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Understanding affine Deligne-Lusztig varieties using the quantum Bruhat graph

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Abstract

Affine Deligne-Lusztig varieties capture the delicate interplay between the Iwahori-Bruhat decomposition of an algebraic group and its decomposition into σ-conjugacy classes. Our four main results express geometric properties of these decompositions in terms of combinatorial properties of the quantum Bruhat graph.

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Contents

1. Introduction

To keep the introduction concise, we refer to Section 2.1 for a detailed description of our technical setup and notation. For now, let us summarize that G denotes an algebraic group over a local field F, whose maximal unramified extension we denote by $L = \breve{F}$. We are interested in two important decompositions of the topological space $G(L)$.

The first is the *Iwahori-Bruhat decomposition* of the topological space $G(L)$. For an Iwahori subgroup $I \subseteq G(L)$ and the extended affine Weyl group $\widetilde{W}$, we have

$$G(L) = \bigsqcup_{x \in \widetilde{W}} IxI.$$

The closure of an Iwahori double coset IxI is naturally a union of Iwahori double cosets, with closure relations given by the Bruhat order $\leqslant$ on $\widetilde{W}$.

$$\overline{IxI} = \bigsqcup_{y \leqslant x} IyI.$$

The Bruhat order has an alternative, purely Coxeter-theoretic description. However, both these approaches can be tricky to work with. In Section 4, we present a new description of the Bruhat order on $\widetilde{W}$ that is amenable to both theoretical reasoning and practical computation.

Theorem 1.1. *Let $x_1, x_2 \in \widetilde{W}$, and write them as $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2}$. Then $x_1 \leqslant x_2$ in the Bruhat order if and only if for each $v_1 \in W$, there exists some $v_2 \in W$ satisfying*

$$v_1^{-1}\mu_1 + \mathrm{wt}(v_2 \Rightarrow v_1) + \mathrm{wt}(w_1v_1 \Rightarrow w_2v_2) \leqslant v_2^{-1}\mu_2.$$

Here, wt denotes the weight function of the quantum Bruhat graph. This function will be studied in detail in Section 3.

For more refined descriptions of the Bruhat order, we refer to Theorems 4.2 and 4.36 as well as Remark 5.23.

As an application, we give a new description of the *admissible sets* in $\widetilde{W}$ as introduced by Kottwitz and Rapoport [KR00; Rap02] (Propositions 4.12 and 4.38).

The product of two Iwahori double cosets is in general not an Iwahori double coset. After passing to closures however, we do find for each $x, y \in \widetilde{W}$ a uniquely determined $z = x * y \in \widetilde{W}$ such that

$$\overline{IxI \cdot IyI} = \overline{IzI}.$$

The Demazure product also has a purely Coxeter-theoretic description, namely

$$x * y = \max\{x'y' \mid x' \leqslant x, y' \leqslant y\}.$$

Using our previously established result on the Bruhat order, we give a new description of the Demazure product $*$ on $\widetilde{W}$ in Section 5.

Theorem 1.2 (Cf. Theorem 5.11)**.** *Let $x_1, x_2 \in \widetilde{W}$, and write them as $x_1 = w_1\varepsilon^{\mu_1}$ and $x_2 = w_2\varepsilon^{\mu_2}$. Then for explicitly described $v_1, v_2 \in W$, we have*

$$x_1 * x_2 = w_1 v_1 v_2^{-1} \varepsilon^{v_2 v_1^{-1}\mu_1 + \mu_2 - v_2 \operatorname{wt}(v_1 \Rightarrow w_2 v_2)}.$$

This description of Demazure products will then shed some light both on our previous result on the Bruhat order and the next result on generic σ-conjugacy classes.

There is a second important stratification on $G(L)$, namely the decomposition into σ-conjugacy classes. Denoting by σ the Frobenius of L/F, it acts on $G(L)$ and we define for $g_1, g_2 \in G(L)$:

$$g_1 \sim_\sigma g_2 \iff \exists h \in G(L) :\ g_1 = h^{-1} g_2 \sigma(h).$$

The set of σ-conjugacy classes in $G(L)$ is denoted $B(G)$. The σ-conjugacy class of an element $g \in G(L)$ is determined by two invariants, as proved by Kottwitz in [Kot85; Kot97]. These invariants are called the *Newton point* $\nu(g)$ and the *Kottwitz point* $\kappa(g)$.

The closure of a σ-conjugacy class $[b]_\sigma$ is again a union of σ-conjugacy classes, so we can write

$$\overline{[b]_\sigma} = \bigsqcup_{[b']_\sigma \leqslant [b]_\sigma} [b']_\sigma.$$

The order $\leqslant$ on σ-conjugacy classes is easily described as $\kappa(b') = \kappa(b)$ and $\nu(b') \leqslant \nu(b)$ in the dominance order. This result is proved by Rapoport-Richartz [RR96] and Viehmann [Vie13] for split groups and by He [He16] for general groups.

We are interested in the intersections $IxI \cap [b]$ for $x \in \widetilde{W}$ and $[b] \in B(G)$, called *Newton strata*. It is an important open question which Newton strata are non-empty, i.e. to describe the set

$$B(G)_x := \{[b] \in B(G) \mid IxI \cap [b] \neq \varnothing\}.$$

Related to these intersections are the *affine Deligne-Lusztig varieties* (cf. [Rap02]), defined by

$$X_x(b)(\overline{\mathbb{F}_q}) = \{g \in G(\breve{F})/I \mid g^{-1} b \sigma(g) \in IxI\}.$$

The dimension and the question of equi-dimensionality of $X_x(b)$ have been intensively studied in the past, yet both problems remain largely open [GHKR06; GHKR10; GH10; He14; MST19]. Affine Deligne-Lusztig varieties for certain groups of small rank have been studied explicitly [Reu02; Bea09; Yan14].

Affine Deligne-Lusztig varieties have been introduced by Rapoport [Rap02] to define Rapoport-Zink moduli spaces, which play an important role for the study of Shimura varieties.

The construction of affine Deligne-Lusztig varieties resembles a classical construction of certain varieties due to Deligne-Lusztig [DL76]. They used the cohomology of these

Deligne-Lusztig varieties to describe all complex representations of finite groups of Lie type.

If one replaces the Iwahori subgroup by a hyperspecial subgroup, the resulting affine Deligne-Lusztig varieties have been well-understood after concentrated effort by many researchers, e.g. [Kot06; GHKR06; Vie06; Ham15].

For the affine Deligne-Lusztig varieties considered in this paper, there are a number of important partial results describing their geometry.

It is proved by Görtz-He-Nie [GHN15] and Viehmann [Vie21] that $B(G)_x$ always contains a uniquely determined smallest element, which is explicitly described. Moreover, $B(G)_x$ always contains a uniquely determined largest element. This follows from the specialization theorem of Rapoport-Richartz [RR96, Theorem 3.6], as explained by Viehmann [Vie14, Proof of Corollary 5.6]. Rapoport-Richartz also prove a version of *Mazur's inequality*, which states that for $[b] \in B(G)_x$ with $x = w\varepsilon^\mu$, we must have an identity of Kottwitz points $\kappa(b) = \kappa(x)$ and the inequality $\nu(b) \leqslant \mu^{\mathrm{dom}} \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.

While the dimension $\dim X_x(b)$ is difficult to compute, the *virtual dimension* $d_x(b)$ introduced by He [He14] is easy to evaluate and always an upper bound for $\dim X_x(b)$. Moreover, we have $\dim X_x(b) = d_x(b)$ for a number of cases, but not always. Cf. [He14; MV20; He21b], affirming conjectures of Reuman and others [Reu02; GHKR06]. The virtual dimension is defined as

$$d_x(b) = \frac{1}{2}\left(\ell(x) + \ell(\eta_\sigma(x)) - \langle \nu(b), 2\rho\rangle - \mathrm{def}(b)\right).$$

Here, $\ell(x)$ denotes the length of x in $\widetilde{W}$, as explained in Section 2.1. By $\eta_\sigma(x)$, we denote a certain element in the finite Weyl group associated with x, as explained in Section 2.2. These two terms only depend on the element $x \in \widetilde{W}$.

The *defect* of a σ-conjugacy class is a non-negative integer that is bounded by the rank of the root system. We will focus on this invariant in Section 6.2.

The uniquely determined largest element of $B(G)_x$ is called *generic σ-conjugacy class* $[b_x]_\sigma$. It is the unique σ-conjugacy class such that $[b_x]_\sigma \cap IxI$ is dense in IxI. The Kottwitz point of b_x coincides with the Kottwitz point of x, which is easy to compute. The calculation of its Newton point, i.e. the *generic Newton point* of x, is less straightforward. We are able to prove the following:

Theorem 1.3 (Cf. Theorem 7.2)**.** *Let $x = w\varepsilon^\mu \in \widetilde{W}$. We can give an explicit closed formula for the generic Newton point $\nu_x = \nu(b_x)$ in terms of μ and the weight function of the quantum Bruhat graph.*

This theorem may be seen as a refinement of the aforementioned Mazur inequality, as it gives a sharp upper bound for $\{\nu(b) \mid [b] \in B(G)_x\}$. We also give a concise formula for the *λ-invariant* $\lambda_G([b_x])$ as introduced by Hamacher-Viehmann [HV18]. This result is useful for proving our second main result.

If the dimension coincides with the virtual dimension for the generic σ-conjugacy class, i.e. $\dim X_x(b_x) = d_x(b_x)$, the element x is called *cordial* following Milićević-Viehmann [MV20]. They prove in [MV20, Corollary 3.17, Theorem 1.1] that cordial elements satisfy the most desirable properties. In particular, the set $B(G)_x$ is explicitly described as a closed interval in $B(G)$, and for each $b \in B(G)_x$, the affine Deligne-Lusztig varieties $X_x(b)$ is equi-dimensional of dimension $d_x(b)$. Using our result on generic Newton points, we are able to fully classify the cordial elements in $\widetilde{W}$.

Theorem 1.4 (Cf. Corollary 7.9)**.** *Let $x \in \widetilde{W}$. Then x is cordial if and only if two conditions are satisfied, that we can summarize as a genericness condition on x and an extremality condition on certain vertices in the quantum Bruhat graph.*

The theory of cordial elements has been used by He [He21b] to compute the dimensions of many affine Deligne-Lusztig varieties, even for non-cordial elements $x \in \widetilde{W}$.

Our main results were known previously only for elements $x \in \widetilde{W}$ satisfying certain regularity conditions: Our result on the Bruhat order was previously only known for superregular elements, as a result of Lam-Shimozono [LS10], as well as groups of type A_n and C_n (cf. Chapter 8 of the textbook of Brenti-Björner on Coxeter groups [BB05]). Our result on Demazure products generalizes the ones from He-Nie [HN21].

A description of generic Newton points for superregular elements is originally due to Milićević [Mil21]. Sadhukhan [Sad21] proved a version with a weaker superregularity constraint. More generally, for shrunken elements in the extended affine Weyl group, a description of generic Newton points is due to He-Nie [HN21]. Each of these results also gives a criterion to check which of the respective regular elements are cordial, as the proof of [MV20, Proposition 4.2] can be easily adapted.

While it is true that most elements in $\widetilde{W}$ lie in a shrunken Weyl chamber, the most interesting ones typically do not. e.g. when one is interested in applications to Shimura varieties, one would be interested in *minuscule* elements in $\widetilde{W}$, of which only very few are also shrunken.

The backbone on our results on the affine flag variety are new combinatorial methods developed in Sections 2 and 3. To each element $x \in \widetilde{W}$, we associate the *length functional* $\ell(x, \cdot)$ and the set of *length positive elements* $\mathrm{LP}(x) \subseteq W$. The set $\mathrm{LP}(x)$ consists of only one element if and only if x lies in a shrunken Weyl chamber. This is one reason why previous approaches, that did not have this language available, failed for non-shrunken elements $x \in \widetilde{W}$. A number of crucial results on the quantum Bruhat graph, as introduced and proved in Section 4, complement our machinery to prove our main theorems. A newly introduced *semi-affine weight function* in Section 3.4 yields a generalization of our description of the Bruhat order, which also generalizes the previously known criteria for types A_n and C_n. Moreover, this semi-affine weight function precisely describes the admissible sets from [Rap02], cf. Proposition 4.38.

As a preparation for the more geometric aspects of our proofs, we review and refine a number of known results on the set of σ-conjugacy classes in Section 6. Our main results hold true whenever G is connected and reductive. Following Görtz-He-Nie [GHN15], we can prove this via a reduction to the case where G is quasi-split. However, many important foundational results have been proved only under the somewhat stricter assumption that G should be unramified. We show how to generalize these classical results to the quasi-split case, allowing us to prove our main results in this setting (Corollaries 7.4 and 7.9). This enables us to conclude them for arbitrary connected reductive groups (Theorem 7.18 and Proposition 7.19).

The main results of this dissertation have been made accessible to the academic community in the form of preprints and have been submitted for publication in peer-reviewed journals. Summarizing broadly, Sections 2, 6 and 7 constitute the paper [Sch22b], and Sections 3, 4 and 5 constitute the paper [Sch22a].

2. The affine root system

2.1. Group-theoretic setup

We fix a non-archimedian local field F whose completion of the maximal unramified extension will be denoted $L = \breve{F}$. We write $\mathcal{O}_F$ and $\mathcal{O}_L$ for the respective rings of integers. Let $\varepsilon \in F$ be a uniformizer. The Galois group $\Gamma = \mathrm{Gal}(L/F)$ is generated by the Frobenius σ.

Concretely, this means we have one of the following situations:

- Mixed characteristic case: $F/\mathbb{Q}_p$ is a finite extension for some prime p. Then $\mathcal{O}_F$ is the set of integral elements of F.

- Equal characteristic case: $\mathcal{O}_F$ is a ring of formal power series $\mathbb{F}_q[\![\varepsilon]\!]$, $F = \mathbb{F}_q(\!(\varepsilon)\!)$ is its fraction field, $\mathcal{O}_L = \overline{\mathbb{F}_q}[\![\varepsilon]\!]$ and $L = \overline{\mathbb{F}_q}(\!(\varepsilon)\!)$. The Frobenius σ acts on L via
$$\sigma\left(\sum a_n \varepsilon^n\right) = \sum a_n^q \varepsilon^n.$$

We consider a connected and reductive group G over F. We construct its associated affine root system and affine Weyl group following Haines-Rapoport [HR08] and Tits [Tit79].

Fix a maximal L-split torus $S \subseteq G_L$ and write T for its centralizer in G_L, so T is a maximal torus of G_L. Write $\mathcal{A} = \mathcal{A}(G_L, S)$ for the apartment of the Bruhat-Tits building of G_L associated with S. We pick a σ-invariant alcove $\mathfrak{a}$ in $\mathcal{A}$. This yields a σ-stable Iwahori subgroup $I \subset G(L)$.

Denote the normalizer of T in G by $N(T)$. Then the quotient
$$\widetilde{W} = N_G(T)(L)/(T(L) \cap I)$$
is called *extended affine Weyl group*, and $W = N_G(T)(L)/T(L)$ is the *(finite) Weyl group.* The Weyl group W is naturally a quotient of $\widetilde{W}$.

The affine roots as constructed in [Tit79, Section 1.6] are denoted Φ_{af}. Each of these roots $a \in \Phi_{\mathrm{af}}$ defines an affine function $a : \mathcal{A} \to \mathbb{R}$. The vector part of this function is denoted $\mathrm{cl}(a) \in V^*$, where $V = X_*(S) \otimes \mathbb{R} = X_*(T)_{\Gamma_0} \otimes \mathbb{R}$. Here, $\Gamma_0 = \mathrm{Gal}(\overline{L}/L)$ is the absolute Galois group of L, i.e. the inertia group of $\Gamma = \mathrm{Gal}(\overline{F}/F)$. The set of *(finite) roots* is[1] $\Phi := \mathrm{cl}(\Phi_{\mathrm{af}})$.

The affine roots in Φ_{af} whose associated hyperplane is adjacent to our fixed alcove $\mathfrak{a}$ are called *simple affine roots* and denoted $\Delta_{\mathrm{af}} \subseteq \Phi_{\mathrm{af}}$.

Writing W_{af} for the extended affine Weyl group of the simply connected quotient of G, we get a natural σ-equivariant short exact sequence (cf. [HR08, Lemma 14])
$$1 \to W_{\mathrm{af}} \to \widetilde{W} \to \pi_1(G)_{\Gamma_0} \to 1.$$

Here, $\pi_1(G) := X_*(T)/\mathbb{Z}\Phi^\vee$ denotes the Borovoi fundamental group.

[1] This is different from the root system that [Tit79] and [HR08] denote by Φ; it coincides with the root system called Σ in [HR08].

For each $x \in \widetilde{W}$, we denote by $\ell(x) \in \mathbb{Z}_{\geqslant 0}$ the length of a shortest alcove path from $\mathfrak{a}$ to $x\mathfrak{a}$. The elements of length zero are denoted Ω. The above short exact sequence yields an isomorphism of Ω with $\pi_1(G)_{\Gamma_0}$, realizing $\widetilde{W}$ as semidirect product $\widetilde{W} = \Omega \ltimes W_{\mathrm{af}}$.

Each affine root $a \in \Phi_{\mathrm{af}}$ defines an affine reflection r_a on $\mathcal{A}$. The group generated by these reflections is naturally isomorphic to W_{af} (cf. [HR08]), so by abuse of notation, we also write $r_a \in W_{\mathrm{af}}$ for the corresponding element. We define $S_{\mathrm{af}} := \{r_a \mid a \in \Delta_{\mathrm{af}}\}$, called the set of *simple affine reflections*. The pair $(W_{\mathrm{af}}, S_{\mathrm{af}})$ is a Coxeter group with length function ℓ as defined above.

We pick a special vertex $\mathfrak{x} \in \mathcal{A}$ that is adjacent to $\mathfrak{a}$. We identify $\mathcal{A}$ with V via $\mathfrak{x} \mapsto 0$. This allows us to decompose $\Phi_{\mathrm{af}} = \Phi \times \mathbb{Z}$, where $a = (\alpha, k)$ corresponds to the function

$$V \to \mathbb{R}, v \mapsto \alpha(v) + k.$$

From [HR08, Proposition 13], we moreover get decompositions $\widetilde{W} = W \ltimes X_*(T)_{\Gamma_0}$ and $W_{\mathrm{af}} = W \ltimes \mathbb{Z}\Phi^\vee$. Using this decomposition, we write elements $x \in \widetilde{W}$ as $x = w\varepsilon^\mu$ with $w \in W$ and $\mu \in X_*(T)_{\Gamma_0}$. For $a = (\alpha, k) \in \Phi_{\mathrm{af}}$, we have $r_a = s_\alpha \varepsilon^{k\alpha^\vee} \in W_{\mathrm{af}}$, where $s_\alpha \in W$ is the reflection associated with α. The natural action of $\widetilde{W}$ on Φ_{af} can be expressed as

$$(w\varepsilon^\mu)(\alpha, k) = (w\alpha, k - \langle \mu, \alpha \rangle).$$

We define the *dominant chamber* $C \subseteq V$ to be the Weyl chamber containing our fixed alcove $\mathfrak{a}$. This gives a Borel subgroup $B \subseteq G$, and corresponding sets of positive/negative/simple roots $\Phi^+, \Phi^-, \Delta \subseteq \Phi$.

By abuse of notation, we denote by Φ^+ also the indicator function of the set of positive roots, i.e.

$$\Phi^+ : \Phi \to \{0,1\}, \quad \alpha \mapsto \begin{cases} 1, & \alpha \in \Phi^+, \\ 0, & \alpha \in \Phi^-. \end{cases}$$

The following easy facts will be used often, usually without further reference:

Lemma 2.1. *Let $\alpha \in \Phi$.*

(a) $\Phi^+(\alpha) + \Phi^+(-\alpha) = 1$.

(b) If $\beta \in \Phi$ and $k, \ell \geqslant 1$ are such that $k\alpha + \ell\beta \in \Phi$, we have

$$0 \leqslant \Phi^+(\alpha) + \Phi^+(\beta) - \Phi^+(k\alpha + \ell\beta) \leqslant 1. \qquad \square$$

The sets of positive and negative affine roots can be defined as

$$\begin{aligned} \Phi_{\mathrm{af}}^+ :=& (\Phi^+ \times \mathbb{Z}_{\geqslant 0}) \sqcup (\Phi^- \times \mathbb{Z}_{\geqslant 1}) = \{(\alpha, k) \in \Phi_{\mathrm{af}} \mid k \geqslant \Phi^+(-\alpha)\}, \\ \Phi_{\mathrm{af}}^- :=& -\Phi_{\mathrm{af}}^+ = \Phi_{\mathrm{af}} \backslash \Phi_{\mathrm{af}}^+ = \{(\alpha, k) \in \Phi_{\mathrm{af}} \mid k < \Phi^+(-\alpha)\}. \end{aligned}$$

One checks that Φ_{af}^+ are precisely the affine roots that are sums of simple affine roots.

Decompose Φ as a direct sum of irreducible root systems, $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$. Each irreducible factor contains a uniquely determined longest root $\theta_i \in \Phi_i^+$. Now the set of simple affine roots is

$$\Delta_{\mathrm{af}} = \{(\alpha, 0) \mid \alpha \in \Delta\} \sqcup \{(-\theta_i, 1) \mid i = 1, \ldots, r\} \subset \Phi_{\mathrm{af}}^+.$$

The *Bruhat order* on W_{af} is the usual Coxeter-theoretic notion. The Bruhat order on $\widetilde{W}$ can be defined as $\omega x \leqslant \omega' x'$ iff $\omega = \omega'$ and $x \leqslant x'$ for $\omega, \omega' \in \Omega$ and $x, x' \in W_{\mathrm{af}}$.

We call an element $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ *dominant* if $\langle \mu, \alpha \rangle \geqslant 0$ for all $\alpha \in \Phi^+$. For elements μ, μ' in $X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ (resp. $X_*(T)_{\Gamma_0}$ or $X_*(T)_{\Gamma}$), we write $\mu \leqslant \mu'$ if the difference $\mu' - \mu$ is a $\mathbb{Q}_{\geqslant 0}$-linear combination of positive coroots.

The induced action of Γ_0 on $\mathcal{A}, \Phi_{\mathrm{af}}, \widetilde{W}, W_{\mathrm{af}}$ and W is trivial by construction. The Frobenius action on $\mathcal{A}, X_*(T)_{\Gamma_0}, \Phi_{\mathrm{af}}$ and Φ will be denoted by σ. Note that σ preserves the set of simple affine roots. The Frobenius action on $W, \widetilde{W}$ and W_{af} will be denoted by $x \mapsto {}^\sigma x$. Then the action of ${}^\sigma x$ on $X_*(T)_{\Gamma_0}$ is the same as the composed action $\sigma \circ x \circ \sigma^{-1}$ ($x \in W$ or $\widetilde{W}$).

For the most part, we consider the case where G is quasi-split over F. This is a convenient assumption that lightens the notational burden significantly. In Section 7.2, we return to the more general setting of connected reductive G and generalize our main results via a reduction to the quasi-split case.

If G is quasi-split, we may and do choose the vertex $\mathfrak{x}$ to be σ-invariant. With this choice, the decompositions $\Phi_{\mathrm{af}} = \Phi \times \mathbb{Z}$ and $\widetilde{W} = W \ltimes X_*(T)_{\Gamma_0}$ are Frobenius equivariant. This means

$$\begin{aligned} \forall (\alpha, k) \in \Phi_{\mathrm{af}} &: \sigma(\alpha, k) = (\sigma(\alpha), k), \\ \forall w\varepsilon^\mu \in \widetilde{W} &: {}^\sigma(w\varepsilon^\mu) = ({}^\sigma w)\varepsilon^{\sigma(\mu)}. \end{aligned}$$

In particular, σ preserves the set of simple roots Δ.

The case where G is unramified has often been studied in the literature. In this case, S is a maximal torus of G_L, so $S = T$ and Φ is the usual root system of (G, T). Each root system Φ together with a Frobenius action comes from such an unramified group. However, care has to be taken when using results proved for unramified groups in the quasi-split setting, as $X_*(T)_{\Gamma_0}$ may have a torsion part if G is not unramified. In particular, the map $X_*(T)_{\Gamma_0} \to X_*(T)_{\Gamma_0} \otimes \mathbb{R} = V \cong \mathcal{A}$ might fail to be injective.

2.2. Root functionals

For every coweight μ, there exists a uniquely determined dominant coweight in the W-orbit of μ. In other words, there exists some $w \in W$ such that $\mu(w\alpha) \geqslant 0$ for all $\alpha \in \Phi^+$.

In this section, we introduce and study certain functions $\varphi : \Phi \to \mathbb{Z}$ which are more general than coweights, but still enjoy this property.

Definition 2.2. (a) A *root functional* is a function $\varphi : \Phi \to \mathbb{Z}$ satisfying the following two conditions for all $\alpha, \beta \in \Phi$:

(1) $|\varphi(\alpha) + \varphi(-\alpha)| \leqslant 1$.

(2) If $\alpha + \beta \in \Phi$, then

$$|\varphi(\alpha + \beta) - \varphi(\alpha) - \varphi(\beta)| \leqslant 1.$$

(b) If φ is a root functional, the *dual root functional* is defined by $\varphi^\vee(\alpha) = -\varphi(-\alpha)$.

(c) Let $v \in W$. The set of *inversions* of v with respect to φ is

$$\mathrm{inv}_\varphi(v) = \{\alpha \in \Phi^+ \mid \varphi(v\alpha) < 0\} \cup \{\alpha \in \Phi^- \mid \varphi(v\alpha) > 0\}.$$

We call v *positive* for φ if $\mathrm{inv}_\varphi(v) = \varnothing$. If $\alpha \in \mathrm{inv}_\varphi(v)$, we call $vs_\alpha \in W$ an *adjustment* of v for φ.

Lemma 2.3. *Let $\varphi : \Phi \to \mathbb{Z}$ be a root functional and $v \in W$ be* not *positive for φ. If v' is an adjustment of v for φ, then*

$$\#\,\mathrm{inv}_\varphi(v') < \#\,\mathrm{inv}_\varphi(v).$$

Proof. Let $\alpha \in \mathrm{inv}_\varphi(v)$ with $v' = vs_\alpha$. Up to replacing (α, φ) by $(-\alpha, \varphi^\vee)$, we may assume $\alpha \in \Phi^+$, so $\varphi(v\alpha) < 0$. Define

$$I := \{\beta \in \Phi^+ \backslash \{\alpha\} \mid s_\alpha(\beta) \in \Phi^-\}.$$

We write

$$\begin{aligned}\#\,\mathrm{inv}_\varphi(v') =& \#\{\beta \in \Phi^+\backslash I \mid \varphi(v'\beta) < 0\} + \#\{\beta \in I \mid \varphi(v'\beta) < 0\} \\ &+ \#\{\beta \in \Phi^-\backslash(-I) \mid \varphi(v'\beta) > 0\} + \#\{\beta \in -I \mid \varphi(v'\beta) > 0\}\end{aligned}$$

Note that $\varphi(v'\alpha) = \varphi(-v\alpha) \geqslant -1 - \varphi(v\alpha) \geqslant 0$ and $s_\alpha(\Phi^+\backslash(I \cup \{\alpha\})) = \Phi^+\backslash(I \cup \{\alpha\})$. Thus

$$\begin{aligned}\#\{\beta \in \Phi^+\backslash I \mid \varphi(v'\beta) < 0\} =& \#\{\beta \in \Phi^+\backslash(I \cup \{\alpha\}) \mid \varphi(vs_\alpha\beta) < 0\} \\ =& \#\{\beta \in \Phi^+\backslash(I \cup \{\alpha\}) \mid \varphi(v\beta) < 0\} \\ =& \#\{\beta \in \Phi^+\backslash I \mid \varphi(v\beta) < 0\} - 1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\#\{\beta \in \Phi^-\backslash(-I) \mid \varphi(v'\beta) > 0\} =& \#\{\beta \in \Phi^-\backslash(-I \cup \{-\alpha\}) \mid \varphi(v'\beta) > 0\} \\ =& \#\{\beta \in \Phi^-\backslash(-I \cup \{-\alpha\}) \mid \varphi(v\beta) > 0\} \\ \leqslant& \#\{\beta \in \Phi^-\backslash(-I) \mid \varphi(v\beta) > 0\}.\end{aligned}$$

Therefore, it suffices to prove the following estimates:

$$\#\{\beta \in I \mid \varphi(v'\beta) < 0\} \leqslant \#\{\beta \in I \mid \varphi(v\beta) < 0\}, \tag{1}$$
$$\#\{\beta \in -I \mid \varphi(v'\beta) > 0\} \leqslant \#\{\beta \in -I \mid \varphi(v\beta) > 0\}. \tag{2}$$

We only prove (1), as the proof of (2) is similar.

In order to prove (1), we consider the involution $\beta \mapsto -s_\alpha(\beta)$, which acts freely on I. Let $o = \{\beta, -s_\alpha(\beta)\} \subseteq I$ be an orbit for this involution. It suffices to show

$$\#\{\beta \in o \mid \varphi(v'\beta) < 0\} \leqslant \#\{\beta \in o \mid \varphi(v\beta) < 0\}. \tag{$*$}$$

In order to prove this, we calculate

$$\begin{aligned}\#\{\beta \in o \mid \varphi(v'\beta) < 0\} =& \#\{\beta \in -s_\alpha(o) \mid \varphi(v'\beta) < 0\}\\ =& \#\{\beta \in o \mid \varphi(-v\beta) < 0\}\\ \leqslant& \#\{\beta \in o \mid \varphi(v\beta) \geqslant 0\}\\ =& 2 - \#\{\beta \in o \mid \varphi(v\beta) < 0\}.\end{aligned}$$

If $\#\{\beta \in o \mid \varphi(v\beta) < 0\} \geqslant 1$, we immediately get $(*)$.

Now suppose that $\varphi(v\beta) \geqslant 0$ for all $\beta \in o$. Fix an element $\beta \in o$ and write

$$\beta' := -s_\alpha(\beta) = \langle \alpha^\vee, \beta\rangle\alpha - \beta.$$

Note that $k\alpha - \beta \in \Phi$ for $k = 0, \dots, \langle\alpha^\vee, \beta\rangle$. Thus

$$\begin{aligned}\left|\varphi(v\beta') - \langle\alpha^\vee,\beta\rangle\varphi(v\alpha) - \varphi(-v\beta)\right| \leqslant& \sum_{k=1}^{\langle\alpha^\vee,\beta\rangle} |\varphi(v(k\alpha-\beta)) - \varphi(v\alpha) - \varphi(v(k-1)\alpha - \beta)|\\ \leqslant& \langle\alpha^\vee,\beta\rangle.\end{aligned}$$

In particular, we get

$$\varphi(v\beta') - \varphi(-v\beta) \leqslant \langle\alpha^\vee,\beta\rangle(1 + \varphi(v\alpha)) \leqslant 0.$$

Thus $\varphi(-v\beta) \geqslant \varphi(v\beta') \geqslant 0$.

Since $\beta \in o$ was arbitrary, we get $\varphi(v'\beta) = \varphi(-v(-s_\alpha)\beta) \geqslant 0$ for all $\beta \in o$. This proves $(*)$, which finishes the proof of the lemma. ☐

Corollary 2.4. *If $\varphi : \Phi \to \mathbb{Z}$ is a root functional and $v \in W$ is any element, there is a sequence*

$$v = v_1, \dots, v_k \in W$$

such that v_{i+1} is an adjustment for v_i for φ (where $i = 1, \dots, k-1$), and v_k is positive for φ. In particular, positive elements exist for each root functional. ☐

The most important root functional for us will be the length functional associated to an element $x \in \widetilde{W}$, which we introduce now.

Definition 2.5. Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $\alpha \in \Phi$. We define

$$\ell(x,\alpha) := \langle\mu,\alpha\rangle + \Phi^+(\alpha) - \Phi^+(w\alpha).$$

The absolute value $|\ell(x,\alpha)|$ can be understood as counting affine root hyperplanes between the base alcove and $x\mathfrak{a}$, while the sign accounts for the orientations (cf. Lemma 2.9).

Lemma 2.6. *Let $x = w\varepsilon^\mu \in \widetilde{W}$. Then $\ell(x,\cdot)$ is a root functional. For each $\alpha \in \Phi$, we have*

$$\ell(x,\alpha) + \ell(x,-\alpha) = 0.$$

Proof. Let $\alpha, \beta \in \Phi$.

(1) We have

$$\begin{aligned}\ell(x,\alpha) + \ell(x,-\alpha) =& \langle \mu, \alpha\rangle + \Phi^+(\alpha) - \Phi^+(w\alpha) + \langle \mu, -\alpha\rangle + \Phi^+(-\alpha) - \Phi^+(-w\alpha)\\ =& \Phi^+(\alpha) + \Phi^+(-\alpha) - (\Phi^+(w\alpha) + \Phi^+(-w\alpha)) = 1 - 1 = 0.\end{aligned}$$

(2) Suppose $\alpha + \beta \in \Phi$. We know that

$$0 \leqslant \Phi^+(\alpha) + \Phi^+(\beta) - \Phi^+(\alpha+\beta) \leqslant 1.$$

Thus, we obtain

$$\begin{aligned}&|\ell(x,\alpha+\beta) - \ell(x,\alpha) - \ell(x,\beta)|\\ =&|\underbrace{\Phi^+(\alpha+\beta) - \Phi^+(\alpha) - \Phi^+(\beta)}_{\in\{-1,0\}} \underbrace{-\Phi^+(w(\alpha+\beta)) + \Phi^+(w\alpha) + \Phi^+(w\beta)}_{\in\{0,1\}}| \leqslant 1.\end{aligned}$$

This finishes the proof. $\square$

Definition 2.7. Let $x \in \widetilde{W}$ and $v \in W$. We say that v is *length positive for x* and write $v \in \mathrm{LP}(x)$ if v is positive for the length functional $\ell(x,\cdot)$. Explicitly, v is length positive for x if $\ell(x, v\alpha) \geqslant 0$ for all $\alpha \in \Phi^+$.

