Evolution of Social Power over Influence Networks Containing Antagonistic Interactions

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Abstract

Individual social power in the opinion formation process over social influence networks has been under intense scientific investigation. Most related works assume explicitly or implicitly that the interpersonal influence weights are always non-negative. In sharp comparison, we argue that such influence weights can be both positive and negative since there exist various contrasting relationships in real-world social networks. Hence, this article studies the evolution of opinion dynamics and social power on cooperative-competitive networks whose influence structure changes via a reflected appraisal mechanism along a sequence of issue discussions. Of particular focus is on identifying the pathways and effects of social power on shaping public opinions from a graph-theoretic perspective. Then, we propose a dynamic model for the reflected self-appraisal process, which enables us to discuss how the individual social power evolves over sequential issue discussions. By accommodating differential Lyapunov theory, we show the global exponential convergence of the self-appraisal model for almost all network topologies. Finally, we conclude that the self-appraisals and social powers are eventually dependent only on an interpersonal appraisal profile.

Keywords: Opinion dynamics, Reflected/interpersonal appraisal, Signed graphs, Differential Lyapunov theory

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1. Introduction

Opinion dynamics have always been the prominent focus subject in socio-cybernetics [1, 2, 3] wherein social entities share and aggregate thoughts, ideas, feelings, experiences, and observations over social networks, and generate new concepts, trends, and reflections at the same time. Such social activities among humans can also find their similar counterparts in engineered systems, e.g., robot and sensor networks, and natural communities, e.g., bacteria, neurons, and fireflies [4]. One central line of research in the modeling of opinion pooling can trace back to the early influential works of French and DeGroot, nowadays known as the French-DeGroot (FG) model [1]. The basis for these models is an empirical observation that individuals update their opinions as a convex combination of their own and neighbors' opinions; this observation is a historical milestone of "cognitive and behavioral algebra" in experimental social psychology [2].

Related to the field of opinion dynamics, it is of particular interest to evaluate the *social influence* or *power* of individuals in a collective debate on a given issue. Indeed, the seminal work [5] of French initiated the investigation of the total (direct and indirect) influence of an individual's initial idea on the final collective opinion outcome. Especially, the individual social powers may change over time in a social group due to the interconnections and interactions among group members. Recent focus is shifting from the single-issue opinion evolution to the opinion formation process on a sequence of different issues [6, 7]. Among others, significant research efforts [8] have been conducted in studying the co-evolution of opinion dynamics and social power along the issue sequence [9]. Specifically, individuals can modify the relative influence structure before discussion on the next issue in response to their perceived social impact on the opinion outcome of the ongoing issue discussion. Such a self-modification of the influence network across an issue sequence is rooted in the theory of *reflected appraisal* [10]. The recent work [11] therefore introduces the so-called DeGroot-Friedkin (DF) model to describe the evolution of an opinion dynamics process and an accompanying self-appraisal process. Fundamentally speaking, the prin-

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cipal objective of the DF model is to explore how the individual social powers evolve through sequential discussion and reflected self-appraisal. Other research efforts on developing the DF model include the validation by empirical data [8], the relaxation to reducible influence networks [9] and single-timescale model [7], the extension to dynamic interaction topology [12], distributed modeling in both continuous- [13] and discrete-time [14], the connection with nonlinear Markov chain [15], and the variational interpretation [16].

The afore-cited works on studying the dynamic evolution of social power often postulate explicitly or implicitly that individuals cooperatively interact with each other. Using graph-theoretic modeling of social networks, such pairwise cooperative interactions are conventionally represented by edges of non-negative weights in the graph. This assumption, however, is not always appropriate since there exist various antagonism, rebellion, and betrayal in many real-life networks [17, 18]. In the graph representation, those competitive interrelations appear in the form of negatively weighted edges in the so-called signed graphs [19]. In the past few years, opinion dynamics in cooperative-competitive (coopetitive) social networks have been studied extensively in the literature; see, e.g., [20, 21, 22]. Compared with cooperative networks, the opinion-forming process on signed graphs may exhibit not only opinion consent, but also other complex outcomes, including neutrality, polarization, and separation [23]. Nevertheless, the qualitative and quantitative analysis for individual social power in coopetitive networks has been less studied than the unsigned cases.

In this article, we aim to study the evolution of opinion dynamics and social power in social networks containing antagonistic interconnections over the sequence of issues. This article first extends on the opinion formation process of sequential issue discussions, which takes place on a coopetitive network. Along with the opinion discussions, we describes the updating rule of the interpersonal influence structure via a reflected appraisal mechanism. For the influence mechanism, each individual accords weights to others' opinions proportionally according to her/his positive or negative appraisals of them. As such, the specification of group influence embodies the formal definition of social power in

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signed networks, which characterizes the oriented effects of individual impact on shaping the collective opinion outcome. Different from the pioneering work, this article further studies the topological characterization of social power from a graph-theoretic perspective rather than algebraic expression, which also paves the way for general extensions to broader network topologies. Then, we develop a concise mathematical treatment for the self-appraisal process of individual social powers and its explicit formulation. A rigorous theoretical analysis is then conducted to examine the convergence and stability of the developed model by taking into account multiple structural properties of the appraisal networks. Especially, we employ differential Lyapunov theory to study the incremental stability of nonlinear dynamical systems. More specifically, we show that group members forget exponentially fast their initial perception of social influence, and the long-term configuration of individual social powers is completely determined by an interpersonal appraisal network.

The remainder of the article is organized as follows. In Section 2, we fix the notation and introduce some basic concepts of graph representation and control theory. Section 3 discusses the coevolution of opinion dynamics and influence networks along issue sequence. The graph-theoretic description of social power over signed networks and explicit mathematical formulation of self-appraisal dynamics can also be found in Section 3. Section 4 contains a complete analysis of the convergence properties of the proposed models. Further theoretical extensions can be found in Section 5. Simulations and conclusions are respectively provided in Section 6 and Section 7. All technical proofs are given in the appendices.

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2. Preliminaries

This section is dedicated to fixing the notations and to offering a recap of basic concepts in algebraic graph and control theory.

2.1. Notations

Let ($\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$) $\mathbb{R}$ be the set of (positive, non-negative) real numbers. Vector $\mathbf{1}_n$ ($\mathbf{0}_n$) represents the n-dimensional column vector of all ones (zeros) with appropriate dimensions. The canonical basis of $\mathbb{R}^n$ is defined by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and the $n \times n$ identity matrix is given by $\boldsymbol{I}_n := [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]$. The notation $|a|$ denotes the absolute value of a scale a and $|\boldsymbol{z}|$ implies the entry-wise absolute value of a vector $\boldsymbol{z} = [z_1, \ldots, z_n]$, i.e., $|\boldsymbol{z}| = [|z_1|, \ldots, |z_n|]$. Similarly, $\boldsymbol{z} \geq 0$ and ($\boldsymbol{z} > 0$) indicate n component-wise inequalities $z_i \geq 0$ ($z_i > 0$). We denote the *tangent space* of an n-dimensional manifold $\mathbb{M}^n$ at $\boldsymbol{z} \in \mathbb{M}^n$ by $T_{\boldsymbol{z}}\mathbb{M}^n$, and the *tangent bundle* of $\mathbb{M}^n$ by $T\mathbb{M}^n = \bigcup_{\boldsymbol{z} \in \mathbb{M}^n} \{\boldsymbol{z}\} \times T_{\boldsymbol{z}}\mathbb{M}^n$. A distance (or metric) $\mathsf{d}_{\mathbb{M}} : \mathbb{M}^n \times \mathbb{M}^n \to \mathbb{R}_{\geq 0}$ on a manifold $\mathbb{M}^n$ is a non-negative function and satisfies $\mathsf{d}_{\mathbb{M}}(\boldsymbol{z}_1, \boldsymbol{z}_2) = 0$ iff (if and only if) $\boldsymbol{z}_1 = \boldsymbol{z}_2$, and $\mathsf{d}_{\mathbb{M}}(\boldsymbol{z}_1, \boldsymbol{z}_2) \leq \mathsf{d}_{\mathbb{M}}(\boldsymbol{z}_1, \boldsymbol{z}_3) + \mathsf{d}_{\mathbb{M}}(\boldsymbol{z}_3, \boldsymbol{z}_2)$ for arbitrary $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \in \mathbb{M}^n$. The l-norm of a vector and their induced matrix norms are denoted by $\|\cdot\|_l$, where $l \geq 1$. The set of n-dimensional vectors whose 1-norm is 1 forms the surface of an n-dimensional *cross-polytope* or *orthoplex*, i.e., $\mathbb{C}^n := \{\boldsymbol{z} \in \mathbb{R}^n | -1 \leq \boldsymbol{z} \leq 1, \|\boldsymbol{z}\|_1 = 1\}$, which has an interior $\mathsf{int}(\mathbb{C}^n) := \{\boldsymbol{z} \in \mathbb{R}^n | -1 < \boldsymbol{z} < 1, \|\boldsymbol{z}\|_1 = 1\}$. To save triviality, the orthoplex manifold with the exclusion of vertices is given by $\nabla\mathbb{C}^n := \mathbb{C}^n \backslash \{\pm\boldsymbol{e}_1, \ldots, \pm\boldsymbol{e}_n\}$. The n-dimensional simplex is given by $\mathbb{S}^n := \{\boldsymbol{z} \in \mathbb{R}^n | \boldsymbol{z} \geq 0, \mathbf{1}_n^\mathsf{T}\boldsymbol{z} = 1\}$ with the interior $\mathsf{int}(\mathbb{S}^n) = \{\boldsymbol{z} \in \mathbb{R}^n | \boldsymbol{z} > 0, \mathbf{1}_n^\mathsf{T}\boldsymbol{z} = 1\}$ and the tangent space $T_{\boldsymbol{z}}\mathbb{S}^n = \{\delta\boldsymbol{z} \in \mathbb{R}^n | \mathbf{1}_n^\mathsf{T}\delta\boldsymbol{z} = 0\}$. Finally, we denote $\nabla\mathbb{S}^n := \mathbb{S}^n \backslash \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$.

2.2. Graph Theory

A signed directed graph (digraph) is given by a triple $\mathcal{G} = (\mathbb{V}, \mathbb{E}, \boldsymbol{A})$ where $\mathbb{V} = \{1, \ldots, n\}$ stands for the set of nodes, $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix, and $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ is a set of edges having elements as ordered pairs (j, i) (an arc from node j to i) if the coupling weight $a_{ij} \neq 0$. Throughout this article, we confine ourselves to the digon sign-symmetric graphs in which any pair of opposite edges (if exists) is identically signed, i.e. $a_{ij}a_{ji} \geq 0$. A signed graph is called *balanced* if $\sum_{j=1, j\neq i}^{n} |a_{ij}| = \sum_{j=1, j\neq i}^{n} |a_{ji}|$ for all $i \in \mathbb{V}$. A signed digraph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is *structurally balanced* (SB) if $\mathbb{V}$ can be split

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into two disjoint subsets (i.e., $\mathbb{V}^+ \cup \mathbb{V}^- = \mathbb{V}$, $\mathbb{V}^+ \cap \mathbb{V}^- = \emptyset$) such that $a_{ij} > 0$ if $i \in \mathbb{V}^+, j \in \mathbb{V}^+$ or $i \in \mathbb{V}^-, j \in \mathbb{V}^-$, and $a_{ij} < 0$ if $i \in \mathbb{V}^+, j \in \mathbb{V}^-$ or $j \in \mathbb{V}^+, i \in \mathbb{V}^-$. Without loss of generality, a SB graph entails an n-dimensional vector $\boldsymbol{\rho} := [\rho_1, \ldots, \rho_n]^\top \in \{\pm 1\}^n$ such that $\rho_i = 1$ if $i \in \mathbb{V}^+$ and $\rho_i = -1$ if $i \in \mathbb{V}^-$. Structural balance epitomizes the famous sociological aphorism: "my friend's friend is my friend", "my friend's enemy is my enemy," "my enemy's enemy is my friend," and "my enemy's friend is my enemy."

A node that can reach any other nodes of the graph through a path is called the *root*. A digraph is *quasi-strongly connected* (QSC) or has a spanning tree if it contains at least one root and is *strongly connected* (SC) if every node is a root. A digraph $\mathcal{G}$ is called a star graph if there exists a unique node, called the *center node*, such that the edges of $\mathcal{G}$ pointing either all to or all away from this center node. Moreover, a subgraph of graph $\mathcal{G} = (\mathbb{V}, \mathbb{E}, \boldsymbol{A})$ is given by $\mathcal{G}^s = (\mathbb{V}^s, \mathbb{E}^s)$ where $\mathbb{V}^s \subseteq \mathbb{V}$ and $\mathbb{E}^s \subseteq (\mathbb{V}^s \times \mathbb{V}^s) \cap \mathbb{E}$. We say a subgraph is *in-isolated* if no edge comes from $\mathbb{V} \backslash \mathbb{V}^s$ to $\mathbb{V}^s$. A subgraph $\mathcal{G}^s$ is an in-isolated structurally balanced (ISB) component of a signed digraph $\mathcal{G}$ if it is an in-isolated subgraph of $\mathcal{G}$ and SB, and any other subgraph of $\mathcal{G}$ strictly containing $\mathcal{G}^s$ is not in-isolated and SB.

For a matrix $\boldsymbol{C} := [c_{ij}] \in \mathbb{R}^{n \times n}$, we define the *associated graph* $\mathcal{G}(\boldsymbol{C})$ of $\boldsymbol{C}$ to be a directed graph with node set $\{1, \ldots, n\}$ and edge set $\mathbb{E}$ which contains a directed edge $(j, i) \in \mathbb{E}$ if $c_{ij} \neq 0$ and $j \neq i$.

2.3. Contraction Analysis and Differential Lyapunov Theory

Before closing this section, we present the needed tools for convergence and stability analysis of nonlinear dynamical systems.

Regarding nonlinear dynamical systems, the prior knowledge of specific solutions (equilibrium points) or reference signals is a significant obstruction to the applicability of linear techniques and Lyapunov stability theory for convergence analysis. Instead of studying the convergence to a specific equilibrium or an unknown reference trajectory, one can look into the evolution of the distance between a pair of trajectories. Along with this line of research, differential Lya-

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punov theory [24] has been well recognized within the control community, central of which is the introduction of Finsler geometry and the lifting of Lyapunov functions to the tangent bundle. For more details on incremental stability and contraction theory, we refer the interested reader to the tutorial articles [24, 25] and references therein.

