Distributed Optimal Control over Bit-rate Constrained Networks with Communication delay

V. Causevic and S. Hirche

Abstract— **In this paper we address the problem of optimal co-design of control and quantization policies for a physically-interconnected system, where each subsystem has a local quantizer. The controllers are assumed to communicate with delay and cooperate in minimizing global quadratic cost. We show that for quantizers that act on the estimation error of the estimator conditioned on common information between controllers, separation holds. In other words, both quantizers can be optimally designed by minimizing a distortion function that is control-independent. Finally, for general class of quantizers we provide structural properties of the optimal control policy.**

I. INTRODUCTION

For many systems, due to the distributed nature, a communication is necessary between individual subsystems (e.g. transport industry, robotics etc). Therefore, complete state information is not necessarily instantly accessible to each individual subsystem, but rather with certain delay. Additionally, in the case of data-rate limited communication links, the information is transmitted with a finite number of bits. The latter two aspects introduce constraints on the admissible control actions each subsystem is able to apply, herein referred to as "information constraints".

In general, the design of optimal control laws for distributed systems with communication delays is a difficult problem (see e.g. [18]). Depending on how fast the decision makers (DMs) communicate with each other, the optimal control policy might be linear or nonlinear, even in settings describing linear systems with a quadratic cost function [1]. A lot of attention has been on the design of optimal control laws for fixed information structures that have the property of being partially nested [2]. Some of them include first explicit solutions to linear Quadratic Gaussian team problems e.g. [4], [15] under the assumption that information between DMs is communicated at the exact speed at which it travels through the plant. Furthermore, in [16] the extension is made to the case where the information between DMs propagates faster then it travels through the plant, with application to distributed control of a vehicle platoon. Although previously mentioned results provide insight into the structure of optimal control policy, they do not consider network channels with constraints induced by finite bit-rate.

In the realm of feedback control under data-rate constraints, an excellent survey can be found in [11]. Most works on control over channels with finite bit-rate have analyzed stability aspect, while optimality has been less explored. It is well-known [12] that feedback control, state estimation and quantization can not be fully separated in general even for the linear quadratic regulator problem. An exception is the case of the full state being available at the quantizer, with certain conditions in [13] which also imply the optimality of the certainty equivalent control laws, for channels with finite bit-rate. Finally, in [14] an extension is provided to the case where quantizer has access to noisy sensor measurements. In general, the optimal co-design of quantizer and controller is difficult because of the presence of the dual effect which makes this networked control problem with two decision makers hard i.e. one cannot get the simplifications that are obtainable for classical single-agent linear quadratic (LQ) problem [8]. Although previous results give important insight into co-design of control and quantization for optimal control problems, they all address single-loop systems. The optimal control design for a distributed control system with delayed information sharing, and bit-rate constraints is a largely open problem.

In this paper we derive the optimal control policy for a physically interconnected system, where each subsystem has a local quantizer and neighboring controllers communicate with a maximum delay of one step. We prove that if the local quantizers belong to certain class, i.e. if they quantize the estimation error of state estimator conditioned a common information between controllers, they can be designed independently of the control inputs of both loops.

The remainder of the paper is outlined as follows. We start with problem setup in section II. The structure of optimal policy for an interconnected system given arbitrary local quantization policies is derived based on information decomposition in section III . Furthermore, the class of quantizers is characterized which is proven to result in control-independent design of optimal coding. Finally conclusions are given in section IV.

Notation: In this paper, for matrices C_i of appropriate dimensions the matrix $D = \mathrm{blkdiag}(C_1, C_2, \ldots, C_n)$ is the block-diagonal matrix such that $D_{ii} = C_i$ and $D_{ij} = 0$ for $i \neq j$. Given a matrix A, $[A]_{ij}$ denotes its element with position (i, j). For a time-varying vector $x(k)$ we denote by x^{k_1} vector ${x^{k_1}}^\top = [x^\top(0), \ldots, x^\top(k_1)]$, where $k_1 > 0$. We denote by $\sigma(x,y)$ the $\sigma-$field generated by random variables x,y. For a vector $y \in \mathbb{R}^p$, $\|y\|_\Omega := y^\top \Omega y$ where $\Omega \in \mathbb{R}^{p\times p}$.

This project has received funding from the German Research Foundation (DFG) within the Priority Program SPP 1914 "Cyber-Physical Networking".

[1] V. Causevic and S. Hirche are with the Chair of Information-oriented Control, Technical University of Munich, Germany; http://www.itr.ei.tum.de, {vedad.causevic, hirche}@tum.de

II. Problem setting

In this section we define the problem addressed here i.e. we explain individual blocks illustrated in Figure 1. Due to the difficulty of the defined problem for the case of arbitrary number of decision makers, we initiate the analysis and demonstrate the result on a system composed of two physically-coupled subsystems. The solution for the arbitrary number of interconnected loops is outside of the scope of this paper.

A. Interconnected Plant

Consider a dynamical system composed of two physically-coupled linear time-invariant (LTI) subsystems, denoted in the Figure 1 as P_1 and P_2. The dynamics of two subsytems is given by first order stochastic difference equations

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_{11}u_1(k) + w_1(k), \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_{22}u_2(k) + w_2(k), \end{aligned} \quad (1)$$

where $A_{11} \in \mathbb{R}^{n_1 \times n_1}$, $A_{12} \in \mathbb{R}^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, $B_{11} \in \mathbb{R}^{n_1 \times m_1}$, $B_{22} \in \mathbb{R}^{n_2 \times m_2}$. Variables $x_1(k) \in \mathbb{R}^{n_1}, x_2(k) \in \mathbb{R}^{n_2}$ are states of subsystems 1 and 2 respectively and $u_1(k) \in \mathbb{R}^{m_1}, u_2(k) \in \mathbb{R}^{m_2}$ are the respective control signals. The noise process $w_i(k) \in \mathbb{R}^{n_i}, i = 1,2$ is zero-mean i.i.d. Gaussian noise with covariance matrix Σ_{w_i} i.e. $w_i(k) \sim \mathscr{N}(0, \Sigma_{w_i})$. Similary, for the initial state it holds $x_i(0) \sim \mathscr{N}(0, \Sigma_{x_i})$. Moreover, $x_i(0)$ and $w_i(k)$ are assumed to be pair-wise independent at each time instant k and for every i. Equations (1) are written as

$$x(k+1) = Ax(k) + Bu(k) + w(k) \quad (2)$$

where the stacked vectors are $x(k) = (x_1^\top(k), x_2^\top(k))^\top \in \mathbb{R}^n$, $w(k) = (w_1^\top(k), w_2^\top(k))^\top \in \mathbb{R}^n$, $u(k) = (u_1^\top(k), u_2^\top(k))^\top \in \mathbb{R}^m$, $n = n_1 + n_2$ and $m = m_1 + m_2$. Additionally, we define $\Sigma_w = \text{blkdiag}(\Sigma_{w_1}, \Sigma_{w_2})$ and $\Sigma_x = \text{blkdiag}(\Sigma_{x_1}, \Sigma_{x_2})$.

