A Suboptimality Approach to Distributed Linear Quadratic Optimal Control

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Abstract—**This note is concerned with a suboptimal version of the distributed linear quadratic optimal control problem for multiagent systems. Given a multiagent system with identical agent dynamics and an associated global quadratic cost functional, our objective is to design distributed control laws that achieve consensus and whose cost is smaller than an *a priori* given upper bound, for all initial states of the network that are bounded in norm by a given radius. A centralized design method is provided to compute such suboptimal controllers, involving the solution of a single Riccati inequality of dimension equal to the dimension of the agent dynamics, and the smallest nonzero and the largest eigenvalue of the Laplacian matrix. Furthermore, we relax the requirement of exact knowledge of the smallest nonzero and largest eigenvalue of the Laplacian matrix by using only lower and upper bounds on these eigenvalues. Finally, a simulation example is provided to illustrate our design method.**

Index Terms—**Consensus, distributed control, linear quadratic optimal control, multiagent systems, suboptimality.**

I. INTRODUCTION

In this note, we study the distributed linear quadratic optimal control problem for multiagent networks. This problem deals with a number of identical agents represented by a finite-dimensional linear input-state system, and an undirected graph representing the communication between these agents. Given is also a quadratic cost functional that penalizes the differences between the states of neighboring agents and the size of the local control inputs. The distributed linear quadratic control problem is the problem of finding a distributed diffusive control law that minimizes this cost functional while achieving consensus for the controlled network. This problem is nonconvex and difficult to solve, and a closed-form solution has not been provided in the literature up to now. It is also unknown under what conditions an optimal distributed diffusive control law exists in general [1]. Therefore, instead of addressing the problem formulated above, in the present note, we will study a *suboptimal* version of this optimal control problem. In other words, our aim will be to design suboptimal distributed diffusive control laws that guarantee the controlled network to reach consensus.

The distributed linear quadratic control problem has attracted extensive attention in the last decade, and has been studied from many different angles. For example, in [2]–[4], it was shown that if the quadratic cost functional involves the differences of states of

neighboring agents, then, necessarily, the optimal control laws must be distributed and diffusive. However, these references do not address the problem of designing the optimal control laws. In [5], a design method was introduced for computing suboptimal distributed *stabilizing* controllers for decoupled linear systems. In this reference, the authors consider a global linear quadratic cost functional that contains terms that penalize the states and inputs of each agent and also the relative states between each agent and its neighboring agents. In [6] and [7], methods were established for designing distributed *synchronizing* control laws for linear multiagent systems, where the control laws are derived from the solution of an algebraic Riccati equation of dimension equal to the state-space dimension of the agents. However, in these references, cost functionals were not taken explicitly into consideration.

The distributed linear quadratic optimal control problem was also addressed in [8] for multiagent systems with *single-integrator* agent dynamics. The authors obtained an expression for the optimal control law, with the optimal feedback gain given in terms of the initial conditions of *all* agents. In addition, in [9], a distributed optimal control problem was considered from the perspective of cooperative game theory. In that paper, the problem being studied was solved by transforming it into a maximization problem for linear matrix inequalities, taking into consideration the structure of the Laplacian matrix. For related work, we also mention [10]–[13] to name a few.

Also, in [14], a hierarchical control approach was introduced for linear leader–follower multiagent systems. For the case that the weighting matrices in the cost functional are chosen to be of a special form, two suboptimal controller design methods were given in this reference. In addition, in [15], an inverse optimal control problem was addressed both for leader–follower and leaderless multiagent systems. For a particular class of digraphs, the authors showed that distributed optimal controllers exist and can be obtained if the weighting matrices are assumed to be of a special form, capturing the graph information. For other work related to distributed inverse optimal control, we refer to [16] and [17].

In the present note, our objective is to design distributed diffusive control laws that guarantee the controlled network to reach consensus and to provide conditions under which the associated cost is smaller than an *a priori* given upper bound. The main contributions of this note are the following.

1) We present a design method for computing suboptimal distributed diffusive control laws, based on computing a positive definite solution of a single Riccati inequality of dimension equal to the dimension of the agent dynamics. In the computation of the local control gain, the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix are involved.

2) For the case that exact information on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix is not available, we establish a design method using only lower and upper bounds on these Laplacian eigenvalues.

The remainder of this note is organized as follows. In Section II, we introduce the required basic notation and formulate the suboptimal

distributed linear quadratic control problem. Section III presents the analysis and design of suboptimal linear quadratic control for linear systems, collecting preliminary classical results for treating the actual suboptimal distributed control problem for multiagent systems. Then, in Section IV, we study the suboptimal distributed control problem for linear multiagent systems. To illustrate our results, a simulation example is provided in Section V. Finally, in Section VI, we formulate some conclusions.

II. NOTATION AND PROBLEM FORMULATION

A. Notation

We denote by $\mathbb{R}$ the field of real numbers, and by $\mathbb{R}^n$ the n-dimensional real Euclidean space. For $x \in \mathbb{R}^n$, its Euclidean norm is defined by $\|x\| := \sqrt{x^\top x}$. For a given $r > 0$, we denote by $B(r) := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ the closed ball of radius r. We denote by $\mathbb{R}^{n \times m}$ the set of real $n \times m$ matrices. For a given matrix A, its transpose and inverse (if it exists) are denoted by $A^\top$ and A^{-1}, respectively. The identity matrix of dimension $n \times n$ is denoted by I_n. We denote the Kronecker product of two matrices A and B by $A \otimes B$, which has the property that $(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1 A_2 \otimes B_1 B_2$. For a given symmetric matrix P we denote $P > 0$ if it is positive definite and $P \geq 0$ if it is positive semidefinite. By $\mathrm{diag}(a_1, a_2, \ldots, a_n)$, we denote the $n \times n$ diagonal matrix with $a_1, a_2, \ldots, a_n$ on the diagonal. The column vector $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector whose components are all equal to 1.