Consider a deterministic discrete-time nonlinear system described by the difference equation

$$\boldsymbol{y}(t+1) = \boldsymbol{g}(\boldsymbol{y}, t), \qquad \text{and } \boldsymbol{y}(0) := \boldsymbol{y}_0 \in \mathbb{M}^n \tag{1}$$

where $\boldsymbol{g}$ is a continuously differentiable vector field on an n-dimensional manifold $\mathbb{M}^n$. Let $\boldsymbol{\phi}(\cdot; t_0, \boldsymbol{y}_0)$ be the semi-flow of system (1) starting from the initial condition $\boldsymbol{y}_0 \in \mathbb{M}^n$ at time t_0, i.e., $\boldsymbol{\phi}(t_0; t_0, \boldsymbol{y}_0) = \boldsymbol{y}_0$. Specifically, following the work [24], we consider the *forward invariant* and *connected* subset $\mathbb{Y}^n \subset \mathbb{M}^n$ for (1), on which $\boldsymbol{\phi}(t; t_0, \boldsymbol{y}_0)$ is *forward complete* for every $\boldsymbol{y}_0 \in \mathbb{Y}^n$.

To make this article self-contained, we recall the following definition of incrementally exponential stability which is a discrete-time analog to that in [24, Definition 1].

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Definition 1. Consider system (1) on a given manifold $\mathbb{M}^n$. Let $\mathbb{Y}^n \subset \mathbb{M}^n$ be a connected and forward invariant set and $\mathsf{d}_{\mathbb{M}} : \mathbb{M}^n \times \mathbb{M}^n \to \mathbb{R}$ be a continuous distance metric on $\mathbb{M}^n$. System (1) is *incrementally exponentially stable* (IES) on $\mathbb{Y}^n$ if there exists a metric $\mathsf{d}_{\mathbb{M}}$, $c_1 \geq 1$, and $c_2 > 1$ such that

$$\mathsf{d}_{\mathbb{M}}\left(\boldsymbol{\phi}(t; t_0, \boldsymbol{y}_1), \boldsymbol{\phi}(t; t_0, \boldsymbol{y}_2)\right) \leq c_1 c_2^{-(t-t_0)} \mathsf{d}_{\mathbb{M}}(\boldsymbol{y}_1, \boldsymbol{y}_2). \tag{2}$$

holds for all $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Y}^n$ and $t \geq t_0 \in \mathbb{R}_{\geq 0}$.

Note that the IES property is uniform since the right-hand side of (2) depends only on the elapsed time $t - t_0$ and thus in the case of $\mathbb{Y}^n = \mathbb{M}^n$, we can say that system (1) is uniformly globally IES.

The dynamics of form (1) has an associated *variational system* as

$$\delta\boldsymbol{y}(t+1) = \frac{\partial \boldsymbol{g}(\boldsymbol{y}, t)}{\partial \boldsymbol{y}} \delta\boldsymbol{y}(t), \tag{3}$$

where $\delta \boldsymbol{y}(t) : \mathbb{R}_{\geq 0} \rightarrow T_{\boldsymbol{y}}\mathbb{M}^n$ is a *virtual displacement*. System (1) together with (3) is referred to as the *prolonged system*. Furthermore, the Finsler geometry is important in the deduction of incremental stability.

Definition 2. A *Finsler* structure $F(\boldsymbol{y}, \delta\boldsymbol{y}) \in T\mathbb{M}^n \rightarrow \mathbb{R}_{\geq 0}$ on the manifold $\mathbb{M}^n$ satisfies the following conditions:

(i) F is a smooth function on $T\mathbb{M}^n \backslash \{\boldsymbol{0}\}$;

(ii) $F(\boldsymbol{y}, \delta\boldsymbol{y}) \geq 0$ where the equality holds iff $\delta\boldsymbol{y} = \boldsymbol{0}$;

(iii) $F(\boldsymbol{y}, a\delta\boldsymbol{y}) = aF(\boldsymbol{y}, \delta\boldsymbol{y})$ on $T\mathbb{M}^n$ for $a \geq 0$;

(iv) $F(\boldsymbol{y}, \delta\boldsymbol{y}_1 + \delta\boldsymbol{y}_2) \leq F(\boldsymbol{y}, \delta\boldsymbol{y}_1) + F(\boldsymbol{y}, \delta\boldsymbol{y}_2)$ for all $\delta\boldsymbol{y}_1, \delta\boldsymbol{y}_1 \in T_{\boldsymbol{y}}\mathbb{M}^n$.

The development of the Finsler structure in the tangent bundle enables us to induce a well-defined distance on $\mathbb{M}^n$

$$\mathsf{d}_{\mathbb{M}}(\boldsymbol{y}_1, \boldsymbol{y}_2) := \inf_{\Gamma(\boldsymbol{y}_1, \boldsymbol{y}_2)} \int_0^1 F(\gamma(s), \dot{\gamma}(s))ds, \tag{4}$$

where $\gamma : [0, 1] \rightarrow \mathbb{Y}^n$ is a curve on $\mathbb{Y}^n$ satisfying $\gamma(0) = \boldsymbol{y}_1$, $\gamma(1) = \boldsymbol{y}_2$, and $\Gamma(\boldsymbol{y}_1, \boldsymbol{y}_2)$ is the collection of those piece-wise continuous curves.

In relation to continuous-time analogue in [24], the following theorem states the differential Lyapunov framework for discrete-time systems.

Theorem 1. *Consider system* (1) *on a smooth manifold* $\mathbb{M}^n \subseteq \mathbb{R}^n$ *with a continuously differentiable vector-valued function* $\boldsymbol{g}$ *in a connected and positively invariant set* $\mathbb{Y}^n \subset \mathbb{M}^n$*. If there exist a Finsler structure* $F(\boldsymbol{y}, \delta\boldsymbol{y}) \in T\mathbb{M}^n \rightarrow \mathbb{R}_{\geq 0}$*, scalars* $c_1, c_2 \in \mathbb{R}_{\geq 0}$*,* $c_3 \in]0, 1[$*,* $l \in \mathbb{R}_{\geq 1}$ *and a candidate differential Lyapunov function* $V(\boldsymbol{y}, \delta\boldsymbol{y}) \in T\mathbb{M}^n \rightarrow \mathbb{R}_{\geq 0}$ *being Lipschitz continuous, such that, in coordinates,*

$$\begin{aligned} c_1 F\big(\boldsymbol{y}(t), \delta\boldsymbol{y}(t)\big)^l \leq V\big(\boldsymbol{y}(t), \delta\boldsymbol{y}(t)\big) \leq c_2 F\big(\boldsymbol{y}(t), \delta\boldsymbol{y}(t)\big)^l \\ V\big(\boldsymbol{y}(t+1), \delta\boldsymbol{y}(t+1)\big) - V\big(\boldsymbol{y}(t), \delta\boldsymbol{y}(t)\big) \leq -c_3 V\big(\boldsymbol{y}(t), \delta\boldsymbol{y}(t)\big) \end{aligned} \tag{5}$$

for $t \in \mathbb{R}$*,* $\boldsymbol{y} \in \mathbb{Y}^n \subset \mathbb{M}^n$*, and* $\delta\boldsymbol{y} \in T_{\boldsymbol{y}}\mathbb{M}^n$*, then* (1) *is IES on the contraction region* $\mathbb{Y}^n$*.*

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The above theorem describes a Lyapunov function characterization for a contracting system and establishes the equivalence between incrementally exponential stability and contraction analysis.

3. Coevolution of Opinion Dynamics and Social Power

In this section, we study the dynamical evolution of opinion dynamics and social power in the context of sequential issue discussion. Our particular emphasis is on the influence network whose underlying graph may involve both positive and negative links and its self-regulation across the issue sequence via a reflected appraisal mechanism.

3.1. Motivation and Mathematical Description

The starting point of this work is the extension of Altafini's model [20] of opinion formation processes on a single issue to opinion discussions on a sequence of issues $\mathbb{I} = \{0, 1, 2, \ldots\}$. For any given issue $s \in \mathbb{I}$, each agent $i \in \mathbb{V}$ ($n \geq 2$) is associated with a time- and issue-dependent variable $x_i(s,t) \in \mathbb{R}$ that represents his/her attitude on issue s at time t. With the definition $\boldsymbol{x} = [x_1, \ldots, x_n]^\mathsf{T}$, the opinion dynamics of the entire group is given in a compact form

$$\boldsymbol{x}(s, t+1) = \boldsymbol{P}(s)\boldsymbol{x}(s,t), \quad \boldsymbol{x}(s,0) \in \mathbb{R}^n \tag{6}$$

where $\boldsymbol{P}(s)$ is referred to as *influence matrix*. More specifically, each agent updates his/her opinion according to the following rule

$$x_i(s,t+1) = p_{ii}(s)x_i(s,t) + \sum_{j \neq i}^{n} p_{ij}(s)x_j(s,t)$$

where $p_{ii}(s) \in [0,1]$ is the self-weight and $p_{ij}(s) \in [-1,1]$ is the interpersonal influence weight that agent i attaches to the opinion of agent j such that $\sum_{j=1}^{n} |p_{ij}(s)| = 1$ on each issue s. For an easy exposition, the shorthand $z_i(s) \in [0,1]$ is used to denote the self-weight $p_{ii}(s)$ for all $i \in \mathbb{V}$.

From a psychological perspective, the diagonal and non-diagonal entries of the influence matrix have distinct roles. Particularly, the self-weight $z_i(s)$ is designated as the indicator of his/her self-appraisal (self-worth or self-confidence)

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corresponding to the degree of assertiveness to his/her own opinion, whereas the interpersonal weight $p_{ij}(s)$ $(j \neq i)$ represents his/her extent of trust-distrust to the displayed opinion of individual j. More importantly, the topology of influence networks evolves from issue to issue via the so-called reflected appraisal mechanism [10]. This mechanism illustrates that individuals intentionally revise their influence structure based on the prior issue negotiation, thereby adjusting the allocation of influence weights [26]. Before discussing on the next issue, each individual therefore estimates his/her own influence on outcome of prior issue discussion and regulates the interpersonal influence weights by

$$p_{ij}(s) = (1 - z_i(s))q_{ij}(s), \quad i, j \in \mathbb{V}, \tag{7}$$

by which individuals allocate the aggregate relative influence $1 - z_i$ by scaling using the interpersonal appraisal scores $q_{ij} \in [-1, 1]$ which represents individual i's appraisal of individual j and satisfies $q_{ii} = 0$ and $\sum_j^n |q_{ij}(s)| = 1$ for all $i \in \{1, \ldots, n\}$, thus ensuring $\sum_j^n |p_{ij}(s)| = 1$.

Remark 1. In addition to the self-regulation of influence matrix along issue sequence, individuals naturally prefer to revise their (positive or negative) appraisals – friendships and enmities – of others [27]. Therefore, the interpersonal appraisal structure is encoded by the zero-diagonal matrix $\boldsymbol{Q}(s) := [q_{ij}(s)] \in \mathbb{R}^{n \times n}$ which updates from one issue to the next. Examples of variant appraisal structures include a congress in the governance system where the representatives of different nations or constituent states regularly assemble to manage issues in multiple domains involving political, economic and cultural matters. Stemming from the common political and social benefit orientation, participants may form conglomerates on some fixed issues while contesting with other opposition factions. These "stable" relationships, however, varies as the discussed topic changes. For instance, conventioneers may realign themselves with others, or possibly even cooperate with the opponents on prior issues. The consideration of dynamic appraisal topology is consistent with the political maxim: "no eternal allies, no perpetual enemies, only eternal and perpetual interests." To intrinsically understand how the opinion dynamics and individual social powers

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evolve through sequential discussion and reflected self-appraisal, the matrix $\boldsymbol{Q}$ and its associated spectral information are only regarded as an exogenous signal in this article. Nevertheless, works on the evolution of interpersonal appraisals have recently appeared in [28, 29], the developments of which seem to have a great possibility for incorporation into this work.

From (7), the influence matrix in the sequence has the compact form

$$\boldsymbol{P}(\boldsymbol{z}, \boldsymbol{Q}) = \mathsf{diag}\left(\boldsymbol{z}(s)\right) + \left(\boldsymbol{I}_n - \mathsf{diag}(\boldsymbol{z}(s))\right)\boldsymbol{Q}(s). \tag{8}$$

If there is no confusion, we drop the explicit dependence of matrix $\boldsymbol{P}$ on $\boldsymbol{z}$ and $\boldsymbol{Q}$, and still write $\boldsymbol{P}(s)$ for the simplicity of notation. Regarding the scenario of social networks without antagonistic interactions, the influence matrix $\boldsymbol{P}(s) \in \mathbb{R}_{\geq 0}^{n\times n}$ satisfies the row-stochasticity for a given issue and thus, the strong connectedness of graph $\mathcal{G}(\boldsymbol{P}(s))$ implies the existence of a unique normalized left eigenvector $\boldsymbol{p}(s) \in \mathbb{R}_{>0}^{n}$ associated with the dominant eigenvalue 1 such that $\lim_{t\to\infty} \boldsymbol{P}^t(s) = \mathbf{1}_n \boldsymbol{p}^\mathsf{T}(s)$. This is a direct application of the Perron-Frobenius theorem to irreducible non-negative matrices. Therefore, the issue discussion process (6) on issue s asymptotically reaches an opinion consensus

$$\lim_{t\to\infty} \boldsymbol{x}(s, t+1) = \big(\lim_{t\to\infty} \boldsymbol{P}^t(s)\big)\boldsymbol{x}(s, 0) = \big(\boldsymbol{p}^\mathsf{T}(s)\boldsymbol{x}(s, 0)\big)\mathbf{1}_n.$$

Namely, the opinions of social actors converge to a common value as time progresses, which is equal to some convex combination of individuals' initial thoughts. However, $\boldsymbol{P}(s)$ usually needs not to be row-stochastic when there coexist positive and negative non-diagonal elements, and it even does not have a dominant eigenvalue 1 whatever $\mathcal{G}(\boldsymbol{P}(s))$ is SC or not. Different from opinion agreement on cooperative networks, outcomes of the opinion dynamics process (6) may involve rich opinion behaviors including neutrality, consensus and polarity. Therefore, we need to characterize the properties of the influence matrix.

