Squeezed-State Generation by a DC Pumped Degenerate Josephson Parametric Amplifier

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Dedicated to Professor Dr. phil. Hans W. Pötzl on the occasion of his 60th birthday.

Squeezed-states of the radiation field can be generated by degenerate parametric amplification. In this paper, an analysis of a DC pumped degenerate Josephson parametric amplifier is presented. The quantum Langevin equations for the fully quantized system are derived. The degree of squeezing that could be obtained below the threshold for parametric oscillations is calculated by linearizing the corresponding quantum Langevin equations.

Die Erzeugung von Squeezed-States durch gleichspannungsgepumpte degenerierte parametrische Verstärker mit Josephsonelement

Mit Hilfe von degenerierten parametrischen Verstärkern ist es möglich neuartige rauscharme Zustände des Strahlungsfeldes, sogenannte Squeezed-States zu erzeugen. In dieser Arbeit wird untersucht inwieweit dies mit einem gleichspannungsgepumpten degenerierten parametrischen Verstärker mit Josephsonelement möglich ist. Dazu werden die Quanten-Langevin-Gleichungen welche die Dynamik des Verstärkers beschreiben abgeleitet. Diese nichtlinearen Langevin-Gleichungen werden linearisiert. Für den Unterhalb der Schwelle zur parametrisch angeregten Schwingung betriebenen Verstärker wird der Grad der Rauschunterdrückung berechnet.

1. Introduction

Generation of squeezed states at optical frequencies by means of degenerate parametric amplification has been demonstrated by Wu et al. [1]. In the microwave regime the Josephson junction provides a nonlinear inductance appropriate for parametric amplification and therefore for the generation of squeezed states [2]. Recently squeezed thermal noise at 4.2 K and 19.4 GHz was produced via a Josephson parametric amplifier (JPA) operating in the three photon mode and therefore pumped at 38.8 GHz, the doubled signal frequency [3]. It is well known that the JPA can also be pumped by an applied dc voltage [4], [5]. Such a device, called a dc pumped Josephson parametric amplifier (DCPJPA), will be investigated in the following. In Section 2 we will derive the classical equations of motion of a DCPJPA and the Lagrange function generating these equations. In Section 3 we obtain the Heisenberg equations of the quantized system by applying the method of canonical quantisation. As shown in Section 4, in the rotating wave approximation the Heisenberg equations have the same form as the quantum Langevin equations discussed by Gardiner and Collet [6]. Finally in Section 5 we study the squeezing behaviour of the DCPJPA below the threshold for oscillation by a linearised analysis.

2. Equations of Motion and Lagrange Formalism of the Classical System

In the following we investigate the circuit of a dc pumped degenerate Josephson parametric amplifier shown in Fig. 1. The Josephson junction is modeled by the resistively shunted junction model. Following the quantum network theory of Yurke and Denker [7],

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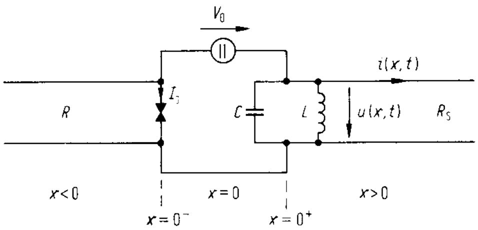

Fig. 1. Schematic circuit diagram of the dc pumped degenerate Josephson parametric amplifier.

the shunt resistance R of the junction is modeled by a transmission line with characteristic impedance R. By this way we overcome the difficulties in the quantum mechanical modeling of the ohmic resistor. The lumped inductance L and capacitance C model the resonator which is coupled to the left transmission line with characteristic impedance R_S.The observable radiation produced by the Josephson junction propagates outward of this transmission line. The pump source supplies a dc voltage V_0 in series to the Josephson junction and the resonant circuit. In the homogeneous regions $x \lessgtr 0$ voltages and currents in the transmission lines obey the transmission line equations

$$\frac{\partial u}{\partial x} = -L_R \frac{\partial i}{\partial t} \text{ if } x<0, \quad \frac{\partial u}{\partial x} = -L_S \frac{\partial i}{\partial t} \text{ if } x>0, \tag{1 a, b}$$

$$\frac{\partial i}{\partial x} = -C_R \frac{\partial u}{\partial t} \text{ if } x<0, \quad \frac{\partial i}{\partial x} = -C_S \frac{\partial u}{\partial t} \text{ if } x>0. \tag{2 a, b}$$

If we introduce as independent variable the total magnetic flux stored to the right of point x at time t

$$\Phi(x,t) = \int_{-\infty}^{t} u(x,t')\,\mathrm{d}t' \tag{3}$$

this variable satisfies in the homogeneous regions, according to eqs. (1 a) to (3), the scalar wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} = c_R^2 \frac{\partial^2 \Phi}{\partial x^2} \text{ if } x<0, \quad \frac{\partial^2 \Phi}{\partial t^2} = c_S^2 \frac{\partial^2 \Phi}{\partial x^2} \text{ if } x>0, \tag{4 a, b}$$

with propagation velocities

$$c_R = 1/\sqrt{L_R C_R}, \quad c_S = 1/\sqrt{L_S C_S}. \tag{5 a, b}$$

The characteristic impedances of the transmission lines are given by

$$R = \sqrt{L_R/C_R}, \quad R_S = \sqrt{L_S/C_S}. \tag{5 c, d}$$

Furthermore, it follows from eqs. (1 a) to (3) that the voltages and currents in both transmission lines at $x = 0^-$ and $x = 0^+$, respectively are given by

$$u(x=0^-,t) = \left.\frac{\partial \Phi}{\partial t}\right|_{x=0^-}, \quad u(x=0^+,t) = \left.\frac{\partial \Phi}{\partial t}\right|_{x=0^+}, \tag{6 a, b}$$

and

$$i(x=0^-,t) = -\frac{1}{L_R}\left.\frac{\partial \Phi}{\partial x}\right|_{x=0^-}, \tag{7 a}$$

$$i(x=0^+,t) = -\frac{1}{L_S}\left.\frac{\partial \Phi}{\partial x}\right|_{x=0^+}. \tag{7 b}$$

From Kirchhoff's voltage law it follows

$$u(x=0^-,t) = u(x=0^+,t) + V_0 \tag{8}$$

and therefore the magnetic flux has to satisfy the boundary condition

$$\left.\frac{\partial \Phi}{\partial t}\right|_{x=0^-} = \left.\frac{\partial \Phi}{\partial t}\right|_{x=0^+} + V_0 \tag{9}$$

or

$$\Phi(x=0^-,t) = \Phi(x=0^+,t) + V_0 t + \Phi'_{\text{ext}}. \tag{10}$$

where the integration constant Φ'_{ext} is the external flux trapped in the loop which is formed by the Josephson junction, the resonator and the dc voltage source. If we choose that the flux variable at point $x = 0$ to be equal to the flux through the inductance L

$$\Phi(x=0,t) = \Phi_L(t) = L i_L(t) \tag{11}$$

than it follows from Kirchhoff's voltage law for the resonator loop formed by the LC circuit

$$\Phi(x=0,t) = \Phi_L(t) = \Phi(x=0^+,t) + \Phi''_{\text{ext}} \tag{12}$$

where the constant of integration Φ''_{ext} is the external flux trapped in the loop formed by the inductor L and capacitor C. In the following we choose the constants of integration equal to zero

