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Collective mode damping and viscosity in a 1D unitary Fermi gas

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Abstract. We calculate the damping of the Bogoliubov–Anderson mode in a one-dimensional (1D) two-component attractive Fermi gas for arbitrary coupling strength within a quantum hydrodynamic approach. Using the Bethe-ansatz solution of the 1D BCS-BEC crossover problem, we derive analytic results for the viscosity covering the full range from a Luther–Emery liquid of weakly bound pairs to a Lieb–Liniger gas of strongly bound bosonic dimers. At the unitarity point, the system is a Tonks–Girardeau gas with a universal constant $\alpha_\zeta = 0.38$ in the viscosity $\zeta = \alpha_\zeta \hbar n$ for $T = 0$. For the trapped case, we calculate the Q-factor of the breathing mode and show that the damping provides a sensitive measure of temperature in 1D Fermi gases.

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Contents

1. Introduction

In the past few years, ultracold gases have entered a new regime, where strong correlation effects appear even in extremely dilute gases. Prominent examples for this new area in atomic physics are the study of the crossover from a BCS-type superfluid of extended Cooper pairs to a BEC of strongly bound molecules [1]–[3] or the realization of a Tonks–Girardeau gas of hard-core bosons in one-dimensional (1D) atomic wires [4, 5]. In the first case, the strong interaction regime is reached in a direct manner because the scattering length a near a Feshbach resonance becomes of the same order or even larger than the average interparticle spacing k_{F}^{-1}. In the second case, it is the squeezing of the kinetic energy in an optical lattice which enhances the role of interactions [6]. A unique role in the context of strongly interacting ultracold gases is played by the so-called unitary Fermi gas, where the dimensionless interaction strength parameter $k_{\mathrm{F}}a$ is infinite. This problem was originally discussed in nuclear physics [7, 8]. In its simplest form, it consists of a two component Fermi gas with a zero range attractive interaction which is just about to bind a state at the two-particle level. Such a situation is realizable with cold gases at a Feshbach resonance, where the scattering length diverges [9]. Precisely at this point and for broad Feshbach resonances, where the range of the effective interaction is much smaller than the mean interparticle spacing, the full many-body problem has the bare Fermi-energy ϵ_{F} as the only energy scale. As a result, the complete thermodynamics is a *universal* function of the ratio $k_{\mathrm{B}}T/\epsilon_{\mathrm{F}}$ [10]. While a quantitatively reliable description of the many-body problem near a Feshbach resonance at finite temperature is still an open problem [11, 12], the situation near zero temperature may be understood in a straightforward manner. Indeed, at low temperatures, a two-component Fermi gas will be in a superfluid state, independent of the strength of the attractive interaction. On quite general grounds therefore, the low lying excitations above the ground state are sound modes of the Bogoliubov–Anderson type, which are the Goldstone modes of the broken gauge symmetry in a neutral superfluid. In this regime, an effective low energy description is possible in terms of a quantum hydrodynamic (QHD) approach [13]. For a Fermi gas with a short range attractive interaction, the associated effective field theory was recently discussed by Son and Wingate [14]. Starting from a Lagrangian formulation of the many-body problem, they realized that in the particular case of a unitary Fermi gas, there is an additional conformal symmetry with respect to arbitrary reparameterizations of the time. Remarkably, the effective field theory can be extended to non-equilibrium problems within the framework of linear, irreversible thermodynamics. In particular, conformal invariance of the unitary Fermi gas applied to the dissipative part of the stress tensor requires that two of the bulk viscosity coefficients vanish [15]. As a result, no entropy is produced in a uniform expansion. The extension of the effective field theory to irreversible processes makes evident that not only the thermodynamics but also dynamical properties like the kinetic coefficients are universal at the unitarity point, a fact, first emphasized by Gelman *et al* [16]. An example of particular interest is the shear viscosity η which determines the damping of sound and collective oscillations in trapped gases [17, 18]. At unitarity, its dependence on density n and temperature T is fixed by dimensional arguments to be $\eta = \hbar n \alpha(T/\mu)$, where μ is the chemical potential and $\alpha(x)$ a dimensionless universal function [15]. At zero temperature, in particular, $\eta(T=0) = \alpha_\eta \hbar n$ is linear in the density with a universal coefficient α_η. Using a simple fluctuation–dissipation type argument in the *normal* phase, a lower bound of the form $\alpha_\eta \geqslant 1/6\pi$ has been derived by Gelman *et al* [16], in analogy to rigorous bounds for the ratio $\eta/\hbar s \geqslant 1/4\pi$ between the viscosity η and the entropy density s in supersymmetric pure gauge Yang–Mills theories [19, 20]. Based on these results, it has been speculated that ultracold atoms near a Feshbach resonance are a nearly perfect liquid [16].

In the present study, we calculate the viscosity ζ of a strongly interacting Fermi gas in the whole regime of coupling strengths for the particular case of one dimension, where an exact solution of the BCS-BEC crossover problem has recently been given using the Bethe ansatz [21]–[23]. It is shown that, at $T = 0$, the viscosity has the form $\zeta = \alpha_\zeta \hbar n$ with a coupling-dependent constant α_ζ, which takes the universal value $\alpha_\zeta = 0.38$ at the unitarity point. At finite temperature, the sound damping does not have a hydrodynamic form and increases like $\sqrt{T}$. We determine the resulting damping of the breathing mode in a trapped gas and show that its Q-factor provides a sensitive measure of temperature in strongly interacting 1D gases.