Lemma 1. *For each issue $s \in \mathbb{I}$, consider an interpersonal appraisal matrix $\boldsymbol{Q}(s) = [q_{ij}(s)] \in \mathbb{R}^{n\times n}$ with $q_{ii}(s) = 0$ and $\sum_{j=1}^{n} |q_{ij}(s)| = 1$ for all $i \in \mathbb{V}$. If*

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the associated graph $\mathcal{G}(\boldsymbol{Q}(s))$ is SC and SB, then the following claims hold for the influence matrix $\boldsymbol{P}(s)$ defined in (8)*:*

(i) The matrix $\boldsymbol{P}(s)$ has a simple dominant eigenvalue 1*;*

(ii) There exists a unique pair of vectors $\boldsymbol{p}(s) \in \mathbb{C}^n$ and $\boldsymbol{\rho}(s) = [\rho_i(s)] \in \{\pm 1\}^n$ satisfying $\boldsymbol{Q}(s)\boldsymbol{\rho}(s) = \boldsymbol{\rho}(s)$, such that $\boldsymbol{p}^{\mathsf{T}}(s)\boldsymbol{P}(s) = \boldsymbol{p}^{\mathsf{T}}(s)$, $\boldsymbol{P}(s)\boldsymbol{\rho}(s) = \boldsymbol{\rho}(s)$, and $\lim_{t\to\infty} \boldsymbol{P}^t(s) = \boldsymbol{\rho}(s)\boldsymbol{p}^{\mathsf{T}}(s)$;

(iii) For $\boldsymbol{z}(s) = \mathbf{1}/n$, $\boldsymbol{p}(s) = \boldsymbol{\rho}(s)/n$ iff $\mathcal{G}(\boldsymbol{Q}(s))$ is balanced;

(iv) Influence network $\mathcal{G}(\boldsymbol{P}(s))$ is SB;

(v) If $\boldsymbol{z}(s) = \boldsymbol{e}_i$ for some $i \in \mathbb{V}$, then $\mathcal{G}(\boldsymbol{P}(s))$ has only one root at node i and $\boldsymbol{p}(s) = \rho_i(s)\boldsymbol{e}_i$;

(vi) The graph $\mathcal{G}(\boldsymbol{P}(s))$ is SC and $\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{p}(s) > 0$ for $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n$.

In the above lemma, we focus on the networks $\mathcal{G}(\boldsymbol{Q}(s))$ that are SC and SB on each issue. As one will see, networks with weaker topological constraints are also appreciated in the sequel of this article, thus making the developments applicable to a wider range of real networks. An immediate consequence of claim (ii) in Lemma 1 is that the opinion dynamics (6) on signed influence networks converge after each issue discussion

$$\lim_{t\to\infty} \boldsymbol{x}(s,t) = \left(\boldsymbol{p}^{\mathsf{T}}(s)\boldsymbol{x}(s,0)\right)\boldsymbol{\rho}(s) \tag{9}$$

where $\boldsymbol{\rho}(s) \in \{\pm 1\}^n$ and $\boldsymbol{p}(s) \in \mathbb{C}^n$ are the dominant right- and left-eigenvector of the influence matrix $\boldsymbol{P}(s)$ on issue s, respectively.

Now, we are in a position to formally state how the self-confidence level $s \mapsto \boldsymbol{z}(s)$ evolves along the issue sequence via the reflected appraisal mechanism, thereby adjusting the interpersonal weights in terms of (7). The most important message of the convergence limit (9) is that the coefficient vector $\boldsymbol{p}(s)$ mathematically specifies the true social contribution of individuals made to the final decision making. In other words, $\boldsymbol{p}(s)$ can be regarded as a social metric that measures the ability of individuals to relatively control the outcome

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of opinion discussion processes [30]. Therefore, the entry $p_i(s)$ is referred to as the social power of individual $i \in \mathbb{V}$. Recalling the reflected appraisal mechanism that each agent tends to perceive his/her individual social power over the sequence of issues, we therefore provide the following mathematical model for the self-appraisal process in an antagonistic social network

$$\boldsymbol{z}(s+1) = \mathsf{diag}\big(\boldsymbol{\rho}(s)\big)\boldsymbol{p}(s) \tag{10}$$

where the normalized vector $\boldsymbol{p}(s)$ is used, i.e., $||\boldsymbol{p}(s)||_1 = 1$, so the elements of self-confidence level $\boldsymbol{z}(s)$ are nonnegative and have unit sum, i.e., $\boldsymbol{z}(s) \in \mathbb{S}^n$. From (7), adjusting the self-weights $z_i(s+1)$ using $\rho_i(s)p_i(s)$, the interpersonal weights $p_{ij}(s+1)$ are also updated with $(1 - z_i(s+1))q_{ij}(s+1)$ (10), thereby engendering the evolution of individual social power $p_i(s+1)$.

Remark 2. The development in (10) reflects the psychological fact that there may be dramatically opposite between self-perceived and veridical appraisals of individual social influence [10]. Indeed, the self-appraisal vector of individuals takes values in $\mathbb{S}^n$ for all issues, while the social power metric $\boldsymbol{p}(s)$ allows a negative vector in $\mathbb{C}^n$ using Lemma 1. Such deviation in the sign agrees with the psychological fact that self-appraisal entails an individual's subjective but not necessarily objective assessment of his/her own power [31]. On the one hand, the individual generally has a positive self-impression $\boldsymbol{z}$. On the other hand, the network-wide quantification $\boldsymbol{p}$ concerning social power represents what the actual appraisals of others on the individual are, and is an objective study of the net effect.

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Remark 3. For cooperative networks associated with a constant matrix $\boldsymbol{Q}$, the self-appraisal model (10) degenerates to the traditional DF model examined in [11, 8], wherein the dominant right- and left-eigenvectors here reduce to $\mathbf{1}_n$ and a non-negative vector, respectively. Hence, the DF framework that unfolds on a non-competitive network can be treated as a special case of this work. In the spirit similar to the DF model, we also assume that the opinion formation process and the self-appraisal process evolve on separate timescales for the simplicity of modeling and analysis. More specifically, opinion dynamics reach convergence before updating the influence weights. The recent literature [7] modifies the original DF model in the case of a single time-scale and provides some interesting insights into future work.

As such, the self-appraisal process (10) aims to adaptively modify the status of individuals (assertion vs. reconciliation, confidence vs. uncertainty) in response to their absolute power over prior issue outcomes. In order to provide a deeper insight into the reflected self-appraisal process, the next subsection is therefore dedicated to studying the dynamical and transient characterization of individual social powers in a cooperative-competitive context.

3.2. Dynamical and Graphical Description of Social Power

According to claim (ii) in Lemma 1, the stack vector form of social power is given by

$$\boldsymbol{p}(s) = \left(\lim_{t\to\infty} \left(\boldsymbol{P}^t(s)\right)\right)^{\mathsf{T}} \boldsymbol{\rho}(s)/n, \tag{11}$$

by which the self-regulation process of influence network using appraisal mechanism, as shown in (7) and (10), leads to the evolution of social power along the sequence of issue.

Slightly different from the notion of social power arising in the cooperative context [30], coopetitive individuals' social power in (11) may appear identical magnitudes but with distinct signs according to Lemma 1. As such, the social power admits an orientation system such that the relative control exerting along the forward direction leads to a positive effect on discussions, while a negative influence along the backward direction, as addressed, e.g., in [32]. Sometimes, the sign pattern of social power could be more significant than their exact values in practical scenarios. For instance, media industries including traditional mass media, e.g., TV, radio, and newspaper, and the recently emerged socio-technical platforms, e.g., blogs, Facebook and Twitter, are of fundamental importance in information distribution. On some occasions, egoistic media with high audience rating may release misleading reports for political or commercial reasons, and manage to persuade people to believe a perceived but false truth. The social

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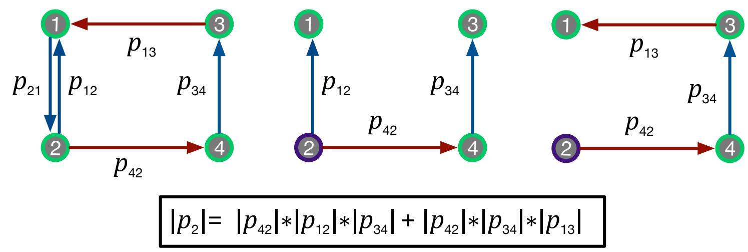

Figure 1: An illustrative calculation of social power in an SC and SB signed graph: two spanning trees rooted from node 2.

power entraining negative influence accounts for this phenomenon. An example is the shifting of U.S. public attitudes from “unjustified” to “justified” on the 2003 invasion of Iraq after Powel’s speech [33]. From this point of view, the dominant right eigenvector $\boldsymbol{\rho}$ is as informative as the dominant left eigenvector $\boldsymbol{p}$. Moreover, note that the oriented effect of social power is argued on a relative scale; that is, the positiveness and negativeness of individual power are defined in a relative coordinate but do not imply its absolute direction. For instance, although Powel’s speech fulfills a passive function on shaping the public opinion of the Iraq invasion, it plays an active role from the perspective of a few politicians.

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Next, we characterize the transient properties of the social power during each issue discussion $s \in \mathbb{I}$ by accommodating *Kirchhoff’s matrix tree theorem* [34] to the signed case.

Lemma 2. *On a given issue $s \in \mathbb{I}$, assume the associated graph $\mathcal{G}(\boldsymbol{P}(s))$ is QSC and SB. Let $|\boldsymbol{P}(s)| = \mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{P}(s)\,\mathsf{diag}(\boldsymbol{\rho}(s))$ where $\boldsymbol{\rho}(s)$ and $\boldsymbol{\rho}(s)$ are the dominant left- and right-eigenvector of matrix $\boldsymbol{P}(s)$. For each $i \in \mathbb{V}$, $\rho_i(s)p_i(s)$ is equal to the sum, over all spanning tree rooted at node i in $\mathcal{G}(|\boldsymbol{P}(s)|)$, of the products of weights of edges traversing each spanning tree.*

Lemma 2 provides several insights into the perception of individual social power. The first hint is that when the SB influence network $\mathcal{G}(\boldsymbol{P}(s))$ is SC for a given issue $s \in \mathbb{I}$, all individuals have a non-zero social power, since each node on $\mathcal{G}(|\boldsymbol{P}(s)|)$ has at least one spanning tree rooted at it. Especially, the absolute social power $|p_i(s)|$ for $s \in \mathbb{I}$ is equal to the sum of the products of absolute weights $|p_{ij}(s)|$ of all the spanning trees starting from i in the graph $\mathcal{G}(\boldsymbol{P}(s))$ without self-loop. A paradigm is provided in Figure 1. Notably, the positive and negative weighted interactions are treated equally without discrimination in the evaluation of social power, although the total effect of social power may be positive or negative. Moreover, the individual influence is usually not imposed via a single (direct) pathway, but through all available (indirect) paths reaching others. In reference to $\boldsymbol{p}(s) \in \mathbb{C}^n$, the social power $p_i(s)$ (up to sign) represents the ratio of the amount of spanning tree products that start from i to the total number of the spanning tree products in the influence network.

Lastly, we specify two distinct configurations of social power: *autocracy* and *democracy*. The former features the existence of a dictator-like individual who dominantly holds all the absolute social power and other members of the organization are dramatically vulnerable to the interpersonal influence. Instead, the democratic specification means the members of social networks equally involving in making the final decision.

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3.3. Explicit Formulation of Self-Appraisal Process over Signed Networks

Before ending this section, we explore an equivalent and explicit expression for the dynamics of self-appraisal process (11) in terms of the interpersonal appraisal network.

Proposition 1. *For each issue $s \in \mathbb{I}$, let the graph $\mathcal{G}(\boldsymbol{Q}(s))$ associated to the per-issue zero-diagonal matrix $\boldsymbol{Q}(s) \in \mathbb{R}^{n\times n}$ be SC and SB. The dynamics of self-appraisal* (10) *are equivalent to the following discrete-time system*

$$\boldsymbol{z}(s+1) = \boldsymbol{f}(\boldsymbol{z}, s), \tag{12}$$

where $f : \mathbb{S}^n \times \mathbb{I} \to \mathbb{S}^n$ is a smooth map defined by

$$\boldsymbol{f}(\boldsymbol{z}, s) = \theta(\boldsymbol{z}, s)\Big[\frac{\rho_1(s)q_1(s)}{1 - z_1(s)}, \cdots, \frac{\rho_n(s)q_n(s)}{1 - z_n(s)}\Big]^\mathsf{T} \tag{13}$$

where $\theta(\boldsymbol{z}, s) = 1/\sum_{i=1}^n \frac{\rho_i(s)q_i(s)}{1-z_i(s)}$ is a scaling factor, the vectors $\boldsymbol{q}(s) := [q_i] \in \mathbb{C}^n$ and $\boldsymbol{\rho}(s) := [\rho_i] \in \{\pm 1\}^n$ satisfy $\boldsymbol{q}^\mathsf{T}(s)\boldsymbol{Q}(s) = \boldsymbol{q}^\mathsf{T}(s)$ and $\boldsymbol{Q}(s)\boldsymbol{\rho}(s) = \boldsymbol{\rho}(s)$.

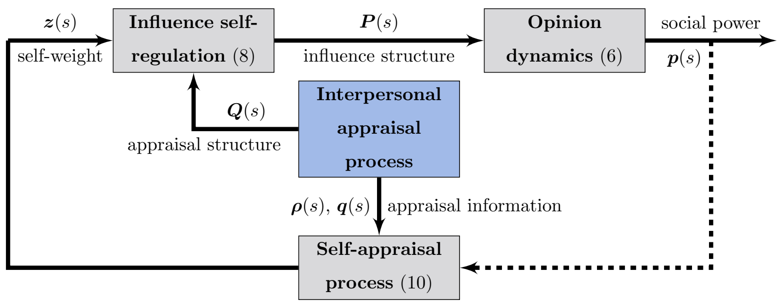

Figure 2: The schematic diagram of the coevolution of opinion dynamics and adaptive influence networks.

The result of Proposition 1 that $\boldsymbol{f}$ is continuous and smooth is useful for the convergence analysis of self-appraisal dynamics, as well as of the social power evolving process. Moreover, Proposition 1 implies that the dominant right- and left-eigenvector of the interpersonal appraisal matrix $\boldsymbol{Q}$ appears an important role in the modeling and analysis of the self-appraisal process and the true social power $\boldsymbol{p}(s)$ plays no direct role. As such, the dynamical evolution of the reflected appraisal process combining (8) and (12) completely depends on an interpersonal appraisal mechanism. In conjunction with the original model (10), Proposition 1 undertakes to study the evolution and the convergence properties of social power on coopetitive topologies combined with reflected appraisals. Figure 2 describes the design philosophy of the theoretical framework, in which the outputs of the interpersonal appraisal systems push forward the coevolutionary networks concerning with the feedback loop of influence structure and opinion dynamics.

4. Convergence Analysis

In this section, we study the theoretical analysis of the proposed self-appraisal model in coopetitive social networks.

4.1. Constant Interpersonal Appraisal with Structural Balance

In this subsection, we consider the invariant interpersonal appraisal structure $\boldsymbol{Q}$ along the issue sequence. That is to say, issue discussants stick to their

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original impression of others and thus the social ties are solid from issue to issue. The self-appraisal dynamics (12) therefore degenerate to a nonlinear autonomous system.

First, we consider the special case when the underlying graph $\mathcal{G}(\boldsymbol{Q})$ has a star topology.

Lemma 3. *Suppose that $n \geq 3$ and the digraph $\mathcal{G}(\boldsymbol{Q})$ is a SC and SB. Let $\boldsymbol{q}$ be the dominant left eigenvector of $\boldsymbol{Q}$ associated to eigenvalue 1. Then, the following statements hold*

(i) $|q_i| \leq 1/2$ for all $i \in \{1, \ldots, n\}$;

(ii) there exists a node i with $|q_i| = 1/2$ iff $\mathcal{G}(\boldsymbol{Q})$ is a star centered at node i. For any initial condition $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$, the self-weight vector $\boldsymbol{z}(s)$ governed by the dynamics (12) converges to $\boldsymbol{e}_i$ and the social power $\boldsymbol{p}(s)$ asymptotically reaches to $\rho_i \boldsymbol{e}_i$ according to (10).

