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BERICHTE
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ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

AN EFFICIENT NUMERICAL SOLUTION TO THE MULTINORMAL INTEGRAL

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Abstract

The multinormal integral is evaluated by application of a recent, asymptotic formula for the probability content of general intersections in the standard normal space together with a recursive scheme proposed earlier by Hohenbichler. It is suitable for arbitrary correlations and computationally efficient even for higher dimensions. It is very accurate for small probabilities.

Introduction

The numerical determination of the multinormal integral is required in many problems of statistics and, recently, turned out to be a key problem in structural reliability [1]. For example, an event tree-type of analysis of redundant structural systems requires the evaluation of the probability of the intersection of the componental dependent state change events along a given path to structural collapse [2]. Other more general applications may be found in [3]. Let $\underline{U} = (U_1, \ldots, U_n)^T$ be an independent standard normal vector and a domain V be given as $V = \bigcap_{j=1}^{m} \langle g_j(\underline{U}) \leq 0 \rangle$ with $g_j(\underline{0}) > 0$ for at least one j and the functions g_j at least piecewise twice differentiable. In [4] an asymptotic approximation for the probability content of V has been derived which is given below for further reference:

$$P(V) \sim \Phi_k(\underline{c}; \underline{\underline{R}}) \, (\det(\underline{\underline{I}} - \underline{\underline{D}}))^{-1/2} = \Phi_k(\underline{c}; \underline{\underline{R}}) \, C \qquad (1)$$

Herein, Φ_k is the k-dimensional standard normal integral, $\underline{c} = \underline{\underline{A}}^T \underline{u}^*$, $\underline{\underline{R}} = \underline{\underline{A}}^T \underline{\underline{A}}$, $C = (\det(\underline{\underline{I}} - \underline{\underline{D}}))^{-1/2}$ a second-order correction term, and $\underline{u}^*$ a point defined by

$$\underline{u}^* = \min \langle \|\underline{u}\| \rangle \text{ for } \langle \underline{u} : \bigcap_{j=1}^{m} g_j(\underline{u}) \leq 0 \rangle \qquad (2)$$

k ($1 \leq k \leq m$) is the size of the index set J for "active" constraints, i. e. for which $g_j(\underline{u}^*) = 0$. $\underline{\underline{A}}$ collects as columns the linearly independent, normalized gradients ($\|\underline{a}_i\| = \|\text{grad } g_j(\underline{u}^*)\| = 1$)

of the active constraints. For simplicity of notation, an orthogonal transformation $\underline{w} = \underline{\underline{T}}^T \underline{u}$ may be performed such that $w^*_{k+1} = \ldots = w^*_n = 0$ and $\underline{b}_i = \underline{\underline{T}}^T \underline{a}_i$, $b_{ij} = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq m$. The first column of $\underline{\underline{T}}$ is formed by any of the $\underline{a}_i$, $i \in J$. The other columns are found by a suitable orthogonalisation algorithm. Further, a negative vector $\underline{\gamma} = (\gamma_1, \ldots, \gamma_k)$ to be determined from

$$\underline{w}^* = \underline{\underline{B}} \, \underline{\gamma} \quad ; \quad \underline{\underline{B}} = (\underline{b}_i) \qquad (1 \leq i \leq k) \qquad (3)$$

exists. Then, the elements of the matrix $\underline{\underline{D}}_{(m-k)\times(m-k)}$ in eq (1) are given by

$$d_{rs} = \sum_{t=1}^{k} \gamma_t \frac{\partial^2 g_t(\underline{u}^*)}{\partial \underline{b}_r \, \partial \underline{b}_s} \quad ; \qquad k+1 \leq r,s \leq m \qquad (4)$$

Formula (1) provides high numerical accuracy for sufficiently large $\|\underline{u}^*\|$ [1,4].

For the multinormal integral in eq. (1)

$$\Phi_n(\underline{c}; \underline{\underline{R}}) = P(\bigcap_{i=1}^{n} \langle X_i \leq c_i \rangle) = P(\underline{X} \leq \underline{c}) \qquad (5)$$

an asymptotic formula given in [5] and some approximations [6] as well as numerical integration schemes [7] have been proposed. Another computationally efficient approximation is derived in [8].