2. Sound damping and viscosity of a 1D superfluid

2.1. BCS-BEC crossover in 1D

Our calculations are based on an exactly solvable model of the BCS-BEC crossover in one dimension proposed by Fuchs *et al* [22, 23] and by Tokatly [21]. The underlying microscopic Hamiltonian is that of the Gaudin–Yang model, see [24]

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_1\sum_{i<j}\delta(x_i - x_j) \qquad (1)$$

of a spin-1/2 Fermi gas interacting via a short range potential $g_1\delta(x)$. Here, N is the total number of Fermions and m their mass. At zero temperature, the model is characterized by a single dimensionless coupling constant $\gamma \equiv mg_1/\hbar^2 n$, where $n \equiv N/L$ is the 1D density. For attractive interactions, the Hamiltonian (1) describes a so-called Luther–Emery liquid. Its ground state at $\gamma \to 0^-$ is a BCS-like state with Cooper pairs, whose size is much larger than the average inter-particle spacing. With increasing magnitude of γ, one reaches the strong coupling regime of tightly bound molecules which behave like a hard core Bose gas as $\gamma \to -\infty$. As shown by Girardeau [25], the hard core Bose gas in one dimension is equivalent—for densities diagonal in real space—to a gas of *non*-interacting fermions. Within a strictly 1D model, the BEC-limit of strongly bound pairs is thus a Tonks–Girardeau gas. Now in practice, the atoms are trapped in a harmonic waveguide with radial frequency $\omega_\perp/2\pi$. The associated transverse oscillator length

$a_\perp \equiv \sqrt{\hbar/m\omega_\perp}$ then defines an additional length, not present in the Gaudin–Yang model (1). As shown by Bergeman *et al* [26], the exact solution of the scattering problem for two particles in such a waveguide, interacting with a 3D pseudo-potential with scattering length a, always exhibits a two-body bound state, *whatever* the sign and magnitude of the scattering length a. It appears at an energy $\hbar\omega_\perp - \tilde{\epsilon}_b$, which is below the continuum threshold at $\hbar\omega_\perp$ of the transverse confining potential. Apart from this bound state, all the scattering properties can be described by an effective 1D delta potential $g_1^{aa}\delta(x)$ for atom–atom interactions with strength [27]

$$g_1^{aa} = 2\hbar\omega_\perp a\,(1 - Aa/a_\perp)^{-1}\,. \tag{2}$$

As naively expected, an attractive 3D interaction $a < 0$ implies a negative value of g_1^{aa}. The associated binding energy $\epsilon_b = m(g_1^{aa})^2/4\hbar^2$ in the 1D delta potential coincides with the exact value $\tilde{\epsilon}_b$ in the weak confinement limit $|a| \ll a_\perp$. Remarkably, the strength g_1^{aa} of the 1D pseudo-potential remains finite at a Feshbach resonance where $a = \pm\infty$. The corresponding exact value of the binding energy is $\tilde{\epsilon}_b \simeq 0.6\,\hbar\omega_\perp$ [21, 26]. Entering the positive side $a > 0$ of the Feshbach resonance, the vanishing of the denominator in (2) at $a_\perp/a = A \simeq 1.0326$[3] leads to a confinement induced resonance (CIR), where g_1^{aa} jumps from $-\infty$ to $+\infty$. The exact bound state at this point has binding energy $\tilde{\epsilon}_b = 2\hbar\omega_\perp$ and a spatial extension along the x-axis, which is of the order of the transverse oscillator length $a_\perp$. With decreasing values $a \lesssim a_\perp$ of the 3D scattering length, $\tilde{\epsilon}_b$ increases monotonically beyond $2\hbar\omega_\perp$ and finally approaches the standard 3D result $\tilde{\epsilon}_b \to \hbar^2/ma^2$ in the weak confinement limit $a \ll a_\perp$ [21, 26]. Since $\hbar\omega_\perp \gg \varepsilon_F$ in the limit of a singly occupied transverse channel, the true bound state energy $\tilde{\epsilon}_b$ is the largest energy scale in the problem in the regime after the CIR where $g_1^{aa} > 0$. In this regime, the appropriate degrees of freedom are no longer the single atoms but instead are strongly bound fermion pairs, which are essentially unbreakable. An exact solution of the four-body problem in a quasi 1D geometry with tight harmonic confinement shows, that these dimers have a *repulsive* interaction in the regime beyond the CIR [28]. The related constant $g_1^{dd} > 0$ in the effective dimer–dimer interaction $g_1^{dd}\delta(x)$ can be calculated as a function of the 3D scattering length [28]. It approaches $g_1^{dd} \to 2\hbar\omega_\perp \times 0.6\,a \to 0$ in the weak confinement limit, where the dimer–dimer scattering length $a^{dd} \approx 0.6\,a$ is identical with the one in free space [29]. Sufficiently far from the CIR, one thus recovers a weakly interacting gas of dimers.

At the many-body level, the situation after the CIR is described by a Lieb–Liniger model [30] of repulsive bosons. Its dimensionless coupling constant $\gamma \equiv mg_1^{dd}/\hbar^2 n$ is now positive and vanishes in the weak confinement limit. It diverges at a value of the 3D scattering length a of order $a_\perp$. Now although the divergence of g_1^{dd} does not exactly coincide with that of g_1^{aa} at the CIR [28], the range of inverse dimensionless coupling constants where this mismatch appears is of order $1/\gamma \approx na_\perp$ [23]. It is thus negligible in the relevant low density limit $(na_\perp)^2 \ll 1$. Indeed, at a fixed density n, the quasi-1D condition $\hbar\omega_\perp \gg \epsilon_F$ that only the lowest transverse mode is occupied is equivalent to $(na_\perp)^2 \ll 1$. In the limit $na_\perp \to 0$, there is thus a continuous evolution from the Gaudin–Yang model of attractive fermions to the Lieb–Liniger model of repulsive bosons which completely describes the BCS-BEC crossover in one dimension [21, 22]. The associated spectrum of elementary excitations is straightforward to understand: in the BCS limit $1/\gamma \to -\infty$, the system consists of weakly bound Cooper pairs. Their breaking is associated with a finite excitation gap and the corresponding spectrum exhibits a relativistic dispersion

[3] Note the $\sqrt{2}$ difference in our definition of $a_\perp$ compared with [26, 27], which accounts for the difference in the value of $A \equiv 1.0326$.

