Distributed Linear Quadratic Tracking Control for Leader-Follower Multi-Agent Systems: A Suboptimality Approach ⋆

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Abstract: In this paper, we extend the results from Jiao et al. (2019) on distributed linear quadratic control for leaderless multi-agent systems to the case of distributed linear quadratic tracking control for leader-follower multi-agent systems. Given one autonomous leader and a number of homogeneous followers, we introduce an associated global quadratic cost functional. We assume that the leader shares its state information with at least one of the followers and the communication between the followers is represented by a connected simple undirected graph. Our objective is to design distributed control laws such that the controlled network reaches tracking consensus and, moreover, the associated cost is smaller than a given tolerance for all initial states bounded in norm by a given radius. We establish a centralized design method for computing such suboptimal control laws, involving the solution of a single Riccati inequality of dimension equal to the dimension of the local agent dynamics, and the smallest and the largest eigenvalue of a given positive definite matrix involving the underlying graph. The proposed design method is illustrated by a simulation example.

Keywords: Distributed control, tracking consensus, linear quadratic control, multi-agent systems, leader-follower systems, suboptimality.

1. INTRODUCTION

Distributed control for multi-agent systems has drawn much attention in the past two decades due to its practical applications, e.g., formation control, intelligent transportation systems and power grids. In the literature, basically two types of multi-agent systems are considered, namely leaderless multi-agent systems and leader-follower multi-agent systems. In the leaderless case, the local agents reach agreement which depends on the dynamics of all agents (Olfati-Saber and Murray (2004), Trentelman et al. (2013)). In the leader-follower case, the states or the outputs of the followers track that of the leader (Hong et al. (2006), Ni and Cheng (2010)). One of the attractive directions in distributed control for multi-agent systems is to design distributed control laws that minimize certain global or local performances, while reaching an agreement for the controlled network.

In the past, quite some work has been devoted to distributed linear quadratic (LQ) optimal control for leaderless multi-agent systems. In Tuna (2008), an LQR based method was used to design distributed synchronizing control laws for a multi-agent system, without taking any performance into consideration. In Borrelli and Keviczky (2008), suboptimal distributed stabilizing control laws were established for a multi-agent system with general agent dynamics with respect to an associated global cost functional, while in Cao and Ren (2010), the optimal synchronizing control gain was computed for leaderless multi-agent systems with single integrator agent dynamics. In the meantime, the distributed LQ control problem was also considered in Semsar-Kazerooni and Khorasani (2009) by utilizing a game theoretic approach, in Movric and Lewis (2014) by adopting an inverse optimal approach, and later in Jiao et al. (2019) by employing a suboptimality approach. For other papers related to this topic, see also Zhang et al. (2015) and Jiao et al. (2020).

On the other hand, distributed LQ tracking control for leader-follower multi-agent systems has also attracted much attention. In Zhang et al. (2011), distributed synchronizing control laws were established using an LQR based approach without optimizing any performance. Later on, in Cheng and Ugrinovskii (2015), suboptimal distributed control laws were proposed for achieving guaranteed cost. In Nguyen (2015), a hierarchical LQR based method was presented to design suboptimal synchronizing control laws for leader-follower systems, and an inverse optimal approach was introduced in Movric and Lewis (2014), see also Zhang et al. (2015).

In the present paper we extend the results from Jiao et al. (2019) on distributed LQ control for leaderless multi-agent systems to the case of distributed LQ tracking control for leader-follower multi-agent systems. Given a leader-follower system with one autonomous leader and a number of followers, we introduce an associated global quadratic cost functional. We assume that the leader shares its state information with at least one of the followers, and the

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communication between the followers is represented by a connected simple undirected graph. Our aim is then to design distributed diffusive control laws such that the controlled network reaches tracking consensus, i.e., the states of the followers track the state of the leader asymptotically and the associated cost is smaller than an a priori given upper bound.

The outline of this paper is as follows. Section 2 provides some preliminaries on graph theory and quadratic performance analysis for linear systems. In Section 3, we formulate the suboptimal distributed linear quadratic tracking control problem for leader-follower multi-agent systems. We then address this suboptimal distributed tracking control problem in Section 4. A simulation example is presented in Section 5 to illustrate our design method. Finally, Section 6 concludes this paper.