Derivation of main result

In this paper the multinormal integral is approximated by applying the foregoing general result (1) for the probability content of intersections to the evaluation of eq. (5) on the basis of a recursive scheme suggested in [8]. Without loss of generality let $\underline{X}$ be a standard normal vector with correlation matrix $\underline{\underline{R}} = (\rho_{ij})$. It can be represented in terms of an independent standard normal vector

$$X_i = \sum_{j=1}^{i} \alpha_{ij} U_j \qquad (1 \le i \le n) \tag{6}$$

with $\alpha_{11} = 1$ and the other coefficients determined such that the left-hand and the right-hand side have the same correlation matrix, e.g. by Cholesky's triangularisation procedure for positive definite matrices [6]. Now, the multinormal integral can be rewritten as

$$P(\underline{X} \le \underline{c}) = P(X_1 \le c_1)\, P\Big(\bigcap_{i=2}^{n} (X_i \le c_i \,|\, X_1 \le c_1)\Big)$$

$$= \Phi(c_1)\, P\Big(\bigcap_{i=2}^{n} \Big(\alpha_{i1} \overline{X}_1 + \sum_{j=2}^{i} \alpha_{ij} U_j - c_i \le 0\Big)\Big)$$

$$= \Phi(c_1)\, P\Big(\bigcap_{i=2}^{n} (g_i(\underline{U}) \le 0)\Big) \tag{7}$$

where the variable $\overline{X}_1$ is again standard normal but truncated at c_1 and independent of the other U's. $\overline{X}_1$ can be represented by its

Rosenblatt-transformation [9] as follows:

$$\overline{X}_1 = \Phi^{-1}[\Phi(c_1)\, \Phi(U_1)] \tag{8}$$

Hence, the g-functions in eq. (7) are:

$$g_i(\underline{U}) = \alpha_{i1}\, \Phi^{-1}[\Phi(c_1)\, \Phi(U_1)] + \sum_{j=2}^{i} \alpha_{ij} U_j - c_i \tag{9}$$

It is seen that the dimension of the intersection in eq. (5) is reduced by one in eq. (7). Let the point $\underline{u}^*$ defined by eq. (2) be found by some suitable search algorithm. An approximation for the second factor in eq. (7) then is by using eq. (1)

$$P\Big(\bigcap_{i=2}^{n} (g_i(\underline{U}) \le 0)\Big) \sim P\Big(\bigcap_{i \in J_2} (\underline{a}_i^T(\underline{u} - \underline{u}^*) \le 0)\Big)\, C_2 \tag{10}$$

where, as before, $J_2 \subseteq \{2,\ldots,n\}$ is the subset of $k \le n-1$ active constraints at $\underline{u}^*$ and C_2 the correction factor as in eq. (1). Therefore, eq. (7) can be rewritten as

$$P(\underline{X} \le \underline{c}) = \Phi(c_1)\, C_2\, P\Big(\bigcap_{i \in J_2} (\underline{a}_i^T(\underline{u} - \underline{u}^*) \le 0)\Big) \tag{11}$$

which is the main improvement as compared to [8].

For the last factor one, in turn, proceeds as described in eq. (7) with new X_i in eq. (6) given by $X_i = \underline{a}_i^T \underline{U}$ being standard normal with correlations $\rho_{ij} = \underline{a}_i^T \underline{a}_j$ and the new c_i defined by $c_i = \underline{a}_i^T \underline{u}^*$. Repeated at most (n-1)-fold application of this scheme finally yields the result.

Technicalities

The computation of the correction factors $C_t (t \geq 2)$ is facilitated by the following observations. Eq. (9) shows that the g-functions are non-linear only in the first variables. Their gradients have the elements:

$$\frac{\partial g_i(\underline{u})}{\partial u_1} = \frac{\alpha_{i1}}{\varphi(\bar{x}_1)} \phi(c_1) \varphi(u_1) \tag{12a}$$

$$\frac{\partial g_i(\underline{u})}{\partial u_j} = \alpha_{ij} \qquad (2 \leq j \leq i) \tag{12b}$$

The Hessian matrix of second derivatives has only one non-zero entry, i.e.:

$$\frac{\partial^2 g_i(\underline{u})}{\partial^2 u_1} = \alpha_{i1} \phi(c_1) \frac{\varphi(u_1)}{\varphi^2(\bar{x}_1)} [\, \phi(c_1)\bar{x}_1 \varphi(u_1) - u_1 \varphi(\bar{x}_1) \,] \tag{13}$$

In particular, the (directional) second-order derivatives at $\underline{w}^* = \underline{\underline{T}}^T \underline{u}^*$ in the directions orthogonal to the active gradients to be used in eq. (4) can be given analytically as:

$$\frac{\partial^2 g_i(\underline{w}^*)}{\partial \underline{b}_r \, \partial \underline{b}_s} = b_{1r} b_{1s} \frac{\partial^2 g_i(\underline{u}^*)}{\partial^2 u_1} \tag{14}$$

