psi_L L_2(T)}{1 - \psi_L L_1(T)}.$$

Since $\bar{U}'_\omega\left(\frac{\psi_L L_2(T)}{1-\psi_L L_1(T)}\right) = \infty$, the inverse of $\bar{U}'_\omega$ is given by

$$\bar{I}_\omega(y) = \frac{1}{1 - \psi_L L_1(T)}\left(\left(\frac{y}{1 - \psi_L L_1(T)}\right)^{\frac{1}{\alpha-1}} + \psi_L L_2(T)\right). \tag{4.7}$$

Hence, given $v_0 \geq \mathbb{E}\left[\tilde{Z}(T)\frac{\psi_L L_2(T)}{1-\psi_L L_1(T)}\right]$, the optimal solution $V^*(T)$ from Theorem 4.2.4 is given by $V^*(T) = \bar{I}_\omega(Y(v_0)\tilde{Z}(T))$, which reads as in the statement. Furthermore,

$$H(0, y, X(0)) = \mathbb{E}\left[\frac{\tilde{Z}(T)}{1 - \psi_L L_1(T)}\left(\left(\frac{y\tilde{Z}(T)}{1 - \psi_L L_1(T)}\right)^{\frac{1}{\alpha-1}} + \psi_L L_2(T)\right)\right]$$

and

$$\begin{aligned} H(0, Y(v_0), X(0)) &\overset{!}{=} v_0 \\ \Leftrightarrow Y(v_0) &= \left(\frac{v_0 - \mathbb{E}\left[\tilde{Z}(T)\frac{\psi_L L_2(T,X(T))}{1-\psi_L L_1(T,X(T))}\right]}{\mathbb{E}\left[\left(\frac{\tilde{Z}(T)}{1-\psi_L L_1(T,X(T))}\right)^{\frac{\alpha}{\alpha-1}}\right]}\right)^{\alpha-1}. \end{aligned}$$

To obtain the optimal investment strategy, we apply Theorem 4.2.5 and calculate

$$\begin{aligned} V^*(t) =& H(t, \mathcal{Y}(t), X(t)) = \mathbb{E}\left[\tilde{Z}(t,T)\bar{I}_\omega(\mathcal{Y}(T))|\mathcal{F}_t\right] \\ =& \mathbb{E}\left[\frac{\tilde{Z}(t,T)}{1 - \psi_L L_1(T, X(T))}\left(\left(\frac{\mathcal{Y}(t)\tilde{Z}(t,T)}{1 - \psi_L L_1(T, X(T))}\right)^{\frac{1}{\alpha-1}} + \psi_L L_2(T, X(T))\right)\middle|\mathcal{F}_t\right] \\ =& \mathcal{Y}(t)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\left(\frac{\tilde{Z}(t,T)}{1 - \psi_L L_1(T, g(X(t), \xi(t,T)))}\right)^{\frac{\alpha}{\alpha-1}}\middle|\mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\tilde{Z}(t,T)\frac{\psi_L L_2(T, g(X(t), \xi(t,T)))}{1 - \psi_L L_1(T, g(X(t), \xi(t,T)))}\middle|\mathcal{F}_t\right]. \end{aligned}$$

Then,

$$
\begin{aligned}
&H_x(t,\mathcal{Y}(t),X(t))\\
&=\mathbb{E}\left[\tilde{Z}(t,T)\left(\frac{\alpha}{1-\alpha}\right)(-\psi_L L_{1,x}(T,X(T)))g_x(X(t),\xi(t,T)))\right.\\
&\cdot\left[1-\psi_L L_1(T,g(X(t),\xi(t,T)))\right]^{\frac{1-2\alpha}{\alpha-1}}\left(\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}}\left|\mathcal{F}_t\right]\\
&+\mathbb{E}\left[\tilde{Z}(t,T)\frac{\psi_L L_{2,x}(T,X(T)g_x(X(t),\xi(t,T))}{1-\psi_L L_1(T,X(T))}\right.\\
&\left.+\tilde{Z}(t,T)\frac{\psi_L L_2(T,X(T))\psi_L L_{1,x}(T,X(T))g_x(X(t),\xi(t,T)))}{(1-\psi_L L_1(T,X(T)))^2}\right|\mathcal{F}_t\right],
\end{aligned}
\tag{4.8}
$$

where expectation and differentiation can be interchanged due to dominated convergence with Assumption (L2). Finally,

$$
\begin{aligned}
&-H_y(t,\mathcal{Y}(t),X(t))\mathcal{Y}(t)\gamma(t)=\mathbb{E}\left[\tilde{Z}(t,T)\frac{\partial}{\partial\mathcal{Y}(t)}\hat{I}_\omega(\mathcal{Y}(T))\middle|\mathcal{F}_t\right]\mathcal{Y}(t)\gamma(t)\\
&=\frac{\gamma(t)}{1-\alpha}\mathcal{Y}(t)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\left(\frac{\tilde{Z}(t,T)}{1-\psi_L L_1(T,g(X(t),\xi(t,T)))}\right)^{\frac{\alpha}{\alpha-1}}\middle|\mathcal{F}_t\right]\\
&=\frac{\gamma(t)}{1-\alpha}\left(V^*(t)-\psi_L\mathbb{E}\left[\tilde{Z}(t,T)\frac{L_2(T,X(T))}{1-\psi_L L_1(T,X(T))}\middle|\mathcal{F}_t\right]\right).
\end{aligned}
\tag{4.9}
$$

Applying Theorem 4.2.5 and inserting (4.8) as well as (4.9), we have

$$
\begin{aligned}
\pi^*(t)=&\frac{1}{\sigma(t)V^*(t)}\left[-H_y\left(t,\mathcal{Y}(t),X(t)\right)\mathcal{Y}(t)\gamma(t)+H_x\left(t,\mathcal{Y}(t),X(t)\right)\sigma_X\left(t,X(t)\right)\right]\\
=&\pi_M(t)-\frac{1}{\sigma(t)V^*(t)}\cdot\frac{\gamma(t)\psi_L}{1-\alpha}\mathbb{E}\left[\tilde{Z}(t,T)\frac{L_2(T,X(T))}{1-\psi_L L_1(T,X(T))}\middle|\mathcal{F}_t\right]\\
&+\pi_{PL}(t)+\frac{\sigma_X(t,X(t))}{\sigma(t)V^*(t)}\mathbb{E}\left[\tilde{Z}(t,T)\frac{\psi_L L_{2,x}(T,X(T)g_x(X(t),\xi(t,T))}{1-\psi_L L_1(T,X(T))}\middle|\mathcal{F}_t\right]+\pi_{mi}(t)\\
=&\pi_M(t)+\pi_{PL}(t)-\frac{1}{\sigma(t)V^*(t)}\cdot\frac{\gamma(t)\psi_L}{1-\alpha}\mathbb{E}\left[\tilde{Z}(t,T)\frac{L_2(T,X(T))}{1-\psi_L L_1(T,X(T))}\middle|\mathcal{F}_t\right]\\
&+\frac{\psi_L}{\sigma(t)V^*(t)}\mathbb{E}\left[\tilde{Z}(t,T)\frac{\sigma_X(t,X(t))L_{2,x}(T,X(T)g_x(X(t),\xi(t,T))}{1-\psi_L L_1(T,X(T))}\middle|\mathcal{F}_t\right]+\pi_{mi}(t)\\
=&\pi_M(t)+\pi_{PL}(t)+\pi_{IL}(t)+\pi_{mi}(t),
\end{aligned}
$$

with $\pi_M(t)$, $\pi_{PL}(t)$, $\pi_{IL}(t)$ and $\pi_{mi}(t)$ as in the statement. □

In this corollary, the portfolios π_{PL} (respectively π_{IL}) are zero when $L_1(T,X(T))$ (respectively $L_2(T,X(T))$) is zero. In each of the two cases, π_{mi} is also zero. We will particularly examine these special cases. The case $L_2(X(T))=0$ will be covered in Corollary 4.4.5 and the case $L_1(T,X(T))=0$ is treated in the following remark.

Remark 4.3.2. *In case $L_1(T, X(T)) = 0$ and with Corollary 4.3.1,*

$$v_0 = \mathbb{E}\left[\tilde{Z}(T)V^*(T)\right] = \mathbb{E}\left[\tilde{Z}(T)\left(Y(v_0)\tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right] + \mathbb{E}\left[\tilde{Z}(T)\psi_L L_2(T)\right].$$

The interpretation is that the optimal terminal surplus is the same as the optimal wealth of an investor with initial capital $v_0 - \mathbb{E}\left[\tilde{Z}(T)\psi_L L_2(T)\right]$ who maximizes the expected utility of terminal wealth only. Moreover, $\pi_{PL}(t) = \pi_{mi}(t) = 0$ and

$$\begin{aligned}\pi^*(t) =& \frac{\gamma(t)}{(1-\alpha)\sigma(t)} \cdot \frac{V^*(t) - \psi_L \mathbb{E}\left[\tilde{Z}(t,T)L_2(T, X(t))|\mathcal{F}_t\right]}{V^*(t)} \\ &+ \frac{\sigma_X(t, X(t))}{\sigma(t)V^*(t)}\psi_L \mathbb{E}\left[\tilde{Z}(t,T)L_{2,x}(T, X(T)g_x(X(t), \xi(t,T))|\mathcal{F}_t\right].\end{aligned}$$

Thus, the optimal investment strategy is as in Chapter 3, given by a three-fund separation and can also be seen as a generalization of a CPPI strategy. The Merton portfolio $\frac{\gamma(t)}{(1-\alpha)\sigma(t)}$ is scaled by the relative surplus at time t. The second term represents the liability hedging portfolio and the remaining wealth is invested in the risk-free asset. A more detailed comparison of the funding ratio optimization and surplus optimization is provided in Section 4.6.

For this case, i.e. liabilities as in (RL) with $L_1(T, X(T)) = 0$, we consider a particular application which allows for a closed-form solution for the investment strategy in the following section.

4.3.2 Index-Linked Liabilities with Capped Maximum Benefits

For the first example with a specific liability model which admits a closed-form solution, we assume constant coefficients μ, σ and r and liabilities as in (ILCB). As a consequence of Theorem 4.2.5 , we obtain the following result:

Corollary 4.3.3 (Index-Linked Liabilities with Capped Maximum Benefits (ILCB))**.** *The optimal investment strategy for an investor with power utility function* (2.6) *and liabilities as in* (ILCB) *is given by*

$$\begin{aligned}\pi(t) = \frac{1}{\sigma(t)V^*(t)}&\left[\frac{\gamma(t)}{1-\alpha}\left(V^*(t) - \psi_L L(0)e^{-r(T-t)}\left(K\Phi(d_2(t)) + X(t)e^{(\hat{\mu}_X - \hat{\sigma}_X\gamma)(T-t)}\Phi(-d_1(t))\right)\right)\right. \\ &\left. + \psi_L L(0)e^{-r(T-t)}e^{(\hat{\mu}_X - \hat{\sigma}_X\gamma)(T-t)}\Phi(-d_1(t))\hat{\sigma}_X X(t)\right],\end{aligned}$$

with

$$d_1(t) = \frac{\log\left(\frac{X(t)}{K}\right) + \left(\hat{\mu}_X - \hat{\sigma}_X\gamma + \frac{1}{2}\hat{\sigma}_X^2\right)(T-t)}{\hat{\sigma}_X\sqrt{T-t}}$$

and

$$d_2(t) = d_1(t) - \hat{\sigma}_X \sqrt{T-t}.$$

Proof. We apply Theorem 4.2.5 and compute

$$\begin{aligned}
&H\left(t, \mathcal{Y}(t), X(t)\right) \\
&= \mathcal{Y}(t)^{\frac{1}{\alpha-1}} \mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\alpha}{\alpha-1}} | \mathcal{F}_t\right] + \psi_L \mathbb{E}\left[\tilde{Z}(t,T) L(T, g(X(t), \xi(t,T))) | \mathcal{F}_t\right] \\
&= \mathcal{Y}(t)^{\frac{1}{\alpha-1}} \mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\alpha}{\alpha-1}} | \mathcal{F}_t\right] + \psi_L L(0) \mathbb{E}\left[\tilde{Z}(t,T) \min(X(T), K) | \mathcal{F}_t\right].
\end{aligned}$$

With

$$\begin{aligned}
&& K \geq X(T) = X(t) e^{\left(\hat{\mu}_X - \frac{1}{2}\hat{\sigma}_X^2\right)(T-t) + \hat{\sigma}_X (W(T) - W(t))} \\
\Leftrightarrow && \frac{W(T) - W(t)}{\sqrt{T-t}} \leq \frac{\log\left(\frac{K}{X(t)}\right) - \left(\hat{\mu}_X - \frac{1}{2}\hat{\sigma}_X^2\right)(T-t)}{\hat{\sigma}_X \sqrt{T-t}} =: -\bar{d}_2(t),
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left[\tilde{Z}(t,T) \min(X(T), K) | \mathcal{F}_t\right] \\
&= e^{-r(T-t)} K - \mathbb{E}\left[\tilde{Z}(t,T) \max\left(K - X(T), 0\right) | \mathcal{F}_t\right] \\
&= e^{-r(T-t)} \left(K - \int_{-\infty}^{-\bar{d}_2(t)} e^{-\frac{\gamma^2}{2}(T-t) - \gamma\sqrt{T-t}u} K \phi(u) du \right. \\
&\quad \left. + X(t) \int_{-\infty}^{-\bar{d}_2(t)} e^{-\frac{\gamma^2}{2}(T-t) - \gamma\sqrt{T-t}u} e^{\left(\hat{\mu}_X - \frac{1}{2}\hat{\sigma}_X^2\right)(T-t) + \hat{\sigma}_X \sqrt{T-t}u} \phi(u) du \right) \\
&= e^{-r(T-t)} \left(K - \int_{-\infty}^{-\bar{d}_2(t)} K \phi(u + \gamma\sqrt{T-t}) du \right. \\
&\quad \left. + X(t) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \int_{-\infty}^{-\bar{d}_2(t)} \phi\left(u + (\gamma - \hat{\sigma}_X)\sqrt{T-t}\right) du \right) \\
&= e^{-r(T-t)} \left(K - \int_{-\infty}^{-d_2(t)} K \phi(u) du + X(t) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \int_{-\infty}^{-d_2(t)} \phi\left(u - \hat{\sigma}_X \sqrt{T-t}\right) du \right) \\
&= e^{-r(T-t)} \left(K - K\Phi(-d_2(t)) + X(t) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \Phi(-d_1(t))\right) \\
&= e^{-r(T-t)} \left(K\Phi(d_2(t)) + X(t) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \Phi(-d_1(t))\right),
\end{aligned}$$

where $d_1(t)$ and $d_2(t)$ are as stated in the theorem. Using

$$\frac{\partial}{\partial X(t)} d_1(t) = \frac{\partial}{\partial X(t)} d_2(t) = \frac{1}{X(t) \hat{\sigma}_X \sqrt{T-t}},$$

we receive

$$
\begin{aligned}
& H_x\left(t, \mathcal{Y}(t), X(t)\right) \\
=& \frac{\partial}{\partial X(t)} \psi_L L(0) e^{-r(T-t)} \left(K\Phi(d_2(t)) + X(t) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \Phi(-d_1(t)) \right) \\
=& \psi_L L(0) e^{-r(T-t)} \left(\frac{K\phi(d_2(t))}{X(t)\hat{\sigma}_X \sqrt{T-t}} + e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \Phi(-d_1(t)) - \frac{e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \phi(-d_1(t))}{\hat{\sigma}_X \sqrt{T-t}} \right) \\
=& \psi_L L(0) e^{-r(T-t)} e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} \Phi(-d_1(t)),
\end{aligned}
$$

since $\phi(x) = \phi(-x)$ and

$$
\begin{aligned}
\phi(d_2(t)) =& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2(t)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1(t) - \hat{\sigma}_X \sqrt{T-t})^2}{2}} \\
=& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1(t)^2 - 2d_1(t)\hat{\sigma}_X \sqrt{T-t} + \hat{\sigma}_X^2 (T-t)}{2}} \\
=& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1(t)^2}{2}} e^{\log\left(\frac{X(t)}{K}\right) + \left(\hat{\mu}_X - \hat{\sigma}_X \gamma + \frac{1}{2}\hat{\sigma}_X^2\right)(T-t) - \frac{1}{2}\hat{\sigma}_X^2 (T-t)} \\
=& \frac{X(t)}{K} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1(t)^2}{2}} e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)} = \frac{X(t)}{K} \phi(d_1(t)) e^{(\hat{\mu}_X - \hat{\sigma}_X \gamma)(T-t)}.
\end{aligned}
$$

Using (4.9), we receive

$$
- H_y(t, \mathcal{Y}(t), X(t)) \mathcal{Y}(t) \gamma(t) = \frac{\gamma(t)}{1-\alpha} \left(V^*(t) - \psi_L \mathbb{E}\left[\tilde{Z}(t,T) \min(X(T), K) \middle| \mathcal{F}_t \right] \right).
$$

The statement then follows directly from Theorem 4.2.5. □

We conclude that this specification of the liability process leads to an optimal investment strategy that can be stated in closed form. However, non-hedgeable risks are not included in this liability specification and the resulting investment strategy does not have an impact on the value of the liabilities. We address these extensions in the next section.

4.4 Performance-Linked Liabilities

In this section, we derive an optimal strategy for an investor with liabilities that include an asset performance participation under the assumption of a power utility function (2.6) and liabilities of the form (PLU). The following theorem, which is a consequence of the general setting dealt with in Theorem 4.2.4, gives the solution for the optimal terminal wealth in the setting with performance-linked liabilities:

Theorem 4.4.1 (Optimal Terminal Wealth, Performance-Linked Liabilities)**.** *The optimal terminal wealth for an investor with power utility function* (2.6) *and performance-linked liabilities as in* (PLU) *is given by*

$$V^*(T) = \left(\Delta_\omega Y(v_0)\tilde{Z}(T)\right)^{\frac{1}{\alpha-1}},$$

with

$$\Delta_\omega := \left(\mathbb{E}[(1-\psi_L L_1(T))^\alpha \,|\mathcal{F}_T]\right)^{-1}$$

and

$$Y(v_0) := \left(\frac{1}{v_0}\mathbb{E}\left[\tilde{Z}(T)\left(\Delta_\omega \tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right]\right)^{1-\alpha}.$$

Proof. For the power utility case with asset participation type liabilities, we have

$$\hat{U}_\omega(v) = \mathbb{E}\left[\tfrac{1}{\alpha}\left(v - \psi_L v L_1(T)\right)^\alpha \middle|\mathcal{F}_T\right] = \frac{v^\alpha}{\alpha}\mathbb{E}\left[(1-\psi_L L_1(T))^\alpha \middle|\mathcal{F}_T\right]$$

and thus, we can directly calculate

$$\hat{U}'_\omega(v) = v^{\alpha-1}\mathbb{E}\left[(1-\psi_L L_1(T))^\alpha \middle|\mathcal{F}_T\right].$$

Using Assumption (L2.1) and $k_1, \psi_L \in (0,1)$, we get $\hat{v}_0(\omega) = 0$ and thus $\hat{U}'_\omega(\hat{v}_0(\omega)) = \hat{U}'_\omega(0) = 0$. Hence,

$$\hat{I}_\omega(y) = (\Delta_\omega y)^{\frac{1}{\alpha-1}},$$

for all $y > 0$ with

$$\Delta_\omega = \left(\mathbb{E}\left[(1-\psi_L L_1(T))^\alpha \,|\mathcal{F}_T\right]\right)^{-1}$$

and

$$H(0,y,X(0)) = \mathbb{E}\left[\tilde{Z}(T)\hat{I}_\omega(y\tilde{Z}(T))\right] = \mathbb{E}\left[\tilde{Z}(T)\left(\Delta_\omega y\tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right].$$

Thus, $Y(v_0)$ is given by

$$\begin{aligned} H(0,Y(v_0),X(0)) = \mathbb{E}\left[\tilde{Z}(T)V^*(T)\right] &= \mathbb{E}\left[\tilde{Z}(T)\left(\Delta_\omega Y(v_0)\tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right] \overset{!}{=} v_0 \\ \Leftrightarrow v_0 &= Y(v_0)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\tilde{Z}(T)\left(\Delta_\omega \tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right] \\ \Leftrightarrow Y(v_0) &= \left(\frac{1}{v_0}\mathbb{E}\left[\tilde{Z}(T)\left(\Delta_\omega \tilde{Z}(T)\right)^{\frac{1}{\alpha-1}}\right]\right)^{1-\alpha}. \end{aligned}$$

Finally, $V^*(T) = \hat{I}(Y(v_0)\tilde{Z}(T))$. □

The structure of the terminal wealth is similar to the terminal wealth of an investor with power utility and without considering the surplus. The surplus is taken into account by the adjustment factor

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\mathbb{E}[(1-\psi_L L_1(T))^\alpha \,|\mathcal{F}_T]\right)^{\frac{1}{1-\alpha}},$$

where the term

$$1-\psi_L L_1(T) = \frac{S(T)}{V(T)}$$

can be interpreted as the terminal surplus relative to the terminal wealth of the asset portfolio. From the structure of Δ_ω, we can also deduce that the unhedgeable risks (i.e. risks independent of $\mathcal{F}_T$) are considered in the optimal terminal wealth (and subsequently in the investment strategy leading to the terminal wealth) only in expectation. The actual realization of the unhedgeable risks only has an impact on the terminal surplus, but not the terminal wealth. However, the hedgeable parts of $L_1(T)$ lead to the following impact of the adjustment factor Δ_ω: for $\alpha < 0$, an $\omega \in \Omega$ resulting in a larger relative surplus leads to a smaller adjustment factor $\Delta_\omega^{\frac{1}{\alpha-1}}$, whereas for $\alpha > 0$, a larger relative surplus leads to a larger adjustment factor. This means that the adjustment factor $\Delta_\omega^{\frac{1}{\alpha-1}}$ has a dampening effect on the volatility of the terminal wealth for $\alpha < 0$, which results from the high risk aversion. On the other hand, a less risk-averse investor with $\alpha > 0$ uses a large capital buffer to increase the final wealth. Interpreting the limit $\alpha \to 0$ as an investor with logarithmic utility in terms of the relative risk aversion, we have for such an investor

$$\begin{aligned}\mathbb{E}\left[U(S(T))\right] &= \mathbb{E}\left[\log\left(V(T) - \psi_L V(T) L_1(T)\right)\right] \\ &= \mathbb{E}\left[\log(V(T))\right] + \mathbb{E}\left[(1-\psi_L L_1(T))\right].\end{aligned}$$

Hence, the solution to $(\mathrm{P_S})$ is independent of $L_1(T)$, the relative surplus does not have an impact on the terminal wealth and therefore, this limit separates the cases $\alpha > 0$ and $\alpha < 0$.

We can also deduce optimal investment strategies in the setting with performance-linked liabilities.

Theorem 4.4.2 (Optimal Investment Strategy, Performance-Linked Liabilities)**.** *Let Assumption (LS) be satisfied. The optimal terminal wealth for an investor with power utility function* (2.6) *and performance-linked liabilities as in* (PLU) *is given by*

$$\pi^*(t) = \frac{\gamma(t)}{(1-\alpha)\sigma(t)} + \frac{1}{\sigma(t)V^*(t)} H_x\left(t, \mathcal{Y}(t), X(t)\right) \sigma_X(t, X(t)),$$

with

$$\mathcal{Y}(t) = Y(v_0)\tilde{Z}(t),$$

$Y(v_0)$ *as in Theorem 4.4.1 and*

$$H_x(t, \mathcal{Y}(t), X(t)) = \mathcal{Y}(t)^{\frac{1}{\alpha-1}} \mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\alpha}{\alpha-1}} \frac{\partial}{\partial X(t)} \left(\Delta_\omega(g(X(t), \xi(t,T)))\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right].$$

Furthermore,

$$V^*(t) = \mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega\left(g(X(t), \xi(t,T))\right)\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right].$$

Proof. We apply Theorem 4.2.5 to the optimal wealth from Theorem 4.4.1. We have

$$\begin{aligned} V^*(t) = H(t, \mathcal{Y}(t), X(t)) =& \mathbb{E}\left[\tilde{Z}(t,T)\hat{I}_\omega(\mathcal{Y}(T))|\mathcal{F}_t\right] \\ =& \mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega(X(T))\mathcal{Y}(T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right] \\ =& \mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega\left(g(X(t), \xi(t,T))\right)\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right]. \end{aligned}$$

For the calculation of H_x and H_y, the expectation and differentiation can be interchanged by dominated convergence due to Assumption (L2). Hence,

$$\begin{aligned} &- H_y(t, \mathcal{Y}(t), X(t))\mathcal{Y}(t)\gamma(t) \\ &= -\mathcal{Y}(t)\gamma(t)\frac{\partial}{\partial \mathcal{Y}(t)}\mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega\left(g(X(t), \xi(t,T))\right)\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right] \\ &= \frac{\gamma(t)}{1-\alpha}\mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega\left(g(X(t), \xi(t,T))\right)\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right] \\ &= \frac{\gamma(t)}{1-\alpha}V^*(t) \end{aligned}$$

and

$$\begin{aligned} H_x(t, \mathcal{Y}(t), X(t)) =& \frac{\partial}{\partial X(t)}\mathbb{E}\left[\tilde{Z}(t,T)\left(\Delta_\omega\left(g(X(t), \xi(t,T))\right)\mathcal{Y}(t)\tilde{Z}(t,T)\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right] \\ =& \mathcal{Y}(t)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\tilde{Z}(t,T)^{\frac{\alpha}{\alpha-1}} \frac{\partial}{\partial X(t)} \left(\Delta_\omega(g(X(t), \xi(t,T)))\right)^{\frac{1}{\alpha-1}} |\mathcal{F}_t\right] \end{aligned}$$

The statement follows with Theorem 4.2.5. □

In Theorem 4.4.2, we observe again a three-fund solution. In contrast to Corollary 4.3.1, where the performance seeking part was scaled by the relative surplus at time t, it has the structure of a constant-mix strategy plus liability hedging term here. In the case of constant coefficients, which will be of particular relevance for our numerical examples, we obtain

$$\begin{aligned}
&H_x(t,\mathcal{Y}(t),X(t)) \\
&= \mathcal{Y}(t)^{\frac{1}{\alpha-1}}\mathbb{E}\left[\left(e^{-\left(r+\frac{1}{2}\gamma^2\right)(T-t)-\gamma(W(T)-W(t))}\right)^{\frac{\alpha}{\alpha-1}}\frac{\partial}{\partial X(t)}\left(\Delta_\omega(g(X(t),\xi(t,T)))\right)^{\frac{1}{\alpha-1}}|\mathcal{F}_t\right] \\
&= \mathcal{Y}(t)^{\frac{1}{\alpha-1}}e^{\frac{\alpha}{1-\alpha}\left(r+\frac{1}{2}\gamma^2\right)(T-t)}\mathbb{E}\left[e^{\frac{\gamma\alpha}{1-\alpha}(W(T)-W(t))}\frac{\partial}{\partial X(t)}\left(\Delta_\omega(g(X(t),\xi(t,T)))\right)^{\frac{1}{\alpha-1}}|\mathcal{F}_t\right].
\end{aligned}$$

4.4.1 Performance-Linked Liabilities with Capped Benefits and Unhedgeable Risks

The preceding theorem allows for the consideration of liability factors $L_1(T)$ which are only partially hedgeable. To illustrate that, we use a two-factor model with one factor representing the hedgeable risks and one factor representing the unhedgeable risks. For the liabilities presented here, we can explicitly calculate

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\mathbb{E}\left[(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2\mathcal{U}_1\right))^\alpha|\mathcal{F}_T\right]\right)^{\frac{1}{1-\alpha}}$$

and $\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}}$ as shown in the following corollary.

