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Optimal design under time-variant reliability constraints

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Abstract

While reliability-oriented structural optimization under time-independent loading and resistances is sufficiently well known, the same problem under time-dependent loads and resistances has found at most, grossly simplified solutions. The main reason is that reliability calculations are far more complicated than for time-invariant loading. In this paper, a first attempt is made to use the out-crossing approach for the reliability part in structural optimization. In particular, reliabilities will be determined by the out-crossing approach in the context of FORM. Two types of load models, stationary rectangular wave renewal processes and differentiable processes, respectively, will be dealt with. As in time-invariant optimal design, a one-level approach is pursued.

Keywords: Reliability-oriented structural optimization; Time-variant reliability; Random processes; Out-crossing rates; One-level optimization

1. Introduction

Optimal reliability-oriented structural design or structural optimization with reliability constraints has been the subject of research for many years. Different approaches have been studied, all essentially assuming that the structural facility potentially will fail once loaded for the first time (for example, see [1–4] for typical approaches). It turned out that this problem is numerically much more involved than simple reliability analysis if standard methods of structural reliability are maintained. Among the few contributions known to the authors which also deal with time-variant aspects is one due to Rosenblueth/Mendoza [5] who, however, chose a rather simple reliability model. They made it clear that capitalization aspects as well as the reconstruction policy are important factors. In this paper a first attempt is made to formulate the time-variant case using standard reliability methodology. In particular, reliabilities will be determined by the out-crossing approach in the context of FORM. Two types of load models, rectangular wave renewal processes and differentiable processes, respectively, will be used. Only stationary cases will be

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dealt with. Both systematic reconstruction and "one mission" structures will be considered. A one-level approach for the cost optimization will be investigated. This paper is an extended version of a keynote lecture given by the second author at the 8th IFIP WG 7.5 working conference in Cracow, 1998 [6].

2. Basic formulations and classification of reliability problems

2.1. Time-invariant component failure probability

In order to clarify terminology it is necessary to start with the well-known time-invariant component reliability problem in the context of FORM. Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a n-dimensional vector of random variables with distribution law $P_\mathbf{X}$ and distribution function $F_\mathbf{X}(\mathbf{x})$ and $\mathbf{p}$ a d-dimensional vector of design and cost parameters. It can involve deterministic parameters but also parameters of the distribution function $F_\mathbf{X}(\mathbf{x})$. Further, define by $g(\mathbf{x}, \mathbf{p})$ a state (performance) function so that $g(\mathbf{x}, \mathbf{p}) > 0$ denotes the safe state, $g(\mathbf{x}, \mathbf{p}) = 0$ the limit state and $g(\mathbf{x}, \mathbf{p}) < 0$ the failure state. $g(\mathbf{x}, \mathbf{p}) = 0$ will also be denoted by failure surface. The time-invariant failure probability then is

$$P_f(\mathbf{p}) = \int_{\{\mathbf{x}: g(\mathbf{x},\mathbf{p}) \leqslant 0\}} P_\mathbf{X}(d\mathbf{x}) = \int_{g(\mathbf{x},\mathbf{p}) \leqslant 0} f_\mathbf{X}(\mathbf{x}) d\mathbf{x} \tag{1}$$

provided that the probability density $f_\mathbf{X}(\mathbf{x})$, exists which is assumed throughout. Moreover, it is assumed that the probability distribution functions are continuously differentiable. Let a probability distribution transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ exist which maps an arbitrary n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ into an independent standard normal vector $\mathbf{U} = (U_1, \ldots, U_n)^T$ (see, for example, Hohenbichler and Rackwitz, [7], Der Kiureghian and Liu [8], Winterstein and Bjerager [9]). With $g(\mathbf{x}, \mathbf{p}) = \tilde{g}(\mathbf{T}(\mathbf{x}), \mathbf{p}) = \tilde{g}(\mathbf{u}, \mathbf{p})$ and the failure domain $\mathcal{F}_\mathbf{p} = \{\mathbf{u} : \tilde{g}(\mathbf{u}, \mathbf{p}) \leqslant 0\}$, it is:

$$P_f(\mathbf{p}) = \int_{\mathcal{F}_\mathbf{p}} P_\mathbf{u}(d\mathbf{u}) = \int_{\tilde{g}(\mathbf{u},\mathbf{p}) \leqslant 0} \varphi_\mathbf{u}(\mathbf{u}) d\mathbf{u} \tag{2}$$

where $P_\mathbf{u}(.)$ is the standard normal distribution law and $\varphi_\mathbf{u}(\mathbf{u})$ is the standard normal density. Then, a first-order reliability estimate [10,11], is

$$P_f(\mathbf{p}) \approx \Phi(-\beta_\mathbf{p}), \quad \text{with } \beta_\mathbf{p} = \min\{\|\mathbf{u}\| : \tilde{g}(\mathrm{u}, \mathrm{p}) \leqslant 0\} \tag{3}$$

$\Phi(.)$ is the standard normal integral. This approximation method is denoted by first-order reliability method (FORM). In principle, this result is valid for any state function but the estimated failure probability is sufficiently accurate only for differentiable limit state functions. If the failure surface is differentiable the solution point of 3, that is $\mathbf{u}^*$ is a Kuhn–Tucker-point for which the following theorem can be proved [3]:

Theorem 1 (β-Point-Theorem in Time-Invariant Case). *If* $\mathbf{u}^*$, *with* $\mathbf{u} \neq 0$, *is the solution point of the optimization problem in 3, then the following two statements hold for each* $\mathbf{p}$:

(a) $\tilde{g}(\mathbf{u}^*, \mathbf{p}) = 0$,

(b) $\mathbf{u}^{*T} \nabla_u \tilde{g}(\mathbf{u}^*, \mathbf{p}) + \|\mathbf{u}^*\| \|\nabla_u \tilde{g}(\mathbf{u}^*, \mathbf{p})\| = 0$

The solution point $\mathbf{u}^*$ is commonly denoted by β-point. Its justification as a suitable expansion point has been demonstrated by higher order methods (see, for example, [12]). Various methods exist to improve this first-order estimate. In practice, such improvements are rarely necessary.

2.2. Time-variant component failure probability

Time-variant reliability is more difficult to compute than time-invariant component reliability. Note that one is hardly interested in the time-dependent failure probability function $P_f(\mathbf{p}, t)$ where t is treated as a parameter but in quantities like the probability of first passage into the failure domain, the total duration of exceedances into the failure domain, the duration of individual exceedances and other related criteria.

Let T be the random time of first exit into the failure domain. Then,

$$P_f(t, \mathbf{p}) = \mathcal{P}(\mathrm{T} \leqslant t | \mathbf{p})$$

where $[0, t]$ is the considered reference time interval. If the component does not fail at time $t = 0$ failure occurs at a random time. The distribution function of the random time T must be known. Unfortunately, this is rarely the case in structural reliability. However, if it is possible to determine the crossing rate of the time-variant random process into the failure domain some useful asymptotic results are available which are the basis for the formulations to come. For the purpose of versatile modelling and with important consequences for the subsequent derivations we distinguish between three types of variables

- $\boldsymbol{R}$ is a n_R-dimensional random vector as in time-invariant reliability. This vector is used to model resistance variables and its most important characteristic is that it is non-ergodic.
- $\boldsymbol{Q}$ is a n_Q-dimensional vector of stationary and ergodic sequences. It is used to model long term variations in time, e.g. traffic and sea states. These variables determine the fluctuating parameters of the random process variables described next.
- $\boldsymbol{S}$ is a n_S-dimensional vector of sufficiently mixing random process variables whose parameters can depend on $\boldsymbol{Q}$ and/or $\boldsymbol{R}$.

As before, the safe state is defined for $g(\mathbf{r}, \mathbf{q}, \mathrm{s}(\mathrm{t}), t, \mathbf{p}) < 0$, the limit state for $g(\mathbf{r}, \mathbf{q}, \mathbf{s}(t), t, \mathbf{p}) = 0$ and the failure state for $g(\mathbf{r}, \mathbf{q}, \mathbf{s}(t), t, \mathbf{p}) \leqslant 0$, respectively.

The **rate of out-crossings** into the failure domain conditional on $\mathbf{r}$, $\mathbf{q}$ and $\mathbf{p}$ can be defined as

$$\nu^+(\mathcal{F}, \tau | \mathbf{r}, \mathbf{q}, \mathbf{p}) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathcal{P}\Big(\{g(\mathbf{r}, \mathbf{q}, \mathrm{S}(\tau), \tau, \mathbf{p}) > 0\} \bigcap \{g(\mathbf{r}, \mathbf{q}, \mathrm{S}(\tau + \Delta), \tau + \Delta, \mathbf{p}) \leqslant 0\}\Big)$$

where $\mathcal{F} = \{(\mathrm{S}(\mathrm{t}), \mathrm{t}) : g(\mathbf{r}, \mathbf{q}, \mathrm{S}(t), t, \mathbf{p}) \leqslant 0\}$ denotes the failure domain conditional on $\mathbf{r}, \mathbf{q}$ and $\mathbf{p}$. The rate of out-crossings exists if the limiting operation can be performed. This is the case when the random process of crossings is a **regular point process**, i.e. if the probability of more than one crossings is negligibly small in a small time interval Δ.

Let $N^+(t_1, t_2)|\mathbf{r}, \mathbf{q}, \mathbf{p}$ denote the number of crossings in the time interval $[t_1, t_2]$ conditional on **rq** and **p**. The mean number of stationary, regular crossings $N^+(t_1, t_2)|\mathbf{r}, \mathbf{q}, \mathbf{p}$ in the time interval $[t_1, t_2]$ conditional on **r**, **q** and **p** can then be determined from

$$E(N^+(t_1, t_2) \mid \mathbf{r}, \mathbf{q}, \mathbf{p}) = \int_{t_1}^{t_2} \nu^+(\mathcal{F}, \tau|\mathbf{r}, \mathbf{q}, \mathbf{p})d\tau = \nu^+(\mathcal{F}|\mathbf{r}, \mathbf{q}, \mathbf{p})(t_2 - t_1)$$