Example 2.8. Let $x = w\varepsilon^\mu \in \widetilde{W}$. The W-orbit of μ contains a unique dominant element of $X_*(T)_{\Gamma_0}$, and there is a unique $v \in W$ of minimal length such that $v^{-1}\mu$ is dominant. The element v is uniquely determined by the following condition for each positive root α:

$$\langle v^{-1}\mu, \alpha\rangle \geqslant \Phi^+(-v\alpha).$$

It follows that

$$\ell(x, v\alpha) = \langle v^{-1}\mu, \alpha\rangle - \Phi^+(-v\alpha) + \Phi^+(-wv\alpha) \geqslant 0.$$

We see that this particular v is length positive. This gives an alternative proof that length positive elements always exist.

Recall the definition of the virtual dimension for $x \in \widetilde{W}$ and $b \in B(G)$.

$$d_x(b) = \frac{1}{2}\left(\ell(x) + \ell(\eta_\sigma(x)) - \langle \nu(b), 2\rho\rangle - \mathrm{def}(b)\right).$$

Here, $2\rho \in X_*(T)^\Gamma$ denotes the sum of positive roots. With $v \in W$ constructed as above, we have

$$\eta_\sigma(x) = {}^{\sigma^{-1}}(v)^{-1} w v \in W.$$

Because of the importance of the virtual dimension, the specific v constructed in this example is of particular interest.

However, the construction of this $v \in W$ is not quite natural in terms of $x \in \widetilde{W}$, e.g. in view of certain automorphisms of $\widetilde{W}$ that preserve dimensions of affine Deligne-Lusztig varieties.

Studying the group GL_3 for example, there are three simple affine reflections $s_0, s_1, s_2 \in \widetilde{W}$. Each of these satisfies $\ell(s_i) = \dim X_{s_i}(1) = 1$. The two simple affine reflections that come from W also satisfy $\ell(\eta_\sigma(s_1)) = \ell(\eta_\sigma(s_2)) = 1$, so that

$$d_{s_i}([1]_\sigma) = \frac{1}{2}(1 + 1 - 0 - 0) = 1 = \dim X_{s_i}(1), \qquad i = 1, 2.$$

For the remaining affine simple reflection s_0, we do however have $\ell(\eta_\sigma(s_0)) = 3$. Thus $d_{s_0}(1) = 2 > \dim X_{s_0}(1)$.

We see that s_1, s_2 satisfy $\dim X_{s_i}(1) = d_{s_i}(1)$ (so both are cordial), whereas s_0 does not have this property. This is problematic insofar as there exists an automorphism of the affine Dynkin diagram sending s_1 to s_0, hence naturally $X_{s_0}(1) \cong X_{s_1}(1)$. This natural isomorphism is not reflected in the corresponding virtual dimensions, which comes precisely from the term $\ell(\eta_\sigma(x))$.

Searching for a replacement of this specific v that is invariant under such automorphisms, we found the notion of length positive elements. The set of length positive elements is well-behaved under such automorphisms, as it allows the following root-theoretic interpretation.

Lemma 2.9 (cf. [Len+15, Lemma 3.12])**.** *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $\alpha \in \Phi$. Then*

$$\#\{k \in \mathbb{Z} \mid (\alpha, k) \in \Phi_{\mathrm{af}}^+ \text{ and } x(\alpha, k) \in \Phi_{\mathrm{af}}^-\} = \max(0, \ell(x, \alpha)).$$

Proof. We have

$$\begin{aligned}
&\{(\alpha, k) \in \Phi_{\mathrm{af}}^+ \mid x(\alpha, k) \in \Phi_{\mathrm{af}}^-\} \\
=&\{(\alpha, k) \in \Phi_{\mathrm{af}} \mid k \geqslant \Phi^+(-\alpha) \text{ and } (w\alpha, k - \langle \mu, \alpha\rangle) \in \Phi_{\mathrm{af}}^-\} \\
=&\{(\alpha, k) \in \Phi_{\mathrm{af}} \mid k \geqslant \Phi^+(-\alpha) \text{ and } k - \langle \mu, \alpha\rangle \leqslant -\Phi^+(w\alpha)\}. \\
\cong&\{k \in \mathbb{Z} \mid \Phi^+(-\alpha) \leqslant k \leqslant \langle \mu, \alpha\rangle - \Phi^+(w\alpha)\}.
\end{aligned}$$

The cardinality of this set is given by

$$\max(0, \langle \mu, \alpha\rangle + 1 - \Phi^+(w\alpha) - \Phi^+(-\alpha)) = \max(0, \ell(x, \alpha)). \qquad \square$$

Corollary 2.10 ([IM65, Proposition 1.23])**.** *Let $x = w\varepsilon^\mu \in \widetilde{W}$. Then*

$$\ell(x) = \sum_{\alpha \in \Phi} \max(0, \ell(x, \alpha)).$$

Proof. Use that

$$\ell(x) = \#\{(\alpha, k) \in \Phi_{\mathrm{af}}^+ \mid x\alpha \in \Phi_{\mathrm{af}}^-\}$$

and decompose the latter set depending on the $\alpha \in \Phi$. □

Corollary 2.11. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in W$. Then*

$$\ell(x) \geqslant \langle v^{-1}\mu, 2\rho\rangle - \ell(v) + \ell(wv).$$

Equality holds if and only if v is length positive for x.

Proof. We calculate

$$\begin{aligned}\ell(x) \geqslant& \sum_{\alpha\in\Phi^+} \ell(x, v\alpha)\\ =& \sum_{\alpha\in\Phi^+} \big(\langle \mu, v\alpha\rangle - \Phi^+(-v\alpha) + \Phi^+(-wv\alpha)\big)\\ =&\langle v^{-1}\mu, 2\rho\rangle - \ell(v) + \ell(wv).\end{aligned}$$

□

Lemma 2.12. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$ and $\alpha \in \Phi$.*

(a) $\ell(xx', \alpha) = \ell(x, w'\alpha) + \ell(x', \alpha)$.

(b) $\ell(x^{-1}, \alpha) = -\ell(x, w^{-1}\alpha)$ and $\mathrm{LP}(x^{-1}) = w\,\mathrm{LP}(x)w_0$.

Proof. (a) Note that $xx' = ww'\varepsilon^{(w')^{-1}\mu+\mu'}$ such that

$$\begin{aligned}&\ell(x, w'\alpha) + \ell(x', \alpha)\\ =&\langle \mu, w'\alpha\rangle + \langle \mu', \alpha\rangle - \Phi^+(ww'\alpha) + \Phi^+(w'\alpha) - \Phi^+(w'\alpha) + \Phi^+(\alpha)\\ =&\langle (w')^{-1}\mu + \mu', \alpha\rangle - \Phi^+(ww'\alpha) + \Phi^+(\alpha) = \ell(xx', \alpha).\end{aligned}$$

(b) By (a), we have

$$0 = \ell(1, \alpha) = \ell(xx^{-1}, \alpha) = \ell(x, w^{-1}\alpha) + \ell(x^{-1}, \alpha).$$

Now observe that for $v \in W$,

$$\begin{aligned}v \in \mathrm{LP}(x^{-1}) &\iff \forall \beta \in \Phi^+ :\ \ell(x^{-1}, v\beta) \geqslant 0\\ &\iff \forall \beta \in \Phi^+ :\ \ell(x^{-1}, v(-w_0\beta)) \geqslant 0\\ &\iff \forall \beta \in \Phi^+ :\ \ell(x, w^{-1}vw_0\beta) \geqslant 0 \iff v \in w\,\mathrm{LP}(x)w_0.\end{aligned}$$

□

Lemma 2.13. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$. The following are equivalent:*

(i) $\ell(xx') = \ell(x) + \ell(x')$.

(ii) *For each root $\alpha \in \Phi$, the values $\ell(x, w'\alpha)$ and $\ell(x', \alpha) \in \mathbb{Z}$ never have opposite signs, i.e.*

$$\ell(x, w'\alpha) \cdot \ell(x', \alpha) \geqslant 0.$$

(iii) $\big((w')^{-1}\,\mathrm{LP}(x)\big) \cap \mathrm{LP}(x') \neq \varnothing$.

In this case, $\mathrm{LP}(xx') = \big((w')^{-1}\,\mathrm{LP}(x)\big) \cap \mathrm{LP}(x')$.

Proof. (i) $\iff$ (ii): By Corollary 2.10 and the equation $\ell(x,\alpha) = -\ell(x,-\alpha)$, we get

$$\begin{aligned}\ell(xx') &= \sum_{\alpha\in\Phi^+} \left|\ell(xx',\alpha)\right| \\ &\underset{\text{L2.12(a)}}{=} \sum_{\alpha\in\Phi^+} \left|\ell(x, w'\alpha) + \ell(x',\beta)\right| \\ &\underset{(*)}{\leqslant} \sum_{\alpha\in\Phi^+} \left|\ell(x, w'\alpha)\right| + \left|\ell(x',\alpha)\right| \\ &= \ell(x) + \ell(x').\end{aligned}$$

Equality holds at $(*)$ iff the values $\ell(x, w'\alpha)$ and $\ell(x', \alpha)$ never have opposite signs. We see that (i) $\iff$ (ii).

(iii) $\Rightarrow$ (ii): Pick $v \in \big((w')^{-1}\,\mathrm{LP}(x)\big) \cap \mathrm{LP}(x')$. If $\alpha \in \Phi^+$, then both $\ell(x, w'v\alpha)$ and $\ell(x', v\alpha)$ must be non-negative by length positivity. If $\alpha \in \Phi^-$, then both $\ell(x, w'v\alpha)$ and $\ell(x', v\alpha)$ must be non-positive. We see that (ii) must hold true.

Finally, let us assume that (ii) holds. It suffices to show that

$$\mathrm{LP}(xx') = \big((w')^{-1}\,\mathrm{LP}(x)\big) \cap \mathrm{LP}(x'),$$

as (iii) follows from this identity. Now for $v \in W$, we have

$$\begin{aligned} v \in \mathrm{LP}(xx') &\iff \forall \alpha \in \Phi^+ :\ \ell(xx', v\alpha) \geqslant 0 \\ &\underset{\text{L2.12(a)}}{\iff} \forall \alpha \in \Phi^+ :\ \ell(x, w'v\alpha) + \ell(x', v\alpha) \geqslant 0 \\ &\underset{(ii)}{\iff} \forall \alpha \in \Phi^+ :\ \ell(x, w'v\alpha) \geqslant 0 \text{ and } \ell(x', v\alpha) \geqslant 0 \\ &\iff v \in \big((w')^{-1}\,\mathrm{LP}(x)\big) \cap \mathrm{LP}(x'). \end{aligned}$$

□

Given one element $v \in \mathrm{LP}(x)$, one can use it to iteratively enumerate all length positive elements for x.

Lemma 2.14. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in \mathrm{LP}(x)$.*

(a) *For every simple root $\alpha \in \Delta$, we have*

$$\ell(x, v\alpha) = 0 \iff vs_\alpha \in \mathrm{LP}(x).$$

(b) *If the root $\alpha \in \Phi^+$ satisfies $\ell(x, v\alpha) = 0$, then there also exists a simple root with this property.*

(c) *Consider the undirected graph $G_{\mathrm{LP}(x)}$ whose vertices are given by* $\mathrm{LP}(x)$ *and whose edges are of the form* (v, vs_α) *for* $\alpha \in \Delta$ *and* $v, vs_\alpha \in \mathrm{LP}(x)$. *Then* $G_{\mathrm{LP}(x)}$ *is connected.*

Proof. (a) If $vs_\alpha \in \mathrm{LP}(x)$, then $\ell(x, v\alpha)$ and $\ell(x, vs_\alpha\alpha) = -\ell(x, v\alpha)$ must both be non-negative. This is only possible if $\ell(x, v\alpha) = 0$.

If $\ell(x, v\alpha) = 0$, confirm that $\ell(x, v\beta) \geqslant 0$ for all $\beta \in \Phi^+ \cup \{-\alpha\}$. The latter set is preserved by s_α.

(b) Suppose $\alpha \in \Phi^+\backslash\Delta$ satisfies $\ell(x, v\alpha) = 0$. We can write $\alpha = \beta + \gamma$ for positive roots $\beta, \gamma \in \Phi^+$. By length positivity, $\ell(x, v\beta), \ell(x, v\gamma) \geqslant 0$. If both of these values are $\geqslant 1$, we get $\ell(x, v\alpha) \geqslant 1$ by the root functional property. Hence $\ell(x, v\beta) = 0$ or $\ell(x, v\gamma) = 0$. We can iterate this argument.

(c) Let $C \subseteq \mathrm{LP}(x)$ denote the connected component that contains v. Among all $v' \in C$, pick one such that $\ell(wv')$ is minimal.

We claim that

$$\forall \alpha \in \Delta : \langle \mu, v'\alpha \rangle + \Phi^+(v'\alpha) \geqslant 1. \qquad (*)$$

- If $\ell(x, v'\alpha) = 0$, then $v's_\alpha \in C$. The minimality of $\ell(wv')$ ensures that $\ell(wv's_\alpha) \geqslant \ell(wv')$, i.e. $wv'\alpha \in \Phi^+$. The definition of $\ell(xv'\alpha) = 0$ implies $\langle \mu, v'\alpha \rangle + \Phi^+(v'\alpha) = 1$.
- If $\ell(x, v'\alpha) \geqslant 1$, we get

$$\langle \mu, v'\alpha \rangle + \Phi^+(v'\alpha) \geqslant \ell(x, v'\alpha) \geqslant 1.$$

Let us re-read condition $(*)$: not only is $(v')^{-1}\mu$ dominant, we have $v'\alpha \in \Phi^+$ for all $\alpha \in \Delta$ with $\langle (v')^{-1}\mu, \alpha \rangle = 0$. This describes exactly the length positive element constructed in Example 2.8.

To summarize: No matter which connected component of $G_{\mathrm{LP}(x)}$ we consider, it will always contain the one length positive element from Example 2.8. Hence $G_{\mathrm{LP}(x)}$ is connected. □

We obtain the following description of the shrunken Weyl chambers:

Proposition 2.15. *For* $x \in \widetilde{W}$, *the following are equivalent:*

(i) x *lies in the lowest two-sided Kazhdan-Lusztig cell of* $\widetilde{W}$.

(ii) For all $\alpha \in \Phi$, $\ell(x, \alpha) \neq 0$.

(iii) The set $\mathrm{LP}(x)$ *contains only one element.*

In this case, we say that x lies in a shrunken Weyl chamber.

Proof. The equivalence (i) $\iff$ (ii) is well known, cf. [HN21, Section 3.1].

The equivalence (ii) $\iff$ (iii) follows directly from Lemma 2.14. □

Remark 2.16. The length functional presented here is related to the k-function from [Shi87a]. For $w \in W, \mu \in X^*(T)$ and $\alpha \in \Phi$, Shi proves

$$k(wt^\mu, \alpha) = \langle \mu, \alpha^\vee \rangle + \Phi^+((\alpha)(w^{-1})) - \Phi^+(\alpha).$$

This result is a translation of [Shi87a, Lemma 3.1] and [Shi87a, Theorem 3.3] into our "$\Phi^+(\cdot)$"-notation. Up to a few changes of conventions, this recovers exactly our length functional. We will make these changes to express a few of Shi's ideas in terms of the length functional.

Shi classifies the functions $\Phi \to \mathbb{Z}$ that are of the form $\ell(x, \cdot)$ in [Shi87a, Proposition 5.1].

Associated to each element $x \in \widetilde{W}$ and root $\alpha \in \Phi$, he defines the value $X(x, \alpha) \in \{+, \bigcirc, -\}$ as

$$X(x,\alpha) = \begin{cases} +, & \ell(x,\alpha) > 0, \\ \bigcirc, & \ell(x,\alpha) = 0, \\ -, & \ell(x,\alpha) < 0. \end{cases}$$

The *sign type* of x is defined as $\zeta(x) = (X(x,\alpha))_{\alpha\in\Phi}$. The *admissible* sign types, i.e. the image of $\zeta : \widetilde{W} \to \{+, \bigcirc, -\}^\Phi$, is explicitly described in [Shi87b, Theorem 2.1]. Shi also computes the number of sign types and canonical representatives in W_a for each.

For root systems of type A_n, the preimages $\zeta^{-1}(S)$ for the different admissible sign types S form exactly the set of left Kazhdan-Lusztig cells for W_a [Shi86]. An explicitly described equivalence relation of sign types then classifies the two-sided Kazhdan-Lusztig cells.

The question to fully describe the Kazhdan-Lusztig cells for all affine Weyl groups seems to be open.

The sign type $\zeta(x)$ determines the set of length positive elements for x. The converse is not true, i.e. it is possible to find groups G and elements $x, y \in \widetilde{W}$ with $\mathrm{LP}(x) = \mathrm{LP}(y)$ but $\zeta(x) \neq \zeta(y)$. Computer searches have revealed such counterexamples for root systems of types G_2 and B_2, thus for every non simply-laced root system. For simply-laced root systems, we can prove that the set $\mathrm{LP}(x)$ determines the sign type $\zeta(x)$.

Proposition 2.17. *Assume that Φ is simply laced, $x \in \widetilde{W}$ and $\alpha \in \Phi$. Then the following are equivalent:*

(i) $\ell(x,\alpha) > 0$.

(ii) For all $v \in \mathrm{LP}(x)$, we have $v^{-1}\alpha \in \Phi^+$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from the definition of length positivity.

Now assume (ii). The condition $v^{-1}\alpha \in \Phi^+$ for one $v \in \mathrm{LP}(x)$ already implies $\ell(x,\alpha) \geqslant 0$. Aiming for a contradiction, we thus assume that $\ell(x,\alpha) = 0$.

Recall from Example 2.8 that there exists an element $v \in \mathrm{LP}(x)$ such that

$$\forall \beta \in \Phi^+ :\ \langle \mu, v\beta\rangle + \Phi^+(v\beta) \geqslant 1.$$

Considering the case $\beta = v^{-1}\alpha \in \Phi^+$ (by (ii)), we see

$$\ell(x,\alpha) = \langle \mu, v\beta\rangle + \Phi^+(v\beta) - \Phi^+(w\alpha) \geqslant 1 - \Phi^+(w\alpha).$$

So if $w\alpha \in \Phi^-$, we conclude (i).

Considering the same situation for x^{-1} by Lemma 2.12, we find an element $v \in \mathrm{LP}(x)$ such that

$$\forall \beta \in \Phi^+ :\ \langle \mu, v\beta\rangle - \Phi^+(wv\beta) \geqslant 0.$$

Considering the case $\beta = v^{-1}\alpha \in \Phi^+$, we see

$$\ell(x,\alpha) = \langle \mu, v\beta\rangle + \Phi^+(\alpha) - \Phi^+(wv\beta) \geqslant \Phi^+(\alpha).$$

So if $\alpha \in \Phi^+$, we are done again.

Let us thus assume from now on that $\alpha \in \Phi^-$ and $w\alpha \in \Phi^+$. In light of the assumption $\ell(x,\alpha) = 0$, we can restate this as $\langle \mu, \alpha\rangle = -1$.

For roots $\beta, \gamma \in \Phi$, we write $\beta \leqslant \gamma$ if the difference $\gamma - \beta$ is a sum of positive roots, and we write $\beta < \gamma$ is moreover $\beta \neq \gamma$.

We define a *root sequence* associated to an element $v \in \mathrm{LP}(x)$ to be a sequence

$$v^{-1}\alpha = \beta_1 > \cdots > \beta_\ell \in \Phi^+$$

such that $\beta_{i+1} - \beta_i \in \Phi^+$ for $i = 1, \dots, \ell - 1$ and $\langle \mu, v\beta_i\rangle = -1$ for $i = 1, \dots, \ell$.

Certainly, we can find a root sequence for each $v \in \mathrm{LP}(x)$ of length 1 by setting $\beta_1 = v^{-1}\alpha$.

We order the set of root sequences lexicographically. Explicitly, let $(\beta_1, \dots, \beta_\ell)$ be a root sequence associated with $v \in \mathrm{LP}(x)$ and $(\beta'_1, \dots, \beta'_{\ell'})$ associated with $v' \in \mathrm{LP}(x)$. We write $(\beta_1, \dots, \beta_\ell) < (\beta'_1, \dots, \beta'_{\ell'})$ if one of the following conditions is satisfied:

- There is $i \in \{1, \dots, \min\{\ell, \ell'\}\}$ with $\beta_{i'} = \beta'_{i'}$ for $i' = 1, \dots, i-1$ and $\beta_i < \beta'_i$.
- We have $\ell > \ell'$ and $\beta_i = \beta'_i$ for $i = 1, \dots, \ell'$.

Among all possible $v \in \mathrm{LP}(x)$ and root sequences $(\beta_1, \dots, \beta_\ell)$ associated with them, we choose a pair such that the root sequence becomes minimal with respect to the above order.

We first claim that β_ℓ is simple: Indeed, if we had $\beta_\ell = \gamma_1 + \gamma_2$ for positive roots γ_1, γ_2, then $\ell(x, v\gamma_1), \ell(v, \gamma_2) \geqslant 0$ by length positivity. Thus

$$\langle \mu, v\gamma_1\rangle \geqslant -1, \quad \langle \mu, v\gamma_2\rangle \geqslant -1, \quad \langle \mu, v\gamma_1 + v\gamma_2\rangle = -1.$$

Hence $\langle \mu, v\gamma_i\rangle = -1$ for one of the roots γ_1, γ_2. We see that we can extend the root sequence $(\beta_1, \dots, \beta_\ell)$, which contradicts minimality by definition.

Note that $\langle \mu, v\beta_\ell \rangle = -1$ and $\ell(x, v\beta_\ell) \geqslant 0$ implies $\ell(x, v\beta_\ell) = 0$. By Lemma 2.14, this means $v' = vs_{\beta_\ell} \in \mathrm{LP}(x)$.

If $\ell = 1$, then $(v')^{-1}\alpha = -v^{-1}\alpha$, so we get the desired contradiction to (ii). Therefore, $\ell > 1$.

We claim that $\langle \beta_\ell^\vee, \beta_i \rangle \geqslant 0$ for $i = 1, \ldots, \ell$: Indeed, if we had $\langle \beta_\ell^\vee, \beta_i \rangle < 0$, then $\beta_i + \beta_\ell \in \Phi^+$. So we get

$$\ell(x, v(\beta_i + \beta_\ell)) \geqslant 0 \text{ and } \langle \mu, v\beta_i + v\beta_\ell \rangle = -2.$$

This is impossible.

Note that $\langle \beta_\ell^\vee, \beta_{\ell-1} \rangle = 1$, as $\beta_{\ell-1}$ is the sum of β_ℓ with another root, and Φ is simply laced.

We thus may pick $\ell' \in \{1, \ldots, \ell - 1\}$ minimally such that $\langle \beta_\ell^\vee, \beta_{\ell'} \rangle > 0$. Consider the root sequence

$$\beta_i' = s_{\beta_\ell}(\beta_i), \quad i = 1, \ldots, \ell'.$$

This is a root sequence associated with $v' = vs_{\beta_\ell} \in \mathrm{LP}(x)$. Since $\beta_i' = \beta_i$ for $i = 1, \ldots, \ell' - 1$ (by choice of ℓ'), and $\beta_{\ell'}' < \beta_{\ell'}$, it is a smaller root sequence.

This is finally a contradiction to minimality. $\square$

The above proof encodes an algorithm, which finds for each root $\alpha \in \Phi$ with $\ell(x, \alpha) = 0$ and each $v \in \mathrm{LP}(x)$ a sequence of elements in $\mathrm{LP}(x)$ as in Lemma 2.14. The sequence starts at v and ending in an element $v' \in \mathrm{LP}(x)$ satisfying $(v')^{-1}\alpha \in \Phi^-$. As noted before, this Proposition is false for every non simply laced root system.

3. Quantum Bruhat graph

In this section, we recall the definition of quantum Bruhat graphs and study its weight functions. Before turning to the abstract theory of these graphs, we will discuss the situation of root systems of type A_n as a motivational example.

For each simple affine root $a = (\alpha, k) \in \Delta_{\mathrm{af}}$, we define a coweight $\omega_a \in \mathbb{Q}\Phi^\vee$ as follows: For $\beta \in \Delta$, we define

$$\langle \omega_a, \beta\rangle = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

In particular, $\omega_a = 0$ if $\alpha \notin \Delta$.

Let now $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$. By [BB05, Theorem 8.3.7], we have

$$x_1 \leqslant x_2 \iff \forall a, a' \in \Delta_{\mathrm{af}}:\ (\mu_1 + \omega_a - w_1^{-1}\omega_{a'})^{\mathrm{dom}} \leqslant (\mu_2 + \omega_a - w_2^{-1}\omega_{a'})^{\mathrm{dom}}.$$

Here, we write $\nu^{\mathrm{dom}} \in X_*$ for the unique dominant element in the W-orbit of $\nu \in X_*$.

Suppose that μ_1 and μ_2 are sufficiently regular, such that we find $v_1, v_2 \in W$ with

$$\forall a, a' \in \Delta_{\mathrm{af}}:\ (\mu_i + \omega_a - w_i^{-1}\omega_{a'})^{\mathrm{dom}} = v_i^{-1}(\mu_i + \omega_a - w_i^{-1}\omega_{a'}).$$

Then we conclude

$$\begin{aligned} x_1 \leqslant x_2 &\iff \forall a, a':\ v_1^{-1}(\mu_1 + \omega_a - w_1^{-1}\omega_{a'}) \leqslant v_2^{-1}(\mu_2 + \omega_a - w_2^{-1}\omega_{a'}) \\ &\iff v_1^{-1}\mu_1 + \sup_{a\in\Delta_{\mathrm{af}}}(v_1^{-1}\omega_a - v_2^{-1}\omega_a) + \sup_{a'\in\Delta_{\mathrm{af}}}\left((w_2v_2)^{-1}\omega_{a'} - (w_1v_1)^{-1}\omega_{a'}\right) \leqslant v_2^{-1}\mu_2. \end{aligned}$$

So if we define

$$\mathrm{wt}(v_1 \Rightarrow v_2) := \sup_{a\in\Delta_{\mathrm{af}}}(v_2^{-1}\omega_a - v_1^{-1}\omega_a), \tag{3.1}$$

we can conclude a version of our result on the Bruhat order (Theorem 1.1).

Indeed, formula (3.1) holds true for root systems of type A_n, but not for any other root system. Many properties of the weight function are easier to prove for type A_n, where an explicit formula exists, so it is helpful to keep this example in mind.

We refer to a paper of Ishii [Ish21] for explicit formulas for the weight functions of all classical root systems (while he discusses explicit criteria for the semi-infinite order, these can be translated to explicit formulas for the weight function as outlined above in the A_n case).

3.1. (Parabolic) quantum Bruhat graph

We start with a discussion of the quantum roots in Φ^+.

Lemma 3.2. *Let $\alpha \in \Phi^+$. Then*

$$\ell(s_\alpha) \leqslant \langle \alpha^\vee, 2\rho\rangle - 1.$$

Equality holds if and only if for all $\alpha \neq \beta \in \Phi^+$ with $s_\alpha(\beta) \in \Phi^-$, we have $\langle \alpha^\vee, \beta\rangle = 1$.

Roots satisfying the equivalent properties of Lemma 3.2 are called *quantum roots*. We see that all long roots are quantum (so in a simply laced root system, all roots are quantum). Moreover, all simple roots are quantum.

The first inequality of Lemma 3.2 is due to [BFP98, Lemma 4.3]. By [BMO11, Lemma 7.2], we have the following more explicit (but somehow less useful for us) result: A short root α is quantum if and only if α is a sum of short simple roots.

Proof of Lemma 3.2. We calculate

$$\langle\alpha^\vee, 2\rho\rangle = \frac{1}{2}\left(\langle\alpha^\vee, 2\rho\rangle + \langle s_\alpha(\alpha^\vee), s_\alpha(2\rho)\rangle\right) = \frac{1}{2}\langle\alpha^\vee, 2\rho - s_\alpha(2\rho)\rangle.$$

Let

$$I := \{\beta \in \Phi^+ \mid s_\alpha(\beta) \in \Phi^-\}.$$

Then $s_\alpha(I) = -I$ and $s_\alpha(\Phi^+\backslash I) = \Phi^+\backslash I$. It follows that

$$\begin{aligned} 2\rho - s_\alpha(2\rho) =& \sum_{\beta\in I}(\beta - s_\alpha(\beta)) + \sum_{\beta\in\Phi^+\backslash I}(\beta - s_\alpha(\beta)) \\ =& 2\sum_{\beta\in I}\beta. \end{aligned}$$

Therefore, we obtain

$$\langle\alpha^\vee, 2\rho\rangle = \sum_{\beta\in I}\langle\alpha^\vee, \beta\rangle.$$

Certainly, $\alpha \in I$. Hence

$$\langle\alpha^\vee, 2\rho\rangle = 2 + \sum_{\substack{\alpha\neq\beta\in\Phi^+ \\ s_\alpha(\beta)\in\Phi^-}}\langle\alpha^\vee, \beta\rangle.$$

Now if $\alpha, \beta \in \Phi^+$ and $s_\alpha(\beta) = \beta - \langle\alpha^\vee, \beta\rangle\alpha \in \Phi^-$, we get $\langle\alpha^\vee, \beta\rangle \geqslant 1$. We conclude

$$\langle\alpha^\vee, 2\rho\rangle = 2 + \sum_{\substack{\alpha\neq\beta\in\Phi^+ \\ s_\alpha(\beta)\in\Phi^-}}\langle\alpha^\vee, \beta\rangle \geqslant 2 + \#\{\beta \in \Phi^+\backslash\{\alpha\} \mid s_\alpha(\beta) \in \Phi^-\} = 1 + \ell(s_\alpha).$$

All claims of the lemma follow immediately from this. □

The parabolic quantum Bruhat graph as introduced by Lenart-Naito-Sagaki-Schilling-Schimozono [Len+15] is a generalization of the classical construction of the quantum Bruhat graph by Brenti-Fomin-Postnikov [BFP98]. To avoid redundancy, we directly state the definition of the parabolic quantum Bruhat graph, even though we will be mostly concerned with the (ordinary) quantum Bruhat graph.

Fix a subset $J \subseteq \Delta$. We denote by W_J the Coxeter subgroup of W generated by the reflections s_α for $\alpha \in J$. We let

$$W^J = \{w \in W \mid w(J) \subseteq \Phi^+\}.$$

For each $w \in W$, let $w^J \in W^J$ and $w_J \in W_J$ be the uniquely determined elements with $w = w^J \cdot w_J$ [BB05, Proposition 2.4.4].

We write $\Phi_J = W_J(J)$ for the root system generated by J. The sum of positive roots in Φ_J is denoted $2\rho_J$. The quotient lattice $\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$ is ordered by declaring $\mu_1 + \Phi_J^\vee \leqslant \mu_2 + \Phi_J^\vee$ if the difference $\mu_2 - \mu_1 + \Phi_J^\vee$ is equal to a sum of positive coroots modulo $\Phi_J^\vee$.

Definition 3.3. (a) The *parabolic quantum Bruhat graph* associated with W^J is a directed and $(\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee)$-weighted graph, denoted $\mathrm{QB}(W^J)$. The set of vertices is given by W^J. For $w_1, w_2 \in W^J$, we have an edge $w_1 \to w_2$ if there is a root $\alpha \in \Phi^+ \backslash \Phi_J$ such that $w_2 = (w_1 s_\alpha)^J$ and one of the following conditions is satisfied:

(B) $\ell(w_2) = \ell(w_1) + 1$ or

(Q) $\ell(w_2) = \ell(w_1) + 1 - \langle \alpha^\vee, 2\rho - 2\rho_J\rangle$.

Edges of type (B) are *Bruhat edges* and have weight $0 \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$. Edges of type (Q) are *quantum edges* and have weight $\alpha^\vee \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$.

(b) A *path* in $\mathrm{QB}(W^J)$ is a sequence of adjacent edges

$$p : w = w_1 \to w_2 \to \cdots \to w_{\ell+1} = w'.$$

The *length* of p is the number of edges, denoted $\ell(p) \in \mathbb{Z}_{\geqslant 0}$. The *weight* of p is the sum of its edges' weights, denoted $\mathrm{wt}(p) \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$. We say that p is a path *from w to w'*.

(c) If $w, w' \in W^J$, we define the *distance function* by

$$d_{\mathrm{QB}(W^J)}(w \Rightarrow w') = \inf\{\ell(p) \mid p \text{ is a path in } \mathrm{QB}(W^J) \text{ from } w \text{ to } w'\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

A path p from w to w' of length $d_{\mathrm{QB}(W^J)}(w \Rightarrow w')$ is called *shortest*.

(d) The *quantum Bruhat graph* of W is the parabolic quantum Bruhat graph associated with $J = \varnothing$, denoted $\mathrm{QB}(W) := \mathrm{QB}(W^\varnothing)$. We also shorten our notation to

$$d(w \Rightarrow w') := d_{\mathrm{QB}(W)}(w \Rightarrow w').$$

Remark 3.4. Let us consider the case $J = \varnothing$, i.e. the quantum Bruhat graph. If $w \in W$ and $\alpha \in \Delta$, then $w \to ws_\alpha$ is always an edge of weight $\alpha^\vee\Phi^+(-w\alpha)$.

The quantum edges are precisely the edges of the form $w \to ws_\alpha$ where α is a quantum root and $\ell(ws_\alpha) = \ell(w) - \ell(s_\alpha)$.

Proposition 3.5 ([Len+15, Proposition 8.1] and [Len+17, Lemma 7.2])**.** *Consider $w, w' \in W^J$.*

(a) The graph $\mathrm{QB}(W^J)$ *is strongly connected, i.e. there exists a path from* w *to* w' *in* $\mathrm{QB}(W^J)$*.*

(b) Any two shortest paths from w *to* w' *have the same weight, denoted*

$$\mathrm{wt}_{\mathrm{QB}(W^J)}(w \Rightarrow w') \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee.$$

(c) Any path p *from* w *to* w' *has weight* $\mathrm{wt}(p) \geqslant \mathrm{wt}_{\mathrm{QB}(W^J)}(w \Rightarrow w') \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$*.*

(d) The image of

$$\mathrm{wt}(w \Rightarrow w') := \mathrm{wt}_{\mathrm{QB}(W)}(w \Rightarrow w') \in \mathbb{Z}\Phi^\vee$$

under the canonical projection $\mathbb{Z}\Phi^\vee \to \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$ *is given by* $\mathrm{wt}_{\mathrm{QB}(W^J)}(w \Rightarrow w')$*.* □

One interpretation of the weight function is that it measures the failure of the inequality $w_1W_J \leqslant w_2W_J$ in the Bruhat order on W/W_J (cf. [BB05, Section 2.5]): Indeed, $w_1W_J \leqslant w_2W_J$ if and only if $\mathrm{wt}_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2) = 0$.