B. Admissible quantization (encoding) policies

Each subsystem $i = 1,2$ has a local quantizer E_i (herein often referred to as encoder) that has access to state x_i. Furthermore, we assume all the encoders to know the parameters of the plant in (2) i.e. matrices A, B, Σ_w, Σ_x. Formally, the following class of encoder mappings is admissible

$$r_i(k) = \mathscr{E}_i(k, x_i^k, r_1^{k-1}, r_2^{k-1}, u_1^{k-1}, u_2^{k-1}) \quad (3)$$

i.e. each encoder has access to local measurement history, local encoding history, one-step delayed encoding history of the other encoder and a one-step delayed history of control inputs u_1, u_2. The map $\mathscr{E}_i$ has as a codomain a discrete set of symbols whose cardinality is determined by a bit-rate of the channels from encoders to the respective controllers. Furthermore $\mathscr{E}_i$ are allowed to be time-varying.

Remark 1: The equation (3) implies that there is a bi-directional communication channel between encoders E_1, E_2 i.e. encoder E_1 has access to the output of encoder E_2 and vice-versa, however with one-step delay.

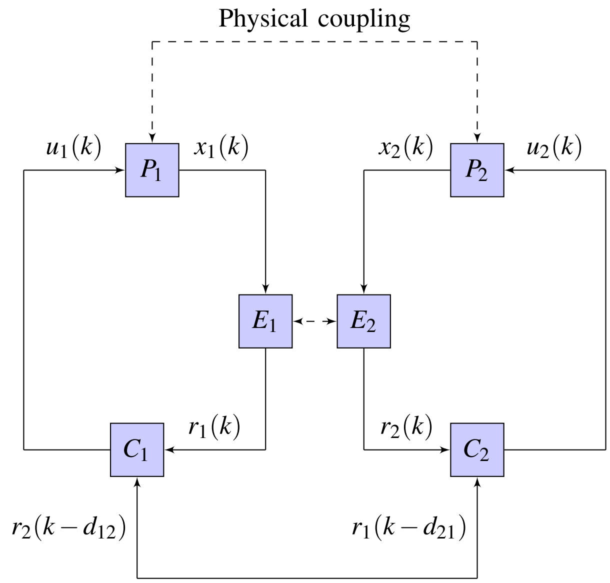

Fig. 1. Control and communication system: P_1, P_2 represent linear plants with physical coupling, E_1, E_2 represent local quantizers and C_1, C_2 are respective controllers that communicate with potential delay

C. Admissible control policies

Between encoder E_i and the corresponding local controller C_i we assume a discrete, error-free and memoryless communication channel. Therefore, each controller C_i, at time instant k, has direct access to a local encoder output history r_i^k. Additionally, controllers of two loops communicate between themselves with a delay. Let d_{12}, d_{21} denote respectively communication delay from controller C_1 to controller C_2 and vice-versa. We consider that $d_{12}, d_{21} \in \{0,1\}$ i.e. controllers communicate with a constant and known delay, with a maximum value of one step. The admissible control policies $\gamma_i(k), i = 1,2$ at time instant k are then measurable functions of the available information $\mathscr{I}_k^i$ to each controller C_i i.e.

$$u_i(k) = \gamma_i(k, \mathscr{I}_k^i) \quad (4)$$

where $\mathscr{I}_k^i,\ k = 0, \dots, T-1$, is defined as

$$\mathscr{I}_k^i = \{\mathscr{I}_{k-1}^i, r_i(k), u(k-1)\} \bigcup \{r_j(k - d_{ji})\}, \quad k > 0, \quad (5)$$

where j denotes the other subsystem and $\mathscr{I}_0^i = \{r_0^i\}$. In other words, the information set of each controller i is updated at time instant k by the current output from encoder E_i and the d_{ji}-step delayed information from the encoder j.

Remark 2: In practice, the validity of the assumption that the controllers communicate with a maximum delay of one sampling interval depends on the communication technology and sampling rates. In case of wired communication between subsystems, communication delay is typically low and can be assumed within the range of one sampling interval for: systems with low and high sampling rates. Wireless communication typically induces larger delays, but technology developments such as 5G aim to reduce this delay.

D. Problem statement

The objective is to minimize the following global control cost

$$J_{\mathscr{C}} = \mathbb{E}\left[\sum_{k=0}^{T-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^{\top} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + x(T)^{\top} Q_T x(T)\right] \quad (6)$$

where matrix R is assumed to be positive-definite matrix, while Q and Q_T are assumed to be semi-definite positive. We also assume controllability of pair (A,B) as well as detectability of $(Q^{\frac{1}{2}}, A)$. Ultimately, the optimization problem is formally written with respect to admissible control and encoding policies as

$$\begin{aligned} \min_{\gamma_i^{T-1}, \mathscr{E}_i^{T-1}, i=1,2} \quad & J_{\mathscr{C}} \\ \text{s.t.} \quad & (2),(3),(4). \end{aligned} \quad (7)$$

Note that the posed problem has four decision makers, namely C_1, E_1, C_2 and E_2.

III. Role of Information Structure

We first compute the solution of the addressed problem under the assumption of infinite bit rate which will motivate the approach for the originally posed problem. To this end, the information available to each controller C_i is obtained from (5) by replacing $r_i(k)$ with $x_i(k)$ and the constraint in (3) is ignored. Therefore the only decision makers in the problem are controllers C_1, C_2. In other words, the measured value of state x_i at subsystem i, is directly sent (without being encoded) to a local controller C_i.

The solution to this problem under the assumption that $d_{12} = 1, d_{21} = 1$ i.e. controllers communicate at the exact speed at which information propagates through the plant (2) is given in [3]. We first slightly generalize their result to accomodate any case where $d_{ij} \leq 1$. For convenience, recalling (1), A and B are partitioned as

$$A = \begin{bmatrix} A_1 | A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 | B_2 \end{bmatrix}.$$

where $A_1 \in \mathbb{R}^{n \times n_1}$, $A_2 \in \mathbb{R}^{n \times n_2}$, $B_1 \in \mathbb{R}^{n \times m_1}$ and $B_2 \in \mathbb{R}^{n \times m_2}$. Similarly, referring to (6), matrix R is partitioned as

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

where $R_{11} \in \mathbb{R}^{m_1 \times m_1}$, $R_{12} \in \mathbb{R}^{m_1 \times m_2}$, $R_{21} \in \mathbb{R}^{m_2 \times m_1}$, and $R_{22} \in \mathbb{R}^{m_2 \times m_2}$. To this end we give the following Theorem.