A directed graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nonempty set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. A pair $(i, j) \in \mathcal{E}$, with $i, j \in \mathcal{V}$, represents an edge from node i to node j. We assume that the graph is *simple*, meaning that the edge set only contains edges of the form (i, j) with $i \neq j$. The graph is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. In this note, we will restrict ourselves to simple, undirected graphs. We denote the neighboring set of node i by $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The adjacency matrix of $\mathcal{G}$ is defined as $\mathcal{A} = [a_{ij}]$ with $a_{ij} = 1$ whenever there is an edge between the nodes i and j, and $a_{ij} = 0$ otherwise. Obviously, for simple graphs, $a_{ii} = 0$ for all i. Furthermore, a graph $\mathcal{G}$ is undirected if and only if $\mathcal{A}$ is symmetric. The Laplacian matrix is defined as $L = D - \mathcal{A}$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ with $d_i = \sum_{j=1}^N a_{ij}$ the degree matrix of $\mathcal{G}$. The Laplacian matrix L of an undirected graph is symmetric and consequently only has real eigenvalues. Furthermore, all eigenvalues are nonnegative and 0 is an eigenvalue of L. The graph is connected if and only if 0 is a simple eigenvalue of L. In the sequel we will assume that $\mathcal{G}$ is connected. In that case the eigenvalues of L can be ordered in increasing order as $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ and there exists an orthogonal matrix U such that $U^\top L U = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$. Moreover, we have $U = (\frac{1}{\sqrt{N}} \mathbf{1}_N \; U_2)$ and $U_2 U_2^\top = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^\top$.

B. Problem Formulation

In this note, we consider a multiagent system consisting of N identical agents. It will be a standing assumption that the underlying graph is simple, undirected, and connected. The corresponding Laplacian matrix is denoted by L. The dynamics of the identical agents is represented by the continuous-time linear time-invariant (LTI) system given by

$$\dot{x}_i(t) = A x_i(t) + B u_i(t), x_i(0) = x_{i0}, i = 1, 2, \ldots, N \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state and input of the agent i, respectively, and x_{i0} is its initial state. Throughout this note, we assume that the pair (A, B) is stabilizable.

We consider the infinite horizon distributed linear quadratic optimal control problem for multiagent system (1), where the global cost functional integrates the weighted quadratic difference of states between every agent and its neighbors, and also penalizes the inputs in a quadratic form. Thus, the cost functional considered in this note is given by

$$J(u) = \int_0^\infty \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (x_i - x_j)^\top Q (x_i - x_j) + \sum_{i=1}^N u_i^\top R u_i \, dt \quad (2)$$

where $Q \geq 0$ and $R > 0$ are given real weighting matrices.

We can rewrite multiagent system (1) in compact form as

$$\dot{x} = (I_N \otimes A) x + (I_N \otimes B) u, x(0) = x_0 \quad (3)$$

with $x = (x_1^\top, \ldots, x_N^\top)^\top$, $u = (u_1^\top, \ldots, u_N^\top)^\top$, where $x \in \mathbb{R}^{nN}$, $u \in \mathbb{R}^{mN}$ contain the states and inputs of all agents, respectively. Note that, although the agents have identical dynamics, we allow the initial states of the individual agents to differ. These initial states are collected in the joint vector of initial states $x_0 = (x_{10}^\top, \ldots, x_{N0}^\top)^\top$. Moreover, we can also write the cost functional (2) in compact form as

$$J(u) = \int_0^\infty x^\top (L \otimes Q) x + u^\top (I_N \otimes R) u \, dt. \quad (4)$$

The distributed linear quadratic problem is the problem of minimizing for all initial states x_0 the cost functional (4) over all distributed diffusive control laws that achieve consensus. By a distributed diffusive control law, we mean a control law of the form

$$u = (L \otimes K) x \quad (5)$$

where $K \in \mathbb{R}^{m \times n}$ is an identical feedback gain for all agents.

The adjective diffusive refers to the fact that the input of each agent depends on the relative state variables with respect to its neighbors. The control law (5) is distributed in the sense that the local gains for all agents are identical.

By interconnecting the agents using this control law, we obtain the overall network dynamics

$$\dot{x} = (I_N \otimes A + L \otimes BK) x. \quad (6)$$

Foremost, we want the control law to achieve consensus:

Definition 1: We say the network reaches consensus using control law (5) if for all $i, j = 1, 2, \ldots, N$ and for all initial states x_{i0} and x_{j0}, we have

$$x_i(t) - x_j(t) \to 0 \text{ as } t \to \infty.$$

As a function of the to-be-designed local feedback gain K, the cost functional (4) can be rewritten as

$$J(K) = \int_0^\infty x^\top (L \otimes Q + L^2 \otimes K^\top RK) x \, dt. \quad (7)$$

In other words, the distributed linear quadratic control problem is the problem of minimizing the cost functional (7) over all $K \in \mathbb{R}^{m \times n}$ such that the controlled network (6) reaches consensus.

Due to the distributed nature of the control law (5) as imposed by the network topology, the distributed linear quadratic problem is a nonconvex optimization problem. It is therefore difficult, if not impossible, to find a closed form solution for an optimal controller, or such optimal controller may not even exist. Therefore, as mentioned in Section I, in this note, we will study and resolve a version of this problem involving the design of suboptimal distributed control laws.