Corollary 4.4.3 (Two-Factor Performance-Linked Liabilities)**.** *The optimal terminal wealth for an investor with power utility function* (2.6) *and performance-linked liabilities as in* (PLCBU) *with general f is given by Theorem 4.4.1 and the optimal investment strategy is given by Theorem 4.4.2 with*

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\frac{(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_1))^{\alpha+1}-(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_2))^{\alpha+1}}{(c_2-c_1)(\psi_L L(0)\beta_2)(\alpha+1)}\right)^{\frac{1}{1-\alpha}}$$

and

$$\begin{aligned}
\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}} =& \frac{1}{1-\alpha}\left(\frac{-\psi_L L(0)\beta_1 f'(X(T))g_x(X(t),\xi(t,T))}{\left((c_2-c_1)\left(\psi_L L(0)\beta_2\right)(\alpha+1)\right)^{\frac{1}{1-\alpha}}}\right)\left[(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right))^\alpha\right. \\
&\left.-\left(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)\right)^\alpha\right](\alpha+1)\cdot \\
&\cdot\left[(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right))^{\alpha+1}\right. \\
&\left.-\left(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)\right)^{\alpha+1}\right]^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

for $\alpha \neq -1$. For $\alpha = -1$, we have

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\frac{1}{(c_2-c_1)(\psi_L L(0)\beta_2)}\log\left(\frac{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right)}{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)}\right)\right)^{\frac{1}{2}}$$

and

$$\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}} = \frac{-\psi_L L(0)\beta_1 f'(X(T))g_x(X(t),\xi(t,T))}{(1-\alpha)\left((c_2-c_1)(\psi_L L(0)\beta_2)\right)^{\frac{1}{2}}}\left(\log\left(\frac{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right)}{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)}\right)\right)^{-\frac{1}{2}}$$
$$\cdot\left(\frac{1}{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right)}-\frac{1}{1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)}\right).$$

Proof. For liabilities of the form (PLCBU), the corresponding random utility function is given by

$$\hat{U}_\omega(v) = \mathbb{E}\left[U\left(v-\psi_L v L(0)\left(\beta_1 f(X(T))+\beta_2\mathcal{U}_1\right)\right)|\mathcal{F}_T\right].$$

For $K < \frac{1}{\beta_1}\left(\frac{1}{\psi_L L(0)}-\beta_2 c_2\right)$, we obtain with $f(X(T)) \leq K$ that

$$L_1(T,X(T),\mathcal{U}_1)) = \psi_L L(0)\left(\beta_1 f(X(T))+\beta_2\mathcal{U}_1\right) < \psi_L L(0)(\beta_1 K+\beta_2 c_2) =: k_1 < 1.$$

Hence, (L2.1) holds and $\hat{U}$ is well-defined for all $v > 0$, i.e. $\hat{v}_0(\omega) \equiv 0$. The optimal terminal wealth is given by Theorem 4.1 with $\Delta_\omega = \left(\mathbb{E}[(1-\psi_L L_1(T))^\alpha|\mathcal{F}_T]\right)^{-1}$. First, we consider the case $\alpha \neq -1$. Since $X(T)$ is $\mathcal{F}_T$-measurable and $\mathcal{U}_1$ is independent of $\mathcal{F}_T$, it follows that

$$\mathbb{E}\left[(1-\psi_L L_1(T))^\alpha|\mathcal{F}_T\right] = \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\left(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 u\right)\right)^\alpha du,$$

and consequently

$$\begin{aligned}\frac{1}{\Delta_\omega} &= \frac{-1}{(c_2-c_1)(\psi_L L(0)\beta_2)}\int_{1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_1)}^{1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_2)} u^\alpha du\\ &= \frac{(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_1))^{\alpha+1}-(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_2))^{\alpha+1}}{(c_2-c_1)(\psi_L L(0)\beta_2)(\alpha+1)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}}\\ &= \frac{\partial}{\partial X(t)}\left(\frac{(1-\psi_L L(0)(\beta_1 f(g(X(t),\xi(t,T)))+\beta_2 c_1))^{\alpha+1}-(1-\psi_L L(0)(\beta_1 f(g(X(t),\xi(t,T)))+\beta_2 c_2))^{\alpha+1}}{(c_2-c_1)(\psi_L L(0)\beta_2)(\alpha+1)}\right)^{\frac{1}{1-\alpha}}\\ &= \frac{1}{1-\alpha}\left(\frac{-\psi_L L(0)\beta_1 f'(X(T))g_x(X(t),\xi(t,T))}{(c_2-c_1)\left(\psi_L L(0)\beta_2\right)(\alpha+1)}\right)\left[(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_1\right))^\alpha\right.\\ &\quad\left.-\left(1-\psi_L L(0)\left(\beta_1 f(X(T))+\beta_2 c_2\right)\right)^\alpha\right]\\ &\quad\cdot(\alpha+1)\left(\frac{(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_1))^{\alpha+1}-(1-\psi_L L(0)(\beta_1 f(X(T))+\beta_2 c_2))^{\alpha+1}}{(c_2-c_1)(\psi_L L(0)\beta_2)(\alpha+1)}\right)^{\frac{\alpha}{1-\alpha}},\end{aligned}$$

which can be written as in the statement. In case $\alpha = -1$, the proof works analogously, but we have

$$\frac{1}{\Delta_\omega} = \frac{1}{(c_2 - c_1)(\psi_L L(0)\beta_2)} \log\left(\frac{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_1\right)}{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_2\right)}\right),$$

respectively

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\frac{1}{(c_2 - c_1)(\psi_L L(0)\beta_2)} \log\left(\frac{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_1\right)}{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_2\right)}\right)\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}} \\
&= \frac{-\psi_L L(0)\beta_1 f'(X(T)) g_x(X(t),\xi(t,T))}{(1-\alpha)\left((c_2 - c_1)(\psi_L L(0)\beta_2)\right)^{\frac{1}{2}}} \left(\log\left(\frac{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_1\right)}{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_2\right)}\right)\right)^{-\frac{1}{2}} \\
&\cdot \left(\frac{1}{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_1\right)} - \frac{1}{1 - \psi_L L(0)\left(\beta_1 f(X(T)) + \beta_2 c_2\right)}\right).
\end{aligned}$$

□

With $\mathcal{U}_1$ being independent of $\mathcal{F}_T$ and $X(T)$ being $\mathcal{F}_T$-measurable, the distortion factor $\Delta_\omega^{\frac{1}{\alpha-1}}$ is expressed in terms of an expected value of a function of $\mathcal{U}_1$ and the actual realization of the unhedgeable risks does not influence the optimal investment strategy.

Remark 4.4.4. *For f as in* (4.4)*, we have $f'(x) = 1$ for $x < K$ and $f'(x) = 0$ for $x > K$. This property together with the fact that f is constant for $x \geq K$ simplifies the numerical calculation of the expectation from Theorem 4.4.2. The case of f from* (4.5) *is very similar. The same holds true for the two subsequent examples in this section.*

In the case $\beta_2 = 0$, i.e. if there are no unhedgeable risks, the optimal investment strategy simplifies and is given by the following corollary.

Corollary 4.4.5 (Performance-Linked Liabilities with Capped Benefits (PLCB))**.** *The optimal terminal wealth for an investor with power utility function* (2.6) *and performance-linked liabilities as in* (PLCBU) *with general f and $\beta_2 = 0$ is given by Theorem 4.4.1 and the optimal investment strategy is given by Theorem 4.4.2 with*

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(1 - \psi_L L(0)\beta_1 f(X(T))\right)^{\frac{\alpha}{1-\alpha}}$$

and

$$\frac{\partial}{\partial X(t)}\Delta_\omega^{\frac{1}{\alpha-1}} = \left(1 - \psi_L L(0)\beta_1 f(X(T))\right)^{\frac{2\alpha-1}{1-\alpha}} \left(\frac{\alpha}{1-\alpha}\right) \left(-\psi_L L(0)\beta_1 f'(X(T))\right) g_x(X(t),\xi(t,T)).$$

Proof. The proof works along the proof of Corollary 4.4.3 with

$$\Delta_\omega = (\mathbb{E}[(1-\psi_L L_1(T))^\alpha \,|\mathcal{F}_T])^{-1} = (1-\psi_L L_1(T))^{-\alpha}$$

since the liabilities are $\mathcal{F}_T$-measurable here. In this case, we obtain

$$\begin{aligned}
\frac{\partial}{\partial X(t)}\Delta_\omega^{-\frac{1}{b}} &= \frac{\partial}{\partial X(t)}\left((1-\psi_L L(0)\beta_1 f(X(T)))^{-\alpha}\right)^{\frac{1}{\alpha-1}} \\
&= \frac{\partial}{\partial X(t)}\,(1-\psi_L L(0)\beta_1 f(g(X(t),\xi(t,T))))^{\frac{\alpha}{1-\alpha}} \\
&= (1-\psi_L L(0)\beta_1 f(X(T)))^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{\alpha}{1-\alpha}\right)(-\psi_L L(0)\beta_1 f'(X(T)))\,g_x(X(t),\xi(t,T)).
\end{aligned}$$

□

It is clearly visible in this example that the factor $\Delta_\omega^{\frac{1}{\alpha-1}}$ only depends on the relative surplus and the risk aversion here. As described earlier, Corollary 4.4.5 describes a special case of Corollary 4.3.1 (with $L_2(T,X(T))=0$).

4.4.2 Liabilities Driven by Geometric Brownian Motion

In this section, we consider the liability model (PLCBU*).

Remark 4.4.6. *In the case of completely hedgeable liabilities, i.e. $\hat{\sigma}_\epsilon = 0$, the liabilities correspond to the setting from Corollary 4.4.5 for the choice $\beta_1 = 1$.*

We examine the optimal terminal wealth and optimal investment strategy, in particular

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\mathbb{E}\left[(1-\psi_L L(0) f(X(T)\mathcal{U}_1(T)))^\alpha \,\middle|\mathcal{F}_T\right]\right)^{\frac{1}{1-\alpha}}$$

in the following corollary.

Corollary 4.4.7 (Liabilities Driven by Geometric Brownian Motion)**.** *The optimal terminal wealth for an investor with power utility function* (2.6) *and performance-linked liabilities as in* (PLCBU*) *with general f is given by Theorem 4.4.1 and the optimal investment strategy is given by Theorem 4.4.2 with*

$$\frac{1}{\Delta_\omega} = \int_{-\infty}^{\infty}\left(1-\psi_L L(0) f\left(X(T)e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T+\hat{\sigma}_\epsilon\sqrt{T}u}\right)\right)^\alpha \phi(u)du$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial X(t)} \Delta_\omega^{\frac{1}{\alpha-1}} = \\
&\frac{1}{1-\alpha} \left(\int_{-\infty}^{\infty} \left(1 - \psi_L L(0) f \left(g(X(t), \xi(t,T)) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \right) \right)^{\alpha} \phi(u) du \right)^{\frac{\alpha}{1-\alpha}} \\
&\cdot \int_{-\infty}^{\infty} \alpha \left(1 - \psi_L L(0) f \left(g(X(t), \xi(t,T)) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \right) \right)^{\alpha-1} \\
&\cdot \left(-\psi_L L(0) f' \left(g(X(t), \xi(t,T)) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \right) \right) \\
&\cdot g_x(X(t), \xi(t,T)) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \phi(u) du.
\end{aligned}
$$

Proof. We proceed in a similar way as in the proof of Corollary 4.4.3. The random utility function for liabilities of the form (PLCBU*) is given by

$$\hat{U}_\omega(v) = \mathbb{E}\left[U(v - \psi_L v L(0) f(X(T)\mathcal{U}_1(T))) | \mathcal{F}_T \right].$$

For $K < \frac{1}{\psi_L L(0)}$, we have with $f(X(T)\mathcal{U}_1(T)) \leq K$

$$L_1(T, X(T), \mathcal{U}_1)) = \psi_L L(0) f(X(T)\mathcal{U}_1) \leq \psi_L L(0) K =: k_1 < 1.$$

Hence, (L2.1) holds and $\hat{U}$ is well-defined for all $v > 0$, i.e. $\hat{v}_0(\omega) \equiv 0$. Again, we apply Theorem 4.4.1 and Theorem 4.4.2. It follows that

$$\frac{1}{\Delta_\omega} = \int_{-\infty}^{\infty} \left(1 - \psi_L L(0) f \left(X(T) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \right) \right)^{\alpha} \phi(u) du,$$

Thus,

$$\Delta_\omega^{\frac{1}{\alpha-1}} = \left(\int_{-\infty}^{\infty} \left(1 - \psi_L L(0) f \left(X(T) e^{-\frac{1}{2}\hat{\sigma}_\epsilon^2 T + \hat{\sigma}_\epsilon \sqrt{T} u} \right) \right)^{\alpha} \phi(u) du \right)^{\frac{1}{1-\alpha}}$$

and the statement follows by a straight-forward derivation using that expectation and differentiation can be interchanged here by dominated convergence due to (L2.1) and the properties of f as introduced in (PLCBU*). $\square$

4.5 Comparison of Optimal Investment Strategies

In this section, we use the different derived results to assess the impact of the liabilities, of the performance participation and of the unhedgeable risks on the investment strategy in the surplus optimization framework. In particular, we use the liability models specified in Examples 4.1.3 (ILCB), 4.1.5 (PLCBU), 4.1.6 (PLCBU*) and Corollary 4.4.5 (PLCB) as well as a strategy with mixed, hedgeable liabilities. In Section 4.5.1, we analyze the influence of general parameters on the optimal investment strategy. Compared to the Merton portfolio $\frac{\mu-r}{(1-\alpha)\sigma^2}$, the replicable liabilities from Corollary 4.3.3 (Example 4.1.3 (ILCB)) illustrate the influence of the hedgeable, index-linked liabilities with capped benefits. On the other hand, compared to the results from Corollary 4.4.5 (performance-linked, replicable (PLCB)), this setting also serves as a reference point for the assessment of the impact of the performance participation with hedgeable liabilities. In addition, we consider replicable liabilities as in Corollary 4.3.1 (mixed liabilities), with liabilities consisting of an equally-weighted average of the replicable performance-linked and index-linked type. In Section 4.5.2, the impact of the unhedgeable component in the liability risk is studied when comparing the results from Corollary 4.4.3 (performance-linked liabilities in a two-factor model, liabilities as in Example 4.1.5 (PLCBU)) and Corollary 4.4.5 (performance-linked, replicable (PLCB)). Finally, comparing the results from Corollary 4.4.3 (performance-linked liabilities in a two-factor model (PLCBU)) and Corollary 4.4.7 (performance-linked liabilities, setting as in Example 4.1.6 (PLCBU*)), we compare the influence of the different types of models with unhedgeable risks for the liabilities.

For these comparisons, we compute the optimal allocation in the risky asset in $t = 0$ numerically. Unless otherwise mentioned, we consider an investor with power utility function with $\alpha = -1$ and a risky asset representing equity with $\mu = 0.06$ and $\sigma = 0.3$. Furthermore, we choose $r = 0.01$, $T = 10$, $\psi_L = 1$, $V(0) = 1$ and $L(0) = 0.5$. X is always modeled as in (4.3) with $\hat{\mu}_X = 0$, $\hat{\sigma}_X = 0.1$. For the specific parameters of the liability models, we choose $K = 1.7$, $\beta_1 = \beta_2 = 0.5$, $c_1 = 0$, $c_2 = 0.1$, and $\hat{\sigma}_\epsilon = 0.48$. To ensure comparability of the models, $\hat{\sigma}_\epsilon$ is chosen such that the optimal allocations of both models with unhedgeable risks, i.e. the models from Corollary 4.4.3 (PLCBU) and Corollary 4.4.7 (PLCBU*), match for the setting in Section 4.5.2, whereas $L(0)$ and K are chosen such that the conditions for Corollary 4.4.3, Corollary 4.4.5 and Corollary 4.4.7 hold. Due to the typical properties of insurance products, liabilities which are partially linked to the development of an index or the portfolio of the insurance company itself are often most interesting. For f as in (4.4), the index-linked liabilities exhibit such a behavior. However, the performance-linked liabilities are stronger linked to the development of the risky asset than the asset portfolio itself. Therefore, we use f from (4.4) only to analyze the optimal investment strategy with index-linked liabilities and to illustrate the impact of the performance participation in the first part. In the second part, we choose f from (4.5) to get a more realistic assessment of the performance-linked liabilities. In this setting, we compare the different models for the performance-linked liabilities and analyze the impact of unhedgeable risks.

4.5.1 Impact of the Type of the Liabilities

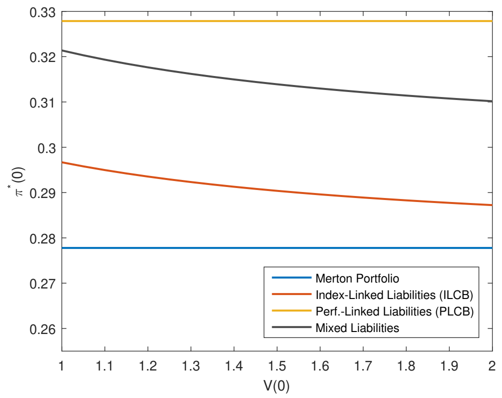

Figure 4.1: Optimal investment strategy depending on the initial wealth, f as in (4.4).

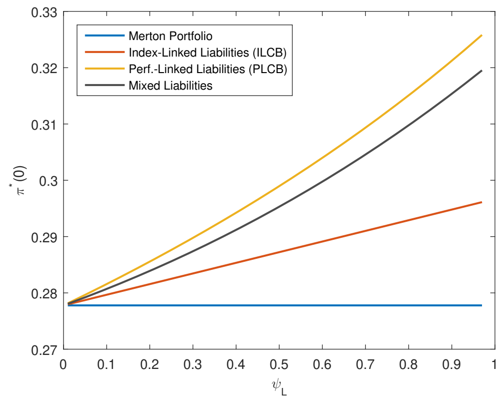

Figure 4.2: Optimal investment strategy depending on ψ_L, f as in (4.4).

In Figure 4.1, we see that the optimal allocation in the risky asset is independent of the initial wealth for the model with performance-linked liabilities (PLCB) as both, the assets and liabilities, increase with $V(0)$, whereas the allocation is decreasing in the initial wealth for the model with index-linked liabilities (ILCB) as a lower initial wealth requires a hedge of the liability risk which makes up a larger part of the portfolio. As $f(X(T))$ is positively correlated to the asset portfolio here, the higher investment for the index-linked and performance-linked liabilities compared to the Merton portfolio can be interpreted as an additional effort to hedge the liabilities.

The level to which the liabilities are considered influences the optimal investment strategy as Figure 4.2 shows. As $\psi_L \to 0$, the extent to which the liabilities are considered is decreasing and the strategies converge to the Merton portfolio. Again, we observe that the performance-linked liabilities lead to the highest allocation in the risky asset. Similar to the observation in Figure 4.1, the positive correlation of $X(T)$ and the asset portfolio leads for the index-linked and performance-linked liabilities to a larger allocation as the surplus decreases, which can be interpreted as an additional effort to hedge the liabilities.

To examine the effect of the risk aversion, Figure 4.3 shows the optimal allocation depending on the relative risk aversion $1-\alpha$. The allocation in the risky asset is decreasing with a higher level of risk aversion in all cases. However, the allocation in the performance-linked case is higher than the allocation of the Merton portfolio for $1-\alpha > 1$ and lower than the Merton portfolio for $1-\alpha < 1$. The higher allocation of the investor being more risk-averse compared to the Merton portfolio can again be interpreted as an additional attempt to hedge the liability risk inherent in $L_1(t)$. As described before, the limit $\alpha \to 0$, can be interpreted as an investor with logarithmic utility and the optimal investment strategy is independent of $L_1(T)$ for such an investor. Therefore, the performance-linked investment strategy converges to the Merton portfolio as $1 - \alpha \to 1$. For the index-linked liabilities, the difference of the allocation between a high risk aversion and a low risk aversion is smaller. The reason is that the investor having index-linked liabilities hedges the liabilities regardless of the risk aversion. Only the investment of the remaining wealth is subject to the risk aversion (see the explanation after Corollary 4.3.1). In Figures 4.1-4.3, the optimal allocation for the mixed liabilities is always between the allocation of the performance-linked liabilities and the index-linked liabilities. To summarize, we see that in the case of index-linked or performance-linked liabilities, an investor invests considerably more in the risky asset than the investor without performance-linked liabilities. Furthermore, we observe that if the surplus is lower or the investor is more risk-averse, additional efforts are made to hedge the liabilities in the case of performance-linked liabilities. The different liability models show that a close link to the insurance company's portfolio value leads to higher allocations in the risky asset, which should be considered for the product design if, e.g., a high share of risky assets is desired for long-term products. Furthermore, the larger difference in the allocation in Figure 4.3 for the performance-linked liabilities compared to the index-linked liabilities shows that in case of performance-linked liabilities, the allocation is more sensitive to a change in α, so the insurance company's risk aversion is more important to the policy holder. This can be explained by the fact that the insurance company's risk aversion does not have an impact on the performance of the index underlying the index-linked product. However, if the policy holder participates in the insurance company's asset portfolio, the performance of the policy is directly influenced by the insurance company's risk aversion. As a consequence, the risk aversion of the insurance company and the policy holder should be consistent for performance-linked products.

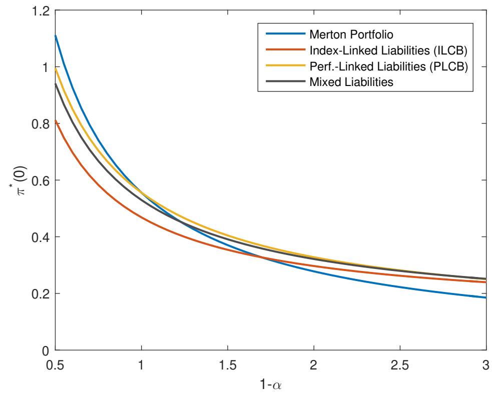

Figure 4.3: Optimal investment strategy depending on the relative risk aversion $1-\alpha$, f as in (4.4).

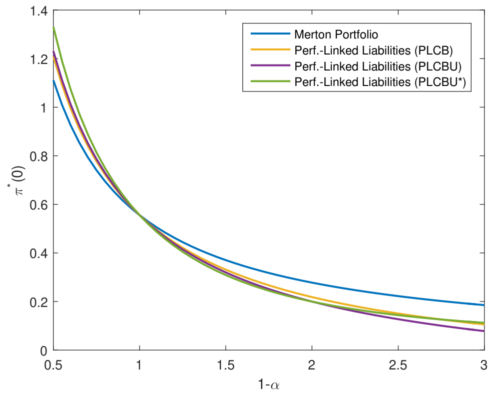

Figure 4.4: Optimal investment strategy depending on the relative risk aversion $1-\alpha$, f as in (4.5).

4.5.2 Comparison of Investment Strategies with Performance-Linked Liabilities and Non-Hedgeable Risks

In this section, we compare the settings for performance-linked liabilities (Corollaries 4.4.5 (PLCB), 4.4.3 (PLCBU) and 4.4.7 (PLCBU*)). In particular, we analyze the impact of unhedgeable risks on the optimal investment strategy. For a more realistic model of the liabilities, we use f from (4.5) in this section.

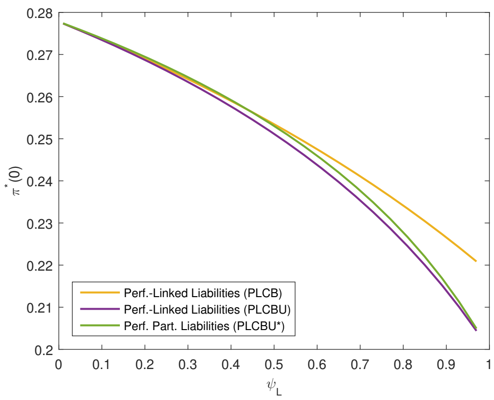

Figure 4.5: Optimal investment strategy depending on ψ_L, f as in (4.5).

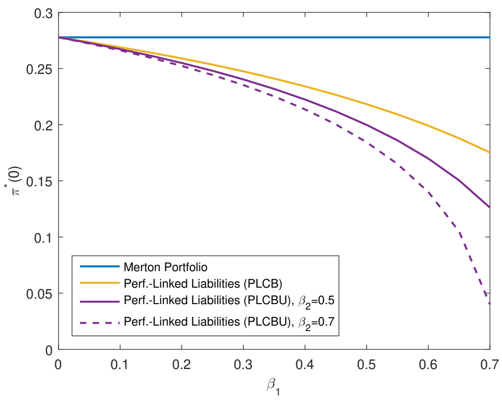

Figure 4.6: Optimal investment strategy depending on β_1, f as in (4.5).

By the choice of the parameters, both cases with performance-linked liabilities and additional unhedgeable risks have the same allocation in the base case with the parameters as described above. It is observable from Figure 4.4 that for $1 - \alpha > 1$, performance-linked liabilities without unhedgeable risks (PLCB) admit a higher allocation compared to the corresponding liability model with unhedgeable risks (PLCBU) as no buffer for these risks must be provided. For an increasing portion in ψ_L, i.e. as the weight of the liabilities increases and the surplus decreases, the allocation in the risky asset is decreasing. This is in contrast to the results presented in Figure 4.3, where the difference occurs as $f(X(T))$ is negatively correlated to the risky asset in the setting of Figure 4.4. Moreover, we notice that the allocation in the risky asset for the liabilities (PLCBU*) is slightly higher than the allocation in the the liability model (PLCBU) for $\psi_L \in (0, 1)$, while both allocations are the same for $\psi_L = 1$ (by construction of the paremter set) and for $\psi_L = 0$ (as the liabilities are not at all taken into account). The weight for the consideration of the hedgeable part of the liability risk in the liability model (PLCBU), represented by β_1, varies in Figure 4.6 and is shown for different levels of β_2, $\beta_2 = 0.7$, $\beta_2 = 0.5$ and $\beta_2 = 0$ (hedgeable performance-linked liabilities). We see that for larger values of β_1, the optimal allocation decreases as well as for larger values of β_2, so an increase in the liability risk leads to a decrease in the allocation. Independent of the level of β_2, the optimal investment strategy converges to the Merton portfolio as $\beta_1 \to 0$. This can be explained by the observation that for $\beta_1 = 0$ and any β_2, the surplus can be represented as a product of the portfolio value and the relative surplus $1 - \psi_L L_1(t)$, with $L_1(t)$ being independent of the portfolio value. In summary, Figures 4.4-4.6 show that unhedgeable risks such as additional mortality risks lead to a reduction in the risky asset allocation. The effect increases as the risk aversion increases (Figure 4.4), the surplus decreases (stronger consideration of the liabilities, Figure 4.5) or the portion of the unhedgeable risks increases (Figure 4.6). For the design of products, this means that the policy holder suffers form the insurance company's unhedgeable risks through a lower allocation in risky assets, in particular in case the insurance company has little own funds or is very risk-averse.

4.6 Comparison of Funding Ratio Optimization and Surplus Optimization

As stated in Remark 4.3.2, the optimal investment strategy for the surplus optimization framework can also be interpreted in terms of a three-fund separation. On the other hand, we have observed in Chapter 3 that the CPT funding ratio optimization can be interpreted as a generalization of the expected utility optimization. To compare the funding ratio optimization and the surplus optimization, we first establish a connection between the CPT funding ratio optimization and the surplus optimization approach. Then, we examine the three-fund separation of the surplus optimization approach further. Most aspects can be nicely interpreted directly from the closed-form representations of the investment strategies. In addition, Figures 4.7-4.10 illustrate the quantitative impact of changes in wealth and the risk aversion. The parameters chosen here are $V(0) = 1.1$, $L(0) = 1$, $\alpha = -1$, $B = \psi_L = 0.5$ and $\delta_+ = 0.1$.

In the CPT context, we optimize the utility of $\bar{F} = F^\pi(T) - B = \frac{V^\pi(T)}{L(T)} - B$ and we have

$$\frac{V^\pi(T)}{L(T)} - B \geq 0 \;\Leftrightarrow\; V^\pi(T) \geq BL(T) \;\Leftrightarrow\; V^\pi(T) - BL(T) \geq 0.$$

We observe that the terminal surplus is positive if and only if the funding ratio is above the reference point. Hence, the reference point B, indicating to which extent the liabilities are considered, corresponds to ψ_L in the partial surplus optimization.

To be able to compare the three-fund separation for the surplus optimization to the results for the funding ratio optimization, we expand the analysis from Remark 4.3.2 by using a liability model as in Chapter 3, i.e. we assume liabilities modeled by a geometric Brownian motion. In Example 4.1.1, we already showed that the liability model from Chapter 3 can be included in the surplus optimization framework by choosing $L_1(T, X(T)) = 0$, $L_2(T, X(T)) = X(T)$ and X is given by a Geometric Brownian motion as in (4.3). In Chapter 3, the discounted liability process was assumed to be a $\tilde{\mathbb{Q}}$-martingale to be able to use the liability process as a numéraire, so we assume that $\tilde{Z}(t)X(t)$ is a martingale here. Then, we have with Remark 4.3.2 for the optimal investment strategy

$$\begin{aligned}
\pi^*(t) =& \frac{\gamma(t)}{(1-\alpha)\sigma(t)} \cdot \frac{V^*(t) - \psi_L \mathbb{E}\left[\tilde{Z}(t,T)L_2(T, X(t))|\mathcal{F}_t\right]}{V^*(t)} \\
&+ \frac{\sigma_X(t, X(t))}{\sigma(t)V^*(t)} \psi_L \mathbb{E}\left[\tilde{Z}(t,T)L_{2,x}(T, X(T)g_x(X(t), \xi(t,T))|\mathcal{F}_t\right] \\
=& \frac{\gamma(t)}{(1-\alpha)\sigma(t)} \cdot \frac{V^*(t) - \psi_L \mathbb{E}\left[\tilde{Z}(t,T)X(T)|\mathcal{F}_t\right]}{V^*(t)} + \frac{X(t)\hat{\sigma}_X}{\sigma(t)V^*(t)} \psi_L \mathbb{E}\left[\tilde{Z}(t,T)\xi(t,T)|\mathcal{F}_t\right] \\
=& \frac{1}{1-\alpha} \frac{V^*(t) - \psi_L X(t)}{V^*(t)} \pi^{PS} + \frac{\psi_L X(t)}{V^*(t)} \sigma^{-1}\hat{\sigma}_X,
\end{aligned} \tag{4.10}$$

with the martingale property of X and subsequently ξ and π^{PS} as in (2.11). Here, $\psi_L X(t)$ is the value of the part of the liabilities considered, and $\sigma^{-1}\hat{\sigma}_X$ corresponds to the liability hedging portfolio from Chapter 3. Thus, the part $\frac{\psi_L X(t)}{V^*(t)}\sigma^{-1}\hat{\sigma}_X$ hedges the liabilities completely. It is visible that the investment in the performance seeking portfolio is scaled by the relative surplus $\lambda^S(V^*(t), X(t)) := \frac{V^*(t) - \psi_L X(t)}{V^*(t)}$ and the relative risk tolerance $\frac{1}{1-\alpha}$. The strategy is therefore a generalization of the CPPI strategy with the power utility of the surplus being a generalization of the usual HARA utility for a stochastic floor. Furthermore, (4.10) can also be written as

$$\pi^*(t) = \frac{1}{1-\alpha}\lambda^S(V^*(t), X(t))\pi^{PS} + (1 - \lambda^S(V^*(t), X(t)))\sigma^{-1}\hat{\sigma}_X.$$

As the investor becomes more risk-averse i.e. α decreases, the investment in the performance seeking portfolio is reduced, the liabilities are still completely hedged, and the investment in the risk-free asset is therefore increased. As the surplus is decreased, i.e. $V^*(t) - \psi_L X(t) \to 0$ and for arbitrary, but fixed α, the allocation in the risky asset is also reduced. As the liabilities are still completely hedged, this means the investor only holds the liability hedging portfolio in the limit.