relation $\varepsilon_s(k) = \sqrt{(\Delta/2\hbar)^2 + (v_s k)^2}$ similar to the standard quasiparticle spectrum of the BCS theory. The associated energy gap Δ and the spin velocity $v_s > v_F$ increase monotonically with $1/\gamma$, both diverging in the strong coupling limit $1/\gamma = 0$ at the CIR [22]. In addition, there are gapless density fluctuations describing the Bogoliubov–Anderson mode of a neutral superfluid. These excitations exist for arbitrary coupling, both before and after the CIR. Their spectrum is $\varepsilon(k) = v_c k$ at low momenta, with a (zero) sound velocity v_c,[4] which monotonically decreases from the ideal Fermi gas value $v_c = v_F$ at $\gamma \to 0^-$ to the weak coupling BEC result $v_c = \sqrt{\gamma} v_F/\pi$ as $\gamma \to 0^+$ [22]. At the CIR, $1/\gamma = 0$, the system is a Tonks–Girardeau gas [25] of tightly bound dimers. The value v_c $(1/\gamma = 0) = v_F/2$ simply reflects the fact that the unitary Fermi gas in 1D is a hard core Bose gas which—in turn—behaves like an ideal gas of Fermions at half the original density. The universal parameter β, which follows from $v_c = v_F\sqrt{1+\beta}$ [10], thus has the exact value $\beta = -3/4$ in one dimension.

2.2. *Quantum hydrodynamic theory (QHD)*

In order to calculate the damping of long-wavelength phonons, we use a 1D version of QHD. Restricting the attention to the gapless Bogoliubov–Anderson mode in the superfluid regime, the QHD Hamiltonian has the same form on both the fermionic (before the CIR) and the bosonic (after the CIR) side of the 1D BCS-BEC crossover. In a harmonic approximation, which is valid at low energies, the sound mode is described by the quadratic Hamiltonian

$$H_0 = \frac{v_c}{2} \int_0^L \mathrm{d}x \left\{ \frac{\rho_0}{v_c} (\partial_x \varphi)^2 + \frac{v_c}{\rho_0} \Pi^2 \right\}, \tag{3}$$

where the conjugate fields $\varphi(x)$ and $\Pi(x)$ describe phase and density fluctuations respectively. The only input parameters are the equilibrium mass-density $\rho_0 = mn$ and the sound velocity v_c. From the Bethe ansatz, the velocity v_c is known as a function of the dimensionless inverse coupling constant $1/\gamma$, which ranges between $1/\gamma = -\infty$ in the BCS- to $1/\gamma = +\infty$ in the BEC-limit [22]. To determine the damping due to the interaction of phonons, the energy functional of a 1D quantum liquid needs to be expanded beyond quadratic order in the fields $\varphi(x)$ and $\Pi(x)$. It is a crucial advantage of the QHD approach, that the coefficients of the leading nonlinear terms are completely determined by thermodynamic quantities [13]. Specifically, the lowest (third) order terms give rise to a contribution H_{int} to the total Hamiltonian of the form

$$H_{\mathrm{int}} = \frac{1}{6} \int_0^L \mathrm{d}x \left\{ (\partial_x \varphi) \Pi (\partial_x \varphi) + (\partial_x \varphi)^2 \Pi + \Pi (\partial_x \varphi)^2 + \frac{\mathrm{d}}{\mathrm{d}\rho_0} \left(\frac{v_c^2}{\rho_0} \right) \Pi^3 \right\}. \tag{4}$$

The quadratic Hamiltonian (3) is diagonalized by the standard mode expansions

$$\varphi(x) = \mathrm{i}\frac{\hbar}{m}\sqrt{\frac{\pi}{2}}\sqrt{\frac{v_c}{2L\, v_F}} \sum_{q\neq 0} \frac{1}{\sqrt{|q|}} (b_q\, \mathrm{e}^{\mathrm{i}qx} - b_q^\dagger\, \mathrm{e}^{-\mathrm{i}qx}),$$

$$\Pi(x) = m\sqrt{\frac{2}{\pi}}\sqrt{\frac{v_F}{2L\, v_c}} \sum_{q\neq 0} \sqrt{|q|} (b_q\, \mathrm{e}^{\mathrm{i}qx} + b_q^\dagger\, \mathrm{e}^{-\mathrm{i}qx}),$$

[4] The notation v_c for the velocity of the Bogoliubov–Anderson mode reflects the usual notation in the Luttinger liquid context as a 'charge' velocity, yet in the present case of neutral atoms, it is a zero sound type density oscillation in a superfluid system.

where $b_q^\dagger$ and b_q denote the usual bosonic creation and annihilation operators respectively and $v_{\mathrm{F}} = \pi\hbar n/2m$ is the Fermi velocity of the non interacting gas. After inserting the mode expansions in (4), we obtain

$$H_{\mathrm{int}} = \sum_{q_1,q_2,q_3} \frac{1}{\sqrt{L}} V(q_1,q_2,q_3)\,\{b_{q_1}b_{q_2}b_{q_3}\delta(q_1+q_2+q_3) + b_{q_1}b_{q_2}b^\dagger_{q_3}\delta(q_1+q_2-q_3)$$
$$+b_{q_1}b^\dagger_{q_2}b_{q_3}\delta(q_1-q_2+q_3) + b_{q_1}b^\dagger_{q_2}b^\dagger_{q_3}\delta(q_1-q_2-q_3) + \mathrm{h.c.}\},$$

with the vertex

$$V(q_1,q_2,q_3) = \frac{\hbar^2}{6m}\sqrt{\frac{\pi v_{\mathrm{c}}|q_1q_2q_3|}{16 v_{\mathrm{F}}}}\left\{\mathrm{sign}(q_1q_3) + \mathrm{sign}(q_1q_2) + \mathrm{sign}(q_2q_3) + \frac{v_{\mathrm{F}}^2}{v_{\mathrm{c}}^2}\frac{\mathrm{d}}{\mathrm{d}v_{\mathrm{F}}}\left(\frac{v_{\mathrm{c}}^2}{v_{\mathrm{F}}}\right)\right\}.$$

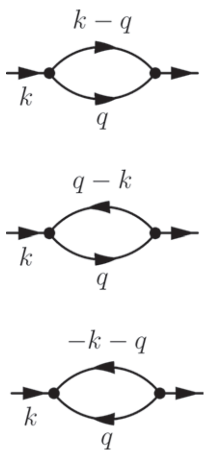
The momentum dependence of the vertex makes perturbation theory applicable for long wavelength phonons. More precisely, it applies as long as the imaginary part of their energy is much smaller than the real part. As will be shown below, this requires the phonon wavelength λ to be much larger than the interparticle spacing on the Fermi side of the crossover, while in the BEC limit the more restrictive condition $\lambda n \gg (1/\gamma)^{1/4} \gg 1$ is required. It should be remarked that an approach to evaluate the nonlinear terms in the Hamiltonian via bosonization leads to additional coupling terms between phonons and spin excitations on the BCS side of the crossover which are not accounted for in the QHD approach. As will be shown in the appendix, however, these terms do not contribute to the phonon damping at long wavelengths.