More specifically, let $B(r) = \{x \in \mathbb{R}^{nN} \mid \|x\| \leq r\}$ be the closed ball of radius r in the joint state-space $\mathbb{R}^{nN}$ of the network (3). Then, for system (3) with initial states in such a closed ball of a given radius, we want to design a distributed diffusive controller such that consensus is achieved and, for all initial states in the given ball, the associated cost

is smaller than an *a priori* given upper bound. Thus, we will consider the following problem:

Problem 1: Consider the multiagent system (3) and associated cost functional given by (7). Let $r > 0$ be a given radius and let $\gamma > 0$ be an *a priori* given upper bound for the cost. The problem is to find a distributed diffusive controller of the form (5) such that the controlled network (6) reaches consensus, and for all $x_0 \in B(r)$ the associated cost (7) is smaller than the given upper bound, i.e., $J(K) < \gamma$.

Remark 2: Note that we could also have formulated the alternative problem of finding a suboptimal controller for a *single, given, initial state* x_0. In fact, this would be closer to the classical linear quadratic problem, which is usually formulated as the problem of minimizing the cost functional for a *given* initial state x_0. In that context, however, the optimal controller is a state feedback that turns out to be optimal *for all initial states*. In order to capture in our problem formulation this property of being optimal for all initial states, we have formulated Problem 1 in terms of initial states contained in a ball of a given radius.

Before we address Problem 1, we will first briefly discuss the suboptimal linear quadratic problem for a single linear system. This will be the subject of the next section.

III. SUBOPTIMAL CONTROL FOR LINEAR SYSTEMS

In this section, we consider a suboptimal linear quadratic control problem for single linear systems. The results presented in this section are standard and can be found scattered over the literature, see e.g., [18]–[20]. Exact references are however hard to give and therefore, in order to make this note self-contained, we will collect the required results here and provide their proofs.

We will first analyze the quadratic performance of a given autonomous system. Subsequently, we will discuss how to design suboptimal control laws for a linear system with inputs.

A. Suboptimality Analysis for Autonomous Systems

Consider the autonomous system

$$\dot{x}(t) = \bar{A}x(t), x(0) = x_0 \tag{8}$$

where $\bar{A} \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ is the state. We consider the quadratic performance of system (8), given by

$$J = \int_0^\infty x^\top \bar{Q}x \, dt \tag{9}$$

where $\bar{Q} \geq 0$ is a given real weighting matrix. Note that the performance J is finite if system (8) is stable, i.e., $\bar{A}$ is Hurwitz.

We are interested in finding conditions such that the performance (9) of system (8) is smaller than a given upper bound. For this, we have the following lemma.

Lemma 3: Consider system (8) with the corresponding quadratic performance (9). The performance (9) is finite if system (8) is stable, i.e., $\bar{A}$ is Hurwitz. In this case, it is given by

$$J = x_0^\top Y x_0 \tag{10}$$

where Y is the unique positive semidefinite solution of

$$\bar{A}^\top Y + Y\bar{A} + \bar{Q} = 0. \tag{11}$$

Alternatively

$$J = \inf\{x_0^\top P x_0 \mid P > 0 \text{ and } \bar{A}^\top P + P\bar{A} + \bar{Q} < 0\}. \tag{12}$$

Proof: The fact that the quadratic performance (9) is given by the quadratic expression (10) involving the Lyapunov (11) is well-known.

We will now prove (12). Let Y be the solution to Lyapunov (11) and let P be a positive definite solution to the Lyapunov inequality in (12). Define $X := P - Y$. Then, we have

$$\bar{A}^\top(X + Y) + (X + Y)\bar{A} + \bar{Q} < 0.$$

So consequently

$$\bar{A}^\top X + X\bar{A} < 0.$$

Since $\bar{A}$ is Hurwitz, it follows that $X > 0$. Thus, we have $P > Y$ and hence $J \leq x_0^\top P x_0$ for any positive definite solution P to the Lyapunov inequality.

Next we will show that for any $\epsilon > 0$ there exists a positive definite matrix P_ϵ satisfying the Lyapunov inequality such that $P_\epsilon < Y + \epsilon I$, and consequently $x_0^\top P_\epsilon x_0 \leq J + \epsilon \|x_0\|^2$. Indeed, for given ϵ, take P_ϵ equal to the unique positive definite solution of

$$\bar{A}^\top P + P\bar{A} + \bar{Q} + \epsilon I = 0. \tag{13}$$

Clearly then, $P_\epsilon = \int_0^\infty e^{\bar{A}^\top t}(\bar{Q} + \epsilon I)e^{\bar{A}t} \, dt$, so $P_\epsilon \downarrow Y$ as $\epsilon \downarrow 0$. This proves our claim. ∎

The following theorem now yields *necessary* and *sufficient* conditions such that, for a given upper bound $\gamma > 0$, the quadratic performance (9) satisfies $J < \gamma$.

Theorem 4: Consider system (8) with the associated quadratic performance (9). For given $\gamma > 0$, we have that $\bar{A}$ is Hurwitz and $J < \gamma$ if and only if there exists a positive definite matrix P satisfying

$$\bar{A}^\top P + P\bar{A} + \bar{Q} < 0 \tag{14}$$

$$x_0^\top P x_0 < \gamma. \tag{15}$$

Proof: (if) Since there exists a positive definite solution to the Lyapunov inequality (14), it follows that $\bar{A}$ is Hurwitz. Take a positive definite matrix P satisfying the inequalities (14) and (15). By Lemma 3, we then immediately have $J \leq x_0^\top P x_0 < \gamma$.

(only if) If $\bar{A}$ is Hurwitz and $J < \gamma$, then, again by Lemma 3, there exists a positive definite solution P to the Lyapunov inequality (14) such that $J \leq x_0^\top P x_0 < \gamma$. ∎

In the next section, we will discuss the suboptimal control problem for a linear system with inputs.