The directional Hessian matrix in the directions $(\underline{b}_{k+1}, \ldots, \underline{b}_n)$ is

$$\underline{\underline{H}}_i = \frac{\partial^2 g_i(\underline{u}^*)}{\partial^2 u_1} \hat{\underline{\underline{H}}} \tag{15}$$

with

$$\hat{\underline{\underline{H}}} = \underline{b}_1 \underline{b}_1^T \tag{16}$$

It is easy to show that $\underline{\underline{H}}_i$ has the n-k Eigenvalues:

$$\lambda_{i,k+1} = \frac{\partial^2 g_i(\underline{u}^*)}{\partial^2 u_1} \sum_{j=k+1}^{n} b_{1j}^2 \tag{17a}$$

$$\lambda_{i,k+2} = \ldots . \lambda_{i,n} = 0 \tag{17b}$$

Further, if there are several active constraints $g_i(\underline{u}) = 0$ $(i = 2, \ldots, k)$, it is according to eq. (4) with the γ_i's determined from eq. (3):

$$\lambda_{k+1} = \sum_{i=1}^{k} \gamma_i \left[\frac{\partial^2 g_i(\underline{u}^*)}{\partial^2 u_1} \sum_{j=k+1}^{n} b_{1j}^2 \right] \tag{18a}$$

$$\lambda_{k+2} = \ldots . = \lambda_n = 0 \tag{18b}$$

Therefore, the asymptotic correction factor in eq. (10) becomes:

$$C_2 = (1 - \lambda_{k+1})^{-1/2} \tag{19}$$

It can be shown that for smaller $\|\underline{u}^*\|$ the correction factor can be

improved especially for not very small probabilities by:

$$C_2 = \left(1 - \frac{\lambda_{k+1}}{\|\underline{u}^*\|} \frac{\varphi(\|\underline{u}^*\|)}{\Phi(-\|\underline{u}^*\|)}\right)^{-1/2} \qquad (20)$$

Remarks

- The analytical form of first and second-order derivatives suggest to use a search algorithm for $\underline{u}^*$ (eq. (2)) which uses this information (See, for example, [10]).

- It can be shown that $\underline{\underline{B}}$ in eq. (3) can always be made a lower triangular matrix which facilitates very much the solution for $\underline{y}$.

- The correction factor in eq. (1) can also be given as:

$$C = \frac{P(\bigcap_i \{g_i(\underline{U}) \le 0\})}{P(\bigcap_i \{\underline{a}_i^T(\underline{U}-\underline{u}^*) \le 0\})} \sim \frac{P(\bigcap_i \{q_i(\underline{U}) \le 0\})}{P(\bigcap_i \{\underline{a}_i^T(\underline{U}-\underline{u}^*) \le 0\})} \qquad \text{for large } \|\underline{u}^*\|$$

where $q_i(\underline{U}) \le 0$ is the parabolic approximation to $g_i(\underline{U}) \le 0$. In our application to the multinormal integral, the principal form of the paraboloid can easily be given since the Eigenvalues of the Hessian matrix are known by eq. (18). The paraboloid can be written as:

$$q(\underline{U}) = U_1 + \frac{1}{2}\lambda U_2^2 + \|\underline{u}^*\|$$

Then, an improved correction factor in making use of the exact

probability content of the prarboloid is:

$$\breve{C} = \frac{2\int_0^\infty \Phi(-\|\underline{u}^*\| - \frac{1}{2}\lambda x^2)\varphi(x)\,dx}{\Phi(-\|\underline{u}^*\|)} \qquad (21)$$

- The integral needs to be determined numerically. In eq. (10) the inactive constraints are omitted and similarly in all consecutive steps. It is possible to retain those constraints if one can justify reasonable expansion points. The simplest choice is to use $\underline{u}^*$ but better results are likely to be obtained by determining points defined by

$$\|\underline{u}^i\| = \min\{\|\underline{u}\|\} \qquad \text{for } \{\underline{u}: \{\bigcap_{\substack{j=2 \\ j\neq i}}^{n} g_j(\underline{u}) \le 0\} \cap \{g_i(\underline{u}) = 0\}\}$$

where i runs over all indices not contained in J_2.

- A crucial assumption for the foregoing results to hold is that the coordinate origin is not contained in V. If, however, $\underline{0} \in V$ or $g_i(\underline{0}) < 0$ for all i which implies that V is a "large" probability domain it is proposed to use the so-called equivalent plane concept outlined in [1,8,11].

- Another assumption is the linear independence of the active gradients $\underline{a}_i$ ($i \in J$) at $\underline{u}^*$. In the case of the multinormal integral this can be achieved by checking in each step of the procedure whether there are "components" in the intersection (see eq. 11) for which $|\rho_{ij}| = 1$. For $\rho_{ij}=+1$ the component with $\min\{c_i, c_j\}$ is omitted; if $\rho_{ij}=-1$, there is $\Phi_n(\ldots)=0$ for

$c_i \leq -c_j$. In each step (p) there are at most p-1 active components in a p-dimensional $\underline{U}$-space which will have linear independent gradients provided this checking is done.

- The accuracy of the method described here has been tested at the examples given in [8]. The quality of the results is at least as good as the best approximation in [8]. The computation times of the method presented herein are significantly smaller (roughly a factor of 10, but still proportional to n^2). It has to be pointed out, however, that the computation time is strongly influenced by the speed of the minimization routine searching for the point(s) $\underline{u}^*$ in eq. (2) resp. eq. (10).

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