Theorem 1: The control law that minimizes (6) under constraints (1),(4) and given that $r_i(k) = x_i(k), \forall i = 1,2$ is

$$u(k) = K(k)\omega(k) + \begin{bmatrix} K_{1L}(k)\omega_1(k) \\ K_{2L}(k)\omega_2(k) \end{bmatrix} \quad (8)$$

where $\omega(k), \omega_1(k), \omega_2(k)$ represent mutually ortoghonal components of state $x(k)$ computed by

$$\begin{aligned} \omega(k) &= Ax(k-1) + Bu(k-1) + \begin{bmatrix} (1-d_{12})w_1(k-1) \\ (1-d_{21})w_2(k-1) \end{bmatrix} \\ \omega_1(k) &= d_{12}w_1(k-1) \\ \omega_2(k) &= d_{21}w_2(k-1) \end{aligned}$$

and the controller gains are computed as

$$\begin{aligned} K(k) &= (R + B^{\top}S(k+1)B)^{-1}B^{\top}S(k+1)A \quad (9) \\ K_{1L}(k) &= (R_{11} + B_1^{\top}S(k+1)B_1)^{-1}B_1^{\top}S(k+1)A_1 \\ K_{2L}(k) &= (R_{22} + B_2^{\top}S(k+1)B_2)^{-1}B_2^{\top}S(k+1)A_2 \\ S(k) &= Q + A^{\top}S(k+1)A - A^{\top}S(k+1)BK(k) \end{aligned}$$

where $S(N) = Q$.

Proof:

1) Estimator based on common information: There are four possible delay combinations for (d_{12}, d_{21}) i.e. $(d_{12}, d_{21}) \in \{(0,0),(1,0),(0,1),(1,1)\}$ and each of them induces a different information structure. From (5) one can write explicitly information sets available at controllers C_1 and C_2 at time k, respectively as

$$\begin{aligned} \mathscr{I}_k^1 &= \left\{x_1^k, x_2^{k-d_{21}}, u_1^{k-1}, u_2^{k-1}\right\} \quad (10) \\ \mathscr{I}_k^2 &= \left\{x_2^k, x_1^{k-d_{12}}, u_1^{k-1}, u_2^{k-1}\right\} \end{aligned}$$

The common information history between two decision makers C_1, C_2 is then written as

$$\mathscr{I}_k^c = \left\{x_1^{k-d_{12}}, x_2^{k-d_{21}}, u_1^{k-1}, u_2^{k-1}\right\} \quad (11)$$

Based on the common information an estimator of global state $x(k)$, from equation (2) is defined as

$$\begin{aligned} \omega(k) &:= \mathbb{E}\left\{x(k) | \mathscr{I}_k^c\right\} \quad (12) \\ &= Ax(k-1) + Bu(k-1) + \begin{bmatrix} (1-d_{12})w_1(k-1) \\ (1-d_{21})w_2(k-1) \end{bmatrix} \end{aligned}$$

The errors in estimating states $x_1(k), x_2(k)$ by latter estimators are given by

$$\begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix} := x(k) - \omega(k) = \begin{bmatrix} d_{12}w_1(k-1) \\ d_{21}w_2(k-1) \end{bmatrix}$$

Due to the independency of $w_1(k-1), w_2(k-1)$ and the assumption that they are zero-mean Gaussian, they are mutually orthogonal. Additionally it holds

$$\begin{aligned} &\mathbb{E}\left[\omega(k)^{\top} \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix}\right] = \\ &\mathbb{E}\left[(Ax(k-1) + Bu(k-1))^{\top} \begin{bmatrix} d_{12}w_1(k-1) \\ d_{21}w_2(k-1) \end{bmatrix}\right] \\ &+ \mathbb{E}\left[\begin{bmatrix} (1-d_{12})w_1(k-1) \\ (1-d_{21})w_2(k-1) \end{bmatrix}^{\top} \begin{bmatrix} d_{12}w_1(k-1) \\ d_{21}w_2(k-1) \end{bmatrix}\right] = 0. \end{aligned}$$

The first term on the right side of the equality sign is zero due to the fact that noise terms $w_1(k-1), w_2(k-1)$ are zero-mean Gaussian variables that are independent from $u(k-1), x(k-1)$. The second term is zero since inside the expectation the corresponding elements of two factors form a convex combination. Therefore as a conclusion $\omega(k), \omega_1(k), \omega_2(k)$ are mutually orthogonal.

2) Optimal Control Policy: Since the following conditions: $d_{12} \leq 1, d_{21} \leq 1$ are satisfied, the information structure $(\mathscr{I}_k^1, \mathscr{I}_k^2)$ and system (2) are partially nested [5] i.e. it holds $\mathscr{I}_{k-1}^1 \subset \mathscr{I}_k^2$ and $\mathscr{I}_{k-1}^2 \subset \mathscr{I}_k^1$. Due to the quadratic form of cost function in (6) this implies that optimal control policies are linear in the respective information [2] i.e. the optimal control inputs $u_1(k), u_2(k)$ are

$$\begin{aligned} u_1(k) &= f_1(k, x_2^{k-d_{21}}, x_1^{k-1}, x_1(k)) \\ u_2(k) &= f_2(k, x_1^{k-d_{12}}, x_2^{k-1}, x_2(k)) \end{aligned}$$

where $f_1(k), f_2(k)$ represent linear functions in respective arguments. To this end we write

$$\begin{aligned} u_1(k) &= K_{11}(k)x_1^{k-d_{12}} + K_{12}(k)x_2^{k-d_{21}} + d_{12}K_{1L}(k)x_1(k) \\ u_2(k) &= K_{21}(k)x_1^{k-d_{12}} + K_{22}(k)x_2^{k-d_{21}} + d_{21}K_{2L}(k)x_2(k) \end{aligned}$$

where $K_{11} \in \mathbb{R}^{m_1 \times (k-d_{12}+1)n_1}, K_{12} \in \mathbb{R}^{m_1 \times (k-d_{21}+1)}, K_{21} \in \mathbb{R}^{m_2 \times (k-d_{12}+1)n_1}, K_{22} \in \mathbb{R}^{m_2 \times n_2(k-d_{21}+1)}, K_{1L} \in \mathbb{R}^{m_1 \times n_1}, K_{2L} \in \mathbb{R}^{m_2 \times n_2}$ represent the control gains. Notice that in the expressions for the optimal control law policies above, the terms multiplying $K_{11}(k), K_{12}(k), K_{21}(k), K_{22}(k)$ represent the common measurement history of the two controllers (11), whose dimension is increasing in time. Correspondingly, this means that the dimensions of $K_{11}, K_{12}, K_{21}, K_{22}$ increase in time. However, as control strategies u_1, u_2 are linear, and the information structure is partially nested, the computation of