$$\Phi'_{\text{ext}} = \Phi''_{\text{ext}} = 0. \tag{13}$$

This may be done, since it causes only a shift in the time coordinate. With this choice of the independent variable $\Phi(x,t)$ and the constants of integration we see from eq. (10) and (12), respectively, that $\Phi(x,t)$ at point $x = 0$ is continuous from the right and not continuous from the left. The current through the Josephson junction [8] is given by

$$I_J = I_c \sin\left(\frac{2e_0}{\hbar}\Phi(x=0^-,t)\right). \tag{14}$$

According to Kirchhoffs current law at point $x - 0$ we obtain with eqs. (10) to (14)

$$C\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Phi_L + \frac{1}{L}\Phi_L + I_c \sin\left(\omega_0 t + \frac{2e_0}{\hbar}\Phi_L\right) + \\ + \frac{1}{L_R}\left.\frac{\partial \Phi}{\partial x}\right|_{x=0^-} - \frac{1}{L_S}\left.\frac{\partial \Phi}{\partial x}\right|_{x=0^+} = 0, \tag{15}$$

with

$$\omega_0 = \frac{2e_0}{\hbar}V_0 \quad \text{and} \quad \Phi_L = \Phi(x=0,t).$$

Eqs. (4 a), (4 b) and (15) constitute the equations of motion for the variable $\Phi(x,t)$, which uniquely determines the state of the system. Defining the distributions

$$H_+(x) = \lim_{\varepsilon \to 0^+} \begin{cases} 1 - e^{-x/\varepsilon} & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0, \end{cases} \tag{16a}$$

$$H_-(x) = H_+(-x) \tag{16b}$$

and

$$\delta_+(x) = \frac{d}{dx} H_+(x) = \lim_{\varepsilon \to 0^+} \begin{cases} \frac{1}{\varepsilon} e^{-x/\varepsilon} & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0, \end{cases} \tag{17a}$$

$$\delta_-(x) = -\frac{d}{dx} H_-(x) = \delta_+(-x), \tag{17b}$$

one can show that the distributions $\delta_+(x)$ and $\delta_-(x)$ pick up the right and left side limit of a function $f(x)$

$$f(x = 0^-, t) = \int_{-\infty}^{+\infty} \delta_-(x) f(x) \, dx, \tag{18a}$$

$$f(x = 0^+, t) = \int_{-\infty}^{+\infty} \delta_+(x) f(x) \, dx. \tag{18b}$$

With these distributions the Lagrangian describing the system may be written as

$$\begin{aligned} L[\Phi] = \int_{-\infty}^{+\infty} \delta_+(x) \Bigg\{ \frac{C}{2} \left(\frac{\partial \Phi(x,t)}{\partial t} \right)^2 - \frac{1}{2L} \Phi(x,t)^2 - \\ - \frac{\hbar}{2e_0} I_c \left(1 - \cos\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi(x,t) \right] \right) \Bigg\} dx + \\ + \int_{-\infty}^{+\infty} H_-(x) \left\{ \frac{C_R}{2} \left(\frac{\partial \Phi(x,t)}{\partial t} \right)^2 - \frac{1}{2L_R} \left(\frac{\partial \Phi(x,t)}{\partial x} \right)^2 \right\} dx + \\ + \int_{-\infty}^{+\infty} H_+(x) \left\{ \frac{C_S}{2} \left(\frac{\partial \Phi(x,t)}{\partial t} \right)^2 - \frac{1}{2L_S} \left(\frac{\partial \Phi(x,t)}{\partial x} \right)^2 \right\} dx. \end{aligned} \tag{19}$$

From this Lagrangian we obtain

$$\frac{\delta L}{\delta \left(\frac{\partial \Phi}{\partial t} \right)} = \{ C \delta_+(x) + C_R H_-(x) + C_S H_+(x) \} \frac{\partial \Phi(x,t)}{\partial t} \tag{20a}$$

and

$$\begin{aligned} \frac{\delta L}{\delta \Phi} = \delta_+(x) \Bigg\{ - \frac{1}{2L} \Phi(x,t)^2 - \\ - I_c \sin\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi(x,t) \right] \Bigg\} - \\ - \delta_-(x) \frac{1}{L_R} \frac{\partial \Phi(x,t)}{\partial x} + \delta_+(x) \frac{1}{L_S} \frac{\partial \Phi(x,t)}{\partial x} + \\ + H_-(x) \frac{1}{L_R} \frac{\partial^2 \Phi(x,t)}{\partial x^2} + H_+(x) \frac{1}{L_S} \frac{\partial^2 \Phi(x,t)}{\partial x^2}. \end{aligned} \tag{20b}$$

Therefore, the Euler Lagrange equation

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \left(\frac{\partial \Phi}{\partial t} \right)} - \frac{\delta L}{\delta \Phi} = 0 \tag{21}$$

for the above system is given by

$$\begin{aligned} H_-(x) \left\{ C_R \frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{L_R} \frac{\partial^2 \Phi}{\partial x^2} \right\} + \\ + H_+(x) \left\{ C_S \frac{\partial^2 \Phi}{\partial t^2} - \frac{1}{L_S} \frac{\partial^2 \Phi}{\partial x^2} \right\} + \\ + \delta_+(x) \Bigg\{ C \frac{\partial^2 \Phi(x,t)}{\partial t^2} + \frac{1}{L} \Phi(x,t) + \\ + I_c \sin\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi(x,t) \right] \Bigg\} + \\ + \delta_-(x) \frac{1}{L_R} \frac{\partial \Phi(x,t)}{\partial x} - \delta_+(x) \frac{1}{L_S} \frac{\partial \Phi(x,t)}{\partial x} = 0. \end{aligned} \tag{22}$$

According to the first two terms in this equation $\Phi(x,t)$ obeys eqs. (4a) and (4b) in the homogenious domains. Integrating the Euler Lagrange equation over the interval $-\varepsilon' < x < \varepsilon'$ we obtain eq. (15) in the limit $\varepsilon' \to 0$. Thus it is shown that the Lagrangian given by eq. (19) correctly describes the DCPJPA.

Because $\Phi(x,t)$ obeys in the domains $x \lessgtr 0$ the wave equations (4a) and (4b), the flux Φ can be decomposed in flux waves propagating inward and outward on the transmission lines according to

$$\Phi(x,t) = \begin{cases} \Phi_R^{in}(x - v_R t) + \Phi_R^{out}(x + v_R t) & \text{if } x < 0 \\ \Phi_S^{in}(x + v_S t) + \Phi_S^{out}(x - v_S t) & \text{if } x > 0. \end{cases} \tag{23}$$

Therefore the partial derivatives with respect to x occuring in eq. (15) can be transformed into derivatives with respect to time

$$\begin{aligned} \frac{1}{L_R} \frac{\partial \Phi}{\partial x} = \frac{1}{R} \left(\frac{\partial \Phi_R^{out}}{\partial t} - \frac{\partial \Phi_R^{in}}{\partial t} \right) = \\ = \frac{1}{R} \left(\frac{\partial \Phi}{\partial t} - 2 \frac{\partial \Phi_R^{in}}{\partial t} \right) \quad \text{if } x < 0, \end{aligned} \tag{24a}$$

$$\begin{aligned} \frac{1}{L_S} \frac{\partial \Phi}{\partial x} = \frac{1}{R_S} \left(\frac{\partial \Phi_S^{in}}{\partial t} - \frac{\partial \Phi_S^{out}}{\partial t} \right) = \\ = -\frac{1}{R_S} \left(\frac{\partial \Phi}{\partial t} - 2 \frac{\partial \Phi_S^{in}}{\partial t} \right) \quad \text{if } x > 0. \end{aligned} \tag{24b}$$