B. Suboptimal Control Design for Linear Systems With Inputs

In this section, we consider the finite dimensional LTI system given by

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \tag{16}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$, and $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ are the state and the input, respectively, and x_0 is a given initial state. Assume that the pair (A, B) is stabilizable. The associated cost functional is given by

$$J(u) = \int_0^\infty x^\top Q x + u^\top R u \, dt \tag{17}$$

where $Q \geq 0$ and $R > 0$ are given weighting matrices that penalize the state and input, respectively.

Given $\gamma > 0$ and initial state x_0, we want to find a state feedback control law $u = Kx$ such that the closed system

$$\dot{x}(t) = (A + BK)x(t) \tag{18}$$

is stable and the corresponding cost

$$J(K) = \int_0^\infty x^\top(Q + K^\top RK)x \, dt \tag{19}$$

satisfies $J(K) < \gamma$.

The following theorem gives a sufficient condition for the existence of such control law.

Theorem 5: Consider the system (16) with initial state x_0 and associated cost functional (17). Assume that the pair (A, B) is stabilizable. Let $\gamma > 0$. Suppose that there exists a positive definite P satisfying

$$A^\top P + PA - PBR^{-1}B^\top P + Q < 0 \qquad (20)$$

$$x_0^\top P x_0 < \gamma. \qquad (21)$$

Let $K := -R^{-1}B^\top P$. Then, the controlled system (18) is stable and the control law $u = Kx$ is suboptimal, i.e., $J(K) < \gamma$.

Proof: Substituting $K := -R^{-1}B^\top P$ into (18) yields

$$\dot{x}(t) = (A - BR^{-1}B^\top P)x(t), x(0) = x_0. \qquad (22)$$

Since P satisfies (20), it should also satisfy

$$(A - BR^{-1}B^\top P)^\top P + P(A - BR^{-1}B^\top P)$$
$$+ Q + PBR^{-1}B^\top P < 0$$

which implies that $A - BR^{-1}B^\top P$ is Hurwitz, i.e., the closed system (22) is stable. Consequently, the corresponding cost is finite and equal to

$$J(K) = \int_0^\infty x^\top (Q + PBR^{-1}B^\top P)x \, dt.$$

Since (21) holds, by taking $\bar{A} = A - BR^{-1}B^\top P$ and $\bar{Q} = Q + PBR^{-1}B^\top P$ in Theorem 4, we immediately have $J(K) < \gamma$. ∎

In the next section, we will apply the above results to tackle the suboptimal distributed linear quadratic control problem for multiagent systems as formulated in Problem 1.

IV. SUBOPTIMAL CONTROL DESIGN FOR LINEAR MULTIAGENT SYSTEMS

Again consider the multiagent system with the dynamics of the identical agents represented by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), x_i(0) = x_{i0}, i = 1, 2, \ldots, N \qquad (23)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state and input of the ith agent, respectively, and x_{i0} its initial state. We assume that the pair (A, B) is stabilizable.

Denoting $x = \left(x_1^\top, \ldots, x_N^\top\right)^\top, u = \left(u_1^\top, \ldots, u_N^\top\right)^\top$, we can rewrite the multiagent system in compact form as

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u, x(0) = x_0. \qquad (24)$$

The cost functional we consider was already introduced in (4). We repeat it here for convenience:

$$J(u) = \int_0^\infty x^\top (L \otimes Q)x + u^\top (I_N \otimes R)u \, dt \qquad (25)$$

where $Q \geq 0$ and $R > 0$ are given real weighting matrices.

As formulated in Problem 1, given a desired upper bound $\gamma > 0$, for multiagent system (24) with initial states contained in the closed ball $B(r)$ of given radius r we want to design a control law of the form

$$u = (L \otimes K)x \qquad (26)$$

such that the controlled network

$$\dot{x} = (I_N \otimes A + L \otimes BK)x \qquad (27)$$

reaches consensus and, moreover, for all $x_0 \in B(r)$ the associated cost

$$J(K) = \int_0^\infty x^\top \left(L \otimes Q + L^2 \otimes K^\top RK\right)x \, dt \qquad (28)$$

is smaller than the given upper bound, i.e., $J(K) < \gamma$.

Let the matrix $U \in \mathbb{R}^{N \times N}$ be an orthogonal matrix that diagonalizes the Laplacian L. Define $\Lambda := U^\top LU = \text{diag}(0, \lambda_2, \ldots, \lambda_N)$. To simplify the problem given above, by applying the state and input transformations $\bar{x} = (U^\top \otimes I_n)x$ and $\bar{u} = (U^\top \otimes I_m)u$ with $\bar{x} = \left(\bar{x}_1^\top, \ldots, \bar{x}_N^\top\right)^\top, \bar{u} = \left(\bar{u}_1^\top, \ldots, \bar{u}_N^\top\right)^\top$, system (24) becomes

$$\dot{\bar{x}} = (I_N \otimes A)\bar{x} + (I_N \otimes B)\bar{u}, \bar{x}(0) = \bar{x}_0 \qquad (29)$$

with $\bar{x}_0 = (U^\top \otimes I_n)x_0$. Clearly, (26) is transformed to

$$\bar{u} = (\Lambda \otimes K)\bar{x} \qquad (30)$$

and the controlled network (27) transforms to

$$\dot{\bar{x}} = (I_N \otimes A + \Lambda \otimes BK)\bar{x}. \qquad (31)$$

In terms of the transformed variables, the cost (28) is given by

$$J(K) = \int_0^\infty \sum_{i=1}^N \bar{x}_i^\top (\lambda_i Q + \lambda_i^2 K^\top RK)\bar{x}_i \, dt. \qquad (32)$$

Note that the transformed states $\bar{x}_i$ and inputs $\bar{u}_i$, $i = 2, 3, \ldots, N$ appearing in system (31) and cost (32) are decoupled from each other, so that we can write system (31) and cost (32) as

$$\dot{\bar{x}}_1 = A\bar{x}_1 \qquad (33)$$

$$\dot{\bar{x}}_i = (A + \lambda_i BK)\bar{x}_i, i = 2, 3, \ldots, N \qquad (34)$$

and

$$J(K) = \sum_{i=2}^N J_i(K) \qquad (35)$$

with

$$J_i(K) = \int_0^\infty \bar{x}_i^\top (\lambda_i Q + \lambda_i^2 K^\top RK)\bar{x}_i \, dt, i = 2, 3, \ldots, N. \qquad (36)$$

Note that $\lambda_1 = 0$, and that therefore (33) does not contribute to the cost $J(K)$.