Notation

We denote by $\mathbb{R}$ the field of real numbers, and by $\mathbb{R}^n$ the n-dimensional real Euclidean space. The column vector $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector whose entries are all equal to 1. For $x \in \mathbb{R}^n$, we define its Euclidean norm $\|x\| := \sqrt{x^\top x}$. For a given $r > 0$, we denote by $B(r) := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ the closed ball of radius r. We denote by $\mathbb{R}^{n\times m}$ the space of real $n \times m$ matrices. For a given matrix A, its transpose and inverse (if it exists) are denoted by $A^\top$ and A^{-1}, respectively. We denote by I_n the identity matrix of dimension $n \times n$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$, which has the property that $(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2$. For a given symmetric matrix P we denote $P > 0$ if it is positive definite and $P < 0$ if it is negative definite. By $\mathrm{diag}(a_1, a_2, \ldots, a_n)$, we denote the $n \times n$ diagonal matrix with $a_1, a_2, \ldots, a_n$ on the diagonal.

2. PRELIMINARIES

2.1 Graph Theory

In this paper, a (directed) graph is a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nonempty node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$. The edge from node i to node j is represented by the pair $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{V}$. We say the graph is simple if the edge set $\mathcal{E}$ only contains edges of the form (i, j) with $i \neq j$. The graph is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The adjacency matrix of the graph $\mathcal{G}$ is defined as $\mathcal{A} = [a_{ij}]$ with $a_{ij} = 1$ if there is an edge between the nodes i and j, and $a_{ij} = 0$ otherwise. For simple graphs, $a_{ii} = 0$ for all i. Furthermore, a graph $\mathcal{G}$ is undirected if and only if $\mathcal{A}$ is symmetric. The Laplacian matrix is defined as $L = D - \mathcal{A}$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ with $d_i = \sum_{j=1}^N a_{ij}$ the degree matrix of $\mathcal{G}$. The Laplacian matrix L of an undirected graph is symmetric and consequently only has real eigenvalues. Furthermore, all eigenvalues are nonnegative and 0 is an eigenvalue of L. The graph is connected if and only if 0 is a simple eigenvalue of L.

For a connected simple undirected graph $\mathcal{G}$, we review the following result:

Lemma 1. (Hong et al. (2006)). Let $\mathcal{G}$ be a connected simple undirected graph with Laplacian matrix L. Let $g_1, g_2, \ldots, g_N$ be non-negative real numbers with at least one $g_i > 0$. Define $G = \mathrm{diag}(g_1, g_2, \ldots, g_N)$. Then the matrix $L + G$ is positive definite.

2.2 Quadratic Performance of Linear Autonomous Systems

In this subsection, we analyze the quadratic performance of a linear autonomous system. Consider the autonomous system

$$\dot{x}(t) = \bar{A}x(t), \quad x(0) = x_0 \tag{1}$$

where $\bar{A} \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$ is the state. We consider the quadratic performance of system (1), given by

$$J = \int_0^\infty x^\top(t)\bar{Q}x(t)\ dt \tag{2}$$

where $\bar{Q} \geq 0$ is a given real weighting matrix. Note that the performance J is finite if system (1) is asymptotically stable, i.e., $\bar{A}$ is Hurwitz.

The following well-known result (Skelton et al. (1997) and Jiao et al. (2019)) provides a *necessary* and *sufficient* condition such that, for a given tolerance $\gamma > 0$, the performance (2) satisfies $J < \gamma$.

Theorem 2. Consider system (1) with associated performance (2). For given $\gamma > 0$, we have that $\bar{A}$ is Hurwitz and $J < \gamma$ if and only if there exists $P > 0$ satisfying

$$\bar{A}^\top P + P\bar{A} + \bar{Q} < 0, \tag{3}$$

$$x_0^\top P x_0 < \gamma. \tag{4}$$

In the next section, we formulate the problem that we will address in this paper.

3. PROBLEM FORMULATION

In this paper, we consider a leader-follower multi-agent system, consisting of one leader and N followers. The dynamics of the leader is represented by the linear time-invariant autonomous system

$$\dot{x}_r(t) = Ax_r(t), \quad x_r(0) = x_{r0}. \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$, $x_r \in \mathbb{R}^n$ is the state of the leader and x_{r0} is its initial state. The dynamics of the followers are identical and represented by the linear time-invariant systems

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, N \tag{6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input of follower i, respectively, and x_{i0} is its initial state. Throughout this paper, we assume that the pair (A, B) is stabilizable. Moreover, we make the following two standard assumptions regarding the communication between the leader and the followers:

Assumption 1. We assume that at least one follower receives the state information of the leader.

Assumption 2. We also assume that the underlying graph $\mathcal{G}$ of the communication between the followers is a connected simple undirected graph.

We consider the infinite horizon distributed linear quadratic tracking control problem for the leader-follower system (5) and (6), where the global cost functional integrates the weighted quadratic difference of states between every follower and its neighbors and the weighted quadratic difference of states between the leader and the followers communicating with the leader, and where the cost functional also penalizes the inputs in a quadratic form.