With these relations and eqs. (9) and (12) we can eliminate the partial derivatives in eq. (15) to obtain

$$\begin{aligned} C \frac{d^2}{dt^2} \Phi_L + \left(\frac{1}{R} + \frac{1}{R_S} \right) \frac{d}{dt} \Phi_L + \frac{1}{L} \Phi_L + \\ + I_c \sin(\omega_0 t + \Phi_L) + \frac{V_0}{R} = \\ = \frac{2}{R} \frac{d}{dt} \Phi_R^{in}(0^-, t) + \frac{2}{R_S} \frac{d}{dt} \Phi_S^{in}(0^+, t). \end{aligned} \tag{25}$$

The elimination of the outward propagating waves introduces a damping term into the differential equation describing the dynamics of the flux Φ_L. The remaining inward propagating waves will describe the inhomogenious quantum fluctuation terms related to the dissipation terms. Since the incident flux waves are completely defined by the initial conditions of the transmission lines at time $t = 0$, eq. (25) allows the computation of the flux through the lumped inductance L for $t > 0$. With that result the observable outward moving flux wave $\Phi_S^{out}(x, t)$ can be calculated according to eq. (23)

$$\Phi_S^{out}(0^+, t) = \Phi_L(t) - \Phi_S^{in}(0^+, t). \tag{26}$$

Thus the classical equations for the DCPJPA are formulated in terms of the Euler Lagrange formalism and can be solved for arbitrary input fields. We will now proceed with the quantum mechanical description of this system.

3. Canonical Quantization and the Heisenberg Equations of Motion

Following the procedure of canonical quantization of a classical system we obtain the canonically conjugate impulse density $\Pi(x, t)$ belonging to the flux variable $\Phi(x, t)$ according to eq. (20a)

$$\Pi(x, t) = \frac{\delta L}{\delta\left(\frac{\partial \Phi}{\partial t}\right)} = C_{inh} \frac{\partial \Phi(x, t)}{\partial t} \tag{27a}$$

where C_{inh} is the capacitance per unit length of our inhomogeneous transmission line with

$$C_{inh} = C\,\delta_+(x) + C_R H_-(x) + C_S H_+(x). \tag{27b}$$

From eqs. (19) and (27a, b) we obtain the Hamiltonian of the DCPJPA by

$$H = \int_{-\infty}^{+\infty} \Pi(x, t) \frac{\partial \Phi(x, t)}{\partial t} \mathrm{d}x - L[\Phi], \tag{28}$$

$$\begin{aligned} H = \int_{-\infty}^{+\infty} \Bigg\{ & \frac{1}{2C_{inh}} \Pi(x, t)^2 + \delta_+(x) \left(\frac{1}{2L} \Phi(x, t)^2 + \right. \\ & \left. + \frac{\hbar}{2e_0} I_c \left(1 - \cos\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi(x, t) \right] \right) \right) \Bigg\} \mathrm{d}x + \\ & + \int_{-\infty}^{+\infty} H_-(x) \frac{1}{2L_R} \left(\frac{\partial \Phi(x, t)}{\partial x} \right)^2 \mathrm{d}x + \\ & + \int_{-\infty}^{+\infty} H_+(x) \frac{1}{2L_S} \left(\frac{\partial \Phi(x, t)}{\partial x} \right)^2 \mathrm{d}x. \end{aligned} \tag{29}$$

The first expression in this equation which has a δ_+ distribution in the denominator is not defined in the limit $\varepsilon \to 0$ of eq. (17a), therefore in eq. (28) we tacitly assume that the limiting process $\varepsilon \to 0$ is not yet carried out, so that the capacitance per unit length in eq. (27a) is not singular and we can divide by it. The limit $\varepsilon \to 0$ will be taken after we have derived the Heisenberg equations of motion for the flux and the corresponding impulse density from eq. (29). It can be easily shown that the Hamilton equations of motion following from eq. (29) agree with those given by eqs. (4a), (4b), and (15).

Now we quantize the system by interpreting the flux $\Phi(x, t)$ and its conjugate impulse density $\Pi(x, t)$ as field operators, satisfying the commutation relation

$$[\Phi(x, t), \Pi(x', t)] = \mathrm{i}\hbar\,\delta(x - x') \tag{30}$$

and H given by eq. (29) is the Hamiltonian of the quantized system.

Thus we obtain for the Heisenberg equations of motion

$$\frac{\partial}{\partial t} \Phi(x, t) = \frac{1}{\mathrm{i}\hbar} [\Phi(x, t), H] = \frac{1}{C_{inh}} \Pi(x, t) \tag{31a}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \Pi(x, t) &= \frac{1}{\mathrm{i}\hbar} [\Pi(x, t), H] = \\ &= -\delta_+(x) \left\{ \frac{1}{L} \Phi(x, t) + I_c \sin\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi(x, t) \right] \right\} - \\ &\quad - \delta_-(x) \frac{1}{L_R} \frac{\partial \Phi(x, t)}{\partial x} + H_-(x) \frac{1}{L_R} \frac{\partial^2 \Phi(x, t)}{\partial x^2} + \\ &\quad + \delta_+(x) \frac{1}{L_S} \frac{\partial \Phi(x, t)}{\partial x} + H_+(x) \frac{1}{L_S} \frac{\partial^2 \Phi(x, t)}{\partial x^2}. \end{aligned} \tag{31b}$$

Differentiating eq. (31a) with respect to time and inserting it into eq. (31b) we show that the Heisenberg equations of motion are equivalent to the classical equations of motion (4a), (4b) and (15). Defining the discrete impulse π via

$$\pi(t) = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \Pi(x, t)\,\mathrm{d}x \tag{32}$$

we obtain from eq. (31a) by multiplication with C_{inh} and integration over the interval $[-\varepsilon, \varepsilon]$

$$\pi(t) = C \frac{\mathrm{d}}{\mathrm{d}t} \Phi_L \tag{33a}$$

and in the same way from eq. (31b)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \pi = &-\frac{1}{L} \Phi_L - I_c \sin\left[\omega_0 t + \frac{2e_0}{\hbar} \Phi_L \right] - \\ &- \frac{1}{L_R} \frac{\partial \Phi(0^-, t)}{\partial x} + \frac{1}{L_S} \frac{\partial \Phi(0^+, t)}{\partial x}. \end{aligned} \tag{33b}$$

For the regions $x \lessgtr 0$ the eqs. (31a, b) imply

$$\frac{\partial \Phi(x, t)}{\partial t} = \frac{\Pi(x, t)}{C_R}, \quad \frac{\partial \Pi(x, t)}{\partial t} = \frac{1}{L_R} \frac{\partial^2 \Phi(x, t)}{\partial x^2} \quad \text{for } x < 0, \tag{34a}$$

$$\frac{\partial \Phi(x, t)}{\partial t} = \frac{\Pi(x, t)}{C_S}, \quad \frac{\partial \Pi(x, t)}{\partial t} = \frac{1}{L_S} \frac{\partial^2 \Phi(x, t)}{\partial x^2} \quad \text{for } x > 0, \tag{34b}$$

which restates that Φ and Π must obey the scalar wave equation in the two regions with different propagation velocities. Therefore, like in Section 2 for the flux Φ, we can also decompose the impulse density in input and output components

$$\Pi(x, t) = \begin{cases} \Pi_R^{in}(x - v_R t) + \Pi_R^{out}(x + v_R t) & \text{if } x < 0 \\ \Pi_S^{in}(x + v_S t) + \Pi_S^{out}(x - v_S t) & \text{if } x > 0. \end{cases} \tag{35}$$