We evaluate the damping by calculating the imaginary part of the phonon self-energy $\Sigma(k,\omega)$ which is defined as the analytic continuation of the corresponding self-energy Σ^{th} in the exact thermodynamic Green function

$$\mathcal{G}(k, \mathrm{i}\omega_n) = \frac{1}{\mathrm{i}\omega_n - v_{\mathrm{c}}|k| - \Sigma^{\mathrm{th}}(k,\mathrm{i}\omega_n)}.$$

Here, $\omega_n = 2\pi n/\beta$ with $n \in \mathbb{Z}$ are the standard Bosonic Matsubara frequencies. (We set $\hbar = k_{\mathrm{B}} = 1$ from now on, except in final results.)

The main contribution to the damping rate comes from the three self-energy diagrams shown below, corresponding to spontaneous decay, absorption of a phonon and three-wave annihilation:



Diagrams of this type have been considered before by Andreev [31] who studied the sound absorption in 1D Bose liquids for $T > 0$ and by Samokhin [32] in the context of the damping of zero sound in a 1D liquid of repulsive fermions.

After taking the limit $L \to \infty$ and analytic continuation, the retarded self energy $\Sigma^{\mathrm{R}}(k,\omega)$ is given by the sum of the three diagrams

$$\Sigma^{\mathrm{R}}(k,\omega) = \Sigma_1^{\mathrm{R}}(k,\omega) + 2\Sigma_2^{\mathrm{R}}(k,\omega) + \Sigma_3^{\mathrm{R}}(k,\omega), \tag{5}$$

with

$$\Sigma_1^{\mathrm{R}}(k,\omega) = -18\int \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\Omega}{2\pi}\coth\left(\frac{\beta\Omega}{2}\right) V^2(k,q,k-q)$$
$$\times\{-G^{\mathrm{R}}(q,\Omega+\omega)\,\mathrm{Im}\,G^{\mathrm{R}}(k-q,-\Omega) + G^{\mathrm{R}}(k-q,\omega-\Omega)\,\mathrm{Im}\,G^{\mathrm{R}}(q,\Omega)\},$$

$$\Sigma_2^{\mathrm{R}}(k,\omega) = -18\int \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\Omega}{2\pi}\coth\left(\frac{\beta\Omega}{2}\right) V^2(k,q,q-k)$$
$$\times\{G^{\mathrm{R}}(q,\Omega+\omega)\,\mathrm{Im}\,G^{\mathrm{R}}(q-k,\Omega) + G^{\mathrm{A}}(k-q,\Omega-\omega)\,\mathrm{Im}\,G^{\mathrm{R}}(q,\Omega)\},$$

$$\Sigma_3^{\mathrm{R}}(k,\omega) = -18\int \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\Omega}{2\pi}\coth\left(\frac{\beta\Omega}{2}\right) V^2(k,q,-k-q)$$
$$\times\{-G^{\mathrm{A}}(q,\Omega-\omega)\mathrm{Im}G^{\mathrm{R}}(-k-q,-\Omega) + G^{\mathrm{A}}(-k-q,-\omega-\Omega)\,\mathrm{Im}G^{\mathrm{R}}(q,\Omega)\}.$$

Here $G^{\mathrm{R}}$and $G^{\mathrm{A}}$are the usual retarded and advanced Green functions respectively. The combinatorial factor 18 arises from the three possible ways of choosing the creation–annihilation operators for the initial and final phonon and two possibilities of pairing the phonons in between. As was already pointed out by Andreev and Samokhin, the fact that for linearly dispersing phonons in 1D, momentum and energy conservation are simultaneously satisfied, requires to go beyond second order perturbation theory which would give an infinite damping rate. Following the approach of Andreev [31], we calculate the self-energy by using the fact that $\Sigma^{\mathrm{R}}(k,\omega) \ll \varepsilon_k = v_{\mathrm{c}}|k|$ at long wavelengths. The precise condition on k for which this holds, has to be determined afterwards and will be discussed below. Since we are interested in the quasiparticle pole of $G^{\mathrm{R}}(k,\omega) = 1/(\omega - \varepsilon_k - \Sigma^{\mathrm{R}}(k,\omega))$we can use the approximation

$$\omega = \varepsilon_k + \Sigma^{\mathrm{R}}(k,\omega) \approx \varepsilon_k + \Sigma^{\mathrm{R}}(k,\varepsilon_k) = \varepsilon_k + \Sigma_k^{\mathrm{R}}$$

leading to

$$G^{\mathrm{R}}(k,\omega) \approx \frac{1}{\omega - \varepsilon_k - \Sigma_k^{\mathrm{R}}} \tag{6}$$

The damping rate of phonons with wavevector $k > 0$ can now be determined from the imaginary part $\Gamma_k = \mathrm{Im}\,\Sigma^{\mathrm{R}}(k,\varepsilon_k) \equiv \mathrm{Im}\,\Sigma_k^{\mathrm{R}}$ of the (on-shell) self-energy.