Note that, as mentioned in Assumption 1, at least one follower receives the state information of the leader. Thus, the leader-follower system (5) and (6) can be interconnected by a distributed diffusive control law of the form

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + Kg_i(x_i(t) - x_r(t)) \quad (7)$$

where a_{ij} is the ij-th entry of the adjacency matrix $\mathcal{A}$ of the underlying graph $\mathcal{G}$, $K \in \mathbb{R}^{m\times n}$ is an identical feedback gain for all followers and we have $g_i > 0$ for at least one $i = 1, 2, \ldots, N$. Accordingly, the cost functional considered in this paper is given by

$$J(u) = \int_0^\infty \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)^\top Q(x_i - x_j) + \sum_{i=1}^{N} g_i(x_i - x_r)^\top Q(x_i - x_r) + \sum_{i=1}^{N} u_i^\top R u_i \ dt \quad (8)$$

where $Q \geq 0$ and $R > 0$ are given real weighting matrices of suitable dimensions.

The distributed linear quadratic tracking problem is then the problem of minimizing the cost functional (8) for all initial states x_{r0} and x_{i0}, $i = 1, 2, \ldots, N$ over all distributed diffusive control laws (7) such that the states of all followers track the state of the leader asymptotically. In that case we say the network reaches *tracking consensus*:

Definition 3. We say the control law (7) achieves tracking consensus for the leader-follower system (5) and (6) if for all $i = 1, 2, \ldots, N$ and for all initial states x_{r0} and x_{i0}, we have

$$x_i(t) - x_r(t) \to 0 \text{ as } t \to \infty.$$

Due to the distributed nature of the control law (7) as imposed by the network topology, the distributed linear quadratic tracking problem is a non-convex optimization problem (Mosebach and Lunze (2014)). It is therefore difficult, if not impossible, to find a closed form solution for an optimal controller, or such optimal controller may not even exist. Therefore, in this paper we will design distributed control laws which solve a *suboptimal* version of this problem.

To proceed, for the ith follower we introduce the following error state

$$e_i = x_i - x_r,$$

for $i = 1, 2, \ldots, N$. Subsequently, the dynamics of e_i is given by

$$\dot{e}_i = Ae_i + Bu_i, \quad i = 1, 2, \ldots, N. \quad (9)$$

Denoting $x = \left(x_1^\top, \ldots, x_N^\top\right)^\top$, $u = \left(u_1^\top, \ldots, u_N^\top\right)^\top$, and $e = \left(e_1^\top, \ldots, e_N^\top\right)^\top$, we can then rewrite the error system (9) in compact form as

$$\dot{e} = (I_N \otimes A)e + (I_N \otimes B)u, \quad e(0) = e_0. \quad (10)$$

Note that

$$e = x - \mathbf{1}_N \otimes x_r.$$

Correspondingly, by using the fact $(L \otimes K)(\mathbf{1} \otimes x_r) = 0$, the control law (7) can be given by

$$u(t) = (\Gamma \otimes K)e \quad (11)$$

where $\Gamma = L + G$ and $G = \text{diag}(g_1, g_2, \ldots, g_N)$. Similarly, the cost functional (8) can be written in terms of e and u as

$$J(u) = \int_0^\infty e^\top(\Gamma \otimes Q)e + u^\top(I_N \otimes R)u \ dt. \quad (12)$$

Now, by substituting the control law (11) into the error dynamics (10), we obtain the closed-loop error system

$$\dot{e} = (I_N \otimes A + \Gamma \otimes BK)e, \quad e(0) = e_0. \quad (13)$$

and the associated cost is now given by

$$J(K) = \int_0^\infty e^\top\Big(\Gamma \otimes Q + \Gamma^2 \otimes K^\top RK\Big)e \ dt \quad (14)$$

Note that the controlled leader-follower system (5) and (6) reaches tracking consensus, i.e., the states of all followers track the state of the leader asymptotically, if and only if the error dynamics (13) is stable.

Let

$$B(r) = \{e_0 \in \mathbb{R}^{nN} \mid \|e_0\| \leq r\} \quad (15)$$

be the closed ball of radius r in the state space $\mathbb{R}^{nN}$ of the error system (13). Then, for the leader-follower system (5) and (6) with initial states such that the error initial state is contained in a closed ball of a given radius, we want to design a distributed diffusive controller such that tracking consensus is achieved and, for all initial states satisfying (15), the associated cost is smaller than an a priori given upper bound. Thus, the problem that we will address is the following:

Problem 1. Consider the leader-follower multi-agent system (5) and (6) and the associated cost functional (8). Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. The problem is to find a distributed diffusive control law of the form (7) such that the controlled leader-follower system reaches tracking consensus and, for all initial conditions x_0 and x_{r0} such that $e_0 = x_0 - x_{r0}$ satisfies (15), the associated cost (8) is smaller than the given upper bound, i.e., $J(K) < \gamma$.