Further, if the random process $\mathbf{S}(t)$ is strongly mixing (i.e. asymptotic independence of $\mathbf{S}(t)$ and $\mathbf{S}(t+\tau)$ for $\tau \rightarrow \infty$) it has been shown that the failure time distribution is asymptotically exponential implying a stationary Poisson process of outcrossings conditional on the vector $\mathbf{R} = \mathbf{r}$ [18,22] leading to the asymptotic failure time distribution

$$P_f(t_1, t_2|\mathbf{r}, \mathbf{p}) \sim 1 - \mathrm{E}_{\mathbf{R}}[\exp[-E_{\mathbf{Q}}[\nu^+(\mathcal{F}|\mathbf{R}, \mathbf{Q}, \mathbf{p})](t_2 - t_1)]] \tag{4}$$

This is the key result to be used later.

Assume now that a probability distribution transformation is performed such that the initial vector of variables $(\mathbf{R}, \mathbf{Q}, \mathbf{S})^T$ is mapped into a standard normal vector $(\mathbf{u}^r, \mathbf{u}^q, \mathbf{u}^s)^T$. Of utmost importance for optimal reliability-oriented structural design, then, is the fact that under quite general conditions the β-point is also the "critical" point in time-variant problems as pointed out already by Veneziano et al. [13] and proven in [14] for Gaussian processes and in [15] and [16] for rectangular wave renewal processes. The β-point may be identified as the point of maximum outcrossing rate implying that the distance of the failure surface to the origin also dominates the local outcrossing rate. This statement is valid at least asymptotically. The β-point-problem in the time-variant, stationary case has the following form:

$$(\beta P - TV - ST) \quad \textbf{minimize} \quad \left\|(\mathbf{u}^r, \mathbf{u}^q, \mathbf{u}^s)^T\right\| \quad \text{subject to} \quad \tilde{g}(\mathbf{u}^r, \mathbf{u}^q, \mathbf{u}^s, \mathbf{p}) \leqslant 0$$

The state function $g(\mathbf{r}, \mathbf{q}, \mathbf{s}, \mathbf{p}) = g(\mathbf{T}(\mathbf{u}^r, \mathbf{u}^q, \mathbf{u}^s), \mathbf{p}) = \tilde{g}(\mathbf{u}^r, \mathbf{u}^q, \mathbf{u}^s, \mathbf{p})$ does not contain time as a parameter. The solution point $\mathbf{u}^* = (\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})$ of the time-variant and stationary problem $(\beta P - TV - ST)$ defines the β-point. The first-order reliability index β_P is defined analogously

$$\beta_\mathrm{P} = \|\mathbf{u}^*\| = \left\|(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})\right\|.$$

$\mathbf{u}^*$ is a Kuhn–Tucker-point, as well. The following important corollary defines sufficient conditions for the optimality of the solution $\mathbf{u}^*$. It parallels the time-invariant β-point-theorem. Its proof is, in fact, identical to the proof of the time-invariant β-point-theorem.

Corollary 2 (β-point-conditions in the time-variant, stationary case). *If* $\mathbf{u}^* = (\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})$, *with* $\mathbf{u} \neq 0$, *is the solution point of optimization problem* $(\beta P - TV - ST)$, *then the following two statements hold for each* **p**:

(a) $\tilde{g}(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*}, \mathbf{p}) = 0$,

(b) $(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})^T \nabla_u \tilde{g}(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*}, \mathbf{p}) + \left\|(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})\right\| \left\|\nabla_u \tilde{g}(\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*}, \mathbf{p}\right\| = 0$

Note that the gradient of the state function surface is defined in the entire $\mathbf{R} - \mathbf{Q} - \mathbf{S}$-space, i.e. $\nabla_u() = \nabla_{u^r, u^q, u^s}()$, and similarly the norm of the optimal vector $\mathbf{u}^* = (\mathbf{u}^{r^*}, \mathbf{u}^{q^*}, \mathbf{u}^{s^*})^T$.

3. Out-crossing rates for two important, stationary random processes

3.1. Outcrossing rates and failure probabilities for rectangular wave renewal vector processes

Breitung and Rackwitz [15] have shown that stationary **out-crossing rate** can be calculated as the product of the jump rate λ_i and the probability that a component of the rectangular wave jumps from the safe domain $\bar{\mathcal{F}}$ into the failure domain $\mathcal{F}$, summed up over all n_s components of the rectangular wave renewal process. Ignoring for the moment all **R**- and **Q**-variables the mean outcrossing rate is:

$$\begin{aligned}\nu^+(\mathcal{F}(\mathbf{p})) &= \sum_{i=1}^{n_s} \lambda_i \cdot \mathcal{P}\left(\left\{\mathbf{S}_i^- \in \bar{\mathcal{F}}\right\} \bigcap \{\mathbf{S}_i^+ \in \mathcal{F}\}\right) \\ &= \sum_{i=1}^{n_s} \lambda_i \cdot (\mathcal{P}(\mathbf{S}_i^+ \in \mathcal{F}) - \mathcal{P}(\{\mathbf{S}_i^- \in \mathcal{F}\} \bigcap \{\mathbf{S}_i^+ \in \mathcal{F}\})).\end{aligned}$$