Starting with the case of zero temperature $T = 0$, the only contribution to Σ_k^R comes from spontaneus decay (first diagram). Applying the approximation (6) in the integrand and doing the Ω- integration, one ends up with

$$\Sigma_k^{\mathrm{R}} = 9\int \frac{\mathrm{d}q}{2\pi} V^2(k,q,k-q)\ \left\{G^{\mathrm{R}}\left(q,\varepsilon_k - \varepsilon_{k-q} - \Sigma_{k-q}^{\mathrm{R}}\right) + G^{\mathrm{R}}\left(k-q,\varepsilon_k - \varepsilon_q - \Sigma_q^{\mathrm{R}}\right)\right\}.$$

The major contribution to the integral comes from $0 < q < k$ where $\varepsilon_k - \varepsilon_{k-q} - \varepsilon_q = 0$. Thus we arrive at the equation

$$\Sigma_k^{\mathrm{R}} = -\frac{\hbar^4 \pi v_{\mathrm{c}}}{32\, m^2 v_{\mathrm{F}}} \left\{3 + \frac{v_{\mathrm{F}}^2}{v_{\mathrm{c}}^2} \frac{\mathrm{d}}{\mathrm{d}v_{\mathrm{F}}} \left(\frac{v_{\mathrm{c}}^2}{v_{\mathrm{F}}}\right)\right\}^2 \int_0^k \frac{\mathrm{d}q}{2\pi} kq(k-q)\, \frac{1}{\Sigma_{k-q}^{\mathrm{R}} + \Sigma_q^{\mathrm{R}}},$$

for the retarded self energy. It is solved with a purely imaginary ansatz $\Sigma_q^{\mathrm{R}} = -\mathrm{i}\mu q^2$ where

$$\mu = \frac{\hbar^2}{4m}\sqrt{\frac{\pi}{2}\frac{v_{\mathrm{c}}}{v_{\mathrm{F}}}\, f_1(a=2)} \left\{3 + \frac{v_{\mathrm{F}}^2}{v_{\mathrm{c}}^2} \frac{\mathrm{d}}{\mathrm{d}v_{\mathrm{F}}} \left(\frac{v_{\mathrm{c}}^2}{v_{\mathrm{F}}}\right)\right\} \qquad f_1(a) = \frac{1}{2\pi}\int_0^1 \mathrm{d}x \frac{x(1-x)}{(1-x)^a + x^a}.$$

At zero temperature, the resulting damping rate

$$\Gamma_k^0 = \frac{\hbar}{8m}\sqrt{\frac{v_{\mathrm{c}}}{v_{\mathrm{F}}}\left(\frac{\pi}{4} - \frac{1}{2}\right)} \left\{3 + \frac{v_{\mathrm{F}}^2}{v_{\mathrm{c}}^2} \frac{\mathrm{d}}{\mathrm{d}v_{\mathrm{F}}} \left(\frac{v_{\mathrm{c}}^2}{v_{\mathrm{F}}}\right)\right\} k^2, \tag{7}$$

of the Bogoliubov–Anderson mode in one dimension is therefore quadratic in the wavevector. Formally, this is precisely the behaviour of a hydrodynamic mode. It allows to define a zero temperature viscosity ζ by the relation

$$\omega_k = v_{\mathrm{c}} k - \mathrm{i}\, \frac{\zeta}{2mn}\, k^2 \tag{8}$$

which is completely analogous to sound damping in three dimensions. In that case, ζ is replaced by the combination $\zeta_2 + 4\eta/3$ involving one of the superfluid bulk viscosities ζ_2 and the shear viscosity η [16, 33] and one has $\zeta_2 = 0$ at unitarity [15]. From the result (7), we see that the viscosity at zero temperature has the form

$$\zeta = \alpha_\zeta \hbar n, \tag{9}$$

with a constant

$$\alpha_\zeta = \frac{1}{4}\sqrt{\left(\frac{\pi}{4} - \frac{1}{2}\right)\frac{v_{\mathrm{c}}}{v_{\mathrm{F}}}} \left\{3 + \frac{v_{\mathrm{F}}^2}{v_{\mathrm{c}}^2} \frac{\mathrm{d}}{\mathrm{d}v_{\mathrm{F}}} \left(\frac{v_{\mathrm{c}}^2}{v_{\mathrm{F}}}\right)\right\}. \tag{10}$$

A plot of α_ζ is given in figure 1, where the exact result for $v_{\mathrm{c}}/v_{\mathrm{F}}$ from the Bethe–Ansatz solution was used. Evidently, the dimensionless viscosity coefficient α_ζ depends on the inverse coupling constant $1/\gamma$ of the BCS-BEC crossover. It is thus in general dependent on the particle density n. At the unitarity point, however, where $1/\gamma = 0$, this dependence vanishes and α_ζ takes the universal value

$$\alpha_\zeta(\gamma^{-1} = 0) = \sqrt{\frac{\pi}{8} - \frac{1}{4}} \approx 0.38$$

which is just $1/\sqrt{2}$ of the value $\alpha_\zeta(\gamma = 0^-) = 0.54$ attained in the weak coupling limit of the 1D noninteracting Fermi gas. Concerning the range of applicability of the perturbative calculation, it is obvious that the approximation leading to (6) is satisfied for small wavenumbers $k \ll mnv_{\mathrm{c}}/\zeta$. Based on the explicit result for ζ this condition translates into phonon wavelengths much larger

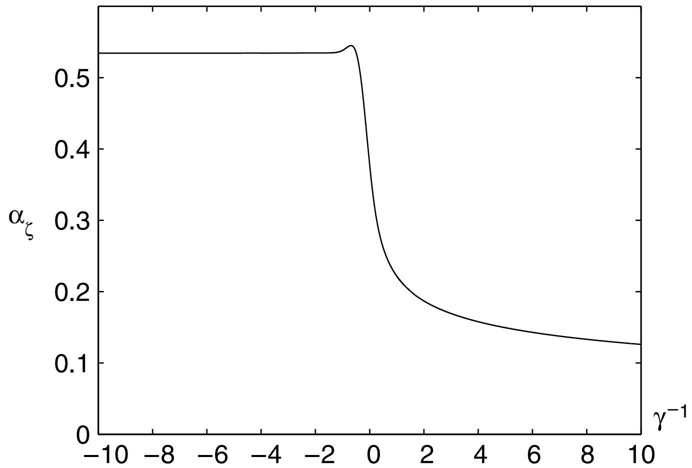


Figure 1. Viscosity-parameter α_ζ as a function of the inverse coupling constant γ^{-1}.

than the mean interparticle spacing on the BCS side of the crossover, including the unitarity point. On the BEC side, the ratio v_c/ζ vanishes like $\sqrt{v_c} \sim \gamma^{1/4}$. The phonon wavelengths have thus to obey the more restrictive condition $\lambda n \gg (1/\gamma)^{1/4} \gg 1$ mentioned above. It is interesting to note, that this condition is less restrictive than the naive estimate $\lambda \gg \xi_1$, where $\xi_1 = n^{-1}(1/\gamma)^{1/2}$ is the 1D healing length.