4. SUBOPTIMAL CONTROL DESIGN FOR LEADER-FOLLOWER MULTI-AGENT SYSTEMS

In this section, we will address Problem 1 and provide a suitable control design method. As mentioned before, the distributed control law (7) achieves tracking consensus and suboptimal performance for the leader-follower system (5) and (6) with respect to the given tolerance on the cost functional (8) if and only if the error dynamics (13) is stable and $J(K) < \gamma$.

Now, let $U \in \mathbb{R}^{N\times N}$ be an orthogonal matrix that diagonalizes $\Gamma = L + G$. Define

$$U^\top \Gamma U := \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).$$

It follows from Lemma 1 that $\lambda_i > 0$ for all $i = 1, 2, \ldots, N$. To simplify the problem formulated in the previous section, by applying the state transformation $\bar{e} = (U^\top \otimes I_n)e$, system (13) becomes

$$\dot{\bar{e}} = (I_N \otimes A + \Lambda \otimes BK)\,\bar{e}, \quad \bar{e}(0) = \bar{e}_0 \quad (16)$$

where $\bar{e} = \left(\bar{e}_1^\top, \ldots, \bar{e}_N^\top\right)^\top$. In terms of the transformed variable, the cost (14) is then given by

$$J(K) = \int_0^\infty \sum_{i=1}^{N} \bar{e}_i^\top(\lambda_i Q + \lambda_i^2 K^\top RK)\bar{e}_i \ dt. \quad (17)$$

Note that the transformed states $\bar{e}_i$, $i = 1, 2, \ldots, N$ appearing in system (16) and cost (17) are decoupled from each other. Then we can write system (16) as

$$\dot{\bar{e}}_i = (A + \lambda_i BK)\bar{e}_i, \quad i = 1, 2, \ldots, N. \tag{18}$$

Also, the cost (17) equals

$$J(K) = \sum_{i=1}^{N} J_i(K) \tag{19}$$

with

$$J_i(K) = \int_0^\infty \bar{e}_i^\top (\lambda_i Q + \lambda_i^2 K^\top RK)\bar{e}_i \, dt, \quad i = 1, 2, \ldots, N. \tag{20}$$

Clearly, the controlled leader-follower system (5) and (6) reaches tracking consensus with control law (7) if and only if, for $i = 1, 2, \ldots, N$, the systems (18) are stable. In addition, the control law (7) is suboptimal if $J(K) < \gamma$.

So far, we have transformed the problem of distributed suboptimal control for the leader-follower system (5) and (6) into the problem of finding one single static feedback gain $K \in \mathbb{R}^{m\times n}$ such that the systems (18) are stable for $i = 1, 2, \ldots, N$ and $J(K) < \gamma$. Since the pair (A, B) is stabilizable, there exists such a feedback gain K (Li et al. (2010), Zhang et al. (2011)).

The following lemma then provides a necessary and sufficient condition for a given feedback gain K to stabilize all systems (18) and for given initial states guarantee that $J(K) < \gamma$.

Lemma 4. Let K be a feedback gain. Consider the systems (18) with given initial states $\bar{e}_{10}, \bar{e}_{20}, \ldots, \bar{e}_{N0}$ and associated cost functionals (19) and (20). Let $\gamma > 0$. Then all systems (18) are stable and $J(K) < \gamma$ if and only if there exist $P_i > 0$ satisfying

$$(A + \lambda_i BK)^\top P_i + P_i(A + \lambda_i BK) + \lambda_i Q + \lambda_i^2 K^\top RK < 0 \tag{21}$$

and

$$\sum_{i=1}^{N} \bar{e}_{i0}^\top P_i \bar{e}_{i0} < \gamma, \tag{22}$$

for $i = 1, 2, \ldots, N$, respectively.

Proof. ($\Leftarrow$) Since (22) holds, there exist $\gamma_i := \bar{e}_{i0}^\top P_i \bar{e}_{i0} + \epsilon_i$ with sufficiently small $\epsilon_i > 0$, $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \gamma_i < \gamma$. Because there exists $P_i > 0$ such that (21) and $\bar{e}_{i0}^\top P_i \bar{e}_{i0} < \gamma_i$ holds for all $i = 1, 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it follows from Theorem 2 that all systems (18) are stable and $J_i(K) < \gamma_i$ for $i = 1, 2, \ldots, N$. Since $J(K) = \sum_{i=1}^N J_i(K)$, this implies that $J(K) < \sum_{i=1}^N \gamma_i < \gamma$.