Recall that the three fund separations, which we obtained in Chapter 3 is given by

$$\pi^*(t) = \frac{1}{1-\alpha}\pi^{PS} + \left(1 - \frac{1}{1-\alpha}\right)\pi^{LH}$$

for the expected utility funding ratio optimization with power utility (see Corollary 3.1.2) and

$$\pi^*(t) = \lambda^{CPT}(F(t), B)\pi^{PS} + (1 - \lambda^{CPT}(F(t), B))\pi^{LH},$$

with

$$\lambda^{CPT}(F(t), B) = \frac{F(t) - B}{F(t)} \cdot \frac{1 - \delta_+}{1 - \alpha}$$

for the CPT optimization with initial well-funding (see Corollary 3.2.7). Comparing these results to (4.10), we observe two differences. First, the allocation of the liability hedging portfolio in the surplus optimization is independent of the the relative risk tolerance in (4.10). The power utility applied to the surplus establishes a terminal wealth constraint $V^*(T) \geq \psi_L L_2(T, X(T))$. Hence, the liabilities $\psi_L L_2(T, X(T))$ have to be completely hedged regardless of the investor's risk aversion. As we observed, the optimal terminal surplus is the same as the optimal terminal wealth from an investor with initial capital $v_0 - \mathbb{E}\left[\tilde{Z}(T)\psi_L L_2(T)\right]$ (see Remark 4.3.2). Depending on the risk aversion, the capital not needed to hedge the liabilties is distributed only between the risky asset and the risk-free asset. This explains why the allocation in the liability hedging portfolio is independent of α (see also Figure 4.7). Furthermore, this means that a very risk-averse investor would hedge the liabilities and invest the remaining capital in a way such that there is little risk for a loss in surplus, i.e. in the risk-free asset. In Figure 4.8, we observe the decrease in the allocation in the risky asset as the risk aversion increases, where the difference in the allocation between the surplus optimization approach and the CPT approach results from the distortion in the CPT approach. For the funding ratio optimization in an expected utility framework with power utility (Corollary 3.1.2), the utility function is defined for any non-negative terminal funding ratio, i.e. there is no constraint on the terminal funding ratio and therefore no need to hedge the liabilities completely. Moreover, the allocations in the liability hedging portfolio and the performance seeking portfolio are independent of the actual wealth and funding ratio (see Figures 4.9 and 4.10). For the CPT funding ratio optimization with initial well-funding and $\eta_+ = 1$ (Corollary 3.2.7), we have $\lambda^{CPT}(F(t), B) = \frac{F(t)-B}{F(t)} \cdot \frac{1-\delta_+}{1-\alpha}$. Inserting $F(t) = \frac{V(t)}{L(t)}$, we obtain

$$\frac{F(t) - B}{F(t)} = \frac{V(t) - BL(t)}{V(t)}.$$

Thus, the allocation in the performance seeking portfolio is, except for the distorting factor, the same in the CPT case and the surplus optimization case, where the reference point B again corresponds to ψ_L in the surplus optimization. In contrast to the surplus optimization framework, the investment in the liability hedging portfolio is increased as the risk aversion increases for the expected utility funding ratio optimization and CPT funding ratio optimization (see Figure 4.7) or as $F(t) \to B$ in the CPT funding ratio optimization (see Figure 4.9). This can be explained as the investor within the surplus optimization framework reduces risk by reducing the volatility of the surplus (i.e. hedging the liabilities completely and investing the remaining capital in the risk-free asset) while an investor considering the funding ratio reduces the volatility by investing in the liability hedging portfolio.

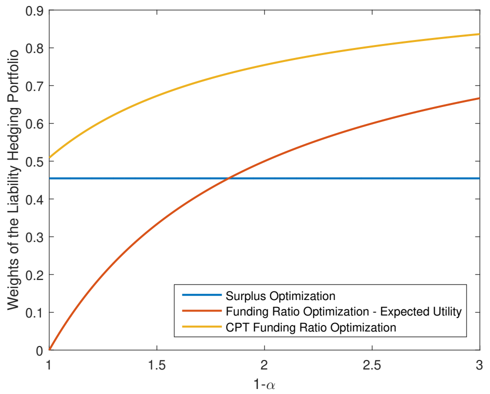

Figure 4.7: Optimal strategy depending on the relative risk aversion (liability hedging part).

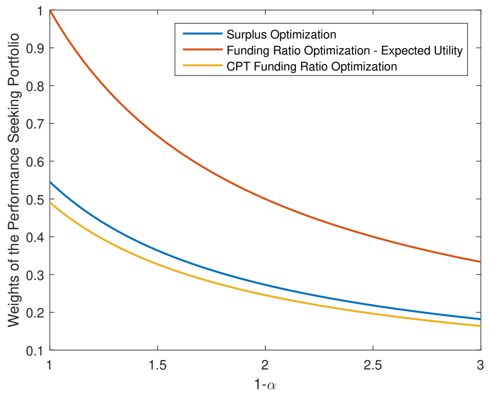

Figure 4.8: Optimal strategy depending on the relative risk aversion (performance seeking part).

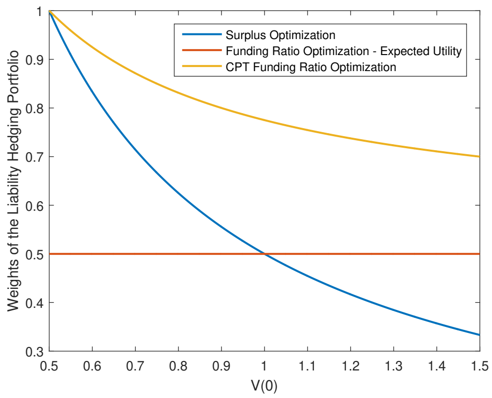

Figure 4.9: Optimal strategy depending on wealth (liability hedging part).

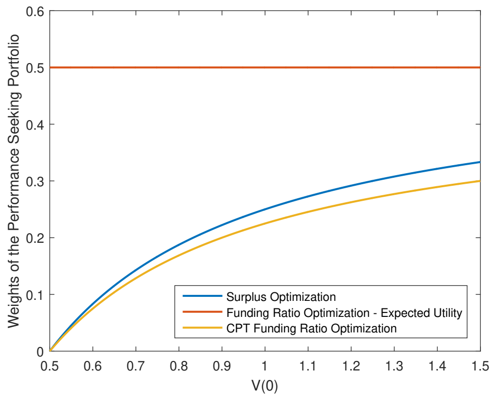

Figure 4.10: Optimal strategy depending on wealth (performance seeking part).

As observed in this chapter, considering the utility of the surplus imposes a constraint on the terminal wealth, which ensures that the wealth does not fall below the value of the liabilities. In practice, regulatory constraints such as the Solvency II regulatory regime for insurance companies aim at protecting the policy holders by imposing constraints on insurance companies. These constraints limit the decisions insurance companies can make. We deal with Solvency II constraints on the investment strategy in Chapter 5 and Chapter 6.

5 An Approximation to Wealth-Dependent Risk Constraints

Motivated by the Solvency II regulatory framework, we develop solution methods for portfolio optimization problems with wealth-dependent risk constraints in this chapter and in Chapter 6. In this chapter, we begin by introducing a general portfolio optimization problem with wealth-dependent risk constraints in Section 5.1. In Section 5.2, we introduce those parts of the convex duality approach from Cvitanić and Karatzas (1992), which are transferable to our framework with wealth-dependent portfolio constraints and we describe why their solution approach is not applicable to our optimization problem. For constraint sets independent of wealth, we state optimal investment strategies based on known results in Section 5.3. These results will be used in the iterative two-step approach to the problem with wealth-dependent constraint sets and serve as an associate problem in the solution approach in Chapter 6. In Section 5.4, we introduce the Solvency II constraint set, for which we develop the iterative two-step approach in Section 5.5. We assess the impact of the Solvency II constraints on the optimal investment strategy in Section 5.6 and compare our approach to an optimal approach on a discrete grid in Section 5.7. Large parts of this chapter coincide with Escobar et al. (2019) or Escobar et al. (2020).

5.1 The Constrained Optimization Problem

We consider a closed and convex wealth-dependent constraint set $\mathbb{K} = (K(t, V(t))_{t\in[0,T]}$ and the set of admissible strategies that satisfy the constraints

$$\begin{aligned}\Lambda(v_0) := \{\pi : \pi(t) \in K(t, V^\pi(t)) \ \mathbb{Q}\text{-a.s.} \\ \text{and} \ \ V^\pi(t) \in \mathbb{V} = (L, \infty) \ \ \mathbb{Q}\text{-a.s. for all } t \in [0,T]\,,\ V^\pi(0) = v_0\}\,.\end{aligned}$$

As in Chapter 2.3, for a strictly increasing, strictly concave utility function, the constrained optimization problem with utility from terminal wealth is defined as

$$\Phi(v_0) := \sup_{\pi\in\Lambda'(v_0)} \mathbb{E}\left[U\left(V^\pi(T)\right)\right], \tag{P}$$

where $\Lambda'(v_0) \subset \Lambda(v_0)$ consists of all $\pi \in \Lambda(v_0)$ satisfying $\mathbb{E}\left[U^-(V^\pi(T))\right] < \infty$.

It should be noted that (P) does not fall under the framework established in Cvitanić and Karatzas (1992). Although the authors mention their proofs can be extended to random constraint sets, i.e. constraint sets on the investment strategy π that depend on a random variable, the case of a constraint set which depends also on wealth as a function of π is not covered as we see later.

5.2 Convex Duality Framework

This section provides the theoretical background for the constrained portfolio optimization. In Sections 5.2.1 and 5.2.2, we introduce the auxiliary market and optimality condition in the context of wealth-dependent and investment constraints. In Cvitanić and Karatzas (1992), the authors show that solving the constrained optimization problem is equivalent to solving an unconstrained optimization problem within an appropriately chosen auxiliary market. The appropriate auxiliary market is chosen via a dual problem. In the following, we adapt parts of this approach to the more general case of wealth-dependent constraints.

5.2.1 Auxiliary Markets

To create the auxiliary markets, we introduce d-dimensional processes $(\lambda(t))_{t\in[0,T]}$, which are $\mathbb{F}$-progressively measurable. For $t \in [0,T]$ and $V(t) \in \mathbb{V}$, the support function of the set $K(t, V(t))$ is defined as

$$\delta(t,\lambda(t),V(t)) := \sup_{x\in K(t,V(t))} (-x^T\lambda(t)) = -\inf_{x\in K(t,V(t))} x^T\lambda(t),\ \ \lambda(t)\in\mathbb{R}^d, \tag{5.1}$$

and

$$\begin{aligned} X_{K(t,V(t))} &:= \left\{\lambda(t)\in\mathbb{R}^d : \delta(t,\lambda(t),V(t)) < \infty\right\} \\ &= \left\{\lambda(t)\in\mathbb{R}^d : \inf_{x\in K(t,V(t))} x^T\lambda(t) > -\infty\right\}. \end{aligned}$$

The following lemma illustrates the connection between the support function and the constraint set and will be needed later.

Lemma 5.2.1 (Rockafellar (1970), p. 112)**.**

$$\pi(t)\in K(t,V(t)) \ \Leftrightarrow\ \delta\left(t,\lambda(t),V(t)\right)+\pi(t)^T\lambda(t)\geq 0 \ \ \textit{for all} \ \ \lambda(t)\in X_{K(t,V(t))}.$$

The auxiliary markets $\mathcal{M}_\lambda$ with $d+1$ assets and price processes $(P_{\lambda,i}(t))_{t\in[0,T]}$, $i=0,...,d$ are derived from the original market by modifying the drift and the interest rate:

$$r_\lambda(t) := r(t)+\delta\left(t,\lambda(t),V_\lambda(t)\right),\ \ \mu_\lambda(t) := \mu(t)+\lambda(t)+\delta\left(t,\lambda(t),V_\lambda(t)\right)\mathbf{1},$$

$$\gamma_\lambda(t) := \sigma^{-1}(t)\left(\mu(t)+\lambda(t)-r(t)\mathbf{1}\right) = \gamma(t)+\sigma^{-1}(t)\lambda(t).$$

The wealth process in $\mathcal{M}_\lambda$ is given by

$$\begin{aligned} dV_\lambda^\pi(t) =& V_\lambda^\pi(t)\left[\pi^T(t)\left(\left(\mu(t)+\lambda(t)+\delta(t,\lambda(t),V_\lambda^\pi(t))\mathbf{1}\right)dt+\sigma(t)dW(t)\right)\right] \\ &+V_\lambda^\pi(t)\left[\left(1-\pi(t)^T\mathbf{1}\right)\left(r(t)+\delta(t,\lambda(t),V_\lambda^\pi(t))\right)dt\right] \\ =& V_\lambda^\pi(t)\left[\pi^T(t)\left(\mu(t)dt+\sigma(t)dW(t)\right)\right] \\ &+V_\lambda^\pi(t)\left[\delta(t,\lambda(t),V_\lambda^\pi(t))+\pi(t)^T\lambda(t)+(1-\pi(t)^T\mathbf{1})r(t)\right]dt. \end{aligned} \tag{5.2}$$

Note that the wealth processes in $\mathcal{M}$ and $\mathcal{M}_\lambda$ are the same if

$$\delta(t,\lambda(t),V_\lambda^\pi(t))+\pi(t)^T\lambda(t)=0 \text{ for } \mathcal{L}\otimes\mathbb{Q}-a.e.\ (t,\omega)\in[0,T]\times\Omega,$$

with $\mathcal{L}$ denoting the Lebesque measure. In the auxiliary market $\mathcal{M}_\lambda$, we consider admissible unconstrained strategies satisfying

$$\Lambda_\lambda(v_0) := \{\pi : V_\lambda^\pi(t) \in \mathbb{V} \text{ } \mathbb{Q}\text{-a.s. for all } t \in [0,T], V_\lambda^\pi(0) = v_0\}$$

and especially those processes λ which are of the form

$$D := \left\{\lambda : \mathbb{E}\left(\int_0^T \|\lambda(t)\|^2 dt\right) < \infty, \ \mathbb{E}\left(\int_0^T \delta\left(t, \lambda(t), V_\lambda^\pi(t)\right) dt\right) < \infty, \text{ for all } \pi \in \Lambda_\lambda(v_0)\right\}.$$

For the optimization problem, we consider the subset $\Lambda'_\lambda(v_0) \subset \Lambda_\lambda(v_0)$ consisting of the strategies in $\Lambda_\lambda(v_0)$, for which

$$\mathbb{E}\left[U^-(V_\lambda^\pi(T))\right] < \infty$$

holds additionally. The auxiliary problem in $\mathcal{M}_\lambda$ is defined as:

$$\Phi_\lambda(v_0) := \sup_{\pi \in \Lambda'_\lambda(v_0)} \mathbb{E}\left[U\left(V_\lambda^\pi(T)\right)\right]. \tag{P_{AUX}}$$

The optimal investment strategy for (P_{AUX}) is denoted by π_λ, the corresponding wealth at time $t \in [0,T]$ is denoted by $V_\lambda^{\pi_\lambda}(t)$. Note that the investment strategy in (P_{AUX}) is, in contrast to (P), not directly restricted by the constraint set $\mathbb{K}$, so (P_{AUX}) might be easier to solve.

5.2.2 Optimality Condition for Wealth-Dependent Constraint Sets

As a result of the wealth-dependent constraint set, several parts of the main results from Cvitanić and Karatzas (1992) are compromised. For once, the support function depends on wealth as well, or alternatively on the integrated strategy, hence jeopardizing the construction of the auxiliary and dual problems, i.e. $\tilde{Z}_\lambda$ depends on V. Moreover, in Cvitanić and Karatzas (1992), the appropriate auxiliary market is obtained from optimizing the dual problem. The dual problem in their case does not depend on primal variables, i.e. control π or wealth V, hence it can be decoupled from the primal problem. As also stated in Cvitanić and Karatzas (1992) this is, in principal, still possible for stochastic constraints (with such decoupling property). For our constraint set, however, the situation is more involved, as both, the investment strategy and wealth, are mixed in the constraint set. The question if and how a dual for such a type of dependence can be established and solved is, to the best of our knowledge, not answered yet. The only part from the approach in Cvitanić and Karatzas (1992) which can be transferred to our setting is their Proposition 8.3. The following proposition provides a condition for the choice of an appropriate auxiliary market. It generalizes Proposition 8.3 from Cvitanić and Karatzas (1992) to a setting including wealth-dependent constraints.

Proposition 5.2.2 (Optimality Condition)**.** *Suppose that, for some $\lambda^* \in \mathcal{D}$*

$$\pi_{\lambda^*}(t) \in K(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t)) \tag{5.3}$$

$$\delta\left(t, \lambda^*(t), V_{\lambda^*}^{\pi_{\lambda^*}}(t)\right) + \pi_{\lambda^*}(t)^\top \lambda^*(t) = 0 \tag{5.4}$$

hold $\mathcal{L} \otimes \mathbb{Q}$-almost everywhere, with π_{λ^*} being the optimal investment strategy for (P_{AUX}) in $\mathcal{M}_{\lambda^*}$. Then, π_{λ^*} belongs to $\Lambda'(v_0)$ and is optimal for the original constrained optimization problem (P)*.*

Proof. With (5.4) and (5.2), we conclude that $V_{\lambda^*}^{\pi_{\lambda^*}}$ is also the wealth process corresponding to π_{λ^*} in the original market $\mathcal{M}$, i.e. $V_{\lambda^*}^{\pi_{\lambda^*}} = V^{\pi_{\lambda^*}}$, and with (5.3) that $\pi_{\lambda^*} \in \Lambda'(v_0)$. Therefore, we have $\Phi_{\lambda^*}(v_0) \leq \Phi(v_0)$.

On the other hand, with Lemma 5.2.1, we have for an arbitrary investment strategy $\pi \in \Lambda'(v_0)$ that $\pi(t) \in K(t, V^\pi(t))$ and thus

$$\delta\left(t, \lambda^*(t), V^\pi(t)\right) + \pi(t)^\top \lambda^*(t) \geq 0$$

$\mathbb{Q}$-almost surely and $t \in [0,T]$ and subsequently since $V_{\lambda^*}^\pi(0) = V^\pi(0) = v_0$ with (5.2)

$$V_{\lambda^*}^\pi(t) \geq V^\pi(t) \qquad \text{for all } t \in [0,T]$$

$\mathbb{Q}$-almost surely and as U is (strictly) increasing, also

$$\mathbb{E}\Big[U^-\big(V_{\lambda^*}^\pi(T)\big)\Big] \leq \mathbb{E}\Big[U^-\big(V^\pi(T)\big)\Big] < \infty$$

and

$$\mathbb{E}\Big[U\big(V^\pi(T)\big)\Big] \leq \mathbb{E}\Big[U\big(V_{\lambda^*}^\pi(T)\big)\Big].$$

We deduce $\Lambda'(v_0) \subset \Lambda'_{\lambda^*}(v_0)$ and

$$\Phi(v_0) \leq \Phi_{\lambda^*}(v_0).$$

Hence, π_{λ^*} is optimal for (P). □

While the previous proposition provides a condition for λ^* in a setting of wealth-dependent constraints, it does not show a practical way how to obtain λ^*. As stated above, the method from Cvitanić and Karatzas (1992) to obtain λ^* through a dual problem is not applicable in our context. In the next section, we deal with constraint sets which are independent of wealth, which will be the basis for the approximative two-step approach in this chapter as well as the treatment of truly wealth-dependent constraints in Chapter 6.

5.3 Dual Optimization Problem for Constraint Sets Independent of Wealth with Logarithmic Utility and Power Utility

The following results provide the solution for the special case of a problem with constraints independent of wealth, i.e. we assume a constraint set $\mathbb{K} \equiv K$, K being nonempty, convex and closed. Furthermore, we consider the corresponding optimization problem (P) with this constraint set. Since the constraint set K is independent of wealth, the setting is inside the framework by Cvitanić and Karatzas (1992). The process λ^* satisfying (5.3) and (5.4) can be obtained by solving a dual problem (Theorem 10.1. from Cvitanić and Karatzas (1992)) which can be used, e.g., to derive the solution for logarithmic utility and power utility. In our setting, the dual problem to (P) reads

$$\tilde{\Phi} := \inf_{\lambda \in D} \mathbb{E}\left[\tilde{U}\left(\tilde{Z}_\lambda(T)\right)\right], \tag{$\hat{D}$}$$

with the pricing kernel in the auxiliary market $\mathcal{M}_\lambda$

$$\tilde{Z}_\lambda(t) = \exp\left(-\int_0^t r(s) + \delta(s, \lambda(s)) + \frac{1}{2}\|\gamma_\lambda(s)\|^2 ds - \int_0^t \gamma_\lambda(s)^T dW(s)\right)$$

and the convex conjugate of the utility function is defined as

$$\tilde{U}(y) := \sup_{v>0} \left[U(v) - vy\right], \ 0 < y < \infty.$$

For the logarithmic utility and the power utility, we present a version of the result in Cvitanić and Karatzas (1992) but without consumption. We consider a logarithmic utility function of the form (2.4) with the convex conjugate of the utility function given by

$$\tilde{U}(y) = \sup_{v>0} \left[U(v) - vy\right] = -(1 + \log(y)), \ 0 < y < \infty$$

and a power utility function of the form (2.6) with the convex conjugate

$$\tilde{U}(y) = \sup_{v>0} \left[U(v) - vy\right] = \frac{1-\alpha}{\alpha} y^{\frac{\alpha}{\alpha-1}}, \ 0 < y < \infty.$$

For both, the logarithmic utility and the power utility, the optimal control λ^* for $(\hat{D})$ and the optimal investment strategy can be represented in the following way.

Proposition 5.3.1. *For U being the logarithmic utility function from* (2.4) *(case $\alpha = 0$) or the power utility function from* (2.6) *(case $\alpha < 1, \ \alpha \neq 0$), the optimal investment strategy with constant constraint set K is given by*

$$\pi^*(t) = \pi_{\lambda^*}(t) = \frac{1}{1-\alpha}(\sigma(t)\sigma(t)^T)^{-1}(\mu(t) + \lambda^*(t) - r(t)\mathbf{1}),$$

with deterministic

$$\lambda^*(t) = \arg \inf_{\lambda \in X_K} \left\{ \frac{1}{2(1-\alpha)} \|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 + \delta(\lambda) \right\}.$$

Furthermore, we have

$$\pi_{\lambda^*}(t) \in K \tag{5.5}$$

$$\lambda^*(t)^T \pi_{\lambda^*}(t) + \delta(\lambda^*(t)) = 0. \tag{5.6}$$

Proof. The proof is provided in Appendix C. □

Remark 5.3.2. *Note that the proof of Proposition 5.3.1 is heuristic. For mathematical rigorosity, the conditions of Theorem 2.3.1 have to be satisfied. This is the case for $\alpha > 0$ and bounded $\lambda^*(t)$ as in most of our later applications.*

5.4 The Solvency II Constraint Set

The constraint sets we use are inspired by the standard formula for the Solvency II regulation for insurance companies, which can be found in European Union (2015) (especially Chapter V). The idea of this risk based regulatory regime is that insurance companies must provide enough own funds to cover the 99.5%-Value-at-Risk on a time horizon of one year. While insurance companies can choose between using the standard formula, an internal model or a partial internal model, we only deal with the standard formula here as internal models might be company specific and not publicly available. An important impact of the regulatory Solvency II rules is that the investment strategy is constrained, with different asset classes requiring different amounts of capital to cover the corresponding risks. Under Solvency II, the following risk categories are considered for the market risk: interest-rate risk, equity risk, property risk, spread risk, concentration risk and currency risk. Since concentration risk and currency risk might be very company-specific, we neglect these two categories. Government bonds are considered to have interest-rate risk only. Equity is only considered in equity risk, real estate only in property risk. Corporate bonds are assumed to have interest-rate risk and spread risk. Thus, we consider a market model with four risky assets (i.e. $d = 4$), which we identify with these asset classes. The standard formula for the market risk of the solvency capital requirements at time t under Solvency II reads

$$SCR^{mkt}(t) = \sqrt{\sum_{i,j=1}^{4} c_{ij} SCR_i^{mkt}(t) SCR_j^{mkt}(t)},$$

with the captial requirements $SCR_i^{mkt}(t)$, $i \in \{\text{interest rate, equity, property, spread}\}$ and the correlation c_{ij} between the risk categories i and j. The correlation matrix is given by (see European Union (2015), Chapter V, Section 5)

$$C = \begin{array}{c} \\ \text{Interest-Rate} \\ \text{Equity} \\ \text{Property} \\ \text{Spread} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Interest-Rate} & \text{Equity} & \text{Property} & \text{Spread} \end{array} \\ \begin{pmatrix} 1 & A & A & A \\ A & 1 & 0.75 & 0.75 \\ A & 0.75 & 1 & 0.5 \\ A & 0.75 & 0.5 & 1 \end{pmatrix} \end{array}. \tag{5.7}$$

In case an increase (decrease) in interest rates represents a risk to the company, the factor A is chosen as 0 (0.5). In both cases, the correlation matrix is positive definite. As insurance companies must provide enough own funds to cover the capital requirements, we can establish a constraint set given by

$$K(t, V(t)) = \left\{\pi(t) \in \mathbb{R}^d : V(t) - L \geq SCR^{mkt}(t)\right\}, \tag{5.8}$$

where we describe the own funds as the difference between the assets (represented by the portfolio value $V(t)$) and the liabilities (represented by a constant $L \geq 0$). This constraint set depends jointly on the investment strategy and on wealth. For equity risk, property risk and spread risk, the capital requirements for each risk category are determined as

$$\begin{aligned} SCR^{mkt}_{equity}(t) =& \pi_2(t)k_2V(t),\\ SCR^{mkt}_{property}(t) =& \pi_3(t)k_3V(t),\\ \text{and } SCR^{mkt}_{spread}(t) =& \pi_4(t)k_4V(t), \end{aligned}$$

with constants $k_i \in (0,1]$, $i = 2,3,4$ representing shocks which are specified within Solvency II. While these shocks represent direct shocks on the value invested in a particular asset class, the shocks for the interest-rate risk are applied to the term structure of interest rates and may affect different asset classes and the value of the liabilities as well. The required capital which must be provided is determined by the loss in own funds suffered from the interest-rate shock $k_1 \in (0,1]$. Here, we assume that the impact of a change in interest rates to the liabilities is higher than to the assets. Therefore, a decrease in interest rates is a risk to the investor. Hence, the value of A in the correlation matrix C is set to 0.5. We approximate the capital requirements for interest-rate risk using the durations of assets and liabilities (see e.g. Zagst (2002), Chapter 6.1)

$$SCR^{mkt}_{interest}(t) = k_1\left(d_L L - d_1\pi_1 V(t) - d_4\pi_4 V(t)\right), \tag{5.9}$$

where d_L denotes the duration of the liabilities, d_1 denotes the duration of the government bonds and d_4 denotes the duration of the corporate bonds. We choose the liabilities to be constant in this chapter and in Chapter 6 as we want to examine the pure effect of the constraints on the investment strategy and to allow for closed-form solutions. Furthermore, liabilities are very company specific. Despite this simplification, the capital charges on interest-rate risk associated with the liabilities can still be calculated. This is comparable to the simplifying assumption of a constant volatility in the Black-Scholes model, where the vega can still be used for option risk management. For this setting, the constraint set (5.8) can be written as

$$K(t,V(t)) = \left\{\pi(t) : \frac{V(t)-L}{V(t)} \geq \sqrt{\left(B\pi(t)+v\right)^T WCW\left(B\pi(t)+v\right)}\right\}, \tag{5.10}$$

with

$$B := \begin{pmatrix} -d_1 & 0 & 0 & -d_4\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, v := \begin{pmatrix} \frac{L}{V(t)}d_L\\ 0\\ 0\\ 0 \end{pmatrix}, W := diag(k_1,...,k_4)$$

and the correlation matrix $C = (c_{ij})_{i,j=1,\dots,4}$ with $A = 0.5$. Similar as in Chapter 4, we denote the process $\frac{V(t)-L}{V(t)}$ as the relative surplus (RS).

5.5 Construction of the Two-Step Approach

To overcome the above described lack of a solution approach via a dual problem for our type of constraints, we establish an approximative two-step approach in this chapter. For the whole two-step approach, we assume constant market coefficients μ, σ and r. We approximate $K(t,V(t))$

by

$$\tilde{K}(t, c(t)) := \left\{ \pi(t) : c(t) \geq \sqrt{(B\pi(t) + \tilde{v})^T WCW(B\pi(t) + \tilde{v})} \right\}, \tag{5.11}$$

with a reasonable deterministic function $c(t)$ and $\tilde{v} := ((1 - c(t))d_L, 0, 0, 0)^T$. The reason for the replacement of v by $\tilde{v}$ is that $c(t)$ will serves as a lower bound for $RS(t)$ and

$$\begin{aligned} c(t) &\leq \frac{V(t) - L}{V(t)} = 1 - \frac{L}{V(t)} \\ \Leftrightarrow \frac{L}{V(t)} &\leq 1 - c(t). \end{aligned} \tag{5.12}$$

Proposition 5.5.1. *Let $t \in [0, T]$ and $\pi(t)$ such that $SCR_i(t) \geq 0$ for all i. If $c(t) \leq \frac{V(t)-L}{V(t)}$,*

$$\pi(t) \in \tilde{K}(t, c(t)) \Rightarrow \pi(t) \in K(t, V(t)).$$

Proof. See Appendix C. □

The two-step approach consists of the following steps:

Step 1: Using Proposition (5.3.1), the optimal investment strategy is determined depending on $c(t)$ and we receive the dynamics of the wealth process depending on $c(t)$.

Step 2: $c(t)$ is chosen such that the probability that the RS falls below $c(t)$ does not exceed a predefined level.

5.5.1 Step 1: Optimal Investment Strategy Depending on $c(t)$

For $\tilde{K}(t, c(t))$, we can calculate the support function analytically as stated in the following result.

Proposition 5.5.2. *The support function of $\tilde{K}(t, c(t))$ can be written as*

$$\begin{aligned} \delta(\lambda(t), c(t)) = c(t)\sqrt{\lambda(t)^T B^{-1}(WCW)^{-1}(B^T)^{-1}\lambda(t)} + \lambda(t)^T B^{-1}\tilde{v}, \\ X_{\tilde{K}} := X_{\tilde{K}(t,c(t))} = \mathbb{R}^d \end{aligned}$$

Proof. See Appendix C. □

Note that the function $\delta(\lambda(t), c(t))$ is continuous and convex on $X_{\tilde{K}} = \mathbb{R}^d$ for all $t \in [0, T]$ and $c(t) > 0$. Moreover, $\delta(\lambda(t), c(t))$ is $\mathbb{F}$-progressively measurable for every $\mathbb{R}^d$-valued process $\lambda(t) \in X_{\tilde{K}}$.