$$u^c(k) := \begin{bmatrix} K_{11}(k)x_1^{k-d_{12}} + K_{12}(k)x_2^{k-d_{21}} \\ K_{21}(k)x_1^{k-d_{12}} + K_{22}(k)x_2^{k-d_{21}} \end{bmatrix} \tag{13}$$

can be done by considering sufficient statistics for it [6]

$$u^c(k) = K^S(k)\mathbb{E}[x(k)|\mathscr{I}_k^c]$$

where the gain $K^S \in \mathbb{R}^{m \times n}$ has fixed dimension and estimator $\omega(k) = \mathbb{E}[x(k)|\mathscr{I}_k^c]$ is based on the common state and input history as introduced in (12). So we can write

$$u(k) = K^S(k)\omega(k) + \begin{bmatrix} d_{12}K_{1L} & 0 \\ 0 & d_{21}K_{2L} \end{bmatrix} \left(\omega(k) + \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix} \right) \tag{14}$$

Grouping the terms proportional to $\omega(k)$ we get (8), where

$$K(k) = K^S(k) + \begin{bmatrix} d_{12}K_{1L} & 0 \\ 0 & d_{21}K_{2L} \end{bmatrix}$$

The computation of optimal gains $K(k), K_{1L}(k), K_{2L}(k)$ can be done similarly to [3] by decomposing the Bellman equation, on the basis of the introduced structure of the optimal control law (14). It is omitted for the sake of brevity. ■

Remark 3: Theorem 1 is derived using the fact that delays d_{12}, d_{21} satisfy $d_{12} \leq 1, d_{21} \leq 1$ which implies optimal control inputs u_1, u_2 to be linear in the associated information. It can be easily verified that in the case of communication delays larger than one step, the system (1) and information structure (10) are not partially nested. This means that optimal control policies might be nonlinear in case of more than one-step delay.

A. Main Result

We now derive the main result of the paper. According to the assumptions on the information structure in (5) controllers C_1 and C_2 have access to the following information

$$\begin{aligned} \mathscr{I}^{c_1}(k) &= \{r_1^k, r_2^{k-d_{21}}, u_1^{k-1}, u_2^{k-1}\} \\ \mathscr{I}^{c_2}(k) &= \{r_2^k, r_1^{k-d_{12}}, u_1^{k-1}, u_2^{k-1}\} \end{aligned} \tag{15}$$

From (3) the encoder information sets $\mathscr{I}^{e_1}(k), \mathscr{I}^{e_2}(k)$ are

$$\mathscr{I}^{e_i}(k) := \{x_i^k, r_1^{k-1}, r_2^{k-1}, u_1^{k-1}, u_2^{k-1}\}, \quad \forall i = 1,2 \tag{16}$$

Notice that unlike the controllers, the encoders E_1, E_2 have access to the values of locally measured variables i.e. x_1^k, x_2^k respectively. Additionally, we define

$$\mathscr{I}^e(k) = \mathscr{I}^{e_1}(k) \cap \mathscr{I}^{e_2}(k) = \{r_1^{k-1}, r_2^{k-1}, u_1^{k-1}, u_2^{k-1}\} \tag{17}$$

i.e. common history between encoders E_1 and E_2 that excludes directly measured signal histories x_1^k, x_2^k . Note that it holds $\mathscr{I}^e(k) \subset \mathscr{I}^{c_1}(k) \cap \mathscr{I}^{c_2}(k)$ since $d_{12}, d_{21} \in \{0,1\}$ i.e. both encoders know the subset of common information between controllers C_1 and C_2.

Remark 4: Information sets $\mathscr{I}^{e_1}(k), \mathscr{I}^{e_2}(k)$, defined in (16), imply that the encoders have access to a one-step delayed information from both subsystems. More precisely, they exchange their outputs r_1, r_2 with a one-step delay and locally, at time k, compute $u_1(k-1), u_2(k-1)$. This computation is possible since $u_1(k-1), u_2(k-1)$ are functions of $\mathscr{I}^{c_1}(k-1), \mathscr{I}^{c_2}(k-1)$ respectively and $\mathscr{I}^{c_i}(k-1) \subset \mathscr{I}^{e_j}(k)$ for any tuple $(i,j) \in \{(1,1),(1,2),(2,1),(2,2)\}$. Such computation is important as it reduces the communication load and increases applicability of the scheme in Figure 1.
Before providing the structure of controllers that minimize (6) subject to dynamics in (1) and subject to bit-rate constraints in (3) as well as information constraints (4), we introduce some notation, motivated by the Theorem 1. We define the estimate of global state $x(k)$ based on the set $\mathscr{I}^e(k)$ (which can be computed at both quantizers)

$$\omega(k) = \mathbb{E}[x(k)|\mathscr{I}^e(k)] \tag{18}$$

At the side of controller C_1, after receiving $r_1(k)$ and possibly $r_2(k)$ a conditional estimate $\hat{x}_1(k)$ is defined

$$\hat{x}_1(k) = \mathbb{E}[x_1(k)|\mathscr{I}^{c_1}(k)] \tag{19}$$

Similary, for controller C_2 we define

$$\hat{x}_2(k) = \mathbb{E}[x_2(k)|\mathscr{I}^{c_2}(k)] \tag{20}$$

We note that it holds

$$\begin{aligned} \omega(k+1) &= \mathbb{E}[x(k+1)|\mathscr{I}^e(k+1)] \\ &= \mathbb{E}[Ax(k) + Bu(k) + w(k)|r_1^k, r_2^k, u_1^k, u_2^k] \\ &= A\mathbb{E}[x(k)|r_1^k, r_2^k, u_1^k, u_2^k] + Bu(k) \\ &= A\hat{x}(k) + Bu(k) \end{aligned} \tag{21}$$

where the last line is due to the fact that $x_1(k), x_2(k)$ are influenced respectively by $u_1(k), u_2(k)$, with one-step delay. The difficulty of the problem addressed here is two-fold:

- Due to communication delay between loops there is unsymmetry in the information available to C_1 and C_2
- Each subsystem could possess the dual effect [7] where controller has the role of reducing the estimation error in future, which in general, might not allow for separated design of control and coding