We have the following converse to part (c) of Proposition 3.5:

Lemma 3.6 (Cf. [MV20, Formula 4.3])**.** *Let* $w_1, w_2 \in W^J$*. For any path* p *from* w_1 *to* w_2 *in* $\mathrm{QB}(W^J)$*, we have*

$$\langle \mathrm{wt}(p), 2\rho - 2\rho_J\rangle = \ell(p) + \ell(w_1) - \ell(w_2).$$

In particular,

$$\langle \mathrm{wt}_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2), 2\rho - 2\rho_J\rangle = d_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2) + \ell(w_1) - \ell(w_2),$$

and p *is shortest if and only if* $\mathrm{wt}(p) = \mathrm{wt}_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2)$*.*

Proof. Note that if $p : w_1 \to w_2 = (w_1 s_\alpha)^J$ is an edge in $\mathrm{QB}(W^J)$, then by definition,

$$\ell(w_2) = \ell(w_1) + 1 - \langle \mathrm{wt}(p), 2\rho - 2\rho_J\rangle.$$

In general, iterate this observation for all edges of p. □

The weights of non-shortest paths do not add more information:

Lemma 3.7. *Let* $\mu \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$ *and* $w_1, w_2 \in W$*. Then* $\mu \geqslant \mathrm{wt}_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2)$ *if and only if there is a path* p *from* w_1 *to* w_2 *in* $\mathrm{QB}(W^J)$ *of weight* μ*.*

Proof. By part (d) of Proposition 3.5, it suffices to consider the case $J = \varnothing$, i.e. the quantum Bruhat graph.

The *if* condition is part (c) of Proposition 3.5. It remains to show the *only if* condition. Note that for each $w \in W$ and $\alpha \in \Delta$, we get a "silly path" of the form

$$w \to ws_\alpha \to w$$

in $\mathrm{QB}(W)$. Precisely one of the edges is quantum with weight $\alpha^\vee$, and the other one is Bruhat with weight 0.

If $\mu \geqslant \mathrm{wt}(w_1 \Rightarrow w_2)$, we may compose a shortest path from w_1 to w_2 with suitably chosen silly paths as above to obtain a path from w_1 to w_2 of weight μ. □

Lemma 3.8 ([Len+15, Lemma 7.7])**.** *Let $J \subseteq \Delta$, $w_1, w_2 \in W^J$ and $a = (\alpha, k) \in \Delta_{\mathrm{af}}$ such that $w_2^{-1}\alpha \in \Phi^-$.*

(a) We have an edge $(s_\alpha w_2)^J \to w_2$ in $\mathrm{QB}(W^J)$ of weight $-kw_2^{-1}\alpha^\vee \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$.

(b) If $w_1^{-1}\alpha \in \Phi^+$, then the above edge is part of a shortest path from w_1 to w_2, i.e.

$$d_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2) = d_{\mathrm{QB}(W^J)}(w_1 \Rightarrow (s_\alpha w_2)^J) + 1.$$

(c) If $w_1^{-1}\alpha \in \Phi^-$, we have

$$\begin{aligned} d_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2) =& d_{\mathrm{QB}(W^J)}((s_\alpha w_1)^J \Rightarrow (s_\alpha w_2)^J), \\ \mathrm{wt}_{\mathrm{QB}(W^J)}(w_1 \Rightarrow w_2) =& \mathrm{wt}_{\mathrm{QB}(W^J)}((s_\alpha w_1)^J \Rightarrow (s_\alpha w_2)^J) + k(w_1^{-1}\alpha^\vee - w_2^{-1}\alpha^\vee). \square \end{aligned}$$

We can use this lemma to reduce the calculation of weights $\mathrm{wt}(w_1 \Rightarrow w_2)$ to weights of the form $\mathrm{wt}(w \Rightarrow 1)$: If $w_2 \neq 1$, we find a simple root $\alpha \in \Delta$ with $w_2^{-1}\alpha \in \Phi^-$. Then

$$\begin{aligned} \mathrm{wt}(w_1 \Rightarrow w_2) &= \begin{cases} \mathrm{wt}(w_1 \Rightarrow s_\alpha w_2), & w_1^{-1}\alpha \in \Phi^+, \\ \mathrm{wt}(s_\alpha w_1 \Rightarrow s_\alpha w_2), & w_1^{-1}\alpha \in \Phi^-, \end{cases} \\ &= \mathrm{wt}(\min(w_1, s_\alpha w_1), s_\alpha w_2). \end{aligned}$$

For an alternative proof of this reduction, cf. [Sad21, Corollary 3.3].

The quantum Bruhat graph has a number of useful automorphisms.

Lemma 3.9. *Let $w_1, w_2 \in W$, and let $w_0 \in W$ be the longest element.*

(a) $\mathrm{wt}(w_0 w_1 \Rightarrow w_0 w_2) = \mathrm{wt}(w_2 \Rightarrow w_1)$.

(b) $\mathrm{wt}(w_0 w_1 w_0 \Rightarrow w_0 w_2 w_0) = -w_0 \,\mathrm{wt}(w_1 \Rightarrow w_2)$.

(c) $\mathrm{wt}(w_1 \Rightarrow 1) = \mathrm{wt}(w_1^{-1} \Rightarrow 1)$.

Proof. Part (a) follows from [Len+15, Proposition 4.3].

For part (b), observe that we have an automorphism of Φ given by $\alpha \mapsto -w_0\alpha$. The induced automorphism of W is given by $w \mapsto w_0 w w_0$. Since the function $\mathrm{wt}(\cdot \Rightarrow \cdot)$ is compatible with automorphisms of Φ, we get the claim.

Now for (c), consider a reduced expression

$$w_0 w_1 = s_1 \cdots s_q.$$

Then, iterating Lemma 3.8, we get

$$\begin{aligned} \mathrm{wt}(w_1 \Rightarrow 1) &\underset{\text{(a)}}{=} \mathrm{wt}(w_0 \Rightarrow w_0 w_1) = \mathrm{wt}(w_0 \Rightarrow s_1 \cdots s_q) \\ &= \mathrm{wt}(s_1 w_0 \Rightarrow s_2 \cdots s_q) = \cdots = \mathrm{wt}(s_q \cdots s_1 w_0 \Rightarrow 1) \\ &= \mathrm{wt}((w_0 w_1)^{-1} w_0 \Rightarrow 1) = \mathrm{wt}(w_1^{-1} \Rightarrow 1). \end{aligned}$$

$\square$

Given elements $w_1, w_2 \in W$, there are generally several shortest paths from w_1 to w_2 in $\mathrm{QB}(W)$. However, one can make a somewhat canonical choice:

Proposition 3.10 ([BFP98, Theorem 6.4])**.** *Let $u, v \in W$ and $\prec$ a reflection order on Φ^+. There is a uniquely determined path*

$$p : u = w_1 \to \cdots \to w_{\ell+1} = v, \qquad w_{i+1} = w_i s_{\alpha_i},\ \alpha_i \in \Phi^+$$

in $\mathrm{QB}(W)$ such that $\alpha_1 \prec \cdots \prec \alpha_\ell$ with respect to the fixed reflection order. Moreover, p is shortest. □

Corollary 3.11. *Let $J \subseteq \Delta$ and $w_1, w_2 \in W_J$. Then*

$$\mathrm{wt}_{\mathrm{QB}(W)}(w_1 \Rightarrow w_2) = \mathrm{wt}_{\mathrm{QB}(W_J)}(w_1 \Rightarrow w_2) \in \mathbb{Z}\Phi_J.$$

Proof. Pick a reflection order $\prec$ on Φ_J and extend it to a reflection order on Φ. Now if p is the unique path from w_1 to w_2 in $\mathrm{QB}(W_J)$ that is increasing with respect to this order, p is shortest both in $\mathrm{QB}(W_J)$ and $\mathrm{QB}(W)$ by the proposition. □

Remark 3.12. As an application and illustration of the introduced methods, we show how to compute the weight $\mathrm{wt}(w_0 \Rightarrow 1)$ where $w_0 \in W$ is the longest element.

Denote by $\theta \in \Phi^+$ the longest root of some irreducible component of Φ. Then $(-\theta, 1) \in \Delta_{\mathrm{af}}$. By Lemma 3.8,

$$\begin{aligned}\mathrm{wt}(w_0 \Rightarrow 1) &= -\, w_0\theta^\vee + \mathrm{wt}(s_\theta w_0 \Rightarrow 1)\\ &= \theta^\vee + \mathrm{wt}(s_\theta w_0 \Rightarrow 1).\end{aligned}$$

Define $J = \{\alpha \in \Delta \mid \langle \theta^\vee, \alpha\rangle = 0\}$. Then

$$\begin{aligned}\{\alpha \in \Phi^+ \mid (s_\theta w_0)(\alpha) \in \Phi^-\} &= w_0\{\alpha \in \Phi^- \mid s_\theta(\alpha) \in \Phi^-\}\\ &= w_0\{\alpha \in \Phi^- \mid \langle \theta^\vee, \alpha\rangle = 0\}\\ &= w_0(\Phi_J \cap \Phi^-) = \Phi_J \cap \Phi^+.\end{aligned}$$

We see that $s_\theta w_0 \in W_J$ is the longest element, so

$$\mathrm{wt}(w_0 \Rightarrow 1) = \theta^\vee + \mathrm{wt}_{\mathrm{QB}(W_J)}(w_0(J) \Rightarrow 1).$$

We can iterate this process for the smaller root system Φ_J to compute $\mathrm{wt}(w_0 \Rightarrow 1)$. For explicit results, we refer to [Sad21, Section 5].

3.2. Lifting the parabolic quantum Bruhat graph

For sufficiently regular elements of the extended affine Weyl group, the Bruhat covers in $\widetilde{W}$ are in a one-to-one correspondence with edges in the quantum Bruhat graph [LS10, Proposition 4.4]. This result is very useful for deriving properties about the quantum Bruhat graph. Moreover, our strategy to prove our results on the Bruhat order will be to reduce to this superregular case.

The result of Lam and Shimozono has been generalized by Lenart et. al. [Len+15, Theorem 5.2], and the extra generality of the latter result will be useful for us. Throughout this section, let $J \subseteq \Delta$ be any subset.

Definition 3.13 ([Len+15])**.** (a) Define

$$\begin{aligned}(W^J)_{\mathrm{af}} :=& \{x \in W_{\mathrm{af}} \mid \forall \alpha \in \Phi_J :\ \ell(x,\alpha) = 0\},\\ \widetilde{(W^J)} :=& \{x \in \widetilde{W} \mid \forall \alpha \in \Phi_J :\ \ell(x,\alpha) = 0\}.\end{aligned}$$

(b) Let $C > 0$ be any real number. We define Ω_J^{-C} to be the set of all elements $x = w\varepsilon^\mu \in \widetilde{(W^J)}$ such that

$$\forall \alpha \in \Phi^+ \backslash \Phi_J :\ \langle \mu, \alpha\rangle \leqslant -C.$$

Similarly, we say $x \in \Omega_J^C$ if

$$\forall \alpha \in \Phi^+ \backslash \Phi_J :\ \langle \mu, \alpha\rangle \geqslant C.$$

(c) For elements $x, x' \in \widetilde{W}$, we write $x \lessdot x'$ and call x' a *Bruhat cover* of x if $\ell(x') = \ell(x) + 1$ and $x^{-1}x'$ is an affine reflection in $\widetilde{W}$.

Theorem 3.14 ([Len+15, Theorem 5.2])**.** *There is a constant $C > 0$ depending only on Φ such that the following holds:*

(a) If $x = w\varepsilon^\mu \lessdot x' = w'\varepsilon^{\mu'}$ is a Bruhat cover with $x \in \Omega_J^{-C}$ and $x' \in \widetilde{(W^J)}$, there exists an edge $(w')^J \to w^J$ in $\mathrm{QB}(W^J)$ of weight $\mu - \mu' + \mathbb{Z}\Phi_J^\vee$.

(b) If $x = w\varepsilon^\mu \in \Omega_J^{-C}$ and $\tilde{w}' \to w^J$ is an edge in $\mathrm{QB}(W^J)$ of weight ω, then there exists a unique element $x \lessdot x' = w'\varepsilon^{\mu'} \in \widetilde{(W^J)}$ with $\tilde{w}' = (w')^J$ and $\mu \equiv \mu' + \omega \pmod{\mathbb{Z}\Phi_J^\vee}$. □

This theorem "lifts" $\mathrm{QB}(W^J)$ into the Bruhat covers of Ω_J^{-C} for sufficiently large C.

The theorem is originally formulated only for $(W^J)_{\mathrm{af}}$, but the generalization to $\widetilde{(W^J)}$ is straightforward.

With a bit of book-keeping, we can compare paths in $\mathrm{QB}(W^J)$ (i.e. sequences of edges) with the Bruhat order on Ω_J^{-C} (i.e. sequences of Bruhat covers).

Lemma 3.15. *Let $C_1 > 0$ be any real number. Then there exists some $C_2 > 0$ such that for all $x = w\varepsilon^\mu \in \Omega_J^{C_2}$ and $x' = w'\varepsilon^{\mu'} \in \widetilde{(W^J)}$ with $\ell(x^{-1}x') \leqslant C_1$, we have*

$$x \leqslant x' \iff \mu - \mathrm{wt}(w' \Rightarrow w) \leqslant \mu' \pmod{\Phi_J^\vee}.$$

The latter condition is shorthand for

$$\mu - \mathrm{wt}(w' \Rightarrow w) - \mu' + \mathbb{Z}\Phi_J^\vee \leqslant 0 + \mathbb{Z}\Phi_J^\vee \in \mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee.$$

Proof. Let $C > 0$ be a constant sufficiently large for the conclusion of Theorem 3.14 to hold. We see that if $x_1 \lessdot x_2$ is any cover in Ω_J^{-C}, then there are only finitely many possibilities for $x_1^{-1}x_2$, so the length $\ell(x_1^{-1}x_2)$ is bounded. We fix a bound $C' > 0$ for this length.

We can pick $C_2 > 0$ such that for all $x_1 = w\varepsilon^\mu \in \Omega_J^{-C_2}$ and $x_2 \in \widetilde{W}^J$ with $\ell(x_1^{-1}x_2) \leqslant C_1C'$, we must at least have $x_2 \in \Omega_J^{-C}$.

We now consider elements $x = w\varepsilon^\mu \in \Omega_J^{-C_2}$ and $x' = w'\varepsilon^{\mu'} \in \widetilde{W}^J$ with $\ell(x^{-1}x') \leqslant C_1$. First suppose that $x \leqslant x'$. We find elements $x = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k = x'$. Note that $k = \ell(x') - \ell(x) \leqslant \ell(x^{-1}x') \leqslant C_1$. By choice of C', we conclude that $\ell(x^{-1}x_i) \leqslant C'i \leqslant C'C_1$ for $i = 1, \dots, k$. Thus $x_i \in \Omega_J^{-C}$.

By Theorem 3.14, we get a path from $(w')^J$ to w^J of weight $\mu - \mu' + \mathbb{Z}\Phi_J^\vee$. Thus

$$\mathrm{wt}(w_1 \Rightarrow w_2) \leqslant \mu - \mu' \pmod{\Phi_J^\vee},$$

which is the estimate we wanted to prove.

Now suppose conversely that we are given $\mu - \mathrm{wt}(w' \Rightarrow w) \geqslant \mu' \pmod{\Phi_J^\vee}$. By Lemma 3.7, we find a path $(w')^J = w_1 \to w_2 \to \cdots \to w_k = w^J$ in $\mathrm{QB}(W^J)$ of weight $\mu - \mu' + \mathbb{Z}\Phi_J^\vee$. Since $\mu - \mu'$ is bounded in terms of C_1, the length k of this path is bounded in terms of C_1 as well. By adding another lower bound for C_2, we can guarantee that each such path $w_1 \to \cdots \to w_k$ can indeed be lifted to Ω_J^{-C}, proving that $x \leqslant x'$. □

We find working with superdominant instead superantidominant coweights a bit easier, so let us restate the lemma for Ω_J^C instead of Ω_J^{-C}.

Corollary 3.16. *Let $C_1 > 0$ be any real number. Then there exists some $C_2 > 0$ such that for all $x = w\varepsilon^\mu \in \Omega_J^{C_2}$ and $x' = w'\varepsilon^{\mu'} \in \widetilde{(W^J)}$ with $\ell(x^{-1}x') \leqslant C_1$, we have*

$$x \leqslant x' \iff \mu + \mathrm{wt}(w \Rightarrow w') \leqslant \mu' \pmod{\Phi_J^\vee}.$$

Proof. Let $w_0(J) \in W_J$ be the longest element. Let $C_2 > 0$ such that the conclusion of the previous Lemma is satisfied.

If $x \in \Omega_J^{C_2}$, then $xw_0(J)w_0 \in \Omega_{-w_0(J)}^{-C_2}$. Moreover, $w_0(J)w_0$ is a length positive element for x, so $\ell(xw_0(J)w_0) = \ell(x) + \ell(w_0(J)w_0)$. Choosing C_2 appropriately, we similarly may assume $x' \in \Omega_J^C$ for some $C > 0$ and obtain $\ell(x'w_0(J)w_0) = \ell(x') + \ell(w_0(J)w_0)$. Then, with the right choice of constants and using the automorphism $\alpha \mapsto -w_0\alpha$ of Φ, we get

$$\begin{aligned}
x \leqslant x' &\iff xw_0(J)w_0 \leqslant x'w_0(J)w_0\\
&\iff w_0w_0(J)\mu - \mathrm{wt}(w'w_0(J)w_0 \Rightarrow ww_0(J)w_0) \geqslant w_0w_0(J)\mu' \pmod{\Phi_{-w_0(J)}^\vee}\\
&\iff w_0(J)\mu + \mathrm{wt}(w_0w'w_0(J) \Rightarrow w_0ww_0(J)) \leqslant w_0(J)\mu' \pmod{\Phi_J^\vee}\\
&\underset{\text{[Len+15, Pr. 4.3]}}{\iff} w_0(J)\mu + \mathrm{wt}(w \Rightarrow w') \leqslant w_0(J)\mu' \pmod{\Phi_J^\vee}
\end{aligned}$$

Since $w_0(J)\mu \equiv \mu \pmod{\Phi_J^\vee}$, we get the desired conclusion. □

As an immediate consequence, we obtain a crucial estimate on the weight function.

Corollary 3.17. *Let $w \in W$ and $\alpha \in \Phi^+$. Then*

$$\mathrm{wt}(ws_\alpha \Rightarrow w) \leqslant \Phi^+(w\alpha)\alpha^\vee.$$

Proof. The claim is clear if $w\alpha \in \Phi^-$, as then $ws_\alpha < w$ in the Bruhat order, and we find a path from ws_α to w consisting solely of Bruhat edges.

Now suppose that $w\alpha \in \Phi^+$. Let $\mu \in Q^\vee$ be dominant and superregular. Put $x := w\varepsilon^\mu$. Then $x(\alpha, \langle \mu, \alpha\rangle - 1) \in \Phi_{\mathrm{af}}^-$, so that

$$w\varepsilon^\mu = x > w\varepsilon^\mu s_\alpha \varepsilon^{(\langle \mu,\alpha\rangle - 1)\alpha^\vee} = ws_\alpha \varepsilon^{\mu - \alpha^\vee}.$$

With the superregularity constant for μ sufficiently large, we get

$$\mu - \alpha^\vee + \mathrm{wt}(ws_\alpha \Rightarrow w) \leqslant \mu,$$

showing the desired claim. □

3.3. Computing the weight function

We already saw in Lemma 3.8 how to find for all $w_1, w_2 \in W$ an element $w \in W$ such that $\mathrm{wt}(w_1 \Rightarrow w_2) = \mathrm{wt}(w \Rightarrow 1)$. It remains to find a method to compute these weights. First, we note that we only need to consider quantum edges for this task.

Lemma 3.18 ([MV20, Proposition 4.11])**.** *For each $w \in W$, there is a shortest path from w to 1 in $\mathrm{QB}(W)$ consisting only of quantum edges.* □

So we only need to find for each $w \in W\backslash\{1\}$ a quantum edge $w \to w'$ in $\mathrm{QB}(W)$ with $d(w' \Rightarrow 1) = d(w \Rightarrow 1) - 1$. In this section, we present a new method to obtain such edges. If it happens that $w^{-1}\theta \in \Phi^-$ for the longest root θ of an irreducible component of Φ, we can use the quantum edge $w \to s_\theta w$ by Lemma 3.8. We even might strengthen this a bit using Corollary 3.11. If this method would always work, we could compute the weight $\mathrm{wt}(w \Rightarrow 1)$ as in Remark 3.12. However, there are in general elements $w \in W$ where this strategy is not applicable.

In this section, we show that the aforementioned strategy will still work whenever θ is any maximal element in $\{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$. This yields a general algorithm and useful theoretical method to describe some quantum edges $w \to w'$ with the desired property $d(w' \Rightarrow 1) = d(w \Rightarrow 1) - 1$.

We remark that not every shortest path $w \Rightarrow 1$ will consist only of quantum edges, nor will every shortest path that does be obtainable by our method of maximal inversions.

Definition 3.19. Let $w \in W$.

(a) The set of *inversions* of w is

$$\mathrm{inv}(w) := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}.$$

(b) An inversion $\gamma \in \mathrm{inv}(w)$ is a *maximal inversion* if there is no $\alpha \in \mathrm{inv}(w)$ with $\alpha \neq \gamma \leqslant \alpha$. Here, $\gamma \leqslant \alpha$ means that $\alpha - \gamma$ is a sum of positive roots.

We write $\max \mathrm{inv}(w)$ for the set of maximal inversions of w.

Remark 3.20. If $\theta \in \operatorname{inv}(w)$ is the longest root of an irreducible component of Φ, then certainly $\theta \in \max\operatorname{inv}(w)$. In this case, everything we want to prove is already shown in [Len+15, Section 5.5]. Our strategy is to follow their arguments as closely as possible while keeping the generality of maximal inversions.

Lemma 3.21. *Let $w \in W$ and $\gamma \in \max\operatorname{inv}(w)$. Then $w \to s_\gamma w$ is a quantum edge.*

Proof. Note that $s_\gamma w = w s_{-w^{-1}\gamma}$. We have to show that $-w^{-1}\gamma$ is a quantum root and that

$$\ell(w s_{-w^{-1}\gamma}) = \ell(w) - \ell(s_{-w^{-1}\gamma}).$$

Step 1. We show that $-w^{-1}\gamma$ is a quantum root using Lemma 3.2. So pick an element $-w^{-1}\gamma \neq \beta \in \Phi^+$ with $s_{-w^{-1}\gamma}(\beta) \in \Phi^-$. We want to show that $\langle -w^{-1}\gamma^\vee, \beta\rangle = 1$.

Note that

$$s_{-w^{-1}\gamma}(\beta) = \beta + \langle -w^{-1}\gamma^\vee, \beta\rangle w^{-1}\gamma.$$

In particular, $k := \langle -w^{-1}\gamma^\vee, \beta\rangle > 0$. It follows from the theory of root systems that

$$\beta_i := \beta + i w^{-1}\gamma \in \Phi, \qquad i = 0, \dots, k.$$

Since $\beta_0 = \beta \in \Phi^+$ and $\beta_k = s_{-w^{-1}\gamma}(\beta) \in \Phi^-$, we find some $i \in \{0, \dots, k-1\}$ with $\beta_i \in \Phi^+$ and $\beta_{i+1} \in \Phi^-$. We show that $k \leqslant 1$ as follows:

- Suppose $w\beta_i \in \Phi^+$. Then $w\beta_{i+1} = w\beta_i + \gamma > \gamma$. In particular, $w\beta_{i+1} \in \Phi^+$. We see that $w\beta_{i+1} \in \operatorname{inv}(w)$, contradicting maximality of γ.
- Suppose $w\beta_{i+1} \in \Phi^-$. Then $-w\beta_i = -w\beta_{i+1} + \gamma > \gamma$. In particular, $-w\beta_i \in \Phi^+$. We see that $-w\beta_i \in \operatorname{inv}(w)$, contradicting maximality of γ.
- Suppose $i \geqslant 1$. Then $\gamma - w\beta_i = -w\beta_{i-1} \in \Phi$. We already proved $w\beta_i \in \Phi^-$, so $-w\beta_i \in \operatorname{inv}(w)$. Since also $\gamma \in \operatorname{inv}(w)$, we conclude $\gamma < -w\beta_{i-1} \in \operatorname{inv}(w)$, contradicting the maximality of γ.
- Suppose $i \leqslant k-2$. Then $w\beta_{i+2} = w\beta_{i+1} + \gamma \in \Phi$. Since both γ and $w\beta_{i+1}$ are in $\operatorname{inv}(w)$, we conclude that $\gamma < w\beta_{i+2} \in \operatorname{inv}(w)$, which is a contradiction to the maximality of γ.

In summary, we conclude $0 = i \geqslant k-1$, thus $k \leqslant 1$. This shows $\langle -w^{-1}\gamma^\vee, \beta\rangle = 1$.

Step 2. We show that

$$\ell(w s_{-w^{-1}\gamma}) = \ell(w) - \ell(s_{-w^{-1}\gamma}).$$

Suppose this is not the case. Then we find some $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^+$ and $s_{-w^{-1}\gamma}(\alpha) \in \Phi^-$. As we saw before, $\langle -w^{-1}\gamma^\vee, \alpha\rangle = 1$, so $s_{-w^{-1}\gamma}(\alpha) = \alpha + w^{-1}\gamma \in \Phi^-$. Now consider the element $w s_{-w^{-1}\gamma}(\alpha) = w\alpha + \gamma \in \Phi$. Since $w\alpha \in \Phi^+$ by assumption, we have $w s_{-w^{-1}\gamma}(\alpha) > \gamma$, in particular $w s_{-w^{-1}\gamma}(\alpha) \in \Phi^+$. We conclude $w s_{-w^{-1}\gamma}(\alpha) \in \operatorname{inv}(w)$, yielding a final contradiction to the maximality of γ. $\square$

Lemma 3.22. *Let $w \in W$ and $\alpha \in \Phi^+$ such that $w \to ws_\alpha$ is a quantum edge. Let moreover $-w\alpha \neq \gamma \in \max \operatorname{inv}(w)$. Then $\gamma \in \max \operatorname{inv}(ws_\alpha)$ and $\langle -w^{-1}\gamma^\vee, \alpha\rangle \geqslant 0$.*

Proof. We first show $\gamma \in \operatorname{inv}(ws_\alpha)$, i.e. $s_\alpha w^{-1}\gamma \in \Phi^-$.

Aiming for a contradiction, we thus suppose that

$$s_\alpha(-w^{-1}\gamma) = \langle \alpha^\vee, w^{-1}\gamma\rangle\alpha - w^{-1}\gamma \in \Phi^-.$$

Then $-w^{-1}\gamma$ is a positive root whose image under s_α is negative. Since α is quantum, we conclude $\langle \alpha^\vee, -w^{-1}\gamma\rangle = 1$. Thus $-\alpha - w^{-1}\gamma \in \Phi^-$. Consider the element

$$w(\alpha + w^{-1}\gamma) = \gamma + w\alpha \in \Phi.$$

We distinguish the following cases:

- If $\gamma + w\alpha \in \Phi^-$, we get $\gamma < -w\alpha \in \operatorname{inv}(w)$, contradicting maximality of γ.
- If $\gamma + w\alpha \in \Phi^+$, we compute

  $$ws_\alpha(-w^{-1}\gamma) = -(ws_\alpha w^{-1})\gamma = -s_{w\alpha}(\gamma) = -(\gamma + w\alpha) \in \Phi^-.$$

  In other words, the positive root $-w^{-1}\gamma \in \Phi^+$ gets mapped to negative roots both by s_α and by $ws_\alpha \in W$. This is a contradiction to $\ell(w) = \ell(ws_\alpha) + \ell(s_\alpha)$ (since $w \to ws_\alpha$ was supposed to be a quantum edge).

In any case, we get a contradiction. Thus $\gamma \in \operatorname{inv}(ws_\alpha)$.

The quantum edge condition $w \to ws_\alpha$ implies $\ell(w) = \ell(ws_\alpha) + \ell(s_\alpha)$, so $\operatorname{inv}(ws_\alpha) \subset \operatorname{inv}(w)$. Because γ is maximal in $\operatorname{inv}(w)$ and $\gamma \in \operatorname{inv}(ws_\alpha) \subseteq \operatorname{inv}(w)$, it follows that γ must be maximal in $\operatorname{inv}(ws_\alpha)$ as well.

Finally, we have to show $\langle -w^{-1}\gamma^\vee, \alpha\rangle \geqslant 0$. If this was not the case, then we would get

$$\gamma < s_\gamma(-w\alpha) = -w\alpha + \langle w^{-1}\gamma^\vee, \alpha\rangle\gamma \in \operatorname{inv}(w),$$

again contradicting maximality of γ. □

Proposition 3.23. *Let $w \in W$ and $\gamma \in \max \operatorname{inv}(w)$. Then*

$$\operatorname{wt}(w \Rightarrow 1) = \operatorname{wt}(s_\gamma w \Rightarrow 1) - w^{-1}\gamma^\vee.$$

Proof. Since the estimate

$$\begin{aligned}\operatorname{wt}(w \Rightarrow 1) &\leqslant \operatorname{wt}(w \Rightarrow s_\gamma w) + \operatorname{wt}(s_\gamma w \Rightarrow 1)\\ &\leqslant -w^{-1}\gamma^\vee + \operatorname{wt}(s_\gamma w \Rightarrow 1)\end{aligned}$$

follows from Corollary 3.17, all we have to show is the inequality "$\geqslant$".

For this, we use induction on $\ell(w)$. If $1 \neq w \in W$, we find by Lemma 3.18 some quantum edge $w \to ws_\alpha$ with $\operatorname{wt}(w \Rightarrow 1) = \operatorname{wt}(ws_\alpha \Rightarrow 1) + \alpha^\vee$. If $\alpha = -w^{-1}\gamma$, we are done.

Otherwise, $\gamma \in \max \operatorname{inv}(ws_\alpha)$ and $\langle -w^{-1}\gamma^\vee, \alpha\rangle \geqslant 0$ by the previous lemma. By induction, we have

$$\begin{aligned} \mathrm{wt}(w \Rightarrow 1) &= \mathrm{wt}(ws_\alpha \Rightarrow 1) + \alpha^\vee \\ &= \mathrm{wt}(s_\gamma w s_\alpha \Rightarrow 1) + \alpha^\vee - (ws_\alpha)^{-1}\gamma^\vee. \end{aligned} \tag{3.24}$$

By Lemma 3.21, we get the following three quantum edges:

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ ws_\alpha \arrow[dr] & & s_\gamma w \\ & s_\gamma w s_\alpha & \end{tikzcd}$$

This allows for the following computation:

$$\begin{aligned} \ell(s_\gamma w s_\alpha) &= \ell(ws_\alpha) + 1 - \langle -(ws_\alpha)^{-1}\gamma^\vee, 2\rho\rangle \\ &= \ell(w) + 2 - \langle \alpha^\vee, 2\rho\rangle - \langle -w^{-1}\gamma^\vee - \langle -w^{-1}\gamma^\vee, \alpha\rangle\alpha^\vee, 2\rho\rangle \\ &= \ell(s_\gamma w) + 1 + (\langle -w^{-1}\gamma^\vee, \alpha\rangle - 1)\langle \alpha^\vee, 2\rho\rangle. \end{aligned} \tag{3.25}$$

We now distinguish several cases depending on the value of $\langle -w^{-1}\gamma^\vee, \alpha\rangle \in \mathbb{Z}_{\geqslant 0}$.

- Case $\langle -w^{-1}\gamma^\vee, \alpha\rangle = 0$. In this case, we get a quantum edge $s_\gamma w \to s_\gamma w s_\alpha$ by (3.25). Evaluating this in (3.24), we get

$$\begin{aligned} \mathrm{wt}(w \Rightarrow 1) &= \mathrm{wt}(s_\gamma w s_\alpha \Rightarrow 1) + \alpha^\vee - (ws_\alpha)^{-1}\gamma^\vee \\ &\geqslant \mathrm{wt}(s_\gamma w \Rightarrow 1) - s_\alpha w^{-1}\gamma^\vee \\ &= \mathrm{wt}(s_\gamma w \Rightarrow 1) - w^{-1}\gamma^\vee. \end{aligned}$$

- Case $\langle -w^{-1}\gamma^\vee, \alpha\rangle = 1$. In this case, we get a Bruhat edge $s_\gamma w \to s_\gamma w s_\alpha$ by (3.25). Evaluating this in (3.24), we get

$$\begin{aligned} \mathrm{wt}(w \Rightarrow 1) &= \mathrm{wt}(s_\gamma w s_\alpha \Rightarrow 1) + \alpha^\vee - (ws_\alpha)^{-1}\gamma^\vee \\ &\geqslant \mathrm{wt}(s_\gamma w \Rightarrow 1) + \alpha^\vee - s_\alpha w^{-1}\gamma^\vee \\ &= \mathrm{wt}(s_\gamma w \Rightarrow 1) - w^{-1}\gamma^\vee. \end{aligned}$$

- Case $\langle -w^{-1}\gamma^\vee, \alpha\rangle \geqslant 2$. We get

$$\begin{aligned} \ell(s_\gamma w s_\alpha) &\leqslant \ell(s_\gamma w) + \ell(s_\alpha) \underset{\text{L3.2}}{\leqslant} \ell(s_\gamma w) + \langle \alpha^\vee, 2\rho\rangle - 1 \\ &< \ell(s_\gamma w) + \ell(s_\alpha) \leqslant \ell(s_\gamma w) + 1 + \big(\langle -w^{-1}\gamma^\vee, \alpha\rangle - 1\big)\langle \alpha^\vee, 2\rho\rangle. \end{aligned}$$

  This is a contradiction to (3.25).