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Lemma 3 features the predictable emergence of autocratic configuration in individuals' absolute social power when the graph $\mathcal{G}(\boldsymbol{Q})$ is with a star topology. More intuitively, social power tends to accumulate at the center node corresponding to the dictator-like individual, for almost every initial conditions except for the vertices of the simplex. Since then, the discussion outcomes of subsequent issues are determined exclusively by the initial attitudes of this autocrat actor.

Next, we provide the following theorem regarding the convergence and stability of the self-appraisal mechanism (12) with a non-star graph $\mathcal{G}(\boldsymbol{Q})$.

Theorem 2. *For $n \geq 3$, consider the interpersonal appraisal graph $\mathcal{G}(\boldsymbol{Q})$ of a non-star topology. If $\mathcal{G}(\boldsymbol{Q})$ is SC and SB, then the following claims hold for the self-appraisal system (12):*

(i) Fixed points: the set of equilibrium points of $\boldsymbol{f}$ is $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \boldsymbol{z}^*\}$, where $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$;

(ii) Convergence: for any non-autocratic initial conditions $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$, the self-weight vector $\boldsymbol{z}(s)$ converges exponentially to the equilibrium configu-

ration $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$ *and thus the social power* $\boldsymbol{p}(s)$ *converges exponentially to* $\mathsf{diag}(\boldsymbol{\rho})\boldsymbol{z}^* \in \mathsf{int}(\mathbb{C}^n)$ *as* $s \to \infty$*;*

(iii) Stability: the fixed point in the interior of simplex is the unique stable equilibrium for dynamics (12) *in* $\mathbb{S}^n$*;*

(iv) Democracy: if $\mathcal{G}(\boldsymbol{Q})$ *is further balanced, then for all* $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$*,* $\boldsymbol{z}(s)$ *converges exponentially to the democratic configuration* $\mathbf{1}/n$ *as* $s \to \infty$*.*

The results of Theorem 2 provide several important indications. First, the convergence and stability analysis benefits substantially from the aid of the differential Lyapunov framework. The differential structure results in the exponential convergence of nonlinear dynamics in question rather than asymptotic results explored in [11, 9]. The convergence property of contractive systems, which is independent of initial conditions, gives that individuals exponentially forget their original self-appraisal of relative control along the issue sequence. Put differently, sequential issue discussion combined with the reflected appraisal mechanism eliminates the initial perception of social power and the true social influence only depends on the interpersonal appraisal network.

Moreover, Theorem 2 implies that the graph eigenvector centrality $|\boldsymbol{q}|$ for the graph $\mathcal{G}(\boldsymbol{Q})$ gives some ranking implications for the configuration of individual social power at the equilibrium point: $z_i^* < z_j^*$ $(|p_i^*| < |p_i^*|)$ iff $|q_i| < |q_j|$ for any pair of i and j, and $z_i^* = z_j^*$ $(|p_i^*| = |p_j^*|)$ iff $|q_i| = |q_j|$. Recalling the proof of Theorem 2, the set $\mathbb{A}^n = \{\boldsymbol{z} \in \mathbb{S}^n | 0 \leq z_i \leq 1 - r\} \subset \nabla\mathbb{S}$ with $r \leq \min_{i\in\mathbb{V}} \frac{1-2\rho_i q_i}{1-\rho_i q_i}$ is forward $\boldsymbol{f}$-invariant and the stable equilibrium point $\boldsymbol{z}^*$ exists in this contraction region. Therefore, it is reasonable to obtain an upper bound for the absolute social power of individuals at the equilibrium point, i.e., $0 < |p_i^*| < |q_i|/(1 - |q_i|)$, by which a smaller $|q_i|$ gives a tighter upper bound for $|p_i^*|$. In the meanwhile, a threshold value $q_{\text{threshold}} = 1/3$ can be attained such that if $|q_i| < q_{\text{threshold}}$ for all $i \in \mathbb{V}$, then $|p_i^*| < 1/2$ for all $i \in \mathbb{V}$. In other words, there is no such agent who holds more than half of the total absolute power after each issue discussion.

Regarding the proof of Lemma 3, a Lyapunov-based method is applied to

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conduct the convergence analysis of self-appraisal dynamics when $\mathcal{G}(\boldsymbol{Q})$ is a star graph. The implicit prediction or prior knowledge of the equilibrium point at the autocratic state allows for the applicability of Lyapunov theory. However, the first challenge encountered in using the Lyapunov methodology to non-star networks is that the explicit calculation of the equilibrium may be an intractable task due to the nonlinear nature of dynamics. The customized remedy in the proof of Theorem 2 does not directly seek to find a Lyapunov-based metric on the state space. Instead, a differential framework by lifting the Lyapunov function to the tangent bundle is employed to investigate the contraction of infinitesimal dynamics, thus establishing the equilibrium-independent convergence of the self-appraisal dynamics.

Finally, we note that the topological interpretation of social power in Section 3.2 enables us to extend the results of Theorem 2 to the case that $\mathcal{G}(\boldsymbol{Q})$ is not SC but with multiple root nodes. On this occasion, only individuals possessing spanning trees could exert their social powers on the issue discussions. More importantly, the individuals corresponding to non-root nodes of $\mathcal{G}(\boldsymbol{Q})$ belong to the vulnerable groups in social activities. Since such individuals have few network-scale interpersonal relationships, sequential discussion together with reflected appraisal mechanism removes their social influence, even they are initially empowered the supreme power in an autocratic configuration, i.e., $p_i(s) \to 0$ as $s \to \infty$ when $|\boldsymbol{p}(0)| = \boldsymbol{e}_i$ and i has no spanning tree on $\mathcal{G}(\boldsymbol{Q})$. We omit the detailed extension to save the risk of overlap.

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4.2. Extension to Dynamic Interpersonal Appraisal Structure

In this subsection, we begin to examine the convergence behavior of the proposed self-appraisal mechanism in a general context in which the interpersonal appraisal structure does not remain unchanged along the issue sequence.

The paradigm shift from a static appraisal structure to a dynamic appraisal network makes the self-appraisal dynamics (12) become a nonlinear nonautonomous system. To clarify the presentation, the following set encapsulates all interpersonal appraisal matrices under consideration

$$\mathbb{Q} := \{\boldsymbol{Q} \in \mathbb{R}^{n\times n} | \text{ the non-star graph } \mathcal{G}(\boldsymbol{Q}) \text{ is SC and SB}\},$$

where we assume $\mathbb{Q}$ is a finite set for theoretical rigor.

Theorem 3. *For $n \geq 3$, consider the self-appraisal system* (12) *on $\mathbb{S}^n$ and the interpersonal appraisal matrix $\boldsymbol{Q}(s) \in \mathbb{Q}$ at issue $s \in \mathbb{I}$ wherein $\boldsymbol{Q}(s)$ is independent of $\boldsymbol{z}(s)$. For any $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$, the self-weight vector $\boldsymbol{z}(s) \in \mathbb{S}^n$ governed by the nonlinear map $\boldsymbol{f}$ in* (13) *converges exponentially to a steady-state trajectory $\boldsymbol{z}^*(s) \in \mathsf{int}(\mathbb{S}^n)$. The network-scale social power $\boldsymbol{p}(s) \in \mathbb{C}^n$ converges exponentially to the trajectory $\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}^*(s) \in \mathsf{int}(\mathbb{C}^n)$ as $s \to \infty$.*

As illustrated in Figure 2, the left- $\boldsymbol{q}(s)$ and right-eigenvector $\boldsymbol{\rho}(s)$ can be treated as external inputs for the self-appraisal dynamics (12), which encode the topologically structural information of graph $\mathcal{G}(\boldsymbol{Q}(s))$ on issue $s \in \mathbb{I}$. Therefore, the steady-state solution $\boldsymbol{z}^*(s)$, in some sense, is specified implicitly by the interpersonal appraisal mechanism.

From Theorem 2, the equilibrium ordering of self-weight/social-power for the constant $\boldsymbol{Q}$ is related to the dominant eigenvectors $\boldsymbol{q}$ and $\boldsymbol{\rho}$ rather than waiting until the end of the issue sequence. In the case of issue-varying $\boldsymbol{Q}(s)$, the self-appraisal system exhibits, however, the non-equilibrium asymptotic behavior because of the dynamic interpersonal appraisal mechanism. In fact, this interrelated correspondence usually fails to preserve for the non-constant $\boldsymbol{Q}(s)$. Nevertheless, the self-weights in some special scenarios may coincidentally reach a stationary fixed point. Let $\sim_e$ be an equivalence relation in the set $\mathbb{Q}$ such that for arbitrary $\boldsymbol{Q}_1, \boldsymbol{Q}_2 \in \mathbb{Q}$, it holds $\boldsymbol{Q}_1 \sim_e \boldsymbol{Q}_2$ if $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ share the same left-eigenvector $\boldsymbol{q}$ and right-eigenvector $\boldsymbol{\rho}$. Such relation enables us to define an equivalent class $[\boldsymbol{Q}]_{\sim_e}$ of matrix $\boldsymbol{Q} \in \mathbb{Q}$.

Corollary 1. *For arbitrary initial appraisal matrix $\boldsymbol{Q}(0) \in \mathbb{Q}$ which has the left eigenvector $\boldsymbol{q}(0)$ and right eigenvector $\boldsymbol{\rho}(0)$ associated to eigenvalue* 1*, if it holds $\boldsymbol{Q}(s) \in [\boldsymbol{Q}(0)]_{\sim_e}$ over the sequence of issue $\mathbb{I}$, then the self-weight vector $\boldsymbol{z}(s)$ governed by the vector field* (13) *converges exponentially to a static equilibrium*

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point $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$ *and the social power* $\boldsymbol{p}(s)$ *has an equilibrium configuration* $\boldsymbol{p}^* \in \mathsf{int}(\mathbb{C}^n)$*. Moreover, the equilibrium self-weights (social powers) of individuals satisfies:* $z_i^* < z_j^*$ *(*$|p_i^*| < |p_i^*|$*) iff* $|q_i(0)| < |q_j(0)|$ *for any pair of* i *and* j*, and* $z_i^* = z_j^*$ *(*$|p_i^*| = |p_j^*|$*) iff* $|q_i(0)| = |q_j(0)|$.

The proof can be easily derived from Theorem 3 and therefore is omitted in this article. Although the developed results of Corollary 1 provide preferable ranking implication for the equilibrium social power, we have to admit such static configuration of relative control is a rare case, while self-appraisal mechanism of social power exposes mostly the stationary non-equilibrium dynamical behavior.

5. Structural Unbalanced Interpersonal Appraisal Mechanism

Until now, we usually postulate explicitly or implicitly the structural balance for the interpersonal appraisal structure in the study of opinion dynamics and reflected appraisal mechanism. This condition, however, may not always be satisfied in many real-life social networks [23]. For example, large-scale online social networks typically have complex and multidimensional appraisal structures. Hence, the interpersonal appraisal graphs arising from such networks hardly satisfy the structural balance condition. So in what follows, we investigate the evolution of social power with structurally unbalanced interpersonal appraisal mechanism.

For any issue $s \in \mathbb{I}$, the opinion forming process on $\mathcal{G}(\boldsymbol{P}(s))$ which is SC and structurally unbalanced, tends towards neutrality no matter what individuals' initial ideas are, i.e., $\lim 1_{t\to\infty}\boldsymbol{x}(s,t) = \mathbf{0}$, for any $\boldsymbol{x}(s,0) \in \mathbb{R}^n$. In reference to (11), social power for opinion neutrality leads to $\boldsymbol{p}(s) = \mathbf{0}$ as $\lim_{t\to\infty} \boldsymbol{P}^t(s) = \mathbf{0}$. Note that the magnitudes of all eigenvalues of $\boldsymbol{P}(s)$ are strictly smaller than 1. In other words, the neutral opinion dynamics represent that all individuals make no direct contribution to the issue discussion and hence, the self-weights are accordingly set to zero for everyone. An intuitive interpretation for this situation would be that all individuals have no desire for power.

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Theorem 4. *For $n \geq 3$, consider the self-appraisal system* (12) *and* (10)*. Assume the associated graph $\mathcal{G}(\boldsymbol{Q})$ of the constant interpersonal appraisal matrix $\boldsymbol{Q}$ is aperiodic, strongly connected, and structurally unbalanced. The map $\boldsymbol{f}$ on $\mathbb{S}^n \cup \{\mathbf{0}\}$ in* (12) *is then defined by*

$$\boldsymbol{f}(\boldsymbol{z}) = \begin{cases} \boldsymbol{e}_i & \text{if } \boldsymbol{z}(s) = \boldsymbol{e}_i, \text{ for all } i \in \mathbb{V}, \\ \mathbf{0} & \text{if } \boldsymbol{z} \in \nabla\mathbb{S}^n \cup \{\mathbf{0}\}. \end{cases}$$

Furthermore, the equilibrium points of $\boldsymbol{f}$ belong to $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \mathbf{0}\}$. For arbitrary initial condition $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n \cup \{\mathbf{0}\}$, the self-weight vector $\boldsymbol{z}(s)$ (social power vector $\boldsymbol{p}(s)$) are constantly $\mathbf{0}$ for $s \in \{1, 2, \ldots\}$.

Regarding Theorem 4, there are several interesting consequences. First, the non-autocratic configuration of initial conditions gives rise to that individuals do not take sides on any issue. Second, the proof of Theorem 4 shows that the presence of autocratic social power (e.g., $\boldsymbol{z}(0) = \boldsymbol{e}_i$) can generate different collective behavior in opinion forming including a consensus outcome, two opposite settled opinions, or a set of unreconciled views. This finding may open up avenues for driving the occurrence of clustering in human populations [35] using self-perception of their relative control.

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6. Numerical Simulation and Further Discussion

This section serves to demonstrate the proposed mathematical model and the theoretical analysis via numerical tests.