To address the first point, notice that estimates $\omega(k), \hat{x}_1(k), \hat{x}_2(k)$ correspond to information decomposition into common information history between C_1 and C_2 (defined by set $\mathscr{I}_e(k)$) and only locally available information $(\mathscr{I}^{c_1}(k), \mathscr{I}^{c_2}(k))$. For the second point, we restrict ourselves to study the class of encoders E_1, E_2 that instead of encoding directly locally measured states x_1, x_2, respectively, they subtract the effects of both controls $u_1(k), u_2(k)$, before encoding. In particular for encoder outputs $r_1(k), r_2(k)$ we assume the following

$$r_i(k) = \mathscr{E}_i(x_i(k) - \mathbb{E}[x_i(k)|\mathscr{I}^e(k)]) \quad \forall i = 1,2 \tag{22}$$

In other words, we consider the class of encoders $\mathscr{E}_1(\cdot), \mathscr{E}_2(\cdot)$, each of which is applied on the most recent error of estimator of local state, conditioned on the information set $\mathscr{I}^e(k)$. As proven later, this class of encoders will enable separated design of optimal control $u(k)$ and optimal encoding policies $\mathscr{E}_1(\cdot), \mathscr{E}_2(\cdot)$. To this end, we introduce the definition of absence of dual effect in a distributed setting as such effect has typically been defined for a single-loop two agents' system (controller and encoder) [7], [8]. Indeed, each subsytems' state can be written as

$$x_i(k) = \begin{bmatrix} \phi_{i1,k} & \phi_{i2,k} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \sum_{l=0}^{k-1} \begin{bmatrix} \phi_{i1,k-l-1} & \phi_{i2,k-l-1} \end{bmatrix} \begin{bmatrix} B_{11}u_1(l) + w_1(l) \\ B_{22}u_2(l) + w_2(l) \end{bmatrix}, \quad \forall i = 1,2$$

where

$$\phi(k) := A^k = \begin{bmatrix} \phi_{11,k} & \phi_{12,k} \\ \phi_{21,k} & \phi_{22,k} \end{bmatrix} \tag{23}$$

i.e. $\phi_{11,k}, \phi_{12,k}, \phi_{21,k}, \phi_{22,k}$ represent the partitions of matrix A^k according to the subsystems 1,2 i.e. according to vector $[x_1(k)^\top x_2^\top(k)]^\top$. Notice that state $x_1(k)$ depends not only on the local control history of u_1^{k-1}, but also on history of control input u_2^{k-1} and similarly for state x_2. This is to be considered for adapted notion of absence of dual effect in Definition 1. Finally, we define input-free states for subystems $i = 1,2$ as

$$\bar{x}_i(k) = x_i(k) - \sum_{l=0}^{k-1} \phi_{i1,k-l-1} B_{11} u_1(l) + \phi_{i2,k-l-1} B_{22} u_2(l)$$

The corresponding encoder outputs are then denoted as

$$\bar{r}_i(k) = \mathscr{E}_i(\bar{x}_i(k) - \mathbb{E}[\bar{x}_i(k)|\bar{r}_1^{k-1}, \bar{r}_2^{k-1}]), \quad \forall i = 1,2 \tag{24}$$

Denoting by $e_1(k) = x_1(k) - \hat{x}_1(k), e_2(k) = x_2(k) - \hat{x}_2(k)$ the estimation errors at the controller sides, and by $\bar{e}_1(k) = \bar{x}_1(k) - \mathbb{E}[\bar{x}_1(k)|\bar{r}_1^k, \bar{r}_2^{k-d_{21}}], \bar{e}_2(k) = \bar{x}_2(k) - \mathbb{E}[\bar{x}_2(k)|\bar{r}_2^k, \bar{r}_1^{k-d_{12}}]$ the estimation errors of the input-free system, we adapt the definition of [13] as follows.

Definition 1 (Absence of dual effect): For the system (2) and information structure in (15) the control has no dual effect if for subsystems 1,2 it holds

$$\mathbb{E}[e_1(k)e_1(k)^\top|\mathscr{I}^{c_1}(k)] = \mathbb{E}[\bar{e}_1(k)\bar{e}_1(k)^\top|\bar{r}_1^k, \bar{r}_2^{k-d_{21}}]$$
$$\mathbb{E}[e_2(k)e_2(k)^\top|\mathscr{I}^{c_2}(k)] = \mathbb{E}[\bar{e}_2(k)\bar{e}_2(k)^\top|\bar{r}_2^k, \bar{r}_1^{k-d_{12}}]$$

Basically, if there is no dual effect then variances of the estimation errors (from controller sides) are independent of control inputs from both subsystems. We next give a lemma that gives a sufficient condition for the absence of dual effect according to Definition 1.

Remark 5: Intuitively, according to Definition 1 controls u_1, u_2 do not affect the inputs e_1, e_2 to the respective encoders E_1, E_2, as e_1, e_2 correspond to input-free variables $\bar{e}_1, \bar{e}_2$. This leads to the behaviour of encoders E_1, E_2 which is not influenced by control policies. This is the main mechanism which will enable separated design of optimal encoding policies as provided in Theorem 2.

Lemma 1: If following conditions are satisfied

$$I)\, \sigma(\bar{r}_1^k, \bar{r}_2^{k-d_{21}}) \subset \sigma(\mathscr{I}^{c_1}(k))$$
$$II)\, \sigma(\bar{r}_1^{k-d_{12}}, \bar{r}_2^k) \subset \sigma(\mathscr{I}^{c_2}(k))$$
$$III)\, \mathbb{E}[\bar{x}_1(k)|\bar{r}_1^k, \bar{r}_2^{k-d_{21}}] = \mathbb{E}[\bar{x}_1(k)|\mathscr{I}^{c_1}(k), \bar{r}_1^k, \bar{r}_2^{k-d_{21}}]$$
$$IV)\, \mathbb{E}[\bar{x}_2(k)|\bar{r}_2^k, \bar{r}_1^{k-d_{12}}] = \mathbb{E}[\bar{x}_2(k)|\mathscr{I}^{c_2}(k), \bar{r}_2^k, \bar{r}_1^{k-d_{12}}]$$

then there is no dual effect.