In any case, we get a contradiction or the required conclusion, finishing the proof. □

Remark 3.26. (a) By Lemma 3.6, it follows that concatenating the quantum edge $w \to s_\gamma w$ with a shortest path $s_\gamma w \Rightarrow 1$ yields indeed a shortest path from w to 1. Thus, iterating Proposition 3.23, we get a shortest path from w to 1.

(b) If $w \in W^J$ and $\gamma \in \max\operatorname{inv}(w)$, we do not in general have a quantum edge $w \to (s_\gamma w)^J$ in $\mathrm{QB}(W^J)$. However, we can concatenate a shortest path from w to $(s_\gamma w)^J$ (which will have weight $-w^{-1}\gamma^\vee + \mathbb{Z}\Phi_J^\vee$) with a shortest path from $(s_\gamma w)^J$ to 1 in $\mathrm{QB}(W^J)$ to get a shortest path from w to 1.

3.4. Semi-affine quotients

We saw that for $w_1, w_2 \in W$ and $J \subseteq \Delta$, we can assign a weight to the cosets $w_1 W_J$ and $w_2 W_J$ in $\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$. In this section, we consider left cosets $W_J w$ instead. This is pretty straightforward if $J \subseteq \Delta$; however, it is more interesting if J is instead allowed to be a subset of Δ_{af}. The quotient of the finite Weyl group by a set of simple affine roots will be called *semi-affine* quotient.

In this section, we introduce the resulting *semi-affine weight function.* This new function generalizes properties of the ordinary weight function. We have the following two motivations to study it:

- For root systems of type A_n, we can explicitly express the weight function using formula (3.1):

$$\mathrm{wt}(v_2 \Rightarrow v_1) = \sup_{a \in \Delta_{\mathrm{af}}} (v_2^{-1}\omega_a - v_1^{-1}\omega_a).$$

  Using the semi-affine weight function, we can prove a generalization of this formula, expressing the weight $\mathrm{wt}(v_2 \Rightarrow v_1)$ as a supremum of semi-affine weights (Lemmas 3.36 and 4.37)

- There is a close relationship between the quantum Bruhat graph and the Bruhat order of the extended affine Weyl group $\widetilde{W}$. Now *Deodhar's lemma* [Deo77] is an important result on the Bruhat order of general Coxeter groups. Translating the statement of Deodhar's lemma to the quantum Bruhat graph yields exactly the semi-affine weight function.

  Conversely, using the semi-affine weight function and Deodhar's lemma, we can generalize our result on the Bruhat order in Section 4.3.

In this thesis, the results of this section are only used in Section 4.3, whose results are not used later. A reader who is not interested in the aforementioned applications is thus invited to skip these two sections.

Definition 3.27. Let $J \subseteq \Delta_{\mathrm{af}}$ be any subset.

(a) If $\mathbf{a} = (\alpha, k) \in \Delta_{\mathrm{af}}$, we define $\omega_{\mathbf{a}} : \mathbb{Z}\Phi \to \mathbb{Z}$ to be the $\mathbb{Z}$-linear function with

$$\forall \beta \in \Delta : \ \omega_{\mathbf{a}}(\beta) = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}$$

(b) We denote by Φ_J the root system generated by the roots

$$\mathrm{cl}\, J := \{\mathrm{cl}(a) \mid a \in J\} = \{\alpha \mid (\alpha, k) \in J\}.$$

(c) We denote by W_J the Weyl group of the root system Φ_J, i.e. the subgroup of W generated by $\{s_\alpha \mid \alpha \in \mathrm{cl}\, J\}$.

(d) Similarly, we denote by $(\Phi_{\mathrm{af}})_J \subseteq \Phi_J$ the (affine) root system generated by J, and by $\widetilde{W}_J$ the Coxeter subgroup of W_{af} generated by the reflections r_a with $a \in J$.

(e) We say that J is a *regular* subset of Δ_{af} if no connected component of the affine Dynkin diagram of Φ_{af} is contained in J, i.e. if $\widetilde{W}_J$ is finite.

Lemma 3.28. *Let $J \subseteq \Delta_{\mathrm{af}}$ be a regular subset.*

(a) $\mathrm{cl}\, J$ *is a basis of Φ_J. The map $(\Phi_{\mathrm{af}})_J \to \Phi_J, (\alpha, k) \mapsto \alpha$ is bijective.*

(b) Writing Φ_J^+ for the positive roots of Φ_J with respect to the basis $\mathrm{cl}\, J$*, we get a bijection*

$$\Phi_J^+ \to (\Phi_{\mathrm{af}})_J^+, \quad \alpha \mapsto (\alpha, \Phi^+(-\alpha)).$$

Proof. (a) Consider the Cartan matrix

$$C_{\alpha,\beta} := \langle \alpha^\vee, \beta \rangle, \qquad \alpha, \beta \in \mathrm{cl}\, J.$$

This must be the Cartan matrix associated to a certain Dynkin diagram, namely the subdiagram of the affine Dynkin diagram of Φ_{af} with set of nodes given by J. We know that this must coincide with the Dynkin diagram of a finite root system by regularity of J. Hence, $C_{\bullet,\bullet}$ is the Cartan matrix of a finite root system. Both claims follow immediately from this observation.

(b) Let φ denote the map

$$\varphi : \Phi_J^+ \to \Phi_{\mathrm{af}}^+, \qquad \alpha \mapsto (\alpha, \Phi^+(-\alpha)).$$

By (a), the map is injective. For each root $\alpha \in \mathrm{cl}(J)$, we certainly have $\varphi(\alpha) \in \Phi_J^+$.

Now, for an inductive argument, suppose that $\alpha \in \Phi_J^+, \beta \in \mathrm{cl}(J)$ and $\alpha + \beta \in \Phi$ satisfy $\varphi(\alpha) \in \Phi_J^+$. We want to show that $\varphi(\alpha + \beta) \in \Phi_J^+$.

We have $(\alpha, \Phi^+(-\alpha)), (\beta, \Phi^+(-\beta)) \in \Phi_J^+$, hence

$$(\alpha + \beta, \Phi^+(-\alpha) + \Phi^+(-\beta)) \in \Phi_J^+.$$

Hence it suffices to show that $\Phi^+(-\alpha) + \Phi^+(-\beta) = \Phi^+(-\alpha - \beta)$.

If $\beta \in \Delta$, this is clear. Hence we may assume that $\beta = -\theta$, where θ is the longest root of the irreducible component of Φ containing α, β. Then $\alpha - \theta \in \Phi$ implies $\alpha \in \Phi^+$ and $\alpha - \theta \in \Phi^-$. We see that $\Phi^+(-\alpha) + \Phi^+(\theta) = \Phi^+(-\alpha + \theta)$ holds true. $\square$

The parabolic subgroup $\widetilde{W}_J \subseteq W_{\mathrm{af}}$ allows the convenient decomposition of W_{af} as $W_{\mathrm{af}} = \widetilde{W}_J \cdot {}^J W_{\mathrm{af}}$ [BB05, Proposition 2.4.4]. We get something similar for $W_J \subseteq W$.

Definition 3.29. Let $J \subseteq \Delta_{\mathrm{af}}$.

(a) By Φ_J^+, we denote the set of positive roots in Φ_J with respect to the basis $\mathrm{cl}(J)$. By abuse of notation, we also use Φ_J^+ as the symbol for the indicator function of Φ_J^+, i.e.

$$\Phi_J^+(\alpha) := \begin{cases} 1, & \alpha \in \Phi_J^+, \\ 0, & \alpha \in \Phi \backslash \Phi_J^+. \end{cases}.$$

(b) We define

$$\begin{aligned} {}^J W :=& \{w \in W \mid \forall b \in J : w^{-1}\,\mathrm{cl}(b) \in \Phi^+\} \\ =& \{w \in W \mid \forall \beta \in \Phi_J^+ : w^{-1}\beta \in \Phi^+\}. \end{aligned}$$

(c) For $w \in W$, we put

$${}^J\ell(w) := \#\{\beta \in \Phi_J^+ \mid w^{-1}\beta \in \Phi^-\}.$$

Lemma 3.30. *If $w \in W$ and $\beta \in \Phi_J^+$ satisfy $w^{-1}\beta \in \Phi^-$, then*

$${}^J\ell(s_\beta w) < {}^J\ell(w).$$

Proof. Write

$$I := \{\beta \neq \gamma \in \Phi_J^+ \mid s_\beta(\gamma) \notin \Phi_J^+\}.$$

Then

$$\begin{aligned} {}^J\ell(s_\beta w) &= \#\{\gamma \in \Phi_J^+ \mid w^{-1}s_\beta(\gamma) \in \Phi^-\} \\ &= \#\{\gamma \in \Phi_J^+ \backslash (I \cup \{\beta\}) \mid w^{-1}s_\beta(\gamma) \in \Phi^-\} + \#\{\gamma \in I \mid w^{-1}s_\beta(\gamma) \in \Phi^-\}. \end{aligned}$$

Since s_β permutes the set $\Phi_J^+ \backslash (I \cup \{\beta\})$, we get

$$\ldots = \#\{\gamma \in \Phi_J^+ \backslash (I \cup \{\beta\}) \mid w^{-1}\gamma \in \Phi^-\} + \#\{\gamma \in I \mid w^{-1}s_\beta(\gamma) \in \Phi^-\}.$$

Note that if $\gamma \in I$, then $\langle \beta^\vee, \gamma\rangle > 0$ and thus

$$w^{-1}s_\beta(\gamma) = w^{-1}\gamma - \langle \beta^\vee, \gamma\rangle w^{-1}\beta > w^{-1}\gamma.$$

We obtain

$$\begin{aligned} &\#\{\gamma \in \Phi_J^+ \backslash (I \cup \{\beta\}) \mid w^{-1}\gamma \in \Phi^-\} + \#\{\gamma \in I \mid w^{-1}s_\beta(\gamma) \in \Phi^-\} \\ \leqslant &\#\{\gamma \in \Phi_J^+ \backslash (I \cup \{\beta\}) \mid w^{-1}\gamma \in \Phi^-\} + \#\{\gamma \in I \mid w^{-1}\gamma \in \Phi^-\} \\ = &\ {}^J\ell(w) - 1. \end{aligned}$$

□

Lemma 3.31. *Let $J \subseteq \Delta_{\mathrm{af}}$ be a regular subset. Then there exists a uniquely determined map ${}^J\pi : W \to {}^JW \times \mathbb{Z}\Phi^\vee$ with the following two properties:*

(1) For all $w \in {}^JW$, we have ${}^J\pi(w) = (w, 0)$.

(2) For all $w \in W$ and $\beta \in \Phi_J^+$ where we write ${}^J\pi(w) = (w', \mu)$, we have

$$ {}^J\pi(s_\beta w) = (w', \mu + \Phi^+(-\beta)w^{-1}\beta^\vee) $$

and $w\mu \in \mathbb{Z}\,\mathrm{cl}(J)$.

Proof. We fix an element $\lambda \in \mathbb{Z}\Phi^\vee$ that is dominant and sufficiently regular (the required regularity constant follows from the remaining proof).

For $w \in W$, we consider the element $w\varepsilon^\lambda \in \widetilde{W}$. Then there exist uniquely determined elements $w'\varepsilon^{\lambda'} \in {}^JW_{\mathrm{af}}$ and $y \in \widetilde{W}_J$ such that

$$ w\varepsilon^\lambda = y \cdot w'\varepsilon^{\lambda'}. $$

We define ${}^J\pi(w) := (w', \lambda - \lambda')$ and check that it has the required properties.

(0) $w' \in {}^JW$: Since $\widetilde{W}_J$ is a finite group, we may assume that λ' is superregular and dominant as well. For $(\alpha, k) \in J$, we have

$$ (w'\varepsilon^{\lambda'})^{-1}(\alpha, k) = ((w')^{-1}\alpha, k + \langle \lambda', (w')^{-1}\alpha\rangle) \in \Phi_{\mathrm{af}}^+, $$

because $w'\varepsilon^{\lambda'} \in {}^JW_{\mathrm{af}}$. Since λ' is superregular and dominant, we have

$$ ((w')^{-1}\alpha, k + \langle \lambda', (w')^{-1}\alpha\rangle) \in \Phi_{\mathrm{af}}^+ \iff (w')^{-1}\alpha \in \Phi^+. $$

This proves $w' \in {}^JW$.

(1) If $w \in {}^JW$, then ${}^J\pi(w) = (w, 0)$: The proof of (0) shows that $w\varepsilon^\lambda \in {}^JW_{\mathrm{af}}$, so that $w\varepsilon^\lambda = w'\varepsilon^{\lambda'}$.

(2) Let $w \in W$ and $\beta \in \Phi_J^+$. We have to show

$$ {}^J\pi(s_\beta w) = (w', \lambda - \lambda' + \Phi^+(-\beta)w^{-1}\beta^\vee). $$

Put

$$ b := (\beta, \Phi^+(-\beta)) \in \Phi_{\mathrm{af}}^+. $$

By Lemma 3.28, we have $b \in (\Phi_{\mathrm{af}})_J^+$. The projection of

$$ r_b w\varepsilon^\lambda = s_\beta w \varepsilon^{\lambda + \Phi^+(-\beta)w^{-1}\beta^\vee} \in \widetilde{W}_J \cdot w\varepsilon^\lambda $$

onto ${}^JW_{\mathrm{af}}$ must again be $w'\varepsilon^{\lambda'}$. We obtain

$$ {}^J\pi(s_\beta w) = (w', \lambda + \Phi^+(-\beta)w^{-1}\beta^\vee - \lambda') $$

as desired.

For the second claim, it suffices to observe that

$$ \varepsilon^{w(\lambda - \lambda')} = w\varepsilon^\lambda\varepsilon^{-\lambda'}w^{-1} = yw'\varepsilon^{\lambda'}\varepsilon^{-\lambda'}w^{-1} = y\,\underbrace{w'w^{-1}}_{\in W_J} \in \widetilde{W}_J. $$

The fact that ${}^J\pi$ is uniquely determined (in particular, independent of the choice of λ) can be seen as follows: If $w \in {}^JW$, then ${}^J\pi(w)$ is determined by (1). Otherwise, we find $\beta \in \Phi_J^+$ with $w^{-1}\beta \in \Phi^-$. We multiply w on the left with s_β, and iterate this process, until we obtain an element in ${}^JW$. This process will terminate after at most ${}^J\ell(w)$ steps with an element in ${}^JW$. Now for each of these steps, we can use property (2) to determine the value of ${}^J\pi(w)$. $\square$

We call the set ${}^JW$ a *semi-affine quotient* of W, as it is a quotient of a finite Weyl group by a set of affine roots. The map ${}^J\pi$ is the *semi-affine projection.* We now introduce the semi-affine weight function.

Lemma 3.32. *Let $w_1, w_2 \in W$ and $J \subseteq \Delta$ be a regular subset. Write*

$$ {}^J\pi(w_1) = (w_1', \mu_1), \qquad {}^J\pi(w_2) = (w_2', \mu_2). $$

Then

$$ \mathrm{wt}(w_1' \Rightarrow w_2') - \mu_1 + \mu_2 = \mathrm{wt}(w_1' \Rightarrow w_2) - \mu_1 \leqslant \mathrm{wt}(w_1 \Rightarrow w_2). $$

Proof. We first show the equation

$$ \mathrm{wt}(w_1' \Rightarrow w_2') + \mu_2 = \mathrm{wt}(w_1' \Rightarrow w_2). $$

Induction by ${}^J\ell(w_2)$. The statement is trivial if $w_2 \in {}^JW$. Otherwise, we find some $\alpha \in \mathrm{cl}(J)$ with $w_2^{-1}\alpha \in \Phi^-$. Because $(w_1')^{-1}\alpha \in \Phi^+$, we obtain from Lemma 3.8 that

$$ \mathrm{wt}(w_1' \Rightarrow w_2) = \mathrm{wt}(w_1' \Rightarrow s_\alpha w_2) - \Phi^+(-\alpha) w_2^{-1}\alpha^\vee. $$

By Lemma 3.31, we have

$$ {}^J\pi(s_\alpha w_2) = (w_2', \mu_2 + \Phi^+(-\alpha) w_2^{-1}\alpha^\vee). $$

Using the inductive hypothesis, we get

$$ \begin{aligned} \mathrm{wt}(w_1' \Rightarrow w_2) &= \mathrm{wt}(w_1' \Rightarrow s_\alpha w_2) - \Phi^+(-\alpha) w_2^{-1}\alpha^\vee \\ &= \mathrm{wt}(w_1' \Rightarrow w_2') + \mu_2 + \Phi^+(-\alpha) w_2^{-1}\alpha^\vee - \Phi^+(-\alpha) w_2^{-1}\alpha^\vee \\ &= \mathrm{wt}(w_1' \Rightarrow w_2') + \mu_2. \end{aligned} $$

This finishes the induction.

It remains to prove the inequality

$$ \mathrm{wt}(w_1' \Rightarrow w_2) - \mu_1 \leqslant \mathrm{wt}(w_1 \Rightarrow w_2). $$

The argument is entirely analogous, using Corollary 3.17 in place of Lemma 3.8. $\square$

Definition 3.33. Let $w_1, w_2 \in W$ and $J \subseteq \Delta_{\mathrm{af}}$ be a regular subset. We write

$$ {}^J\pi(w_1) = (w_1', \mu_1), \qquad {}^J\pi(w_2) = (w_2', \mu_2). $$

(a) We define the *semi-affine weight function* by

$$^J\mathrm{wt}(w_1 \Rightarrow w_2) := \mathrm{wt}(w_1' \Rightarrow w_2') - \mu_1 + \mu_2 = \mathrm{wt}(w_1' \Rightarrow w_2) - \mu_1 \in \mathbb{Z}\Phi^\vee.$$

(b) If $\beta \in \Phi_J$ and $(\beta, k) \in (\Phi_{\mathrm{af}})_J$ is the image of β under the bijection of Lemma 3.35, we define $\chi_J(\beta) := -k$.

If $\beta \in \Phi \backslash \Phi_J$, we define $\chi_J(\beta) := \Phi^+(\beta)$.

In other words, for $\beta \in \Phi$, we have

$$\chi_J(\beta) = \Phi^+(\beta) - \Phi_J^+(\beta).$$

Example 3.34. Suppose that Φ is irreducible of type A_2 with basis α_1, α_2. Let $J = \{(-\theta, 1)\} = \{(-\alpha_1 - \alpha_2, 1)\}$, such that $\Phi_J^+ = \{-\theta\} = \{-\alpha_1 - \alpha_2\}$. We want to compute $^J\mathrm{wt}(1 \Rightarrow s_1 s_2)$ (writing $s_i := s_{\alpha_i}$).

Observe that $^J\pi(1) = (s_\theta, \theta^\vee)$. Hence

$$\begin{aligned} ^J\mathrm{wt}(1 \Rightarrow s_1) &= \mathrm{wt}(s_\theta \Rightarrow s_1 s_2) - \theta^\vee \\ &= \mathrm{wt}(s_1 s_2 s_1 \Rightarrow s_1 s_2) - \alpha_1^\vee - \alpha_2^\vee = -\alpha_2^\vee. \end{aligned}$$

Unlike the usual weight function, the value $^J\mathrm{wt}(w_1 \Rightarrow w_2)$ no longer needs to be a sum of positive coroots.

Lemma 3.35. *Let $w_1, w_2, w_3 \in W$ and let $J \subseteq \Delta$ be a regular subset.*

(a) The semi-affine weight function satisfies the triangle inequality,

$$^J\mathrm{wt}(w_1 \Rightarrow w_3) \leqslant {}^J\mathrm{wt}(w_1 \Rightarrow w_2) + {}^J\mathrm{wt}(w_2 \Rightarrow w_3).$$

(b) If $\alpha \in \Phi_J$, we have

$$\begin{aligned} ^J\mathrm{wt}(s_\alpha w_1 \Rightarrow w_2) &= {}^J\mathrm{wt}(w_1 \Rightarrow w_2) + \chi_J(\alpha) w_1^{-1}\alpha^\vee, \\ ^J\mathrm{wt}(w_1 \Rightarrow s_\alpha w_2) &= {}^J\mathrm{wt}(w_1 \Rightarrow w_2) - \chi_J(\alpha) w_2^{-1}\alpha^\vee. \end{aligned}$$

(c) If $\beta \in \Phi^+$, we have

$$\begin{aligned} ^J\mathrm{wt}(w_1 s_\beta \Rightarrow w_2) &\leqslant {}^J\mathrm{wt}(w_1 \Rightarrow w_2) + \chi_J(w_1\beta)\beta^\vee, \\ ^J\mathrm{wt}(w_1 \Rightarrow w_2 s_\beta) &\leqslant {}^J\mathrm{wt}(w_1 \Rightarrow w_2) + \chi_J(-w_2\beta)\beta^\vee. \end{aligned}$$

Proof. Part (a) follows readily from the definition. Let us prove part (b). We focus on the first identity, as the proof of the second identity is analogous.

Up to replacing α by $-\alpha$, which does not change the reflection s_α nor the value of

$$\chi_J(\alpha) w_1^{-1}\alpha^\vee,$$

we may assume $\alpha \in \Phi_J^+$. Now write

$$^J\pi(w_1) = (w_1', \mu_1), \qquad {}^J\pi(w_2) = (w_2', \mu_2).$$

Then $^J\pi(s_\alpha w_1) = (w_1', \mu_1 + \Phi^+(-\alpha) w_1^{-1}\alpha^\vee)$. Thus

$$\begin{aligned} {}^J\mathrm{wt}(s_\alpha w_1 \Rightarrow w_2) &= \mathrm{wt}(w_1' \Rightarrow w_2') - \mu_1 - \Phi^+(-\alpha)w_1^{-1}\alpha^\vee + \mu_2 \\ &= {}^J\mathrm{wt}(w_1 \Rightarrow w_2) - \Phi^+(-\alpha)w_1^{-1}\alpha^\vee \\ &= {}^J\mathrm{wt}(w_1 \Rightarrow w_2) + \chi_J(\alpha)w_1^{-1}\alpha^\vee \end{aligned}$$

as $\alpha \in \Phi_J^+$.

Now we prove part (c). Again, we only show the first inequality. If $w_1\beta \in \Phi_J$, the inequality follows from part (b). Otherwise, we use (a) and Corollary 3.17to compute

$$\begin{aligned} {}^J\mathrm{wt}(w_1 s_\beta \Rightarrow w_2) &\leqslant {}^J\mathrm{wt}(w_1 s_\alpha \Rightarrow w_1) + {}^J\mathrm{wt}(w_1 \Rightarrow w_2) \\ &\underset{\text{L3.32}}{\leqslant} \mathrm{wt}(w_1 s_\alpha \Rightarrow w_1) + {}^J\mathrm{wt}(w_1 \Rightarrow w_2) \\ &\leqslant \Phi^+(w\alpha)\alpha^\vee + {}^J\mathrm{wt}(w_1 \Rightarrow w_2) \\ &= \chi_J(w\alpha)\alpha^\vee + {}^J\mathrm{wt}(w_1 \Rightarrow w_2). \end{aligned}$$

This finishes the proof. □

Lemma 3.36. *Let $w_1, w_2 \in W$ and $J \subseteq \Delta_{\mathrm{af}}$ be regular. Suppose that for all $\alpha \in \Phi_J^+$, at least one of the following conditions is satisfied:*

$$w_1^{-1}\alpha \in \Phi^+ \text{ or } w_2^{-1}\alpha \in \Phi^-.$$

Then $^J\mathrm{wt}(w_1 \Rightarrow w_2) = \mathrm{wt}(w_1 \Rightarrow w_2)$.

Proof. We show the claim via induction on $^J\ell(w_1)$. If $w_1 \in {}^JW$, then the claim follows from Lemma 3.32.

Otherwise, we find some $\alpha \in \mathrm{cl}(J)$ with $w_1^{-1}\alpha \in \Phi^-$. By assumption, also $w_2^{-1}\alpha \in \Phi^-$. Using Lemma 3.8, we get

$$\mathrm{wt}(w_1 \Rightarrow w_2) = \mathrm{wt}(s_\alpha w_1 \Rightarrow s_\alpha w_2) + \chi_J(\alpha)w_1^{-1}\alpha^\vee - \chi_J(\alpha)w_2^{-1}\alpha^\vee.$$

Since $^J\ell(s_\alpha w_1) < {}^J\ell(w_1)$ by Lemma 3.30, we want to show that $(s_\alpha w_1, s_\alpha w_2)$ also satisfy the condition stated in the lemma.

For this, let $\beta \in \Phi_J^+$. If $\beta = \alpha$, then $(s_\alpha w_1)^{-1}\alpha = -w_1^{-1}\alpha \in \Phi^+$ by choice of α. Now assume that $\beta \neq \alpha$, so that $s_\alpha\beta \in \Phi_J^+$. By the assumption on w_1 and w_2, we must have $w_1^{-1}s_\alpha(\beta) \in \Phi^+$ or $w_2^{-1}s_\alpha(\beta) \in \Phi^-$. In other words, we have

$$(s_\alpha w_1)^{-1}\beta \in \Phi^+ \text{ or } (s_\alpha w_2)^{-1}\beta \in \Phi^-.$$

This shows that $(s_\alpha w_1, s_\alpha w_2)$ satisfy the desired properties.

By the inductive hypothesis and Lemma 3.35, we get

$$\begin{aligned}
&\mathrm{wt}(s_\alpha w_1 \Rightarrow s_\alpha w_2) + \chi_J(\alpha) w_1^{-1}\alpha^\vee - \chi_J(\alpha) w_2^{-1}\alpha^\vee \\
=&\, {}^J\mathrm{wt}(s_\alpha w_1 \Rightarrow s_\alpha w_2) + \chi_J(\alpha) w_1^{-1}\alpha^\vee - \chi_J(\alpha) w_2^{-1}\alpha^\vee \\
=&\, {}^J\mathrm{wt}(w_1 \Rightarrow w_2).
\end{aligned}$$

This completes the induction and the proof. □

Corollary 3.37. *Let $w_1, w_2 \in W$ and let $J \subseteq \Delta_{\mathrm{af}}$ be regular. Denote by $w_0 \in W$ the longest element. Then*

$$^J\mathrm{wt}(w_1 w_0 \Rightarrow w_2 w_0) = -w_0\, {}^J\mathrm{wt}(w_2 \Rightarrow w_1).$$

Proof. Both sides of the equation behave identically when multiplying w_1 or w_2 on the left by a reflection in W_J: For $\alpha \in \Phi_J$, we use Lemma 3.35 to see

$$\begin{aligned}
&{}^J\mathrm{wt}(s_\alpha w_1 w_0 \Rightarrow w_2 w_0) - {}^J\mathrm{wt}(w_1 w_0 \Rightarrow w_2 w_0) = -\chi_J(\alpha)(w_1 w_0)^{-1}\alpha^\vee \\
=&- w_0 \left({}^J\mathrm{wt}(w_2 \Rightarrow s_\alpha w_1) - \mathrm{wt}(w_2 \Rightarrow w_1)\right). \\
&{}^J\mathrm{wt}(w_1 w_0 \Rightarrow s_\alpha w_2 w_0) - {}^J\mathrm{wt}(w_1 w_0 \Rightarrow w_2 w_0) = \chi_J(\alpha)(w_2 w_0)^{-1}\alpha^\vee \\
=&- w_0 \left({}^J\mathrm{wt}(s_\alpha w_2 \Rightarrow w_1) - \mathrm{wt}(w_2 \Rightarrow w_1)\right).
\end{aligned}$$

Therefore, it suffices to show the desired equality in case $w_1, w_2 \in {}^J W$. By Lemma 3.36, we get

$$\begin{aligned}
&{}^J\mathrm{wt}(w_1 w_0 \Rightarrow w_2 w_0) = \mathrm{wt}(w_1 w_0 \Rightarrow w_2 w_0), \\
&{}^J\mathrm{wt}(w_2 \Rightarrow w_1) = \mathrm{wt}(w_2 \Rightarrow w_1).
\end{aligned}$$

Now the claim follows from Lemma 3.9. □

3.5. Maximal subsets

In this section, we specialize to the situation where Φ is irreducible and $J = \Delta_{\mathrm{af}} \setminus \{a\}$ for some $a \in \Delta_{\mathrm{af}}$. As we saw before, the calculation of weight functions can be reduced to this situation.

We define the *fundamental coweight* $\omega_a \in \mathbb{Q}\Phi^\vee$ by declaring for each $\beta \in \Delta$ that

$$\langle \omega_a, \beta \rangle = \begin{cases} 1, & a = (\beta, 0), \\ 0, & a \neq (\beta, 0). \end{cases}$$

Denote the longest root of Φ by θ. Then we define the *normalized coweight* $\widetilde{\omega}_a \in \mathbb{Q}\Phi^\vee$ by

$$\widetilde{\omega}_a = \begin{cases} \omega_a = 0, & a = (-\theta, 1), \\ \frac{1}{\omega_a(\theta)}\omega_a, & a \neq (-\theta, 1). \end{cases}$$

Lemma 3.38. *If $w \in W$, there exists a uniquely determined element $w' \in W_J w \cap {}^J W$, and it satisfies*

$$ {}^J\pi(w) = (w', w^{-1}\widetilde{\omega}_a - (w')^{-1}\widetilde{\omega}_a). $$

Proof. From the definition of ${}^J\pi(w)$ in Lemma 3.31, it follows that the intersection $W_J w \cap {}^J W$ contains exactly one element w', and that ${}^J\pi(w)$ has the form (w', μ) for some μ.

Define a function $\varphi : W \to {}^J W \times \mathbb{Q}\Phi^\vee$ via

$$ \varphi(w) = (w', w^{-1}\widetilde{\omega}_a - (w')^{-1}\widetilde{\omega}_a) \text{ if } {}^J\pi(w) = (w', \mu). $$

We show that $\varphi = {}^J\pi$ by verifying the recursive definition of ${}^J\pi$.

If $w \in {}^J W$, then certainly $w' = w$ and $\varphi(w) = (w, 0)$.

Now suppose that $w \in W$ is any element with $\varphi(w) = {}^J\pi(w)$ and pick $\beta \in \Phi_J^+$. We have

$$ {}^J\pi(s_\beta w) = (w', \mu + \Phi^+(-\beta) w^{-1}\beta^\vee). $$

Now we calculate

$$ \begin{aligned} \varphi(s_\beta w) =& (w', (s_\beta w)^{-1}\widetilde{\omega}_a - (w')^{-1}\omega_a) \\ =& (w', w^{-1}\widetilde{\omega}_a - \langle \widetilde{\omega}_a, \beta\rangle w^{-1}\beta^\vee - (w')^{-1}\omega_a) \\ =& (w', \mu - \langle \widetilde{\omega}_a, \beta\rangle w^{-1}\beta^\vee). \end{aligned} $$

In order to conclude $\varphi(s_\beta w) = {}^J\pi(s_\beta w)$ (completing the induction and the proof), it remains to show

$$ -\langle \widetilde{\omega}_a, \beta\rangle = \Phi^+(-\beta). $$

If $a = (-\theta, 1)$, both sides are trivially zero (as $\beta \in \Phi_J^+ = \Phi^+$). Thus let us assume that $a \neq (-\theta, 1)$.

We use the condition $\beta \in \Phi_J^+$. By Lemma 3.28, we have $(\beta, \Phi^+(-\beta)) \in \Phi_J^+$, so

$$ \begin{aligned} \omega_a(\beta + \Phi^+(-\beta)\theta) = 0 &\implies \langle \widetilde{\omega}_a, \beta + \Phi^+(-\beta)\theta\rangle = 0 \\ &\implies -\langle \widetilde{\omega}_a, \beta\rangle = \Phi^+(-\beta)\langle \widetilde{\omega}_a, \theta\rangle = \Phi^+(-\beta). \end{aligned} $$

The proof is finished. □

Lemma 3.39. *For $w_1, w_2 \in W$, we have*

$$ {}^J\mathrm{wt}(w_1 \Rightarrow w_2) \geqslant w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a. $$

In case $W_J w_1 = W_J w_2$, we have equality.

Note that ${}^J\mathrm{wt}(w_1 \Rightarrow w_2)$ is an integral sum of coroots, whereas $w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a$ will, in general, be only a rational linear combination of coroots.

Proof of Lemma 3.39. We want to use the definition of the semi-affine weight function, so let us write

$$^J\pi(w_1) = (w_1', \mu_1), \qquad ^J\pi(w_2) = (w_2', \mu_2).$$

Then, by definition,

$$\begin{aligned} ^J\mathrm{wt}(w_1 \Rightarrow w_2) &= \mathrm{wt}(w_1' \Rightarrow w_2') - \mu_1 + \mu_2 \\ &\underset{\text{L3.38}}{=} \mathrm{wt}(w_1' \Rightarrow w_2') - w_1^{-1}\widetilde{\omega}_a + (w_1')^{-1}\widetilde{\omega}_a + w_2^{-1}\widetilde{\omega}_a - (w_2')^{-1}\widetilde{\omega}_a. \end{aligned}$$

If $W_J w_1 = W_J w_2$, we get $w_1' = w_2'$ and the claim follows. In general, we need to prove

$$\forall w_1', w_2' \in W: \ \mathrm{wt}(w_1' \Rightarrow w_2') \geqslant (w_2')^{-1}\widetilde{\omega}_a - (w_1')^{-1}\widetilde{\omega}_a. \tag{3.40}$$

This is clear if $a = (-\theta, 1)$, as then the right-hand side vanishes. Hence let us assume that $a \neq (-\theta, 1)$. It suffices to show the inequality (3.40) for edges $w_1' \to w_2'$ in $\mathrm{QB}(W)$, as we then can use induction on $d(w_1' \Rightarrow w_2')$.