where $\mathbf{S}_i^-$ is the vector of jumping components just before a jump of the i-th component $\mathbf{S}_i^-$ and $\mathbf{S}_i^+$ after a jump. It is assumed that at a jump of the component $\mathbf{S}_i$ changes its position from a random value to a new random value. As before, it is assumed that an appropriate probability distribution transformation has been performed. Then, the out-crossing rate can be given for a linear failure surfaces $\partial\mathcal{F} = \alpha_s^T s + \beta_\mathrm{P} = 0$ as

$$\begin{aligned}\nu^+(\mathcal{F}(\mathbf{p})) &= \sum_{i=1}^{n_s} \lambda_i \cdot (\Phi(-\beta_\mathrm{p}) - \Phi_2(-\beta_\mathrm{p}, -\beta_\mathrm{p}; \rho_i)) \\ &= \Phi(-\beta_\mathrm{p}) \cdot \sum_{i=1}^{n_s} \lambda_i \cdot \left(1 - \frac{\Phi_2(-\beta_\mathrm{p}, -\beta_\mathrm{p}; \rho_i)}{\Phi(-\beta_\mathrm{p})}\right) \\ &= \Phi(-\beta_\mathrm{p}) \cdot \sum_{i=1}^{n_s} \lambda_i' \\ &\leqslant \Phi(-\beta_\mathrm{p}) \cdot \sum_{i=1}^{n_s} \lambda_i\end{aligned} \tag{5}$$

with $\Phi(.,.;.)$ the two-dimensional normal integral. The correlation coefficient of the two state variables before and after a jump equals $\rho_i = 1 - \alpha_{si}^2$. Formally, the last factor in the second line can be interpreted as a first-order correction to the jump rates then denoted by λ' in the third line. For large β the probability of jumps from the failure domain into the failure domain can be neglected and the correction term for λ_i' vanishes. This may be called a FORM-approximation which is used later. The same result holds as a first order approximation if the failure surface is linearized as $\partial\mathcal{F} \approx \alpha_s^T s + \beta_\mathrm{p} = 0$. The process of outcrossings conditional on $\mathbf{R} = \mathbf{r}$ can be shown to be asymptotically a Poisson process [16].

3.2. Out-crossing rates and failure probabilities for Gaussian processes

The determination of out-crossing rates for Gaussian processes is well known by Rice's formula [17,18] and extensions to vector processes exist. For Gaussian vector processes $\mathbf{S}(t)$ their stochastic characteristics are fully defined by the mean vector $m_S(t)$ and a symmetric, positive definite matrix $C_S(t_1, t_2) = \{\sigma_{ij}(t_1, t_2); i, j = 1, \ldots, n\}$ of covariance functions. By a suitable orthogonal transformation the matrix of auto-correlation functions $R(t_1, t_2)$ can always be made a diagonal matrix. The correlation matrix of the derivative processes $\dot{R}(t_1, t_2)$ can be obtained by twice differentiating the auto-correlation matrix.

The crossing rates are computed according to the generalization of Rice's formula put forward by Belyaev [23]. Belyaev's formula for the stationary case is

$$\nu^+(\mathcal{F}(\mathbf{p})) = \int_{\partial\mathcal{F}} E\left(-\alpha_s^T(\mathbf{s})\dot{\mathbf{S}}|\mathbf{S}(\tau) = \mathbf{s}\right)\cdot\varphi_n(\mathbf{s})\mathrm{d}s_{\partial\mathcal{F}}$$

where $\partial\mathcal{F} = \{(\mathbf{s}) : g(\mathbf{s}) \leqslant 0\}$, $\alpha_s(\mathbf{s}) = \nabla g(\mathbf{s})/\|\nabla g(\mathbf{s})\|$ the surface normal and $\mathrm{d}s_{\partial\mathcal{F}}$ means surface integration. The critical point is the usual β-point which has to be located by an appropriate search algorithm. If it is assumed that the integration with respect to the **Q**-variables can be performed simultaneously with the integration for the **S**-variables and it is admissible to integrate over the **R**-variables together with the other variables and the failure surface is linear, the outcrossing rate can be given as

$$\nu^+(\mathcal{F}(\mathbf{p})) = \varphi(\beta_\mathbf{p})\cdot\omega_0\cdot\frac{1}{\sqrt{2\pi}} \quad (6)$$

with

$$\omega_0^2 = -\alpha_s^T(\mathbf{s})\cdot\ddot{\mathbf{R}}\cdot\alpha_s(\mathbf{s})$$

Here again, (6) can be used as a first order approximation if the failure surface is linearized as $\partial\mathcal{F} \approx \alpha_s^T s + \beta_\mathbf{p} = 0$. The process of outcrossings conditional on $\mathbf{R} = \mathbf{r}$ can be shown to be asymptotically a Poisson process [18].