Proof: We start from the definition of estimation error $e_1(k)$ at the side of controller C_1 i.e.

$$\begin{aligned} e_1(k) &= x_1(k) - \mathbb{E}[x_1(k)|\mathscr{I}^{c_1}(k)] \\ &= \bar{x}_1(k) + \sum_{i=0}^{k-1} \phi_{11,k-i-1}B_{11}u_1(i) + \phi_{12,k-i-1}B_{22}u_2(i) \\ &\quad - \mathbb{E}[\bar{x}_1(k)|\mathscr{I}^{c_1}(k)] \\ &\quad - \sum_{i=0}^{k-1} \mathbb{E}[\phi_{11,k-i-1}B_{11}u_1(i) + \phi_{12,k-i-1}B_{22}u_2(i)|\mathscr{I}^{c_1}(k)] \\ &= \bar{x}_1(k) - \mathbb{E}[\bar{x}_1(k)|\mathscr{I}^{c_1}(k)] \end{aligned}$$

where the last equality is due to the fact that set $\mathscr{I}^{c_1}(k)$ as defined in (15) contains input histories u_1^{k-1}, u_2^{k-1}. Finally, assuming conditions I and III we have

$$\begin{aligned} e_1(k) &= \bar{x}_1(k) - \mathbb{E}[\bar{x}_1(k)|I^{c_1}(k), \bar{r}_1^k, \bar{r}_2^{k-d_{21}}] \\ &= \bar{x}_1(k) - \mathbb{E}[\bar{x}_1(k)|\bar{r}_1^k, \bar{r}_2^{k-d_{21}}] = \bar{e}_1(k). \end{aligned}$$

The proof for the subsystem 2 is analogous. ■

Given that Lemma 1 provides a sufficient condition for the absence of dual effect in setting considered here, we now prove that class of encoders in (22) satisfies that condition.

Proposition 1: For the class of quantizers in (22), the state dynamics (2) and information constraints (3), (4) the following conditions are satisfied

$$\bar{r}_1^k = r_1^k$$
$$\bar{r}_2^k = r_2^k$$
$$\bar{x}_1(k) \to \bar{r}_1^k, \bar{r}_2^{k-d_{21}} \to u_1^{k-1}, u_2^{k-1}$$
$$\bar{x}_2(k) \to \bar{r}_2^k, \bar{r}_1^{k-d_{12}} \to u_1^{k-1}, u_2^{k-1}$$

where $\bar{r}_1(k), \bar{r}_2(k)$ are defined in (24). Therefore the conditions in Lemma 1 hold and there is no dual effect.

Proof: First, it is easy to see that assumed conditions are sufficient for Lemma 1 to hold. Indeed, due to the first two conditions, the nestedness of σ-algebras as in I), II) is guaranteed. The equality of expectations in III), IV) is straightforward from the definition of the Markov chains [10]. Now we prove those conditions, similarly to [13], based on the principle of induction.

First, it holds $\bar{r}_1(0) = r_1(0), \bar{r}_2(0) = r_2(0)$ since $\bar{x}_1(0) = x_1(0)$, $\bar{x}_2(0) = x_2(0)$. Since $u_1(0)$ and $u_2(0)$ are functions of $r_1(0), r_2(0)$, when conditioned on $r_1(0), r_2(0)$ they are independent of $\bar{x}_1(0), \bar{x}_2(0), w_1(0), w_2(0)$. Therefore it holds $\bar{x}_1(0), \bar{x}_2(0), w_1(0), w_2(0) \to r_1(0), r_2(0) \to u_1(0), u_2(0)$. As $\bar{x}_1(1) = A_{11}\bar{x}_1(0) + A_{12}\bar{x}_2(0) + w_1(0)$, and $\bar{x}_2(1) = A_{21}\bar{x}_1(0) + A_{22}\bar{x}_2(0) + w_2(0)$ this implies $\bar{x}_1(1) \to r_1(0), r_2(0) \to u_1(0), u_2(0)$ and $\bar{x}_2(1) \to r_1(0), r_2(0) \to u_1(0), u_2(0)$. Thus

$$
\begin{aligned}
r_1(1) &= \mathscr{E}_1(x_1(1) - \mathbb{E}[x_1(1)|\mathscr{I}^e(1)]) \\
&= \mathscr{E}_1(\bar{x}_1(1) + B_{11}u_1(0) - \mathbb{E}[\bar{x}_1(1) + B_{11}u_1(0))|\mathscr{I}^e(1)]) \\
&= \mathscr{E}_1(\bar{x}_1(1) - \mathbb{E}[\bar{x}_1(1)|r_1(0), r_2(0), u_1(0), u_2(0)] \\
&= \mathscr{E}_1(\bar{x}_1(1) - \mathbb{E}[\bar{x}_1(1)|\bar{r}_1(0), \bar{r}_2(0)] = \bar{r}_1(1)
\end{aligned}
$$

Similary, it holds $\bar{r}_2(1) = r_2(1)$. Due to $r_1(1) = \bar{r}_1(1), r_2(1) = \bar{r}_2(1)$, both $r_1(1), r_2(1)$ are independent of $u_1(0), u_2(0)$, which with last two chains proven implies $\bar{x}_1(1) \to r_1^1, r_2^{1-d_{21}} \to u_1(0), u_2(0)$ and $\bar{x}_2(1) \to r_2^1, r_1^{1-d_{12}} \to u_1(0), u_2(0)$, as $d_{12}, d_{21} \in \{0,1\}$. We next prove the induction step.

Assume that $r_1^t = \bar{r}_1^t, r_2^t = \bar{r}_2^t$ and that $\bar{x}_1(k) \to r_1^k, r_2^{k-d_{21}} \to u_1^{k-1}, u_2^{k-1}$, $\bar{x}_2(k) \to r_2^k, r_1^{k-d_{12}} \to u_1^{k-1}, u_2^{k-1}$, for $1 \le k \le t$. It holds that $\bar{x}_1(t), \bar{x}_2(t), w_1(t), w_2(t) \to r_1^t, r_2^t, u_1^{t-1}, u_2^{t-1} \to u_1(t), u_2(t)$ because when conditioned on $r_1^t, r_2^t, u_1^{t-1}, u_2^{t-1}$ the control inputs $u_1(t), u_2(t)$ do not depend on $\bar{x}_1(t), \bar{x}_2(t), w_1(t), w_2(t)$. Since $w_1(t), w_2(t)$ are independent of $r_1^t, r_2^t, u_1^{t-1}, u_2^{t-1}$ and $\bar{x}_1(t), \bar{x}_2(t)$ are input free and satisfy induction assumption, we have that $\bar{x}_1(t), w_1(t), \bar{x}_2(t), w_2(t) \to r_1^t, r_2^t \to u_1^{t-1}, u_2^{t-1}$. Therefore, the last two chains imply $\bar{x}_1(t), w_1(t), \bar{x}_2(t), w_2(t) \to r_1^t, r_2^t \to u_1^t, u_2^t$. Since $\bar{x}_1(t+1) = A_{11}\bar{x}_1(t) + A_{12}\bar{x}_2(t) + w_1(t)$, and $\bar{x}_2(t+1) = A_{21}\bar{x}_1(t) + A_{22}\bar{x}_2(t) + w_2(t)$ this implies $\bar{x}_1(t+1), \bar{x}_2(t+1) \to r_1^t, r_2^t \to u_1^t, u_2^t$. Therefore $r_1(t+1) = \bar{r}_1(t+1), r_2(t+1) = \bar{r}_2(t+1)$. Now it holds $\bar{x}_1(t+1) \to r_1^{t+1}, r_2^{t-d_{21}} \to u_1^t, u_2^t$ since $r_1(t+1) = \bar{r}_1(t+1)$ is independent of u_1^t, u_2^t and $d_{21} \in \{0,1\}$. The same conclusion is made for subsystem 2. This concludes the proof. ■