Applying the usual methods of field quantization as described for example in [7], we express the field amplitudes in the frequency domain by the creation operators $a_R^{in}(\omega)^\dagger$, $a_R^{out}(\omega)^\dagger$, $a_S^{in}(\omega)^\dagger$, $a_S^{out}(\omega)^\dagger$ and annihilation operators $a_R^{in}(\omega)$, $a_R^{out}(\omega)$, $a_S^{in}(\omega)$, $a_S^{out}(\omega)$. The subscripts R, S denote the left (R) and right (S) transmission line in Fig. 1. Since the transmission lines have infinite extension in the onedimensional half spaces $x<0$ and $x>0$, respectively, we obtain a continuous spectrum of states denoted by ω.

$$\Phi_R^{in}(x,t) = \int_0^\infty \sqrt{\frac{\hbar R}{4\pi\omega}} \cdot [a_R^{in}(\omega)\,e^{i(kx-\omega t)} + a_R^{in}(\omega)^\dagger\, e^{-i(kx-\omega t)}]\,d\omega, \tag{36a}$$

$$\Phi_R^{out}(x,t) = \int_0^\infty \sqrt{\frac{\hbar R}{4\pi\omega}} \cdot [a_R^{out}(\omega)\,e^{i(kx+\omega t)} + a_R^{out}(\omega)^\dagger\, e^{i(kx+\omega t)}]\,d\omega, \tag{36b}$$

$$\Phi_S^{in}(x,t) = \int_0^\infty \sqrt{\frac{\hbar R_s}{4\pi\omega}} \cdot [a_S^{in}(\omega)\,e^{-i(kx+\omega t)} + a_S^{in}(\omega)^\dagger\, e^{i(kx+\omega t)}]\,d\omega, \tag{36c}$$

$$\Phi_S^{out}(x,t) = \int_0^\infty \sqrt{\frac{\hbar R_s}{4\pi\omega}} \cdot [a_S^{out}(\omega)\,e^{i(kx-\omega t)} + a_S^{out}(\omega)^\dagger\, e^{-i(kx-\omega t)}]\,d\omega, \tag{36d}$$

$$\Pi_R^{in}(x,t) = -i\int_0^\infty C_R\sqrt{\frac{\hbar\omega R}{4\pi}} \cdot [a_R^{in}(\omega)\,e^{i(kx-\omega t)} - a_R^{in}(\omega)^\dagger\, e^{-i(kx-\omega t)}]\,d\omega, \tag{37a}$$

$$\Pi_R^{out}(x,t) = -i\int_0^\infty C_R\sqrt{\frac{\hbar\omega R}{4\pi}} \cdot [a_R^{out}(\omega)\,e^{-i(kx+\omega t)} - a_R^{out}(\omega)^\dagger\, e^{i(kx+\omega t)}]\,d\omega, \tag{37b}$$

$$\Pi_S^{in}(x,t) = -i\int_0^\infty C_S\sqrt{\frac{\hbar\omega R_s}{4\pi}} \cdot [a_S^{in}(\omega)\,e^{-i(kx+\omega t)} - a_S^{in}(\omega)^\dagger\, e^{i(kx+\omega t)}]\,d\omega, \tag{37c}$$

$$\Pi_S^{out}(x,t) = -i\int_0^\infty C_S\sqrt{\frac{\hbar\omega R_s}{4\pi}} \cdot [a_S^{out}(\omega)\,e^{i(kx-\omega t)} - a_S^{out}(\omega)^\dagger\, e^{-i(kx-\omega t)}]\,d\omega. \tag{37d}$$

These operators have to satisfy the commutation relations

$$[a_k^u(\omega), a_l^u(\omega')^\dagger] = \delta_{k,l}\,\delta(\omega-\omega'), \tag{38a}$$

$$[a_k^u(\omega), a_l^u(\omega')] = 0, \tag{38b}$$

$$[a_k^u(\omega)^\dagger, a_l^u(\omega')^\dagger] = 0 \tag{38c}$$

for $k, l \in \{R, S\}$ and $u \in \{in, out\}$, to ensure the commutation relation (30) for $\Phi(x,t)$ and $\Pi(x,t)$. Like in the classical case we can eliminate the partial derivatives with respect to x in eq. (33b) with help of eqs. (24a, b), which are also valid in the quantum mechanical treatment and eq. (34a, b). We shift the system variable Φ_L by the constant flux produced by the dc current through the inductance L via the loss resistance R

$$\varphi = \Phi_L - L V_0/R, \tag{39}$$

and obtain the Heisenberg equations of motion for the flux φ and impulse π from (33a, b) and (39)

$$\frac{d}{dt}\varphi = \frac{1}{C}\pi \tag{40a}$$

$$\frac{d}{dt}\pi = -\frac{1}{L}\varphi - \frac{1}{C}\left(\frac{1}{R}+\frac{1}{R_S}\right)\pi - I_c \sin\left[\omega_0 t - \varphi_0 + \frac{2e_0}{\hbar}\varphi\right] + \frac{2}{RC_R}\Pi_R^{in}(0^-,t) + \frac{2}{R_S C_S}\Pi_S^{in}(0^+,t) \tag{40b}$$

with the phase delay $\varphi_0 = 2e_0 V_0 L/(\hbar R)$. From eqs. (11), (30), (32) and (39) we obtain

$$[\varphi, \pi] = i\hbar. \tag{41}$$

Thus φ and π are canonically conjugate variables. We introduce the creation operator $a^\dagger$ and annihilation operator a by the transformation

$$a = \sqrt{\frac{\Omega C}{2\hbar}}\varphi + i\sqrt{\frac{1}{2\hbar\Omega C}}\pi, \tag{42a}$$

$$a^\dagger = \sqrt{\frac{\Omega C}{2\hbar}}\varphi - i\sqrt{\frac{1}{2\hbar\Omega C}}\pi, \tag{42a}$$

with the resonant frequency

$$\Omega = 1/\sqrt{LC}. \tag{43}$$

Expressing (40a) and (40b) by creation and annihilation operators we obtain

$$\dot a = -i\Omega a - i\sigma\sin[\omega_0 t - \varphi_0 + \varkappa(a+a^\dagger)] - \gamma(a-a^\dagger)/2 + \sqrt{\gamma_R}\,[b_R^{in}(t) - b_R^{in}(t)^\dagger] + \sqrt{\gamma_S}\,[b_S^{in}(t) - b_S^{in}(t)^\dagger] \tag{44}$$

with the time dependent incident wave operators

$$b_R^{in}(t) = \int_0^\infty \sqrt{\frac{\omega}{2\pi\Omega}}\, a_R^{in}(\omega)\,e^{-i\omega t}\,d\omega, \tag{45a}$$