($\Rightarrow$) Since $J(K) < \gamma$ and $J(K) = \sum_{i=1}^N J_i(K)$, there exist $\gamma_i := J_i(K) + \epsilon_i$ with sufficiently small $\epsilon_i > 0$, $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \gamma_i < \gamma$. Because all systems (18) are stable and $J_i(K) < \gamma_i$ for $i = 1, 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it again follows from Theorem 2 that there exist $P_i > 0$ such that (21) and $\bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma_i$ hold for all $i = 1, 2, \ldots, N$. Since $\sum_{i=1}^N \gamma_i < \gamma$, this implies that $\sum_{i=1}^N \bar{x}_{i0}^\top P_i \bar{x}_{i0} < \sum_{i=1}^N \gamma_i < \gamma$. □

Lemma 4 establishes a necessary and sufficient condition for a given feedback gain K to stabilize all systems (18) and to satisfy, for given initial states of these systems, $J(K) < \gamma$. However, Lemma 4 does not yet provide a design method for computing such K. Therefore, in the following we will provide a method to find such K.

Lemma 5. Consider the leader-follower system (5) and (6) with associated cost functional (8). Let x_{r0} be the given initial state of the leader and x_{i0}, $i = 1, 2 \ldots, N$ be the given initial states of the followers, respectively. Let $\gamma > 0$ be a given tolerance. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:

(a) if
$$\frac{2}{\lambda_1 + \lambda_N} \le c < \frac{2}{\lambda_N}, \tag{23}$$
then there exists $P > 0$ satisfying
$$A^\top P + PA + (c^2\lambda_N^2 - 2c\lambda_N)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{24}$$

(b) if
$$0 < c < \frac{2}{\lambda_1 + \lambda_N}, \tag{25}$$
then there exists $P > 0$ satisfying
$$A^\top P + PA + (c^2\lambda_1^2 - 2c\lambda_1)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{26}$$

In both cases, if in addition P satisfies

$$\sum_{i=1}^{N} (x_{i0} - x_{r0})^\top P(x_{i0} - x_{r0}) < \gamma, \tag{27}$$

then the distributed control law (7) with $K := -cR^{-1}B^\top P$ achieves tracking consensus for the controlled leader-follower system (5) and (6), and with the initial states x_{r0} and x_{i0} we have $J(K) < \gamma$.

Proof. Since the line of the proof for case (b) is analogous to that for case (a), we will only give the proof for case (a). Using the upper and lower bounds on c given by (23), it can be verified that $c^2\lambda_i^2 - 2c\lambda_i \le c^2\lambda_N^2 - 2c\lambda_N < 0$ for $i = 1, 2, \ldots, N$. It is then easily seen that (24) has many positive definite solutions. Since also $\lambda_i \le \lambda_N$, any such solution P is a solution to the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^2 - 2c\lambda_i)PBR^{-1}B^\top P + \lambda_i Q < 0, \quad i = 1, 2, \ldots, N. \tag{28}$$

Equivalently, P also satisfies the Lyapunov inequalities

$$\begin{aligned}(A - c\lambda_i BR^{-1}B^\top P)^\top P + P(A - c\lambda_i BR^{-1}B^\top P)\\ + \lambda_i Q + c^2\lambda_i^2 PBR^{-1}B^\top P < 0, \ i = 1, 2, \ldots, N.\end{aligned} \tag{29}$$

Next, by substituting $\bar{e} = (U^\top \otimes I_n)e$ into (22) we have $\sum_{i=1}^N e_{i0}^\top P e_{i0} < \gamma$, which is equal to (27).

Next, taking $P_i = P$ for $i = 1, 2, \ldots, N$ and $K := -cR^{-1}B^\top P$ in inequalities (21) and (22) immediately gives us inequalities (29) and (27). Then it follows from Lemma 4 that all systems (18) are stable and $J(K) < \gamma$. Subsequently, the controlled leader-follower system reaches tracking consensus and $J(K) < \gamma$. □

We will now apply Lemma 5 to establish a solution to Problem 1. The next theorem provides a condition under which, for given radius r and upper bound γ, suboptimal distributed diffusive control laws exist, and explains how these can be computed.