4. Reliability-oriented structural optimization

4.1. Objective functions for stationary, time-variant systems

Rosenblueth and Mendoza [5] distinguished between two reconstruction policies for Poissonian failure processes, i.e. processes with outcrossing rate $\nu^+(\mathcal{F}(\mathbf{p}))$. On the one hand they assumed that the structural facility is to be used for one mission only and is abandoned after fulfilling its mission or after failure. Otherwise the structure has to fulfill its function continuously and thus will be systematically reconstructed after failure. The reconstruction times will be assumed to be negligibly short as compared to the interarrival times of failure. They started from the general objective function

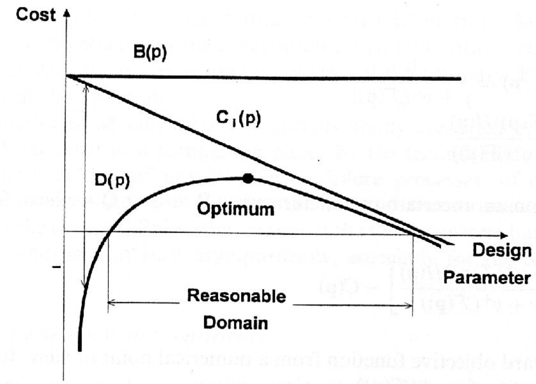

Fig. 1. Initial cost C, expected cost of failure D and benefit B over design parameter (after Rosenblueth and Esteva [19]).

$$Z(\mathbf{p}) = B(\mathbf{p}) - C(\mathbf{p}) - D(\mathbf{p})$$

where $B(\mathbf{p})$ is the benefit function, $C(\mathbf{p})$ is the construction cost and $D(\mathbf{p})$ is the damage cost.

All quantities are expected values. Because time is involved all expected cost terms have to be capitalized down to the decision point at $t = 0$. The (continuous) capitalization or discount function is

$$\mathrm{d}(t) = \exp[-\gamma t]$$

with γ the discount rate. For a yearly rate of discount γ', $\gamma = \ln(1 + \gamma')$.

If the structure is built only for one mission we have

$$B(T, \mathbf{p}) = \int_0^T b(t)\mathrm{d}(t)R(t, \mathbf{p})\mathrm{d}t$$
$$D(T, \mathbf{p}) = \int_0^T f(t, \mathbf{p})\mathrm{d}(t)H(\mathbf{p})H\mathrm{d}t$$

and therefore

$$Z(\mathbf{p}) = \int_0^T b(t)\mathrm{d}(t)R(t, \mathbf{p})\mathrm{d}t - C(\mathbf{p}) - \int_0^T f(t, \mathbf{p})\mathrm{d}(t)H\mathrm{d}t$$

Here, $H(\mathbf{p})$ is the damage cost, $R(t, \mathbf{p})$ the reliability, $b(t)$ the benefit per unit time derived from the existence of the structure and $f(t, \mathbf{p})$ the time to failure. For a Poisson process with intensity $\nu^+(\mathcal{F}(\mathbf{p}))$ [see Eq. (4)], constant benefit per unit time $b(t) = b$ and a given time T_s of anticipated use

$$B(T_s, \mathbf{p}) = \frac{b}{\gamma + \nu^+(\mathcal{F}(\mathbf{p}))}\left(1 - \exp[-(\gamma + \nu^+(\mathcal{F}(\mathbf{p})))T_s]\right)$$
$$D(T_s, \mathbf{p}) = \frac{\nu^+(\mathcal{F}(\mathbf{p}))H(\mathbf{p})}{\gamma + \nu^+(\mathcal{F}(\mathbf{p}))}\left(1 - \exp[-(\gamma + \nu^+(\mathcal{F}(\mathbf{p})))T_s]\right)$$

We have for $T_s \rightarrow \infty$

$$B^*(\mathbf{p}) = B(\infty, \mathbf{p}) = \frac{b}{\gamma + \nu^+(\mathcal{F}(\mathbf{p}))}$$
$$D^*(p) = \frac{\nu^+(\mathcal{F}(\mathbf{p}))H(\mathbf{p})}{\gamma + \nu^+(\mathcal{F}(\mathbf{p}))}$$

If $\nu^+(\mathcal{F}(\mathbf{p}))$ depends on an uncertain parameter vector **R** and/or **Q** we have for $T_s \rightarrow \infty$

$$Z(\mathbf{p}) = \mathrm{E}_{\mathrm{R,Q}}\left[\frac{b - \nu^+(\mathcal{F}(\mathbf{p}))H(\mathbf{p})}{\gamma + \nu^+(\mathcal{F}(\mathbf{p}))}\right] - C(\mathbf{p}) \tag{7}$$

This is a rather awkward objective function from a numerical point of view. It is easy to see that a fairly good approximation for $\nu^+(\mathcal{F}(\mathbf{p})) \ll \gamma$ is

$$Z(\mathbf{p}) \approx \frac{\mathrm{b}}{\gamma} - \mathrm{C}(\mathbf{p}) - \mathrm{H}(\mathbf{p})\frac{\mathrm{E}_{\mathrm{R,Q}}[\nu^+(\mathcal{F}(\mathbf{p}))]}{\gamma} \tag{8}$$