We now turn to the situation at finite temperature $T > 0$, where the long wavelength phonons, for which $T \gg \varepsilon_k$, behave classically. The thermal factor $\coth(\beta\Omega/2)$ may therefore be replaced by its classical limit $2/(\beta\Omega)$. At $T \neq 0$, the second diagram representing the absorption of another phonon also contributes to the damping. An explicit calculation along the lines performed at $T = 0$ gives a phonon damping rate at finite temperature of the form

$$\Gamma_k^T = \frac{\hbar}{4m}\sqrt{\frac{\pi}{2}\frac{k_B T}{\hbar v_F} f_2(3/2)}\left\{3 + \frac{v_F^2}{v_c^2}\frac{\mathrm{d}}{\mathrm{d}v_F}\left(\frac{v_c^2}{v_F}\right)\right\} k^{3/2}. \tag{11}$$

Here, $f_2(a = 3/2) \approx 0.6221$ is a numerical coefficient defined by the integral

$$f_2(a) = \int_0^1 \frac{\mathrm{d}x}{2\pi}\frac{1}{(1-x)^a + x^a} + 2\int_1^\infty \frac{\mathrm{d}x}{2\pi}\frac{1}{(x-1)^a + x^a}.$$

As already noted by Andreev [31], the damping $\propto k^{3/2}$ for $T > 0$ is not of the standard hydrodynamic form, in contrast to the behaviour at zero temperature. The quite different results can be understood from the fact that at any finite temperature, the quasi-long-range superfluid order present at $T = 0$ is destroyed by phase fluctuations on a characteristic length scale $\xi_T = \hbar v_c/k_B T$. Depending on the ratio $y = k\xi_T$ between this length scale and the phonon wavelength, the behaviour is either essentially superfluid for $y \gg 1$ or normal for $y \ll 1$. Similar to the formulation used in dynamical scaling laws near critical points [34, 35], the crossover between the two different types of behaviour may be described by an Ansatz of the form

$$\Gamma_k = \frac{\hbar k^2}{2m}\,\Phi(\xi_T k). \tag{12}$$

The associated crossover function has the limiting behaviour

$$\Phi(y) \xrightarrow{y\to\infty} \alpha_\zeta, \qquad \Phi(y) \xrightarrow{y\ll 1} \frac{3.70\,\alpha_\zeta}{\sqrt{y}}.$$

with the parameter α_ζ defined in equation (10). It should be pointed out, that the dependence of the damping rate on temperature is a simple power law $\sim T^{1/2}$ only to the extent that the temperature dependence of the velocity v_c itself can be neglected. Moreover, note that for nonzero temperature, the damping remains finite in the BEC limit $1/\gamma \to \infty$ in contrast to the $T = 0$ case.

2.3. Harmonically trapped gas

Finally we extend our results on damping in a homogeneous gas to the experimentally accessible case of a harmonically trapped system, using essentially a local density approximation. Our main interest is to calculate the damping of the so-called breathing modes, which have already been measured in 1D Bose gases in a regime near the Tonks–Girardeau limit [36]. Assuming a standard type of viscous damping in a classical fluid, the damping rate in a 1D inhomogeneous case is given by [37]

$$\Gamma = \left|\frac{\langle \dot{E}_{\mathrm{mech}}\rangle_t}{2\langle E\rangle_t}\right| = \left|\frac{\int \mathrm{d}z\ \zeta(z)\langle(\partial_z v)^2\rangle_t}{2m\int \mathrm{d}z\ n(z)\langle v^2\rangle_t}\right| \tag{13}$$

where $\langle . \rangle_t$ denotes the time average, z is the spatial coordinate and the last equation holds for harmonically oscillating perturbations, where $\langle E\rangle_t = 2\langle E_{\mathrm{kin}}\rangle_t$.

Breathing modes in a harmonic trap are characterized by a velocity profile of the form $v(z,t) = \mathrm{const.}z e^{-i\omega_B t}$. Since the damping of the Bogoliubov–Anderson mode at zero temperature has precisely the form of a standard viscous fluid one obtains using equation (9)

$$\Gamma_B^0 = \frac{\hbar}{2m}\,\frac{\langle\alpha_\zeta\rangle}{\langle z^2\rangle}, \tag{14}$$

where the brackets denote the spatial average defined by

$$\langle f(z)\rangle = \frac{1}{N}\int \mathrm{d}z\ n(z) f(z)$$

and N is the total number of particles. Since the constant α_ζ depends on density except at the unitarity point, the damping also involves a spatial average $\langle\alpha_\zeta\rangle$. A plot of the coupling-dependent Q-factor $Q = \omega_B/\Gamma_B^0$ of the breathing mode at zero temperature is given in figure 2 together with the ratio of its frequency in units of the axial trap frequency ω_z. The required density profiles were calculated numerically using a local density approximation and the exact results for the chemical potential from the Bethe–ansatz solution. As shown by Menotti and Stringari [38], the