6.1. Self-appraisal Dynamics with Constant $\boldsymbol{Q}$

Thurman's informal network of interpersonal ties among 15 staffs in the office of an oversea branch of an international corporation is reported in [36]. To fit our work, we make a slight modification such that the antagonism is also involved as shown in Figure 3, such that {Ann, Tina, Katy, Lisa, Pete, Amy} and {Presendent, Rose, Mary, Mike, Emma, Peg, Minna, Andy, Bill} are two hostile cliques. The associated interpersonal appraisal matrix is given by $\boldsymbol{Q}_1$ associated with a SC and SB graph $\mathcal{G}(\boldsymbol{Q}_1)$, which has a dominant right-eigenvector

$$\boldsymbol{\rho} = [1, 1, -1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1]^\mathsf{T}$$

and a dominant left-eigenvector

$$\boldsymbol{q} = [0.027, 0.026, -0.106, 0.027, 0.164, 0.111, -0.148, -0.066, \dots, \\ 0.027, 0.027, -0.14, -0.048, 0.014, 0.009, -0.058]^\mathsf{T}.$$

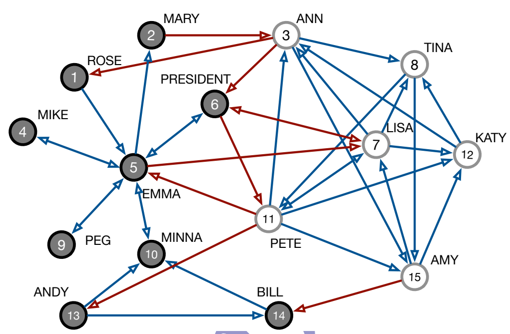

Figure 3: The signed Thurman's informal network: The cooperation interrelation is drawn in blue arrow line and competition is in red.

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We numerically study the proposed self-appraisal framework on this modified Thurman's social network. For illustrative purpose, we conduct the simulation in a Monte-Carlo (MC) trial of 200 initial conditions. The dynamical trajectories of the self-appraisals of individuals are illustrated in Figure 4(a) which shows the self-weights $\boldsymbol{z}(s)$ converge exponentially to an equilibrium point, independent of initially perceived appraisals $\boldsymbol{z}(0)$. Especially, all self-weights strictly belong to the domain $]0, 1[$, evidencing that the equilibrium self-confidence lies in the interior of the simplex $\mathbb{S}_n$. In other words, individual social powers $\boldsymbol{p}(s)$ in (10) exponentially forget their initial configuration $\boldsymbol{p}(0)$ as a consequence of sequential opinion discussion combined with the reflected-appraisal mechanism. The numerical test is consistent with the statements in Theorem 2.

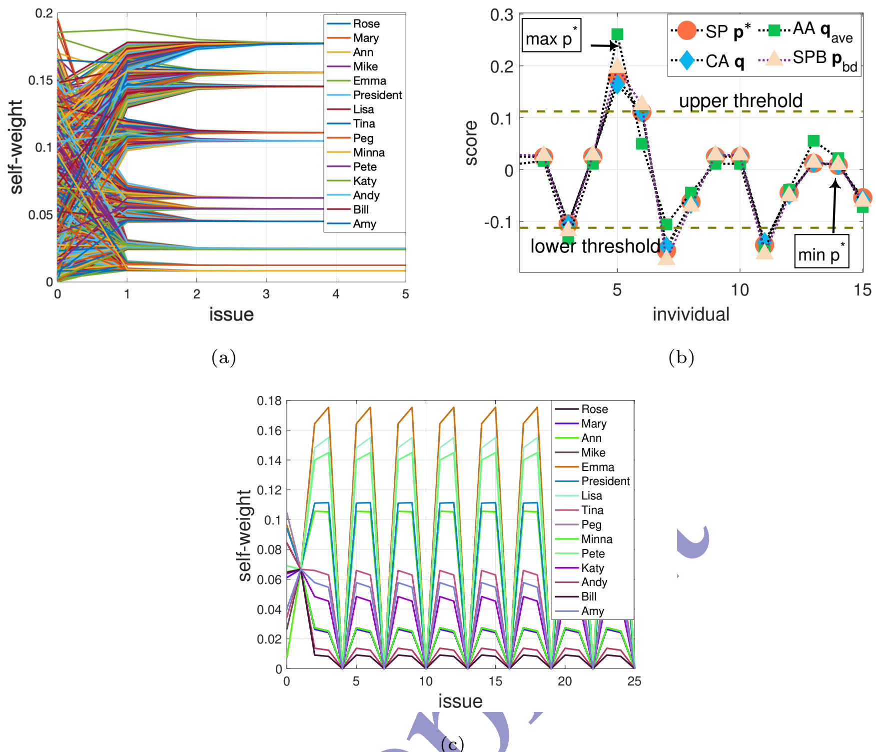

Figure 4: Monte Carlo trial of self-weight dynamics: (a). Static case; (b). Comparison of evaluation metrics: social power (SP), consensual appraisal (CA), average appraisal (AA) and social power bound (SPB); (c). Dynamic case.

6.2. Comparisons of Social Power Metrics

Next, we compare different metrics of individual social powers. First, we introduce two additional measures of individual relative influence. Since each column of $\boldsymbol{Q}$ collects others' assessments of the corresponding individual, the product $\sum_{j=1}^{n} q_j q_{ij}$ can be treated as the collective appraisal of others on the individual $i \in \mathbb{V}$. Thereby, one can also refer to the eigenvector centrality $\boldsymbol{q}$ as the consensual appraisal of individuals based on the fact $q_i = \sum_{j=1}^{n} q_j q_{ij}$. Another vector-based index arising from this context is an average interpersonal appraisal $\boldsymbol{q}_{\text{ave}} := \boldsymbol{Q}^{\mathsf{T}} \boldsymbol{\rho}/n$. Those influence metrics provide profound implications to the equilibrium ranking of individual social power.

To this end, Figure 4(b) shows the comparison results among different social power metrics, where $\boldsymbol{p}^*$ is the steady-state social power, $\boldsymbol{q}$ is the consensual appraisal, $\boldsymbol{q}_{\text{ave}} = \boldsymbol{Q}_1^{\top}\boldsymbol{\rho}/n$ is the average interpersonal appraisal, and $\boldsymbol{p}_{\text{bd}} = [\frac{q_i}{1-\rho_i q_i}]$ is the social power bound for $i \in \mathbb{V}$. The first observation is that Emma who has the most spanning tree starting from her, reaches the maximal personal influence; While Bill who has the least spanning tree, lies at the lowest power layer in the office. This demonstrates Lemma 2. Second, as discussed in Section 4.1, the absolute social power of individuals is strictly upper bounded by $\boldsymbol{p}_{\text{bd}}$, i.e., $|p_i^*| < \frac{|q_i|}{1-|q_i|}$ for all $i \in \mathbb{V}$. Moreover, although those influence metrics (final social power $\boldsymbol{p}^*$, consensual appraisal $\boldsymbol{q}$, average appraisal $\boldsymbol{q}_{\text{ave}}$) are different in the exact value, they share the same ordering of the importance ranking, that is $|p_i^*| > |p_i^*|$ iff $|q_i| > |q_j|$ ($|q_{\text{ave}}^i| > |q_{\text{ave}}^j|$) for $i, j \in \mathbb{V}$. Moreover, since all eigenvector centralities are lower than 1/3 in modulus, as discussed in Section 4.1, nobody in the office possesses more than half of the total social power at equilibrium. Even though there is no predominant actor in this organization, the experiment retains the "iron law of oligarchy" in sociological study [37]. By giving a threshold $1 - \theta(\boldsymbol{z}^*) = 0.1123$ at the equilibrium, we can observe from Figure 4(b) that social powers are accumulated in the individuals with $|q_i| > 0.1123$. More specifically, an oligarchic hierarchy is formed by Emma, Pete and Lisa, since their social power satisfies $|p_i^*| > |q_i|$.

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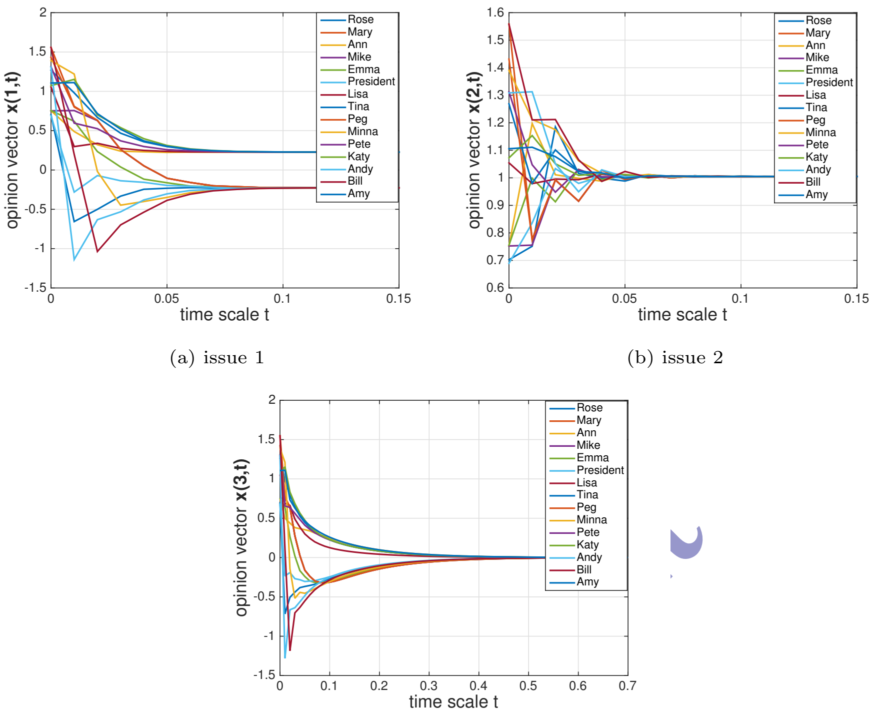


(a) issue 1

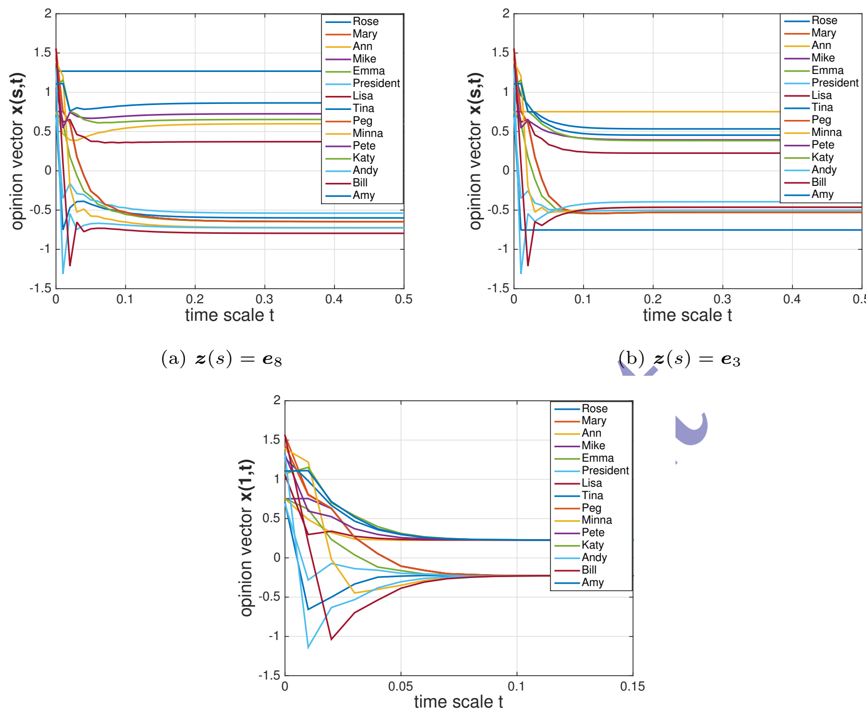
(b) issue 2

(c) issue 3

Figure 5: Evolution of opinion dynamics on first three issues: (a). Opinion polarity; (b). Opinion consensus; (c). Opinion neutrality.

6.3. Self-appraisal Dynamics with Dynamic Appraisal Topology

Next, we study the self-appraisal system (13) on a dynamic appraisal network. Let $\mathcal{G}(\boldsymbol{Q}_2)$ be the graph by converting all edges with negative weights of graph $\mathcal{G}(\boldsymbol{Q}_1)$ into the positively weighted edges. Obviously, the graph $\mathcal{G}(\boldsymbol{Q}_2)$ is unsigned and SC. By converting the cooperative link $(7, 5)$ in graph $\mathcal{G}(\boldsymbol{Q}_1)$ to an antagonistic one, we can explore a SC and structurally unbalanced graph $\mathcal{G}(\boldsymbol{Q}_3)$. Then, we implement the self-appraisal dynamics on a periodically switching appraisal networks $\{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3\}$, the Monte Carlo trial in Figure 4(c) exposes that the self-weights asymptotically fall into an attractor system which relies on the setup of the interpersonal appraisal mechanism, independent of the setup of the initially perceived states $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$. Note that it is generally difficult to draw any conclusion on the ordering of social power in the case of dynamic topology, since there does not exist a static equilibrium point.



(a) $\boldsymbol{z}(s) = \boldsymbol{e}_8$ (b) $\boldsymbol{z}(s) = \boldsymbol{e}_3$

(c) $\boldsymbol{z}(s) = \boldsymbol{e}_7$

Figure 6: Evolution of opinion dynamics with autocratic social power: (a). Opinion clustering; (b). Opinion separation; (c). Opinion polarization.

In addition, we examine the evolution of opinion dynamics over sequential issue discussion and the periodically switching appraisal structure $\{\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3\}$. Figure 5 presents the opinion formation on the first three issues, exhibiting polarization in Figure 5(a), consensus in Figure 5(b), and neutrality in Figure 5(c). Using the interpersonal appraisal matrix $\boldsymbol{Q}_3$, we also study the forming process of opinions under autocratic configuration of social power. Allocation of autocratic power at a specific individual results in community cleavages of opinion on issues as shown in Figure 6. What is intriguing is the case $\boldsymbol{z}(s) = \boldsymbol{e}_3$ depicted in Figure 6(b) in which the opinions of Ann and Rose polarize at the exact opposite values and the opinions of all other members in the office lie in between these two polarized values. The perception of social power $\boldsymbol{z}(s) = \boldsymbol{e}_7$ gives rise to that the attitudes of the entire office evolve into two polarized camps.

Finally, we modify the graph $\mathcal{G}(\boldsymbol{Q}_1)$ in Figure 3 by breaking up the existing links $(4,5)$, $(10,14)$, $(10,13)$ and building up positively weighted links $(14,10)$, $(13,10)$ such that there is no spanning tree starting from the members {Mike, Andy, Bill}. Therefore, let $\mathcal{G}(\boldsymbol{Q}_4)$ present the resulting graph which is QSC but still SB. The associated interpersonal appraisal matrix $\boldsymbol{Q}_4$ from $\mathcal{G}(\boldsymbol{Q}_4)$ gives a dominant right-eigenvector by

$$\boldsymbol{\rho} = [1,\ 1,\ -1,\ 1,\ 1,\ 1,\ -1,\ -1,\ 1,\ 1,\ -1,\ -1,\ 1,\ 1,\ -1]^{\mathsf{T}},$$

and a unique dominant left-eigenvector

$$\boldsymbol{q} = [0.0353,\ 0.0286,\ -0.1143,\ 0,\ 0.1767,\ 0.1187,\ -0.1514,\ -0.0637, \ldots$$
$$0.0353,\ 0.0353,\ -0.1365,\ -0.0498,\ 0,\ 0,\ -0.0545]^{\mathsf{T}}.$$

Then, the dynamical trajectories of the self-weights with 200 randomly initial conditions are illustrated in Figure 7(a). Likewise, individuals exponentially forget their originally perceived social influence. In analogy with the strongly connected case, Emma still has the maximum equilibrium social power and the statements on the ordering of the social power ranking at equilibrium point hold, as illustrated in Figure 7(b). The observation that Mike, Andy and Bill have zero social power in the equilibrium configuration verifies that statement that the non-root individuals lose social power in the limit.