Using Proposition (5.3.1), a suitable auxiliary market can be constructed with the optimal dual $\lambda^*(c(t))$ given by

$$\begin{aligned}\lambda^*(c(t)) &= \arg\inf_{\lambda \in X_{\tilde{K}}} \left\{ \frac{1}{2(1-\alpha)} (\|\gamma + \sigma^{-1}\lambda\|^2) + \delta(\lambda, c(t)) \right\} \\ &= \arg\inf_{\lambda \in \mathbb{R}^d} \left\{ \frac{1}{2(1-\alpha)} (\|\gamma + \sigma^{-1}\lambda\|^2) + c(t)\sqrt{\lambda^T B^{-1}(WCW)^{-1}(B^T)^{-1}\lambda} + \lambda^T B^{-1}\tilde{v} \right\}\end{aligned} \tag{5.13}$$

While we are not aware of an analytical representation for $\lambda^*(c(t))$ in the case of more than one risky asset, $\lambda^*(c(t))$ can easily be computed numerically. The explicit representation of $\lambda^*(c(t))$ for the case of only one risky asset is presented in the following example.

Example 5.5.3. *Let $d = 1$ with the risky asset representing, e.g., equity. We consider Solvency II capital requirements of the form*

$$SCR^{mkt}(t) = \pi(t)kV(t), \ k \in (0,1] \, .$$

The corresponding Solvency II constraint set $K(t, c(t))$ can be written as

$$K(t, c(t)) = \left\{ \pi(t) : c(t) \geq \sqrt{\pi(t)^2 k^2} \right\} = \{\pi(t) : c(t) \geq k|\pi(t)|\} \, .$$

and the support function is given by

$$\delta(\lambda(t), c(t)) = \frac{c(t)}{k} |\lambda(t)|.$$

In this case, we have

$$\begin{aligned}\lambda^*(c(t)) &= \arg\inf_{\lambda \in \mathbb{R}} \left\{ \frac{1}{2(1-\alpha)} \|\gamma + \sigma^{-1}\lambda\|^2 + \frac{c(t)}{k}|\lambda| \right\} \\ &= \min\left(\frac{c(t)(1-\alpha)\sigma^2}{k} - (\mu - r), 0 \right)\end{aligned}$$

and

$$\pi^*_{\lambda^*}(c(t)) = \min\left(\frac{c(t)}{k}, \frac{1}{1-\alpha}\frac{\mu - r}{\sigma^2} \right).$$

Proof. See Appendix C. □

5.5.2 Step 2: Determination of $c(t)$

In this section, we choose a constant c as the largest possible number for which the two-step Solvency II capital requirements (5.11) hold for all points in time $t \in [0, T]$ with a probability of

$1-\beta$, i.e. we find

$$\begin{aligned}&\max c\\ &\text{s.t. } \mathbb{Q}\left[\min_{0\le t\le T}\frac{V^*(t,c)-L}{V^*(t,c)}\ge c\right]\ge 1-\beta, \quad c\in\left[0,\frac{v_0-L}{v_0}\right].\end{aligned}$$

We determine this probability using methods similar to the ones from barrier option pricing. In particular, we have

$$\begin{aligned}\frac{V^*(t,c)-L}{V^*(t,c)}&\ge c\\ \Leftrightarrow 1-\frac{L}{V^*(t,c)}&\ge c\\ \Leftrightarrow \qquad 1-c&\ge\frac{L}{V^*(t,c)}\\ \Leftrightarrow \qquad V^*(t,c)&\ge\frac{L}{1-c}.\end{aligned}$$

Thus, we compute the probability that the optimal wealth process $V^*(t,c)$ hits the barrier $\frac{L}{1-c}$ from above. For a given probability $1-\beta$, the constant c can be obtained numerically from this result.

Proposition 5.5.4. *For the optimal wealth process $V^*(t,c)$ corresponding to the optimal investment strategy $\pi^*(c,t)$, with the constraint set from* (5.11)*,*

$$\begin{aligned}&\mathbb{Q}\left[\min_{0\le t\le T}\frac{V^*(t,c)-L}{V^*(t,c)}\ge c\right]\\ &=1-\Phi\left(\frac{\log\left(\frac{L}{(1-c)v_0}\right)-\tilde{\mu}T}{\tilde{\sigma}\sqrt{T}}\right)-e^{2\frac{\log\left(\frac{L}{(1-c)v_0}\right)\tilde{\mu}}{\tilde{\sigma}^2}}\Phi\left(\frac{\log\left(\frac{L}{(1-c)v_0}\right)+\tilde{\mu}T}{\tilde{\sigma}\sqrt{T}}\right)\end{aligned}$$

with

$$\tilde{\mu}=\pi^*(c)^T(\mu-r)+r-\frac{\tilde{\sigma}^2}{2} \qquad \textit{and} \qquad \tilde{\sigma}=\|\pi^*(c)^T\sigma\|.$$

Proof. With Proposition 5.3.1 and $c(t)\equiv c$ being constant in (5.13), the wealth process $V^*(t,c)$ evolves like a geometric Brownian motion with constant drift and diffusion as $\pi^*(c(t))\equiv\pi^*(c)$ is a constant mix strategy. The SDE can be written as

$$dV^*(t,c)=V^*(t,c)\left[(\pi^*(c)^T(\mu-r)+r)dt+\pi^*(c)^T\sigma dW(t)\right].$$

Then, the minimum of the wealth process is for a fixed $t\in[0,T]$

$$\min V^*(t,c)=V^*(0,c)e^{\min R(t)}=v_0e^{\min R(t)},$$

with $R(t) := \tilde{\mu}t + \tilde{\sigma}W(t)$ being a Brownian motion with drift $\tilde{\mu}$ and diffusion $\tilde{\sigma}$

$$\tilde{\mu} = \pi^*(c)^T(\mu - r) + r - \frac{\tilde{\sigma}^2}{2} \quad \text{and} \quad \tilde{\sigma} = \|\pi^*(c)^T\sigma\|.$$

For the distribution of the RS process, we find

$$\begin{aligned}
&\mathbb{Q}\left[\min_{0\le t\le T}\frac{V^*(t,c)-L}{V^*(t,c)} \ge c\right] = \mathbb{Q}\left[\min_{0\le t\le T}\left(1 - L\frac{1}{V^*(t,c)}\right) \ge c\right] = \mathbb{Q}\left[1 - L\max_{0\le t\le T}\left(\frac{1}{V^*(t,c)}\right) \ge c\right]\\
&= \mathbb{Q}\left[\max_{0\le t\le T}\left(\frac{1}{V^*(t,c)}\right) \le \frac{1-c}{L}\right] = \mathbb{Q}\left[\frac{1}{\min\limits_{0\le t\le T}(V^*(t,c))} \le \frac{1-c}{L}\right] = \mathbb{Q}\left[\frac{L}{1-c} \le \min_{0\le t\le T} V^*(t,c)\right]\\
&= 1 - \mathbb{Q}\left[\min_{0\le t\le T} V^*(t,c) \le \frac{L}{1-c}\right] = 1 - \mathbb{Q}\left[v_0 e^{\min R(t)} \le \frac{L}{1-c}\right]\\
&= 1 - \mathbb{Q}\left[\min_{0\le t\le T} R(t) \le \log\left(\frac{L}{(1-c)v_0}\right)\right]\\
&= 1 - \Phi\left(\frac{\log\left(\frac{L}{(1-c)v_0}\right) - \tilde{\mu}T}{\tilde{\sigma}\sqrt{T}}\right) - e^{2\frac{\log\left(\frac{L}{(1-c)v_0}\right)\tilde{\mu}}{\tilde{\sigma}^2}}\Phi\left(\frac{\log\left(\frac{L}{(1-c)v_0}\right) + \tilde{\mu}T}{\tilde{\sigma}\sqrt{T}}\right),
\end{aligned}$$

where the distribution function of the running minimum of $R(t)$ can be found in Björk (2009), p.266. □

5.5.3 Iterative approach

In this section, we approximate the optimal investment strategy under a wealth-dependent constraint set further by applying the two-step approach to smaller intervals and recalculating the optimal investment strategy at an increasing number of fixed points t_i in $[0, T]$ with a decreasing length of the intervals $\Delta t = t_{i+1} - t_i$. This means, we take into account that the optimal investment strategy may depend on the stochastic outcome of the investment decisions.

We consider the set of $n_I + 1$ points in time t_i with

$$0 = t_0 < t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_{n_I} = T \text{ and } t_{i+1} - t_i = \Delta t.$$

For $t \in [t_i, t_{i+1})$, we denote the optimal iterative investment strategy by $\pi_I^*(t, c_i)$. It is found using the two-step probability approach with a constant c_i, so considering the probability

$$\mathbb{Q}\left[\min_{t_i\le t<t_{i+1}}\frac{V^{\pi_I^*}(t,c_i)-L}{V^{\pi_I^*}(t,c_i)} \ge c_i\right] \ge (1-\beta)^{\Delta t}, \tag{5.14}$$

where for $t \in [t_i, t_{i+1})$, the optimal wealth process evolves according to

$$dV^{\pi_I^*}(t,c_i) = V^{\pi_I^*}(t,c_i)\left(\pi_I^*(t,c_i)^T(\mu - r) + r\right)dt + \pi_I^*(t,c_i)^T\sigma dW(t),$$

with a known realized wealth $V^{\pi_I^*}(t_i, c_i))$ at time t_i.

In contrast to the two-step probability approach, we scale the probability from the right-hand side of (5.14) to the length of the sub-intervals Δt for the iterative approach. As the wealth process $V^{\pi_I^*}(t, c_i))$ follows a Geometric Brownian motion with independent increments and the investment strategy as well as $c(t)$ are recalculated in every point t_i, a probability of $(1-\beta)^{\Delta t}$ corresponds to a probability for the whole investment horizon of

$$\left((1-\beta)^{\Delta t}\right)^{\frac{1}{\Delta t}} = 1 - \beta.$$

The function $c(t)$ is then a stepwise-constant function with values c_i for $t \in [t_i, t_{i+1})$ depending on the actual wealth $V^{\pi_I^*}(t_i, c_i))$ at the beginning of the time period $[t_i, t_{i+1})$.

Compared to the probability approach on $[0, T]$ without recalculation of the optimal investment strategy, the constraints are relaxed in the iterative approach as there is the possibility of a more dynamic risk reduction. In Figure 5.1, the constant c for the regular approach and the time-dependent $c(t)$ for $n_I = 10$ and $n_I = 100$ in the iterative approach are shown for a sample path of the RS. The higher value of $c(t)$ for the iterative approach is observable. The same parameters as in Section 5.6 are used with an initial RS of 0.5.

Comparing both versions of the iterative approach, for a higher number of readjustments, the probabilities $(1-\beta)^{\frac{1}{\Delta t}}$ increase on each sub-interval on the one hand. On the other hand, the length of the intervals on which $c(t)$ is constant, decreases. In total, $c(t)$ for $n_I = 100$ is closer to the RS process most of the time. This is only violated in cases in which the RS process drops close to $c(t)$ for $n_I = 10$ near the next adjustment point of $c(t)$ with $n_I = 10$. In general, a shorter interval for the readjustments allows for higher values of $c(t)$, which lead to less strict constraints.

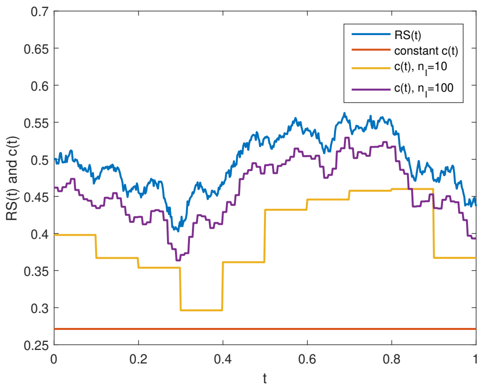

Figure 5.1: c(t) for different numbers of readjustments.

In the following sections, we assess the numerical impact of the Solvency II constraints in a multi-dimensional example using the established iterative two-step approach. In Section 5.6, we study the optimal investment strategy under the impact of the constraints and analyze its sensitivities. In Section 5.7, we show that, as the number of increments increases, the investment strategy converges numerically to the optimal investment strategy with wealth-dependent constraint set. As we do not know the optimal investment strategy with wealth-dependent constraint set in closed form, we approximate this strategy on a discrete grid for the one-asset example. Due to the increased computational effort of the numerically optimal strategy, we only consider one risky asset here.

5.6 Numerical Evaluation of the Two-Step Approach in a Multidimensional Example

We begin by describing the example with three risky assets. The first risky asset represents government bonds, the second equity and the third corporate bonds. In contrast to the constraint set presented in Section 5.4, we neglect investment in real estate and the corresponding property risk constraints as the holdings in this asset class might be very company specific. The market parameters μ, σ and r are chosen as

$$r = 0.01,\ \mu = \begin{pmatrix} 0.011 \\ 0.06 \\ 0.018 \end{pmatrix},\ \sigma = \begin{pmatrix} 0.06 & 0 & 0 \\ 0.0167 & 0.2995 & 0 \\ 0.0050 & 0.0298 & 0.0953 \end{pmatrix}. \tag{5.15}$$

These parameters are determined based on historical yield curve data from the European Central Bank as well as historical time series of equity indices, government bond indices and corporate bond indices. Historical risk premiums were used for the asset representing equity. The investment horizon T of the investor is chosen to be one year. Larger investment horizons would lead to a more volatile terminal wealth, which requires significantly more computational effort to obtain a good approximation to the optimal investment strategy for the Bellman approach on the discrete grid. The investor is assumed to have a power utility function as in (2.6) with $\alpha = 0.2$. The probability $1 - \beta$ of the iterative approach is chosen to be 95%. The intuitive interpretation is that the Solvency II constraints will be fulfilled for all $t \in [0, T]$ in 19 out of 20 years on average.

In the constraint set, we include the three risk categories, which apply to the chosen asset classes: interest-rate risk, equity risk and spread risk according to Section 5.4. The total solvency capital requirements are then determined by (5.10), with

$$C = \begin{array}{l} \\ \text{Interest-Rate} \\ \text{Equity} \\ \text{Spread} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Interest-Rate} & \text{Equity} & \text{Spread} \end{array} \\ \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.75 \\ 0.5 & 0.75 & 1 \end{pmatrix} \end{array},\ B = \begin{pmatrix} -d_1 & 0 & -d_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ v := \begin{pmatrix} \frac{L}{V(t)} d_L \\ 0 \\ 0 \end{pmatrix}.$$

The matrix W is defined according to European Union (2015) as

$$W = \begin{pmatrix} 0.35\mu_1 & 0 & 0 \\ 0 & 0.39 & 0 \\ 0 & 0 & 0.091 \end{pmatrix}.$$

Under Solvency II, the capital requirements for equity risk, spread risk and property risk are determined as the losses occurring from the application of shocks to the value of the affected assets. In contrast, the interest-rate shock is determined by a shift in the interest rates. In line with European Union (2015), we choose a 35% decrease in the level of interest rates. In our model, the level of interest rates can be approximated by μ_1, the drift of the first risky asset representing government bonds. This shock is applied to government bonds, corporate bonds and the liabilities in our example, where the loss triggered by the shock is calculated with (5.9). Therefore, the first entry of the diagonal of W is chosen as $k_1 = 0.35\mu_1$. For the choice of C, we also use the Solvency II parameters specified for companies, for which a decrease in interest rates provides a risk, which is consistent with the definition of $SCR^{mkt}_{interest}(t)$ as well as with the durations we choose in the following. The duration of the liabilities d_L is set to 12 years, whereas the duration of the government bonds d_1 is set to 8.9 years and the duration of the corporate bonds d_3 is set to 6.7 years. The values are determined using mean durations in EIOPA (2013).

Optimal Investment Strategy

For the parametrization of the iterative approach, we assume that the investor readjusts the approach on a daily basis. However, we only calculate the optimal investment strategy in $t = 0$. Since the optimal allocation at later points is only depending on the RS, we consider different values of initial RS. In Figure 5.2, the optimal λ^* in $t = 0$ is shown for different initial RSs and the corresponding optimal investment strategy in $t = 0$ is shown in Figure 5.3 for different values of the initial RS. For all values of the initial RS which we consider, the optimal investment strategy leads to positive SCRs for the different risk categories. Thus, Proposition 5.5.1 is applicable and the Solvency II requirements hold with a probability of 95%.

For an initial RS higher than 0.28, the unconstrained optimal investment strategy can be implemented. In this case, a significant share of wealth is allocated in all three different asset classes, with the highest proportion (65.17%) being invested in equity. For an initial RS less than 0.28, the optimal unconstrained investment strategy is not admissible and the risk in the investment strategy is reduced. As the initial RS falls below 0.28, the allocation in equity decreases sharply and the allocation in corporate bonds is also reduced, whereas the allocation in government bonds increases. This observation can be explained by the fact that the most severe constraints are imposed on equity, which is the most risky asset class in our example, The constraints on corporate bonds also contribute considerably which leads to a substantial decrease in the allocation.

On the other hand, government bonds become more attractive since they hedge the liability risk better than corporate bonds due to the higher duration and they are assumed to have no spread risk under the Solvency II standard formula. For an initial RS equal to zero, the only investment strategy which satisfies the constraints is to hedge the interest-rate risk of the liabilities completely with government bonds. It is observable in Figure 5.3 that the optimal investment strategy converges to this allocation as the initial RS converges to 0.

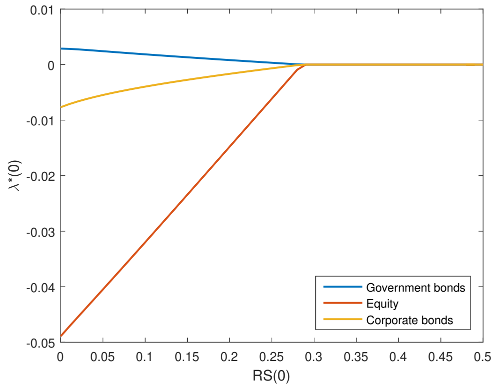

Figure 5.2: Optimal dual λ^* in t=0 for different values of the initial RS.

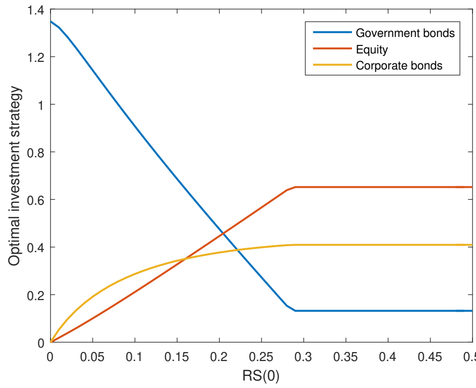

Figure 5.3: Optimal investment strategy π^* in t=0 for different values of the initial RS. The remaining wealth is invested in the risk-free asset.

Sensitivities of the Optimal Investment Strategy

In this section, we analyze the sensitivity of the investment strategy with respect to the initial RS, the insurance company's risk preferences and the duration of the liabilities. Figures 5.4, 5.6 and 5.8 show the optimal investments in the three different assets for different levels of α. For all three risky asset classes, a higher risk aversion (i.e. a lower value of α) leads to a reduction of the investment. This means, a higher portion of wealth is invested in the risk-free asset. Consequently, for lower values of α, the constraints only have an impact for lower values of the initial RS. Figure 5.4 exhibits a convergence towards government bonds for very small values of the initial RS as a way to minimize the capital requirements as described before. For all levels of risk aversion, the allocations in Figures 5.4, 5.6 and 5.8 converge to the same values as the initial RS approaches 0 due to this effect. Since α has a significant impact on the unconstrained portfolio, the differences in the allocation in equity and corporate bonds are large for higher values of the initial RS (representing a situation in which the constraints have little or no effect on the investment strategy). In conclusion, these observed effects have the following consequences: First, the allocations in equity and corporate bonds are most sensitive to α for high values of the initial RS (where the constraints do not have an impact). Second, the allocation in government bonds is most sensitive to changes in α for values of the RS, for which a high risk aversion leads to an optimal investment strategy which is not impacted by the constraints, whereas a low risk aversion leads to a constrained optimal investment strategy.

As the duration of the liabilities is, as the level of risk aversion, company-specific, we analyze the optimal investment strategy for various values of this parameter to analyze how the structure of the liability influences the optimal investment strategy. The optimal allocation in each risky asset class can be found in Figures 5.5, 5.7 and 5.9 for various values of the liability duration. The liability duration has an impact on the interest-rate SCR module. A higher liability duration requires a larger investment in bonds to hedge the greater sensitivity of the insurance company's value of the liabilities to changes in interest rates. As described above, the allocation in government bonds is increased as RS approaches zero to hedge the liability risk. In this situation, the sensitivity of the allocation to changes in the liability duration is especially high. For all values of the RS, we observe that liabilities with a larger duration lead to lower allocations in equity and corporate bonds. This can be explained by the fact that more capital is needed to cover the higher interest-rate risk for higher liability durations, so a smaller portion of wealth is available to cover spread risk and equity risk. Hence, the allocations are most sensitive with respect to different liability durations in the area where the restrictions have a large impact. In Figures 5.5, 5.7 and 5.9, it can also be observed that, for a fixed level of α, the allocations change monotonically in the duration of the liabilities. Interestingly, the increasing need for liability hedging caused by higher liability durations results in an increase in only the allocation in government bonds, whereas the corporate bond allocation is decreased. At this point, the advantage of a better diversification which would be provided by an increase in corporate bonds as well does not outweigh the disadvantage of the additional risk capital required for spread risk. Due to this opposite effect on government bonds and corporate bonds, changes in the liability duration have only a small impact on the allocation in equity.

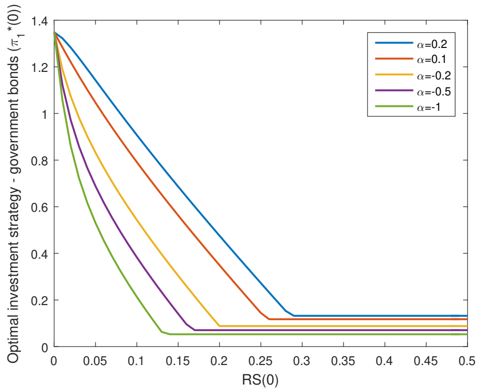

Figure 5.4: Portion of wealth invested in government bonds for different levels of α.

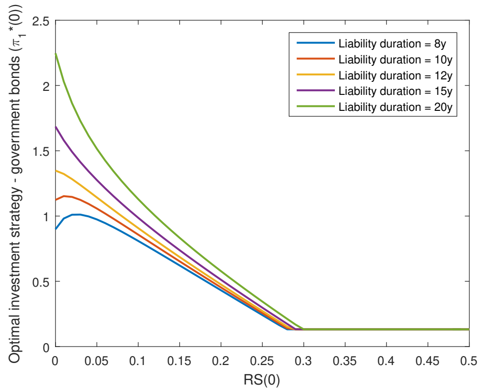

Figure 5.5: Portion of wealth invested in government bonds for different liability durations.

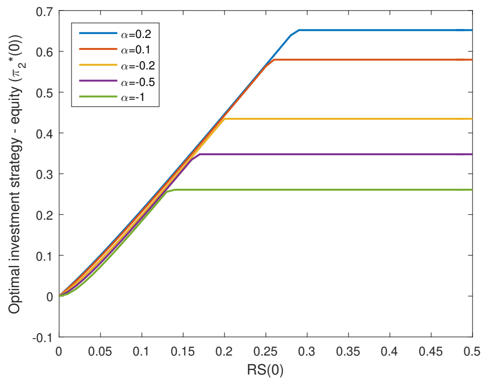

Figure 5.6: Portion of wealth invested in equity for different levels of α.

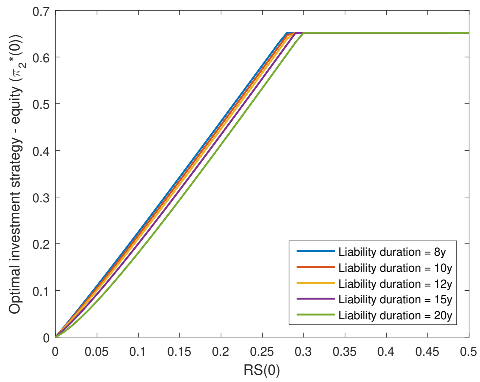

Figure 5.7: Portion of wealth invested in equity for different liability durations.

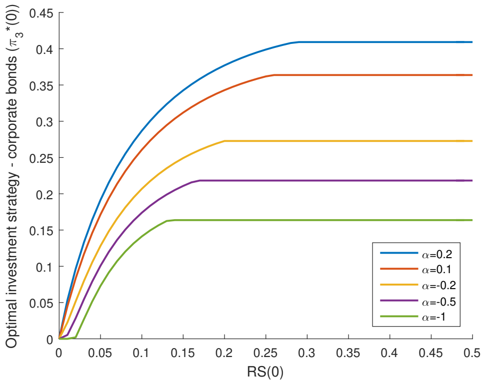

Figure 5.8: Portion of wealth invested in corporate bonds for different levels of α.

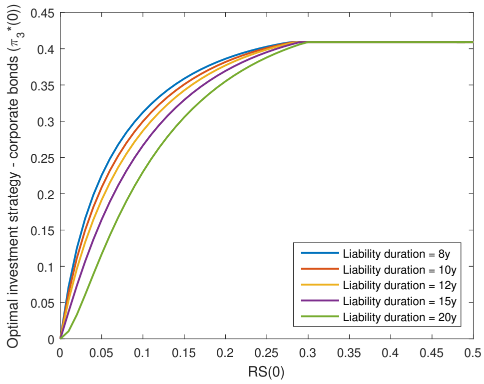

Figure 5.9: Portion of wealth invested in corporate bonds for different liability durations.

5.7 Comparison to a Numerically Optimal Investment Strategy

As we do not have a closed-form solution of the optimal dynamic investment strategy with Solvency II constraints, we construct an approximation on a discrete grid using the Bellman principle. We use this approximative optimal investment strategy as a reference for a comparison to our approaches. First, we explain the approach using the Bellman principle in Section 5.7.1. Then, we compare this approach to the two-step approach with constant c and to the iterative approach with readjustments of $c(t)$ in Section 5.7.2.

5.7.1 Optimal Investment Strategy Using the Bellman Principle

In this section, we construct an approximation to the optimal investment strategy with Solvency II constraints on a discrete grid using a backwards-inductive approach, which is based on the Bellman principle. The use of a discrete approximation in case of the lack of a closed-form solution to a portfolio optimization problem was suggested in He (1990), which also includes a proof of convergence of a discrete model to a continuous model. Bertsekas (1995) as well as Rieder and Zagst (1994) show that the numerical approximation on a discrete grid is also possible for constrained portfolio optimization problems.

We consider a set of $n_B + 1$ points in time with

$$0 = t_0 < t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_{n_B} = T \text{ and } t_{i+1} - t_i = \Delta t_B.$$

The wealth $V(t_i)$ at time t_i can take values on the set

$$\bar{\mathbb{V}} := \left\{ v^0, v^1, v^2, ..., v^{j-1}, v^j, v^{j+1}, ..., v^M \right\} \text{ with } v^{j+1} - v^j = \Delta v,$$

so $\bar{\mathbb{V}}$ is constant over time and approximates the set of all possible values of the wealth with $M + 1$ discrete points.

In our approach, all wealth outcomes $V(t_i) \in \left(v^j - \frac{\Delta v}{2}, v^j + \frac{\Delta v}{2}\right]$ are assigned to $v^j \in \bar{\mathbb{V}}$. For t_0, the initial wealth $V(t_0)$ is known as v_0.

By $\pi(t_i, V(t_i))$, we denote the investment strategy for $[t_i, t_{i+1})$. It is constant on this interval and subject to the portfolio constraints at the beginning of the interval. This constraint set is adapted from (5.11) and for the one-dimensional case given by

$$\tilde{K}_{t_i} = \left\{ \pi(t_i, V(t_i)) : \frac{V(t_i) - L}{V(t_i)} \geq k |\pi(t_i, V(t_i))| \right\}.$$

The discrete optimization problem on the grid reads

$$\max_{\pi(t_0,V(t_0)) \in \tilde{K}_{t_0}, \pi(t_1,V(t_1)) \in \tilde{K}_{t_1}, \ldots, \pi(t_n,V(t_n)) \in \tilde{K}_{t_{n_B}}} \mathbb{E}\left[U(V(T))\right]. \tag{PD}$$

We solve (PD) using the backwards-inductive approach based on Bellman's principle with Algorithm 1. The following result is used in this algorithm.

Proposition 5.7.1. *Let $V(t_i)$ be given in t_i. Then, for the investment $\pi(t_i, V(t_i))$, the probability that the wealth at time $t_{i+1} = t_i + \Delta t$ is assigned to $v^j \in \bar{\mathbb{V}}$ is given by*

$$\begin{aligned}
&q(V(t_i), \pi(t_i, V(t_i)), v^j) \\
&= \mathbb{Q}\left(V(t_i) e^{\left(\pi(t_i,V(t_i))(\mu - r) + r - \frac{\pi(t_i,V(t_i))^2\sigma^2}{2}\right)\Delta t + \pi(t_i,V(t_i))\sigma\sqrt{\Delta t} z_j} \in \left(v^j - \frac{\Delta v}{2}, v^j + \frac{\Delta v}{2}\right] \right) \\
&= \Phi\left(\frac{\log\left(\frac{v^j + \frac{\Delta v}{2}}{V(t_i)}\right) - \left(\pi(t_i, V_{t_i})(\mu - r) + r - \frac{\pi(t_i,V(t_i))^2\sigma^2}{2}\right)\Delta t}{\pi(t_i, V(t_i))\sigma\sqrt{\Delta t}} \right) \\
&- \Phi\left(\frac{\log\left(\frac{v^j - \frac{\Delta v}{2}}{V(t_i)}\right) - \left(\pi(t_i, V(t_i))(\mu - r) + r - \frac{\pi(t_i,V(t_i))^2\sigma^2}{2}\right)\Delta t}{\pi(t_i, V(t_i))\sigma\sqrt{\Delta t}} \right),
\end{aligned}$$

with z_j being a standard-normally distributed random variable.