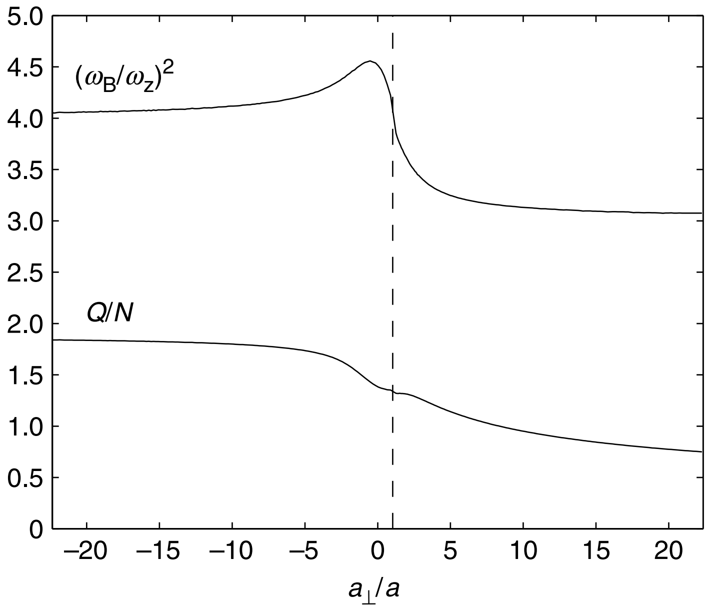


Figure 2. Frequency (ω_B) and quality factor $Q = \omega_B/\Gamma_0$ of breathing modes at $T = 0$ as a function of the 3d scattering length a. The dashed line indicates the CIR (unitarity point). Plot for $\omega_\perp/N\omega_z = 5$ where $\omega_\perp$ and ω_z are the radial and axial trapping frequency, N denotes the number of particles in the trap.

density profiles also determine the frequency from

$$\omega_B^2 = -2\left(\frac{\mathrm{d}\ln\langle z^2\rangle}{\mathrm{d}\omega_z^2}\right)^{-1}.$$

The appearance of a maximum in the breathing frequency just before the confinement induced resonance may qualitatively be understood from the fact that around this point the nature of the pairing changes from overlapping, correlated pairs to individual molecules. Indeed, the size of a molecule is of the order of the average interparticle distance n^{-1} for inverse coupling constants $1/\gamma \approx -0.5$ [22]. Remarkably, this is also close to the point where the effective dimer–dimer interaction g_1^{dd} diverges [28] and where the dimensionless viscosity coefficient (10) exhibits a small maximum as shown in figure 1.

For $T > 0$, the situation is more complicated, because the k-dependence of the damping rate in (11) implies a non-hydrodynamic behaviour. In order to account for the specific k-dependence in a classical calculation, we modify the stress-tensor by introducing an effectively velocity-dependent viscosity. The form of ζ to be used in (13) which leads to the result (12) for the damping in the homogeneous system is

$$\zeta = \hbar n \Phi\left(\xi_T \frac{\langle|\partial_z v|\rangle}{\langle|v|\rangle}\right), \tag{15}$$

with the function Φ as defined in (12).

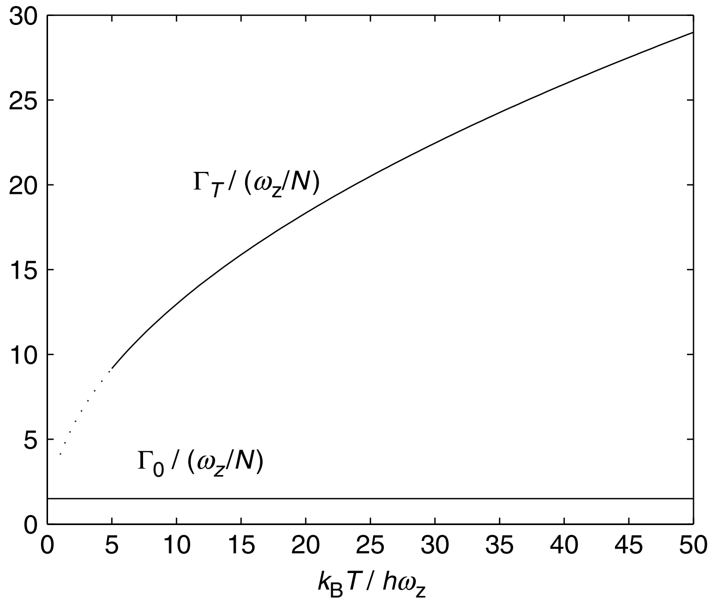


Figure 3. Damping of breathing modes at the point of unitarity as a function of T. The damping at $T = 0$ is pictured for comparison.

Using (15) for the inhomogeneous system, the result for the damping of breathing modes can be expressed as

$$\Gamma_{\mathrm{B}} = \frac{\hbar}{2m} \left\langle \Phi \left(\frac{\xi_T}{\langle |z| \rangle} \right) \right\rangle \frac{1}{\langle z^2 \rangle}. \tag{16}$$

In the particular case of the Tonks–Girardeau limit describing the unitary 1D Fermi gas at the CIR the damping of breathing modes at $T = 0$ and at finite temperatures $T \gg \hbar\omega_z$ is given by

$$\Gamma_{\mathrm{B}}^0 \approx 1.5 \frac{\omega_z}{N} \tag{17}$$

$$\Gamma_{\mathrm{B}}^T \approx 4.1 \frac{\omega_z}{N} \sqrt{\frac{k_{\mathrm{B}} T}{\hbar \omega_z}}. \tag{18}$$

A plot is given in figure 3. Note that in the Tonks–Girardeau limit, the constraint $\xi_T / \langle |z| \rangle \ll 1$ simply translates into $k_{\mathrm{B}} T \gg \hbar\omega_z$. The zero temperature result for the damping is thus only valid in the experimentally hardly accessible regime $T \ll \hbar\omega_z$, while for realistic temperatures the damping is expected to increase like $\sqrt{T}$.