Before stating the main result on the structural properties of optimal control inputs u_1^{T-1}, u_2^{T-1} we define the following

$$
\begin{aligned}
\omega_i(k) &= \mathbb{E}[a_i^\top e(k-1) + w_i(k-1)|\mathscr{I}^{c_i}(k)] \quad \forall i = 1,2 \\
\hat{\delta}^i(k) &= \mathbb{E}[a_{3-i}^\top e(k-1) + w_{3-i}(k-1)|\mathscr{I}^{c_i}(k)] \quad \forall i = 1,2
\end{aligned} \tag{25}
$$

where $a_1^\top = [A_{11}\ A_{12}], a_2^\top = [A_{21}\ A_{22}]$. The gain $K(k)$ computed by (9) is partitioned as follows

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where $K_{11} \in \mathbb{R}^{m_1 \times n_1}, K_{12} \in \mathbb{R}^{m_1 \times n_2}, K_{21} \in \mathbb{R}^{m_2 \times n_1}, K_{22} \in \mathbb{R}^{m_2 \times n_2}$.

Theorem 2: For an arbitrary sequences of quantizers $\mathscr{E}_1^{T-1}, \mathscr{E}_2^{T-1}$ defined by (3), the control inputs u_1^{T-1}, u_2^{T-1} minimizing cost in (6), given constraints on state dynamics in (2) and information constraints in (4) are given by

$$u(k) = K(k)\omega(k) + \begin{bmatrix} K_{11}(k)\omega_1(k) + K_{12}(k)\hat{\delta}^1(k) \\ K_{22}(k)\omega_2(k) + K_{21}(k)\hat{\delta}^2(k) \end{bmatrix} \tag{26}$$

where $\omega_1(k), \omega_2(k), \hat{\delta}^1(k), \hat{\delta}^2(k)$ are defined in (25). Moreover, for the class of encoders in (22), the global smallest cost over all encoder-controller 4-tuples $(\gamma_1^{T-1}, \gamma_2^{T-1}, \mathscr{E}_1^{T-1}, \mathscr{E}_2^{T-1})$ decomposes as

$$
\begin{aligned}
\min_{\gamma_1^{T-1}, \gamma_2^{T-1}, \mathscr{E}_1^{T-1}, \mathscr{E}_2^{T-1}} J_{\mathscr{C}} &= tr(S(0)\Sigma_x) + \sum_{k=0}^{T-1} tr(S(k+1)\Sigma_w) \\
&+ \min_{\mathscr{E}_1^{T-1}, \mathscr{E}_2^{T-1}} D
\end{aligned} \tag{27}
$$

where $D = D(\bar{e}_1^{T-1}, \bar{e}_2^{T-1})$ is a control-independent distortion function defined as

$$D = \sum_{k=0}^{T-1} \mathbb{E}\,\|\mathscr{D}(k)\|_{\Omega_k} \tag{28}$$

and

$$\mathscr{D}(k) = \begin{bmatrix} K_{11}(k)\bar{e}_1(k) + K_{12}(k)(\bar{e}_2(k) + \omega_2(k) - \hat{\delta}^1(k)) \\ K_{21}(k)(\bar{e}_1(k) + \omega_1(k) - \hat{\delta}^2(k)) + K_{22}(k)e_2(k) \end{bmatrix}.$$

Proof: We start by analyzing the relationship between the controllers' estimates $\hat{x}_1(k), \hat{x}_2(k)$ defined in (19), (20) and estimate $\omega(k)$ of global state $x(k)$ produced at encoders' sides and defined by (18). Indeed it holds

$$
\begin{aligned}
\hat{x}(k) := \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} &= \begin{bmatrix} \mathbb{E}[x_1(k)|\mathscr{I}^{c_1}(k)] \\ \mathbb{E}[x_2(k)|\mathscr{I}^{c_2}(k)] \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{E}[a_1^\top x(k-1) + B_1u_1(k-1) + w_1(k-1)|\mathscr{I}^{c_1}(k)] \\ \mathbb{E}[a_2^\top x(k-1) + B_2u_2(k-1) + w_2(k-1)|\mathscr{I}^{c_2}(k)] \end{bmatrix}
\end{aligned}
$$

From the definition of $e_1(k-1), e_2(k-1)$ and the fact that $\mathscr{I}^{c_2}(k-1) \subset \mathscr{I}^{c_1}(k), \mathscr{I}^{c_1}(k-1) \subset \mathscr{I}^{c_2}(k)$ we get

$$\hat{x}(k) = A\hat{x}(k-1) + Bu(k-1) + \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix} = \omega(k) + \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix}$$

where the last equality is due to (21). To this end we write

$$
\begin{aligned}
x(k) = \hat{x}(k) + e(k) &= \omega(k) + (x(k) - \hat{x}(k)) + \begin{bmatrix} \omega_1(k) \\ \omega_2(k) \end{bmatrix} \\
&= \omega(k) + Ae(k-1) + w(k-1)
\end{aligned} \tag{29}
$$

Now we state lemma to help derive the main result.

Lemma 2 ([9]): The cost in (6) can be written as

$$
\begin{aligned}
J_{\mathscr{C}} = &\sum_{k=0}^{T-1} \mathbb{E}\left[(u(k) - K(k)x(k))^\top \Omega(k)(u(k) - K(k)x(k))\right] \\
&+ x(0)^\top S(0)x(0) + \sum_{k=0}^{T-1} tr(S(k+1)\Sigma_w)
\end{aligned}
$$

given that $w(k)$ is zero-mean and independent from $(x(k),u(k))$. The matrices $K(k),S(k)$ are defined in Theorem 1 and $\Omega(k)=R+B^\top S(k+1)B$.