If, however, systematic reconstruction is chosen we have

$$B^* = B(\infty) = \frac{b}{\gamma}$$

and, by considering infinitely many "renewals" (see, Rosenblueth and Mendoza [5])

$$D(\mathbf{p}) = (C(\mathbf{p}) + \mathrm{H}(\mathbf{p}))\frac{\nu^+(\mathcal{F}(\mathbf{p}))}{\gamma}$$

so that for $\nu^+(\mathcal{F}(\mathbf{p}))$ depending on an uncertain vector $(\mathbf{R}, \mathbf{Q})$

$$Z(\mathbf{p}) = \frac{\mathrm{b}}{\gamma} - C(\mathbf{p}) - (\mathrm{C}(\mathbf{p}) + \mathrm{H}(\mathbf{p}))\frac{E_{R,Q}[\nu^+(\mathcal{F}(\mathbf{p}))]}{\gamma} \tag{9}$$

Note that the parameter time, i.e. some usually unknown life time of the structure, has disappeared and the failure cost or the failure rate are formally increased by a factor of $1/\gamma$. Because the benefit is assumed to be independent of **p** it is sufficient to consider the total cost

$$C_{\mathrm{total}}(\mathbf{p}) = \mathrm{C}(\mathbf{p}) + (C(\mathbf{p}) + H(\mathbf{p}))\frac{E_{R,Q}[\nu^+(\mathcal{F}(\mathbf{p}))]}{\gamma} \tag{10}$$

when optimizing the design. It is seen that formulation (10) differs from (8) only by the damage cost term. It should also be noted that the expectation operations with respect to **R** and **Q** in (4) can be performed directly at the out-crossing rate in view of the corollary. Both (8) and (9) must, of course, be positive at the optimum.

Based on Rosenblueth and Mendoza [5] the concepts briefly presented before have been thoroughly reviewed and extended in a companion paper by the second author [20]. In particular, aspects of serviceability failure, of non-Poissonian failure processes, of obsolescence and of inspection and maintenance modify the objective function and are not treated herein. It is important to remember that the assumed Poissonian nature of the failure process has been shown to hold under quite general conditions at least asymptotically, especially for the two special processes considered above.

4.2. Cost optimization with reliability constraints

The reliability-based structural cost optimization problem is a problem where total cost, including initial cost of design and expected cost of failure are minimized subject to the constraint on reliability and constraints on structural performance and on cost parameters. Usually, the reliability-based structural optimization is solved successively by two levels of optimization. The first problem (top-level) is cost optimization. The second problem (sub-level) determines the reliability of the structure which as seen from (3), (5) and (6) together with the corollary is essentially also an optimization task. A new and promising approach for time-invariant componential optimization has been investigated in [3]. Instead of using a two-level approach the two optimizations are combined into one optimization problem. This approach is now extended to the time-variant, stationary case.

The necessary first-order optimality condition for design points from **Corollary 2** are inserted into the cost optimization problem. More precisely, the optimization problem must fullfil the necessary optimality conditions for the β-point-problem in the time-invariant and stationary case.

$$\begin{array}{ll} \textbf{minimize} & C_{\mathrm{total}}(\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s, \boldsymbol{p}) \\ \text{subject to} & g(\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s, \boldsymbol{p}) = 0 \\ & (\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s)^T \nabla_u g(\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s, \boldsymbol{p}) + \|(\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s)\| \, \|\nabla_u g(\boldsymbol{u}^r, \boldsymbol{u}^q, \boldsymbol{u}^s, \boldsymbol{p})\| = 0 \\ & \text{constraints on random and cost vector} \\ & \text{simple bounds for random and cost vector} \end{array} \tag{11}$$

In accordance with Eq. (10) the constraint related to reliability must be given in terms of a failure rate as

$$E_R\big(E_Q(\nu^+(\mathcal{F}(\mathbf{p})))\big) \leqslant \nu_f^{\mathrm{maximum}},$$

where ν_f^{maximum} is some maximum allowable out-crossing rate selected by other criteria than optimization.

Using the total cost formulation including a discount or actualization aspect of building and failure cost of Eq. (10) therefore, leads to:

$$\begin{aligned}
\textbf{minimize} \quad & C(p) + (C(p) + H(p))\big(E_{\mathbf{R}}\big(E_{\mathbf{Q}}(\nu^{+}(\mathcal{F}(p)))\big)\big)/\gamma \\
\text{subject to} \quad & E_R\big(E_Q(\nu^{+}(\mathcal{F}(p)))\big) \leqslant \nu_f^{\text{maximum}} \\
& g(u^r, u^q, u^s, p) = 0 \\
& (u^r, u^q, u^s)^T \nabla_u g(u^r, u^q, u^s, p) + \|(u^r, u^q, u^s)\| \, \|\nabla_u g(u^r, u^q, u^s, p)\| = 0 \qquad (12) \\
& h_i(T(u^r, u^q, u^s), p) = 0, \qquad i = 1, \ldots, m' \\
& \tilde{h}_j(T(u^r, u^q, u^s), p) \leqslant 0, \qquad j = m' + 1, \ldots, m \\
& \big((u^r, u^q, u^s)^l, p^l\big) \leqslant ((u^r, u^q, u^s), p) \leqslant ((u^r, u^q, u^s)^u, p^u),
\end{aligned}$$