We first record a well-known fact (see [21] and [22]), which we will use later.

Lemma 6: Consider the multiagent system (24). Then, the controlled network reaches consensus with control law (26) if and only if, for $i = 2, 3, \ldots, N$, the systems (34) are stable.

Thus, we have transformed the problem of distributed suboptimal control for system (24) into the problem of finding a feedback gain $K \in \mathbb{R}^{m \times n}$ such that the systems (34) are stable and $J(K) < \gamma$. Moreover, since the pair (A, B) is stabilizable, there exists such a feedback gain K [22].

The following lemma gives a necessary and sufficient condition for a given feedback gain K to make all systems (34) stable and such that $J(K) < \gamma$ is satisfied for given initial states.

Lemma 7: Let K be a feedback gain. Consider the systems (34) with given initial states $\bar{x}_{20}, \bar{x}_{30}, \ldots, \bar{x}_{N0}$ and associated cost functionals (35) and (36). Let $\gamma > 0$. Then, all systems (34) are stable and $J(K) < \gamma$ if and only if there exist positive definite matrices P_i satisfying

$$(A + \lambda_i BK)^\top P_i + P_i(A + \lambda_i BK) + \lambda_i Q + \lambda_i^2 K^\top RK < 0 \qquad (37)$$

$$\sum_{i=2}^N \bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma \qquad (38)$$

for $i = 2, 3, \ldots, N$, respectively.

Proof: (if) Since (38) holds, there exist sufficiently small $\epsilon_i > 0$, $i = 2, \ldots, N$ such that $\sum_{i=2}^{N} \gamma_i < \gamma$ where $\gamma_i := \bar{x}_{i0}^{\top} P_i \bar{x}_{i0} + \epsilon_i$. Because there exists P_i such that (37) and $\bar{x}_{i0}^{\top} P_i \bar{x}_{i0} < \gamma_i$ holds for all $i = 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i B K$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^{\top} R K$, it follows from Theorem 4 that all systems (34) are stable and $J_i(K) < \gamma_i$ for $i = 2, \ldots, N$. Since $J(K) = \sum_{i=2}^{N} J_i(K)$, this implies that $J(K) < \sum_{i=2}^{N} \gamma_i < \gamma$.

(only if) Since $J(K) < \gamma$ and $J(K) = \sum_{i=2}^{N} J_i(K)$, there exist sufficiently small $\epsilon_i > 0$, $i = 2, \ldots, N$ such that $\sum_{i=2}^{N} \gamma_i < \gamma$ where $\gamma_i := J_i(K) + \epsilon_i$. Because all systems (34) are stable and $J_i(K) < \gamma_i$ for $i = 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i B K$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^{\top} R K$, it follows from Theorem 4 that there exist positive definite P_i such that (37) and $\bar{x}_{i0}^{\top} P_i \bar{x}_{i0} < \gamma_i$ hold for all $i = 2, \ldots, N$. Since $\sum_{i=2}^{N} \gamma_i < \gamma$, this implies that $\sum_{i=2}^{N} \bar{x}_{i0}^{\top} P_i \bar{x}_{i0} < \sum_{i=2}^{N} \gamma_i < \gamma$. ∎

Lemma 7 establishes a necessary and sufficient condition for a given feedback gain K to stabilize all systems (34) and to satisfy, for given initial states of these systems, $J(K) < \gamma$. However, Lemma 7 does not yet provide a method to compute such K. In the following, we present a method to find such K.

Lemma 8: Consider the multiagent system (24) with associated cost functional (28). Let x_0 be a given initial state for the multiagent system. Let $\gamma > 0$. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish the following two cases:

1) if

$$\frac{2}{\lambda_2 + \lambda_N} \leq c < \frac{2}{\lambda_N} \tag{39}$$

then there exists $P > 0$ satisfying the Riccati inequality

$$A^{\top} P + PA + (c^2 \lambda_N^2 - 2c\lambda_N) PBR^{-1} B^{\top} P + \lambda_N Q < 0. \tag{40}$$

2) if

$$0 < c < \frac{2}{\lambda_2 + \lambda_N} \tag{41}$$

then there exists $P > 0$ satisfying

$$A^{\top} P + PA + (c^2 \lambda_2^2 - 2c\lambda_2) PBR^{-1} B^{\top} P + \lambda_N Q < 0. \tag{42}$$

In both cases, if in addition P satisfies

$$x_0^{\top} \left((I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^{\top}) \otimes P \right) x_0 < \gamma \tag{43}$$

then the controlled network (27) with $K := -cR^{-1} B^{\top} P$ reaches consensus and with the initial state x_0 we have $J(K) < \gamma$. ∎

Proof: We will only give the proof for case (1) above. Using the upper and lower bounds on c given by (39), it can be verified that $c^2 \lambda_i^2 - 2c\lambda_i \leq c^2 \lambda_N^2 - 2c\lambda_N < 0$ for $i = 2, 3, \ldots, N$. It is then easily seen that (40) has many positive definite solutions. Since also $\lambda_i \leq \lambda_N$, any such solution P is a solution to the $N-1$ Riccati inequalities

$$A^{\top} P + PA + (c^2 \lambda_i^2 - 2c\lambda_i) PBR^{-1} B^{\top} P + \lambda_i Q < 0$$
$$i = 2, \ldots, N. \tag{44}$$