With Algorithm 1, the initial optimal investment $\pi^*(t_0, V_0)$, which we will also use in our examples later, can be calculated by solving the problem in $t = 0$ and with known initial wealth v_0.

Remark 5.7.2. *The optimization problems* (5.17) *have to be solved for each point of the grid, i.e. for every $t_i \in \{0, ..., n_B - 1\}$ at all $v^j \in \bar{\mathbb{V}}$, numerically by testing various values for the allocation. Hence, the computational effort grows exponentially in the number of risky assets. This is the reason, why we only consider the example with one risky asset here.*

Algorithm 1 (Bellman Approach)

Step $t_{i+1} \to t_i$ for all $i \in \{n_B - 1, n_B - 2, ..., 1, 0\}$:

We calculate the value function $\Phi(t_i, V(t_i))$, defined as the expected utility of the optimal terminal wealth, given $V(t_i)$ at time t_i, recursively as

$$\Phi(t_i, V(t_i)) = \sum_{j=0}^{M} q(V(t_i), \pi^*(t_i, V(t_i)), v^j)\ \Phi(t_{i+1}, v^j), \tag{5.16}$$

with $\Phi(t_{n_B}, v^j) := U(v^j), V(t_i) \in \bar{\mathbb{V}}$.
For any $V(t_i) \in \bar{\mathbb{V}}$, the optimal investment in the risky asset $\pi^*(t_i, V(t_i))$ is given by

$$\underset{\pi(t_i, V(t_i)) \in K_{t_i}}{\arg\max} \sum_{j=0}^{M} q(V(t_i), \pi(t_i, V(t_i)), v^j)\ \Phi(t_{i+1}, v^j). \tag{5.17}$$

5.7.2 Comparison of the (Iterative) Two-Step Approach and the Bellman Approach

For the comparison of the two-step approach and the iterative approach to the Bellman approach, we use measures, which we introduce in the following. As the iterative approach includes a buffer to keep the risk of violating the constraints between two readjustment points at an acceptable level, we want to examine the costs of this buffer. Compared to the Bellman approach, we want to analyze the loss in utility gain and the difference in the investment strategy, which can be interpreted as a safety margin. Although measures for the loss which are expressed in terms of the wealth or the surplus (as we use in Section 6.3) provide a better interpretability than measures in terms of the utility function, the former are not suitable for the comparison here, since the difference between the iterative approach and the Bellman approach is already very small. This may provide difficulties as we discretize wealth here. Instead, we consider the expected utility gain, defined as

$$\mathbb{E}\left[\Delta U(v_0, \pi, T)\right] := \mathbb{E}\left[U(V^\pi(T))\right] - U(v_0),$$

for initial wealth v_0, investment strategy π and investment horizon T. Furthermore, we define the relative underperformance of the investment strategy π compared to an optimal investment strategy π^* (in our case the Bellman strategy) as

$$RUP(v_0, T, \pi, \pi^*) := 1 - \frac{\mathbb{E}\left[\Delta U(v_0, \pi, T)\right]}{\mathbb{E}\left[\Delta U(v_0, \pi^*, T)\right]}.$$

For the considered investment strategy π, we measure the margin of safety compared to the Bellman strategy π^* at time $t \in [0, T]$ as

$$MOS(t, \pi, \pi^*) := 1 - \frac{\pi(t)}{\pi^*(t)}.$$

In the following examples for the two-step approach without readjustments and the iterative approach, we consider a time period of $T = 1$ due to the increased computational effort for larger T as described above. In this setting, the optimal two-step approach without iterative readjustments will be set constant for one year in a way such that the Solvency II constraints are not violated with a probability of $1 - \beta = 95\%$. Naturally, we can expect a significant underperformance of the two-step approach compared to the Bellman approach. For the Bellman approach, we set $n_B = 500$, which corresponds to more than one readjustment daily. The parameters of the market model are chosen consistently with the risky asset representing equity from Section 5.6, i.e. $\mu = 6\%, \sigma = 30\%$ $r = 1\%$ and the Solvency II SCR parameter k is set to $k = 0.39$. The investor is assumed to have a power utility function with $\alpha = 0.2$. These settings result in an allocation in the risky asset of 69.44% if no constraints are applied at all. Since the optimal investment in the risky asset is non-negative, Proposition 5.5.1 is applicable.

Two-Step Approach without Readjustments

The performance of an investment strategy using the two-step approach without readjustments, which results in a constant investment strategy on the one-year period, is compared to the Bellman approach in Figures 5.10 and 5.11. Figure 5.10 shows that the underperformance in utility gain is small if the RS is high, but it increase up to roughly 25% as the RS decreases. The maximum of the underperformance can be observed for values of the RS close to 0.15. For increasing values of the RS higher than 0.15, the impact of the constraints reduces in general, which leads to a lower underperformace of the two-step approach. For decreasing values of the RS lower than 0.15, both, the Bellman strategy and the two-step approach become sharply constrained, which also reduces the underperformance of the two-step approach.

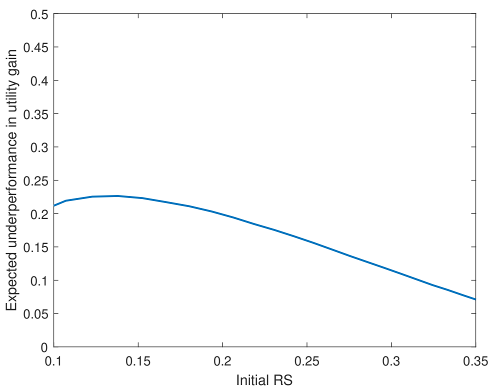

Figure 5.10: Relative underperformance in expected utility gain of the one-year probability approach.

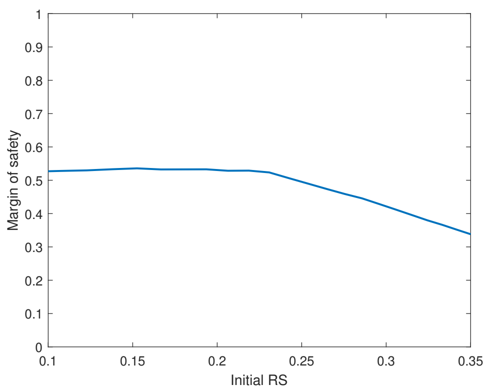

Figure 5.11: Margin of Safety of the one-year probability approach.

Figure 5.11 exhibits a considerable margin of safety, resulting from the requirement to fulfill the Solvency II capital charges in T with probability $1 - \beta$ for the constant investment strategy of the two-step approach compared to the Bellman strategy, which allows for a dynamic readjustment. Only for large values of the RS, a slightly lower margin of safety is possible. Clearly, the considerable losses and margins of safety for small values of the RS occur due to the lack of a dynamic way to adjust the allocation and to prevent violating the Solvency II constraints by reducing risk within $(0, T)$. To mitigate this weakness, we consider the iterative approach as introduced in Section 5.5.3 with allowance for readjustments of the strategy on smaller intervals in the next section.

Iterative Approach with Readjustments

For the iterative approach, with its strategy depending on the current wealth as for the Bellman approach, we also develop an algorithm on a grid, which has the same set of possible outcomes $\bar{\mathbb{V}}$ for the wealth process at each point in time t_i as the grid for the Bellman approach. However, the grid for the iterative approach has a smaller number n_I of readjustments of the strategy throughout $[0, T]$ than the Bellman strategy as it is designed as an approximation to the optimal investment strategy in continuous time. For the calculations of the value function of the iterative approach, we use the following algorithm on the grid.

Algorithm 2 (Iterative Approach)

Step $t_{i+1} \to t_i$ for all $i \in \{n_I - 1, n_I - 2, ..., 1, 0\}$:

For general c_i, the investment strategy on $[t_i, t_{i+1})$ is calculated as in Example 5.5.3:

$$\pi_I^*(c_i) = \min\left(\frac{c_i}{k}, \frac{1}{1-\alpha}\frac{\mu - r}{\sigma^2}\right)$$

and the corresponding wealth on this interval is denoted by $V^{\pi_I^*}(t, c_i)$. We determine c_i^* numercially (using Proposition 5.5.4) as

$$\begin{aligned} &\max c_i^* \\ &\text{s.t. } \mathbb{Q}\left[\underset{t_i \le t < t_{i+1}}{min} \frac{V^{\pi_I^*}(t, c_i) - L}{V^{\pi_I^*}(t, c_i)} \ge c_i\right] \ge 0.95^{\Delta t}. \end{aligned}$$

For $V(t_i) \in \bar{\mathbb{V}}$, the value function $\Phi^I(t_i, V(t_i))$ is calculated recursively as

$$\Phi^I(t_i, V(t_i)) = \sum_{j=0}^{M} q(V(t_i), \pi_I^*(c_i^*), v^j)\ \Phi^I(t_{i+1}, v^j)$$

with $\Phi^I(t_{n_I}, v^j) = U(v^j)$.

Note that the backwards-inductive approach from Algorithm 2 is only necessary to obtain the value function. The optimal investment strategy in t_i can be directly calculated since $V(t_i)$ is known. In particular, c_0^* and $\pi_I^*(c_0^*)$ are calculated with the initial wealth $V(t_0) = V(0) = v_0$.

Remark 5.7.3. *In comparison to the Bellman approach, the optimal investment strategy can be calculated faster here for two reasons: First, if the discretization of the time horizon is equidistant, the optimal strategy is only depending on the wealth at each point of the grid and it is independent of the time. Thus, the calculation of the optimal strategy has to be made only once for each point* *in $\bar{\mathbb{V}}$. Second, the optimal strategy does not have to be found by testing various values as in the Bellman approach, but is available in closed-form. Even the case of a multi-asset setting, λ^* for the optimal investment strategy has to be numerically determined, which is still much faster than the search in the Bellman approach (where the effort increases exponentially with the number of risky assets).*

Performance of the Iterative Approach

In order to compare the iterative approach to the Bellman approach, we calculate the optimal investment strategy and the value function using Algorithm 2. For the discretization $\bar{\mathbb{V}}$, we use the same set as for the Bellman approach. For the iterative approach, we want to examine the impact of regular readjustments, so we choose $n_I = 20 < 500 = n_B$, so we have a finer grid for the Bellman approach.

Figures 5.12 and 5.13 illustrate the better approximation considering the underperformance and the margin of safety, which the iterative approach provides for the optimal dynamic investment strategy. For large values of the RS, the underperformance and margin of safety of the iterative approach is barely noticeable (recall that the measures for the underperformance and the margin of safety are defined in relative terms compared to the Bellman strategy). For small values of RS, the underperformance and margin of safety of the iterative approach range up to roughly 10% and 30%, which is considerably less than for the two-step approach without readjustments.

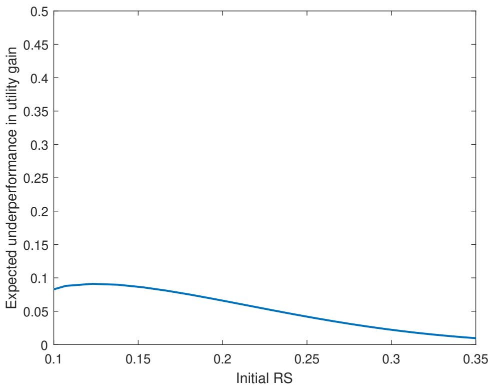

Figure 5.12: Relative underperformance in expected utility gain of the iterative approach.

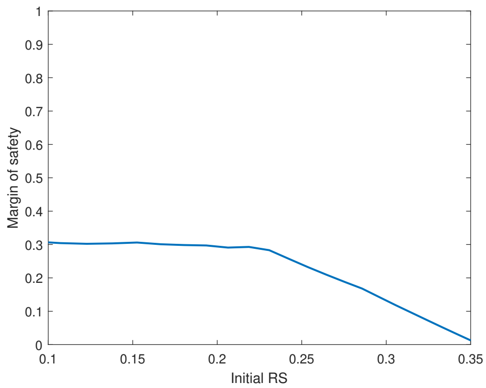

Figure 5.13: Margin of safety of the iterative approach.

Convergence of the Iterative Approach

As the iterative approach is designed to approximate the optimal constrained investment strategy, we analyze its underperformance and the margin of safety as we shorten the intervals between the readjustments in this section. We focus here on lower values of the RS, for which the difference between the iterative approach and the Bellman approach is larger. For an increasing number of readjustments, Figure 5.14 shows the convergence of the underperformance in expected utility gain. As the number of readjustments increases, the safety margin decreases (see Figure 5.15) and the difference between the performance of the iterative approach and the Bellman strategy vanishes. For $n_I = 250$ readjustments per year, the underperformance drops below 2% even for small values of the initial RS. This means that, although the probability for which the constraints need to hold at the end of each sub-interval $(1-\beta)^{\Delta t}$ increases with a shorter frequency of readjustments, i.e. as Δt decreases, a higher portion of wealth can be allocated to the risky asset due to the shorter period until the next readjustment. For insurance companies, who face the trade-off between the increasing operational costs of a more frequent readjustment and the loss caused by the required margin of safety for less frequent updates, this analysis can be used to determine a suitable frequency of readjustments of the investment strategy.

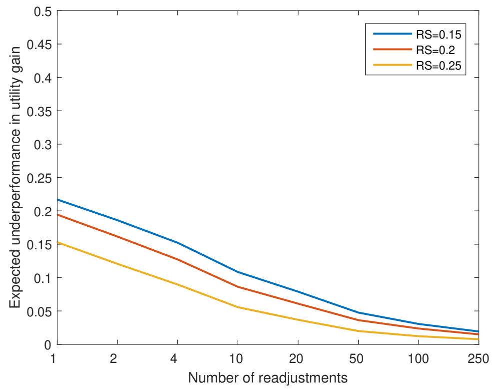

Figure 5.14: Relative underperformance in expected utility gain.

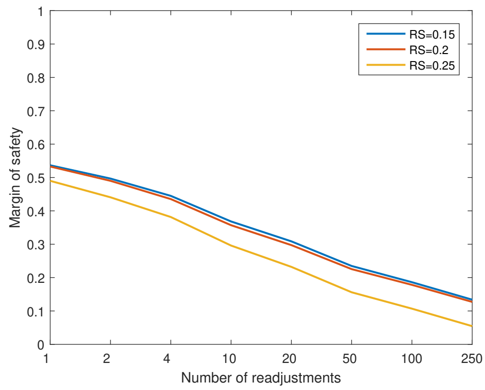

Figure 5.15: Margin of safety.

5.7.3 Conclusion on the Iterative Two-Step Approach

With the two-step approach and the possibility to apply it iteratively, we provide an approach to treat wealth-dependent Solvency II constraints in a setting with a continuous market model and allowance for dynamic investment strategies. In particular, the presented approach is applicable for investors with power utility who also want to take the liability risk in the constraint set into account. Motivated by the lack of an analytical solution to this problem, the iterative two-step approach provides a setting, which is superior to the Bellman approach with respect to the computational effort. Moreover, the approach can handle settings with more risky assets without having the disadvantage of an exponentially growing effort as in the Bellman approach. However, the lack of an analytical solution raises the question if we can find conditions under which portfolio optimization problems in continuous time with wealth-dependent risk constraints can actually be solved. From a practical point of view, such a solution would be better suited to assess investment strategies in settings with long-term investment horizons, in which the discretizations for the approximate approaches become large. Such an improved handling for long-term investment horizons would be of special interest as this perspective coincides with the nature of many insurance products and corresponding liabilities. We deal with these aspects in the next chapter.

6 Wealth-Dependent Risk Constraints

In this chapter, we consider a constraint set, which depends jointly on wealth and the investment strategy. Within the portfolio optimization approach using the convex duality method to cover constraints on the investment strategy, the crucial step is to solve the dual problem to obtain a process λ^* and corresponding optimal investment strategy π_{λ^*} in $\mathcal{M}^*_\lambda$ which jointly satisfy the optimality conditions from Proposition 5.2.2. For constraint sets, which depend jointly on wealth and the investment strategy, the support function generally depends on the wealth as well. As described in Section 5.2.2, the construction of an appropriate dual problem is to the best of our knowledge not solved. Therefore, we show in Section 6.1 that, under certain conditions, a problem with a wealth-dependent constraint set can be transformed such that λ^* can be determined by the duality methods from Cvitanić and Karatzas (1992). With this approach, we overcome the main shortcoming of the approach presented in Chapter 5, where we could only solve the optimization problem using constraint sets, which approximate the original wealth-dependent constraint set. In Section 6.2, we introduce a version of the Solvency II constraint set, which we use for the numerical study in 6.3. Finally, we compare the iterative two-step approach from the previous chapter to the exact approach from this chapter in Section 6.4. Large parts of this chapter coincide with Escobar et al. (2020). For the wealth-dependent constraint sets, we impose the following assumptions:

Assumption (A1). *Let $f : [0, T] \times \mathbb{V} \to \mathbb{R}^+$. $\mathbb{K}$ is of the form*

$$K(t, V(t)) = \left\{\pi(t) \in \mathbb{R}^d : f(t, V(t))^\beta \geq g(\pi(t)),\ V(t) \in \mathbb{V}\right\},\ t \in [0, T], \tag{A1}$$

for a homogeneous function $g : \mathbb{R}^d \to \mathbb{R}$ of order β satisfying

$$g(\alpha\pi(t)) = \alpha^\beta g(\pi(t))$$

for all $\alpha \geq 0$ and a constant $\beta > 0$.

Assumption (A2). *For any $\lambda(t) \in X_{K(t,V(t))}$, the suport function has the separable structure*

$$\delta(t, \lambda(t), V(t)) = f(t, V(t)) \cdot h(\lambda(t)) \tag{A2}$$

for some function $h : \mathbb{R}^d \to \mathbb{R}^+$ and f as in (A1).

Assumption (A1) represents a very general form for wealth-dependent risk constraints, where $g(\pi)$ represents some function describing the risk of the portfolio process π and $f(t, V(t))$ represents the risk budget depending on $V(t)$. Here, $g(\pi)$ is the component, which is inside the setting of Cvitanić and Karatzas (1992), whereas the wealth-dependence through $f(t, V(t))$ is not covered therein. In the following section, we introduce the constraint set derived from the Solvency II standard formula, i.e. $g(\pi)$ is derived from the regulations for calculating the market risk of the asset portfolio of an insurance company and $f(t, V(t))$ represents the relative surplus (own funds relative to the value of the assets). In Assumption (A2), we require the support function to be proportional to the risk budget $f(t, V(t))$.

Associate Problem with Constraint Set Independent of Wealth

In the following, we introduce the associate problem with constraints independent of wealth to which we will reduce our original problem with wealth-dependent constraints (P) later. In addition to the constraint set $K(t, V(t))$ from (A1), we consider an associate constraint set

$$\hat{K} := \left\{\pi \in \mathbb{R}^d : 1 \geq g(\pi)\right\}, \tag{6.1}$$

which can be interpreted as $\mathbb{K}$ from (A1) with $f(t, V(t)) \equiv 1$. The support function is therefore with (A2) given by

$$\hat{\delta}(\lambda(t)) = h(\lambda(t)).$$

We consider the problem (P) with constraint set $\hat{K}$ from (6.1) and utility $\hat{U}$. This problem is called the associate problem. For $\hat{U}$ being the logarithmic utility function from (2.4) (case $\alpha = 0$) or the power utility function from (2.6) (case $\alpha < 1$, $\alpha \neq 0$), Proposition 5.3.1 can be applied to solve the associate problem. The optimal investment strategy is then given by

$$\hat{\pi}_{\hat{\lambda}^*}(t) = \frac{1}{1-\alpha}(\sigma(t)\sigma(t)^T)^{-1}\left(\mu(t) + \hat{\lambda}^*(t) - r(t)\mathbf{1}\right), \tag{6.2}$$

with deterministic optimal dual process

$$\hat{\lambda}^*(t) = \arg\inf_{\lambda \in X_{\hat{K}}}\left\{\frac{1}{2(1-\alpha)}\|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 + \hat{\delta}(\lambda)\right\}. \tag{6.3}$$

Conditions (5.5) and (5.6) hold with Proposition 5.3.1 and read

$$\hat{\pi}_{\hat{\lambda}^*}(t) \in \hat{K} \tag{6.4}$$

$$\hat{\lambda}^*(t)^T\hat{\pi}_{\hat{\lambda}^*}(t) + \hat{\delta}(\hat{\lambda}^*(t)) = 0. \tag{6.5}$$

6.1 Solution Approach for Wealth-Dependent Constraint Sets

In this section, we consider the problem (P) (and subsequently also (P_{AUX})) with general wealth-dependent constraints satisfying (A1)and (A2) and a general utility function U satisfying Definition 2.2.1. We use the following assumption in Theorem 6.1.1, which provides a solution for this problem and conditions under which a reduction of the problem to a problem with constraint set $\hat{K}$ from (6.1) is possible.

Assumption (A3*). *The solution $\Phi \in C^{1,2}([0,T]\times\mathbb{V})$ to (P_{AUX}) within $\mathcal{M}_{\lambda^*}$ satisfies*

$$-\frac{\Phi_v(t,v)}{v\Phi_{vv}(t,v)} = \frac{f(t,v)}{1-\alpha} \Leftrightarrow \Phi_v(t,v) = c_1 e^{\int_{c_2}^{v} \frac{-(1-\alpha)}{xf(t,x)}dx}\tilde{\varphi}(t), \tag{A3*}$$

for a function $\tilde{\varphi}(t)$ and constants c_1, c_2 and $\alpha < 1$.

Theorem 6.1.1. *Let Assumptions* (A1), (A2) *and* (A3*) *be satisfied with $f(t,v) > 0$ for all $(t,v) \in [0,T]\times\mathbb{V}$ and let $\lambda^* = \hat{\lambda}^*$ be as in* (6.3)*. Then, the optimal investment strategy is given by*

$$\pi_{\lambda^*}(t) = \frac{1}{1-\alpha}\cdot f(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t))\cdot(\sigma(t)\sigma(t)^T)^{-1}(\mu(t)+\lambda^*(t)-r(t)\mathbf{1}). \tag{6.6}$$

Furthermore, π_{λ^*} is also the optimal investment strategy for (P)*.*

Proof. The HJB equation corresponding to (P_{AUX}) in $\mathcal{M}_{\lambda^*}$ is given by

$$\begin{aligned}\sup_{\pi(t)\in\mathbb{R}^d}\Big\{ & v\pi(t)^T\left(\mu(t)+\lambda^*(t)-r(t)\mathbf{1}\right)\Phi_v(t,v) + \frac{1}{2}v^2\|\pi^T(t)\sigma(t)\|^2\Phi_{vv}(t,v) \qquad (6.7)\\ & +\Phi_t(t,v) + v(r(t)+\delta(t,\lambda^*(t),v))\Phi_v(t,v)\Big\} = 0\\ & \Phi(T,v) = U(v). \qquad (6.8)\end{aligned}$$

From (A3*) and the first order condition in (6.7), we have

$$\begin{aligned}\pi_{\lambda^*}(t) &= \left(-\frac{\Phi_v(t,v)}{v\Phi_{vv}(t,v)}\right)(\sigma(t)\sigma(t)^T)^{-1}(\mu(t)+\lambda^*(t)-r(t)\mathbf{1})\\ &= \frac{1}{1-\alpha}f(t,v)(\sigma(t)\sigma(t)^T)^{-1}(\mu(t)+\lambda^*(t)-r(t)\mathbf{1}).\end{aligned}$$

Hence, (A1) reads

$$f(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t))^\beta \geq g\left(\pi_{\lambda^*}(t)\right) \Leftrightarrow 1 \geq g\left(\hat{\pi}_{\lambda^*}(t)\right) \tag{6.9}$$

with $\hat{\pi}_{\lambda^*}$ from (6.2). Using (A2), (6.6) and $f(t, V_{\lambda^*}^{\pi_{\lambda^*}}) > 0$, (5.4) can be written as

$$\begin{aligned} f(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t))\cdot h(\lambda^*(t)) + f(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t))\cdot\hat{\pi}_{\lambda^*}(t)^\top\lambda^*(t) &= 0\\ \Leftrightarrow h(\lambda^*(t)) + \hat{\pi}_{\lambda^*}(t)^\top\lambda^*(t) &= 0. \qquad (6.10)\end{aligned}$$

Thus, conditions (5.3) and (5.4) are equivalent to (6.9) and (6.10) in our setting. On the other hand, for $\lambda^* = \hat{\lambda}^*$, (6.9) and (6.10) are equivalent to (6.4) and (6.5). With Proposition 5.3.1, $\hat{\lambda}^*$ and $\hat{\pi}_{\hat{\lambda}^*}$ solve (6.4) and (6.5) and consequently $\lambda^* = \hat{\lambda}^*$ and $\pi_{\lambda^*}(t) = f(t, V_{\lambda^*}^{\pi_{\lambda^*}}(t))\cdot\hat{\pi}_{\hat{\lambda}^*}(t)$ solve (5.3) and (5.4). Thus, π_{λ^*} is optimal for (P) by Proposition 5.2.2. ☐

For the shifted logarithmic utility and the HARA utility in (P), this theorem is applied in the next section.

Logarithmic Utility and HARA Utility

In the sequel, we use the shifted logarithmic utility (2.3) and HARA utility (2.5) with $0 < L < v_0$. For these two utility functions, we provide solutions to (P) using Theorem 6.1.1 under the following assumption:

Assumption (A3). Let

$$f(t, V(t)) = \frac{V(t) - L(t)}{V(t)},\ L(t) := e^{-\int_t^T r(s)ds} L,\ t \in [0, T]. \tag{A3}$$

Corollary 6.1.2 (Logarithmic Utility)**.** *Let Assumptions* (A1), (A2) *and* (A3) *be satisfied. For an investor with shifted logarithmic utility* (2.3)*, the value function to* (P_{AUX}) *in* $\mathcal{M}_{\lambda^*}$ *is*

$$\begin{aligned}\Phi(t, v) =& \log(v - L(t)) + \varphi(t),\\ \varphi(t) =& \int_t^T \frac{1}{2}\|\gamma_{\lambda^*}(s)\|^2 + r(s) + h(\lambda^*(s))ds.\end{aligned}$$

Hence, (A3*) *is fulfilled, the optimal investment strategy to* (P) *is given by*

$$\pi_{\lambda^*}(t) = \left(\frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)}\right)(\sigma(t)\sigma(t)^T)^{-1}(\mu(t) + \lambda^*(t) - r(t)\mathbf{1}),$$

and $\Phi(t, v)$ *is also the value function for* (P)*.*

Proof. See Appendix D. □

Corollary 6.1.3 (HARA Utility)**.** *Let Assumptions* (A1), (A2) *and* (A3) *be satisfied. For an investor with HARA utility* (2.5)*, the value function to* (P_{AUX}) *in* $\mathcal{M}_{\lambda^*}$ *is*

$$\begin{aligned}\Phi(t, v) =& \frac{(v - L(t))^\alpha}{\alpha}\varphi(t),\\ \varphi(t) =& e^{\alpha \int_t^T \frac{1}{2(1-\alpha)}\|\gamma_{\lambda^*}(s)\|^2 + r(s) + h(\lambda^*(s))ds}.\end{aligned}$$

Hence, (A3*) *is fulfilled, the optimal investment strategy to* (P) *is given by*

$$\pi_{\lambda^*}(t) = \frac{1}{(1-\alpha)}\left(\frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)}\right)(\sigma(t)\sigma(t)^T)^{-1}(\mu(t) + \lambda^*(t) - r(t)\mathbf{1}),$$

and $\Phi(t, v)$ *is also the value function for* (P)*.*

Proof. See Appendix D. □

Remark 6.1.4. *Remark 5.3.2 applies to Corollary 6.1.3 as well.*

A closed-form representation for the wealth process of the optimal investment can be calculated in the same way as for a usual HARA strategy, as shown in the following proposition.

Proposition 6.1.5 (Optimal Wealth Process)**.** *The optimal wealth process corresponding to π_{λ^*} from Corollary 6.1.2 or Corollary 6.1.3 has the representation*

$$V^{\pi_{\lambda^*}}(t) = (V^{\pi_{\lambda^*}}(0) - L(t))\, e^{\frac{1}{1-\alpha}\left(\int_0^t \gamma_{\lambda^*}(s)^T\gamma(s)+(1-\alpha)r(s)-\frac{1}{2(1-\alpha)}\|\gamma_{\lambda^*}(s)\|^2 ds\right)} \cdot e^{\frac{1}{1-\alpha}\int_0^t \gamma_{\lambda^*}(s)dW(s)} + L(t),$$

with $\alpha = 0$ for the logarithmic utility.

Proof. See Appendix D. □

6.2 The Solvency II Constraint Set

To apply the previous results to Solvency II constraints, we discuss a version of the constraint set, which we use for the numerical study to the approach with wealth-dependent constraints. In contrast to the applications in Chapter 5 for the (iterative) two-step approach, we cannot consider the interest-rate risk inherent in the liabilities here, as the resulting support function (see Proposition 5.5.2) is not separable as required in Assumption (A2). Without interest-rate risk within the liabilities, an increase in the level of interest rates is a risk for the insurance company. This is the opposite case as in the application in Chapter 5. Both cases are captured by Solvency II with different levels in the interest-rate shocks applied and different correlations to the other risk categories (see (5.7)). While the case of Chapter 5 represents a typical life insurance company with long-term liabilities, the case considered here could be interpreted as a non-life insurance company. As in Chapter 5, we start by considering four risky assets representing government bonds, equity, real estate and corporate bonds. In this setting, the Solvency II capital requirements for the risk categories are therefore determined as

$$\begin{aligned} SCR^{mkt}_{interest}(t) &= k_1(d_1\pi_1 + d_4\pi_4)V(t),\\ SCR^{mkt}_{equity}(t) &= k_2\pi_2(t)V(t),\\ SCR^{mkt}_{property}(t) &= k_3\pi_3(t)V(t),\\ SCR^{mkt}_{spread}(t) &= k_4\pi_4(t)V(t), \end{aligned}$$

with constants $k_i > 0,\ i = 1, ..., 4$ representing the shocks specified by Solvency II for the corresponding risk categories and d_1 and d_4 being the duration of the government bonds and corporate bonds. Adapted to this case, (5.10) reads

$$K(t, V(t)) := \left\{\pi(t) \in \mathbb{R}^d : \frac{V(t) - L(t)}{V(t)} \geq \sqrt{(B\pi(t))^T WCWB\pi(t)}\right\},\ t \in [0, T], \tag{6.11}$$

with C as in (5.7) with $A = 0$, as well as

$$B = \begin{pmatrix} d_1 & 0 & 0 & d_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad W = diag(k_1, k_2, k_3, k_4).$$

Note that WCW is positive definite and (A1) and (A3) hold with $\beta = 1$ as well as $g(\pi(t)) = \sqrt{(B\pi(t))^T WCWB\pi(t)}$. The following result holds for a more general Solvency II-type constraint set with the function f not being further specified. Note that the support function here satisfies the separability Assumption (A2).