where $h_i(\cdot,\cdot)$ denote m' equality constraints and $\tilde{h}_j(\cdot,\cdot)$, denote $m - m'$ inequality constraints for the design vector and the parameters. $((u^r, u^q, u^s)^l, p^l)$, $((u^r, u^q, u^s)^u, p^u)$ are simple lower and upper bounds for the standard normal vector $((u^r, u^q, u^s)^l, p^l)$ and the cost parameter is p. This is the new result. The optimization problem must be solved by a suitable algorithm. The minimization of total cost under time-variant constraints can be carried out conveniently by the non-linear optimization algorithm NLPQL based on sequential quadratic programming (SQP, see [21]). Other algorithms may also work. As in time-invariant problems an optimal solution usually can easily be found if expected failure cost are not included. Then, a failure rate restriction must be imposed. In this case the number of iterations are similar to or a little larger than for simple reliability analysis. The optimization task is numerically more involved if expected failure cost are included and the imposed reliability constraint is not active. The reason simply is that objectives like 10 are extremely flat in the neighborhood of the optimum under realistic conditions. In the following two illustration examples expected failure cost are included.

5. Numerical examples

5.1. Steel column

The first numerical example is a pinned-pinned steel column with cost parameter $p = (\mu_b, \mu_d, \mu_h)$:

Parameter	Symbol	Unit	Bounds
Mean of flange breadth	μ_b	mm	(200, 400)
Mean of flange thickness	μ_d	mm	(10, 30)
Mean of height of steel profile	μ_h	mm	(100, 500)

The steel column has a constant length of 7500 mm. The function of total cost $C_t(p, u)$ includes failure cost of

$$H = 500\,000(CU)(CU = \text{currency unit})$$

discounted continuously with rate $\gamma = 1{,}5$ and 10% per year and parameter-dependent cost:

$$C(p) = (\mu_b \mu_d + 5(\text{mm}) \cdot \mu_h) \cdot (CU/\text{mm}^2)$$

Systematic reconstruction is assumed.

The independent uncertain vector $X = (F_s, P_1, P_2, P_3, B, D, H, F_0, E)$ and its stochastic characteristics are given by:

Variable	Symbol	Distribution	Type	Mean\s.d.	Unit	Jump rate
Yield stress	F_s	Log$N$	R	400/35	Mpa	—
Flange breadth	B	Log$N$	R	$\mu_b/3$	mm	—
Flange thickness	D	Log$N$	R	$\mu_d/2$	mm	—
Height of profile	H	Log$N$	R	$\mu_h/5$	mm	—
Initial deflection	F_0	N	R	30/10	mm	—
Youngs modulus	E	Weibull	R	21 000/4200	Mpa	—
Dead weight load	P_1	N	R	500 000/50 000	N	—
Variable load	P_2	Gumbel	S	600 000/90 000	N	$0.1_{[1/\text{year}]}$
Variable Load	P_3	Gumbel	S	600 000/90 000	N	$10_{[1/\text{year}]}$

Two loads P_2 and P_3, respectively, are modelled by rectangular wave renewal processes. The limit state function in terms of the random vector X, the parameter (μ_b, μ_d, μ_h) and auxiliary functions $\mathcal{A}_s, \mathcal{M}_s, \mathcal{M}_i, \mathcal{E}_b, \mathcal{P} = P_1 + P_2 + P_3$ is defined by:

$$g(X, p) = F_s - \mathcal{P}\left(\frac{1}{\mathcal{A}_s} + \frac{F_0}{\mathcal{M}_s} \cdot \frac{\mathcal{E}_b}{\mathcal{E}_b - \mathcal{P}}\right).$$

where

$\mathcal{A}_s$ = 2BD (area of section)
$\mathcal{M}_s$ = BDH (modulus of section)
$\mathcal{M}_i$ = $\frac{1}{2}$BDH^2 (moment of inertia)
$\mathcal{E}_b$ = $\frac{\pi^2 E \mathcal{M}_i}{s^2}$, (Euler buckling load)

It should be mentioned that the failure criterion is highly non-linear in the original as well as in the transformed space and, therefore, is a good test for the chosen algorithm. The admissible failure rate is $\nu_f^{\text{maximum}} = 10^{-4}$/year. Formally, this condition is replaced by [see Eq. (5)]

$$-\Phi^{-1}\left(\frac{\nu_f^{\text{maximum}}}{\lambda_1 + \lambda_2}\right) \leqslant \|T(x)\|$$

No other constraints on cost and design parameters are given in the example. The results for optimization problem are:

Discount rate γ	0.01	0.05	0.10
Optimal total cost	**5168** CU	**4972** CU	**4888** CU
Optimal design vector $\mathbf{p}^*$	(200, 22.72, 100)	(200, 21.74, 100)	(200, 21.32, 100)
Failure rate at optimum	$2.4 \cdot 10^{-6}$	$1.2 \cdot 10^{-5}$	$2.4 \cdot 10^{-5}$

In this case the reliability constraint is not active. The lower bounds for μ_b and μ_h are retained during iteration. The algorithm essentially iterates only in μ_d. As expected the optimum failure rate is approximately proportional to the discount rate. The same effect could have been achieved by modifying the failure cost appropriately. The objective function is rather flat and small deviations from the optimum result only in small changes of the total cost.