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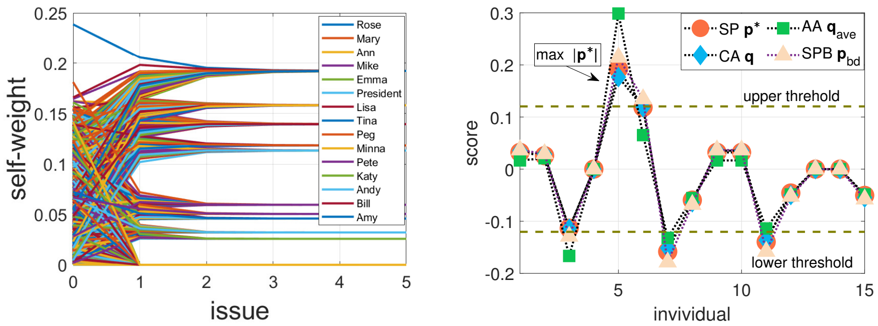

(a) (b)

Figure 7: Self-appraisal dynamics on the QSC and SB graph $\mathcal{G}(\boldsymbol{Q}_4)$: (a). Monte Carlo trial of 200 initial conditions; (b). Comparsion of different evaluation metrics.

6.4. Numerical Tests on The Sampson's Monastery Network

The last test is on a real signed network which is inferred from Sampson's dataset for monastery interactions [38]. The graph $\mathcal{G}(\boldsymbol{Q}_5)$ associated with the fourth time window of Sampson's empirical data on the interpersonal esteem of the monastery's relation is SB and SC; see Figure 8(a). The corresponding two dominant eigenvectors of $\boldsymbol{Q}_5$ are

$$\boldsymbol{\rho} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1]^{\mathsf{T}},$$

$$\boldsymbol{q} = [0.0667, 0.0778, 0.0333, 0.0778, 0.0333, 0.0556, 0.0667, 0.0444, 0.0444, \dots$$
$$0.0667, 0.0444, 0.0778, 0.0444, 0.0778, -0.0444, -0.0556, -0.0444, -0.0444]^{\mathsf{T}}.$$

Similarly, we implement the self-appraisal dynamics (12) in a Monte-Carol trial of 200 randomly initial conditions and the resultant trajectories are drawn in Figure 8(b) which shows that the individual self-appraisals converge to an equilibrium state in an exponential rate without dependence of initial conditions. That is to say, the social power at the equilibrium point is determined only by $\boldsymbol{\rho}$ and $\boldsymbol{q}$. Moreover, we also compare the different metrics of individual social power in this case. The results are shown in Figure 8. In addition to the observations that have been obtained in previous cases, note that some pairs of nodes share the same absolute social powers but with different signs, e.g., $p_6^* = 0.0553$ and $p_6^* = -0.0553$.

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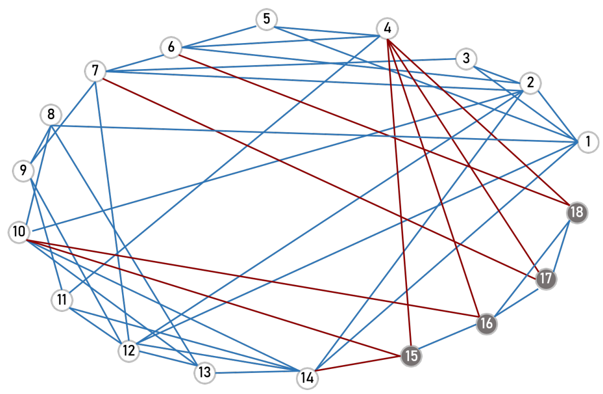
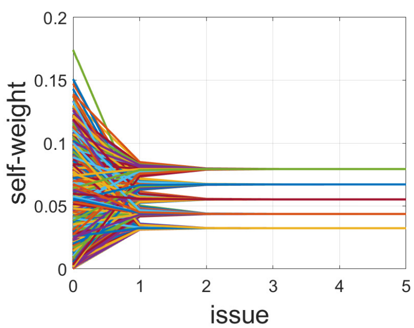
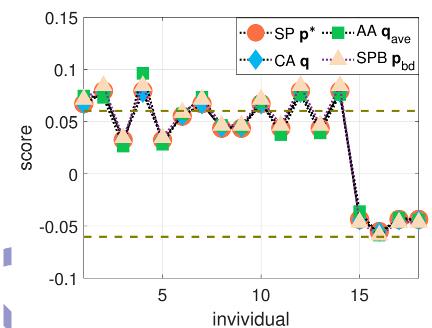


(a)



(b)



(c)

Figure 8: Numerical test on a real social network: (a). Sampson's monastery interaction network consists of two hostile campus: $\{1, 2, \ldots, 14\}$ and $\{15, \ldots, 18\}$; (b). The self-appraisal process; (c). Comparison of different social power metrics.

7. Conclusions and Future Works

In this article, we study the dynamic evolution of social power and self-appraisal in a coopetitive network that holds the opinion formation process along a sequence of issues. First, we explore an interpersonal appraisal mechanism in the configuration of influence relationships among individuals, such that the interpersonal influences are proportional to the interpersonal appraisals. This employment provides a natural, plausible explanation for the emergence of positiveness and negativeness in social interrelations. We also present the algebraic definition and graph-theoretical properties for the social powers of signed networks. The dynamic evolution of social power is then studied by means of the reflected self-appraisal process across the issue sequence. Regarding the theoretical analysis, the accommodation of differential Lyapunov theorem establishes the exponential convergence of the self-appraisal system and the associated dynamics of social power. In more detail, we show that individuals gradually forget their initial perception of relative importance in the networks as the issue sequence enlarges. Individual social powers in the limit therefore rely only on the topological properties of the interpersonal appraisal structure. For better applicability, we also examine the theoretical framework under the consideration of networks with different topological hypotheses. The numerical illustrations confirm the specifications of the theoretical results in this article.

There is much further work remaining to be done along this research line. First, we have not addressed the specific dynamics of the interpersonal appraisal mechanism throughout the article. As presented in Figure 2, we study the coevolution framework as a whole from an open-loop context, whereby the appraisal matrix and its associated dominant spectral properties are regarded as an exogenous signal. Therefore, the mathematical descriptions given in [28, 29] which explain how an appraisal network evolves, are likely to incorporate in a closed-loop sense with our framework. Furthermore, the self-appraisal process of individual importance in sequential opinion-forming is implemented in a centralized manner. From the viewpoint of network systems, we aim to develop the rigorously mathematical methodology of the reflected appraisal mechanism in a distributed way. Potential ideas involve distributed computation of objective eigenvectors [39] and other distributed learning approaches [40]. In support of the obtained models and theoretical results, we need more empirical tests on signed networks inferred from the well-known sociology datasets, e.g., in [41].

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Appendix A. Proofs

Appendix A.1. Proof of Theorem 1

As demonstrated by [42, Theorem 15], contraction analysis is equivalent to uniform exponential incremental stability. The proof closely follows the continuous-time version found in [24, Theorem 1], which can mimic the discrete-time counterpart of incremental Lyapunov stability criterion given by [42, Theorem 9] and contraction analysis shown by [42, Theorem 11] The details are omitted due to space limitations.

Appendix A.2. Proof of Lemma 1

To make this treatment self-contained, the proof is not presented in the same order as the claims appearing in the lemma. For a given issue $s \in \mathbb{I}$, the argument s of vectors or matrices in question will be dropped for the simplicity of this proof.

First, we note that when $\boldsymbol{z} = \boldsymbol{e}_i$ for some $i \in \mathbb{V}$ or more concretely, let $i = n$ without loss of generality, the matrix $\boldsymbol{P}$ can be calculated by

$$\boldsymbol{P} = \mathsf{diag}(0, \ldots, 0, 1) + \mathsf{diag}(1, \ldots, 1, 0)\boldsymbol{Q} = \begin{bmatrix} \tilde{\boldsymbol{Q}} \\ \boldsymbol{e}_n^\mathsf{T} \end{bmatrix} \tag{A.1}$$

where the matrix $\tilde{\boldsymbol{Q}} \in \mathbb{R}^{(n-1)\times n}$ is obtained by removing the n-th row from $\boldsymbol{Q}$. As $\mathcal{G}(\boldsymbol{Q})$ is SC, each node of $\mathcal{G}(\boldsymbol{Q})$ has paths connecting to others. The observation that the associated graph of $\boldsymbol{P}$ given in (A.1) has no other root except for node n, implies $\mathcal{G}(\boldsymbol{P})$ is only QSC in the case $\boldsymbol{z} = \boldsymbol{e}_i$. Especially, simple calculation shows $\boldsymbol{p} = \rho_i \boldsymbol{e}_i$ and the proof of statement (v) is complete. Here, $\mathcal{G}(\boldsymbol{P})$ can be obtained from $\mathcal{G}(\boldsymbol{Q})$ by removing all incoming edges of node i and thus inherits the same structural balance from $\mathcal{G}(\boldsymbol{Q})$.

Regarding claim (vi), condition $\boldsymbol{z} \in \nabla\mathbb{S}^n$ implies the graph $\mathcal{G}(\boldsymbol{P} - \mathsf{diag}(\boldsymbol{z}))$ has the same sign pattern on the edge set as $\mathcal{G}(\boldsymbol{Q})$'s, namely, $\mathcal{G}(\boldsymbol{P} - \mathsf{diag}(\boldsymbol{z}))$ is SC and SB. As the self-appraisals is $z_i \in [0, 1)$ for all $i \in \{1, \dots, n\}$, the graph $\mathcal{G}(\boldsymbol{P})$ is also SC and SB. The statements (iv) and the first part of statement (vi) are proved.

Next, the structural balance of $\mathcal{G}(\boldsymbol{P})$ implies the node set can be split into two disjoint subsets that the negatively weighted edges only exist between nodes belonging to distinct groups. By associating a vector $\boldsymbol{\rho} \in \{\pm 1\}^n$ to graph $\mathcal{G}(\boldsymbol{P})$, an observed relation is $p_{ij} = |p_{ij}|\rho_i\rho_j$ for all $i, j \in \mathbb{V}$, and thus one has $\mathsf{diag}(\boldsymbol{\rho})\boldsymbol{P}\,\mathsf{diag}(\boldsymbol{\rho}) = |\boldsymbol{P}|$. It is not difficult to show that $|\boldsymbol{P}|$ is a non-negative matrix and is row-stochastic. Thanks to the similar transformation, $\boldsymbol{P}$ and $|\boldsymbol{P}|$ share the same spectrum. Therefore, the application of the Perron-Frobenius theorem to $|\boldsymbol{P}|$ shows indirectly the existence, uniqueness, and other properties of the dominant left eigenvector $\boldsymbol{p}$ of matrix $\boldsymbol{P}$, as well as the spectral property. This is the proof of statement (i). Note that $\mathsf{diag}(\rho)\boldsymbol{p}$ is the left-eigenvector associated to eigenvalue 1 of $|\boldsymbol{P}|$, i.e., $\boldsymbol{p}^\mathsf{T}\,\mathsf{diag}(\rho)|\boldsymbol{P}| = \boldsymbol{p}^\mathsf{T}\,\mathsf{diag}(\rho)$. Therefore, if $\mathcal{G}(\boldsymbol{P})$ is SC, as is $\mathcal{G}(|\boldsymbol{P}|)$, then the matrix $|\boldsymbol{P}|$ admits a unique (up to a scaling) left eigenvector $\mathsf{diag}(\rho)\boldsymbol{p} > 0$. The second part of statement (vi) is proved.

Moreover, direct calculation shows that $\sum_j^n p_{ij}\rho_j = \sum_j^n |p_{ij}|\rho_i = \rho_i$, for all $i \in \mathbb{V}$, which implies the 1-norm of each row of $\boldsymbol{P}$ equals to 1, i.e., $\boldsymbol{P\rho} = \boldsymbol{\rho}$. The final asymptotic behavior $\lim_{t\to\infty} \boldsymbol{P}^t = \boldsymbol{\rho}\boldsymbol{p}^\mathsf{T}$ is an immediate consequence of claim (i), so the proof of claim (ii) is completed.

Finally, in regard to claim (iii) where $\boldsymbol{z}(s) = \boldsymbol{1}/n$, the relation (8), by left multiplying $\boldsymbol{\rho}^\mathsf{T}(s)/n$ to both sides, results in $\boldsymbol{\rho}^\mathsf{T}(s)\boldsymbol{P}(s)/n = \boldsymbol{\rho}^\mathsf{T}(s)\,\mathsf{diag}(\boldsymbol{1}_n/n)/n +$

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$(n-1)\boldsymbol{\rho}^{\mathsf{T}}(s)\boldsymbol{Q}(s)/n^2$, thus due to $\boldsymbol{p}(s) = \boldsymbol{\rho}(s)/n$, yielding

$$\boldsymbol{\rho}^{\mathsf{T}}(s)/n = \boldsymbol{\rho}^{\mathsf{T}}(s)\,\mathsf{diag}(\mathbf{1}_n/n)/n + (n-1)\boldsymbol{\rho}^{\mathsf{T}}(s)\boldsymbol{Q}(s)/n^2.$$

Therefore, the equation $\boldsymbol{\rho}_n^{\mathsf{T}}(s)\boldsymbol{Q}(s) = \boldsymbol{\rho}_n^{\mathsf{T}}(s)$ holds, i.e., $\mathcal{G}(\boldsymbol{Q})$ is balanced according to the definition. Meanwhile, if $\mathcal{G}(\boldsymbol{Q})$ is balanced, one can immediately prove that $\boldsymbol{\rho}^{\mathsf{T}}(s)\boldsymbol{P}(s)/n = \boldsymbol{\rho}^{\mathsf{T}}(s)/n$, i.e., $\boldsymbol{p}(s) = \boldsymbol{\rho}(s)/n$.

Appendix A.3. Proof of Lemma 2

Following the proof of Lemma 1, $\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{p}(s)$ is known to be the dominant left eigenvector of matrix $|\boldsymbol{P}(s)|$, from which one can form a Laplacian matrix by $\boldsymbol{L}(s) = \boldsymbol{I} - |\boldsymbol{P}(s)|$. It is obvious that $\mathcal{G}(\boldsymbol{L}(s))$ is QSC, provided that $\mathcal{G}(\boldsymbol{P}(s))$ is QSC and SB, so that $\dim\ker(\boldsymbol{L}^{\mathsf{T}}(s)) = 1$ and $(\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{p}(s))^{\mathsf{T}}\boldsymbol{L}(s) = \mathbf{0}_n$.