Equivalently, P also satisfies the Lyapunov inequalities

$$(A - c\lambda_i BR^{-1} B^{\top} P)^{\top} P + P(A - c\lambda_i BR^{-1} B^{\top} P)$$
$$+ \lambda_i Q + c^2 \lambda_i^2 PBR^{-1} B^{\top} P < 0, i = 2, \ldots, N. \tag{45}$$

Next, recall that $\bar{x} = (U^{\top} \otimes I_n)x$ with $U = (\frac{1}{\sqrt{N}} \mathbf{1}_N \ U_2)$. From this it is easily seen that $(\bar{x}_{20}^{\top}, \bar{x}_{30}^{\top}, \cdots, \bar{x}_{N0}^{\top})^{\top} = (U_2^{\top} \otimes I_n)x_0$. Also,

$U_2 U_2^{\top} = I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^{\top}$. Since (43) holds, we have

$$x_0^{\top} (U_2 U_2^{\top} \otimes P) x_0 < \gamma \Leftrightarrow$$

$$((U_2^{\top} \otimes I_n)x_0)^{\top} (I_{N-1} \otimes P) ((U_2^{\top} \otimes I_n)x_0) < \gamma \Leftrightarrow$$

$$(\bar{x}_{20}^{\top}, \bar{x}_{30}^{\top}, \cdots, \bar{x}_{N0}^{\top}) (I_{N-1} \otimes P) (\bar{x}_{20}^{\top}, \bar{x}_{30}^{\top}, \cdots, \bar{x}_{N0}^{\top})^{\top} < \gamma$$

which is equivalent to

$$\sum_{i=2}^{N} \bar{x}_{i0}^{\top} P \bar{x}_{i0} < \gamma. \tag{46}$$

Taking $P_i = P$ for $i = 2, 3, \ldots, N$ and $K := -cR^{-1} B^{\top} P$ in inequalities (37) and (38) immediately gives us inequalities (45) and (46). Then, it follows from Lemma 7 that all systems (34) are stable and $J(K) < \gamma$. Furthermore, it follows from Lemma 6 that the controlled network (27) reaches consensus. ∎

We will now apply Lemma 8 to establish a solution to Problem 1. Indeed, the next main theorem gives a condition under which, for given radius r and upper bound γ, suboptimal distributed diffusive control laws exist, and explain how these can be computed.

Theorem 9: Consider the multiagent system (24) with associated cost functional (28). Let $r > 0$ be a given radius and let $\gamma > 0$ be an *a priori* given upper bound for the cost. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish the following two cases:

1) if

$$\frac{2}{\lambda_2 + \lambda_N} \leq c < \frac{2}{\lambda_N} \tag{47}$$

then there exists $P > 0$ satisfying the Riccati inequality

$$A^{\top} P + PA + (c^2 \lambda_N^2 - 2c\lambda_N) PBR^{-1} B^{\top} P + \lambda_N Q < 0. \tag{48}$$

2) if

$$0 < c < \frac{2}{\lambda_2 + \lambda_N} \tag{49}$$

then there exists $P > 0$ satisfying

$$A^{\top} P + PA + (c^2 \lambda_2^2 - 2c\lambda_2) PBR^{-1} B^{\top} P + \lambda_N Q < 0. \tag{50}$$

In both cases, if in addition P satisfies

$$P < \frac{\gamma}{r^2} I \tag{51}$$

then the controlled network (27) with $K := -cR^{-1} B^{\top} P$ reaches consensus and $J(K) < \gamma$ for all $x_0 \in B(r)$.

Proof: Again, we only give the proof for case (1) above. Let $P > 0$ satisfy (48) and (51) holds. Our aim is to prove that (43) is satisfied for all $x_0 \in B(r)$. First note that

$$\frac{1}{N} \mathbf{1}_N \mathbf{1}_N^{\top} \otimes P = \frac{1}{N} (\mathbf{1}_N \otimes P^{\frac{1}{2}})(\mathbf{1}_N \otimes P^{\frac{1}{2}})^{\top}$$

which is therefore positive semidefinite. Now, for all $x_0 \in B(r)$ we have

$$x_0^{\top} \left(\left(I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^{\top} \right) \otimes P \right) x_0$$

$$\leq x_0^{\top} (I_N \otimes P) x_0 < \frac{\gamma}{r^2} x_0^{\top} x_0 \leq \gamma.$$

By Lemma 8 then, the controlled network (27) with the given K reaches consensus and $J(K) < \gamma$ for all $x_0 \in B(r)$. ∎

Remark 10: Theorem 9 states that after choosing c satisfying the inequality (47) for case (1) and finding a positive definite P satisfying (48) and (51), the distributed control law with local gain $K = -cR^{-1} B^{\top} P$

is γ-suboptimal for all initial states of the network in the closed ball with radius r. By (51), the smaller the solution P of (48), the smaller the quotient $\frac{\gamma}{r^2}$ is allowed to be, leading to a smaller upper bound and a larger radius. The question then arises: how should we choose the parameter c in (47) so that the Riccati inequality (48) allows a positive definite solution that is as small as possible? In fact, one can find a positive definite solution $P(c, \epsilon)$ to (48) by solving the Riccati equation

$$A^\top P + PA - PB\bar{R}(c)^{-1}B^\top P + \bar{Q}(\epsilon) = 0 \qquad (52)$$

with $\bar{R}(c) = \frac{1}{-c^2\lambda_N^2 + 2c\lambda_N}R$ and $\bar{Q}(\epsilon) = \lambda_N Q + \epsilon I_n$ where c is chosen as in (47) and $\epsilon > 0$. If c_1 and c_2 as in (47) satisfy $c_1 \leq c_2$, then we have $\bar{R}(c_1) \leq \bar{R}(c_2)$, so, clearly, $P(c_1, \epsilon) \leq P(c_2, \epsilon)$. Similarly, if $0 < \epsilon_1 \leq \epsilon_2$, we immediately have $\bar{Q}(\epsilon_1) \leq \bar{Q}(\epsilon_2)$. Again, it follows that $P(c, \epsilon_1) \leq P(c, \epsilon_2)$. Therefore, if we choose $\epsilon > 0$ very close to 0 and $c = \frac{2}{\lambda_2 + \lambda_N}$, we find the "best" solution to the Riccati inequality (48) in the sense explained above.