$$b_S^{in}(t) = \int_0^\infty \sqrt{\frac{\omega}{2\pi\Omega}}\, a_S^{in}(\omega)\,e^{-i\omega t}\,d\omega, \tag{45b}$$

and the constants

$$\gamma_R = \frac{1}{RC}, \quad \gamma_S = \frac{1}{R_S C}, \quad \gamma = \gamma_R + \gamma_S,$$

$$\varkappa = \sqrt{\frac{2e_0^2}{\hbar\Omega C}}, \quad \sigma = \sqrt{\frac{1}{2\hbar\Omega C}}\,I_c, \tag{46}$$

where γ_R denotes the damping constant due to the shunt resistance of the Josephson junction and γ_S is the damping constant due to the coupling to the signal transmission line which carries the observable radiation away from the junction. The dimensionless ratio $\varkappa/2\pi$ is the average number of magnetic flux quanta $h/2e_0$ stored in the inductance L if the resonator is in the vacuum state $|0\rangle$ or roughly speaking $\varkappa$ is 2π times the number of flux quanta created by the creation of one photon in the resonator. σ is a measure for the coupling of the Josephson junction to the resonator. For the time evolution of the annihilation operator we obtain the hermitian conjugate equation to eq. (44).

Those terms in the equation of motion, eq. (44), which arise from the lumped circuit elements in Fig. 1 can be derived from the time dependent Hamiltonian

$$H_{\text{sys}}(a, a^\dagger, t) = \hbar\,\Omega\,(a^\dagger a + \tfrac{1}{2}) + + \hbar\frac{\sigma}{k}[1 - \cos[\omega_0 t - \varphi_0 + \varkappa(a + a^\dagger)]]. \tag{47}$$

H_{sys} is the Hamiltonian of the resonator and the dc biased Josephson junction. The phase φ_0 has been introduced to take into account the flux stored in the inductance L due to the dc current flowing over the shunt resistance R of the Josephson junction. Thus the equation of motion for the annihilation operator a may be written as

$$\dot{a} = -\frac{i}{\hbar}[a, H_{\text{sys}}] - \gamma(a - a^\dagger)/2 + + \sqrt{\gamma_R}\,[b_R^{\text{in}}(t) - b_R^{\text{in}}(t)^\dagger] + \sqrt{\gamma_S}\,[b_S^{\text{in}}(t) - b_S^{\text{in}}(t)^\dagger]. \tag{48}$$

From the time derivative of eq. (26) which is also valid for the corresponding operators in the quantum mechanical treatment and eqs. (33 a, b) and (34 a) we obtain

$$\frac{\pi(t)}{C} = \frac{\Pi_S^{\text{in}}(0^+, t) + \Pi_S^{\text{out}}(0^+, t)}{C_S}, \tag{49}$$

$$b_S^{\text{out}}(t) - b_S^{\text{out}}(t)^\dagger = \sqrt{\gamma_S}(a - a^\dagger) - [b_S^{\text{in}}(t) - b_S^{\text{in}}(t)^\dagger] \tag{50}$$

with the output field operator

$$b_S^{\text{out}}(t) = \int_0^\infty \sqrt{\frac{\omega}{2\pi\Omega}}\, a_S^{\text{out}}(\omega)\, e^{-i\omega t}\, d\omega. \tag{51}$$

If we apply the inverse Fourier transform we obtain the input-output relation

$$a_S^{\text{out}}(\omega) = \sqrt{\gamma_S}\sqrt{\frac{\Omega}{2\pi\omega}} \int_{-\infty}^{\infty} [a(t) - a(t)^\dagger]\, e^{i\omega t}\, dt - a_S^{\text{in}}(\omega). \tag{52}$$

Up to now the Heisenberg eq. (48) describes the behaviour of the system without any approximations and for arbitrary input fields. And from eq. (52) we can compute the output field in terms of the system variables and the input field.

4. Rotating Wave Approximation and the Quantum Langevin Equations

Assuming the system to be weakly damped, that means $\gamma \ll \Omega$, we can perform some approximations usually made in quantum optics, i.e. the rotating wave approximation and the Markov approximation [9]. In the rotating wave approximation the operator $a(t)$ is decomposed into a slowly varying operator $\underset{\sim}{a}(t)$ and a phase factor with frequency Ω, that means

$$\underset{\sim}{a}(t) = a(t)\, e^{i\Omega t}. \tag{53}$$

Inserting eq. (53) into eq. (48) yields the equation of motion for $\underset{\sim}{a}(t)$. Assuming only small changes of $\underset{\sim}{a}(t)$ during one period of oscillation, allows to average the equation of motion over one period. Neglecting the fast varying terms we obtain

$$\dot{\underset{\sim}{a}} = -\frac{i}{\hbar}[\underset{\sim}{a}, \bar{H}_{\text{sys}}(\underset{\sim}{a}, \underset{\sim}{a}^\dagger)] - \frac{\gamma}{2}\underset{\sim}{a} + \sqrt{\gamma_R}\, \underset{\sim}{b}_R^{\text{in}}(t) + \sqrt{\gamma_S}\, \underset{\sim}{b}_S^{\text{in}}(t) \tag{54}$$

with

$$\bar{H}_{\text{sys}}(\underset{\sim}{a}, \underset{\sim}{a}^\dagger) = \frac{1}{T}\int_0^T \hbar\frac{\sigma}{\varkappa}[1 - \cos[\omega_0 t - \varphi_0 + + \varkappa(\underset{\sim}{a}\, e^{-i\Omega t} + \underset{\sim}{a}^\dagger e^{i\Omega t})]]\, dt \tag{55}$$

and

$$\underset{\sim}{b}_S^{\text{in}}(t) = b_S^{\text{in}}(t)\, e^{i\Omega t} = \int_{-\Omega}^{\infty} \sqrt{\frac{\omega + \Omega}{2\pi\Omega}}\, a_S^{\text{in}}(\omega + \Omega)\, e^{-i\omega t}\, d\omega, \tag{56}$$

and the analogous expression for $\underset{\sim}{b}_R^{\text{in}}(t)$. Since the system operators are slowly varying only that part of the integrand in a small surrounding at $\omega = 0$ influence the dynamics of a, as we will see later on. Therefore we may extend the range of integration to $\omega = -\infty$

$$\underset{\sim}{b}_R^{\text{in}}(t) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_R^{\text{in}}(\omega + \Omega)\, e^{-i\omega t}\, d\omega, \tag{57}$$

and we also may approximate the square root by its value at $\omega = 0$. So far we have applied the rotating wave approximation to the system variables and the input operators. We also apply the rotating wave approximation to the output relation (50) and obtain

$$\underset{\sim}{b}_S^{\text{out}}(t) = b_S^{\text{out}}(t)\, e^{+i\Omega t} \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_S^{\text{out}}(\omega + \Omega)\, e^{-i\omega t}\, d\omega. \tag{58}$$

and

$$\underset{\sim}{b}_S^{\text{out}}(t) = \sqrt{\gamma_S}\, \underset{\sim}{a}(t) - \underset{\sim}{b}_S^{\text{in}}(t). \tag{59}$$

Thus the equation of motion of the DCPJPA is now reduced to the form given in eq. (54). This is a quantum Langevin equation of exactly that type discussed extensively by Gardiner and Collet [6], and the relation between input output and system variables (59) is up to a minus sign the same as given there. These equations are the starting point for a complete analysis of the nonlinear behaviour of the DCPJPA, which follows the lines in [6], and will be presented in a forthcoming paper. Here we will perform a linearized analysis of the quantum Langevin equation (54) to show how squeezing occurs in the DCPJPA.