Theorem 6. Consider the leader-follower system (5) and (6) with associated cost functional (8). Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:

(a) if
$$\frac{2}{\lambda_1 + \lambda_N} \leq c < \frac{2}{\lambda_N}, \tag{30}$$
then there exists $P > 0$ satisfying
$$A^\top P + PA + (c^2\lambda_N^2 - 2c\lambda_N)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{31}$$

(b) if
$$0 < c < \frac{2}{\lambda_1 + \lambda_N}, \tag{32}$$
then there exists $P > 0$ satisfying
$$A^\top P + PA + (c^2\lambda_1^2 - 2c\lambda_2)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{33}$$

In both cases, if in addition P satisfies
$$P < \frac{\gamma}{r^2} I, \tag{34}$$
then the distributed control law (7) with $K := -cR^{-1}B^\top P$ achieves tracking consensus for the controlled leader-follower system (5) and (6) and $J(K) < \gamma$ for all initial states x_{r0} and x_0 satisfying
$$x_0 - \mathbf{1}_N \otimes x_{r0} \in B(r). \tag{35}$$

Proof. Again, due to the fact that the line of the proof for case (b) is analogous to that for case (a), we only give the proof for case (a). Let $P > 0$ satisfy (31) and (34). Next, we will show that if the initial states x_{r0} and x_0 satisfy $x_0 - \mathbf{1}_N \otimes x_{r0} \in B(r)$, then (27) holds. Indeed, if $\|x_0 - \mathbf{1}_N \otimes x_{r0}\| \leq r$, then
$$\begin{aligned}&\sum_{i=1}^{N}(x_{i0} - x_{r0})^\top P(x_{i0} - x_{r0})\\ =&(x_0 - \mathbf{1}_N \otimes x_{r0})^\top (I \otimes P)(x_0 - \mathbf{1}_N \otimes x_{r0})\\ <&\frac{\gamma}{r^2}\|x_0 - \mathbf{1}_N \otimes x_{r0}\|^2 \leq \gamma.\end{aligned}$$
It then follows from Lemma 5 that the controlled leader-follower system (5) and (6) reaches tracking consensus with the given K and $J(K) < \gamma$ for all initial states x_{r0} and x_0 satisfying (35). □

Remark 7. Theorem 6 states that after choosing c satisfying the inequality (30) for case (a) and finding $P > 0$ satisfying (31) and (34), the distributed control law with local gain $K = -cR^{-1}B^\top P$ is γ-suboptimal for all initial states of the leader-follower system satisfying the condition (35). According to (34), the smaller the solution P of (31), the smaller the quotient $\frac{\gamma}{r^2}$ is allowed to be, leading to a smaller upper bound and a larger radius. The question then arises: how should we choose the parameter c in (30) so that the Riccati inequality (31) allows a positive definite solution that is as small as possible. In fact, one can find a positive definite solution $P(c, \epsilon)$ to (31) by solving the Riccati equation
$$A^\top P + PA - PB\bar{R}(c)^{-1}B^\top P + \bar{Q}(\epsilon) = 0 \tag{36}$$
with $\bar{R}(c) = \frac{1}{-c^2\lambda_N^2 + 2c\lambda_N} R$ and $\bar{Q}(\epsilon) = \lambda_N Q + \epsilon I_n$ where c is chosen as in (30) and $\epsilon > 0$. If c_1 and c_2 as in (30) satisfy $c_1 \leq c_2$, then we have $\bar{R}(c_1) \leq \bar{R}(c_2)$, so, clearly, $P(c_1, \epsilon) \leq P(c_2, \epsilon)$. Similarly, if $0 < \epsilon_1 \leq \epsilon_2$, we immediately have $\bar{Q}(\epsilon_1) \leq \bar{Q}(\epsilon_2)$. Again, it follows that $P(c, \epsilon_1) \leq P(c, \epsilon_2)$. Therefore, if we choose $\epsilon > 0$ very close to 0 and $c = \frac{2}{\lambda_1 + \lambda_N}$, we find the 'best' solution to the Riccati inequality (31) in the sense explained above.

Likewise, if c satisfies (32) corresponding to case (b), it can be shown that if we choose $\epsilon > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_1 + \lambda_N}$, we find the 'best' solution to the Riccati inequality (33) in the sense explained above.

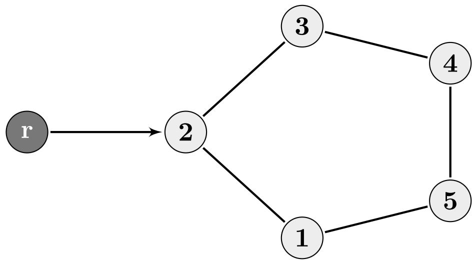

Fig. 1. The underlying graph of the communication between the leader and the followers.

5. SIMULATION EXAMPLE

In this section, we will use a numerical example borrowed from Nguyen (2015) to illustrate the design method for the suboptimal distributed control laws given in Theorem 6.