Proposition 6.2.1 (Generalized Solvency II Constraint Set)**.** *We consider a general Solvency II-type constraint set, which is of the form* (A1) *with*

$$g(\pi(t)) = \sqrt{(B\pi(t))^T RB\pi(t)}, \tag{6.12}$$

a positive definite matrix R, and an invertible matrix B. For $\lambda(t) \in X_{K(t,V(t))} \equiv \mathbb{R}^d$, the support function is given by

$$\delta(t, \lambda(t), V(t)) = f(t, V(t))\sqrt{\lambda(t)^T R^{-1}\lambda(t)}. \tag{6.13}$$

Hence, (A2) *is fulfilled. Furthermore, if f is bounded, $\mathcal{D}$ can be equivalently defined as*

$$\mathcal{D} := \left\{\lambda :\ \mathbb{E}\left(\int_0^T \|\lambda(t)\|^2 dt\right) < \infty\right\}. \tag{6.14}$$

Proof. See Appendix D. □

Remark 6.2.2. *Note that f as in Assumption* (A3) *is bounded for $V(t) \geq L(t)$, so $\mathcal{D}$ can be defined as in Proposition 6.2.1.*

6.3 Numerical Study

In this section, we illustrate the impact of regulations and market conditions using the preceding results and the version of the Solvency II constraint set. Throughout this chapter and for the case of exposition, we assume constant market coefficients μ, σ, r. We conduct two numerical studies, one with one risky asset and one with three risky assets. Our goal is to assess the long-term impact of the regulation.

For all examples, we consider the optimization problem (P) with HARA utility from (2.5) and the constraint set from (6.11), i.e. $f(t, V(t)) = \frac{V(t)-L(t)}{V(t)}$. With the support function from (6.13), Proposition 5.3.1 and Corollary 6.1.3, we have

$$\begin{aligned}
\pi_{\lambda^*}(t) &= \frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)}\hat{\pi}^*_{\hat{\lambda}^*} \\
&= \frac{1}{1-\alpha}\frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)}(\sigma\sigma^T)^{-1}\left(\mu + \lambda^* - r\mathbf{1}\right),
\end{aligned} \tag{6.15}$$

with

$$\lambda^* = \hat{\lambda}^* = \arg\inf_{\lambda\in\mathbb{R}^d}\left\{\sqrt{\lambda^T(B^T RB)^{-1}\lambda} + \frac{1}{2(1-\alpha)}(\|\gamma + \sigma^{-1}\lambda\|^2)\right\}.$$

The value function is given by

$$\Phi(t,v) = \frac{(v-L(t))^{\alpha}}{\alpha}\varphi(t),$$

with $\varphi(t) = e^{\alpha\left(\frac{1}{2(1-\alpha)}\|\sigma^{-1}(\mu+\lambda^*-r\mathbf{1})\|^2+r+\sqrt{(\lambda^*)^T(B^TRB)^{-1}\lambda^*}\right)(T-t)}$. By π_u and Φ_u, we denote the investment strategy and value function for the unconstrained optimization problem in the original market, which we use to assess the loss caused by the constraints. The unconstrained case is given for $\lambda^* = 0$, so inserting in π_{λ^*} and Φ yields

$$\begin{aligned}\pi_u(t) =& \frac{1}{1-\alpha}\frac{V^{\pi_u}(t)-L(t)}{V^{\pi_u}(t)}(\sigma\sigma^T)^{-1}(\mu - r\mathbf{1}),\\ \Phi_u(t,v) =& \frac{(v-L(t))^{\alpha}}{\alpha}\varphi_u(t),\end{aligned}$$

with $\varphi_u(t) = e^{\alpha\left(\frac{1}{2(1-\alpha)}\|\sigma^{-1}(\mu-r\mathbf{1})\|^2+r\right)(T-t)}$. Besides studying the optimal investment strategy $\pi_{\lambda^*}(t)$, we also want to analyze the loss an insurance company suffers due to regulatory constraints under relevant market conditions.

This is commonly measured via a *wealth equivalent utility loss*, which can be defined (see Escobar et al. (2015) and literature therein) as the solution l to

$$\begin{aligned}&\Phi_u(0,(1-l)v) = \Phi(0,v)\\ &\Leftrightarrow l = 1 - \frac{1}{v}\left(\left(\frac{\alpha\Phi(0,v)}{\varphi_u(0)}\right)^{\frac{1}{\alpha}} + L(0)\right)\\ &\quad = \left(1 - \left(\frac{\varphi(0)}{\varphi_u(0)}\right)^{\frac{1}{\alpha}}\right)\left(\frac{v-L(0)}{v}\right).\end{aligned}$$

Note that for an unconstrained strategy, we obtain $l = 0$. Whereas the wealth equivalent loss is independent of wealth for a power utility function (i.e. for $L = 0$), it depends on wealth in our case as we use the HARA utility. Therefore to describe the losses independently of the initial wealth, we introduce the concept of *surplus equivalent losses (SEL)*. The SEL measures the percentage-wise reduction in initial surplus an unconstrained investor requires to reach the expected utility of a constrained investor. In other words, the SEL describes the reduction in "return" caused by the constraints. We denote the surplus of the unconstrained strategy and the constrained strategy by

$$S_u(t) := V^{\pi_u}(t) - L(t) \quad\text{and}\quad S(t) := V^{\pi}_{\lambda^*}(t) - L(t).$$

In particular, we write $S := S(0) = v - L(0)$ to write the value function in terms of the surplus as

$$\Phi^S(0,S) := \frac{S^{\alpha}}{\alpha}\varphi(0) = \Phi(0,V),\ \ \Phi^S_u(0,S) := \frac{S^{\alpha}}{\alpha}\varphi_u(0) = \Phi_u(0,V)$$

and define the SEL l_s as the solution to

$$\Phi^S_u(0,(1-l_s)S) = \Phi^S(0,S),\ \ \text{i.e.}\ \ l_s = 1 - \left(\frac{\varphi(0)}{\varphi_u(0)}\right)^{\frac{1}{\alpha}}.$$

In addition to this measure, we introduce the *surplus equivalent risk reduction (SERR)*. While SEL captures the initial surplus return reduction, SERR is meant to capture the risk reduction due to the constraints. SERR measures the percentage-wise decrease in the terminal surplus a constrained investor can allow for to reach the same VaR level as an unconstrained investor. Therefore, they both capture the trade-off of return (on initial surplus) respectively risk (on terminal surplus) due to constraints: for a given $\epsilon \in (0, 1)$, the surplus equivalent risk reduction is defined as the solution r_s to

$$\mathbb{Q}((1 - r_s)S < c_\epsilon^u) = \mathbb{Q}(S_u < c_\epsilon^u) = \epsilon,$$

with a constant c_ϵ^u being the ϵ-quantile of the distribution of the terminal surplus from the optimal unconstrained strategy $S_u := V^{\pi_u}(T) - L$ and the surplus corresponding to the optimal constrained strategy $S := V^{\pi_{\lambda^*}}(T) - L$. Then, $c_\epsilon := \frac{c_\epsilon^u}{(1-r_s)}$ is the ϵ-quantile of the terminal surplus of the constrained strategy. Hence,

$$r_s = 1 - \frac{c_\epsilon^u}{c_\epsilon} = 1 - \frac{VaR_\epsilon^u}{VaR_\epsilon},$$

with VaR_ϵ^u denoting the Value-at-Risk of the terminal surplus for the unconstrained optimal investment strategy and VaR_ϵ denoting the Value-at-Risk of the terminal surplus for the constrained investment strategy (each at a confidence level of $1 - \epsilon$). For the numerical study, we choose $V(0) = 1$ and $L = 1$ unless stated otherwise. For all numerical examples, we use the parameters from Chapter 5 wherever possible.

When determining the parametrization of the constraints, supervising authorities have to deal with the utility trade-off between the risk reducing effect of the regulatory constraints on the one hand (measured by the SERR here), and the loss in performance on the other hand (measured by the SEL in our case). To examine this trade-off further, we introduce, inspired by the mean-variance principle, the *surplus equivalent risk adjusted loss (SERIAL)* as

$$SERIAL = l_s - a \cdot r_s,$$

with the parameter $a > 0$ determining how many units in additional SEL the investor would accept for one additional unit in SERR.

Example with One Risky Asset

For simplicity, we start with an example with one risky asset representing equity and the risk-free asset representing cash. Hence, this example is the wealth-dependent version of Example 5.5.3. We begin, as in Section 5.7.2, by setting $\mu = 0.06$, $\sigma = 0.3$ and $r = 0.01$. Unless stated otherwise, we use the constraint set (6.16) with $k = 0.39$ and $T = 10$. With this choice, we want to assess the long-term impact of the constraints as this perspective coincides with the nature of many insurance products and the duration of insurance liabilities. The optimal unconstrained investment strategy is given by

$$\pi_u(t) = \frac{1}{1 - \alpha} \frac{V^{\pi_u}(t) - L(t)}{V^{\pi_u}(t)} \frac{\mu - r}{\sigma^2}.$$

For α, we consider different values between $\alpha = 0.4$ and $\alpha = 0.55$. For $\alpha = 0.55$, the terminal surplus for the unconstrained investment strategy exhibits a median very close to the initial surplus. Higher values of α would lead to a median lower than the initial surplus, which is unappealing for insurance companies. The other bound $\alpha = 0.4$ is chosen to represent an investor who does not want to invest more than 100% of the wealth in the risky asset (even in case of a very high surplus). The constraint set (6.11) reads (see Example 5.5.3):

$$\begin{aligned} K(V(t)) &= \left\{ \pi(t) : \frac{V(t) - L(t)}{V(t)} \geq \sqrt{\pi(t)^2 k^2} \right\} \\ &= \left\{ \pi(t) : \frac{V(t) - L(t)}{V(t)} \geq |\pi(t)k| \right\} \end{aligned} \tag{6.16}$$

and the resulting support function from (6.13)

$$\delta(t, \lambda(t), V(t)) = \frac{V(t) - L(t)}{kV(t)} |\lambda(t)|.$$

λ^* can be calculated with Corollary 6.1.3, Theorem 6.1.1 and Example 5.5.3 explicitly as

$$\lambda^* = \min\left(\frac{(1-\alpha)\sigma^2}{k} - (\mu - r), 0 \right).$$

With (6.15), we have

$$\begin{aligned} \pi_{\lambda^*}(t) &= \frac{1}{1-\alpha} \frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)} \frac{(\mu + \lambda^* - r)}{\sigma^2} \\ &= \frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)} \min\left(\frac{1}{k}, \frac{1}{1-\alpha} \frac{\mu - r}{\sigma^2} \right). \end{aligned}$$

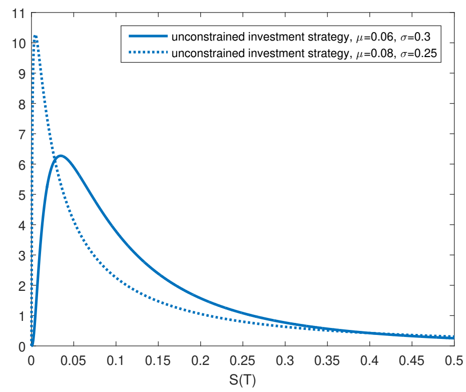

Figure 6.1: Density function of the optimal terminal surplus ($\alpha = 0.5$).

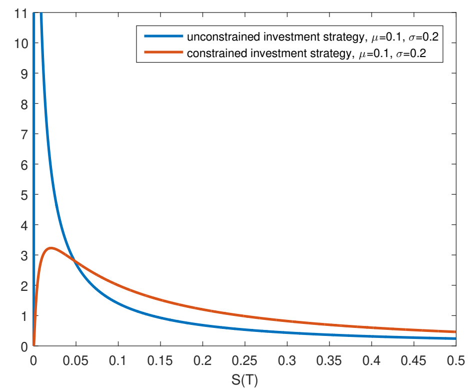

Figure 6.2: Density function of the optimal terminal surplus ($\alpha = 0.5$).

The density function of the terminal surplus for the unconstrained investment strategy can be seen in Figure 6.1. For this parameter set, the optimal unconstrained strategy does not violate the constraints since the optimal HARA strategy naturally includes a reduction of risk as the surplus is decreasing in times of distress. However, if market conditions change such that investors are more optimistic, we show that the constraints play an important role. Figure 6.1 shows a slightly more optimistic scenario with $\mu = 0.08$ and $\sigma = 0.25$. In this case, the unconstrained strategy does not violate the constraints either. However, as we continue to look at an even more optimistic scenario and consider $\mu = 0.1$ and $\sigma = 0.2$, the impact of the constraints becomes visible and is shown in Figure 6.2. In this case, one can observe that the unconstrained investment strategy leads to a higher risk of a loss in the sense of a terminal wealth below the initial surplus (approx 0.1), which is compensated by a higher chance of a very large gain. In the following, we will compare different parameter settings and we will refer to the setting of Figure 6.2 as the reference scenario.

Figure 6.3 shows the impact of a variation of k on the optimal investment strategy for several choices of α. For low values of k ($k \leq 0.2$), the optimal unconstrained investment strategy can be implemented and therefore, the variation of k does not have an influence on the strategy. For small values of α, there is a larger region which allows for the implementation of the unconstrained strategy. The currently prescribed value of k (0.39), however, would even affect very risk-averse investors (low α).

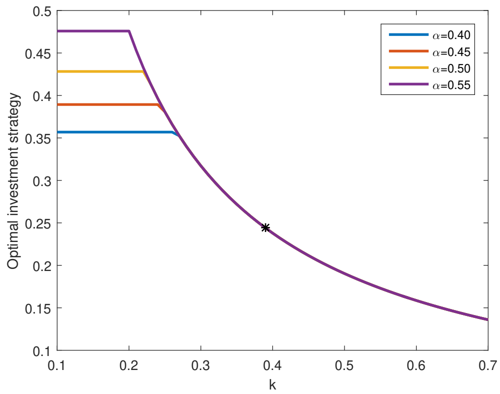

Figure 6.3: Optimal investment strategies as a function of α and k. The asterisk marks the parameter setting from Figure 6.2.

The specific impact of the constraint set with respect to the loss in utility and decrease in risk is displayed in Figures 6.4-6.6. An increase in k, i.e. a more restrictive Solvency II constraint set, results in an increasing SEL (Figure 6.4). The areas exhibiting no SEL correspond to parameter sets for which the unconstrained strategy can be implemented. In Figure 6.5, the surplus equivalent risk reduction is displayed for $\epsilon = 0.1$. As we would expect, an increasing k, i.e. more restrictive Solvency II constraints, lead to an increasing SERR. A key observation from Figure 6.4 and Figure 6.5 is that there is an optimal value of k for given market conditions and investor risk-aversion levels. Such k shall lead to a reasonable combination of low SEL and high SERR. For instance, for $\alpha = 0.4$, a value $k = 0.4$ would keep the SEL below 18% with a large SERR of 70%. In other words, the regulatory value would ensure a proper risk control with minimum initial penalty. To give a more precise answer on this question, it is important to know how investors weigh changes in SEL and SERR. Assuming that the supervising authorities determine k such that it is optimal for an insurance company in our base case ($\alpha = 0.5$), the minimum SERIAL would correspond to the actual value of k (k=0.39) for $a = 3.69$. Therefore, we choose this value of a in Figure 6.6. It is observable that for different values of $\alpha \in [0.4, 0.55]$, the $k(\alpha)$ which leads to a minimum SERIAL ranges from 0.32 to 0.57. In particular, a very risk-averse investor (small α) would have a larger k which minimizes SERIAL. In other words, a more risk-averse investor can deal better with stricter regulation.

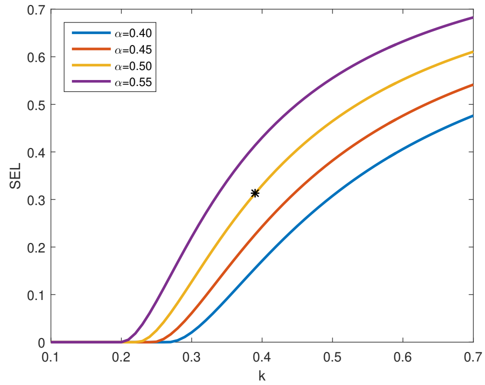

Figure 6.4: SEL versus k.

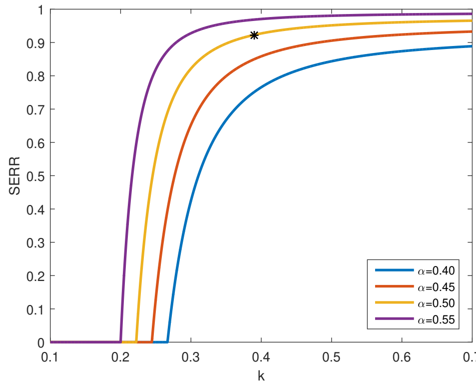

Figure 6.5: SERR versus k.

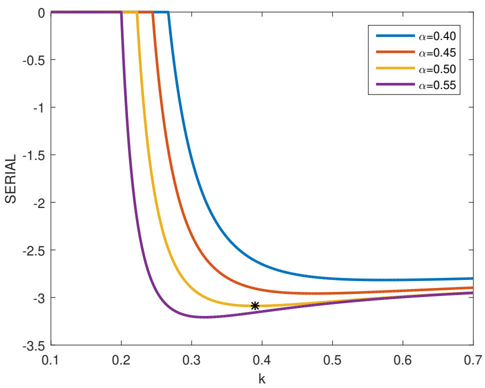

Figure 6.6: SERIAL versus k, $a = 3.69$.

Concluding, our one-dimensional analysis shows that while an investor with HARA utility reduces risk as the surplus decreases by construction even in the case without constraints, the presence of constraints has an important role as it prevents the investor from taking too much risk in good market scenarios. The constraints could lead to losses of up to 30% in the setting of Figure 6.2 for the standard parametrization of Solvency II with $k = 0.39$ (see Figure 6.4). While this one-asset example provides a clear view on the impact of the constraints on the portfolio risk in general, we work with a multi-asset example in the following section to analyze the interaction between the assets.

Example with Three Risky Assets

Extending the setting from the previous section with one asset representing equity, we now add two further risky asset classes. With respect to the assets and risk categories considered as well as the market parameters μ, σ, r, we proceed as in Section 5.6, i.e. we consider the market parameters from (5.15). However, as described above, we cannot include the liabilities in the interest-rate capital requirements due to the lack of the separability of the support function here. Consequently, the interest-rate risk is represented by an increase in the interest rates, so the Solvency II capital requirement differs from the ones in Section 5.6 with respect to the interest-rate shock k_1, the incorporation of the liability risk and the correlation matrix C. As in Section 5.6, we do not consider an investment in real estate and the corresponding property capital requirements. In total, the constraints are in this case determined by (6.11) with

$$B = \begin{pmatrix} 8.9 & 0 & 6.7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad W = diag(k_1, k_2, k_3) = \begin{pmatrix} 0.47\mu_1 & 0 & 0 \\ 0 & 0.39 & 0 \\ 0 & 0 & 0.091 \end{pmatrix},$$

$$\text{and } C = (c_{ij})_{i,j=1,\dots,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.75 \\ 0 & 0.75 & 1 \end{pmatrix},$$

The parameter $0.47\mu_1$ represents a 47% upward shock in the interest rates (corresponding to the drift of the government bonds in our example). The 47% is the upward-shock specified by Solvency II for an interest rate of 8 years, being the closest key rate to the mean of the assumed duration of the government bonds (8.9 years) and corporate bonds (6.7 years). The reasoning behind the choice of $k_1 = 0.47\mu$ is the same as in Section 5.6, where the differences in k_1 and $SCR^{mkt}_{interest}(t)$ occur as an increase in the level of interest rates represents a risk in this section. In Section 5.6, we consider a decrease in interest rates as we additionally take the interest-rate risk in the liabilities into account there. According to the previous section, we choose $\alpha = 0.5$. Again, we observe that the optimal investment strategy corresponds to the unconstrained strategy in this parameter setting (see Figure 6.7). In the example with one asset, the risk premium $\mu - r$ in the optimistic scenario is 1.8 times the risk premium in the original scenario and the volatility σ in the optimistic scenario corresponds to the original volatility multiplied by a factor of $2/3$. To obtain a consistent optimistic scenario for the three asset example, we determine μ such that the risk premium $\mu - r\mathbf{1}$ is scaled by the factor 1.8 compared to the original scenario and we scale the volatility matrix σ by the factor $2/3$. This procedure ensures that the correlations between the assets remain unchanged. The resulting μ and σ are given by

$$\mu = \begin{pmatrix} 0.0118 \\ 0.1 \\ 0.0244 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.04 & 0 & 0 \\ 0.0111 & 0.1997 & 0 \\ 0.0033 & 0.0198 & 0.0636 \end{pmatrix}.$$

In this scenario, the constraints have an impact which is comparable to the one in the example with one risky asset, a density plot of the terminal surplus is given in Figure 6.8).

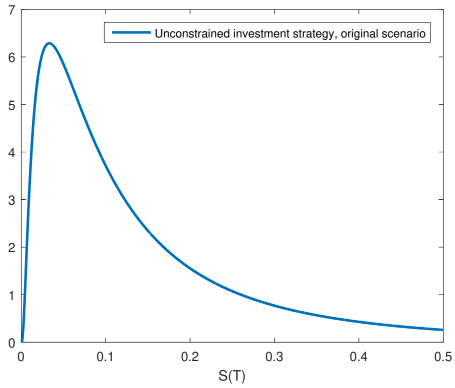

Figure 6.7: Density function of the optimal terminal surplus ($\alpha = 0.5$)

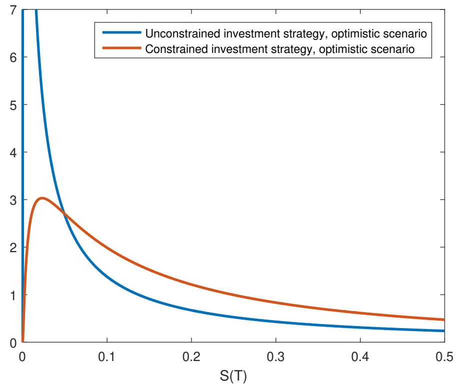

Figure 6.8: Density function of the optimal terminal surplus ($\alpha = 0.5$)

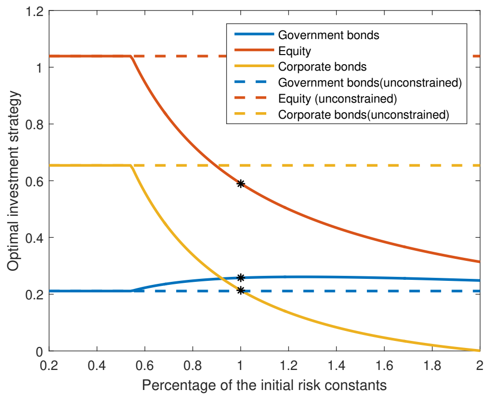

Figure 6.9: Investment strategy versus varying W ($\alpha = 0.5$). The asterisks mark the parameter setting from Figure 6.8.

In order to examine the influence of the regulatory constraints further, we vary W in the following. In Figure 6.9, the optimal investment strategy is shown where W is multiplied by a factor between 0.2 and 2 so we vary the capital requirements for the risk categories. In the area of low values for the factor, the unconstrained investment strategy can be implemented. As the factor increases and the risk constraints become more severe, the decrease is particularly large on equity, but also on corporate bonds, whereas the allocation in government bonds starts to increase. The effect on equity can be explained by the large risk of the asset class which is also represented in the parametrization of W. The different effects on corporate bonds and government bonds result from the fact that corporate bonds are subject to spread risk and interest-rate risk, whereas government bonds are only subject to interest-rate risk. As the risk constraints become more severe, corporate bonds are less attractive and are reduced. Although government bonds are also affected by the increasing severity of the interest-rate risk requirements, they appear to be more attractive compared to the other asset classes, in particular corporate bonds. Figures 6.10-6.12 show the SEL, SERR and SERIAL. As in the example with one risky asset, we choose a such that the SERIAL is minimal for the original capital requirements W. We find a similar structure of minimum SERIAL for various values of α as in the example with one risky asset. The higher value of a ($a = 4.64$ here compared to $a = 3.69$ for one risky asset) means that for the same amount of additional SEL, the investor would be willing to accept less additional SERR than the investor with only one risky asset due to the diversification possibilities.

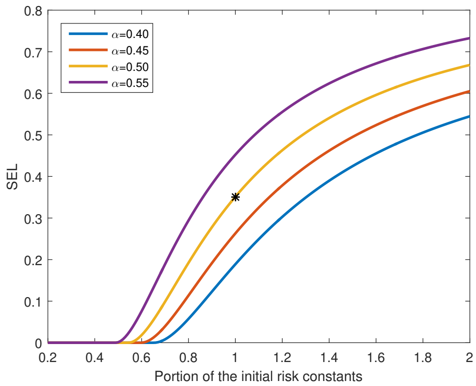

Figure 6.10: SEL versus varying W ($\alpha = 0.5$).

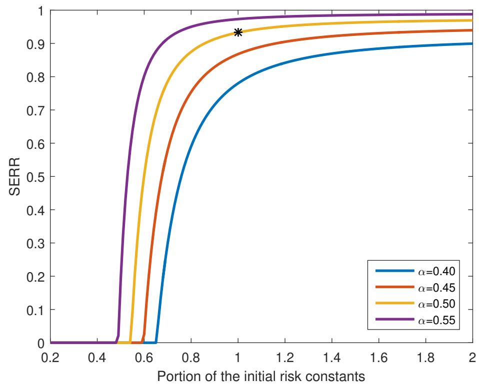

Figure 6.11: SERR versus varying W ($\alpha = 0.5$).

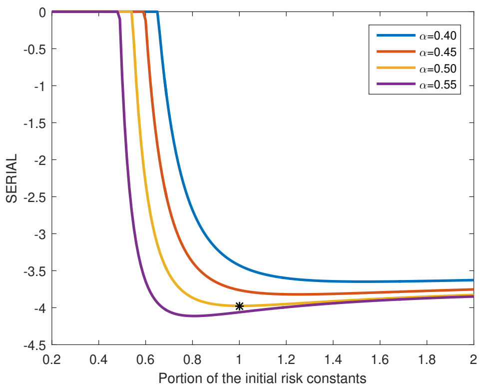

Figure 6.12: SERIAL versus varying W ($\alpha = 0.5$, $a = 4.64$).

6.4 Comparison of the Two-Step Approach and the Approach with Wealth-Dependent Constraints

In Chapter 5 and Chapter 6, we establish the aproximative, iterative two-step approach and an exact method for wealth-dependent constraint sets and analyze the impact of the constraints in Solvency II-examples on the investment strategies derived with these two methods as well as further measures of performance and risk. Due to the specific requirements of each approach, the applicability of the two approaches is slightly different, which directly affects the numerical Solvency II examples for both cases.

	Two-step approach	Wealth-dependent approach
Wealth-dependent constraints	Approximative	Truly wealth-dependent
Utility function	Power utility	HARA utility
Risk capital for liability risk	Included	Not included
Comparison	To Bellman approach	To unconstrained strategy

Table 6.1: Comparison of the two approaches.

In Table 6.1, an overview for the comparison of both approaches is provided. A major difference is of course given by the fact that the constraint set is only approximatively wealth-dependent in the iterative two-step approach, whereas it is really wealth-dependent in the setting Chapter 6. The power utility function can be used for the two-step approach as a solution for the dual problem is available, whereas we are not aware of an analytical solution of the dual problem for the HARA utility with a constraint set independent of wealth. On the other hand, the lack of an exact solution for the problem with wealth-dependent constraint set and power utility provides the motivation for the search for an approach with HARA utility and wealth-dependent constraint sets. The use of the power utility in our approach to wealth-dependent constraint sets is included as a special case for $L = 0$. However, this results in $f(t, V(t))$ to be constant, so the constraint set would have to be independent of the wealth process due to Assumption (A1). These observations lead to the conclusion that the presented iterative two-step approach is not applicable to a setting with HARA utility and the solution for the wealth-dependent constraint sets is not applicable to a setting with power utility. Furthermore, due to the necessity of the separability of the support function as in Assumption (A2) for the reduction to the associate problem, an inclusion of the liability risks in the setting with wealth-dependent constraint set is not possible with the model of the interest-rate capital requirements from Chapter 5. In Chapter 5, however, this type of interest-rate risk requirement can be used as the separability of the support function is not necessary here. With respect to the numerical results, we find that the constraints have a considerable impact on the investment strategy in the base scenario for the investor with power utility whereas the investor with HARA utility reduces the risk even without constraints as the relative surplus (RS) decreases. Thus, the constraints only have an impact in scenarios like the optimistic scenario. For the investor with power utility, the impact of the constraints is depending on the RS and for suitably high values of the RS, the optimal unconstrained investment strategy can be implemented. In contrast, the impact of the constraints on the investor with HARA utility is only depending on the market parameters and the risk aversion.