5. Linearized Analysis of the Device Below Threshold of Parametric Oscillation

To obtain a degenerate parametric amplifier we have to choose the bias voltage V_0 according to $\omega_0 = 2\, e_0 V_0/\hbar$, so that $\omega_0 = 2\,\Omega$, the double resonator frequency. For this choice of the pump voltage we obtain from eqs. (47) and (55) the following averaged system Hamiltonian

$$\bar{H}_{\text{sys}}(\underset{\sim}{a}, \underset{\sim}{a}^\dagger) = \hbar\frac{\sigma\varkappa}{4}[e^{i\varphi_0}\underset{\sim}{a}^2 + e^{-i\varphi_0}\underset{\sim}{a}^{\dagger 2}] + O(\varkappa^3). \tag{60}$$

If we assume the constant $\varkappa$ to be small, we can neglect those terms in eq. (60) which are of higher order in $\varkappa$. It is a well known fact that this Hamiltonian produces squeezed states [9]. With eq. (54) and the Hamiltonian (60) we obtain the linearized quantum Langevin equation

$$\dot{\underset{\sim}{a}} = -\frac{\gamma}{2}\underset{\sim}{a} + \mathrm{i}\,\varepsilon\,\underset{\sim}{a}^{\dagger} + \sqrt{\gamma_{\mathrm{R}}}\,\underset{\sim}{b}_{\mathrm{R}}^{\mathrm{in}}(t) + \sqrt{\gamma_{\mathrm{S}}}\,\underset{\sim}{b}_{\mathrm{S}}^{\mathrm{in}}(t), \quad (61\,\mathrm{a})$$

$$\dot{\underset{\sim}{a}}^{\dagger} = -\frac{\gamma}{2}\underset{\sim}{a}^{\dagger} - \mathrm{i}\,\varepsilon^{*}\,\underset{\sim}{a} + \sqrt{\gamma_{\mathrm{R}}}\,\underset{\sim}{b}_{\mathrm{R}}^{\mathrm{in}}(t)^{\dagger} + \sqrt{\gamma_{\mathrm{S}}}\,\underset{\sim}{b}_{\mathrm{S}}^{\mathrm{in}}(t)^{\dagger}, \quad (61\,\mathrm{b})$$

with

$$\varepsilon = \tfrac{1}{2}\,\sigma\,\varkappa\,\mathrm{e}^{-\mathrm{i}\varphi_0}. \quad (62)$$

These linear equations can be easily solved by means of the Fourier transform as it was done by Collet and Gardiner [11], and we obtain for the Fourier spectra of the creation and annihilation operators

$$\hat{\underset{\sim}{a}}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underset{\sim}{a}(t)\,\mathrm{e}^{+\mathrm{i}\omega t}\,\mathrm{d}\omega. \quad (63)$$

$$\hat{\underset{\sim}{a}}(\omega) = \frac{1}{\varDelta(\omega)}\left[\left(\frac{\gamma}{2} - \mathrm{i}\,\omega\right)\hat{\underset{\sim}{b}}^{\mathrm{in}}(\omega) - \mathrm{i}\,\varepsilon\,\hat{\underset{\sim}{b}}^{\mathrm{in}\dagger}(-\,\omega)\right], \quad (64\,\mathrm{a})$$

$$\hat{\underset{\sim}{a}}^{\dagger}(-\,\omega) = \frac{1}{\varDelta(\omega)}\left[\left(\frac{\gamma}{2} - \mathrm{i}\,\omega\right)\hat{\underset{\sim}{b}}^{\mathrm{in}\dagger}(-\,\omega) + \mathrm{i}\,\varepsilon^{*}\,\hat{\underset{\sim}{b}}^{\mathrm{in}}(\omega)\right], \quad (64\,\mathrm{b})$$

where

$$\varDelta(\omega) = \left(\frac{\gamma}{2} - \mathrm{i}\,\omega\right)^{2} - |\varepsilon|^{2}, \quad (65)$$

and

$$\hat{\underset{\sim}{b}}^{\mathrm{in}}(\omega) = \sqrt{\gamma_{\mathrm{R}}}\,a_{\mathrm{R}}^{\mathrm{in}}(\Omega + \omega) + \sqrt{\gamma_{\mathrm{S}}}\,a_{\mathrm{S}}^{\mathrm{in}}(\Omega + \omega). \quad (66)$$

We now assume that the transmission lines are in thermal equilibrium at temperatures T_{R} and T_{S} respectively. It is possible to compute the averages of the system operators via (64 a, b) in terms of the following averages of the input operators

$$\langle a_k^{\mathrm{in}}(\omega)\,a_k^{\mathrm{in}}(\omega')^{\dagger}\rangle = (n_k(\omega) + 1)\,\delta(\omega - \omega'), \quad (67\,\mathrm{a})$$

$$\langle a_k^{\mathrm{in}}(\omega)^{\dagger}\,a_k^{\mathrm{in}}(\omega')\rangle = n_k(\omega)\,\delta(\omega - \omega') \quad (67\,\mathrm{b})$$

with

$$n_k(\omega) = \frac{1}{\exp(\hbar\,\omega/k\,T_k) - 1}, \quad (68)$$

where $k = \mathrm{R}, \mathrm{S}$. To show the effect of squeezing, we express the field by the quadrature components

$$A_{\Theta} = \mathrm{e}^{\mathrm{i}\Theta}\,\underset{\sim}{a}(t) + \mathrm{e}^{-\mathrm{i}\Theta}\,\underset{\sim}{a}(t)^{\dagger}. \quad (69)$$

With eqs. (63) to (68) we obtain for the variance of this quadrature components

$$\langle \varDelta^2 A_{\Theta}\rangle = \quad (70)$$

$$= \frac{\gamma - 2\,|\varepsilon|\sin(\varphi_0 - 2\,\Theta)}{\gamma^2 - (2\,|\varepsilon|)^2}\,(\gamma + 2\,\gamma_{\mathrm{R}}\,n_{\mathrm{R}}(\Omega) + 2\,\gamma_{\mathrm{S}}\,n_{\mathrm{S}}(\Omega))$$

where we have exploited the fact that due to the resonance denominator $\varDelta(\omega)$ in eqs. (64 a, b), only the thermal noise around the center frequency Ω enters the result if $\gamma \ll \Omega$. This is the reason for allowing the rotating wave approximation as discussed above. From eq. (70) we can see that for $\varphi_0 - 2\,\Theta_{\min} = \pi/2 + 2\,\pi\,n$ the fluctuations in this quadrature component achieves the minimum value

$$\langle \varDelta^2 A_{\Theta_{\min}}\rangle = \frac{1}{1 + \sigma\,\varkappa/\gamma}\left(1 + 2\,\frac{\gamma_{\mathrm{R}}}{\gamma}\,n_{\mathrm{R}}(\Omega) + 2\,\frac{\gamma_{\mathrm{S}}}{\gamma}\,n_{\mathrm{S}}(\Omega)\right). \quad (71)$$

Thus if the temperatures of the transmission lines T_{R}, T_{S} approach zero, we obtain the maximum obtainable intracavity squeezing of a linear degenerate parametric amplifier

$$\langle \varDelta^2 A_{\Theta_{\min}}\rangle = 1/2, \quad (72)$$

if the pump parameter $p = \sigma\,\varkappa/\gamma$ of the device reaches the threshold for parametric oscillation $p = 1$. The fluctuations of the quadrature component $A_{\Theta_{\max}}$ with $\Theta_{\max} = \Theta_{\min} + \pi/2$ go to infinity at that point due the infinite gain of the parametric amplifier at threshold.