Consider a leader-follower multi-agent system, consisting of one leader and five followers. The dynamics of the leader is given by
$$\dot{x}_r(t) = Ax_r(t), \quad x_r(0) = x_{r0},$$
and the dynamics of the followers are identical and represented by
$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, 5$$
where
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
The pair (A, B) is stabilizable. Assume the underlying graph representing the communication between the leader and the followers is given as in Figure 1. The graph representing the communication between the followers is then the undirected cycle graph with the Laplacian matrix
$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Since the leader shares its state information only with follower 2, it follows from Lemma 1 that the associated diagonal matrix $G = \text{diag}(g_1, g_2, \ldots, g_5) = \text{diag}(0, 1, 0, 0, 0)$. Furthermore, we consider the cost functional
$$J(u) = \int_0^\infty \frac{1}{2}\sum_{i=1}^{5}\sum_{j=1}^{5} a_{ij}(x_i - x_j)^\top Q(x_i - x_j) + \sum_{i=1}^{5} g_i(x_i - x_r)^\top Q(x_i - x_r) + \sum_{i=1}^{N} u_i^\top Ru_i \, dt$$
with
$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1.$$

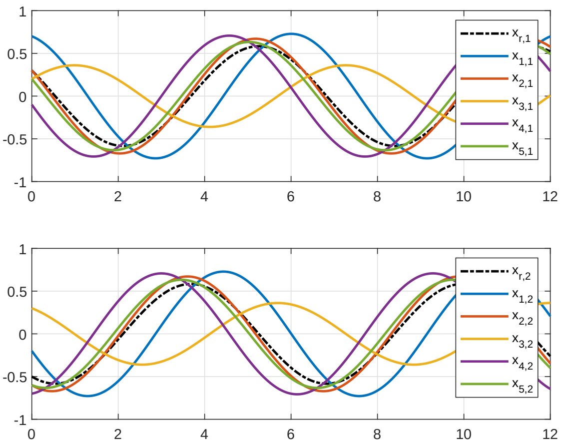

Fig. 2. Plots of the states x_{r1} and $x^1 = (x_{1,1}, \ldots, x_{5,1})$ (upper plot) and x_{r2} and $x^2 = (x_{1,2}, \ldots, x_{5,2})$ (lower plot) of the six decoupled local agents without control

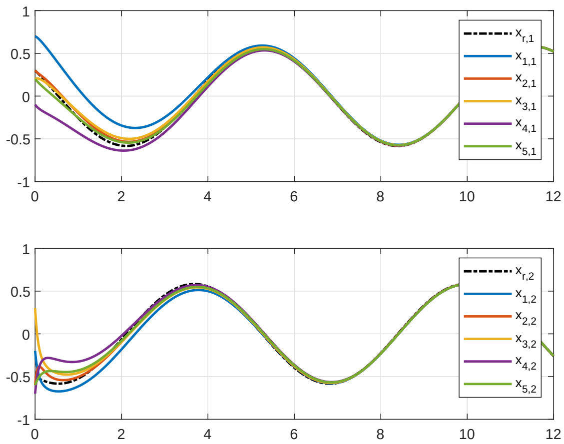

Fig. 3. Plots of the states x_{r1} and $x^1 = (x_{1,1}, \ldots, x_{5,1})$ (upper plot) and x_{r2} and $x^2 = (x_{1,2}, \ldots, x_{5,2})$ (lower plot) of the controlled leader-follower system

Let the desired tolerance for the cost functional be $\gamma = 20$. Our aim is then to design a control law of the form

$$u_i(t) = K \sum_{j=1}^{5} a_{ij}(x_i(t) - x_j(t)) + K g_i(x_i(t) - x_r(t)) \quad (37)$$

such that the controlled leader-follower system reaches tracking consensus and the associated cost satisfies $J(K) < 20$ for all initial states x_0 and x_{r0} satisfying the condition $\|x_0 - \mathbf{1}_5 \otimes x_{r0}\| \leq r$ with radius r to be specified later.

In this simulation example, we will use the design method of case (a) in Theorem 6. For $\Gamma = L + G$ the smallest and largest eigenvalues are $\lambda_1 = 0.1392$ and $\lambda_5 = 4.1149$, respectively. We first compute a solution $P > 0$ to (31) by solving

$$A^\top P + PA + (c^2\lambda_5^2 - 2c\lambda_5)PBR^{-1}B^\top P + \lambda_5 Q + \epsilon I_2 = 0 \quad (38)$$

with ϵ sufficiently small as mentioned in Remark 7. Here we choose $\epsilon = 0.01$. In addition, we choose $c = \frac{2}{\lambda_1 + \lambda_5} = 0.4701$, which is the 'best' choice as mentioned in Remark 7. Then by solving (38) using Matlab, we compute

$$P = \begin{pmatrix} 13.2553 & 3.3886 \\ 3.3886 & 9.2760 \end{pmatrix}.$$

Correspondingly, the control gain is equal to $K = (1.5931 \ 4.3610)$. We now compute the radius r of a ball $B(r)$ of initial states for which the distributed control law (37) is suboptimal, i.e. $J(K) < 20$. We compute that the largest eigenvalue of P is equal to 15.1952. Hence for every radius r such that $\frac{20}{r^2} > 15.1952$ the inequality (34) holds. Thus, the distributed controller with local gain K is suboptimal for all x_{r0} and x_0 satisfying $\|x_0 - \mathbf{1}_5 \otimes x_{r0}\| \leq r$ with $r < 1.1473$.