Now suppose that $w_2' = w_1' s_\alpha$ for some root $\alpha \in \Phi^+$. Then

$$\begin{aligned} (w_2')^{-1}\widetilde{\omega}_a - (w_1')^{-1}\widetilde{\omega}_a =& s_\alpha (w_1')^{-1}\widetilde{\omega}_a - (w_1')^{-1}\widetilde{\omega}_a \\ =& -\langle (w_1)^{-1}\widetilde{\omega}_a, \alpha\rangle \alpha^\vee \\ =& -\langle \widetilde{\omega}_a, w_1'\alpha\rangle \alpha^\vee \\ \leqslant& \Phi^+(-w_1'\alpha)\alpha^\vee = \mathrm{wt}(w_1' \Rightarrow w_2'). \end{aligned}$$

We conclude that inequality (3.40) holds true for edges $w_1' \to w_2'$ in $\mathrm{QB}(W)$, which finishes the last gap in the proof. $\square$

If α is special, the estimate in Lemma 3.39 is an equality:

Lemma 3.41. *Suppose that $a \in \Delta_{\mathrm{af}}$ is a* special node*, i.e. such that $a = (-\theta, 1)$ or $\omega_a(\theta) = 1$. Then ^JW consists only of one element. For $w_1, w_2 \in W$, we have*

$$^J\mathrm{wt}(w_1 \Rightarrow w_2) = w_2^{-1}\omega_a - w_1^{-1}\omega_a.$$

Proof. If $a = (-\theta, 1)$, then $J = \Delta$ and $\Phi_J^+ = \Phi^+$. Now $^JW = \{1\}$, $\omega_a = 0$ and $\chi_J \equiv 0$. It follows that $^J\mathrm{wt}(\cdot \Rightarrow \cdot) = 0$ by Lemma 3.35.

Let us consider the case $a = (\alpha, 0)$. We first show that ^JW consists only of one element: Let $w \in {}^JW$ and $J' := \mathrm{cl}(J) \cap \Delta$. We claim that

$$\forall \beta \in \Phi^+ : \ w^{-1}\beta \in \Phi^+ \iff \beta \in \Phi_{J'}. \tag{3.42}$$

By definition of ^JW, the claim is satisfied for $\beta \in J'$, and hence for sums of those roots. i.e. if $\beta \in \Phi_{J'}^+$, then $w^{-1}\beta \in \Phi^+$.

Now suppose that $\beta \in \Phi^+ \backslash \Phi_{J'}^+$. We write

$$\begin{aligned} \beta &= \sum_{\gamma \in \Delta} c_\gamma \gamma, \\ \theta &= \sum_{\gamma \in \Delta} c_\gamma' \gamma. \end{aligned}$$

As θ is the longest root, $c_\gamma \leqslant c'_\gamma$ for all $\gamma \in \Delta$. By choice of a, we have $c'_\alpha \leqslant 1$, and by choice of β, we have $c_\alpha \geqslant 1$. Thus $c_\alpha = c'_\alpha = 1$.

By definition of $w \in {}^J W$ and $(-\theta, 1) \in J$, we see that $w^{-1}\theta \in \Phi^-$. Now observe that

$$\begin{aligned} w^{-1}\beta =& w^{-1}\theta + \sum_{\alpha \neq \gamma \in \Delta} \underbrace{(c_\gamma - c'_\gamma)}_{\leqslant 0} \underbrace{w^{-1}\gamma}_{\in \Phi^+} \\ \leqslant& w^{-1}\theta \in \Phi^-. \end{aligned}$$

Thus $w^{-1}\beta \in \Phi^-$. The claim (3.42) is proved. It follows that ${}^J W$ consists of only one element.

By Lemma 3.38, we conclude $W = W_J$. Thus $W_J w_1 = W_J w_2$ for all $w_1, w_2 \in W$, such that the final claim follows from Lemma 3.39. □

Remark 3.43. For irreducible root systems of type A_n, all nodes are special and Lemmas 3.36, 4.37 and 3.41 allow an easy way to compute the weight function. This recovers formula (3.1). As mentioned before, this formula fails for all other root systems, precisely because not all nodes are special: Indeed, if $\mathbf{a} = (\alpha, 0)$ is a non-special node, we get

$$\omega_{\mathbf{a}} - s_\theta(\omega_{\mathbf{a}}) = \langle \omega_{\mathbf{a}}, \theta \rangle \theta^\vee = \omega_{\mathbf{a}}(\theta)\theta^\vee > \theta^\vee = \mathrm{wt}(s_\theta \Rightarrow 1).$$

In general, let us write

$$\lceil w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a \rceil \in \mathbb{Z}\Phi^\vee$$

for the smallest element in $\mathbb{Z}\Phi^\vee$ that is $\geqslant w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a \in \mathbb{Q}\Phi^\vee$. We have

$$\mathrm{wt}(w_1 \Rightarrow w_2) \geqslant \sup_a \lceil w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a \rceil. \tag{3.44}$$

and we may ask whether equality holds. In general, we cannot expect equality to hold (the resulting criterion for the Bruhat order of finite Weyl groups would be "too simple"). It is interesting though that the lack of equality in (3.44) explains precisely the difference between the so-called *admissible* and *permissible* subsets as defined in [KR00], cf. Corollary 4.15.

Remark 3.45. The computation of weight functions for non simply laced root systems can be reduced to a calculation for a simply laced root system using the technique of *Dynkin diagram folding*:

Suppose that $\varphi : \Phi \to \Phi$ is an automorphism of the root system with $\varphi(\Delta) = \Delta$. We obtain the folded root system Φ/φ with *coroots*

$$(\Phi/\varphi)^\vee = \left\{ \sum_{\alpha \in o} \alpha \mid o \subseteq \Phi^\vee \text{ is a } \varphi\text{-orbit} \right\} \subseteq \mathbb{Z}\Phi^\vee.$$

Thus, the roots in Φ/φ are in bijection with φ-orbits in Φ. The Weyl group of Φ/φ is given by W^φ, the φ-invariant elements of W. Similarly, the affine Weyl group of Φ/φ is given by $(W_{\mathrm{af}})^\varphi$.

Both W^φ and $(W_{\mathrm{af}})^\varphi$ inherit the Bruhat order from the larger groups W resp. W_{af} (This is a simple Coxeter theoretic fact). Using Corollary 3.16, we see that

$$\forall w_1, w_2 \in W^\varphi : \ \mathrm{wt}_{\mathrm{QB}(W)}(w_1 \Rightarrow w_2) = \mathrm{wt}_{\mathrm{QB}(W^\varphi)}(w_1 \Rightarrow w_2).$$

We observe, using the classification of root systems, that the only irreducible root systems with non-trivial automorphisms are those of type A_n, D_n and E_6. If Φ is of type A_n and $\varphi(\alpha) = -w_0(\alpha)$ is the non-trivial automorphism of order 2, then Φ/φ is of type $C_{\lceil n/2 \rceil}$. The quotient of D_4 by one of the automorphisms of order 3 is G_2; the quotient of any D_n by an automorphism of order 2 is given by B_{n-1}. Finally, the quotient of E_6 by the unique automorphism of order 2 is given by F_4.

For root systems of type C_n, the calculation of the weight function can thus be reduced to this calculation for A_{2n-1}, for which an explicit formula is known. Alternatively, one can compare Corollary 3.16 to the explicit criterion for the Bruhat order on the affine Weyl group given by [BB05, Theorem 8.4.7].

The A_n case gives hope that the semi-affine weight function ${}^{\Delta_{\mathrm{af}}\setminus\{\mathbf{a}\}}\mathrm{wt}(\cdot \Rightarrow \cdot)$ would be easier to compute than the "full" weight function $\mathrm{wt}(\cdot \Rightarrow \cdot)$. While this is true for C_n (due to the aforementioned reduction to A_n), such formulas seem to be unknown for other root systems.

For types B_n and, more generally D_n, the recent paper of Ishii [Ish21] seems promising. Ishii presents explicit criteria for the semi-infinite order, which should yield explicit formulas for the weight function.

For the remaining exceptional root systems, the question of how to compute the weight function can, in principle, be solved by giving a finite list of answers. It is doubtful how feasible or useful such a task would be.

A "simple formula" to describe the weight function for all root systems would, in particular, entail a "simple criterion" for the Bruhat order for all finite Weyl groups, which seems already to be a difficult problem[2].

[2] This leads to the somewhat paradoxical situation that we are able to prove new results for the Bruhat order on the affine Weyl group, but nothing new for the Bruhat order on the finite Weyl group.

4. Bruhat order

The Bruhat order on $\widetilde{W}$ is a fundamental Coxeter-theoretic notion that has been studied with great interest, e.g. [BB95; KR00; Rap02; Len+15]. In this section, we present new characterizations of the Bruhat order on $\widetilde{W}$.

The structure of this section is as follows: In Section 4.1, we state our main criterion for the Bruhat order as Theorem 4.2 and discuss some of its applications. We then prove this criterion in Section 4.2. Finally, Section 4.3 will cover some consequences of Deodhar's lemma (cf. [Deo77]) and feature an even more general criterion.

4.1. A criterion

Definition 4.1. Let $x = w\varepsilon^\mu \in \widetilde{W}$. A *Bruhat-deciding datum* for x is a tuple $(v, J_1, \dots, J_m)$ where $v \in W$ and $J_\bullet$ is a finite collection of arbitrary subsets $J_1, \dots, J_m \subseteq \Delta$ with $m \geqslant 1$, satisfying the following two properties:

(1) The element v is length positive for x, i.e. $\ell(x, v\alpha) \geqslant 0$ for all $\alpha \in \Phi^+$.

(2) Writing $J := J_1 \cap \dots \cap J_m$, we have $\ell(x, v\alpha) = 0$ for all $\alpha \in \Phi_J$.

The name *Bruhat-deciding* is justified by the following result.

Theorem 4.2. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$. Fix a Bruhat-deciding datum $(v, J_1, \dots, J_m)$ for x. Then the following are equivalent:*

(1) $x \leqslant x'$.

(2) For all $i = 1, \dots, m$, there exists an element $v_i' \in W$ such that

$$v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v_i') \leqslant (v_i')^{-1}\mu' \pmod{\Phi_{J_i}^\vee}.$$

We again use the shorthand notation $\mu_1 \leqslant \mu_2 \pmod{\Phi_J^\vee}$ for $\mu_1 - \mu_2 + \mathbb{Z}\Phi_J^\vee \leqslant 0 + \mathbb{Z}\Phi_J^\vee$ in $\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$.

This theorem is the main result of this section. We give a proof in Section 4.2.

First, let us remark that the construction of a Bruhat-deciding datum is easy. It suffices to choose any length positive element v for x, and then $(v, \varnothing)$ is Bruhat-deciding.

The inequality of Theorem 4.2 is only interesting for $v \in \mathrm{LP}(x)$ and $v_i' \in \mathrm{LP}(x')$, as explained by the following lemma in conjunction with Lemma 2.3.

Lemma 4.3. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$. Suppose we are given elements $v, v' \in W$, a subset $J \subseteq \Delta$ and a positive root $\alpha \in \Phi^+$.*

(a) Assume $\ell(x, v\alpha) < 0$. Then the inequality

$$(vs_\alpha)^{-1}\mu + \mathrm{wt}(v' \Rightarrow vs_\alpha) + \mathrm{wt}(wvs_\alpha \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi_J^\vee}$$

implies

$$v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi_J^\vee}.$$

(b) *Assume $\ell(x', v\alpha) < 0$. Then the inequality*

$$v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi_J^\vee}$$

implies

$$v^{-1}\mu + \mathrm{wt}(v's_\alpha \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v's_\alpha) \leqslant (v's_\alpha)^{-1}\mu' \pmod{\Phi_J^\vee}.$$

Proof. (a) We have

$$\begin{aligned}
(v')^{-1}\mu' \geqslant &(vs_\alpha)^{-1}\mu + \mathrm{wt}(v' \Rightarrow vs_\alpha) + \mathrm{wt}(wvs_\alpha \Rightarrow w'v') \\
\geqslant &v^{-1}\mu - \langle v^{-1}\mu, \alpha\rangle\alpha^\vee + \mathrm{wt}(v' \Rightarrow v) - \mathrm{wt}(vs_\alpha \Rightarrow v) \\
&+ \mathrm{wt}(wv \Rightarrow w'v') - \mathrm{wt}(wv \Rightarrow wvs_\alpha) \\
\underset{(*)}{\geqslant} &v^{-1}\mu - \langle v^{-1}\mu, \alpha\rangle\alpha^\vee + \mathrm{wt}(v' \Rightarrow v) - \Phi^+(v\alpha)\alpha^\vee \\
&+ \mathrm{wt}(wv \Rightarrow w'v') - \Phi^+(-wv\alpha)\alpha^\vee \\
= &v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') - (\ell(x, v\alpha) + 1)\alpha^\vee \\
\geqslant &v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \pmod{\Phi_J^\vee}.
\end{aligned}$$

The inequality $(*)$ is Corollary 3.17.

(b) The calculation is completely analogous:

$$\begin{aligned}
(v's_\alpha)^{-1}\mu' = &(v')^{-1}\mu' - \langle (v')^{-1}\mu, \alpha\rangle\alpha^\vee \\
\geqslant &v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') - \langle \mu, v'\alpha\rangle\alpha^\vee \\
\geqslant &v^{-1}\mu + \mathrm{wt}(v's_\alpha \Rightarrow v) - \mathrm{wt}(v's_\alpha \Rightarrow v') \\
&+ \mathrm{wt}(wv \Rightarrow w'v's_\alpha) - \mathrm{wt}(w'v' \Rightarrow w'v's_\alpha) - \langle \mu, v'\alpha\rangle\alpha^\vee \\
\geqslant &v^{-1}\mu + \mathrm{wt}(v's_\alpha \Rightarrow v) - \Phi^+(v'\alpha)\alpha^\vee \\
&+ \mathrm{wt}(wv \Rightarrow w'v's_\alpha) - \Phi^+(-w'v'\alpha)\alpha^\vee - \langle \mu, v'\alpha\rangle\alpha^\vee \\
= &v^{-1}\mu + \mathrm{wt}(v's_\alpha \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v's_\alpha) - (\ell(x', v'\alpha) + 1)\alpha^\vee \\
\geqslant &v^{-1}\mu + \mathrm{wt}(v's_\alpha \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v's_\alpha).
\end{aligned}$$ □

Proof of Theorem 1.1 using Theorem 4.2. We use the notation of Theorem 1.1. In view of Lemma 4.3 and Lemma 2.3, the condition

$$\exists v_2 \in W:\ v_1^{-1}\mu_1 + \mathrm{wt}(v_2 \Rightarrow v_1) + \mathrm{wt}(w_1v_1 \Rightarrow w_2v_2) \leqslant v_2^{-1}\mu_2 \tag{$*$}$$

is true for all $v_1 \in \mathrm{LP}(x)$ iff it is true for all $v_1 \in W$. We see that asking condition $(*)$ for all $v_1 \in W$ is equivalent to asking condition (2) of Theorem 4.2 *for each Bruhat-deciding datum.* In this sense, Theorem 4.2 implies Theorem 1.1. □

If x' is in a shrunken Weyl chamber, there is a canonical choice for v'.

Corollary 4.4. *Let $x = w\varepsilon^{\mu}$ and $x' = w'\varepsilon^{\mu'}$. Assume that x' is in a shrunken Weyl chamber and that v' is the length positive element for x'. Pick any length positive element v for x. Then $x \leqslant x'$ if and only if*

$$v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu'.$$

Proof. $(v, \varnothing)$ is a Bruhat-deciding datum for x. By Lemma 4.3 and Corollary 2.4, the inequality in Theorem 4.2 (2) is satisfied by *some* $v' \in W$ iff it is satisfied by the unique length positive element v' for x'. ☐

We now show how Theorem 4.2 can be used to describe Bruhat covers in $\widetilde{W}$. The following proposition generalizes the previous results of Lam-Shimozono [LS10, Proposition 4.1] and Milićević [Mil21, Proposition 4.2].

Proposition 4.5. *Let $x = w\varepsilon^{\mu}, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$ and $v \in \mathrm{LP}(x)$. Then the following are equivalent:*

(a) $x \lessdot x'$*, i.e.* $x < x'$ *and* $\ell(x) = \ell(x') - 1$.

(b) There exists some $v' \in \mathrm{LP}(x')$ *such that*

- *(b.1)* $v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') = (v')^{-1}\mu'$ *and*
- *(b.2)* $d(v' \Rightarrow v) + d(wv \Rightarrow w'v') = 1$.

(c) There is a root $\alpha \in \Phi^+$ *satisfying at least one of the following conditions:*

- *(c.1) There exists a Bruhat edge* $v' := s_\alpha v \to v$ *in* $\mathrm{QB}(W)$ *with* $x' = xs_\alpha$ *and* $v' \in \mathrm{LP}(x')$.
- *(c.2) There exists a quantum edge* $v' := s_\alpha v \to v$ *in* $\mathrm{QB}(W)$ *with* $v^{-1}\alpha \in \Phi^+, x' = xr_{(-\alpha,1)}$ *and* $v' \in \mathrm{LP}(x')$.
- *(c.3) There exists a Bruhat edge* $wv \to s_\alpha wv$ *in* $\mathrm{QB}(W)$ *such that* $x' = s_\alpha x$ *and* $v \in \mathrm{LP}(x')$.
- *(c.4) There exists a quantum edge* $wv \to s_\alpha wv$ *in* $\mathrm{QB}(W)$ *with* $(wv)^{-1}\alpha \in \Phi^-$, $x' = r_{(-\alpha,1)}x$ *and* $v \in \mathrm{LP}(x')$.

(d) There exists a root $\alpha \in \Phi^+$ *satisfying at least one of the following conditions:*

- *(d.1) We have* $w' = ws_\alpha, \mu' = s_\alpha(\mu), \ell(s_\alpha v) = \ell(v) - 1$ *and for all* $\beta \in \Phi^+$*:*
$$\ell(x, v\beta) + \Phi^+(s_\alpha v\beta) - \Phi^+(v\beta) \geqslant 0.$$
- *(d.2) We have* $w' = ws_\alpha, \mu' = s_\alpha(\mu) - \alpha^\vee, \ell(s_\alpha v) = \ell(v) - 1 + \langle v^{-1}\alpha^\vee, 2\rho\rangle$ *and for all* $\beta \in \Phi^+$*:*
$$\ell(x, v\beta) + \langle \alpha^\vee, v\beta\rangle + \Phi^+(s_\alpha v\beta) - \Phi^+(v\beta) \geqslant 0.$$
- *(d.3) We have* $w' = s_\alpha w, \mu' = \mu, \ell(s_\alpha wv) = \ell(wv) + 1$ *and for all* $\beta \in \Phi^+$*:*
$$\ell(x, v\beta) + \Phi^+(wv\beta) - \Phi^+(s_\alpha wv\beta) \geqslant 0.$$

(d.4) *We have* $w' = s_\alpha w, \mu' = \mu - w^{-1}\alpha^\vee, \ell(s_\alpha wv) = \ell(wv) + 1 + \langle (wv)^{-1}\alpha^\vee, 2\rho\rangle$ *and for all* $\beta \in \Phi^+$*:*

$$\ell(x, v\beta) + \langle \alpha^\vee, wv\beta\rangle + \Phi^+(wv\beta) - \Phi^+(s_\alpha wv\beta) \geqslant 0.$$

Proof. (a) $\iff$ (b): We start with a key calculation for $v' \in \mathrm{LP}(x')$:

$$\begin{aligned}
&\langle (v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow v) - \mathrm{wt}(wv \Rightarrow w'v') - v^{-1}\mu, 2\rho\rangle \\
\underset{\text{L3.6}}{=}&\langle (v')^{-1}\mu, 2\rho\rangle - d(v' \Rightarrow v) - \ell(v') + \ell(v) \\
&\qquad - d(wv \Rightarrow w'v') - \ell(wv) + \ell(w'v') - \langle v^{-1}\mu, 2\rho\rangle \\
\underset{\text{C2.11}}{=}&\ell(x') - \ell(x) - d(v' \Rightarrow v) - d(wv \Rightarrow w'v').
\end{aligned}$$

First assume that (a) holds, i.e. $x \lessdot x'$. By Theorem 4.2 and Lemma 4.3, we find $v' \in \mathrm{LP}(x')$ such that

$$(v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow v) - \mathrm{wt}(wv \Rightarrow w'v') - v^{-1}\mu \geqslant 0$$

By the above key calculation, we see that

$$\ell(x') \geqslant \ell(x) + d(v' \Rightarrow v) + d(wv \Rightarrow w'v'),$$

where equality holds if and only if (b.1) is satisfied. Note that $x \lessdot x'$ implies that $x^{-1}x'$ must be an affine reflection, thus $w \neq w'$. We see that $v \neq v'$ or $wv \neq w'v'$, thus in particular

$$\ell(x) + 1 = \ell(x') \geqslant \ell(x) + d(v' \Rightarrow v) + d(wv \Rightarrow w'v') \geqslant \ell(x) + 1.$$

Since equality must hold, we get (b.1) and (b.2).

Now assume conversely that (b) holds. By (b.1) and Theorem 4.2, we see that $x < x'$. Now using the key calculation and (b.2), we get $\ell(x') = \ell(x) + 1$.

(b) $\iff$ (c): The condition (b.2) means that either $v = v'$ and $wv \to w'v'$ is an edge in $\mathrm{QB}(W)$, or $wv = w'v'$ and $v' \to v$ is an edge. If we now distinguish between Bruhat and quantum edges, we get the explicit conditions of (c) (or (d)).

Let us first assume that (b) holds. We distinguish the following cases:

(1) $wv = w'v'$ and $v' \to v$ is a Bruhat edge: Then we can write $v' = s_\alpha v$ for some $\alpha \in \Phi^+$ with $v^{-1}\alpha \in \Phi^-$. Now the condition $wv = w'v'$ implies $w' = ws_\alpha$. Condition (b.1) implies $v^{-1}\mu = (v')^{-1}\mu'$, so $\mu' = s_\alpha(\mu)$. We get (c.1).

(2) $wv = w'v'$ and $v' \to v$ is a quantum edge: Then we can write $v' = s_\alpha v$ for some $\alpha \in \Phi^+$ with $v^{-1}\alpha \in \Phi^+$. Now the condition $wv = w'v'$ implies $w' = ws_\alpha$. Condition (b.1) implies $v^{-1}\mu + v^{-1}\alpha^\vee = (v')^{-1}\mu'$, so $\mu' = s_\alpha(\mu) - \alpha^\vee$. We get (c.2).

(3) $v = v'$ and $wv \to w'v'$ is a Bruhat edge: Then we can write $w'v' = s_\alpha wv$ for some $\alpha \in \Phi^+$ with $(wv)^{-1}\alpha \in \Phi^-$. Now the condition $v = v'$ implies $w' = s_\alpha w$. Condition (b.1) implies $v^{-1}\mu = (v')^{-1}\mu$, so $\mu' = \mu$. We get (c.3).

(4) $v = v'$ and $wv \to w'v'$ is a quantum edge: Then we can write $w'v' = s_\alpha wv$ for some $\alpha \in \Phi^+$ with $(wv)^{-1}\alpha \in \Phi^-$. Now the condition $v = v'$ implies $w' = s_\alpha w$. Condition (b.1) implies $v^{-1}\mu - (wv)^{-1}\alpha^\vee = (v')^{-1}\mu$, so $\mu' = \mu - w^{-1}\alpha^\vee$. We get (c.4).

Reversing the calculations above shows that (c) $\implies$ (b).

For (c) $\iff$ (d), we just explicitly rewrite the conditions for length positivity of v', and the definition of edges in the quantum Bruhat graph. ☐

Remark 4.6. If the translation part μ of $x = w\varepsilon^\mu$ is sufficiently regular, the estimates for the length function of x in part (d) of Proposition 4.5 are trivially satisfied. Writing $\mathrm{LP}(x) = \{v\}$, we get a one-to-one correspondence

$$\{\text{Bruhat covers of } x\} \leftrightarrow \{\text{edges } ? \to v\} \sqcup \{\text{edges } wv \to ?\}.$$

If Φ is simply laced and x to lies in a shrunken Weyl chamber, then still all the estimated for the length function of x in part (d) are satisfied. I.e. each edge $? \to v$ or $wv \to ?$ yields a Bruhat cover, but different edges might yield the same element x'.

If Φ is not simply laced, being in a shrunken Weyl chamber is not sufficient: Indeed, consider the case where $x = w_0$ (so $\mathrm{LP}(x) = \{1\}$) and α any short simple root. Then $s_\alpha \to 1$ is an edge in $\mathrm{QB}(W)$, but $x < xr_{(-\alpha,1)}$ is not a Bruhat cover.

We obtain the following useful technical observation from Proposition 4.5:

Corollary 4.7. *Let $x \in \widetilde{W}$, $v \in \mathrm{LP}(x)$ and $(\alpha, k) \in \Delta_{\mathrm{af}}$ with $\ell(x, \alpha) = 0$. If $v^{-1}\alpha \in \Phi^+$, then $s_\alpha v \in \mathrm{LP}(x)$.*

Proof. Since $x(\alpha, k) \in \Phi^+$ by Lemma 2.9, we have $x < xr_a$. Since a is a simple affine root, we must have $x \lessdot xr_a$. So one of the four possibilities (c.1) – (c.4) of Proposition 4.5 must be satisfied.

If (c.3) or (c.4) are satisfied, we get $v \in \mathrm{LP}(x')$. Since $x' = xr_a$ is a length additive product, Lemma 2.13 shows $s_\alpha v \in \mathrm{LP}(x)$, finishing the proof.

Now assume that (c.1) is satisfied. Then $x' = xs_\beta$ for some $\beta \in \Phi^+$ means $k = 0$ and $\alpha = \beta$. Now $v^{-1}\alpha \in \Phi^+$ means that $\ell(s_\alpha v) > \ell(v)$, so $s_\alpha v \to v$ cannot be a Bruhat edge.

Finally assume that (c.2) is satisfied. Then $x' = xr_{(-\beta,1)}$ for some $\beta \in \Phi^+$ means that $k = 1$ and $\alpha = -\beta \in \Phi^-$. Then $s_\alpha v \to v$ cannot be a quantum edge, as $\ell(s_\alpha v) < \ell(v)$.

We get the desired claim or a contradiction, finishing the proof. ☐

As a second application, we discuss the semi-infinite order on $\widetilde{W}$ as introduced by Lusztig [Lus80]. It plays a role for certain constructions related to the affine Hecke algebra, cf. [Lus80; NW17].

Definition 4.8. Let $x = w\varepsilon^\mu \in \widetilde{W}$.

(a) We define the *semi-infinite length* of x as

$$\ell^{\frac{\infty}{2}}(x) := \ell(w) + \langle \mu, 2\rho \rangle.$$

(b) We define the *semi-infinite order* on $\widetilde{W}$ to be the order $<^{\frac{\infty}{2}}$ generated by the relations

$$\forall x \in \widetilde{W}, a \in \Phi_{\mathrm{af}}: \ x <^{\frac{\infty}{2}} xr_a \text{ if } \ell^{\frac{\infty}{2}}(x) \leqslant \ell^{\frac{\infty}{2}}(xr_a).$$

We have the following link between the semi-infinite order and the Bruhat order:

Proposition 4.9 ([NW17, Proposition 2.2.2])**.** *Let $x_1, x_2 \in \widetilde{W}$. There exists a number $C > 0$ such that for all $\lambda \in \mathbb{Z}\Phi^\vee$ satisfying the regularity condition $\langle \lambda, \alpha \rangle > C$ for every positive root α, we have*

$$x_1 \leqslant^{\frac{\infty}{2}} x_2 \iff x_1\varepsilon^\lambda \leqslant x_2\varepsilon^\lambda.$$

Corollary 4.10. *Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$. Then $x_1 \leqslant^{\frac{\infty}{2}} x_2$ if and only if*

$$\mu_1 + \mathrm{wt}(w_1 \Rightarrow w_2) \leqslant \mu_2.$$

Proof. Let λ be as in Proposition 4.9. Choosing λ sufficiently large, we may assume that $x_1\varepsilon^\lambda$ and $x_2\varepsilon^\lambda$ are superregular with $\mathrm{LP}(x_1\varepsilon^\lambda) = \mathrm{LP}(x_2\varepsilon^\lambda) = \{1\}$. Now $x_1\varepsilon^\lambda \leqslant x_2\varepsilon^\lambda$ if and only if

$$\mu_1 + \mathrm{wt}(w_1 \Rightarrow w_2) \leqslant \mu_2,$$

by Corollary 4.4. □

We finish this section with another application of our Theorem 4.2, namely a discussion of admissible and permissible sets in $\widetilde{W}$, as introduced by Kottwitz and Rapoport [KR00].

Definition 4.11. Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $\lambda \in X_*(T)_{\Gamma_0}$ a dominant coweight.

(a) We say that x lies in the *admissible* set defined by λ, denoted $x \in \mathrm{Adm}(\lambda)$, if there exists $u \in W$ such that $x \leqslant \varepsilon^{u\lambda}$ with respect to the Bruhat order on $\widetilde{W}$.

(b) The *fundamental coweight* associated with $a = (\alpha, k) \in \Delta_{\mathrm{af}}$ is the uniquely determined element $\omega_a \in \mathbb{Q}\Phi^\vee$ such that for each $\beta \in \Delta$,

$$\langle \omega_a, \beta \rangle = \begin{cases} 1, & a = (\beta, 0), \\ 0, & a \neq (\beta, 0). \end{cases}$$

In particular, $\omega_a = 0$ iff $k \neq 0$.

(c) Let $a = (\alpha, k) \in \Delta_{\mathrm{af}}$, and denote by $\theta \in \Phi^+$ the longest root of the irreducible component of Φ containing α. The *normalized coweight* associated with a is

$$\widetilde{\omega}_a = \begin{cases} 0, & k \neq 0, \\ \frac{1}{\langle \omega_a, \theta \rangle}\omega_a, & k = 0. \end{cases}$$

(d) We say that x lies in the *permissible* set defined by λ, denoted $x \in \mathrm{Perm}(\lambda)$, if $\mu \equiv \lambda \pmod{\Phi^\vee}$ and for every simple affine root $a \in \Delta_{\mathrm{af}}$, we have

$$(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a)^{\mathrm{dom}} \leqslant \lambda \text{ in } X_*(T)_{\Gamma_0} \otimes \mathbb{Q}.$$

It is shown in [KR00] that the admissible set is always contained in the permissible set and that equality holds for the groups GL_n and GSp_{2n} if λ is *minuscule* (i.e. a fundamental coweight of some special node). It is a result of Haines and Ngô [HN02] that $\mathrm{Adm}(\lambda) \neq \mathrm{Perm}(\lambda)$ in general. We show how the latter result can be recovered using our methods.

Proposition 4.12 (Cf. [HY21, Prop. 3.3])**.** *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $\lambda \in X_*(T)_{\Gamma_0}$ a dominant coweight. Then the following are equivalent:*

(1) $x \in \mathrm{Adm}(\lambda)$.

(2) For all $v \in W$, we have

$$v^{-1}\mu + \mathrm{wt}(wv \Rightarrow v) \leqslant \lambda.$$

(3) For some $v \in \mathrm{LP}(x)$, we have

$$v^{-1}\mu + \mathrm{wt}(wv \Rightarrow v) \leqslant \lambda.$$

Proof. (1) $\implies$ (2): Suppose that $x \in \mathrm{Adm}(\lambda)$, so $x \leqslant \varepsilon^{u\lambda}$ for some $u \in W$. Let also $v \in W$. By Lemma 4.17, we find $\tilde{u} \in W$ such that

$$v^{-1}\mu + \mathrm{wt}(\tilde{u} \Rightarrow v) + \mathrm{wt}(wv \Rightarrow \tilde{u}) \leqslant \tilde{u}^{-1}u\lambda.$$

Thus

$$\begin{aligned} v^{-1}\mu + \mathrm{wt}(wv \Rightarrow v) \leqslant& v^{-1}\mu + \mathrm{wt}(\tilde{u} \Rightarrow v) + \mathrm{wt}(wv \Rightarrow \tilde{u}) \\ \leqslant& \tilde{u}^{-1}u\lambda \\ \leqslant& (\tilde{u}^{-1}u\lambda)^{\mathrm{dom}} = \lambda. \end{aligned}$$

Since (2) $\implies$ (3) is trivial, it remains to show (3) $\implies$ (1). So let $v \in \mathrm{LP}(x)$ satisfy $v^{-1}\mu + \mathrm{wt}(wv \Rightarrow v) \leqslant \lambda$. By Theorem 4.2, we immediately get $x \leqslant \varepsilon^{v\lambda}$, showing (1). □

Lemma 4.13. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $\lambda \in X_*(T)_{\Gamma_0}$ a dominant coweight. Then the following are equivalent:*

(1) $x \in \mathrm{Perm}(\lambda)$.

(2) For all $v \in W$, we have

$$v^{-1}\mu + \sup_{a \in \Delta_{\mathrm{af}}} \left(v^{-1}\widetilde{\omega}_a - (wv)^{-1}\widetilde{\omega}_a\right) \leqslant \lambda.$$

If moreover x lies in a shrunken Weyl chamber, the conditions are equivalent to

(3) For the uniquely determined $v \in \mathrm{LP}(x)$, we have

$$v^{-1}\mu + \sup_{a\in\Delta_{\mathrm{af}}} \left(v^{-1}\widetilde{\omega}_a - (wv)^{-1}\widetilde{\omega}_a\right) \leqslant \lambda.$$

Proof. We have

$$\begin{aligned}(1) &\iff \forall a \in \Delta_{\mathrm{af}}: \ \left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right)^{\mathrm{dom}} \leqslant \lambda\\ &\iff \forall a \in \Delta_{\mathrm{af}}, v \in W: \ v^{-1}\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right) \leqslant \lambda\\ &\iff \forall v \in W: \ \sup_{a\in\Delta_{\mathrm{af}}} v^{-1}\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right) \leqslant \lambda\\ &\iff (2).\end{aligned}$$

Now assume that x is in a shrunken Weyl chamber, $\mathrm{LP}(x) = \{v\}$ and $a \in \Delta_{\mathrm{af}}$. We claim that

$$\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right)^{\mathrm{dom}} = v^{-1}\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right).$$

Once this claim is proved, the equivalence (1) $\iff$ (3) follows.