The field that can be observed in an experimental setup is the outward travelling field $\underset{\sim}{b}_{\mathrm{S}}^{\mathrm{out}}(t)$. And we will investigate the squeezing behaviour of this field by a homodyne detection experiment shown in Fig. 2. The analysis of this experimental set up is analogously to that given by Gea-Banacloche et al. [12] in the optical regime. From eqs. (3) and (34 b) the voltage of the outward traveling wave on the signal transmission line is given by

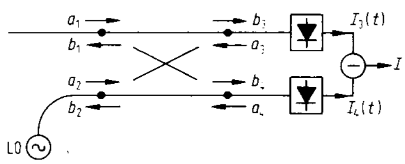

Fig. 2. Balanced homodyne detector.

$$u_{\mathrm{S}}^{\mathrm{out}}(t) = \frac{1}{C_{\mathrm{S}}}\,\Pi_{\mathrm{S}}^{\mathrm{out}}(0, t) \quad (73)$$

and from eq. (37 d) we obtain

$$u_{\mathrm{S}}^{\mathrm{out}}(t) = u_{\mathrm{S}}^{\mathrm{out}}(t)^{(+)} + u_{\mathrm{S}}^{\mathrm{out}}(t)^{(-)} \quad (74)$$

with

$$u_{\mathrm{S}}^{\mathrm{out}}(t)^{(+)} = -\,\mathrm{i}\int_{0}^{\infty}\sqrt{\frac{\hbar\,\omega\,R_{\mathrm{s}}}{4\,\pi}}\,a_{\mathrm{S}}^{\mathrm{out}}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega. \quad (75\,\mathrm{a})$$

and

$$u_{\mathrm{S}}^{\mathrm{out}}(t)^{(-)} = u_{\mathrm{S}}^{\mathrm{out}}(t)^{(+)\dagger}. \quad (75\,\mathrm{b})$$

From eqs. (51), (58) and (75) we obtain in rotating wave approximation

$$u_{\mathrm{S}}^{\mathrm{out}}(t)^{(+)} = -\,\mathrm{i}\sqrt{\frac{\hbar\Omega\,\mathrm{R}_{\mathrm{s}}}{2}}\,\underset{\sim}{b}_{\mathrm{S}}^{\mathrm{out}}(t)\,\mathrm{e}^{-\mathrm{i}\Omega t}. \quad (76)$$

Thus up to an arbitrary phase factor due to the propagation of the signal over the transmission line the signal a_1 entering the input port 1 of the 3 dB coupler in Fig. 2 is given by

$$a_1(t) = u_S^{\text{out}}(t)^{(+)}. \tag{77}$$

Due to the 3 dB coupler the signal at the output ports are related to the input signals according to

$$b_3(t) = [a_1(t) + \mathrm{i}\, a_2(t)]/\sqrt{2}, \tag{78a}$$

$$b_4(t) = [\mathrm{i}\, a_1(t) + a_2(t)]/\sqrt{2}. \tag{78b}$$

If we assume that the detectors shown in Fig. 2 have 100% quantum efficiency the operator corresponding to the photocurrent $I_i(t)$ of detector i is given by

$$I_i(t) = \alpha\, b_i(t)^\dagger b_i(t) \quad \text{for} \quad i = 3, 4, \tag{79}$$

where α is the detector sensitifity. Thus we obtain for the difference current $I(t)$ at the output of the homodyne detector

$$I(t) = I_3(t) - I_4(t) = \mathrm{i}\,\alpha\,[a_1(t)^\dagger a_2(t) - a_2(t)^\dagger a_1(t)], \tag{80}$$

where we have taken into account eqs. (78 a, b). If we now assume that the input signal $a_2(t)$ is a wave at frequency Ω excited by the local oscillator and if it is in the coherent state $|\beta\rangle$ we can write

$$a_2(t) = b_2\, \mathrm{e}^{-\mathrm{i}(\Omega t + \Theta)} \quad \text{with} \quad [b_2, b_2^\dagger] = 1, \tag{81}$$

and

$$b_2|\beta\rangle = \beta|\beta\rangle \quad \text{with} \quad \beta\,\text{real}. \tag{82}$$

Therefore we obtain for the normalized autocorrelation function of the current

$$\begin{aligned} c(t) &= \frac{1}{\beta^2}\langle I(t)\, I(0)\rangle = \\ &= -\alpha^2 \langle [a_1(t)^\dagger \mathrm{e}^{-\mathrm{i}(\Omega t + \Theta)} - a_1(t)\,\mathrm{e}^{\mathrm{i}(\Omega t + \Theta)}] \cdot \\ &\quad \cdot [a_1(0)^\dagger \mathrm{e}^{-\mathrm{i}\Theta} - a_1(0)\,\mathrm{e}^{\mathrm{i}\Theta}]\rangle + \\ &\quad + 4\frac{\alpha^2}{\beta^2}\langle a_1(t)^\dagger \mathrm{e}^{-\mathrm{i}(\Omega t + \Theta)} a_1(0)\rangle. \end{aligned} \tag{83}$$

Thus in the limit $\beta \to \infty$ we obtain with eqs. (76) and (77)

$$c(t) = 2\,\alpha^2 \hbar\,\Omega\, R_S \langle A_\Theta^{\text{out}}(t)\, A_\Theta^{\text{out}}(0)\rangle \tag{84}$$

with the quadrature component of the output field

$$A_\Theta^{\text{out}}(t) = \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}\Theta} b_S^{\text{out}}(t) + \mathrm{e}^{-\mathrm{i}\Theta} b_S^{\text{out}}(t)^\dagger], \tag{85}$$

where the phase Θ now plays the role of the phase of the local oscillator signal used for homodyning. The autocorrelation spectrum $C(\omega)$ is the Fourier transform of $c(t)$.

Introducing the normalized squeezing spectrum $S_\Theta^{\text{out}}(\omega)$ defined by

$$S_\Theta^{\text{out}}(\omega) = 2\int_0^\infty \langle A_\Theta^{\text{out}}(t)\, A_\Theta^{\text{out}}(0)\rangle \cos(\omega t)\, \mathrm{d}t \tag{86a}$$

the autocorrelation spectrum of the detector current $C(\omega)$ may be written as

$$C(\omega) = 2\,\alpha^2 \hbar\,\Omega\, R_S\, S_\Theta^{\text{out}}(\omega). \tag{86b}$$

The normalized squeezing spectrum has the dimension 1. With the Fourier transform of the output field eq. (85) we obtain

$$S_\Theta^{\text{out}}(\omega) = \int_{-\infty}^{\infty} \langle \hat{A}_\Theta^{\text{out}}(\omega)\, \hat{A}_\Theta^{\text{out}}(\omega')\rangle\, \mathrm{d}\omega'. \tag{87}$$