5.2. Air gap design for offshore structures

For offshore structures a sufficiently large air gap above still water level is required. The corresponding state function is

$$g(X, p) = H(p) - W$$

where H is the air gap and W the wave height (half amplitude).

The Gaussian wave height (S-variable) depends on the sea state and the sea spectrum. According to standard theory the standard deviation of wave height is given in terms of the significant wave height by

$$\sigma = \frac{H_s}{4}$$

and the autocorrelation function of wave heights is derived from

$$R(\tau) = \int_0^\infty G(\omega)\cos(\omega\tau)\mathrm{d}\omega$$

with Pierson/Moskowitz's spectrum

$$G(\omega) = \frac{H_s^2}{4\pi} \cdot \left(\frac{2\pi}{T_0}\right)^4 \cdot \frac{1}{\omega^5} \cdot \exp\left[-\frac{1}{\pi} \cdot \left(\frac{2\pi}{T_0}\right)^4 \cdot \frac{1}{\omega^4}\right]$$

The autocorrelation function as well as its second derivative at $\tau = 0$ as required by [Eq. (6)] must be determined numerically. The significant wave height is assumed to be a Weibull distributed sequence (Q-variable) with location parameter $w_{H_s} = 3.5$ m and scale parameter $k_{H_s} = 1.5$. The wave period in [1/s] (Q-variable) is also Weibull distributed with parameters related to significant wave height as

$$w_{T_0} = 6.05\exp(0.07H_s) \text{ and } k_{T_0} = 2.35\exp(0.21H_s)$$

Finally, it is assumed that the height of the structure H (R-variable) can only be built with a standard deviation of $\sigma_H = 0.2$ [m] due to uncertainties in the foundation conditions and construction processes. The random deviations from the target $p = \mu_H$ are normally distributed. Enlarging the air gap can be very costly depending on the type of offshore structure and water depth. Also, failure, i.e. full impact of the wave on the deck structure, can be very costly. Eq. (10) implying systematic reconstruction will be used. If it is assumed that the building cost equal $C_{(p)}/C_0 = 1 + (C_1/C_0)p = 1 + 0.005\mu_H$, the failure costs including environmental impact are $H/C_0 = 2.5$ and the effective interest rate is 0.025, one finds the optimal mean height at $p^* = \mu_H^* \approx 17.5m$ at a yearly failure rate of $6 \cdot 10^{-5}$ which is a little smaller than the admissible rate of $\nu_f^{\text{maximum}} = 10^{-4}$. For differentiable loading this is imposed as [see Eq. (6)]

$$\sqrt{-2\ln\left(\frac{2\pi\nu_f^{\text{maximum}}}{\omega_0}\right)} \geqslant \|T(\mathbf{x})\|$$

The total cost are $C_{\text{Total}}/C_0 \approx 1.09$ implying that in this case about 9% of the total cost should be invested in order to avoid failure. In this example the physical model is simple but the stochastic model is rather sophisticated as it involves all types of variables, partly highly dependent on each other. This example has been made simple because it is an example with only one parameter and this is a distribution parameter. The results obtained can easily be verified in a parameter study for μ_H.

6. Summary and conclusion

Based on an optimization scheme for structural components developed for time-invariant reliability constraints the theory is generalized to time-variant reliability constraints within the context of FORM. Reliabilities are computed by the outcrossing approach for rectangular wave renewal processes and for Gaussian vector processes, at present for non-intermittent, stationary processes, only. A unique $\beta_{\mathbf{p}}$-point must exist and the failure process must be a Poisson process, at least approximately and possibly conditional on other random variables. Therefore, our theory is applicable to high reliability problems. Generalization to intermittent processes should be straightforward but consideration of non-Poissonian failure processes originating, for example, from fatigue or other deterioration requires further research. The optimization scheme is a one-level scheme which requires second order derivatives of the structural limit function, a restriction which limits general applications only very little, and a formulation of the reliability problem in the so-called standard space. Capitalization of failure cost as well as benefits and the reconstruction policy is considered. Reliability constraints must be given in terms of failure rates instead of failure probabilities for arbitrary reference periods. Extension to more accurate second- or higher-order reliability methods requires further development. Direct extension is hardly possible because the optimization problem can no more be formulated as simple as in Eq. (12). Since some second-order reliability results are already available a straightforward but time-consuming approach is by iteration. Also, the presence of multiple failures modes must still be investigated. If the failure modes have different failure rate and different failure cost the objective function is likely to have multiple optima even if unique β_p-points can be found for each mode. Two example applications illustrate theory and methods. Simple cost optimization with a reliability constraint requires approximately the same numerical effort as simple reliability analysis. However, the numerical effort in reliability-oriented optimization both with time-invariant and with time-variant reliability, is significantly larger, i.e. roughly by a factor of 10, than for simple reliability analysis if expected failure cost is included. If it is included the optimization problem often simplifies to an unconstrained optimization problem.

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