7 Conclusion

We present solutions to portfolio optimization problems in continuous time with allowance for dynamic investment strategies including two major components which ensure that insurance companies can meet their liabilities: portfolio optimization under stochastic liabilities and portfolio optimization with risk constraints, which jointly depend on wealth and the investment strategy. The novelties presented extend the existing literature on portfolio optimization in continuous time to the best our our knowledge in the following way:

- The CPT funding ratio optimization provides the most general funding ratio optimization framework with explicit solutions.
- The dynamic surplus optimization is the most comprehensive framework with closed-form solutions for terminal surplus optimization including various types of index- and performance participation as well as unhedgeable risks.
- The approaches to constrained portfolio optimization establish the most far reaching results on portfolio optimization with risk constraints, which depend jointly on wealth and the investment strategy, in particular Solvency II-type constraints.

These results are complemented by numerical studies and subsequent economic conclusions.

In particular, we extend the quantile optimization approach for CPT portfolio optimization to funding ratio optimization and apply it to a modification of the Wang-distortion function in Chapter 3. We obtain optimal investment strategies for well-funded and underfunded investors. The optimal investment strategies are compared to the ones in an expected utility framework. In Chapter 4, we extend a generalized martingale approach to dynamic surplus optimization and derive closed-form solutions for various types of index- and performance-linked liability models, which may also include unhedgeable risks. In numerical studies, we analyze the impact of different liability models on the optimal investment strategy and we compare the results to the ones from the funding ratio optimization. Inspired by the Solvency II standard formula, we formulate convex portfolio constraints, we introduce the setting for wealth-dependent risk constraint sets and we construct an iterative two-step approach for an investor with power utility who is subject to Solvency II constraints in Chapter 5. In Chapter 6, we establish the approach for truly wealth-dependent risk constraints and general utility functions by showing that the problem can, under certain conditions, be reduced to an associate problem with constraint set independent of wealth. The associate problem can then be solved by known convex duality results. Using this approach, we derive closed-form solutions for the optimal investment strategy and terminal wealth for a shifted logarithmic utility and HARA utility. In a numerical study with HARA utility and Solvency II-type constraints, we analyze the impact of the constraints and we examine the trade-off between the effect on the reduction of risk and the loss in utility caused by the constraints.

The results obtained point at several starting points for possible future research. With respect to constraints, which jointly depend on the investment strategy and wealth, the question how an appropriate dual problem can be constructed remains open. Moreover, a combination of LDI and wealth-dependent portfolio constraints may represent an even better framework for insurance companies. A direct way may be the application of our iterative two-step and possibly our framework for truly wealth-dependent risk constraints to an expected utility funding ratio optimization. With respect to a combination of the generalized martingale approach and portfolio constraints, the question arises, how a suitable auxiliary market could be set up and how the liabilities and the stochastic utility function are treated.

A Appendix to Chapter 3

A.1 HJB Approach to Funding Ratio Optimization

Proof of Theorem 3.1.1

The HJB equation associated with (3.4) can be written as:

$$\Phi_t(t,v) + \sup_{\pi\in\mathbb{R}^d}\left[\mu_F^\pi(t)v\Phi_v(t,v) + \frac{1}{2}\sigma_F^\pi(t)^2v^2\Phi_{vv}(t,v)\right] = 0, \tag{A.1}$$
$$\Phi(T,v) = U(v),\ v\in\mathbb{V}.$$

We consider general $\Phi(t,v)$ and receive the optimal investment strategy by computing the supremum in the HJB equation using the help function

$$M(\pi) := \mu_F^\pi(t)\cdot v\cdot\Phi_v(t,v) + \frac{1}{2}\cdot\sigma_F^\pi(t)^2\cdot v^2\cdot\Phi_{vv}(t,v).$$

As the first order condition, we obtain

$$\begin{aligned}
\frac{\partial}{\partial\pi}M(\pi)\bigg|_{\pi=\pi^*} &= \left((\mu - r\mathbf{1}) - \sigma\sigma_L^T\right)v\Phi_v(t,v) + (\sigma\sigma^T\pi^* - \sigma\sigma_L^T)v^2\Phi_{vv}(t,v) = 0\\
\Leftrightarrow\ \sigma\sigma^T\pi^*v^2\Phi_{vv}(t,v) &= -\left((\mu - r\mathbf{1}) - \sigma\sigma_L^T\right)v\Phi_v(t,v) + \sigma\sigma_L^Tv^2\Phi_{vv}(t,v)\\
\Leftrightarrow\qquad \pi^* &= -\left((\sigma\sigma^T)^{-1}(\mu - r\mathbf{1}) - (\sigma^T)^{-1}\sigma_L^T\right)\frac{\Phi_v(t,v)}{v\Phi_{vv}(t,v)} + (\sigma^T)^{-1}\sigma_L^T.
\end{aligned}$$

Since $\frac{\partial^2}{\partial\pi^2}M(\pi)\Big|_{\pi=\pi^*} = v^2\Phi_{vv}(t,v)\sigma\sigma^T$ is negative definite, the optimal investment strategy can be written as

$$\pi^*(t,F(t)) = \left(1 - \lambda^{EU}(t,F(t))\right)\pi^{LH} + \lambda^{EU}(t,F(t))\pi^{PS}.$$

$\square$

Proof of Corollary 3.1.2

We apply Theorem 3.1.1 and use the usual separation approach

$$\Phi(t,F(t)) = U(F(t))\varphi(t), \tag{A.2}$$

where $\varphi : [0,T] \to \mathbb{R}_+$. Thus, Φ is strictly concave in $F(t)$ as U is strictly concave and we can find a unique maximizer π^*. With $U'(F(t)) = F(t)^{\alpha-1}$ and $U''(F(t) = (\alpha-1)F(t)^{\alpha-2}$,

$$\lambda^{EUP}(F(t),t) = -\frac{\Phi_v(t,F(t))}{F(t)\Phi_{vv}(t,F(t))} \cdot \frac{1}{1-\alpha}.$$

Thus, the optimal investment strategy is given by

$$\pi^*(t) = \left(1 - \frac{1}{1-\alpha}\right)\pi^{LH} + \frac{1}{1-\alpha}\pi^{PS},$$

with the remaining wealth $1 - \mathbf{1}^T\pi^*$ being invested in the risk-free asset. Inserting π^*, we have

$$\begin{aligned}
\mu_F^{\pi^*}(t) =& r + \pi^*(t)^T(\mu - r\mathbf{1}) - \mu_L + \sigma_L\sigma_L^T + \sigma_\epsilon^2 - \sigma_L\sigma^T\pi^*(t) \\
=& r - \mu_L + \sigma_L\sigma_L^T + \sigma_\epsilon^2 + \left(1 - \frac{1}{1-\alpha}\right)\sigma_L\sigma^{-1}(\mu - r\mathbf{1}) + \frac{1}{1-\alpha}\|\gamma\|^2 \\
&- \left(1 - \frac{1}{1-\alpha}\right)\sigma_L\sigma^T(\sigma^T)^{-1}\sigma_L^T - \frac{1}{1-\alpha}\sigma_L\sigma^T(\sigma\sigma^T)^{-1}(\mu - r\mathbf{1}) \\
=& r - \mu_L + \|\sigma_L\|^2 + \sigma_\epsilon^2 + \left(1 - \frac{1}{1-\alpha}\right)\sigma_L\gamma + \frac{1}{1-\alpha}\|\gamma\|^2 \\
&- \left(1 - \frac{1}{1-\alpha}\right)\|\sigma_L\|^2 - \frac{1}{1-\alpha}\sigma_L\gamma \\
=& r - \mu_L + \frac{1}{1-\alpha}\|\sigma_L\|^2 + \sigma_\epsilon^2 + \left(1 - \frac{2}{1-\alpha}\right)\sigma_L\gamma + \frac{1}{1-\alpha}\|\gamma\|^2
\end{aligned}$$

and

$$\begin{aligned}
\sigma_F^{\pi^*}(t)^2 &= \|\pi^*(t)^T\sigma - \sigma_L\|^2 + \sigma_\epsilon^2 \\
&= \left\| \left(1 - \frac{1}{1-\alpha}\right)\sigma_L\sigma^{-1}\sigma + \frac{1}{1-\alpha}(\mu - r\mathbf{1})^T(\sigma\sigma^T)^{-1}\sigma - \sigma_L \right\|^2 + \sigma_\epsilon^2 \\
&= \left\| \frac{1}{1-\alpha}\left(\gamma^T - \sigma_L\right) \right\|^2 + \sigma_\epsilon^2.
\end{aligned}$$

With the ansatz (A.2), the HJB equation (A.1) simplifies to an ODE of the form

$$0 = \varphi'(t) + \left[\mu_F^{\pi^*}(t)\alpha + \frac{1}{2}\sigma_F^{\pi^*}(t)^2\alpha(\alpha-1)\right]\varphi(t).$$

The terminal condition implies $\varphi(T) = 1$, so

$$\varphi(t) = \exp\left(\left[\mu_F^{\pi^*}(t)\alpha + \frac{1}{2}\sigma_F^{\pi^*}(t)^2\alpha(\alpha-1)\right](T-t)\right).$$

The conditions of Theorem 2.3.1 hold by the same argument as in Remark 2.3.4. $\square$

A.2 Probability Distortion

Lemma A.2.1. *The function* $\bar{w}_\delta : [0,1] \to [0,1]$ *with*

$$\bar{w}_\delta(p) = \Phi\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right), \ \delta > 0,$$

is convex.

Proof. With

$$\bar{w}_\delta(p) = \int_0^{q_{Z_\delta}(p)} r f_{Z_\delta}(r)dr = \int_0^{q_{Z_\delta}(p)} q_{Z_\delta}(\mathbb{Q}_{Z_\delta}(r)) f_{Z_\delta}(r)dr = \int_0^p q_{Z_\delta}(s)ds,$$

we have

$$\bar{w}'_\delta(p) = q_{Z_\delta}(p) = \exp\left(-\frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T + \delta\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right).$$

Hence,

$$\bar{w}''_\delta(p) = \bar{w}'_\delta(p)\frac{\delta\|\sigma_L - \gamma^T\|\sqrt{T}}{\phi\left(\Phi^{-1}(p)\right)} \geq 0,$$

with ϕ denoting the density function of a standard normally distributed random variable. □

Proof of Lemma 3.2.5

We apply findings from Jin and Zhou (2008), Section 6.2. to our distortion function. For $w(p) = (\bar{w}_\delta(p))^\eta$, we have with $\bar{w}'_\delta(p) = q_{Z_\delta}(p)$ (see the proof of Lemma A.2.1)

$$\begin{aligned}
w'(p) =& \eta(\bar{w}_\delta(p))^{\eta-1} q_{Z_\delta}(p) \\
=& \eta\Phi^{\eta-1}\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right) \\
& \cdot \exp\left(-\frac{1}{2}\delta^2\|\sigma_L - \gamma^T\|^2 T + \delta\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right) \\
=& \eta\Phi^{\eta-1}\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right) \\
& \cdot \exp\left(-\frac{1}{2}\|\sigma_L - \gamma^T\|^2 T + \|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right) \\
& \cdot \exp\left(\frac{1}{2}(1-\delta^2)\|\sigma_L - \gamma^T\|^2 T + (\delta - 1)\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right) \\
=& \eta\Phi^{\eta-1}\left(\Phi^{-1}(p) - \delta\|\sigma_L - \gamma^T\|\sqrt{T}\right) \cdot q_Z(p) \\
& \cdot \exp\left(\frac{1}{2}(1-\delta^2)\|\sigma_L - \gamma^T\|^2 T + (\delta - 1)\|\sigma_L - \gamma^T\|\sqrt{T}\Phi^{-1}(p)\right).
\end{aligned}$$

We consider

$$\frac{q_Z(p)}{w'(p)}$$

to show that (M) is satisfied. With

$$c := q_Z(p), \quad I(c) := \left.\frac{w'(p)}{q_Z(p)}\right|_{p=\mathbb{Q}_Z(c)} = \frac{w'(\mathbb{Q}_Z(c))}{c}, \quad H(c) := w(\mathbb{Q}_Z(c)),$$

and since $H'(c) = w'(\mathbb{Q}_Z(c))\mathbb{Q}'_Z(c)$, we have

$$I(c) = \frac{H'(c)}{c\mathbb{Q}'_Z(c)}$$

and

$$\begin{aligned}
I'(c) = & \frac{H''(c)c\mathbb{Q}'_Z(c) - H'(c)c\mathbb{Q}''_Z(c) - H'(c)\mathbb{Q}'_Z(c)}{(c\mathbb{Q}'_Z(c))^2} \leq 0 \\
\Leftrightarrow \quad & \frac{H''(c)}{c\mathbb{Q}'_Z(c)} - \frac{H'(c)\mathbb{Q}''_Z(c)}{c\mathbb{Q}'_Z(c)^2} \leq \frac{H'(c)}{c^2\mathbb{Q}'_Z(c)} \\
\Leftrightarrow \quad & \frac{cH''(c))}{H'(c)} - \frac{c\mathbb{Q}''_Z(c)}{\mathbb{Q}'_Z(c)} \leq 1 \\
\Leftrightarrow \quad & j(c) := c\left(\frac{H''(c))}{H'(c)} - \frac{\mathbb{Q}''_Z(c)}{\mathbb{Q}'_Z(c)}\right) \leq 1.
\end{aligned}$$

As Z is log-normally distributed, $c > 0$. Furthermore,

$$\frac{q_Z(p)}{w'(p)} \text{ is increasing } \Leftrightarrow \; I(c) \text{ is decreasing } \Leftrightarrow \; j(c) \leq 1, \tag{A.3}$$

since $\mathbb{Q}_Z(c)$ is monotonically increasing. In the following, we set up the function $j(c)$. With (3.6),

$$H(c) = w(\mathbb{Q}_Z(c)) = \Phi^\eta\left(\frac{\log c + \left(\frac{1}{2} - \delta\right)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right),$$

we have

$$\begin{aligned}
H'(c) =& \eta\Phi^{\eta-1}\left(\frac{\log c + (\frac{1}{2} - \delta)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right) \\
& \phi\left(\frac{\log c + (\frac{1}{2} - \delta)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)\frac{1}{c\|\sigma_L - \gamma^T\|\sqrt{T}} > 0
\end{aligned}$$

and, since $\phi'(x) = -x\phi(x)$,

$$\begin{aligned}
H''(c) =& H'(c)(\eta - 1)\frac{\phi\left(\frac{\log c + (\frac{1}{2} - \delta)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)}{\Phi\left(\frac{\log c + (\frac{1}{2} - \delta)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)}\frac{1}{c\|\sigma_L - \gamma^T\|\sqrt{T}} \\
& - H'(c)\left(\frac{\log c + (\frac{1}{2} - \delta)\|\sigma_L - \gamma^T\|^2 T}{\|\sigma_L - \gamma^T\|\sqrt{T}}\right)\frac{1}{c\|\sigma_L - \gamma^T\|\sqrt{T}} - H'(c)\frac{1}{c}.
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{cH''(c)}{H'(c)} =&(\eta-1)\frac{\phi\left(\frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)}{\Phi\left(\frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)}\frac{1}{\|\sigma_L-\gamma^T\|\sqrt{T}}\\ &-\left(\frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)\frac{1}{\|\sigma_L-\gamma^T\|\sqrt{T}}-1. \end{aligned} \tag{A.4}$$

Furthermore, with (3.6),

$$\mathbb{Q}_Z'(c) = \phi\left(\frac{\log c+\frac{1}{2}\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)\frac{1}{c\|\sigma_L-\gamma^T\|\sqrt{T}}$$

and

$$\mathbb{Q}_Z''(c) = -\phi\left(\frac{\log c+\frac{1}{2}\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)\frac{1}{c\|\sigma_L-\gamma^T\|\sqrt{T}}\left(\frac{\log c+\frac{1}{2}\|\sigma_L-\gamma^T\|^2T}{c\|\sigma_L-\gamma^T\|^2T}+\frac{1}{c}\right).$$

Hence, the second term in j is given by

$$\frac{c\mathbb{Q}_Z''(c)}{\mathbb{Q}_Z'(c)} = -\frac{\log c+\frac{1}{2}\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|^2T}-1. \tag{A.5}$$

With (A.4) and (A.5), the function j reads

$$\begin{aligned} j(c) =&\frac{\eta-1}{\|\sigma_L-\gamma^T\|\sqrt{T}}\frac{\phi\left(\frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)}{\Phi\left(\frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}\right)}+\delta\\ =&\frac{\eta-1}{\|\sigma_L-\gamma^T\|\sqrt{T}}\frac{\phi\left(d(c,T,\delta-1)\right)}{\Phi\left(d(c,T,\delta-1)\right)}+\delta, \end{aligned} \tag{A.6}$$

with $d(c,T,\delta) := \frac{\log c-(\delta+\frac{1}{2})\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}$, so $d(c,T,\delta-1) = \frac{\log c+(\frac{1}{2}-\delta)\|\sigma_L-\gamma^T\|^2T}{\|\sigma_L-\gamma^T\|\sqrt{T}}$. Now we want to examine when (M) holds using (A.2). Since $\eta\in(0,1]$, $\frac{\eta-1}{\|\sigma_L-\gamma^T\|\sqrt{T}}\frac{\phi(d(c,T,\delta-1))}{\Phi(d(c,T,\delta-1))}\leq 0$ and if $c\to 0$, this term converges to 0. Hence, $j(c)\leq 1$ for all $c>0 \Leftrightarrow \delta\leq 1$ so w satisfies (M) if and only if $\delta\leq 1$. We proceed by determining when w is reverse-S shaped. By definition of H,

$$H'(c) = w'(\mathbb{Q}_Z(c))\mathbb{Q}_Z'(c) > 0$$

and

$$\begin{aligned} H''(c) &= w''(\mathbb{Q}_Z(c))\mathbb{Q}_Z'(c)^2 + w'(\mathbb{Q}_Z(c))\mathbb{Q}_Z''(c)\\ &= w''(\mathbb{Q}_Z(c))\mathbb{Q}_Z'(c)^2 + \frac{H'(c)}{\mathbb{Q}_Z'(c)}\mathbb{Q}_Z''(c), \end{aligned} \tag{A.7}$$

as well as

$$w(p) = H(q_Z(p)) = H(c).$$

Rewriting (A.7),

$$w''(\mathbb{Q}_Z(c)) = \frac{1}{\mathbb{Q}_Z'(c)^2}\left[H''(c) - \frac{H'(c)}{\mathbb{Q}_Z'(c)}\mathbb{Q}_Z''(c)\right].$$

Finally,

$$w''(\mathbb{Q}_Z(c)) < 0 \Leftrightarrow H''(c) - \frac{H'(c)}{\mathbb{Q}_Z'(c)}\mathbb{Q}_Z''(c) < 0 \Leftrightarrow c\left(\frac{H''(c))}{H'(c)} - \frac{\mathbb{Q}_Z''(c)}{\mathbb{Q}_Z'(c)}\right) = j(c) < 0.$$

Thus, w is reverse-S shaped if and only if j is first negative and then positive. We assume $\eta \in (0,1)$. Furthermore, j as in (A.6) can be written as

$$\begin{aligned} j(c) =&(\eta-1)c\frac{\partial}{\partial c}\left(\log\left(\Phi\left(d(c,T,\delta-1)\right)\right)\right) + \delta \\ =&\delta + \frac{\eta-1}{\|\sigma_L - \gamma^T\|\sqrt{T}}\frac{\partial}{\partial d}\left(\log\left(\Phi\left(d\right)\right)\right)|_{d=d(c,T,\delta-1)} \end{aligned}$$

and therefore, we have

$$j'(c) = \frac{\eta-1}{\|\sigma_L - \gamma^T\|\sqrt{T}}\frac{\partial^2}{\partial d^2}\left(\log\left(\Phi\left(d\right)\right)\right)|_{d=d(c)} \cdot \frac{\partial}{\partial c}d(c,T,\delta-1).$$

As $\eta \in (0,1)$, $\frac{\eta-1}{\|\sigma_L-\gamma^T\|\sqrt{T}} < 0$. Furthermore, $\frac{\partial}{\partial c}d(c,T,\delta-1) = \frac{1}{c\|\sigma_L-\gamma^T\|\sqrt{T}} > 0$. As the normal distribution has a log-concave distribution function (see e.g. Bagnoli and Bergstrom (2005)), $\frac{\partial^2}{\partial d^2}\left(\log\left(\Phi\left(d\right)\right)\right) \leq 0$. Consequently, j is monotonically increasing. Moreover,

$$\lim_{c\to\infty} j(c) = \delta$$

and

$$\begin{aligned} \lim_{c\to 0} j(c) &= \lim_{d\to-\infty} \frac{\eta-1}{\|\sigma_L - \gamma^T\|\sqrt{T}}\frac{\phi\left(d\right)}{\Phi\left(d\right)} + \delta \\ &= \lim_{d\to-\infty} \frac{1-\eta}{\|\sigma_L - \gamma\|\sqrt{T}}d + \delta = -\infty \end{aligned}$$

with l'Hôpital's rule and since $\phi'\left(d\right) = -d\phi\left(d\right)$. Thus, j changes its sign from negative to positive and w is reverse-S shaped, if $\delta > 0$. □

A.3 Replicating Strategies for Selected Pay-Offs

We begin by replicating general funding ratio processes $F^{\pm}(t, Z_L(t))$. Since the funding ratio process can be interpreted as a wealth process, which is discounted by the numéraire L, $F^{\pm}(t, Z_L(t))$ is a $\mathbb{Q}_L$-martingale. Therefore, the SDE can with Itô's lemma be written as

$$dF^{\pm}(t, Z_L(t)) = Z_L(t)\frac{\partial}{\partial z}F^{\pm}(t, Z_L(t))(\sigma_L - \gamma^T)dW_{\mathbb{Q}_L}(t), \tag{A.8}$$

with a $\mathbb{Q}_L$-Brownian motion $W_{\mathbb{Q}_L}$. On the other hand, with (3.2) and using again that $F^{\pm}(t, Z_L(t))$ is a $\mathbb{Q}_L$-martingale,

$$dF^{\pm}(t, Z_L(t)) = F^{\pm}(t, Z_L(t))(\pi^T\sigma - \sigma_L)dW_{\mathbb{Q}_L}(t). \tag{A.9}$$

Using (A.8) and (A.9),

$$F^{\pm}(t, Z_L(t))(\pi(t)^T\sigma - \sigma_L) = Z_L(t)\frac{\partial}{\partial z}F^{\pm}(t, Z_L(t))(\sigma_L - \gamma^T),$$

which we solve for π

$$\begin{aligned}\pi(t) =& \frac{Z_L(t)}{F^{\pm}(t, Z_L(t))}\frac{\partial}{\partial z}F^{\pm}(t, Z_L(t))(\sigma^T)^{-1}(\sigma_L^T - \gamma) + (\sigma^T)^{-1}\sigma_L^T \\ =& \frac{Z_L(t)}{F^{\pm}(t, Z_L(t))}\frac{\partial}{\partial z}F^{\pm}(t, Z_L(t))(\pi^{LH} - \pi^{PS}) + \pi^{LH}.\end{aligned} \tag{A.10}$$

We use this result to derive the replicating portfolios of $(\bar{F}^*)^+$ and $(\bar{F}^*)^-$. The corresponding strategies are denoted by π_+ and π_- respectively. To compute the replicating portfolio π^* of the total terminal funding ratio F^*, further calculations are required since the hedging portfolio of F^* is not the sum of the hedging portfolios of the partial solutions. Instead,

$$dF^*(t, Z_L(t)) = d(\bar{F}^*(t, Z_L(t)) + B) = d((\bar{F}^*)^+(t, Z_L(t))) - d((\bar{F}^*)^-(t, Z_L(t))),$$

since $dB = 0$. With (A.9),

$$\begin{aligned}F^*(t, Z_L(t))(\pi^T(t)\sigma - \sigma_L)dW_{\mathbb{Q}_L}(t) = \big(&(\bar{F}^*)^+(t, Z_L(t))(\pi_+(t)^T\sigma - \sigma_L) \\ &- (\bar{F}^*)^-(t, Z_L(t))(\pi_-(t)^T\sigma - \sigma_L)\big)\,dW_{\mathbb{Q}_L}(t),\end{aligned}$$

which can also be written as

$$\begin{aligned}F^*(t, Z_L(t))\pi^T(t)\sigma dW_{\mathbb{Q}_L}(t) = \big[&(\bar{F}^*)^+(t, Z_L(t))\pi_+(t)^T\sigma - (\bar{F}^*)^-(t, Z_L(t))\pi_-(t)^T\sigma \\ &+(F^*(t, Z_L(t)) - ((\bar{F}^*)^+(t, Z_L(t)) - (\bar{F}^*)^-(t, Z_L(t))))\sigma_L\big]\,dW_{\mathbb{Q}_L}.\end{aligned}$$

Since

$$F^*(t, Z_L(t)) - ((\bar{F}^*)^+(t, Z_L(t)) - (\bar{F}^*)^-(t, Z_L(t))) = B$$

for all $t \in [0, T]$, solving for π leads to

$$\pi^*(t) = \frac{1}{F^*(t, Z_L(t))}\left((\bar{F}^*)^+(t, Z_L(t))\pi_+(t) - (\bar{F}^*)^-(t, Z_L(t))\pi_-(t) + B\pi^{LH}\right). \tag{A.11}$$

With (A.10) for π_+ and π_-,

$$\pi^*(t) = \frac{Z_L(t)}{F^*(t, Z_L(t))}\left(\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t)) - \frac{\partial}{\partial z}(\bar{F}^*)^-(t, Z_L(t))\right)(\pi^{LH} - \pi^{PS}) + \pi^{LH}.$$

In the well-funded case, i.e. $(\bar{F}^*)^- = 0$, we have

$$\pi^*(t) = \frac{Z_L(t)\frac{\partial}{\partial z}(\bar{F}^*)^+(t, Z_L(t))}{F^*(t, Z_L(t))}(\pi^{LH} - \pi^{PS}) + \pi^{LH}.$$

In the following, we derive a more explicit representation for the replication strategy of a terminal funding ratio of the form

$$F(T, Z_L(T)) = Z_L(T)^\nu \mathbb{1}_{Z_L(T)\in(c_1,c_2)}, \ \nu \in \mathbb{R}, \ c_1 \geq 0, c_2 > 0.$$

With $Z_L(t,T) := \frac{Z_L(T)}{Z_L(t)}$, the funding ratio at time t can be calculated as

$$\begin{aligned} F(t, Z_L(t)) =& \mathbb{E}_{\mathbb{Q}_L}\left[Z_L(T)^\nu \mathbb{1}_{Z_L(T)\in(c_1,c_2)} \middle| \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}_L}\left[Z_L(t,T)^\nu Z_L(t)^\nu \mathbb{1}_{Z_L(t)Z_L(t,T)\in(c_1,c_2)} \middle| \mathcal{F}_t\right] \\ =& Z_L(t)^\nu \mathbb{E}\left[Z_L(t,T)^{\nu+1} \mathbb{1}_{Z_L(t,T)\in\left(\frac{c_1}{Z_L(t)}, \frac{c_2}{Z_L(t)}\right)} \middle| \mathcal{F}_t\right]. \end{aligned}$$

Applying Footnote 2, we receive

$$\begin{aligned} F(t, Z_L(t)) =& Z_L(t)^\nu \exp\left(-\frac{1}{2}(\nu+1)\|\sigma_L - \gamma^T\|^2(T-t) + \frac{1}{2}(\nu+1)^2\|\sigma_L - \gamma^T\|^2(T-t)\right) \\ & \cdot \left(\Phi\left(\frac{\log\left(\frac{c_2}{Z_L(t)}\right) + \frac{1}{2}\|\sigma_L - \gamma^T\|^2(T-t) - (\nu+1)\|\sigma_L - \gamma^T\|^2(T-t)}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}\right)\right. \\ & \left. - \Phi\left(\frac{\log\left(\frac{c_1}{Z_L(t)}\right) + \frac{1}{2}\|\sigma_L - \gamma^T\|^2(T-t) - (\nu+1)\|\sigma_L - \gamma^T\|^2(T-t)}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}\right)\right) \\ =& Z_L(t)^\nu \exp\left(\frac{1}{2}(\nu+1)\nu\|\sigma_L - \gamma^T\|^2(T-t)\right) \\ & \cdot \left(\Phi\left(\frac{\log\left(\frac{c_2}{Z_L(t)}\right) - (\nu+\frac{1}{2})\|\sigma_L - \gamma^T\|^2(T-t)}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}\right)\right. \\ & \left. - \Phi\left(\frac{\log\left(\frac{c_1}{Z_L(t)}\right) - (\nu+\frac{1}{2})\|\sigma_L - \gamma^T\|^2(T-t)}{\|\sigma_L - \gamma^T\|\sqrt{T-t}}\right)\right) \\ =& Z_L(t)^\nu \exp\left(\frac{1}{2}(\nu+1)\nu\|\sigma_L - \gamma^T\|^2(T-t)\right) \\ & \cdot \left(\Phi\left(d\left(\frac{c_2}{Z_L(t)}, T-t, \nu\right)\right) - \Phi\left(d\left(\frac{c_1}{Z_L(t)}, T-t, \nu\right)\right)\right), \end{aligned} \tag{A.12}$$

with $d\,(c,s,\nu)$ as in Appendix A.2. For the application of (A.10), we calculate

$$\begin{aligned}\frac{\partial}{\partial z}F(t,z) =& \exp\left(\frac{1}{2}(\nu+1)\nu\|\sigma_L-\gamma^T\|^2(T-t)\right)\\ &\cdot\left(\nu z^{\nu-1}\left(\Phi\left(d\left(\frac{c_2}{z},T-t,\nu\right)\right)-\Phi\left(d\left(\frac{c_1}{z},T-t,\nu\right)\right)\right)\right.\\ &\left.-\left(\phi\left(d\left(\frac{c_2}{z},T-t,\nu\right)\right)-\phi\left(d\left(\frac{c_1}{z},T-t,\nu\right)\right)\right)\frac{z^{\nu-1}}{\|\sigma_L-\gamma^T\|\sqrt{T-t}}\right)\\ =&\frac{\nu}{z}F(t,z)-\exp\left(\frac{1}{2}(\nu+1)\nu\|\sigma_L-\gamma^T\|^2(T-t)\right)\\ &\cdot\left(\phi\left(d\left(\frac{c_2}{z},T-t,\nu\right)\right)-\phi\left(d\left(\frac{c_1}{z},T-t,\nu\right)\right)\right)\frac{z^{\nu-1}}{\|\sigma_L-\gamma^T\|\sqrt{T-t}}.\end{aligned}$$

Inserting $F(t,Z_L(t))$ and $\frac{\partial}{\partial z}F(t,z)$ in (A.10), we see that

$$\begin{aligned}\pi(t) =&\pi^{LH}+\left(\nu-\frac{\exp\left(\frac{1}{2}(\nu+1)\nu\|\sigma_L-\gamma^T\|^2(T-t)\right)}{F(t,Z_L(t))\|\sigma_L-\gamma^T\|\sqrt{T-t}}\cdot\right.\\ &\left.\left(\phi\left(d\left(\frac{c_2}{Z_L(t)},T-t,\nu\right)\right)-\phi\left(d\left(\frac{c_1}{Z_L(t)},T-t,\nu\right)\right)\right)Z_L(t)^\nu\right)(\pi^{LH}-\pi^{PS})\\ =&\pi^{LH}\\ &+\left(\nu-\frac{1}{\|\sigma_L-\gamma^T\|\sqrt{T-t}}\frac{\phi\left(d\left(\frac{c_2}{Z_L(t)},T-t,\nu\right)\right)-\phi\left(d\left(\frac{c_1}{Z_L(t)},T-t,\nu\right)\right)}{\Phi\left(d\left(\frac{c_2}{Z_L(t)},T-t,\nu\right)\right)-\Phi\left(d\left(\frac{c_1}{Z_L(t)},T-t,\nu\right)\right)}\right)\cdot\\ &(\pi^{LH}-\pi^{PS})\\ =&\pi^{LH}\\ &+\left(-\nu+\frac{1}{\|\sigma_L-\gamma^T\|\sqrt{T-t}}\frac{\phi\left(d\left(\frac{c_2}{Z_L(t)},T-t,\nu\right)\right)-\phi\left(d\left(\frac{c_1}{Z_L(t)},T-t,\nu\right)\right)}{\Phi\left(d\left(\frac{c_2}{Z_L(t)},T-t,\nu\right)\right)-\Phi\left(d\left(\frac{c_1}{Z_L(t)},T-t,\nu\right)\right)}\right)\cdot\\ &(\pi^{PS}-\pi^{LH}).\end{aligned} \tag{A.13}$$

B Appendix to Chapter 4

B.1 Proof of Theorem 4.2.4

We adapt the proof of Theorem 3.3 in Desmettre and Seifried (2016) (see also Seifried (2010)) to our setting, in particular to the different continuation of $\hat{I}_\omega$.