And with eqs. (63) to (68) the squeezing spectrum is given by

$$\begin{aligned} S_\Theta^{\text{out}}(\omega) &= \\ &= \frac{1}{4}\frac{1}{|\Delta(\omega)|^2}\{[[\tfrac{1}{4}(\gamma_S^2 - \gamma_R^2) + |\varepsilon|^2 + \omega^2]^2 + \omega^2\gamma_R^2 + |\varepsilon|^2\gamma_S^2 \\ &\quad + 2|\varepsilon|\,\gamma_S[\tfrac{1}{4}(\gamma_S^2 - \gamma_R^2) + |\varepsilon|^2 + \omega^2]\sin(2\Theta - \varphi_0)](2n_S + 1) \\ &\quad + \gamma_R\gamma_S[\tfrac{1}{4}\gamma^2 + |\varepsilon|^2 + \omega^2 + \gamma|\varepsilon|\sin(2\Theta - \varphi_0)](2n_R + 1)\}. \end{aligned} \tag{88}$$

The spectrum for $\Theta = \Theta_{\min} = \varphi_0/2 + 3\pi/4 + n\pi$ receives the minimum value

$$\begin{aligned} S_{\Theta_{\min}}^{\text{out}}(\omega) &= \frac{1}{4}\frac{1}{|(\gamma/2 - \mathrm{i}\,\omega)^2 - |\varepsilon|^2|^2} \cdot \\ &\quad \cdot \left\{\left[\left[\left(\frac{\gamma}{2} - |\varepsilon|\right)^2 + \omega^2\right][(\tfrac{1}{2}(\gamma_S - \gamma_R) - |\varepsilon|)^2 + \omega^2] + \right.\right. \\ &\quad \left. + \omega^2\tfrac{1}{4}(\gamma_S^2 - {\gamma_R}^2)\right]\coth(\hbar\,\Omega/2\,k\,T_S) + \\ &\quad \left. + \gamma_R\gamma_S\left[\left(\frac{\gamma}{2} - |\varepsilon|\right)^2 + \omega^2\right]\coth(\hbar\,\Omega/2\,k\,T_R)\right\}. \end{aligned} \tag{89}$$

In the case $\omega = 0$ we obtain

$$\begin{aligned} S_{\Theta_{\min}}^{\text{out}}(0) &= \\ &= \frac{1}{4}\frac{1}{(\gamma/2 + |\varepsilon|)^2}\{[\tfrac{1}{2}(\gamma_S - \gamma_R) - |\varepsilon|]^2\coth(\hbar\,\Omega/2\,k\,T_S) + \\ &\qquad + \gamma_R\gamma_S\coth(\hbar\,\Omega/2\,k\,T_R)\}. \end{aligned} \tag{90}$$

Setting the temperatures T_R and T_S of the transmission lines equal to T, we obtain for the minimum value at the threshold for parametric oscillations $\gamma = 2|\varepsilon| = p = 1$

$$\begin{aligned} S_{\Theta_{\min}}^{\text{out}}(0) &= \frac{1}{4}\frac{\gamma_R}{\gamma}\coth(\hbar\,\Omega/2\,k\,T) = \\ &= \frac{1}{4}\frac{R_S}{R + R_S}\coth(\hbar\,\Omega/2\,k\,T). \end{aligned} \tag{91}$$

The noise reduction r with respect to the vacuum noise floor which would produce a normalized spectral density equal to 1/4, is given by four times the spectral density in eq. (91) and at $T = 0$ we obtain

$$r = \frac{R_S}{R + R_S} \simeq \frac{R_S}{R} \quad \text{for} \quad R_S \ll R. \tag{92}$$

Thus our result for the DCPJPA agrees with the result for the external pumped JPA analysed by Yurke [2]. If the losses due to the shunt resistance of the Josephson junction are small in comparison with the losses due to the coupling of the signal transmission line, that means $R \gg R_S$, eq. (91) suggests that we can achieve arbitrarily large noise reduction. But at the threshold for oscillation the linear analysis leading to eq. (91) is no longer valid, and thus the maximum value for squeezing that can be achieved with this device requires a nonlinear analysis starting from the quantum Langevin equations derived in Section 4.

As we have seen from the linearized analysis above, we obtain high squeezing at the threshold for oscillation, if $R_S = rR$ with $r \ll 1$. This threshold is reached if the pump parameter defined by

$$p = \sigma \varkappa / \gamma \tag{93}$$

approaches $p = 1$. From eq. (46) we obtain

$$p = \frac{I_c e_0 R R_S}{\hbar \Omega (R + R_S)} = \frac{1}{2} \frac{I_c R}{\Phi_0 f} \frac{r}{1+r} \simeq \frac{r}{2} \frac{I_c R}{\Phi_0 f} \quad \text{for} \quad r \ll 1 \tag{94}$$

with the magnetic flux quantum $\Phi_0 = 2.07 \cdot 10^{-15}$ Vs. The product of the critical current and the shunt resistance is a constant for a given Josephson junction. For a Josephson junction built with usual low temperature superconductors the product $I_c R$ is approximately 1 mV. Since this critical voltage is proportional to the energy gap of the superconductor it must be higher for high-T_c superconductors. For $p < 1$ the parametric amplifier is stable. The threshold condition for parametric oscillation is $p = 1$ and with this value for p we obtain from eq. (94) a relationship between the frequency where squeezing can be observed and the noise reduction rate r

$$f = 241.8\, r I_c R \quad \text{GHz/mV}. \tag{95}$$

Thus for low temperature superconductors the frequency domain where squeezing is possible is therefore restricted to $f < 241.8$ GHz. Since we have neglected higher order terms in $\varkappa$ in the Hamiltonian (60), the results obtained for the linearized system with respect to squeezing below threshold will approach the results of the nonlinear system sufficiently well if we make the parameter $\varkappa$ small. If we introduce the loaded quality factor Q_L of the resonator, given by

$$Q_L = \Omega C R_S, \tag{96}$$

we obtain for the parameter $\varkappa$ according to eq. (46).

$$\varkappa = \sqrt{R_S / (R_\varkappa Q_L)} \tag{97}$$

with

$$R_\varkappa = \hbar / (2 e_0^2) = 2.054\, \text{k}\Omega. \tag{98}$$

Up to now we have not considered any internal losses of the resonator. The internal resonator losses may be taken into account by considering R to represent the paralleled Josephson shunt resistor and the equivalent loss resistor of the LC circuit. Thus the internal losses have the same degrading effects onto the squeezing as the shunt resistance R of the Josephson junction. Therefore we have to guarantee that the internal losses of the resonator are much smaller than those due to the coupling of the signal transmission line, that means that the loaded quality factor of the resonator is much smaller than the unloaded. R_S must be chosen much smaller than the shunt resistance R of the Josephson junction to obtain noise reduction as discussed above. Since the shunt resistance R can be as high as 1 kΩ, the parameter $\varkappa$ can be made much smaller then one.

6. Discussion and Conclusions

We have shown that squeezed states may be generated with the dc pumped degenerate Josephson junction parametric amplifier (DCPJPA). The DCPJPA seems to be an interesting tool for the experimental observation of squeezed states. Compared with optical experiments the values of the device parameters for investigating the performance at threshold for oscillation where maximum squeezing can be obtained are easily achievable. As will be shown in a forthcoming paper above threshold the DCPJPA allows the generation of squeezed states with a nonvanishing amplitude. Such a device can be used as a local oscillator with reduced shot noise.

Acknowledgement

The authors wishes to thank the Deutsche Forschungsgemeinschaft for their support.

(Received January 25, 1990.)

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