Let $\mathsf{cof}(\boldsymbol{L}(s))$ be the cofactor matrix associated to Laplacian $\boldsymbol{L}(s)$ where the (i,j)-th cofactor $[\mathsf{cof}(\boldsymbol{L}(s))]_{ij}$ of $\boldsymbol{L}(s)$ is equal to $(-1)^{i+j}b_{ij}(s)$ where $b_{ij}(s)$ is the determinant of the (i,j)-th minor of $\boldsymbol{L}(s)$. A well-known fact is that $\mathsf{cof}(\boldsymbol{L}(s)) \cdot \boldsymbol{L}^{\mathsf{T}}(s) = \det(\boldsymbol{L}(s))\boldsymbol{I}_n = \mathbf{0}_{n\times n}$. Since the sum of the rows of the Laplacian $\boldsymbol{L}(s)$ is zero for a given issue $s \in \mathbb{I}$, i.e., $\boldsymbol{L}(s)\mathbf{1}_n = \mathbf{0}_n$, the characteristics of the determinant function reveal that the entries of each column of $\mathsf{cof}(\boldsymbol{L}(s))$ are uniform. That is to say, $[\mathsf{cof}(\boldsymbol{L}(s))]_{ij}$ is independent of i and thus, one says $[\mathsf{cof}(\boldsymbol{L}(s))]_{ij} = \rho_j(s)p_j(s) \geq 0$ without loss of generality. Put differently, $\rho_i(s)p_i(s)$ is equal to the sum, over all spanning tree rooted at node i in $\mathcal{G}(\boldsymbol{L}(s))$, of the products of weights of edges traversing each tree according to Kirchhoff matrix tree theorem [43].

Appendix A.4. Proof of Proposition 1

According to statement (v) of Lemma 1, one has known that $\boldsymbol{z}(s+1) = \boldsymbol{e}_i$ if $\boldsymbol{z}(s) = \boldsymbol{e}_i$. For the self-weight vector $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n$ at issue $s \in \mathbb{I}$, an immediate deduction from the fact $\boldsymbol{P}(s)^{\mathsf{T}}\boldsymbol{p}^{\mathsf{T}}(s) = \boldsymbol{p}^{\mathsf{T}}(s)$ and (10) is

$$\boldsymbol{P}^{\mathsf{T}}(s)\,\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}(s+1) = \mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}(s+1).$$

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In conjunction with the forming of influence matrix given in (8), straightforward computation shows

$$\boldsymbol{Q}^{\mathsf{T}}(s)\,\mathsf{diag}(\mathbf{1}_n-\boldsymbol{z}(s))\,\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}(s+1)=\mathsf{diag}(\mathbf{1}_n-\boldsymbol{z}(s))\,\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}(s+1),$$

which means that $\mathsf{diag}(\mathbf{1}_n-\boldsymbol{z}(s))\,\mathsf{diag}(\boldsymbol{\rho}(s))\boldsymbol{z}(s+1)$ is a left eigenvector corresponding to eigenvalue 1 of $\boldsymbol{Q}(s)$. Bearing in mind $\boldsymbol{z}(s+1)\in\mathbb{S}^n$, one can acquire $\rho_i(s)(1-z_i(s))z_i(s+1)=\theta(\boldsymbol{z},s)q_i(s)$, for all $i\in\mathbb{V}$, wherein the scaling coefficient $\theta(\boldsymbol{z},s)=1/\sum_{i=1}^{n}\frac{\rho_i(s)q_i(s)}{1-z_i(s)}$ guarantees $\boldsymbol{z}(s+1)\in\mathbb{S}^n$.

About the smoothness of the map $\boldsymbol{f}:\mathbb{S}^n\times\mathbb{I}\to\mathbb{S}^n$, we first consider the self-appraisal process in a neighborhood of any vertex $\boldsymbol{e}_i$ which is given by $\mathbb{B}_i:=\{\boldsymbol{z}\in\mathbb{S}^n|\mathsf{d}_{\mathbb{S}}(\boldsymbol{z},\boldsymbol{e}_i)\le c,\boldsymbol{z}\neq\boldsymbol{e}_i\}$, where $c>0$ is a constant scalar. For any $\boldsymbol{z}(s)\in\mathbb{B}_i$, we can therefore rewrite the vector field into the following pattern

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{z},s)&=\Big[\frac{\theta(\boldsymbol{z})}{1-z_i(s)}\frac{\rho_1(s)q_1(s)(1-z_i(s))}{1-z_1(s)},\ldots,\frac{\theta(\boldsymbol{z})\rho_i(s)q_i(s)}{1-z_i(s)},\\&\qquad\qquad\ldots,\frac{\theta(\boldsymbol{z})}{1-z_i(s)}\frac{\rho_n(s)q_n(s)(1-z_i(s))}{1-z_n(s)}\Big]^{\mathsf{T}}\\&=\frac{1}{\sum_{j\neq i}^{n}\frac{\rho_jq_j(1-z_i)}{1-z_j}+\rho_iq_i}\Big[\frac{\rho_1q_1(1-z_i)}{1-z_1},\ldots,\rho_iq_i,\ldots,\frac{\rho_nq_n(1-z_i)}{1-z_n}\Big]^{\mathsf{T}},\end{aligned}\tag{A.2}$$

by which one can immediately attain $\boldsymbol{f}\to\boldsymbol{e}_i$ as $\boldsymbol{z}(s)\to\boldsymbol{e}_i$. Thus, the map $\boldsymbol{f}$ is continuous at vertices of the simplex. Namely, the continuity of $\boldsymbol{f}$ on $\mathbb{S}^n$ is clear due the analytic expression for $\boldsymbol{z}\in\mathbb{S}^n$.

Now, the remaining task is to prove the differentiability of $\boldsymbol{f}$ on simplex $\mathbb{S}^n$. For any $\boldsymbol{z}(s)\in\mathbb{B}_i$, entries of $\boldsymbol{z}(s)$ satisfy $z_j(s)<1$ for all $j\in\{1,,\cdots,n\}$ which allows for the computation of the Jacobian of the vector field $\boldsymbol{f}$ in this neighborhood B_i. That is to say, $\boldsymbol{f}$ is differentiable for any $\boldsymbol{z}(s)\in\nabla\mathbb{S}^n$ at issue $s\in\mathbb{I}$. Next, the Jacobian matrix of the vector field $\boldsymbol{f}$ at $\boldsymbol{e}_i$ can be calculated from (A.2) by

$$\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{z}}(\boldsymbol{z},s)=\begin{bmatrix}0&\cdots&-\frac{\rho_1(s)q_1(s)}{\rho_i(s)q_i(s)}&\cdots&0\\\vdots&&\vdots&&\vdots\\0&\cdots&\frac{1-\rho_i(s)q_i(s)}{\rho_i(s)q_i(s)}&\cdots&0\\\vdots&&\vdots&&\vdots\\0&\cdots&-\frac{\rho_n(s)q_n(s)}{\rho_i(s)q_i(s)}&\cdots&0\end{bmatrix}\tag{A.3}$$

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which implies that $\boldsymbol{f}$ is also differentiable at $\boldsymbol{e}_i$. Obviously, the first order partial derivative of $\boldsymbol{f}$ is continuous for $\boldsymbol{z}(s) \in \mathbb{S}^n$. The higher order differentiability of function $\boldsymbol{f}$ can be deduced in the same manner. Therefore, the smoothness of $\boldsymbol{f}$ is the immediate consequence of the differentiability for all orders and thus, the proof is completed.

Appendix A.5. Proof of Lemma 3

We first note that a signed graph with star topology is always SB and it can, in some sense, be treated as an unsigned graph under gauge transformation [20]. Hence, the proof of statement (i) and the first-half part of the statement (ii) closely follows the proof of [11, Lemma 2.3] The details are omitted due to space limitation.

The second-half part of the statement (ii) resembles the proof of [11, Lemma 3.2] by using $|q_{ij}| = \rho_i \rho_j qij$, which again using the SB nature of star graphs. Thus, we can demonstrate the nonexistence of equilibrium in $\nabla\mathbb{S}^n$ when graph $\mathcal{G}(\boldsymbol{Q})$ has a star topology. After assuming agent n be the center node of graph $\mathcal{G}(\boldsymbol{Q})$, without loss of generality, we summarize that $z_n(s+1) - z_n(s) > 0$ for $z_n(s) \in [0, 1[$ and $z_n(s+1) = z_n(s)$ when $z_n(s) = 1$. The specific details are omitted due to space limitations and can be found in [11, Lemma 3.2].

Consider a Lyapunov function candidate by $V(\boldsymbol{z}(s)) = \|\boldsymbol{z}(s) - \boldsymbol{e}_n\|_1/2$, for $\boldsymbol{z} \in \mathbb{S}^n$, which has the difference

$$V(\boldsymbol{z}(s+1)) - V(\boldsymbol{z}(s)) = z_n(s) - \frac{\theta(\boldsymbol{z}(s))\rho_n q_n}{1 - z_n(s)}, \tag{A.4}$$

where $\theta(\boldsymbol{z})$ is well defined and $\theta(\boldsymbol{z}) > 0$ for $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n$.

In the trivial case $z_n(s) = 0$, the difference of Lyapunov function (A.4) leads to $V(\boldsymbol{z}(s+1)) < V(\boldsymbol{z}(s))$. For the nontrivial situation $z_n(s) > 0$, the factor $\theta(\boldsymbol{z})$ has a lower bound as follows

$$\theta(\boldsymbol{z}) = \frac{1}{\frac{\rho_n q_n}{1 - z_n(s)} + \sum_{j=1}^{n-1} \frac{\rho_j q_j}{1 - z_j(s)}} \geq \frac{1}{\frac{\rho_n q_n}{1 - z_n(s)} + \frac{\rho_n q_n}{z_n(s)}} = \frac{z_n(1 - z_n)}{\rho_n q_n},$$

where the inequality is derived from $1 - z_j \geq z_n$ for $j \in \mathbb{V}\backslash\{n\}$ and $\rho_n q_n = \sum_{j=1}^{n-1} \rho_j q_j$. Additionally, we underline the lower bound is given in a strict

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sense. That is, in the case that there exists $k \in \mathbb{V}\backslash\{n\}$ such that $1 - z_k = z_n$, one can obtain that $\sum_j^{n-1} \frac{\rho_j q_j}{1-z_j(s)} = \frac{\rho_k q_k}{1-z_k(s)} + (\rho_n q_n - \rho_k q_k) < \frac{\rho_n q_n}{z_n}$, wherein the property $|q_n| > |q_k|$ for all $k \in \mathbb{V}\backslash\{n\}$ is used. Hence, one can draw conclusion on the difference of Lyapunov function along the issue sequence as $V(\boldsymbol{z}(s+1)) < V(\boldsymbol{z}(s))$, $\forall \boldsymbol{z}(s) \in \nabla\mathbb{S}^n$, where $V(\boldsymbol{z}(s)) > 0$ for all $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n$. In conclusion, we claim that $\boldsymbol{e}_n$ is the asymptotically stable equilibrium point for self-appraisal dynamic (12) in the case of $\mathcal{G}(\boldsymbol{Q})$ having a star topology. This is a direct application of Lyapunov stability theory to discrete-time system [44]. Therefore, given the center node of graph $\mathcal{G}(\boldsymbol{Q})$ being i and $\lim_{s\to\infty} \boldsymbol{z}(s) = \boldsymbol{e}_i$, one can immediately compute that $\rho_i \boldsymbol{e}_i$ is the appropriate dominant left eigenvector of $\boldsymbol{P}(\boldsymbol{e}_i)$, which is equivalent to $\lim_{s\to\infty} \boldsymbol{p}(s) = \rho_i \boldsymbol{e}_i$. The proof is completed.

Appendix A.6. Proof of Theorem 2

From the analytic expression (13), the vertices $\boldsymbol{e}_i$ ($i \in \mathbb{V}$) of $\mathbb{S}^n$ are naturally fixed points of the map $\boldsymbol{f}$. Furthermore, the factor $\theta(\boldsymbol{z})$ is strictly positive for $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n$, which suffices to ensure $\boldsymbol{z}(s+1) > 0$. Namely, no fixed point exists on the boundary of simplex $\mathbb{S}^n$.

We define a compact set by $\mathbb{A}^n = \{\boldsymbol{z} \in \mathbb{S}^n | 0 \leq z_i \leq 1 - r, \forall i \in \mathbb{V}\}$, where $r \in \mathbb{R}_{>0}$ is a extremely small scalar and satisfies $0 < r \leq \min_{i\in\mathbb{V}} \frac{1-2\rho_i q_i}{1-\rho_i q_i}$. Note that the properties of the dominant left eigenvector $\boldsymbol{q}$ developed in Lemma 3 ensures that $(1 - 2\rho_i q_i)/(1 - \rho_i q_i) > 0$ for all $i \in \mathbb{V}$ and graphs $\mathcal{G}(\boldsymbol{Q})$ with a non-star topology.

We first calculate the i-th entry of vector field $\boldsymbol{f}(\boldsymbol{z})$ by

$$f_i(\boldsymbol{z}) = \frac{\rho_i q_i}{(1 - z_i)\sum_j^n \frac{\rho_j q_j}{1-z_j}} = \frac{1}{1 + \frac{\sum_{j\neq i}^n \frac{\rho_j q_j}{1-z_j}}{\rho_i q_i / 1 - z_i}} \leq \frac{1}{1 + \frac{r}{\rho_i q_i}\sum_{j\neq i}^n \frac{\rho_j q_j}{(1-z_j)}}, \qquad \text{(A.5)}$$

as $1 - z_i \geq r$. Due to $1 - z_j < 1$ for $\boldsymbol{z} \in \nabla\mathbb{S}^n$ and $\rho_i q_i = 1 - \sum_{j\neq i}^n \rho_j q_j$, the formula (A.5) further becomes

$$\begin{aligned} f_i(\boldsymbol{z}) < \frac{\rho_i q_i}{r + (1-r)\rho_i q_i} &= \frac{(1-\rho_i q_i)r^2 + (2\rho_i q_i - 1)r}{r + (1-r)\rho_i q_i} + 1 - r \\ &= \frac{r(1-\rho_i q_i)(r - \frac{1-2\rho_i q_i}{1-\rho_i q_i})}{r + (1-r)\rho_i q_i} + 1 - r \leq 1 - r, \end{aligned}$$

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where the last inequality is based on the fact that $r \leq \frac{1-2\rho_i q_i}{1-\rho_i q_i}$ for all $i \in \mathbb{V}$. Hence, one can derive the conclusion that $\boldsymbol{f}(\mathbb{A}^n) \subset \mathbb{A}^n$. In what follows, we restrict the consideration of self-appraisal dynamics to this compact set $\mathbb{A}^n$.