It remains to show that $v^{-1}\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right)$ is dominant. Hence let $\alpha \in \Phi^+$. We obtain

$$\begin{aligned}\left\langle v^{-1}\left(\mu + \widetilde{\omega}_a - w^{-1}\widetilde{\omega}_a\right), \alpha\right\rangle &= \langle \mu, v\alpha\rangle + \langle \widetilde{\omega}_a, v\alpha\rangle - \langle \widetilde{\omega}_a, wv\alpha\rangle\\ &\geqslant \langle \mu, v\alpha\rangle - \Phi^+(-v\alpha) - \Phi^+(wv\alpha)\\ &= \ell(x, v\alpha) - 1 \geqslant 0.\end{aligned}$$

□

Corollary 4.14 ([KR00, Sec. 11.2])**.** *For all dominant $\lambda \in X_*$, the admissible set is contained in the permissible set,* $\mathrm{Adm}(\lambda) \subseteq \mathrm{Perm}(\lambda)$*.*

Proof. Let $x = w\varepsilon^\mu \in \widetilde{W}$. Then

$$\begin{aligned}x \in \mathrm{Adm}(\lambda) &\underset{\text{P4.12}}{\Longrightarrow} \forall v \in W: \ v^{-1}\mu + \mathrm{wt}(wv \Rightarrow v) \leqslant \lambda\\ &\underset{(3.44)}{\Longrightarrow} \forall v \in W: \ v^{-1}\mu + \sup_{a\in\Delta_{\mathrm{af}}} v^{-1}\widetilde{\omega}_a - (wv)^{-1}\widetilde{\omega}_a \leqslant \lambda\\ &\underset{\text{L4.13}}{\Longrightarrow} x \in \mathrm{Perm}(\lambda).\end{aligned}$$

□

Corollary 4.15. *For any fixed root system Φ, the following are equivalent:*

(1) For all dominant $\lambda \in X_*(T)_{\Gamma_0}$, we get the equality $\mathrm{Adm}(\lambda) = \mathrm{Perm}(\lambda)$*.*

(2) For all $w_1, w_2 \in W$, the element

$$\left\lceil \sup_{a\in\Delta_{\mathrm{af}}} w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a \right\rceil := \min\{z \in \mathbb{Z}\Phi^\vee \mid z \geqslant \sup_{a\in\Delta_{\mathrm{af}}} w_2^{-1}\widetilde{\omega}_a - w_1^{-1}\widetilde{\omega}_a \text{ in } \mathbb{Q}\Phi^\vee\}$$

agrees with $\mathrm{wt}(w_1 \Rightarrow w_2)$*.*

(3) Each irreducible component of Φ is of type A_n $(n \geqslant 1)$, B_2, C_3 or G_2.

Proof. (1) $\implies$ (2): Comparing condition (3) of Proposition 4.12 with condition (3) of Lemma 4.13 for superregular elements $x \in \widetilde{W}$ yields the desired claim.

(2) $\implies$ (1): We can directly compare condition (2) of Proposition 4.12 with condition (2) of Lemma 4.13.

(2) $\iff$ (3): Call an irreducible root system Φ' *good* if condition (2) is satisfied for Φ', and *bad* otherwise. Certainly, Φ is good iff each irreducible component of Φ is good. Moreover, root systems of type A_n are good, we saw this in formula (3.1) and again in Remark 3.43.

If $\Phi_J \subseteq \Phi$ is bad for some $J \subseteq \Delta$, then certainly Φ is bad as well (cf. Corollary 3.11). It remains to show that root systems of types C_3 and G_2 are good, and that root systems of types B_3, C_4 and D_4 are bad. Each of these claims is easily verified using the Sagemath computer algebra system [Sage; SaCo]. $\square$

For irreducible root systems of rank $\geqslant 4$, the equivalence (1) $\iff$ (3) is due to [HN02], using a result of Deodhar:

Proposition 4.16 ([Deo78])**.** *For any fixed root system Φ, the following are equivalent:*

(1) For all $w_1, w_2 \in W$, we have

$$w_1 \leqslant w_2 \iff \sup_{a \in \Delta_{\mathrm{af}}} w_2^{-1}\omega_a - w_1^{-1}\omega_a \leqslant 0$$

(2) Each irreducible component of Φ has rank $\leqslant 3$ or is of type A_n $(n \geqslant 1)$. $\square$

4.2. Proof of the criterion

The goal of this section is to prove Theorem 4.2. We start with the direction (1) $\implies$ (2), which is the easier one.

Lemma 4.17. *Let $x = w\varepsilon^{\mu}, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$ and $v \in W$. If $x \leqslant x'$, then there exists an element $v' \in W$ such that*

$$v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu.$$

Proof. First note that the relation

$$x \preceq x' :\iff \forall v \exists v' : \ v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu$$

is transitive. Thus, it suffices to show the implication $x \leqslant x' \implies x \preceq x'$ for generators (x, x') of the Bruhat order.

In other words, we may assume that $x' = xr_{\mathbf{a}}$ for an affine root $\mathbf{a} = (\alpha, k) \in \Phi_{\mathrm{af}}^+$ with

$$x\mathbf{a} = (w\alpha, k - \langle \mu, \alpha \rangle) \in \Phi_{\mathrm{af}}^+.$$

This means that $w' = ws_\alpha$ and $\mu' = \mu + (k - \langle \mu, \alpha\rangle)\alpha^\vee$, where $k - \langle\mu,\alpha\rangle \geqslant \Phi^+(-w\alpha)$. We now do a case distinction depending on whether the root $v^{-1}\alpha$ is positive or negative.

Case $v^{-1}\alpha \in \Phi^-$. Put $v' = s_\alpha v$ such that $wv = w'v'$. Then using Corollary 3.17,

$$\begin{aligned}
&v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v')\\
=\,&v^{-1}\mu + \mathrm{wt}(vs_{-v^{-1}\alpha} \Rightarrow v) + 0\\
\leqslant\,&v^{-1}\mu - \Phi^+(-\alpha)v^{-1}\alpha^\vee\\
\leqslant\,&v^{-1}\mu - kv^{-1}\alpha^\vee\\
=\,&(s_\alpha v)^{-1}(s_\alpha(\mu) + k\alpha^\vee) = (v')^{-1}\mu'.
\end{aligned}$$

Case $v^{-1}\alpha \in \Phi^+$. Put $v' = v$ such that $w'v' = wvs_{v^{-1}\alpha}$. Then using Corollary 3.17,

$$\begin{aligned}
&v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v')\\
=\,&v^{-1}\mu + \mathrm{wt}(wv \Rightarrow wvs_{v^{-1}\alpha})\\
\leqslant\,&v^{-1}\mu + \Phi^+(-w\alpha)v^{-1}\alpha^\vee\\
\leqslant\,&v^{-1}\mu + (k - \langle\mu,\alpha\rangle)\alpha^\vee = (v')^{-1}\mu'.
\end{aligned}$$

This finishes the proof. □

The direction (1) $\implies$ (2) of Theorem 4.2 follows directly from this lemma. We now start the journey to prove (2) $\implies$ (1).

Lemma 4.18. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, and suppose that $(1, J_1, \ldots, J_m)$ is a Bruhat-deciding datum for both x and x'. If the inequality*

$$\mu + \mathrm{wt}(w \Rightarrow w') \leqslant \mu' \pmod{\Phi^\vee_{J_i}}$$

holds for $i = 1, \ldots, m$, then $x \leqslant x'$.

Proof. Let $J = J_1 \cap \cdots \cap J_m$. Then we get

$$\mu + \mathrm{wt}(w \Rightarrow w') \leqslant \mu' \pmod{\Phi^\vee_J}.$$

Let $C_1 := \ell(x^{-1}x')$ and pick $C_2 > 0$ such that the conclusion of Corollary 3.16 holds true. We can find an element $\lambda \in \mathbb{Z}\Phi^\vee$ such that $\langle\lambda,\alpha\rangle = 0$ for all $\alpha \in J$ and

$$\langle\lambda,\alpha\rangle \geqslant C_2$$

for all $\alpha \in \Phi^+ \backslash \Phi_J$. Since $1 \in W$ is length positive for both x and x', it follows from Lemma 2.13 that

$$\ell(x\varepsilon^\lambda) = \ell(x) + \ell(\varepsilon^\lambda), \qquad \ell(x'\varepsilon^\lambda) = \ell(x') + \ell(\varepsilon^\lambda).$$

So it suffices to show $x\varepsilon^\lambda \leqslant x'\varepsilon^\lambda$. Note that $x\varepsilon^\lambda, x'\varepsilon^\lambda \in \Omega_J^{C_2}$ by choice of λ. Moreover, we have

$$\mu + \lambda + \mathrm{wt}(w \Rightarrow w') \leqslant \mu' + \lambda \pmod{\Phi^\vee_J}$$

by assumption. Therefore, the inequality $x\varepsilon^\lambda \leqslant x'\varepsilon^\lambda$ follows from Corollary 3.16. □

Lemma 4.19. *Let $x = w\varepsilon^{\mu}, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, and suppose that $(1, J_1, \dots, J_m)$ is a Bruhat-deciding datum for x. If the inequality*

$$\mu + \mathrm{wt}(w \Rightarrow w') \leqslant \mu' \pmod{\Phi^\vee_{J_i}}$$

holds for $i = 1, \dots, m$, then $x \leqslant x'$.

Proof. Induction on $\ell(x')$.

If $(1, J, \dots, J_m)$ is also Bruhat-deciding for x', we are done by Lemma 4.18. Otherwise, we must have that $1 \in W$ is not length positive for x', or that $J := J_1 \cap \cdots \cap J_m$ allows some $\alpha \in \Phi_J$ with $\ell(x', \alpha) \neq 0$.

First consider the case that $1 \in W$ is not length positive for x'. Then we find a positive root $\alpha \in \Phi^+$ with $\ell(x', \alpha) < 0$. Hence $a := (-\alpha, 1) \in \Phi^+_{\mathrm{af}}$ with $x'a \in \Phi^-$, so that

$$x'' := w''\varepsilon^{\mu''} := x' r_a = w' s_\alpha \varepsilon^{\mu' - (1 + \langle \mu', \alpha\rangle)\alpha^\vee} < x'.$$

We calculate

$$\begin{aligned}
\mu + \mathrm{wt}(w \Rightarrow w'') \leqslant& \mu + \mathrm{wt}(w \Rightarrow w') + \mathrm{wt}(w' \Rightarrow w' s_\alpha)\\
\leqslant& \mu' + \Phi^+(-w'\alpha)\alpha^\vee\\
=& \mu' - (1 + \langle \mu', \alpha\rangle)\alpha^\vee + (\langle \mu', \alpha\rangle + 1 + \Phi^+(-w'\alpha))\alpha^\vee\\
=& \mu'' + (\ell(x', \alpha) + 1)\alpha^\vee \leqslant \mu'' \pmod{\Phi^\vee_J}.
\end{aligned}$$

By induction, $x \leqslant x''$. Since $x'' < x'$, we conclude $x < x'$ and are done.

Next consider the case that $1 \in W$ is indeed length positive for x', but we find some $\alpha \in \Phi_J$ with $\ell(x', \alpha) \neq 0$. We may assume $\alpha \in \Phi^+$, and then $\ell(x', \alpha) > 0$ by length positivity. Then $a = (\alpha, 0) \in \Phi^+_{\mathrm{af}}$ with $x'a \in \Phi^-$. We conclude that

$$x'' := w''\varepsilon^{\mu''} := x' r_a = w' s_\alpha \varepsilon^{\mu' - \langle \mu', \alpha\rangle\alpha^\vee} < x'.$$

We calculate

$$\begin{aligned}
\mu + \mathrm{wt}(w \Rightarrow w'') \leqslant& \mu + \mathrm{wt}(w \Rightarrow w') + \mathrm{wt}(w' \Rightarrow w' s_\alpha)\\
\leqslant& \mu' + \Phi^+(-w'\alpha)\alpha^\vee\\
=& \mu'' + (\Phi^+(-w'\alpha) + \langle \mu', \alpha\rangle)\alpha^\vee\\
\equiv& \mu'' \pmod{\Phi^\vee_J},
\end{aligned}$$

as $\alpha^\vee \in \Phi^\vee_J$. So as in the previous case, we get $x \leqslant x'' < x'$ and are done.

This completes the induction and the proof. ∎

Before we can continue the series of incremental generalizations, we need a technical lemma.

Lemma 4.20. *Let $x = w\varepsilon^{\mu}, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$. Let $J \subseteq \Delta$ and $v' \in W$ be given such that*

$$\mu + \mathrm{wt}(v' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi^\vee_J}.$$

Then there exists an element $v'' \in W$ satisfying the same inequality as v' above, and satisfying moreover the condition $\ell(x', \gamma) < 0$ for all $\gamma \in \max \mathrm{inv}(v'')$.

Proof. Among all $v' \in W$ satisfying the inequality

$$\mu + \mathrm{wt}(v' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi_J^\vee},$$

pick one of minimal length in W. We prove that $\ell(x', \gamma) < 0$ for all $\gamma \in \max \mathrm{inv}(v')$.

Suppose that this was not the case, so $\ell(x', \gamma) \geqslant 0$ for some $\gamma \in \max \mathrm{inv}(v')$. The condition $\gamma \in \mathrm{inv}(v')$ implies $\ell(s_\gamma v') < \ell(v')$. Moreover, $\mathrm{wt}(v' \Rightarrow 1) = \mathrm{wt}(s_\gamma v' \Rightarrow 1) - (v')^{-1}\gamma^\vee$ by Proposition 3.23. We calculate

$$\begin{aligned}
&\mu + \mathrm{wt}(s_\gamma v' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w's_\gamma v') \\
&\qquad =\mu + \mathrm{wt}(v' \Rightarrow 1) + (v')^{-1}\gamma^\vee + \mathrm{wt}(w \Rightarrow w's_\gamma v') \\
&\qquad \leqslant\mu + \mathrm{wt}(v' \Rightarrow 1) + (v')^{-1}\gamma^\vee + \mathrm{wt}(w \Rightarrow w'v') + \mathrm{wt}(w'v' \Rightarrow w's_\gamma v') \\
&\qquad \leqslant(v')^{-1}\mu' + (v')^{-1}\gamma^\vee + \mathrm{wt}(w'v' \Rightarrow w's_\gamma v') \\
&\qquad =(v')^{-1}\mu' + (v')^{-1}\gamma^\vee + \mathrm{wt}(w's_\gamma v's_{-(v')^{-1}(\gamma)} \Rightarrow w's_\gamma v') \\
&\qquad \leqslant(v')^{-1}\mu' + (v')^{-1}\gamma^\vee - \Phi^+(w'\gamma)(v')^{-1}\gamma^\vee \\
&\qquad =(s_\gamma v')^{-1}\mu' + \langle \mu', \gamma\rangle(v')^{-1}\gamma^\vee + (v')^{-1}\gamma^\vee - \Phi^+(w'\gamma)(v')^{-1}\gamma^\vee \\
&\qquad =(s_\gamma v')^{-1}\mu' + \ell(x', \gamma)(v')^{-1}\gamma^\vee \leqslant (s_\gamma v')^{-1}\mu' \pmod{\Phi_J^\vee}.
\end{aligned}$$

This is a contradiction to the choice of v', so we get the desired claim. □

Lemma 4.21. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, and suppose that $(1, J_1, \ldots, J_m)$ is a Bruhat-deciding datum for x. If for each $i = 1, \ldots, m$, there exists some $v_i' \in W$ with*

$$\mu + \mathrm{wt}(v_i' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'v_i') \leqslant (v_i')^{-1}\mu' \pmod{\Phi_{J_i}^\vee},$$

then $x \leqslant x'$.

Proof. Induction on $\ell(x')$.

By Lemma 4.20, we may assume that for each $i \in \{1, \ldots, m\}$ and $\gamma \in \max \mathrm{inv}(v_i')$, we have $\ell(x', \gamma) < 0$.

If $1 \in W$ is length positive for x', i.e. $\ell(x', \alpha) \geqslant 0$ for all $\alpha \in \Phi^+$, then we get $\max \mathrm{inv}(v_i') = \varnothing$ for all $i = 1, \ldots, m$, i.e. $v_i' = 1$. Now the claim follows from Lemma 4.19.

Thus suppose that the set

$$\{\alpha \in \Phi^+ \mid \ell(x', \alpha) < 0\}$$

is non-empty. We fix a root α that is maximal within this set. Now $\mathbf{a} = (-\alpha, 1) \in \Phi_{\mathrm{af}}^+$ satisfies $x'\mathbf{a} \in \Phi_{\mathrm{af}}^-$, as $\ell(x', \alpha) < 0$. Consider

$$x'' := w''\varepsilon^{\mu''} := x'r_{\mathbf{a}} = w's_\alpha \varepsilon^{\mu' - (1 + \langle \mu', \alpha\rangle)\alpha^\vee} < x'.$$

We want to show $x \leqslant x''$ using the inductive assumption. So pick an index $i \in \{1, \ldots, m\}$. We do a case distinction based on whether the root $(v_i')^{-1}\alpha$ is positive or negative.

Case $(v_i')^{-1}\alpha \in \Phi^-$. Then $\alpha \in \mathrm{inv}(v_i')$, so there exists some $\gamma \in \max \mathrm{inv}(v_i')$ with $\alpha \leqslant \gamma$. By choice of v_i', we get $\ell(x', \gamma) < 0$. By maximality of α and $\alpha \leqslant \gamma$, we get $\alpha = \gamma$. In other words, $\alpha \in \max \mathrm{inv}(v_i')$.

Define $v_i'' := s_\alpha v_i'$. Then by Proposition 3.23, $\mathrm{wt}(v_i' \Rightarrow 1) = \mathrm{wt}(v_i'' \Rightarrow 1) - (v_i')^{-1}\alpha^\vee$. We compute

$$\begin{aligned}
&\mu + \mathrm{wt}(v_i'' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'' v_i'')\\
&= \mu + \mathrm{wt}(v_i' \Rightarrow 1) + (v_i')^{-1}\alpha^\vee + \mathrm{wt}(w \Rightarrow w' v_i')\\
&\leqslant (v_i')^{-1}\mu' + (v_i')^{-1}\alpha^\vee\\
&= (s_\alpha v_i')^{-1}(\mu' - (1 + \langle \mu', \alpha\rangle)\alpha^\vee) = (v_i'')^{-1}\mu'' \pmod{\Phi_{J_i}^\vee}.
\end{aligned}$$

Case $(v_i')^{-1}\alpha \in \Phi^+$. We define $v_i'' := v_i'$ and use Corollary 3.17 to compute

$$\begin{aligned}
&\mu + \mathrm{wt}(v_i'' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'' v_i'')\\
&\leqslant \mu + \mathrm{wt}(v_i' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w' v_i') + \mathrm{wt}(w' v_i' \Rightarrow w' v_i' s_{(v_i')^{-1}\alpha})\\
&\leqslant (v_i')^{-1}\mu' + \Phi^+(-w'\alpha)(v_i')^{-1}\alpha^\vee\\
&= (v_i')^{-1}(\mu' - (1 + \langle \mu', \alpha\rangle)\alpha^\vee) + (\langle \mu', \alpha\rangle + 1 + \Phi^+(-w'\alpha))(v_i')^{-1}\alpha^\vee\\
&= (v_i')^{-1}\mu'' + (\ell(x', \alpha) + 1)(v_i')^{-1}\alpha^\vee \leqslant (v_i'')^{-1}\mu'' \pmod{\Phi_{J_i}^\vee}.
\end{aligned}$$

In any case, we get the desired inequality

$$\mu + \mathrm{wt}(v_i'' \Rightarrow 1) + \mathrm{wt}(w \Rightarrow w'' v_i'') \leqslant (v_i'')^{-1}\mu'' \pmod{\Phi_{J_i}^\vee}.$$

By induction, $x \leqslant x'' < x'$, completing the induction and the proof. □

Lemma 4.22. *Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, and suppose that $(v, J_1, \ldots, J_m)$ is a Bruhat-deciding datum for x. If for each $i = 1, \ldots, m$, there exists some $v_i' \in W$ with*

$$v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w' v_i') \leqslant (v_i')^{-1}\mu' \pmod{\Phi_{J_i}^\vee},$$

then $x \leqslant x'$.

Proof. Induction on $\ell(v)$. If $v = 1$, this follows from Lemma 4.21.

Let $J := J_1 \cap \cdots \cap J_m$. If $\alpha \in J$, then vs_α trivially satisfies the same condition as v. So we may assume that $v \in W^J$.

Since $v \neq 1$, we find a simple root $\alpha \in \Delta$ with $v^{-1}\alpha \in \Phi^-$. In particular, $\ell(x, \alpha) \leqslant 0$, such that $x < xs_\alpha$.

We claim that $(s_\alpha v, J_1, \ldots, J_m)$ is a Bruhat-deciding datum for xs_α. Indeed, for $\beta \in \Phi$, we use Lemma 2.12 to compute

$$\begin{aligned}
\ell(xs_\alpha, s_\alpha v\beta) =& \ell(x, v\beta) + \ell(s_\alpha, s_\alpha v\beta)\\
=& \ \ell(x, v\beta) + \begin{cases} 1, & v\beta = -\alpha,\\ -1, & v\beta = \alpha,\\ 0, & v\beta \neq \pm\alpha.\end{cases}
\end{aligned}$$

If $\beta \in \Phi^+$, the condition $v^{-1}\alpha \in \Phi^-$ forces $v\beta \neq \alpha$, showing

$$\ell(xs_\alpha, s_\alpha v\beta) \geqslant \ell(x, v\beta) \geqslant 0.$$

Now consider the case $\beta \in \Phi_J^+$. Then $\ell(x, v\beta) = 0$ by assumption. Moreover, $v\beta \in \Phi^+$ as $v \in W^J$, so that $v\beta \neq -\alpha$. We conclude $\ell(xs_\alpha, s_\alpha v\beta) = \ell(x, v\beta) = 0$ in this case.

This shows that $(s_\alpha v, J_1, \dots, J_m)$ is Bruhat-deciding for xs_α. Since $\ell(s_\alpha v) < \ell(v)$, we may apply the inductive hypothesis to xs_α to prove $xs_\alpha \leqslant \max(x', x's_\alpha)$. We distinguish two cases.

Case $\ell(x', \alpha) \leqslant 0$. This means $x' < x's_\alpha$, so we wish to prove $xs_\alpha < x's_\alpha$, using the inductive hypothesis. So let $i \in \{1, \dots, m\}$. By Lemma 4.3, we may assume that v_i' is length positive for x'.

First assume that $(v_i')^{-1}\alpha \in \Phi^-$. By Lemma 3.8, we get

$$\mathrm{wt}(v_i' \Rightarrow v) = \mathrm{wt}(s_\alpha v_i' \Rightarrow s_\alpha v).$$

Define $v_i'' := s_\alpha v_i'$. Then

$$\begin{aligned}
&(s_\alpha v)^{-1}(s_\alpha \mu) + \mathrm{wt}(v_i'' \Rightarrow s_\alpha v) + \mathrm{wt}(ws_\alpha s_\alpha v \Rightarrow w's_\alpha v_i'') \\
&= v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v_i') \\
&\leqslant (v_i')^{-1}\mu' = (v_i'')(s_\alpha \mu') \pmod{\Phi_{J_i}^\vee}.
\end{aligned}$$

Next, assume that $(v_i')^{-1}\alpha \in \Phi^+$. By length positivity, we must have $\ell(x', \alpha) = 0$. By Lemma 3.8, we get

$$\mathrm{wt}(v_i' \Rightarrow v) = \mathrm{wt}(v_i' \Rightarrow s_\alpha v).$$

Define $v_i'' := v_i'$. Then using Corollary 3.17,

$$\begin{aligned}
&(s_\alpha v)^{-1}(s_\alpha \mu) + \mathrm{wt}(v_i'' \Rightarrow s_\alpha v) + \mathrm{wt}(ws_\alpha s_\alpha v \Rightarrow w's_\alpha v_i'') \\
&= v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w's_\alpha v_i') \\
&\leqslant v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v_i') + \mathrm{wt}(w'v_i' \Rightarrow w'v_i's_{(v_i')^{-1}\alpha}) \\
&\leqslant (v_i')^{-1}\mu' + \Phi^+(-w'\alpha)(v_i')^{-1}\alpha \\
&= (v_i')^{-1}s_\alpha\mu' + (\langle \mu', \alpha\rangle + \Phi^+(-w'\alpha))(v_i')^{-1}\alpha \\
&= (v_i'')^{-1}s_\alpha\mu' + \ell(x', \alpha)(v_i')^{-1}\alpha = (v_i'')^{-1}s_\alpha\mu. \pmod{\Phi_{J_i}^\vee}.
\end{aligned}$$

We see that the inequality

$$(s_\alpha v)^{-1}(s_\alpha \mu) + \mathrm{wt}(v_i'' \Rightarrow s_\alpha v) + \mathrm{wt}(ws_\alpha s_\alpha v \Rightarrow w's_\alpha v_i'') \leqslant (v_i'')^{-1}s_\alpha\mu \pmod{\Phi_{J_i}^\vee}$$

always holds, proving $xs_\alpha \leqslant x's_\alpha$. Since s_α is a simple reflection in $\widetilde{W}$, $x < xs_\alpha$ and $x' < x's_\alpha$, we conclude that $x \leqslant x'$ must hold as well.

Case $\ell(x', \alpha) > 0$. We now wish to show $xs_\alpha \leqslant x'$, as $x' > x's_\alpha$. We prove this using the inductive assumption, so let $i \in \{1, \dots, m\}$. As in the previous case, we assume that v_i' is length positive for x'. In particular, $(v_i')^{-1}\alpha \in \Phi^+$.

By Lemma 3.8, we get

$$\mathrm{wt}(v_i' \Rightarrow v) = \mathrm{wt}(v_i' \Rightarrow s_\alpha v).$$

Define $v_i'' := v_i'$. Then

$$\begin{aligned}&(s_\alpha v)^{-1}(s_\alpha\mu) + \mathrm{wt}(v_i'' \Rightarrow s_\alpha v) + \mathrm{wt}(w s_\alpha s_\alpha v \Rightarrow w' v_i'')\\ =& v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v_i')\\ \leqslant& (v_i')^{-1}\mu' = (v_i'')^{-1}\mu'.\end{aligned}$$

By the inductive assumption, we get $xs_\alpha \leqslant x'$. Thus $x < xs_\alpha \leqslant x'$.

This completes the induction and the proof. □

Proof of Theorem 4.2. The implication (1) ⇒ (2) follows from Lemma 4.17.

The implication (2) ⇒ (1) follows from Lemma 4.22. □

4.3. Deodhar's lemma

In this section, we apply Deodhar's lemma [Deo77] to our Theorem 4.2. We need the semi-affine weight functions and related notions as introduced in Section 3.4. We moreover need a two-sided version of Deodhar's lemma, which seems to be well-known for experts, yet our standard reference [BB05, Theorem 2.6.1] only provides a one-sided version. We thus introduce the two-sided theory briefly. For convenience, we state it for the extended affine Weyl group $\widetilde W$, even though it holds true in a more general Coxeter theoretic context.

Definition 4.23. Let $L, R \subseteq \Phi_{\mathrm{af}}$ be any sets of affine roots (we will mostly be interested in sets of simple affine roots).

(a) By $\widetilde W_L$, we denote the subgroup of $\widetilde W$ generated by the affine reflections r_a for $a \in L$.

(b) We define

$$^L\widetilde W^R := \{x \in \widetilde W : x^{-1}L \subseteq \Phi_{\mathrm{af}}^+ \text{ and } xR \subseteq \Phi_{\mathrm{af}}^+\}.$$

Recall that we called a subset $L \subseteq \Delta_{\mathrm{af}}$ *regular* if $\widetilde W_L$ is finite.

Proposition 4.24. *Let $x, y \in \widetilde W$ and $L, R \subseteq \Delta_{\mathrm{af}}$ be regular.*

(a) The double coset $\widetilde W_L x \widetilde W_R$ contains a unique element of minimal length, denoted $^Lx^R$, and a unique element of maximal length, denoted $^{-L}x^{-R}$. We have

$$\begin{aligned}{}^L\widetilde W^R \cap \left(\widetilde W_L x \widetilde W_R\right) &= \left\{{}^Lx^R\right\},\\ {}^{-L}\widetilde W^{-R} \cap \left(\widetilde W_L x \widetilde W_R\right) &= \left\{{}^{-L}x^{-R}\right\}.\end{aligned}$$

(b) We have

$$^Lx^R \leqslant x \leqslant {}^{-L}x^{-R}$$

in the Bruhat order, and there exist (non-unique) elements $x_L, x'_L \in \widetilde{W}_L$ *and* $x_R, x'_R \in \widetilde{W}_R$ *such that*

$$x = x_L \cdot {}^L x^R \cdot x_R \text{ and } \ell(x) = \ell(x_L) + \ell\left({}^L x^R\right) + \ell(x_R),$$
$${}^{-L}x^{-R} = x'_L \cdot x \cdot x'_R \text{ and } \ell\left({}^{-L}x^{-R}\right) = \ell(x'_L) + \ell(x) + \ell(x'_R).$$

(c) If $x \leqslant y$*, then*

$${}^L x^R \leqslant {}^L y^R \text{ and } {}^{-L}x^{-R} \leqslant {}^{-L}y^{-R}.$$

(d) Suppose $L_1, \dots, L_\ell, R_1, \dots, R_r \subseteq \Delta_{\mathrm{af}}$ *are regular subsets such that* $L = L_1 \cap \cdots \cap L_\ell$ *and* $R = R_1 \cap \cdots \cap R_r$*. Then*

$${}^L x^R \leqslant {}^L y^R \iff \forall i, j : {}^{L_i} x^{R_j} \leqslant {}^{L_i} y^{R_j}.$$

Proof. (a) We only show the claim for ${}^L x^R$, as the proof for ${}^{-L}x^{-R}$ is analogous.

Let $x_1 \in \widetilde{W}_L x \widetilde{W}_R$ an element of minimal length. It is clear that each such element must lie in ${}^L\widetilde{W}^R$.

Let now $x_0 \in {}^L\widetilde{W}^R \cap \left(\widetilde{W}_L x \widetilde{W}_R\right)$ be any element. It suffices to show that $x_0 = x_1$.

Since $x_1 \in \widetilde{W}_L x_0 \widetilde{W}_R$, we find $x_L \in \widetilde{W}_L, x_R \in \widetilde{W}_R$ such that $x_1 = x_L x_0 x_R$. We show $x_1 = x_0$ via induction on $\ell(x_L)$. If $x_L = 1$, the claim is evident.

As $x_0 \in {}^L\widetilde{W}^R$ and $x_R \in \widetilde{W}_R$, it follows that $\ell(x_0 x_R) = \ell(x_0) + \ell(x_R)$, cf. Lemma 2.13 or [BB05, Proposition 2.4.4]. Now

$$\ell(x_0) \geqslant \ell(x_1) = \ell(x_L x_0 x_R) \geqslant \ell(x_0 x_R) - \ell(x_L) = \ell(x_0) + \ell(x_R) - \ell(x_L).$$

We conclude that $\ell(x_L) \geqslant \ell(x_R)$. By an analogous argument, we get $\ell(x_L) \leqslant \ell(x_R)$, such that $\ell(x_L) = \ell(x_R)$. It follows that

$$\ell(x_0) = \ell(x_1) = \ell(x_L x_0 x_R) = \ell(x_0 x_R) - \ell(x_L).$$

Since we may assume $x_L \neq 1$, we find a simple affine root $a \in L$ with $x_L(a) \in \Phi_{\mathrm{af}}^-$, so that $(x_0 x_R)^{-1}(a) \in \Phi_{\mathrm{af}}^-$. Since $x_0 \in {}^L\widetilde{W}^R$, we have $x_0^{-1}(a) \in \Phi_{\mathrm{af}}^+$, so $r_{x_0^{-1}(a)} x_R < x_R$. We see that we can write

$$x_1 = x_L x_0 x_R = \underbrace{(x_L r_a)}_{<x_L} x_0 \underbrace{(r_{x_0^{-1}(a)} x_R)}_{<x_R},$$

finishing the induction and thus the proof.

(b) The claims on the Bruhat order are implied by the claimed existences of length additive products, so it suffices to show the latter. We again focus on ${}^L x^R$.

Among all elements in

$$\{\tilde{x} \in \widetilde{W} \mid \exists x_L \in \widetilde{W}_L, x_R \in \widetilde{W}_R : \ x = x_L \tilde{x} x_R \text{ and } \ell(x) = \ell(x_L) + \ell(\tilde{x}) + \ell(x_R)\},$$

choose an element x_0 of minimal length. As in (a), one shows easily that $x_0 \in {}^L\widetilde{W}^R$. By (a), we get $x_0 = {}^L x^R$, so the claim follows.

(c) This is [BB05, Proposition 2.5.1].

(d) If ${}^L x^R \leqslant {}^L y^R$ and $i \in \{1, \dots, \ell\}, j \in \{1, \dots, r\}$, we get $L \subseteq L_i, R \subseteq R_i$ such that

$${}^{L_i} x^{R_j} = {}^{L_i} \left({}^L x^R\right)^{R_j} \underset{\text{(c)}}{\leqslant} {}^{L_i} \left({}^L y^R\right)^{R_j} = {}^{L_i} y^{R_j}.$$

It remains to show the converse.

In case $R = \varnothing$ and $r = 0$, this is exactly [BB05, Theorem 2.6.1]. Similarly, the claim follows if $L = \varnothing$ and $\ell = 0$. Writing ${}^L x^R = {}^L \left(x^R\right)$ etc. one reduces the claim to applying [BB05, Theorem 2.6.1] twice. ☐

We first describe a replacement for the length functional $\ell(x, \cdot)$ that is well-behaved with passing to ${}^L x^R$.

Definition 4.25. Let $L, R \subseteq \Delta_{\mathrm{af}}$ be regular. Then we define for each $x = w\varepsilon^\mu \in \widetilde{W}$ the *coset length functional*

$$\begin{aligned} &{}^L\ell^R(x, \cdot) : \Phi \to \mathbb{Z}, \quad \alpha \mapsto {}^L\ell^R(x, \alpha), \\ &{}^L\ell^R(x, \alpha) := \langle \mu, \alpha \rangle + \chi_R(\alpha) - \chi_L(w\alpha). \end{aligned}$$

We refer to Definition 3.33 for the definition of χ_L, χ_R.