3. Summary

In summary, we have calculated the damping of the Bogoliubov–Anderson mode for an attractively interacting 1D Fermi gas in the whole regime between the BCS and the BEC limit. At zero temperature, the damping is of a hydrodynamic form with a viscosity $\zeta = \alpha_\zeta \hbar n$. The associated constant α_ζ is a smooth function along the crossover from a BCS-type superfluid to a

BEC of strongly bound pairs of fermions, with a universal value $\alpha_\zeta = 0.38$ at the unitarity point. It is remarkable that a rough analysis [16] of the experiments by Bartenstein *et al* [43] gives a value of 0.3 for the universal viscosity coefficient of the 3D unitary Fermi gas at the lowest attainable temperatures. However, it is obvious that a comparison between this and our result in 1D is not meaningful. Nevertheless, the fact that the Bogoliubov–Anderson mode spectrum and velocity are hardly different between the one and the three dimensional case, suggests that the viscosity in 3D exhibits a similar dependence on the inverse coupling constant $1/\gamma = 1/(k_F a)$. The unitarity point would then define a minimal value of the viscosity on the *Fermi* side of the crossover, yet lower viscosities will be reached by going further into the BEC regime. It is a peculiar property of the 1D BCS-BEC crossover problem, that the boundary between fermionic and bosonic behaviour is sharp and defined by the CIR. A similar sharp separation, however, does not exist in three dimensions. At finite temperature, the sound damping in 1D does not have a hydrodynamic form and behaves like $\Gamma_k \sim k^{3/2}$. The resulting damping of the breathing mode in a trapped gas has been calculated within a simple model, which accounts for the inhomogeneity in the case of a nonstandard damping. In particular, it has been shown that in the experimentally relevant regime $T \gg \hbar\omega_z$, the damping increases like $\sqrt{T}$, thus providing a sensitive measure of temperature in strongly interacting 1D gases. Experimentally, an attractive Fermi gas near a Feshbach and CIR has been realized by Moritz *et al* [39]. Since the typical temperatures in this gas were of order $T \approx 0.2\, T_F$ with $T_F \approx N \cdot \hbar\omega_z$ and typical particle numbers are $N \approx 100$, the condition $T \gg \hbar\omega_z$ is realized. It would be quite interesting therefore, to study the temperature dependence of the breathing mode Q-factor similar to the measurements performed in 1D Bose gases [36]. In this context, it is interesting to note that for the Tonks–Girardeau gas, exact results for the dynamics have been derived at zero temperature by Minguzzi and Gangardt [40]. In particular they imply zero damping of the breathing mode at the unitarity point, i.e. an infinite Q-factor. From our present results, the Q-factor is infinite only in the limit $n \to \infty$ but not for the finite and typically small values $N \approx 50$–100 realized experimentally. This point needs to be studied further.

Appendix. QHD versus bosonization

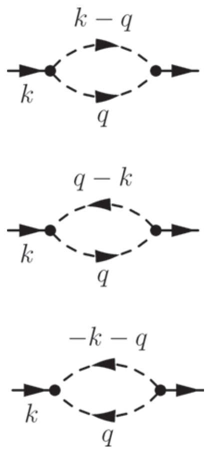
As mentioned earlier, the QHD approach does not give rise to interaction terms between spin and charge excitations. We now use bosonization techniques to construct H_{int} on the BCS side of the crossover and show that these terms do not contribute to the damping rate.

In order to obtain damping in the Tomonaga–Luttinger model, one needs to incorporate the quadratic dispersion relation in the Hamiltonian, leading to third order terms in the fields as was already shown by Haldane for a spinless gas of fermions [41]. Since we are dealing with spin 1/2 particles, third order terms arise which couple charge and spin excitations. Bosonizing the kinetic energy term via point splitting we find

$$\begin{aligned} H_{\mathrm{kin}} &= \frac{\hbar^2}{2m} \sum_{\Sigma=\uparrow,\downarrow} \int_0^L \mathrm{d}x \left\{ \partial_x \psi_\Sigma^\dagger \partial_x \psi_\Sigma + \partial_x \overline{\psi}_\Sigma^\dagger \partial_x \overline{\psi}_\Sigma \right\} \\ &= -\frac{\hbar^2}{6m}\sqrt{\frac{\pi}{2}} \int_0^L \mathrm{d}x \{ (\partial_x\varphi_c)\Pi_c^2 + \Pi_c^2(\partial_x\varphi_c) + \Pi_c(\partial_x\varphi_c)\Pi_c + (\partial_x\varphi_c)^3 \\ &\quad + 3\Pi_c[(\partial_x\varphi_s)\Pi_s + \Pi_s(\partial_x\varphi_s)] + 3(\partial_x\varphi_c)\left[\Pi_s^2 + (\partial_x\varphi_s)^2\right]\}, \end{aligned}$$

where we used the same notation as [42]. To avoid confusion, it should be mentioned that here $\partial_x \varphi$ plays the role of density fluctuations and Π describes phase fluctuations in contrast to (4). The charge–charge-interaction Hamiltonian given above is essentially the same as the one in (4) with one subtle difference: in the QHD approach the Π^3 term has a prefactor $(\mathrm{d}/\mathrm{d}\rho_0)(v_\mathrm{c}^2/\rho_0)$ which is absent in the bosonized counterpart. This factor is important because it prevents α_ζ and thus the damping from going to infinity in the BEC limit. Since the crossover from the BCS to the BEC regime is continuous, the damping must also change continuously when one crosses the point of unitarity. This argument leads us to include this prefactor also in H_int on the BCS side and thus use (4) in the whole crossover regime.

The contribution to the phonon damping rate arising from interaction with spin excitations in second order perturbation theory can be calculated from the following diagrams:



For $T = 0$, the contribution from the first diagram corresponding to spontaneus decay is given by

$$\Gamma_k \propto \int \frac{\mathrm{d}q}{2\pi} V_\mathrm{sc}^2(k, q, k-q)\delta(\underbrace{v_\mathrm{c} k - \omega_s(q) - \omega_s(k-q)}_{\neq 0\ \forall q}) = 0,$$

where V_sc denotes the spin–charge interaction vertex and $\omega_s(q)$ is the spinon dispersion relation. We immediately see that this process is strongly suppressed by energy conservation. The other two diagrams give a small contribution only for $T > 0$. In the strong coupling limit $(1/\gamma \to 0^-)$ we obtain

$$\Gamma_k^\mathrm{sc} \approx \frac{1}{64\pi^3} \frac{\varepsilon_\mathrm{F}}{\hbar} \gamma^4 \xi_T k\, \mathrm{e}^{-\beta\Delta/2} + \mathcal{O}((\xi_T k)^2).$$

This term involves the energy gap $\Delta \sim 2\varepsilon_\mathrm{F}\gamma^2/\pi^2$ and thus is negligible compared to (12).

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