Likewise, if c satisfies (49) corresponding to case (2), it can be shown that if we choose $\epsilon > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_2 + \lambda_N}$, we find the "best" solution to the Riccati inequality (50) in the sense explained above.

In Theorem 9, in order to compute a suitable feedback gain K, one needs to know λ_2 and λ_N, the smallest nonzero eigenvalue (the algebraic connectivity) and the largest eigenvalue of the Laplacian matrix, exactly. This requires so-called global information on the network graph which might not always be available. There exist algorithms to estimate λ_2 in a distributed way, yielding lower and upper bounds, see e.g., [23]. Moreover, also an upper bound for λ_N can be obtained in terms of the maximal node degree of the graph, see [24]. Then, the question arises: can we still find a suboptimal controller reaching consensus, using as information only a *lower bound* for λ_2 and an *upper bound* for λ_N? The answer to this question is affirmative, as shown in the following theorem.

Theorem 11: Let a lower bound for λ_2 be given by $l_2 > 0$ and an upper bound for λ_N be given by L_N. Let $r > 0$ be a given radius and let $\gamma > 0$ be an *a priori* given upper bound for the cost. Choose c such that

$$\frac{2}{l_2 + L_N} \leq c < \frac{2}{L_N}. \qquad (53)$$

Then, there exists $P > 0$ such that

$$A^\top P + PA + (c^2 L_N^2 - 2cL_N)PBR^{-1}B^\top P + L_N Q < 0. \qquad (54)$$

If, in addition, P satisfies

$$P < \frac{\gamma}{r^2}I \qquad (55)$$

then the controlled network with local gain $K = -cR^{-1}B^\top P$ reaches consensus and $J(K) < \gamma$ for all initial states $x_0 \in B(r)$.

Furthermore, if we choose c such that

$$0 < c < \frac{2}{l_2 + L_N} \qquad (56)$$

then there exists $P > 0$ such that

$$A^\top P + PA + (c^2 l_2^2 - 2cl_2)PBR^{-1}B^\top P + L_N Q < 0. \qquad (57)$$

If, in addition, P satisfies (55), then the controlled network with $K := -cR^{-1}B^\top P$ reaches consensus and $J(K) < \gamma$ for all $x_0 \in B(r)$.

Proof: A proof can be given along the lines of the proof of Theorem 9. ∎

Remark 12: Note that also in Theorem 11 the question arises how to choose $c > 0$ such that the Riccati inequalities (54) and (57) admit a positive definite solution that is as small as possible. Following the

same ideas as in Remark 10, if we choose $\epsilon > 0$ very close to 0 and $c > 0$ equal to $\frac{2}{l_2 + L_N}$ in (54) [respectively very close to $\frac{2}{l_2 + L_N}$ in (57)], we find the "best" solution to the Riccati inequalities (54) and (57).

Moreover, one may also ask the question: can we compare, with the same choice for c, solutions to (54) with solutions to (48), and also solutions to (57) with solutions to (50)? The answer is affirmative. Choose c that satisfies both conditions (47) and (53). One can then check that the computed positive definite solution to (54) is indeed "larger" than that to (48) as explained in Remark 10. A similar remark holds for the positive definite solutions to (57) and corresponding solutions to (50) if c satisfies both (49) and (56). We conclude that if, instead of using the exact values λ_2 and λ_N, we use a lower bound, respectively, upper bound for these eigenvalues, then the computed distributed control law is suboptimal for "less" initial states of the agents.

Remark 13: As a final remark, we note that the theory developed in this note carries over unchanged to the case of undirected *weighted* graphs. In that case the expression for cost functional (2) should be changed to

$$J(u) = \int_0^\infty \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x_i - x_j)^\top Q(x_i - x_j) + \sum_{i=1}^N u_i^\top Ru_i \, dt$$

in which $\mathcal{A} = [a_{ij}]$ is the *weighted* adjacency matrix. Denoting the corresponding weighted Laplacian matrix by L, also this cost functional can be represented in compact form by (4), and the subsequent development will remain the same.

V. ILLUSTRATIVE EXAMPLE

In this section, we use a simulation example borrowed from [14] to illustrate the proposed design method for suboptimal distributed controllers. Consider a group of eight linear oscillators with identical dynamics

$$\dot{x}_i = Ax_i + Bu_i, x_i(0) = x_{i0}, i = 1, \ldots, 8 \qquad (58)$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume the underlying graph is the undirected line graph with Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