Lemma 4.26. *Let $K, L, R \subseteq \Delta_{\mathrm{af}}$ be regular subsets and let $x = w\varepsilon^\mu \in \widetilde{W}$.*

(a) For $\alpha \in \Phi$, we have

$$\chi_K(\alpha) + \chi_K(-\alpha) = \begin{cases} 1, & \alpha \in \Phi \backslash \Phi_K, \\ 0, & \alpha \in \Phi_K. \end{cases}$$

If $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta \in \Phi$, then

$$\chi_K(\alpha) + \chi_K(\beta) - \chi_K(\alpha + \beta) \in \{0, 1\}.$$

(b) ${}^L\ell^R(x, \cdot)$ is a root functional, as studied in Section 2.2.

Proof. (a) We have

$$\chi_K(\alpha) + \chi_K(-\alpha) = 1 - \Phi_K^+(\alpha) - \Phi_K^+(-\alpha) = \begin{cases} 1, & \alpha \in \Phi \backslash \Phi_K, \\ 0, & \alpha \in \Phi_K. \end{cases}$$

Now suppose $\alpha + \beta \in \Phi$. Observe that the set

$$R := \Phi_{\mathrm{af}}^- \cup (\Phi_{\mathrm{af}})_K \subseteq \Phi_{\mathrm{af}}$$

is closed under addition, in the sense that for $a, b \in R$ with $a + b \in \Phi_{\mathrm{af}}$, we have $a + b \in R$.

By definition, $(\alpha, -\chi_K(\alpha)), (\beta, -\chi_K(\beta)) \in R$. Thus

$$c := (\alpha + \beta, -\chi_K(\alpha) - \chi_K(\beta)) \in R.$$

If $c \in (\Phi_{\mathrm{af}})_K$, then $\chi_K(\alpha + \beta) = \chi_K(\alpha) + \chi_K(\beta)$ by definition of $\chi_K(\alpha + \beta)$. Hence let us assume that $c \in \Phi_{\mathrm{af}}^- \backslash (\Phi_{\mathrm{af}})_K$.

The condition $c \in \Phi_{\mathrm{af}}^-$ means that

$$-\chi_K(\alpha) - \chi_K(\beta) \leqslant -\Phi^+(\alpha + \beta) \leqslant -\chi_K(\alpha + \beta).$$

This shows $\chi_K(\alpha) + \chi_K(\beta) - \chi_K(\alpha + \beta) \geqslant 0$. We want to show it lies in $\{0, 1\}$, so suppose that

$$\chi_K(\alpha) + \chi_K(\beta) - \chi_K(\alpha + \beta) \geqslant 2.$$

We observe that

$$\underbrace{(\alpha, 1 - \chi_K(\alpha))}_{\in \Phi_{\mathrm{af}} \backslash R} + \underbrace{(\beta, 1 - \chi_K(\beta))}_{\in \Phi_{\mathrm{af}} \backslash R} = \underbrace{(\alpha + \beta, 2 - \chi_K(\alpha) - \chi_K(\beta))}_{\in R}.$$

Since also the set $\Phi_{\mathrm{af}} \backslash R$ is closed under addition, this is impossible. The contradiction shows the claim.

(b) This is immediate from (a):

$$\begin{aligned} {}^L\ell^R(x, \alpha) + {}^L\ell^R(x, -\alpha) =& \langle \mu, \alpha \rangle + \langle \mu, -\alpha \rangle + \underbrace{\chi_R(\alpha) + \chi_R(-\alpha)}_{\in \{0,1\}} \\ & - \underbrace{(\chi_L(w\alpha) + \chi_L(-w\alpha))}_{\in \{0,1\}} \\ \in& \{-1, 0, 1\}. \end{aligned}$$

Now if $\alpha + \beta \in \Phi$, we get

$$\begin{aligned} & {}^L\ell^R(x, \alpha) + {}^L\ell^R(x, \beta) - {}^L\ell^R(x, \alpha + \beta) \\ =& \langle \mu, \alpha \rangle + \langle \mu, \beta \rangle - \langle \mu, \alpha + \beta \rangle + \underbrace{\chi_R(\alpha) + \chi_R(\beta) - \chi_R(\alpha + \beta)}_{\in \{0,1\}} \\ & - \underbrace{(\chi_L(w\alpha) + \chi_L(w\beta) - \chi_L(w\alpha + w\beta))}_{\in \{0,1\}} \\ \in& \{-1, 0, 1\}. \end{aligned}$$

□

We are ready to state our main result for this subsection:

Proposition 4.27. *Let $x = w\varepsilon^{\mu}, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, let $L, R \subseteq \Delta_{\mathrm{af}}$ be regular subsets and $v \in W$ be positive for ${}^L\ell^R(x,\cdot)$. Moreover, fix subsets $J_1, \dots, J_m \subseteq \Delta$ such that $J := J_1 \cap \dots \cap J_m$ satisfies*

$$\forall \alpha \in \Phi_J : \ {}^L\ell^R(x, v\alpha) \geqslant 0.$$

We have ${}^Lx^R \leqslant {}^L(x')^R$ if and only if for each $i = 1, \dots, m$, there exists some $v_i' \in W$ with

$$v^{-1}\mu + {}^R\mathrm{wt}(v_i' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v_i') \leqslant (v_i')^{-1}\mu' \pmod{\Phi_{J_i}^{\vee}}.$$

We remark that this recovers Theorem 4.2 in case $L = R = \emptyset$.

We now start the work towards proving Proposition 4.27.

Lemma 4.28. *Let $K \subseteq \Delta_{\mathrm{af}}$ be regular, $\alpha \in \Phi_K$ and $\beta \in \Phi$. Then*

$$\chi_K(s_\alpha(\beta)) = \chi_K(\beta) - \langle \alpha^\vee, \beta \rangle \chi_K(\alpha).$$

Proof. Consider the affine roots $a = (\alpha, -\chi_K(\alpha)) \in (\Phi_{\mathrm{af}})_K$ and $b = (\beta, -\chi_K(\beta)) \in \Phi_{\mathrm{af}}$.

If $\beta \in \Phi_K$, then $b \in (\Phi_{\mathrm{af}})_K$ such that $r_a(b) \in (\Phi_{\mathrm{af}})_K$. Explicitly,

$$r_a(b) = (s_\alpha(\beta), -\chi_K(\beta) + \langle \alpha^\vee, \beta \rangle \chi_K(\alpha)),$$

such that the claim follows from the definition of $\chi_K(s_\alpha(\beta))$.

Next assume that $\beta \notin \Phi_K$, such that $b \in (\Phi_{\mathrm{af}})^- \backslash (\Phi_{\mathrm{af}})_K$. Since r_a stabilizes the set $(\Phi_{\mathrm{af}})^- \backslash (\Phi_{\mathrm{af}})_K$, we get $r_a(b) \in (\Phi_{\mathrm{af}})^- \backslash (\Phi_{\mathrm{af}})_K$. This proves (together with the above calculation) that

$$-\chi_K(\beta) + \langle \alpha^\vee, \beta \rangle \chi_K(\alpha) \leqslant -\Phi^+(s_\alpha(\beta)) = -\chi_K(s_\alpha(\beta)).$$

If the inequality above was strict, we would get

$$b' := (s_\alpha(\beta), -\chi_K(\beta) + \langle \alpha^\vee, \beta \rangle \chi_K(\alpha) + 1) \in \Phi_{\mathrm{af}}^- \backslash (\Phi_{\mathrm{af}})_K$$

with

$$r_a(b') = (\beta, 1 - \chi_K(\beta)) \in \Phi_{\mathrm{af}}^+,$$

contradiction. □

Lemma 4.29. *Let $x \in \widetilde{W}, x_L \in \widetilde{W}_L$ and $x_R \in \widetilde{W}_R$ where $L, R \subseteq \Delta_{\mathrm{af}}$ are regular subsets. Denoting the image of x_R in W by $\mathrm{cl}(x_R)$, we have the following identity for every $\alpha \in \Phi$:*

$${}^L\ell^R(x_L x x_R, \alpha) = {}^L\ell^R(x, \mathrm{cl}(x_R)(\alpha)).$$

Proof. We start with two special cases:

In case $x_L = r_a$ and $x_R = 1$ for some $(\beta, k) := a \in L$, we obtain

$$\begin{aligned}
{}^L\ell^R(x_L x x_R, \alpha) = {}^L\ell^R\left(s_\beta w \varepsilon^{\mu + k w^{-1}\beta^\vee}, \alpha\right) \\
&= \langle \mu + k w^{-1}\beta^\vee, \alpha\rangle + \chi_R(\alpha) - \chi_L(s_\beta w\alpha) \\
&= \langle \mu, \alpha\rangle - \chi_L(\beta)\langle \beta^\vee, w\alpha\rangle + \chi_R(\alpha) - \chi_L(s_\beta w\alpha) \\
&\underset{\text{L4.28}}{=} \langle \mu, \alpha\rangle + \chi_R(\alpha) - \chi_L(w\alpha) = {}^L\ell^R(x, \alpha).
\end{aligned}$$

In case $x_L = 1$ and $x_R = r_a$ for some $(\beta, k) := a \in R$, we obtain

$$\begin{aligned}
{}^L\ell^R(x_L x x_R, \alpha) = {}^L\ell^R\left(w s_\beta \varepsilon^{s_\beta(\mu) + k\beta^\vee}, \alpha\right) \\
&= \langle s_\beta(\mu) + k\beta^\vee, \alpha\rangle + \chi_R(\alpha) - \chi_L(w s_\beta \alpha) \\
&= \langle \mu, s_\beta(\alpha)\rangle - \chi_R(\beta)\langle \beta^\vee, \alpha\rangle + \chi_R(\alpha) - \chi_L(w s_\beta\alpha) \\
&\underset{\text{L4.28}}{=} \langle \mu, s_\beta(\alpha)\rangle + \chi_R(s_\beta\alpha) - \chi_L(w s_\beta \alpha) \\
&= {}^L\ell^R(x, s_\beta\alpha).
\end{aligned}$$

Now in the general case, pick reduced decompositions for $x_L \in \widetilde{W}_L$ and $x_R \in \widetilde{W}_R$ and iterate the previous arguments. ☐

Definition 4.30. By a *valid tuple*, we mean a seven tuple

$$(x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'}, v, v', L, R, J)$$

consisting of

- elements $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$,
- elements $v, v' \in W$,
- regular subsets $L, R \subseteq \Delta_{\text{af}}$ and
- a subset $J \subseteq \Delta$,

satisfying the condition

$$v^{-1}\mu + {}^R\mathrm{wt}(v' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu' \pmod{\Phi_J^\vee}.$$

The tuple is called *strict* if v is positive for ${}^L\ell^R(x, \cdot)$ and v' is positive for ${}^L\ell^R(x', \cdot)$.

We have the following analogue of Lemma 4.3:

Lemma 4.31. *Let $(x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'}, v, v', L, R, J)$ be a valid tuple. If v' is not positive for ${}^L\ell^R(x', \cdot)$ and v'' is an adjustment in the sense of Definition 2.2, then (x, x', v, v'', L, R, J) is also a valid tuple.*

Proof. This means that there is a root $\alpha \in \Phi^+$ such that $v'' = v's_\alpha$ and either

$$ {}^L\ell^R(x', v'\alpha) < 0 \text{ or } {}^L\ell^R(x', -v'\alpha) > 0. $$

We calculate

$$\begin{aligned} & v^{-1}\mu + {}^R\mathrm{wt}(v'' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v'') \\ =& v^{-1}\mu + {}^R\mathrm{wt}(v's_\alpha \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v's_\alpha) \\ \underset{\text{L3.35}}{\leqslant}& v^{-1}\mu + {}^R\mathrm{wt}(v' \Rightarrow v) + \chi_R(v'\alpha)\alpha^\vee + {}^L\mathrm{wt}(wv \Rightarrow w'v') + \chi_L(-w'v'\alpha)\alpha^\vee \\ \leqslant& (v')^{-1}\mu + (\chi_R(v'\alpha) + \chi_L(-w'v'\alpha))\alpha^\vee \\ =& (v'')^{-1}\mu + \left(\langle\mu, \alpha\rangle + \chi_R(v'\alpha) + \chi_L(-w'v'\alpha)\right)\alpha^\vee \pmod{\Phi_J^\vee} \end{aligned}$$

In case ${}^L\ell^R(x', v'\alpha) < 0$, we use the fact $\chi_L(-w'v'\alpha) \leqslant 1 - \chi_L(w'v'\alpha)$ (cf. Lemma 4.26) to show

$$\begin{aligned} & \langle\mu, \alpha\rangle + \chi_R(v'\alpha) + \chi_L(-w'v'\alpha) \\ \leqslant& \langle\mu, \alpha\rangle + \chi_R(v'\alpha) + 1 - \chi_L(w'v'\alpha) \\ =& {}^L\ell^R(x', \alpha) + 1 \leqslant 0. \end{aligned}$$

Similarly if ${}^L\ell^R(x', -v'\alpha) > 0$, we get

$$\begin{aligned} & \langle\mu, \alpha\rangle + \chi_R(v'\alpha) + \chi_L(-w'v'\alpha) \\ \leqslant& \langle\mu, \alpha\rangle + 1 - \chi_R(-v'\alpha) + \chi_L(-w'v'\alpha) \\ =& 1 - {}^L\ell^R(x', -\alpha) \leqslant 0. \end{aligned}$$

In any case, we see that

$$ \langle\mu, \alpha\rangle + \chi_R(v'\alpha) + \chi_L(-w'v'\alpha) \leqslant 0, $$

from where the desired claim is immediate. ☐

Lemma 4.32. *Let $(x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'}, v, v', L, R, J)$ be a (strict) valid tuple. Let moreover $x_L, x'_L \in \widetilde{W}_L$ and $x_R, x'_R \in \widetilde{W}_R$ be any elements. Then*

$$ (x_L x x_R, x'_L x' x'_R, \mathrm{cl}(x_R)v, \mathrm{cl}(x'_R)v', L, R, J) $$

is a (strict) valid tuple as well.

Proof. Similar to the proof of Lemma 4.29, it suffices to show the claim in case three of the four elements x_L, x'_L, x_R, x'_R are trivial and the remaining one is a simple affine reflection.

We just explain the argument in case $x_L = r_a, x'_L = x_R = x'_R = 1$ for some $a \in L$, as the remaining arguments are very similar. Write $a = (\alpha, k)$ so that $\chi_L(\alpha) = -k$. Then $x_L x = s_\alpha w\varepsilon^{\mu + kw^{-1}\alpha^\vee}$ We calculate

$$\begin{aligned} & v^{-1}\left(\mu + kw^{-1}\alpha^\vee\right) + {}^R\mathrm{wt}(v' \Rightarrow v) + {}^L\mathrm{wt}(s_\alpha wv \Rightarrow w'v') \\ \underset{\text{L3.35}}{=}& v^{-1}\mu + k(wv)^{-1}\alpha^\vee + {}^R\mathrm{wt}(v' \Rightarrow v) + \chi_L(\alpha)(wv)^{-1}\alpha^\vee + {}^L\mathrm{wt}(wv \Rightarrow w'v') \\ =& v^{-1}\mu + {}^R\mathrm{wt}(v' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v'). \end{aligned}$$

It follows that $(x_L x, x', v, v', L, R, J)$ is a valid tuple. The strictness assertion follows from Lemma 4.29. $\square$

Using Lemma 4.32, it will suffice to show Proposition 4.27 only in the case $x \in {}^L\widetilde{W}^R$ and $x' \in {}^{-L}\widetilde{W}^{-R}$.

Lemma 4.33. *Let $(x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'}, v, v', L, R, J)$ be a strict valid tuple.*

(a) If $x \in {}^L\widetilde{W}^R$ and $\alpha \in \Phi$ satisfies ${}^L\ell^R(x,\alpha) \geqslant 0$, then $\ell(x,\alpha) \geqslant 0$.

(b) If $x \in {}^L\widetilde{W}^R$ and $\alpha \in \Phi_L^+$ satisfies $(wv)^{-1}\alpha \in \Phi^-$, then

$$(x, x', s_{w^{-1}\alpha}v, v', L, R, J)$$

is a strict valid tuple as well.

(c) If $x' \in {}^{-L}\widetilde{W}^{-R}$ and $\alpha \in \Phi_R^+$ satisfies $v^{-1}\alpha \in \Phi^-$, then

$$(x, x', v, s_\alpha v', L, R, J)$$

is a strict valid tuple as well.

Proof. We write

$$\begin{aligned} {}^L\ell^R(x,\alpha) =& \langle \mu, \alpha\rangle + \chi_R(\alpha) - \chi_L(w\alpha) \\ =& \langle \mu, \alpha\rangle + \Phi^+(\alpha) - \Phi_R^+(\alpha) - \Phi^+(w\alpha) + \Phi_L^+(w\alpha) \\ =& \ell(x,\alpha) - \Phi_R^+(\alpha) + \Phi_L^+(w\alpha). \end{aligned}$$

(a) If $w\alpha \notin \Phi_L^+$, then

$$\ell(x,\alpha) = {}^L\ell^R(x,\alpha) + \Phi_R^+(\alpha) \geqslant 0.$$

If $w\alpha \in \Phi_L^+$, then the condition $x \in {}^L\widetilde{W}^R$ already implies $\ell(x,\alpha) \geqslant 0$.

(b) The condition $\alpha \in \Phi_L^+$ together with $x \in {}^L\widetilde{W}^R$ yields $\ell(x, w^{-1}\alpha) \geqslant 0$. We have

$${}^L\ell^R(x, -w^{-1}\alpha) = {}^L\ell^R(x, v(-(wv)^{-1}\alpha)) \geqslant 0$$

by the positivity assertion on v. By (a), we conclude $\ell(x, -w^{-1}\alpha) \geqslant 0$, so altogether we get $\ell(x, w^{-1}\alpha) = 0$.

By the above computation, we get

$${}^L\ell^R(x, w^{-1}\alpha) = -\Phi_R^+(w^{-1}\alpha) + \Phi_L^+(\alpha) = 1 - \Phi_R^+(w^{-1}\alpha).$$

On the other hand, we have

$${}^L\ell^R(x, w^{-1}\alpha) = {}^L\ell^R(x, v(wv)^{-1}\alpha) \leqslant 0$$

by the positivity assertion on v. Thus ${}^L\ell^R(x, w^{-1}\alpha) = 0$ and $w^{-1}\alpha \in \Phi_R^+$.

Consider the elements $a = (\alpha, \Phi^+(-\alpha)) \in (\Phi_{\mathrm{af}})^+_L$ and $b = (w^{-1}\alpha, \Phi^+(-w^{-1}\alpha)) \in (\Phi_{\mathrm{af}})^+_R$. We have

$$\begin{aligned} x(b) =&(\alpha, \Phi^+(-w^{-1}\alpha) - \langle \mu, w^{-1}\alpha\rangle) \\ =&(\alpha, \Phi^+(-\alpha) + \ell(x, -w^{-1}\alpha)\rangle) = (\alpha, \Phi^+(-\alpha)) = a. \end{aligned}$$

We see that $x = r_a x r_b$. Now the claim follows from Lemma 4.32.

(c) The proof is analogous to (b): We have $\ell(x', \alpha) > 0$ as $\alpha \in \Phi^+_R$ and $x'(\alpha, \Phi^+(-\alpha)) \in \Phi^-_{\mathrm{af}}$. Now

$$\begin{aligned} 0 \leqslant {}^L\ell^R(x', -\alpha) &= \ell(x', -\alpha) - \Phi^+_R(-\alpha) + \Phi^+_L(-w'\alpha) \\ =&\ell(x', -\alpha) + \Phi^+_L(-w\alpha) \leqslant -1 + \Phi^+_L(-w'\alpha) \leqslant 0. \end{aligned}$$

So equality must hold, hence $\ell(x', \alpha) = 1$ and $-w'\alpha \in \Phi^+_L$.

Writing $b = (\alpha, \Phi^+(-\alpha)) \in (\Phi_{\mathrm{af}})^+_R$ and $a = (-w'\alpha, \Phi^+(w'\alpha))$, we compute

$$\begin{aligned} x'(b) =&(w'\alpha, \Phi^+(-\alpha) - \langle \mu', \alpha\rangle) \\ =&(w'\alpha, \ell(x', -\alpha) + \Phi^+(-w'\alpha)) \\ =&(w'\alpha, -1 + \Phi^+(-w'\alpha)) = (w'\alpha, -\Phi^+(w'\alpha)) = -a. \end{aligned}$$

Hence $r_a x' r_b = x'$ with $r_a \in \widetilde{W}_L$ and $r_b \in \widetilde{W}_R$. The conclusion follows from Lemma 4.32. $\square$

Proof of Proposition 4.27. Let us fix $L, R, J_1, \ldots, J_m, J$ for the entire proof. To keep our notation concise, we make the following convention: We call a triple (x, x', v) *valid* if, for each $i = 1, \ldots, m$, there exists $v'_i \in W$ such that $(x, x', v, v'_i, L, R, J_i)$ is a strict valid tuple.

First assume that ${}^Lx^R \leqslant {}^Lx'^R$. We want to show that (x, x', v) is valid. Write $x = x_L \cdot {}^Lx^R \cdot x_R$ with $x_L \in \widetilde{W}_L, x_R \in \widetilde{W}_R$. It suffices to show that $\left({}^Lx^R, x', \mathrm{cl}(x_R)^{-1}v\right)$ is valid by Lemma 4.32.

In other words, we may assume that $x \in {}^L\widetilde{W}^R$ and $x \leqslant x'$ for proving that (x, x', v) is valid. By Lemma 4.17, we find $v' \in W$ such that

$$v^{-1}\mu + \mathrm{wt}(v' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v') \leqslant (v')^{-1}\mu'.$$

Now recall from Lemma 3.32 that

$$\begin{aligned} {}^R\mathrm{wt}(v' \Rightarrow v) &\leqslant \mathrm{wt}(v' \Rightarrow v), \\ {}^L\mathrm{wt}(wv \Rightarrow w'v') &\leqslant \mathrm{wt}(wv \Rightarrow w'v'). \end{aligned}$$

We conclude that $(x, x', v, v', L, R, J_i)$ is valid for all $i = 1, \ldots, m$. Up to iteratively choosing adjustments for v', we may assume that the tuple is strict valid, so (x, x', v) is indeed valid.

For the converse direction, let us assume that (x, x', v) is valid. We have to show ${}^L x^R \leqslant {}^L(x')^R$. Again, we can use Lemma 4.32 and Lemma 4.29 to reduce this to any other elements in $\widetilde{W}_L x \widetilde{W}_R$ resp. $\widetilde{W}_L x' \widetilde{W}_R$.

Thus, we may and will assume that $x \in {}^L\widetilde{W}^R$ and $x' \in {}^{-L}\widetilde{W}^{-R}$. We then have to show $x \leqslant x'$ using the fact that (x, x', v) is valid for some $v \in W$.

Among all $v \in W$ such that (x, x', v) is valid, choose one such that ${}^L\ell(wv)$ is as small as possible. If $wv \notin {}^L W$, then we find some $\alpha \in \Phi_L^+$ with $(wv)^{-1} \in \Phi^-$. By Lemma 4.33, also $(x, x', s_{w^{-1}\alpha} v)$ is valid and by Lemma 3.30, ${}^L\ell(s_\alpha wv) < {}^L\ell(wv)$. This is a contradiction to the minimality of ${}^L\ell(wv)$.

We see that we always find some $v \in W$ such that (x, x', v) is valid and $wv \in {}^L W$.

We now prove that $x \leqslant x'$ using Theorem 4.2.

By Lemma 4.33 (a), it follows that $v \in W$ is length positive for x and that $\ell(x, v\alpha) \geqslant 0$ for all $\alpha \in \Phi_J$. Since $\Phi_J = -\Phi_J$ and $\ell(x, -v\alpha) = -\ell(x, v\alpha)$, this is only possible if $\ell(x, v\alpha) = 0$ for all $\alpha \in \Phi_J$. We conclude that $(v, J_1, \dots, J_m)$ is a Bruhat-deciding datum for x.

Now for each $i = 1, \dots, m$, by assumption, there exists some $v_i \in W$ such that $(x, x', v, v_i', L, R, J_i)$ is a strict valid tuple. Minimizing ${}^R\ell(v_i')$ as before, we may assume that $v_i' \in {}^R W$ by Lemma 4.33.

We see that $(x, x', v, v_i', L, R, J_i)$ is a strict valid tuple with $wv \in {}^L W$ and $v_i' \in {}^R W$. By definition of the semi-affine weight function, we get

$$\begin{aligned} {}^R\mathrm{wt}(v_i' \Rightarrow v) &= \mathrm{wt}(v_i' \Rightarrow v), \\ {}^L\mathrm{wt}(wv \Rightarrow w'v_i') &= \mathrm{wt}(wv \Rightarrow w'v_i'). \end{aligned}$$

We conclude

$$\begin{aligned} & v^{-1}\mu + \mathrm{wt}(v_i' \Rightarrow v) + \mathrm{wt}(wv \Rightarrow w'v_i') \\ = \ & v^{-1}\mu + {}^R\mathrm{wt}(v_i' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v_i') \\ \underset{\text{valid}}{\leqslant} & (v')^{-1}\mu' \pmod{\Phi_{J_i}^\vee}. \end{aligned}$$

This is exactly the inequality we had to check in order to apply Theorem 4.2. So we conclude $x \leqslant x'$, finishing the proof. □

We finish the section with three applications for this proposition. Our first application re-proves the well-known criterion for type A_n, and even a bit more.

Corollary 4.34. *Suppose that Φ is irreducible and that $a_L, a_R \in \Delta_{\mathrm{af}}$ are special nodes. Let $L = \Delta_{\mathrm{af}} \backslash \{a_L\}, R = \Delta_{\mathrm{af}} \backslash \{a_R\}$ and write $\omega_{a_L}, \omega_{a_R} \in \mathbb{Q}\Phi^\vee$ for the corresponding coweights.*

Let $x = w\varepsilon^\mu, x' = w'\varepsilon^{\mu'} \in \widetilde{W}$, and assume that $\mu \equiv \mu' \pmod{\Phi^\vee}$. Then we have

$$ {}^L x^R \leqslant {}^L(x')^R \iff (\mu + \omega_{a_R} - w^{-1}\omega_{a_L})^{\mathrm{dom}} \leqslant (\mu' + \omega_{a_R} - (w')^{-1}\omega_{a_L})^{\mathrm{dom}}. $$

Proof. For all $\alpha \in \Phi$, we easily verify $\langle \omega_{a_L}, \alpha \rangle = \chi_L(\alpha)$. Thus $v \in W$ is positive for ${}^L\ell^R(x, \cdot)$ if and only if $v^{-1}\left(\mu + \omega_{a_R} - w^{-1}\omega_{a_L}\right)$ is dominant.

Similarly, $v' \in W$ is positive for ${}^L\ell^R(x', \cdot)$ if and only if $(v')^{-1}\left(\mu' + \omega_{a_R} - (w')^{-1}\omega_{a_L}\right)$ is dominant.

Finally observe that for all $v, v' \in W$, we can use Lemma 3.41 to compute

$$\begin{aligned}
&v^{-1}\mu + {}^R\mathrm{wt}(v' \Rightarrow v) + {}^L\mathrm{wt}(wv \Rightarrow w'v') - (v')^{-1}\mu' \\
=&v^{-1}\mu + v^{-1}\omega_{a_R} - (v')^{-1}\omega_{a_R} + (w'v')^{-1}\omega_{a_L} - (wv)^{-1}\omega_{a_L} - (v')^{-1}\mu' \\
=&v^{-1}(\mu + \omega_{a_R} - w^{-1}\omega_{a_L}) - (v')^{-1}(\mu + \omega_{a_R} - (w')^{-1}\omega_{a_L}).
\end{aligned}$$

The conclusion follows in light of Proposition 4.27. □

For irreducible root systems of type A, this recovers the Bruhat order criterion presented at the beginning of Section 3.

As another application, we present our most general criterion for the Bruhat order on affine Weyl groups.

Definition 4.35. Let $x \in \widetilde{W}$. A *Deodhar datum for x* consists of the following:

- Regular subsets $L_1, \dots, L_\ell, R_1, \dots, R_r \subseteq \Delta_{\mathrm{af}}$ with $\ell, r \geqslant 1$ such that $L := L_1 \cap \dots \cap L_\ell$ and $R := R_1 \cap \dots \cap R_r$ satisfy $x \in {}^L\widetilde{W}^R$.
- For each $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, r\}$ an element $v_{i,j} \in W$ that is positive for ${}^{L_i}\ell^{R_j}(x, \cdot)$.
- For each $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, r\}$ a collection of subsets
$$J(i,j)_1, \dots, J(i,j)_{m(i,j)} \subseteq \Delta$$
such that $m(i,j) \geqslant 1$ and $J(i,j) := J(i,j)_1 \cap \dots \cap J(i,j)_{m(i,j)}$ satisfies
$$\forall \alpha \in \Phi_{J(i,j)} : \ {}^{L_i}\ell^{R_j}(x, v_{i,j}\alpha) \geqslant 0.$$

Theorem 4.36. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and fix a Deodhar datum*

$$L_1, \dots, L_\ell, \quad R_1, \dots, R_r, \quad (v_{\bullet,\bullet}), \quad (J(\bullet,\bullet)_\bullet).$$

Let $x' = w'\varepsilon^{\mu'} \in \widetilde{W}$. Then $x \leqslant x'$ if and only if for each $i \in \{1, \dots, \ell\}, j \in \{1, \dots, r\}$ and $k \in \{1, \dots, m(i,j)\}$, there exists some $v'_{i,j,k} \in W$ such that

$$v_{i,j}^{-1}\mu + {}^{R_j}\mathrm{wt}(v'_{i,j,k} \Rightarrow v_{i,j}) + {}^{L_i}\mathrm{wt}(wv_{i,j} \Rightarrow w'v'_{i,j,k}) \leqslant (v'_{i,j,k})^{-1}\mu' \pmod{\Phi^\vee_{J(i,j)_k}}.$$

Proof. In view of Proposition 4.27, the existence of the $v'_{i,j,k}$ for fixed i, j means precisely

$${}^{L_i}x^{R_j} \leqslant {}^{L_i}(x')^{R_j}.$$

By Deodhar's lemma, i.e. Proposition 4.24, this is equivalent to $x = {}^Lx^R \leqslant x'$. □

Lemma 4.37. *Let $w_1, w_2 \in W$. Let moreover $R_1, \dots, R_k \subseteq \Delta_{\mathrm{af}}$ be regular subsets with $k \geqslant 1$ and $R := R_1 \cap \dots \cap R_k$. Then we have the following equality in $\mathbb{Z}\Phi^\vee$:*

$$^R\mathrm{wt}(w_1 \Rightarrow w_2) = \sup_{i=1,\dots,k} {}^{R_i}\mathrm{wt}(w_1 \Rightarrow w_2).$$

Proof. Consider Proposition 4.27 for μ and μ' sufficiently regular, with $L = \varnothing$ and $(J_1, \dots, J_m) = (\varnothing)$. Then by Proposition 4.24,

$$x^R \leqslant (x')^R \iff \forall i \in \{1, \dots, k\} : \; x^{R_i} \leqslant (x')^{R_i}.$$

The claim follows from Proposition 4.27 with little effort. □

Together with Lemma 3.36, this result allows us to express the weight function of the quantum Bruhat graph $\mathrm{wt} : W \times W \to \mathbb{Z}\Phi^\vee$ as a supremum of semi-affine weight functions.

As our final application of Proposition 4.27, we generalize Proposition 4.12 to the admissible subsets considered in [Rap02].

Proposition 4.38. *Let $K \subseteq \Delta_{\mathrm{af}}$ be regular, $x = w\varepsilon^\mu \in \widetilde{W}$ and $\lambda \in X_*(T)_{\Gamma_0}$ dominant. Then the following are equivalent:*

(i) $x \in \widetilde{W}_K \operatorname{Adm}(\lambda)\widetilde{W}_K$.

(ii) For every $v \in W$, we have

$$v^{-1}\mu + {}^K\mathrm{wt}(wv \Rightarrow v) \leqslant \lambda.$$

(iii) There exists some $v \in W$ that is positive for ${}^K\ell^K(x, \cdot)$ and satisfies

$$v^{-1}\mu + {}^K\mathrm{wt}(wv \Rightarrow v) \leqslant \lambda.$$

Proof. By definition, (i) means that there exists $u \in W$ such that

$${}^K x^K \leqslant {}^K(\varepsilon^{u\lambda})^K.$$

By Proposition 4.27, we get condition (ii) for every $v \in W$ that is positive for ${}^K\ell^K(x, \cdot)$. Now a simple adjustment argument, similar to Lemma 4.31, shows that (ii) holds for every $v \in W$.

(ii) $\implies$ (iii) is clear, as we always find a positive element for each root functional Corollary 2.4.

(iii) $\implies$ (i): It suffices to show that ${}^K x^K \leqslant \varepsilon^{v\lambda}$. This follows immediately from Proposition 4.27. □

5. Demazure product

The Demazure product $*$ is another operation on the extended affine Weyl group $\widetilde{W}$. In the context of the Iwahori-Bruhat decomposition of a reductive group, the Demazure product describes the closure of the product of two Iwahori double cosets, cf. [HN21, Section 2.2]. In a more Coxeter-theoretic style, we can define the Demazure product of $\widetilde{W}$ as follows:

Proposition 5.1 ([He09, Lemma 1])**.** *Let $x_1, x_2 \in \widetilde{W}$. Then each of the following three sets contains a unique maximum (with respect to the Bruhat order), and the maxima agree:*

$$\{x_1 x_2' \mid x_2' \leqslant x_2\}, \quad \{x_1' x_2 \mid x_1' \leqslant x_1\}, \quad \{x_1' x_2' \mid x_1' \leqslant x_1,\ x_2' \leqslant x_2\}.$$

The common maximum is denoted $x_1 * x_2$. If we write $x_1 * x_2 = x_1 x_2' = x_1' x_2$, then

$$\ell(x_1 * x_2) = \ell(x_1) + \ell(x_2') = \ell(x_1') + \ell(x_2). \qquad \square$$

Demazure products have recently been studied in the context of affine Deligne-Lusztig varieties [Sad21; He21a; HN21]. While the Demazure product is a somewhat simple Coxeter-theoretic notion, it is connected to the question of generic Newton points of elements in $\widetilde{W}$. He [He21a] shows how to compute generic Newton points in terms of iterated Demazure products, a method that we will review in Section 7.3. Conversely, He and Nie [HN21] use the Milićević's formula for generic Newton points [Mil21] to show new properties of the Demazure product.