As an example, the following initial states of the agents satisfy this norm bound: $x_{r0}^\top = (0.3 \ -0.5)$, $x_{10}^\top = (0.7 \ -0.2)$, $x_{20}^\top = (0.3 \ -0.6)$, $x_{30}^\top = (0.2 \ 0.3)$, $x_{40}^\top = (-0.1 \ -0.7)$, $x_{50}^\top = (0.2 \ -0.6)$. The plots of the state of the six local agents without control are shown in Figure 2. Figure 3 shows that the controlled leader-follower system reaches tracking consensus.

6. CONCLUSION

In this paper, we have studied the distributed linear quadratic tracking control problem for leader-follower multi-agent systems. We have considered leader-follower systems consisting of one autonomous leader and N followers, together with an associated global cost functional. We assume that the leader shares its state information with at least one of the followers and the underlying graph connecting the followers is a connected simple undirected graph. For this type of leader-follower systems, we have provided a design method to compute distributed suboptimal control laws such that the controlled network reaches tracking consensus and the associated cost is smaller than a given tolerance for all initial states bounded in norm by a given radius. The computation of the local gain involves the solution of a single Riccati inequality, whose dimension is equal to the dimension of the agent dynamics, and also involves the largest and smallest eigenvalue of a positive definite matrix capturing the underlying graph structure.

REFERENCES

Borrelli, F. and Keviczky, T. (2008). Distributed LQR design for identical dynamically decoupled systems. *IEEE Transactions on Automatic Control*, 53(8), 1901–1912.

Cao, Y. and Ren, W. (2010). Optimal linear-consensus algorithms: An LQR perspective. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 40(3), 819–830.

Cheng, Y. and Ugrinovskii, V. (2015). Leader–follower tracking control with guaranteed consensus performance for interconnected systems with linear dynamic uncertain coupling. *International Journal of Control*, 88(8), 1663–1677.

Hong, Y., Hu, J., and Gao, L. (2006). Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica*, 42(7), 1177–1182.

Jiao, J., Trentelman, H.L., and Camlibel, M.K. (2019). A suboptimality approach to distributed linear quadratic optimal control. *IEEE Transactions on Automatic Control*, to appear.

Jiao, J., Trentelman, H.L., and Camlibel, M.K. (2020). Distributed linear quadratic optimal control: Compute locally and act globally. *IEEE Control Systems Letters*, 4(1), 67–72.

Li, Z., Duan, Z., Chen, G., and Huang, L. (2010). Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 57(1), 213–224.

Mosebach, A. and Lunze, J. (2014). Synchronization of autonomous agents by an optimal networked controller. In *2014 European Control Conference (ECC)*, 208–213.

Movric, K.H. and Lewis, F.L. (2014). Cooperative optimal control for multi-agent systems on directed graph topologies. *IEEE Transactions on Automatic Control*, 59(3), 769–774.

Nguyen, D.H. (2015). A sub-optimal consensus design for multi-agent systems based on hierarchical LQR. *Automatica*, 55, 88–94.

Ni, W. and Cheng, D. (2010). Leader-following consensus of multi-agent systems under fixed and switching topologies. *Systems & Control Letters*, 59(3), 209–217.

Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9), 1520–1533.

Semsar-Kazerooni, E. and Khorasani, K. (2009). Multi-agent team cooperation: A game theory approach. *Automatica*, 45(10), 2205–2213.

Skelton, R.E., Iwasaki, T., and Grigoriadis, D.E. (1997). *A Unified Algebraic Approach to Control Design*. Boca Raton, FL, USA: CRC Press.

Trentelman, H.L., Takaba, K., and Monshizadeh, N. (2013). Robust synchronization of uncertain linear multi-agent systems. *IEEE Transactions on Automatic Control*, 58(6), 1511–1523.

Tuna, S.E. (2008). LQR-based coupling gain for synchronization of linear systems. [Online]. Available: https://arxiv.org/abs/0801.3390.

Zhang, H., Feng, T., Yang, G.H., and Liang, H. (2015). Distributed cooperative optimal control for multiagent systems on directed graphs: An inverse optimal approach. *IEEE Transactions on Cybernetics*, 45(7), 1315–1326.

Zhang, H., Lewis, F.L., and Das, A. (2011). Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback. *IEEE Transactions on Automatic Control*, 56(8), 1948–1952.