Referred to system (12), we attain the prolonged system by

$$\begin{cases} \boldsymbol{z}(s+1) = \boldsymbol{f}(\boldsymbol{z}) \\ \delta\boldsymbol{z}(s+1) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}(\boldsymbol{z})\delta\boldsymbol{z}(s) \end{cases} \tag{A.6}$$

where the infinitesimal displacement is $\delta\boldsymbol{z} \in T_{\boldsymbol{z}}\mathbb{S}^n$ and the Jacobian matrix of vector field $\boldsymbol{f}$ has the form

$$\Big[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}\Big]_{ij}(\boldsymbol{z}) = \begin{cases} \frac{z_i(s+1)(1-z_i(s+1))}{1-z_i(s)} & \text{if } j = i \\ -\frac{z_i(s+1)z_j(s+1)}{1-z_j(s)} & \text{if } j \neq i, \end{cases}$$

where the relation $\frac{\partial \theta(\boldsymbol{z})}{\partial z_i} = -\frac{\rho_i q_i \theta^2(\boldsymbol{z})}{(1-z_i)^2}$ is an intermediate for the computation.

For each $s \in \mathbb{I}$, $\boldsymbol{z} \in \mathbb{A}^n$, and $\delta\boldsymbol{z} \in T_{\boldsymbol{z}}\mathbb{A}^n$, we consider a candidate Finsler-Lyapunov function of the form

$$V(\boldsymbol{z}(s), \delta\boldsymbol{z}(s)) = \sum_{i=1}^{n} \Big| \frac{\delta z_i(s)}{1 - z_i(s)} \Big|, \tag{A.7}$$

which satisfies conditions of Definition 2 by using factors $c_1 = c_2 = l = 1$ and a Finsler structure $F(\boldsymbol{z}(s), \delta\boldsymbol{z}(s)) = V(\boldsymbol{z}(s), \delta\boldsymbol{z}(s))^{\frac{1}{l}}$ in (5).

Denote $\boldsymbol{\Pi}(\boldsymbol{z}(s)) := \mathsf{diag}(1/(1 - z_1(s)), \ldots, 1/(1 - z_n(s)))$ for clarity of presentation. The Finsler-Lyapunov function then can be rewritten to a form $V = \|\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta\boldsymbol{z}(s)\|_1$ in terms of the 1-norm, which has a difference calculation along the issue sequence

$$\begin{aligned} & V(\boldsymbol{z}(s+1), \delta\boldsymbol{z}(s+1)) - V(\boldsymbol{z}(s), \delta\boldsymbol{z}(s)) \\ =& \|\boldsymbol{\Pi}(\boldsymbol{z}(s+1))\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}}(\boldsymbol{z}(s))\delta\boldsymbol{z}(s)\|_1 - \|\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta\boldsymbol{z}(s)\|_1 \\ =& \|\boldsymbol{K}(\boldsymbol{z}(s+1)\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta\boldsymbol{z}(\boldsymbol{z}(s))\|_1 - \|\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta\boldsymbol{z}(s)\|_1 \end{aligned} \tag{A.8}$$

where $\boldsymbol{K}(\boldsymbol{z}(s))$ represents the matrix with entries

$$[\boldsymbol{K}]_{ij}(\boldsymbol{z}(s)) = \begin{cases} z_i(s) & \text{if } j = i \\ -\frac{z_i(s)z_j(s)}{1-z_i(s)} & \text{if } j \neq i. \end{cases}$$

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Due to $0 < z_i \leq 1 - r$ for all $i \in \mathbb{V}$ and $\sum_i z_i = 1$, one can obtain $z_i/1 - z_j < 1$ for arbitrary $j \neq i$. Thus, the 1-norm of each column of the matrix $\boldsymbol{K}(\boldsymbol{z}(s))$ has a strict upper bound, i.e., $z_i(s) + \sum_{j=1, j\neq i}^{n} \frac{z_i(s) z_j(s)}{1 - z_j(s)} < 1$, for all $i \in \mathbb{V}$, which guarantees, as well as the compactness of the set $\mathbb{A}^n$, $\|\boldsymbol{K}(\boldsymbol{z}(s))\|_1 < 1 - \kappa$ for some $0 < \kappa < 1$ for all $\boldsymbol{z}(s) \in \mathbb{A}^n$. The difference inequality (A.8) can be reformulated by

$$\begin{aligned} & V(\boldsymbol{z}(s+1), \delta \boldsymbol{z}(s+1)) - V(\boldsymbol{z}(s), \delta \boldsymbol{z}(s)) \\ < & (1-\kappa)\|\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta \boldsymbol{z}(s)\|_1 - \|\boldsymbol{\Pi}(\boldsymbol{z}(s))\delta \boldsymbol{z}(s)\|_1 = -\kappa V(\boldsymbol{z}(s), \delta \boldsymbol{z}(s)), \end{aligned} \tag{A.9}$$

which means the differential Lyapunov function V decreases non-trivially along the trajectories of the prolong system (A.6). As a consequence of Theorem 1, the self-appraisal system (10) is incrementally exponentially stable on $\nabla\mathbb{S}^n \subset \mathbb{S}^n$ with respect to the contraction measure V given in (A.7).

Then, we prove the existence and uniqueness of the equilibrium in the interior of the simplex. The construction of the distance $\mathsf{d}_{\mathbb{S}}$ concerning curve integration (4) endows $\mathbb{S}^n$ with the structure of metric space. Specifically, the distance function $\mathsf{d}_{\mathbb{S}}$ induced by $F(\boldsymbol{z}, \delta(z)) = V(\boldsymbol{z}, \delta(z))$ in coordinates reads

$$\mathsf{d}_{\mathbb{S}}(\boldsymbol{z}_1, \boldsymbol{z}_2) = \inf_{\Gamma(\boldsymbol{z}_1, \boldsymbol{z}_2)} \int_J V\left(\gamma(\tau), \frac{\partial \gamma(\tau)}{\partial \tau}\right) d\tau$$

where $\Gamma(\boldsymbol{z}_1, \boldsymbol{z}_2)$ is the collection of piece-wisely differential curves $\gamma : J \to \nabla\mathbb{S} \subset \mathbb{S}$, $J := \{\tau \in \mathbb{R} | 0 \leq \tau \leq 1\}$, connecting $\boldsymbol{z}_1$ to $\boldsymbol{z}_2$, namely, $\gamma(0) = \boldsymbol{z}_1$ and $\gamma(1) = \boldsymbol{z}_2$. For any initial conditions $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, and any given converging sequence $\{\chi_1, \ldots, \chi_k, \ldots\} \in \mathbb{R}_{>0}$ with $\lim_{k\to\infty} \chi_k = 0$, one can develop a sequence of continuously differential curves $\gamma_k : J_k \to \mathbb{S}^n$ s.t.

$$\lim_{k\to\infty} \int_{J_k} V\left(\gamma_k(\tau), \frac{\partial \gamma_k(\tau)}{\partial \tau}\right) d\tau \leq \lim_{k\to\infty} (1 + \chi_k)\mathsf{d}_{\mathbb{S}}(\boldsymbol{z}_1, \boldsymbol{z}_2) = d_{\mathbb{S}}(\boldsymbol{z}_1, \boldsymbol{z}_2), \tag{A.10}$$

where J_k follows a reparameterization of $\gamma : J \to \nabla\mathbb{S}$. From (A.9), one can get $V(\boldsymbol{z}(s), \delta \boldsymbol{z}(s)) \leq (1 - \kappa)^s V(\boldsymbol{z}(0), \delta \boldsymbol{z}(0))$, for all $s \geq 0$, which together with (A.10) implies that in the limit of $k \to \infty$, for arbitrary initial conditions $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \nabla\mathbb{S}^n$,

$$\mathsf{d}_{\mathbb{S}}(\boldsymbol{\phi}(s; 0, \boldsymbol{z}_1), \boldsymbol{\phi}(s; 0, \boldsymbol{z}_2)) \leq \int_{J_k} V(\gamma_k(\tau), \frac{\partial \gamma_k(\tau)}{\partial \tau}) d\tau \leq [(1 - \kappa)]^s \mathsf{d}_{\mathbb{S}}(\boldsymbol{z}_1, \boldsymbol{z}_2).$$

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Since the Lipschitz constant $(1-\kappa)^s$ is strictly smaller than 1, the map $\boldsymbol{f}: \mathbb{S}^n \to \mathbb{S}^n$ is a contraction mapping on $\mathbb{S}^n$. Thereby, the employment of *Banach fixed-point theorem* [44] to the complete metric space $(\nabla\mathbb{S}^n, \mathsf{d}_{\mathbb{S}})$ suffices to prove the existence and uniqueness of a fixed point $\boldsymbol{z}^* \in \nabla\mathbb{S}^n$ such that $\boldsymbol{z}^* = \boldsymbol{f}(\boldsymbol{z}^*)$. Since the previous examination has addressed that there is no other fixed-point on the boundary of the simplex, this non-vertex equilibrium $\boldsymbol{z}^*$ only appears in the interior $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$. The proof of statement (i) is achieved.

Hence, one can draw the conclusion that the trajectory of the solutions to $\boldsymbol{z}(s+1) = \boldsymbol{f}(\boldsymbol{z}(s))$ converge exponentially to a unique equilibrium point $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$. As a by-product of the convergence of self-weights, the social power indicators $\boldsymbol{p}(s)$ converges exponentially to a unique fixed point $\boldsymbol{p}^* \in \mathsf{int}(\mathbb{C}^n)$ as the issue sequence progresses. The statement (ii) is finished.

The stability of the fixed point in the interior of simplex has been addressed in the above statement. That is, the fixed point $\boldsymbol{z}^* \in \mathsf{int}(\mathbb{S}^n)$ is an exponentially stable equilibrium point for the self-appraisal dynamics. It remains to elucidate that the vertices $\boldsymbol{e}_i$ ($i \in \{1, \dots, n\}$) are unstable fixed point. The Jacobian matrix evaluated at the vertex of simplex can be found in the proof of Proposition 1. Without loss of generality, for any $i \in \mathbb{V}$, the virtual system with the Jacobian (A.3) at $\boldsymbol{z} = e_i$ characterizes the linearization of (12) about the vertex $\boldsymbol{z} = e_i$. In particular, this Jacobian has a single eigenvalue at $(1-|q_i|)/|q_i|$ and all other eigenvalues are zero. From Lemma 3, the entry q_i of the influence matrix satisfies $|q_i| < 1/2$ if the graph $\mathcal{G}(\boldsymbol{Q})$ has no star topology, thus implying $(1-|q_i|)/|q_i| > 1$. Hence, the vertices of simplex are unstable equilibrium points for the self-appraisal dynamics Proposition 1 according to the adoption of Lyapunov's indirect method [44] to a discrete-time setting, thus claiming the statement (iii). The balancedness of $\mathcal{G}(\boldsymbol{Q})$ implies $\boldsymbol{q} = \boldsymbol{\rho}/n$ and the remaining proof of the statement (iv) is simply the special case of the statement (ii).

Appendix A.7. Proof of Theorem 3

It is straightforward to conduct the convergence analysis following from the proofs of Theorem 2. In particular, one can treat the self-appraisal dynamics

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with vector field (13) as a switching system and then employ the function (A.7) as a common (differential) Lyapunov function in the studying of stability. The contraction region here is modified by $\mathbb{A}^n = \{\boldsymbol{z} \in \mathbb{S}^n | 0 \leq z_i \leq 1 - r, \forall i \in \mathbb{V}\}$, where $r \leq \inf_{i \in \mathbb{V}, s \in \mathbb{I}} \frac{1-2\rho_i(s)q_i(s)}{1-\rho_i(s)q_i(s)}$. The rest of proof can be induced issue-wise from the proof of Theorem 2 and is omitted in order to save triviality. Since $\mathbb{A}^n$ is convex and compact, by incremental exponential stability, the solution $\boldsymbol{z}(s)$ starting from $\boldsymbol{z}(0) \in \nabla\mathbb{S}^n$ exponentially approaches to a limiting trajectory $\boldsymbol{z}^*(s) \in \mathsf{int}(\mathbb{S}^n)$ being independent of its initial conditions.

Finally, the limit set of $\boldsymbol{z}(s)$ is a trajectory in the interior of the simplex and the social power $\boldsymbol{p}(s)$ converges either to a limiting trajectory $|\boldsymbol{z}^*(s)| \in \mathsf{int}(\mathbb{C}^n)$. The proof is completed.

Appendix A.8. Proof of Theorem 4

For a given issue $s \in \mathbb{I}$ and any $\boldsymbol{z}(s) \in \nabla\mathbb{S}^n \cup \{\boldsymbol{0}\}$, the graph $\mathcal{G}(\boldsymbol{P}(s))$ is aperiodic, SC and structurally unbalanced according to the definition (8) since SC graph $\mathcal{G}(\boldsymbol{Q})$ is structurally unbalanced and aperiodic, inducing the formulation $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$.

Moreover, if $\boldsymbol{z}(s) = \boldsymbol{e}_i$ for some $i \in \mathbb{V}$ (without loss of generality, let $i = n$), then graph $\mathcal{G}(\boldsymbol{P}(s))$ is QSC, as node n is the only root vertex in $\mathcal{G}(\boldsymbol{P}(s))$. Two cases are considered. First, if $\mathcal{G}(\boldsymbol{P}(s))$ is SB, it equivalently means that removing one or multiple incoming edges of node n in $\mathcal{G}(\boldsymbol{Q})$ retrieves the structural balance. Thus, the vector field in this case has the same form as in the situation when $\mathcal{G}(\boldsymbol{Q})$ is QSC and SB, i.e., $\boldsymbol{f}(\boldsymbol{e}_i) = \boldsymbol{e}_i$. Second, if $\mathcal{G}(\boldsymbol{P}(s))$ is structurally unbalanced, there exists one, and only one (ISB) component definitely containing n in $\mathcal{G}(\boldsymbol{P}(s))$. By contradiction, assume there exists another ISB component $\mathcal{H}$ in $\mathcal{G}(\boldsymbol{P}(s))$. Since node n as a root has a path to contact any nodes belonging to $\mathcal{H}$, so $\mathcal{H}$ has at least one inward edge, which contradicts the definition of in-isolated subgraph. In this situation, $\boldsymbol{P}(s)$ has a dominant eigenvalue 1 associated with a (up to scaling) left eigenvector $\boldsymbol{e}_i$. Thus, we have $\boldsymbol{f}(\boldsymbol{e}_i) = \boldsymbol{e}_i$ for some $i \in \mathbb{V}$. Finally, following the fact that $\boldsymbol{f}$ keeps constantly zero in $\nabla\mathbb{S}^n \cup \{\boldsymbol{0}\}$, one can immediately show $\lim_{s\to\infty} \boldsymbol{z}(s) = \boldsymbol{0}$ and $\lim_{s\to\infty} \boldsymbol{p}(s) = \boldsymbol{0}$.

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