Lemma B.1.1 (Young's Inequality)**.** *Let $v > \hat{v}_0(\omega)$, $y > 0$. Then,*

$$\hat{U}_\omega(v) \leq \hat{U}_\omega(\hat{I}_\omega(y)) + y\left(v - \hat{I}_\omega(y)\right) \mathbb{Q}\text{-}a.s.$$

Proof. As U is concave and $\hat{U}_\omega$ is $\mathbb{Q}$-a.s. differentiable as stated in Lemma 4.2.2, we have for $v, v_1 > \hat{v}_0(\omega)$

$$\hat{U}_\omega(v) \leq \hat{U}_\omega(v_1) + \hat{U}'_\omega(v_1)\,(v - v_1)\,. \tag{B.1}$$

In the following, we use (B.1) and set $v_1 = \hat{I}_\omega(y)$. If $y < \hat{U}'_\omega(\hat{v}_0(\omega))$, then $\hat{U}'_\omega(\hat{I}_\omega(y)) = y$ and

$$\hat{U}_\omega(v) \leq \hat{U}_\omega(\hat{I}_\omega(y)) + y\left(v - \hat{I}_\omega(y)\right).$$

If $y > \hat{U}'_\omega(\hat{v}_0(\omega))$, $\hat{I}_\omega(y) = \hat{v}_0(\omega)$ and

$$\begin{aligned}\hat{U}_\omega(v) \leq& \hat{U}_\omega(\hat{I}_\omega(y)) + \hat{U}'_\omega(\hat{I}_\omega(y))\left(v - \hat{I}_\omega(y)\right)\\ =& \hat{U}_\omega(\hat{I}_\omega(y)) + \hat{U}'_\omega(\hat{v}_0(\omega))\left(v - \hat{I}_\omega(y)\right)\\ \leq& \hat{U}_\omega(\hat{I}_\omega(y)) + y\left(v - \hat{I}_\omega(y)\right)\end{aligned}$$

since $v > \hat{v}_0(\omega) = \hat{I}_\omega(y)$ by assumption. □

Proof of Theorem 4.2.4

We consider the set

$$\mathcal{V} := \left\{V\,\mathcal{F}_T\text{-measurable: } V \geq \hat{v}_0,\ \mathbb{E}\left[\hat{U}_\omega(V)^-\right] < \infty,\ \mathbb{E}\left[\tilde{Z}(T)V\right] \leq v_0\right\}.$$

By construction of our financial market, $\mathcal{V}$ is the set of all payoffs which can be replicated with an initial wealth of not more than v_0 and which cover the value of the liabilities in T for every realization of the unhedgeable risks. In the following, we show that $V^*(T) := \hat{I}_\omega(Y(v_0)\tilde{Z}(T))$ is the optimal terminal wealth. We observe that $V^*(T) \geq \hat{v}_0$ by the definition of $\hat{I}_\omega$, $V^*(T)$ is

$\mathcal{F}_T$-measurable and $\mathbb{E}\left[\tilde{Z}(T)V^*(T)\right] = v_0$ by the definition of $Y(v_0)$. Furthermore, with arbitrary $v = V \in \mathcal{V}$ and $y = Y(v_0)\tilde{Z}(T)$, Young's inequality (see Lemma B.1.1) reads

$$\begin{aligned}\hat{U}_\omega(V^*(T)) &= \hat{U}_\omega(\hat{I}_\omega(Y(v_0)\tilde{Z}(T))) \\ &\geq \hat{U}_\omega(V) + Y(v_0)\tilde{Z}(T)\left(\hat{I}_\omega(Y(v_0)\tilde{Z}(T)) - V\right) \\ &= \hat{U}_\omega(V) + Y(v_0)\tilde{Z}(T)\left(V^*(T) - V\right).\end{aligned} \tag{B.2}$$

Therefore,

$$\hat{U}_\omega(V^*(T))^- \leq \hat{U}_\omega(V)^- + Y(v_0)\tilde{Z}(T)\left(V^*(T) + V\right)$$

and thus

$$\mathbb{E}\left[\hat{U}_\omega(V^*(T))^-\right] \leq \mathbb{E}\left[\hat{U}_\omega(V)^-\right] + Y(v_0)\left(\mathbb{E}\left[\tilde{Z}(T)V^*(T)\right] + \mathbb{E}\left[\tilde{Z}(T)V\right]\right) < \infty$$

as $V \in \mathcal{V}$ and $\mathbb{E}\left[\tilde{Z}(T)V^*(T)\right] = v_0 < \infty$. Consequently, $V^*(T) \in \mathcal{V}$. Applying (B.2) again and taking the expectation on both sides of this inequality leads to

$$\mathbb{E}[\hat{U}_\omega(V^*(T))] \geq \mathbb{E}[\hat{U}_\omega(V)] + Y(v_0)\left(v_0 - \mathbb{E}[\tilde{Z}(T)V]\right) \geq \mathbb{E}[\hat{U}_\omega(V)],$$

since $\mathbb{E}[\tilde{Z}(T)V^*(T)] = v_0$ and $\mathbb{E}[\tilde{Z}(T)V] \leq v_0$. Since

$$\begin{aligned}\mathbb{E}\left[\hat{U}_\omega(V)\right] =& \mathbb{E}\left[\mathbb{E}\left[U(V - \psi_L L(T,V))|\mathcal{F}_T\right]\right] \\ =& \mathbb{E}\left[U(V - \psi_L L(T,V))\right]\end{aligned}$$

for all $V \in \mathcal{V}$, we see that

$$V^*(T) = \arg\max_{V\in\mathcal{V}} \mathbb{E}\left[U(V - \psi_L L(T,V))\right].$$

□

C Appendix to Chapter 5

Proof of Proposition 5.3.1

The proof is adapted from Cvitanić and Karatzas (1992). For the logarithmic utility, the dual problem reads

$$
\begin{aligned}
\tilde{\Phi} &= \inf_{\lambda \in D} \mathbb{E}\left[\tilde{U}\left(\tilde{Z}_\lambda(T)\right)\right] = \inf_{\lambda \in D} \mathbb{E}\left[-\left(1 + \log\left(\tilde{Z}_\lambda(T)\right)\right)\right] \\
&= -1 + \inf_{\lambda \in D} \mathbb{E}\left[\int_0^T (r(s) + \delta(\lambda(s)) + \frac{1}{2}\|\gamma_\lambda(s)\|^2) ds + \int_0^T \gamma_\lambda(s)^T dW(s)\right] \\
&= -1 + \inf_{\lambda \in D} \mathbb{E}\left[\int_0^T (r(s) + \delta(\lambda(s)) + \frac{1}{2}\|\gamma_\lambda(s)\|^2) ds\right].
\end{aligned}
$$

This expression can be minimized point wise for $t \in [0, T]$ with

$$
\lambda^*(t) := \arg \inf_{\lambda \in X_{\hat{K}}} \left\{\frac{1}{2}\|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 + \delta(\lambda)\right\}.
$$

For the power utility, with the value function being defined as

$$
\tilde{\Phi}(t, z) := \inf_{\lambda \in D} \mathbb{E}\left[\tilde{U}\left(\tilde{Z}_\lambda(T)\right) | \tilde{Z}_\lambda(t) = z\right],
$$

the associated HJB equation is given by

$$
\begin{aligned}
0 = &\inf_{\lambda \in X_K} \left\{-z\delta(\lambda)\tilde{\Phi}_z(t, z) + \frac{1}{2}z^2\|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 \tilde{\Phi}_{zz}(t, z)\right\} \\
&+ \tilde{\Phi}_t(t, z) - zr(t)\tilde{\Phi}_z(t, z) \\
&\tilde{\Phi}(T, z) = \tilde{U}(z), \ z \in (0, \infty).
\end{aligned}
$$

In order to solve it, we use the ansatz

$$
\tilde{\Phi}(t, z) = \frac{1 - \alpha}{\alpha} z^{\frac{\alpha}{\alpha - 1}} \tilde{\varphi}(t).
$$

Then,

$$
\tilde{\Phi}_t(t, z) = \frac{1 - \alpha}{\alpha} z^{\frac{\alpha}{\alpha - 1}} \tilde{\varphi}'(t)
$$

and

$$
\tilde{\Phi}_z(t, z) = -z^{\frac{\alpha}{\alpha - 1} - 1} \tilde{\varphi}(t) = -z^{\frac{1}{\alpha - 1}} \tilde{\varphi}(t)
$$

as well as

$$\tilde{\Phi}_{zz}(t,z) = \frac{-1}{\alpha-1} z^{\frac{\alpha}{\alpha-1}-2}\tilde{\varphi}(t).$$

Inserting into the HJB equation and dividing by $z^{\frac{\alpha}{\alpha-1}}$, we receive

$$0 = \left(\inf_{\lambda \in X_K} \left\{ \frac{1}{2(1-\alpha)} \|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 + \delta(\lambda) \right\} + r(t) \right) \tilde{\varphi(t)} + \frac{1-\alpha}{\alpha}\tilde{\varphi}'(t).$$

The solution to this ODE , which also satisfies the terminal condition is given by

$$\tilde{\varphi}(t) = e^{-\frac{\alpha}{1-\alpha}\int_t^T \frac{1}{2(1-\alpha)}\|\gamma(s)+\sigma^{-1}(s)\lambda^*(s)\|^2+\delta(\lambda^*(s))+r(s)ds},$$
$$\lambda^*(t) = \arg\inf_{\lambda \in X_K} \left\{ \frac{1}{2(1-\alpha)} \|\gamma(t) + \sigma^{-1}(t)\lambda\|^2 + \delta(\lambda) \right\}.$$

Using the result in Cvitanić and Karatzas (1992), the optimal investment strategy in the unconstrained auxiliary market $\mathcal{M}_{\lambda^*}$ is given by replacing μ and r with μ_{λ^*} and r_{λ^*} in Corollary 2.3.2 and Corollary 2.3.3 (see also Theorem 15.3 in Cvitanić and Karatzas (1992)). As λ^* is determined by the dual problem, π_{λ^*} is also the optimal investment strategy for the original constrained problem (see Theorem 10.1 in Cvitanić and Karatzas (1992), in particular the implication $(D) \Rightarrow (A)$). Finally, with the implication $(D) \Rightarrow (B)$ from Theorem 10.1. in Cvitanić and Karatzas (1992), we have (5.5) and (5.6). □

Proof of Proposition 5.5.1

We have $SCR_i(t) \geq 0$ for all i, i.e.

$$\begin{aligned} SCR^{mkt}_{interest}(t) =& k_1\left(d_L L - d_1\pi_1(t)V(t) - d_4\pi_4(t)V(t)\right) \geq 0, \\ SCR^{mkt}_{equity}(t) =& k_2\pi_2(t)V(t) \geq 0, \\ SCR^{mkt}_{property}(t) =& k_3\pi_3(t)V(t) \geq 0 \\ \text{and } SCR^{mkt}_{spread}(t) =& k_4\pi_4(t)V(t) \geq 0. \end{aligned}$$

The right-hand sides of the inequalities in $\tilde{K}(t, c(t))$ and $K(t, V(t))$ differ only in the first component of v and $\tilde{v}$ and

$$v_1 = \frac{L}{V(t)} d_L \leq (1 - c(t))d_L = \tilde{v}_1.$$

Due to the monotonicity of the square-root function, we consider the function $g(v) := (B\pi(t) + v)^T\, WCW\, (B\pi(t) + v)$ and its gradient

$$\nabla g(v) = 2WCW(B\pi(t) + v).$$

Hence, $\nabla g(v)$ is non-negative since

$$B\pi(t) + v \geq 0 \Leftrightarrow \begin{pmatrix} d_L L - d_1\pi_1(t)V(t) - d_4\pi_4(t)V(t) \\ \pi_2(t) \\ \pi_3(t) \\ \pi_4(t) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

by assumption and WCW has non-negative entries. Hence,

$$f(v) := \sqrt{(B\pi(t) + v)^T \, WCW \, (B\pi(t) + v)}$$

is monotonically increasing in the first component of v. With $v_1 < \tilde{v}_1$,

$$f(v) \leq f(\tilde{v}).$$

We conclude

$$c(t) \geq f(\tilde{v}) \Rightarrow c(t) \geq f(v)$$

and as $c(t) \leq \frac{V(t)-L}{V(t)}$ is assumed, the statement follows. □

Proof of Proposition 5.5.2

Let $t \in [0, T]$. We consider the Lagrangian function to (5.1) for $\tilde{K}(t, c(t))$

$$L(x, y) = -\lambda(t)^T x + y \left(c(t)^2 - (Bx + \tilde{v})^T \, WCW \, (Bx + \tilde{v}) \right)$$

with the Lagrange multiplier $y \geq 0$. For $\lambda(t) = 0$,

$$\delta(\lambda(t), c(t)) = 0 = c(t)\sqrt{\lambda(t)^T B^{-1}(WCW)^{-1}(B^T)^{-1}\lambda(t)} + \lambda(t)^T B^{-1}\tilde{v}.$$

Let $\lambda(t) \neq 0$. From the first order condition, we have

$$\nabla_x L(x, y) = -\lambda(t)^T - 2y \, (Bx + \tilde{v})^T \, WCWB = 0$$

and

$$\frac{\partial}{\partial y} L(x, y) = c(t)^2 - (Bx + \tilde{v})^T \, WCW \, (Bx + \tilde{v}) = 0.$$

The first equation yields $y > 0$ and

$$x = B^{-1} \left(-\frac{1}{2y}(WCW)^{-1}(B^T)^{-1}\lambda(t) - \tilde{v} \right).$$

Furthermore, inserting x into the second equation, we have

$$c(t)^2 - \left(-\frac{1}{2y}(WCW)^{-1}(B^T)^{-1}\lambda(t) \right)^T WCW \left(-\frac{1}{2y}(WCW)^{-1}(B^T)^{-1}\lambda(t) \right) = 0$$

and thus

$$y = \frac{1}{2c(t)} \left(\lambda(t)^T B^{-1}(WCW)^{-1}(B^T)^{-1}\lambda(t)\right)^{\frac{1}{2}}.$$

Inserting y into x, $\delta(\lambda(t), c(t)) = -\lambda(t)^T x$ follows as in the statement. $\square$

Proof of Example 5.5.3

For the calculation of the support function, we consider two cases:
Case 1: for $\lambda \geq 0$,

$$\delta(\lambda, c(t)) = \sup_{x \in K(t,c(t))} (-x\lambda) = \frac{c(t)}{k}\lambda.$$

Case 2: for $\lambda < 0$,

$$\delta(\lambda, c(t)) = \sup_{x \in K(t,c(t))} (-x\lambda) = -\frac{c(t)}{k}\lambda.$$

In total, we can write these cases as

$$\delta(\lambda, c(t)) = \frac{c(t)}{k}|\lambda|.$$

In case $\lambda \geq 0$, λ^* is given by

$$\lambda^*(c(t)) = \arg\inf_{\lambda \in \mathbb{R}} \left\{ c(t)\frac{\lambda}{k} + \frac{1}{2}\left(\gamma + \frac{\lambda}{\sigma}\right)^2 \frac{1}{1-\alpha} \right\}.$$

We find λ^* by considering $f(\lambda) := c(t)\frac{\lambda}{k} + \frac{1}{2}\left(\gamma + \frac{\lambda}{\sigma}\right)^2 \frac{1}{1-\alpha}$. For $\lambda > 0$,

$$f'(\lambda) = \frac{c(t)}{k} + \frac{1}{\sigma}\left(\gamma + \frac{1}{\sigma}\lambda\right)\frac{1}{1-\alpha} > 0,$$

since $c(t) \geq 0$ and $\gamma \geq 0$ due to $\mu \geq r$, so f is strictly monotonically increasing and attains its minimum for this case in $\lambda = 0$.

In the second case, i.e. $\lambda < 0$,

$$\lambda^*(c(t)) = \arg\inf_{\lambda \in \mathbb{R}} \left\{ -c(t)\frac{\lambda}{k} + \frac{1}{2}\left(\gamma + \frac{\lambda}{\sigma}\right)^2 \frac{1}{1-\alpha} \right\}.$$

We minimize $f(\lambda) := -c(t)\frac{\lambda}{k} + \frac{1}{2}\left(\gamma + \frac{\lambda}{\sigma}\right)^2 \frac{1}{1-\alpha}$ and consider the first order condition

$$\begin{aligned} f'(\lambda) = -\frac{c(t)}{k} + \frac{1}{\sigma}\left(\gamma + \frac{\lambda}{\sigma}\right)\frac{1}{1-\alpha} &= 0 \\ \Leftrightarrow \lambda &= \frac{c(t)\sigma^2(1-\alpha)}{k} - (\mu - r), \end{aligned}$$

which satisfies $\lambda < 0$, if

$$\frac{c(t)\sigma^2(1-\alpha)}{k} < (\mu - r).$$

Otherwise, $f'(\lambda) < 0$ for $\lambda < 0$ and therefore, f attains its minimum in $\lambda = 0$.

Combining the two cases,

$$\lambda^*(c(t)) = \min\left(\frac{c(t)(1-\alpha)\sigma^2}{k} - (\mu - r), 0\right).$$

Using Proposition (5.3.1),

$$\begin{aligned}
\pi^*_{\lambda^*}(c(t)) =& \frac{1}{1-\alpha}\sigma^{-1}\left(\gamma + \sigma^{-1}\lambda^*(c(t))\right) = \frac{1}{(1-\alpha)\sigma^2}\left(\mu - r + \lambda^*(c(t))\right) \\
=& \frac{1}{(1-\alpha)\sigma^2}\left(\mu - r + \min\left(\frac{c(t)(1-\alpha)\sigma^2}{k} - (\mu - r), 0\right)\right) \\
=& \min\left(\frac{c(t)}{k}, \frac{1}{1-\alpha}\frac{\mu - r}{\sigma^2}\right).
\end{aligned}$$

□

D Appendix to Chapter 6

Proof of Corollary 6.1.2

We use the ansatz

$$\Phi(t, v) = \log(v - L(t)) + \varphi(t).$$

Thus, we have

$$\Phi_t(t, v) = \frac{-1}{v - L(t)} r(t)L(t) + \varphi'(t) \quad \text{and} \quad \Phi_v(t, v) = \frac{1}{v - L(t)}.$$

Remembering that $f(t, v) = \frac{v - L(t)}{v}$ due to (A3), we learn that $\Phi_v(t, v)$ satisfies (A3*) for $\tilde{\varphi}(t) = 1$, $c_1 = 1$, $c_2 = 1 + L(t)$, and $\alpha = 0$. Moreover, inserting π_{λ^*} from the proof of Theorem 6.1.1 into (6.7) and using (A2) , we receive

$$\begin{aligned}
&\sup_{\pi(t)\in\mathbb{R}^d} \Big\{ v\pi(t)^T \left(\mu(t) + \lambda^*(t) - r(t)\mathbf{1}\right) \Phi_v(t, v) + \frac{1}{2} v^2 \|\pi^T(t)\sigma(t)\|^2 \Phi_{vv}(t, v) \\
&+\Phi_t(t, v) + V(r(t) + \delta(t, \lambda^*(t), v))\Phi_v(t, v)\} \\
&= vf(t, v)\|\gamma_{\lambda^*}(t)\|^2 \Phi_v(t, v) + \frac{1}{2} v^2 f(t, v)^2 \|\gamma_{\lambda^*}(t)\|^2 \Phi_{vv}(t, v) \\
&+ \Phi_t(t, v) + v(r(t) + \delta(t, \lambda^*(t), v))\Phi_v(t, v) \\
&= \frac{1}{2} vf(t, v)\|\gamma_{\lambda^*}(t)\|^2 \Phi_v(t, v) + \Phi_t(t, v) + v(r(t) + \delta(t, \lambda^*(t), v))\Phi_v(t, v) \\
&= \frac{1}{2} \|\gamma_{\lambda^*}(t)\|^2 - \frac{1}{v - L(t)} r(t)L(t) + \varphi'(t) + \frac{v}{v - L(t)} r(t) + h(\lambda^*(t)) \\
&= \frac{1}{2} \|\gamma_{\lambda^*}(t)\|^2 + r(t) + h(\lambda^*(t)) + \varphi'(t) = 0
\end{aligned}$$

due to the definition of $\varphi(t)$ in the statement. Hence, (6.7) holds. Finally, the ansatz for $\Phi(t, v)$ also satisfies (6.8) and solves (P_{AUX}) in $\mathcal{M}_{\lambda^*}$. The statement then follows with Theorem 6.1.1. □

Proof of Corollary 6.1.3

We use the ansatz

$$\Phi(t,v) = \frac{(v-L(t))^\alpha}{\alpha}\varphi(t).$$

Thus, we have

$$\Phi_t(t,v) = -(v-L(t))^{\alpha-1}r(t)L(t)\varphi(t) + \frac{(v-L(t))^\alpha}{\alpha}\varphi'(t)$$

and

$$\Phi_v(t,v) = (v-L(t))^{\alpha-1}\varphi(t).$$

$\Phi_v(t,v)$ satisfies (A3*) for $\tilde{\varphi}(t) = \varphi(t)$, $c = 1$ and $c_2 = 1 + L(t)$. Moreover, inserting π_{λ^*} from the proof of Theorem 6.1.1 into (6.7) and using (A2) and (A3), we receive

$$\begin{aligned}
&\sup_{\pi(t)\in\mathbb{R}^d}\Big\{v\pi(t)^T\left(\mu(t)+\lambda^*(t)-r(t)\mathbf{1}\right)\Phi_v(t,v) + \frac{1}{2}v^2\|\pi^T(t)\sigma(t)\|^2\Phi_{vv}(t,v)\\
&+\Phi_t(t,v) + v(r(t)+\delta(t,\lambda^*(t),v))\Phi_v(t,v)\}\\
&= \frac{v}{1-\alpha}f(t,v)\|\gamma_{\lambda^*}(t)\|^2\Phi_v(t,v) + \frac{1}{2}\frac{v^2}{(1-\alpha)^2}f(t,v)^2\|\gamma_{\lambda^*}(t)\|^2\Phi_{vv}(t,v)\\
&+ \Phi_t(t,v) + v(r(t)+\delta(t,\lambda^*(t),v))\Phi_v(t,v)\\
&= \frac{1}{2}\frac{v}{1-\alpha}f(t,v)\|\gamma_{\lambda^*}(t)\|^2\Phi_v(t,v) + \Phi_t(t,v) + v(r(t)+\delta(t,\lambda^*(t),v))\Phi_v(t,v)\\
&= \left(\frac{1}{2}\|\gamma_{\lambda^*}(t)\|^2 + h(\lambda^*(t))\right)(v-L(t))^\alpha\varphi(t) - (v-L(t))^{\alpha-1}r(t)L(t)\varphi(t)\\
&+ \frac{(v-L(t))^\alpha}{\alpha}\varphi'(t) + r(t)v(v-L(t))^{\alpha-1}\varphi(t)\\
&= \left(\frac{1}{2}\|\gamma_{\lambda^*}(t)\|^2 + h(\lambda^*(t))\right)(v-L(t))^\alpha\varphi(t) + \frac{(v-L(t))^\alpha}{\alpha}\varphi'(t) + r(t)(v-L(t))^\alpha\varphi(t)\\
&= \left[\left(\frac{1}{2}\|\gamma_{\lambda^*}(t)\|^2 + h(\lambda^*(t)) + r(t)\right)\varphi(t) + \frac{1}{\alpha}\varphi'(t)\right](v-L(t))^\alpha = 0.
\end{aligned}$$

for $\varphi(t)$ as given in the statement and thus (6.7) holds. Finally, the ansatz for $\Phi(t,v)$ also satisfies (6.8) and solves (P_{AUX}) in $\mathcal{M}_{\lambda^*}$. The statement then follows with Theorem 6.1.1. □

Proof of Proposition 6.1.5

We calculate the dynamics of the surplus corresponding to the optimal investment strategy $S^{\pi_{\lambda^*}}(t) := V^{\pi_{\lambda^*}}(t) - L(t)$. The SDE of the optimal terminal wealth is with (5.2) given by

$$dV^{\pi_{\lambda^*}}(t) = V^{\pi_{\lambda^*}}(t)\left[\frac{1}{1-\alpha}\frac{V^{\pi_{\lambda^*}}(t) - L(t)}{V^{\pi_{\lambda^*}}(t)}\gamma_{\lambda^*}(t)^T\left(\gamma(t)dt + dW(t)\right) + r(t)dt\right].$$

The SDE of $S^{\pi_{\lambda^*}}(t)$ is given by

$$\begin{aligned}
dS^{\pi_{\lambda^*}}(t) &= dV^{\pi_{\lambda^*}}(t) - r(t)L(t)dt \\
&= \left(V^{\pi_{\lambda^*}}(t) - L(t)\right)\left[\frac{1}{1-\alpha}\left(\gamma_{\lambda^*}(t)^T\gamma(t)dt + \gamma_{\lambda^*}(t)^T dW(t)\right) + r(t)dt\right] \\
&= S^{\pi_{\lambda^*}}(t)\left[\left(\frac{1}{1-\alpha}\gamma_{\lambda^*}(t)^T\gamma(t) + r(t)\right)dt + \frac{1}{1-\alpha}\gamma_{\lambda^*}(t)^T dW(t)\right].
\end{aligned}$$

Thus, $S^{\pi_{\lambda^*}}(t)$ can be written explicitly as

$$S^{\pi_{\lambda^*}}(t) = S^{\pi_{\lambda^*}}(0)e^{\frac{1}{1-\alpha}\left(\int_0^t \gamma_{\lambda^*}(s)^T\gamma(s) + (1-\alpha)r(s) - \frac{1}{2(1-\alpha)}\|\gamma_{\lambda^*}(s)\|^2 ds + \int_0^t \gamma_{\lambda^*}(s)dW(s)\right)},$$

which completes the proof with the definition of $S^{\pi_{\lambda^*}}(t)$. □

Proof of Proposition 6.2.1

The representation of $\delta(t, \lambda(t), V(t))$ follows analogue to Proposition 5.5.2 with $c(t) := f(t, V(t))$, $d_L := 0$, $d_1 := -d_1$ and $d_4 := -d_4$. Moreover,

$$\delta\left(t, \lambda(t), V(t)\right) = f(t, V(t))\|\lambda(t)\|_{(B^T RB)^{-1}},$$

with $\|\lambda(t)\|_{(B^T RB)^{-1}} := \sqrt{\lambda(t)^T(B^T RB)^{-1}\lambda(t)}$. In the case that f is bounded and since all norms on $\mathbb{R}^d$ are equivalent, there exists a constant $c > 0$ such that

$$f(t, V(t))\|\lambda(t)\|_{(B^T RB)^{-1}} \leq c \cdot \|\lambda(t)\| \leq c \cdot \max\left\{1, \|\lambda(t)\|^2\right\} \leq c \cdot \left(1 + \|\lambda(t)\|^2\right)$$

and consequently

$$\mathbb{E}\left(\int_0^T \delta\left(t, \lambda(t), V(t)\right)dt\right) \leq c \cdot \mathbb{E}\left(\int_0^T 1 + \|\lambda(t)\|^2 dt\right) = c \cdot T + c \cdot \mathbb{E}\left(\int_0^T \|\lambda(t)\|^2 dt\right),$$

so $\mathcal{D}$ can be defined as in (6.14). □

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