We consider the cost functional

$$J(u) = \int_0^\infty x^\top (L \otimes Q)x + u^\top (I_8 \otimes R)u \, dt \qquad (59)$$

where the matrices Q and R are chosen to be

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, R = 1.$$

Let the desired upper bound for the cost functional (59) be given as $\gamma = 3$. Our goal is to design a control law $u = (L \otimes K)x$ such that the
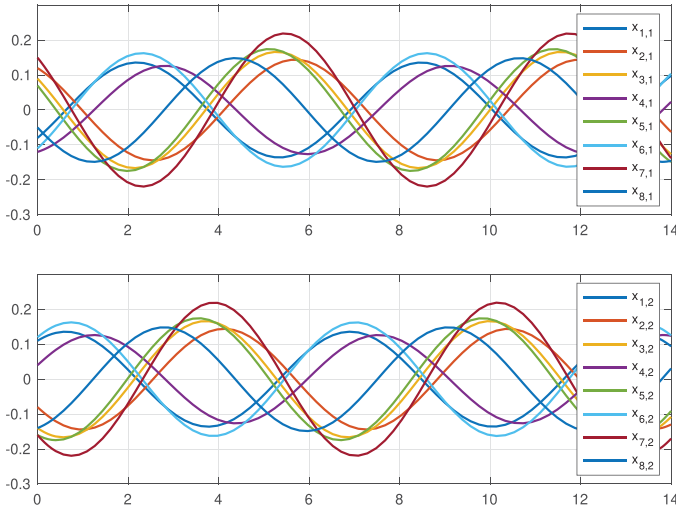
Fig. 1.	Plots of the state vector $x^1 = (x_{1,1}, \ldots, x_{8,1})$ (upper plot) and $x^2 = (x_{1,2}, \ldots, x_{8,2})$ (lower plot) of the eight decoupled oscillators without control.
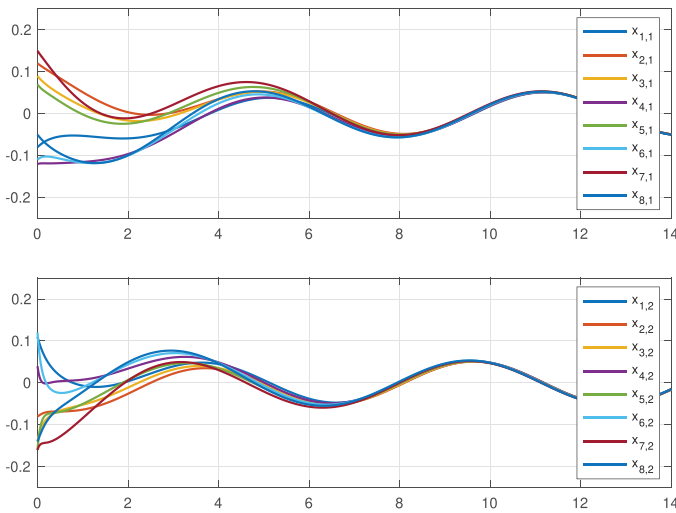

Fig. 2.	Plots of the state vector $x^1 = (x_{1,1}, \ldots, x_{8,1})$ (upper plot) and $x^2 = (x_{1,2}, \ldots, x_{8,2})$ (lower plot) of the controlled oscillator network.

controlled network reaches consensus and the associated cost is less than γ for all initial states x_0 in a closed ball $B(r)$ with radius r. The radius r will be specified later on in this example.

In this example, we adopt the control design method given in case (1) of Theorem 9. The smallest nonzero and largest eigenvalue of L are $\lambda_2 = 0.0979$ and $\lambda_8 = 3.8478$. First, we compute a positive definite solution P to (48) by solving the Riccati equation

$$A^\top P + PA + (c^2\lambda_8^2 - 2c\lambda_8)PBR^{-1}B^\top P + \lambda_8 Q + \epsilon I_2 = 0 \quad (60)$$

with $\epsilon > 0$ chosen small as mentioned in Remark 10. Here, we choose $\epsilon = 0.001$. Moreover, we choose $c = \frac{2}{\lambda_2 + \lambda_8} = 0.5$, which is the "best" choice as mentioned in Remark 10. Then, by solving (60) in Matlab, we obtain

$$P = \begin{pmatrix} 12.1168 & 3.1303 \\ 3.1303 & 8.3081 \end{pmatrix}.$$

Correspondingly, the local feedback gain is then equal to $K = (-1.5652 \quad -4.1541)$. We now compute the radius r of a ball $B(r)$ of initial states for which the distributed control law $u = (L \otimes K)x$ is suboptimal, i.e., $J(K) < 3$. We compute that the largest eigenvalue of P is equal to 13.8765. Hence, for every radius r such that $\frac{3}{r^2} > 13.8765$ the inequality (55) holds. Thus, the distributed controller with local gain K is suboptimal for all x_0 with $\|x_0\| \leq r$ with $r < 0.4650$.

As an example, the following initial states of the agents satisfy this norm bound: $x_{10}^\top = (-0.08 \quad 0.11)$, $x_{20}^\top = (0.12 \quad -0.08)$, $x_{30}^\top = (0.09 \quad -0.14)$, $x_{40}^\top = (-0.12 \quad 0.04)$, $x_{50}^\top = (0.07 \quad -0.16)$, $x_{60}^\top = (-0.11 \quad 0.12)$, $x_{70}^\top = (0.15 \quad -0.16)$, $x_{80}^\top = (-0.05 \quad -0.14)$. The plots of the eight decoupled oscillators without control are shown in Fig. 1.

Fig. 2 shows that the controlled network of oscillators reaches consensus.

VI. Conclusion

In this note, we have studied a suboptimal distributed linear quadratic control problem for undirected linear multiagent networks. We have considered a multiagent system with identical linear agent dynamics and an associated global quadratic cost functional. For these, we have provided a design method to compute distributed diffusive control laws whose cost is bounded by a given upper bound for all initial states in a closed ball of a given radius, and such that the controlled network reaches consensus. The computation of the local gain involves finding solutions of a single Riccati inequality, whose dimension is equal to the dimension of the agent dynamics, and also involves the smallest nonzero and largest eigenvalue of the Laplacian matrix. As an extension, we have removed the requirement of having exact knowledge on the smallest nonzero and largest eigenvalue of the Laplacian matrix by, instead, using only lower and upper bounds for these eigenvalues.

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