Extracting the part that is dependent on control inputs taking into account (29) we write

$$J_u := \sum_{k=0}^{T-1}\mathbb{E}\,\|u(k)-K(k)x(k)||_{\Omega_k}$$
$$=\sum_{k=0}^{T-1}\mathbb{E}\,\|u(k)-K(k)\omega(k)-K(k)(Ae(k-1)+w(k-1))\|_{\Omega_k}$$

Minimizing the last cost with respect to u_1,u_2 conditioned on $\mathscr{I}^{c_1},\mathscr{I}^{c_2}$ respectively, taking into account that $\omega(k)$ is conditioned on $\mathscr{I}^e(k)\subset\mathscr{I}^{c_1}(k)\cap\mathscr{I}^{c_2}(k)$ we get (26). For the class of encoders (22), components of control law in (26) proportional to $\omega_1(k),\omega_2(k),\hat{\delta}_1(k),\hat{\delta}_2(k)$ are only dependent on $e(k-1)$ which is equal to $\bar{e}(k-1)$ as the Proposition 1 holds. Therefore, the distortion produced by (26) in J_u is

$$D(e_1^{T-1},e_2^{T-1})=D(\bar{e}_1^{T-1},\bar{e}_2^{T-1})=$$
$$\sum_{k=0}^{T-1}\mathbb{E}\left\|\begin{bmatrix}K_{11}(k)\bar{e}_1(k)+K_{12}(k)(\bar{e}_2(k)+\omega_2(k)-\hat{\delta}^1(k))\\K_{21}(k)(\bar{e}_1(k)+\omega_1(k)-\hat{\delta}^2(k))+K_{22}(k)\bar{e}_2(k)\end{bmatrix}\right\|_{\Omega_k}$$

since there is no dual effect and this together with Lemma 2 proves (28) and concludes the proof. ■

As it is implied by Theorem 2, the computation of optimal control inputs u_1,u_2 requires knowledge of control gain $K(k)$ defined in (9) that is computed based on global system matrices A,B,Q,R. However, as such computation can be done offline, the control law in (26) is still of distributed nature. The offline computation is particularly important for in-network [17] implementation of derived optimal control law where control functionalities are pushed as close as possible to the controlled process exploiting the computational power of active network components - even if limited.

Remark 6: Theorem 2 indicates that optimal quantizers E_1,E_2 have to minimize a rather complicated distortion function D. Similar to the case of single-loop quantized system in [14], in order to derive recursive and easily implementable estimate $\omega(k)$ additional assumptions need to be introduced such as Gaussianity of predicted densites of states $x_1(k),x_2(k)$ conditioned on information set $\mathscr{I}^e(k)$. However, while resulting in more efficient control implementation such assumption would yield suboptimal solution to the problem addressed. The analysis of this subproblem and performance of suboptimal controller is outside of the scope of this paper.

Remark 7: The optimal control law in Theorem 2 is a superposition of two components. The first component is proportional to the estimator $\omega(k)$ of the global state $x(k)$, conditioned on the common information between two controllers, and thus is computed by both controllers. This common information is not the full information available to C_1, C_2 as each controller, at time instant k, also receives output from local encoder and potentially, output from the other encoder. Thus, second component represents local corrections that are applied to compensate for the discrepancy of $\omega(k)$ and actual state $x(k)$, due to the process noise $w(k-1)$, as well as estimation errors $e_1(k-1),e_2(k-1)$ which affect both systems due to physical coupling.

IV. CONCLUSIONS

In this paper a distributed quantized LQG control problem is addressed, under the assumption that individual controllers communicate with a maximum delay of one step. For a class of quantizers, that encode the estimation errors of local state estimators conditioned on the common information between controllers, a separation is established. The optimal encoders are shown to be those that minimize certain distortion function, that is control-independent and depends on the estimation error for the input-free system.

REFERENCES

[1] H. S. Witsenhausen, *A counterexample in stochastic optimum control*, SIAM J. Control, vol. 6, no. 1, pp. 131147, 1968

[2] Y.-C. Ho and K'ai-Ching Chu, *Team decision theory and information structures in optimal control problems-Part I*, IEEE Transactions on Automatic Control, vol. 17, no. 1, pp. 1522, 1972

[3] A. Lamperski and J. C. Doyle, "On the structure of state-feedback LQG controllers for distributed systems with communication delays," 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, 2011, pp. 6901-6906.

[4] A. Lamperski and J. Doyle, *Dynamic Programming Solutions for Decentralized State-Feedback LQG Problems with Communication Delays*, American Control Conference, 2012

[5] N. Matni, A. Lamperski, J. Doyle, *Optimal Two Player LQR State Feedback With Varying Delay*, In IFAC Proceedings Volumes, 2014

[6] A. Mahajan, A. Nayyar, *Sufficient statistics for linear control strategies in decentralized systems with partial history sharing* , IEEE Transactions on Automatic Control, vol.60, no.8, pp.2046-2056, 2015.

[7] Bar-Shalom, Yaakov, and Edison Tse. "Dual effect, certainty equivalence, and separation in stochastic control." IEEE Transactions on Automatic Control 19.5 (1974): 494-500.

[8] Rabi, Maben, Chithrupa Ramesh, and Karl H. Johansson. "Separated design of encoder and controller for networked linear quadratic optimal control." SIAM Journal on Control and Optimization 54.2 (2016): 662-689.

[9] Astrom, Karl J. Introduction to stochastic control theory. Courier Corporation, 2012.

[10] Cover, Thomas M., and Joy A. Thomas. Elements of information theory. John Wiley Sons, 2012.

[11] G. N. Nair, F. Fagnani, S. Zampieri and R. J. Evans, "Feedback Control Under Data Rate Constraints: An Overview," in Proceedings of the IEEE, vol. 95, no. 1, pp. 108-137, Jan. 2007. doi: 10.1109/JPROC.2006.887294

[12] Minyue Fu, "Linear quadratic Gaussian control with quantized feedback," 2009 American Control Conference, St. Louis, MO, 2009, pp. 2172-2177. doi: 10.1109/ACC.2009.5160058

[13] S. Tatikonda, A. Sahai and S. Mitter, "Stochastic linear control over a communication channel," in IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1549-1561, Sept. 2004.doi: 10.1109/TAC.2004.834430

[14] K. You and L. Xie, "Linear quadratic Gaussian control with quantised innovations Kalman filter over a symmetric channel," in IET Control Theory and Applications, vol. 5, no. 3, pp. 437-446, 17 February 2011. doi: 10.1049/iet-cta.2009.0488

[15] V. Causevic, P. Ugo Abara, and S. Hirche, "Information-Constrained Optimal Control of Distributed Systems with Power Constraints.", European Control Conference (ECC), Limassol, Cyprus, 2018

[16] V. Causevic, Y. Fanger, T. Brüdigam, and S. Hirche, "Information-Constrained Model Predictive Control with Application to Vehicle Platooning.", 21st IFAC World Congress, Berlin, Germany, 2020

[17] Rüth, J., Glebke, R., Wehrle, K., Causevic, V., and Hirche, S. (2018). "Towards in-network industrial feedback control. " In SIGCOMM Workshop on In-Network Computing, NetCompute. ACM. doi:10.1145/3229591.3229592.

[18] P. Ugo Abara, V. Causevic, and S. Hirche. "Quadratic invariance for distributed control system with intermittent observations." 2018 IEEE Conference on Decision and Control (CDC). IEEE, 2018.