In this section, we prove a new description of Demazure products in $\widetilde{W}$, generalizing the aforementioned results of [HN21]. As applications, we obtain new results on the quantum Bruhat graph that shed some light on our previous results on the Bruhat order.

5.1. Computation of Demazure products

If one plays a bit with our Theorem 4.2 or [HN21, Proposition 3.3], one will soon get an idea of how Demazure products should roughly look like. We capture the occurring formulas as follows.

Situation 5.2. Let $x_1 = w_1\varepsilon_1^{\mu}, x_2 = w_2\varepsilon_2^{\mu} \in \widetilde{W}$. Let $v_1, v_2 \in W$ and define

$$\begin{aligned} x_1' :=& w_1'\varepsilon^{\mu_1'} := (w_1v_1)(w_2v_2)^{-1}\varepsilon^{w_2v_2v_1^{-1}\mu_1 - w_2v_2\,\mathrm{wt}(v_1\Rightarrow w_2v_2)},\\ x_2' :=& w_2'\varepsilon^{\mu_2'} := v_1v_2^{-1}\varepsilon^{\mu_2 - v_2\,\mathrm{wt}(v_1\Rightarrow w_2v_2)},\\ x_* :=& w_*\varepsilon^{\mu_*} := w_1v_1v_2^{-1}\varepsilon^{v_2v_1^{-1}\mu_1+\mu_2-v_2\,\mathrm{wt}(v_1\Rightarrow w_2v_2)} = x_1'x_2 = x_1x_2'. \end{aligned}$$

In this situation, we want to compute the Demazure product $x_1 * x_2$, knowing that $x_1 * x_2$ can be written as $\tilde{x}_1 x_2 = x_1\tilde{x}_2$ for some $\tilde{x}_1 \leqslant x_1$ and $\tilde{x}_2 \leqslant x_2$. If x_1 is in a shrunken Weyl chamber with $\mathrm{LP}(x_1) = v_1$, and x_2 is shrunken with $\mathrm{LP}(x_2) = \{v_2\}$, then $x_* = x_1 * x_2$ by [HN21, Proposition 3.3], so $\tilde{x}_1 = x_1'$ and $\tilde{x}_2 = x_2'$.

In the general case, our goal is to find conditions on $v_1, v_2 \in W$ to ensure that $x_* = x_1 * x_2$.

Before examining this situation further, it will be very convenient for our proofs to see that the property

$$(x_1 * x_2)^{-1} = x_2^{-1} * x_1^{-1}$$

is reflected by our construction in Situation 5.2.

Lemma 5.3. *Suppose we are in Situation 5.2. Let us write $y_1 := x_2^{-1}$ and $y_2 := x_1^{-1}$. Define $v_1' := w_2 v_2 w_0$ resp. $v_2' := w_1 v_1 w_0$.*

Construct y_1', y_2', y_* associated with (y_1, y_2, v_1', v_2') as in Situation 5.2. Then

$$y_1' = (x_2')^{-1}, \quad y_2' = (x_1')^{-1}, \quad y_* = x_*^{-1}.$$

Moreover,

- *$v_1 \in \mathrm{LP}(x_1)$ iff $v_2' \in \mathrm{LP}(y_1)$.*
- *$v_2 \in \mathrm{LP}(x_2)$ iff $v_1' \in \mathrm{LP}(y_2)$.*
- *$d_{\mathrm{QB}(W)}(v_1 \Rightarrow w_2 v_2) = d_{\mathrm{QB}(W)}(v_1' \Rightarrow w_1^{-1} v_2')$ and $\mathrm{wt}(v_1 \Rightarrow w_2 v_2) = -w_0 \,\mathrm{wt}(v_1' \Rightarrow w_1^{-1} v_2)$.*

Proof. Write

$$y_1 = w_2^{-1} \varepsilon^{-w_2 \mu_2}, \quad y_2 = w_1^{-1} \varepsilon^{-w_1 \mu_1}$$

and compute

$$\begin{aligned} y_2' =&(w_2 v_2 w_0)(w_1 v_1 w_0)^{-1} \varepsilon^{-w_1\mu_1 - w_1 v_1 w_0 \,\mathrm{wt}(w_2 v_2 w_0 \Rightarrow (w_1)^{-1} w_1 v_1 w_0)} \\ =&(w_2 v_2)(w_1 v_1)^{-1} \varepsilon^{-w_1 \mu_1 + w_1 v_1 \,\mathrm{wt}(v_1 \Rightarrow w_2 v_2)} = (x_1')^{-1}. \end{aligned}$$

A similar computation, or a repetition of this argument for $x_1 = (y_2)^{-1}, x_2 = (y_1)^{-1}$, shows that $y_1' = (x_2')^{-1}$. Then the conclusion $y_* = x_*^{-1}$ is immediate.

For the "Moreover" statements, recall that

$$\mathrm{LP}(y_1) = \mathrm{LP}(x_2^{-1}) \underset{\text{Lemma 2.12}}{=} w_2 \,\mathrm{LP}(x_2) w_0.$$

The same holds for $y_2 = x_1^{-1}$. The final statement is due to the fact that $v_1' = w_2 v_2 w_0$ and $w_1^{-1} v_2' = v_1 w_0$ using the duality anti-automorphism of the quantum Bruhat graph, cf. Lemma 3.9. □

The first step towards proving $x_1 * x_2 = x_*$ is the following estimate:

Lemma 5.4. *Let $x_1, x_2 \in \widetilde{W}$ and $v_2 \in \mathrm{LP}(x_1 * x_2)$. There exists $v_1 \in \mathrm{LP}(x_1)$ such that*

$$\ell(x_1 * x_2) \leqslant \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2 v_2).$$

Proof. Write $x_1 * x_2 = yx_2$ for some element $y = w'\varepsilon^{\mu'} \leqslant x_1$. Observe that $\ell(yx_2) = \ell(y) + \ell(x_2)$, so that v_2 must be length positive for x_2 and w_2v_2 must be length positive for y.

Since $y \leqslant x_1$, using Lemma 4.17, we find a length positive element v_1 for x_1 such that

$$(w_2v_2)^{-1}\mu' + \mathrm{wt}(v_1 \Rightarrow w_2v_2) + \mathrm{wt}(w'w_2v_2 \Rightarrow w_1v_1) \leqslant (v_1)^{-1}\mu_1.$$

Pairing with 2ρ and using Lemma 3.6, we compute

$$\begin{aligned}&\langle 2\rho, (w_2v_2)^{-1}\mu'\rangle + \ell(v_1) - \ell(w_2v_2)\\ &\qquad + d(v_1 \Rightarrow w_2v_2) + \ell(w'w_2v_2) - \ell(w_1v_1) + d(w'w_2v_2 \Rightarrow w_1v_1)\\ &\leqslant \langle 2\rho, (v_1)^{-1}\mu_1\rangle.\end{aligned}$$

Using the length positivity of w_2v_2 for y and v_1 for x_1 (Corollary 2.11), we conclude

$$\ell(y) + d(v_1 \Rightarrow w_2v_2) + d(w'w_2v_2 \Rightarrow w_1v_1) \leqslant \ell(x_2).$$

Thus

$$\ell(x_1 * x_2) = \ell(y) + \ell(x_2) \leqslant \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2v_2) - d(w'w_2v_2 \Rightarrow w_1v_1).$$

We obtain the desired conclusion. □

We now study the Situation 5.2 further.

Lemma 5.5. *Consider Situation 5.2, and assume that $v_1 \in \mathrm{LP}(x_1)$. Then we always have the estimate*

$$\ell(x_1') \geqslant \ell(x_1) - d_{\mathrm{QB}(W)}(v_1 \Rightarrow w_2v_2).$$

The following are equivalent:

(i) Equality holds above:

$$\ell(x_1') = \ell(x_1) - d_{\mathrm{QB}(W)}(v_1 \Rightarrow w_2v_2).$$

(ii) w_2v_2 is length positive for x_1'.

(iii) For any positive root α, we have

$$\ell(x_1, v_1\alpha) - \langle \mathrm{wt}(v_1 \Rightarrow w_2v_2), \alpha\rangle + \Phi^+(w_2v_2\alpha) - \Phi^+(v_1\alpha) \geqslant 0$$

In that case, $x_1' \leqslant x_1$, so that $x_* \leqslant x_1 * x_2$.

Proof. Consider the calculation

$$\begin{aligned}
\ell(x_1') &\underset{\text{C2.11}}{\geqslant} \left\langle (w_2v_2)^{-1}\left(w_2v_2v_1^{-1}\mu_1 - w_2v_2\operatorname{wt}(v_1 \Rightarrow w_2v_2)\right), 2\rho\right\rangle - \ell(w_2v_2) + \ell(w_1v_1)\\
&\underset{\text{L3.6}}{=} \langle v_1^{-1}\mu, 2\rho\rangle - \ell(v_1) + \ell(w_2v_2) - d(v_1 \Rightarrow w_2v_2) - \ell(w_2v_2) + \ell(w_1v_1)\\
&\underset{\text{C2.11}}{=} \ell(x_1) - d(v_1 \Rightarrow w_2v_2).
\end{aligned}$$

This shows the estimate and (i) $\iff$ (ii). In order to show (ii) $\iff$ (iii), we compute

$$\begin{aligned}
\ell(x_1', w_2v_2\alpha) =& \langle w_2v_2\alpha, w_2v_2v_1^{-1}\mu_1 - w_2v_2\operatorname{wt}(v_1 \Rightarrow w_2v_2), \alpha\rangle + \Phi^+(w_2v_2\alpha) - \operatorname{wt}(w_1v_1\alpha)\\
=& \ell(x_1, v_1\alpha) - \Phi^+(v_1\alpha) - \langle \operatorname{wt}(v_1 \Rightarrow w_2v_2), \alpha\rangle + \Phi^+(w_2v_2\alpha).
\end{aligned}$$

Finally, assume that (i) – (iii) are satisfied. We have to show $x_1' \leqslant x_1$. For this, we calculate

$$\begin{aligned}
&(w_2v_2)^{-1}\left(w_2v_2v_1^{-1}\mu_1 - w_2v_2\operatorname{wt}(v_1 \Rightarrow w_2v_2)\right) + \operatorname{wt}(v_1 \Rightarrow w_2v_2)\\
&\qquad + \operatorname{wt}(w_1v_1 \Rightarrow w_1v_1)\\
=\ & v_1^{-1}\mu_1.
\end{aligned}$$

Since we assumed $w_2v_2 \in \mathrm{LP}(x_1')$, we conclude $x_1' \leqslant x_1$ by Theorem 4.2. Now by definition of the Demazure product, we get $x_* = x_1'x_2 \leqslant x_1 * x_2$. □

By the duality presented in Lemma 5.3, we obtain the following:

Lemma 5.6. *Consider Situation 5.2, and assume that $v_2 \in \mathrm{LP}(x_2)$. Then we always have the estimate*

$$\ell(x_2') \geqslant \ell(x_2) - d_{\mathrm{QB}(W)}(v_1 \Rightarrow w_2v_2).$$

The following are equivalent:

(i) Equality holds above:

$$\ell(x_2') = \ell(x_2) - d_{\mathrm{QB}(W)}(v_1 \Rightarrow w_2v_2).$$

(ii) v_2 is length positive for x_2'.

(iii) For any positive root α, we have

$$\ell(x_2, v_2\alpha) - \langle \operatorname{wt}(v_1 \Rightarrow w_2v_2), \alpha\rangle + \Phi^+(w_2v_2\alpha) - \Phi^+(v_1\alpha) \geqslant 0.$$

In that case, $x_2' \leqslant x_2$, so that $x_* \leqslant x_1 * x_2$.

Proof. Under Lemma 5.3, this is precisely Lemma 5.5. □

Lemma 5.7. *Suppose we are given Situation 5.2, and that $v_1 \in \mathrm{LP}(x_1)$ and $v_2 \in \mathrm{LP}(x_2)$. We have the estimate*

$$\ell(x_*) \geqslant \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2 v_2).$$

Equality holds if and only if $v_2 \in \mathrm{LP}(x_*)$.

Proof. Using again Corollary 2.11and Lemma 3.6, we calculate

$$\begin{aligned}\ell(x_*) \geqslant &\left\langle v_2^{-1}\left(v_2 v_1^{-1}\mu_1 + \mu_2 - v_2\,\mathrm{wt}(v_1 \Rightarrow w_2 v_2)\right), 2\rho\right\rangle - \ell(v_2) + \ell(w_1 v_1)\\ =&\langle v_1^{-1}\mu_1, 2\rho\rangle + \langle v_2^{-1}\mu_2, 2\rho\rangle - d(v_1 \Rightarrow w_2 v_2) - \ell(v_1) + \ell(w_2 v_2) + \ell(v_2) + \ell(w_1 v_1)\\ =&\ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2 v_2)\end{aligned}$$

Both claims follow from this calculation. □

Lemma 5.8. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $u \in W$. Among all $v \in \mathrm{LP}(x)$, there is a unique one such that $d(v \Rightarrow u)$ becomes minimal. For this particular v, we have*

$$\forall \alpha \in \Phi^+ :\ \ell(x, v\alpha) - \langle \mathrm{wt}(v \Rightarrow u), \alpha\rangle + \Phi^+(u\alpha) - \Phi^+(v\alpha) \geqslant 0.$$

Proof. Let $x_2 = t^{u\lambda}$ with $\lambda \in X_*(T)_{\Gamma_0}$ superregular and dominant. Let $v = v_1 \in \mathrm{LP}(x)$ such that $d(v \Rightarrow u)$ becomes minimal. Set $v_2 = u$.

Consider Situation 5.2 for $x_1 = x$ and x_2 as above. Now the condition (iii) of Lemma 5.6 is satisfied by superregularity of λ. We conclude that $x_2' \leqslant x_2$, so that $x_* \leqslant x * x_2$.

Combining Lemma 5.4 with Lemma 5.7 shows

$$\ell(x) + \ell(x_2) - d(v \Rightarrow u) \geqslant \ell(x_1 * x_2) \geqslant \ell(x_*) \geqslant \ell(x) + \ell(x_2) - d(v \Rightarrow u).$$

In particular, we get $x_1 * x_2 = x_*$.

The above argument works whenever $v \in \mathrm{LP}(x)$ is chosen such that $d(v \Rightarrow u)$ becomes minimal. Since the value of $x_1 * x_2$ does not depend on the choice of such an element v, nor does $x_* = x_1 * x_2$. In particular, the classical part $\mathrm{cl}(x_*) = wvu^{-1}$ does not depend on v, hence v is uniquely determined.

The formula $x_* = x_1 * x_2 = x_1' x_2$ implies that $\ell(x_*) = \ell(x_1') + \ell(x_2)$. Using the previously computed length of x_*, we conclude $\ell(x_1') = \ell(x_1) - d(v \Rightarrow u)$. Now the estimate follows from Lemma 5.5. □

Considering Lemma 5.8 for the inverse x^{-1}, we obtain the following:

Lemma 5.9. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $u \in W$. Among all $v \in \mathrm{LP}(x)$, there is a unique one such that $d(u \Rightarrow wv)$ becomes minimal. For this particular v, we have*

$$\forall \alpha \in \Phi^+ :\ \ell(x, v\alpha) - \langle \mathrm{wt}(u \Rightarrow wv), \alpha\rangle - \Phi^+(u\alpha) + \Phi^+(wv\alpha) \geqslant 0. \qquad \square$$

Definition 5.10. Let $x \in \widetilde{W}$ and $u \in W$. The uniquely determined $v \in \mathrm{LP}(x)$ such that $d(v \Rightarrow u)$ is minimal will be denoted by $v = \rho_x^\vee(u)$. The uniquely determined $v \in \mathrm{LP}(x)$ such that $d(u \Rightarrow wv)$ is minimal will be denoted by $v = \rho_x(u) = w^{-1}\rho^\vee_{x^{-1}}(uw_0)w_0$.

The functions ρ_x and $\rho_x^\vee$ will be studied in Section 5.2. For now, we state our announced description of Demazure products in $\widetilde{W}$.

Theorem 5.11. *Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$. Among all pairs $(v_1, v_2) \in \mathrm{LP}(x_1) \times \mathrm{LP}(x_2)$, pick one such that the distance $d(v_1 \Rightarrow w_2v_2)$ becomes minimal.*

Construct x_* as in Situation 5.2. Then

$$\begin{aligned}
&x_1 * x_2 = x_* = w_1v_1\varepsilon^{v_1^{-1}\mu_1+v_2^{-1}\mu_2-\mathrm{wt}(v_1\Rightarrow w_2v_2)}v_2^{-1},\\
&\ell(x_1 * x_2) = \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2v_2),\\
&v_2 \in \mathrm{LP}(x_1 * x_2).
\end{aligned}$$

Proof. We have $x_* \leqslant x_1 * x_2$ by Lemmas 5.8 and 5.5. By Lemma 5.4, we find $(v_1', v_2') \in \mathrm{LP}(x_1) \times \mathrm{LP}(x_2)$ such that

$$\ell(x_1) + \ell(x_2) - d(v_1' \Rightarrow w_2v_2') \geqslant \ell(x_1 * x_2) \geqslant \ell(x_*) \geqslant \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2v_2).$$

By choice of (v_1, v_2), the result follows. □

We note the following consequences of Theorem 5.11.

Proposition 5.12. *Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$. Write*

$$\begin{aligned}
M = M(x_1, x_2) := \{&(v_1, v_2) \in \mathrm{LP}(x_1) \times \mathrm{LP}(x_2) \mid\\
&\forall (v_1', v_2') \in \mathrm{LP}(x_1) \times \mathrm{LP}(x_2) :\ d(v_1 \Rightarrow w_2v_2) \leqslant d(v_1' \Rightarrow w_2v_2')\}
\end{aligned}$$

for the set of all pairs (v_1, v_2) such that the theorem's condition is satisfied.

(a) The following two functions on M are both constant:

$$\begin{aligned}
&\varphi_1 : M \to W, \quad (v_1, v_2) \mapsto v_1v_2^{-1},\\
&\varphi_2 : M \to \mathbb{Z}\Phi^\vee, \quad (v_1, v_2) \mapsto v_2\,\mathrm{wt}(v_1 \Rightarrow w_2v_2).
\end{aligned}$$

(b) The following is a well-defined bijective map:

$$M \to \mathrm{LP}(x_1 * x_2), \quad (v_1, v_2) \mapsto v_2.$$

Proof. (a) From the theorem, we get that the function

$$\begin{aligned}
M \to \widetilde{W}, \quad (v_1, v_2) \mapsto &w_1v_1v_2^{-1}\varepsilon^{v_2v_1^{-1}\mu_1+\mu_2-v_2\,\mathrm{wt}(v_1\Rightarrow w_2v_2)}\\
= &w_1\varphi_1(v_1, v_2)\varepsilon^{\varphi_1(v_1,v_2)^{-1}\mu_1+\mu_2-\varphi_2(v_1,v_2)}
\end{aligned}$$

is constant with image $\{x_1 * x_2\}$. This proves that φ_1 and φ_2 are constant.

(b) Injectivity follows from (a). Well-definedness follows from the theorem. For surjectivity, let $v_2 \in \mathrm{LP}(x_1 * x_2)$. Then certainly $v_2 \in \mathrm{LP}(x_2)$. By Lemma 5.4, we find $v_1 \in W$ such that $\ell(x_1 * x_2) \leqslant \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2v_2)$. By the theorem, we find $(v_1', v_2') \in M$ with $\ell(x_1 * x_2) = \ell(x_1) + \ell(x_2) - d(v_1' \Rightarrow w_2v_2')$, such that $d(v_1 \Rightarrow w_2v_2) \leqslant d(v_1' \Rightarrow w_2v_2')$. It follows that $(v_1, v_2) \in M$, finishing the proof of surjectivity. □

Remark 5.13. In case $\ell(x_1x_2) = \ell(x_1) + \ell(x_2)$, we get $x_1x_2 = x_1 * x_2$. In this case, we recover Lemma 2.13.

5.2. Generic action

Studying the Demazure product where one of the factors is superregular induces actions of $(\widetilde{W}, *)$ on W, that we denoted by ρ_x resp. $\rho_x^\vee$ in Definition 5.10. In this section, we study these actions and the consequences for the quantum Bruhat graph.

Lemma 5.14. *Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$. Then*

$$\rho_{x_1 * x_2} = \rho_{x_2} \circ \rho_{x_1}.$$

Proof. Note that if $z \in \widetilde{W}$ is in a shrunken Weyl chamber with $\mathrm{LP}(z) = \{u\}$ and $x \in \widetilde{W}$, then by Proposition 5.12,

$$\mathrm{LP}(z * x) = \{\rho_x(u)\}.$$

Hence we have

$$\{\rho_{x_2}(\rho_{x_1}(u))\} = \mathrm{LP}\left((z * x_1) * x_2\right) = \mathrm{LP}\left(z * (x_1 * x_2)\right) = \{\rho_{x_1 * x_2}(u)\}.$$

This shows the desired claim. □

Remark 5.15. (a) There is a dual, albeit more complicated statement for the dual generic action $\rho^\vee$.

(b) If $x = \omega r_{a_1} \cdots r_{a_n}$ is a reduced decomposition with simple affine roots $a_1, \dots, a_n \in \Delta_{\mathrm{af}}$ and $\omega \in \Omega$ of length zero, then

$$\rho_x = \rho_{\omega * r_{a_1} * \cdots * r_{a_n}} = \rho_{r_{a_n}} \circ \cdots \circ \rho_{r_{a_1}} \circ \rho_\omega.$$

The map ρ_ω is simply given by $\rho_\omega(v) = \mathrm{cl}(\omega)v$, as $\mathrm{LP}(\omega) = W$. We now describe the $\rho_{r_{a_i}}$ as follows:

For a simple affine root $(\alpha, k) \in \Delta_{\mathrm{af}}$, we have

$$\ell(r_{(\alpha,k)}, \beta) = \begin{cases} 1, & \beta = \alpha, \\ -1, & \beta = -\alpha, \\ 0, & \beta \neq \pm\alpha. \end{cases}$$

Thus

$$\mathrm{LP}(r_{(\alpha,k)}) = \{v \in W \mid v^{-1}\alpha \in \Phi^+\}.$$

Let $v \in W$. If $v^{-1}\alpha \in \Phi^-$, then $s_\alpha v \in \mathrm{LP}(r_{(\alpha,k)})$ with $d(v \Rightarrow s_\alpha(s_\alpha v)) = 0$. Hence $\rho_{r_{(\alpha,k)}}(v) = s_\alpha v$.

If $v^{-1}\alpha \in \Phi^+$, then $v \in \mathrm{LP}(r_{(\alpha,k)})$ with $d(v \Rightarrow s_\alpha v) = 1$ by Lemma 3.8. Since there exists no $u \in \mathrm{LP}(r_{(\alpha,k)})$ with $d(v \Rightarrow s_\alpha u) = 0$, a distance of 1 is already minimal. We see that $\rho_{r_{(\alpha,k)}}(v) = v$. Summarizing:

$$\rho_{r_{(\alpha,k)}}(v) = \begin{cases} v, & v^{-1}\alpha \in \Phi^+, \\ s_\alpha v, & v^{-1}\alpha \in \Phi^-. \end{cases}$$

This gives an alternative method to compute ρ_x. One easily obtains a dual method to compute $\rho_x^\vee$ in a similar fashion.

Lemma 5.16. *Let $x \in \widetilde{W}$ and $v, v' \in \mathrm{LP}(x)$ be two length positive elements. There exists a shortest path p from v to v' in the quantum Bruhat graph such that each vertex in p lies in* $\mathrm{LP}(x)$.

Proof. Let us first study the case $v' = 1$.

We do induction on $\ell(v)$. If $\ell(v) = 0$, the statement is clear.

Otherwise, there exists a quantum edge $v \to vs_\alpha$ for some quantum root $\alpha \in \Phi^+$ such that $d(v \Rightarrow v') = d(vs_\alpha \Rightarrow v') + 1$ (Lemma 3.18). In this case, it suffices to show that $vs_\alpha \in \mathrm{LP}(x)$.

The quantum edge condition means that $\ell(vs_\alpha) = \ell(v) - \ell(s_\alpha)$. In other words, every positive root $\beta \in \Phi^+$ with $s_\alpha(\beta) \in \Phi^-$ satisfies $v(\beta) \in \Phi^-$.

Let $\beta \in \Phi^+$, we want to show that $\ell(x, vs_\alpha(\beta)) \geqslant 0$. This follows from length positivity of v if $s_\alpha(\beta) \in \Phi^+$. So let us assume that $s_\alpha(\beta) \in \Phi^-$. Then $vs_\alpha(\beta) \in \Phi^+$, applying the above observation to $-s_\alpha(\beta)$. Hence $\ell(x, vs_\alpha(\beta)) \geqslant 0$, as $1 \in \mathrm{LP}(x)$. This finishes the induction, so the claim is established whenever $v' = 1$.

For the general case, we do induction on $\ell(v')$. If $v' = 1$, we have proved the claim, so let us assume that $\ell(v') > 0$. Then we find a simple root $\alpha \in \Delta$ with $s_\alpha v' < v'$. In particular, $(v')^{-1}\alpha \in \Phi^-$ so that $\ell(x, \alpha) \leqslant 0$. Consider the element $x' := xs_\alpha > x$. We observe that for any $u \in W$ and $\beta \in \Phi$,

$$\ell(x', s_\alpha u\beta) = \ell(x, u\beta) + \ell(s_\alpha, -u\beta) = \begin{cases} \ell(x, u\beta), & u\beta \neq \pm\alpha, \\ -\ell(x, \alpha) + 1 > 0, & u\beta = -\alpha, \\ \ell(x, \alpha) - 1 < 0, & u\beta = \alpha. \end{cases}$$

It follows that

$$\mathrm{LP}(x') = \{s_\alpha u \mid u \in \mathrm{LP}(x) \text{ and } u^{-1}\alpha \in \Phi^-\}.$$

In particular, $s_\alpha v' \in \mathrm{LP}(x')$. Now suppose that $v^{-1}\alpha \in \Phi^-$. Then also $s_\alpha v \in \mathrm{LP}(x')$. We may apply the inductive assumption to get a path p' from $s_\alpha v$ to $s_\alpha v'$ in $\mathrm{LP}(x')$. Multiplying each vertex by s_α on the left, we obtain the desired path p in $\mathrm{LP}(x)$.

Finally assume that $v^{-1}\alpha \in \Phi^+$. Then $s_\alpha v \in \mathrm{LP}(x)$ by Corollary 4.7.

By Lemma 3.8, $v \to s_\alpha v$ is an edge in $\mathrm{QB}(W)$ and

$$d_{\mathrm{QB}(W)}(v \Rightarrow v') = d_{\mathrm{QB}(W)}(v \Rightarrow s_\alpha v') = d_{\mathrm{QB}(W)}(s_\alpha v \Rightarrow v') + 1.$$

We get a path from $s_\alpha v$ to v' in $\mathrm{LP}(x)$ by repeating the above argument, then concatenate it with $v \to s_\alpha v$.

This finishes the induction and the proof. □

Corollary 5.17. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v, v' \in \mathrm{LP}(x)$. Then*

$$v^{-1}\mu - (v')^{-1}\mu - \mathrm{wt}(v \Rightarrow v') + \mathrm{wt}(wv \Rightarrow wv') = 0.$$

In particular, $d(v \Rightarrow v') = d(wv \Rightarrow wv')$.

Proof. Let

$$p : v = v_1 \xrightarrow{\alpha_1} v_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} v_n = v'$$

be a path in $\mathrm{LP}(x)$ of weight $\mathrm{wt}(v \Rightarrow v')$. Now for $i = 1, \dots, n-1$, observe that both v_i and $v_i s_{\alpha_i}$ are in $\mathrm{LP}(x)$. Thus $\ell(x, v_i\alpha_i) = 0$. We conclude that

$$\begin{aligned}
&(v_i)^{-1}\mu - (v_{i+1})^{-1}\mu - \mathrm{wt}(v_i \Rightarrow v_{i+1}) + \mathrm{wt}(wv_i \Rightarrow wv_{i+1}) \\
=&\langle v_i\alpha_i, \mu\rangle\alpha_i^\vee - \Phi^+(-v_i\alpha_i)\alpha_i^\vee + \mathrm{wt}(wv_i \Rightarrow wv_i s_{\alpha_i}) \\
\leqslant&\langle v_i\alpha_i, \mu\rangle\alpha_i^\vee - \Phi^+(-v_i\alpha_i)\alpha_i^\vee + \Phi^+(wv_i\alpha_i)\alpha_i^\vee \\
=&\ell(x, v_i\alpha_i)\alpha_i^\vee = 0.
\end{aligned}$$

Summing these estimates for $i = 1, \dots, n-1$, we conclude

$$v^{-1}\mu - (v')^{-1}\mu - \mathrm{wt}(v \Rightarrow v') + \mathrm{wt}(wv \Rightarrow w'v') \leqslant 0.$$

Considering the same argument for $x^{-1}, wvw_0, wv'w_0$, we get the other inequality.

The "in particular" part follows from inspecting the argument given. Alternatively, pair the identity just proved with 2ρ, then apply Lemma 3.6 and Corollary 2.11. $\square$

Remark 5.18. The corollary can be shown directly by evaluating the Demazure product

$$\varepsilon^{wv'2\rho} * x * \varepsilon^{v2\rho}$$

in two different ways, using the associativity property of Demazure products.

Proposition 5.19. *Let $x = w\varepsilon^\mu \in \widetilde{W}$, $v \in \mathrm{LP}(x)$ and $u \in W$. Then*

$$d(u \Rightarrow wv) = d(u \Rightarrow w\rho_x(u)) + d(w\rho_x(u) \Rightarrow wv).$$

Proof. Let λ be superregular and $y := \varepsilon^{u\lambda}$. Define the element

$$z := y * x = u\rho_x(u)^{-1}\varepsilon^{\rho_x(u)\lambda + \mu - \rho_x(u)\,\mathrm{wt}(u \Rightarrow w\rho_x(u))}.$$

Then z is superregular with $\mathrm{LP}(z) = \{\rho_x(u)\}$. Consider the element

$$\tilde{y}' := u(wv)^{-1}\varepsilon^{wv\lambda - wv\,\mathrm{wt}(u \Rightarrow wv)}.$$

This is superregular with $\mathrm{LP}(\tilde{y}') = \{wv\}$. Note that Theorem 4.2 implies $\tilde{y}' \leqslant y$, as

$$(wv)^{-1}(wv\lambda - wv\,\mathrm{wt}(u \Rightarrow wv)) + \mathrm{wt}(u \Rightarrow wv) + \mathrm{wt}(u \Rightarrow u) = \lambda.$$

Thus $\tilde{z} \leqslant z$, where

$$\tilde{z} = \tilde{y}x = uv^{-1}\varepsilon^{v\lambda + \mu - v\,\mathrm{wt}(u \Rightarrow wv)}.$$

Note that $\tilde{z}$ is superregular with $\mathrm{LP}(\tilde{z}) = \{v\}$. In light of Theorem 4.2, the inequality $\tilde{z} \leqslant z$ means

$$v^{-1}(v\lambda + \mu - v\,\mathrm{wt}(u \Rightarrow wv)) + \mathrm{wt}(\rho_x(u) \Rightarrow v) + \mathrm{wt}(u \Rightarrow u)$$
$$\leqslant \rho_x(u)^{-1}(\rho_x(u)\lambda + \mu - \rho_x(u)\,\mathrm{wt}(u \Rightarrow w\rho_x(u))).$$

Rewriting this, we get

$$v^{-1}\mu - \mathrm{wt}(u \Rightarrow wv) + \mathrm{wt}(\rho_x(u) \Rightarrow v) \leqslant \rho_x(u)^{-1}\mu - \mathrm{wt}(u \Rightarrow w\rho_x(u)).$$

Corollary 5.17 yields the equation

$$v^{-1}\mu - \rho_x(u)^{-1}\mu + \mathrm{wt}(\rho_x(u) \Rightarrow v) = \mathrm{wt}(w\rho_x(u) \Rightarrow wv).$$

We conclude

$$\mathrm{wt}(u \Rightarrow wv) \geqslant \mathrm{wt}(u \Rightarrow w\rho_x(u)) + \mathrm{wt}(w\rho_x(u) \Rightarrow wv).$$

This implies the desired claim. □

By the duality from Lemma 5.3, we obtain the following.

Corollary 5.20. *Let $x = w\varepsilon^\mu \in \widetilde{W}$, $v \in \mathrm{LP}(x)$ and $u \in W$. Then*

$$d(v \Rightarrow u) = d(v \Rightarrow \rho_x^\vee(u)) + d(\rho_x^\vee(u) \Rightarrow u). \qquad \square$$

Remark 5.21. In the language of [BFP98, Section 6], this means that the set $w\,\mathrm{LP}(x)$ contains a unique minimal element with respect to the tilted Bruhat order $\preceq_u$. Since $w\,\mathrm{LP}(x) = \mathrm{LP}(x^{-1})w_0$, it follows that the set $\mathrm{LP}(x)$ contains a unique maximal element with respect to $\preceq_u$. If $x = \varepsilon^\mu$ is a pure translation element, this recovers [Len+15, Theorem 7.1].

The converse statements are generally false, i.e. $\mathrm{LP}(x)$ will in general not contain tilted Bruhat minima, and $w\,\mathrm{LP}(x)$ will not contain maxima. For a concrete example, choose x to be a simple affine reflection of type A_2.

The set $\mathrm{LP}(x)$ satisfies a number of interesting structural properties with respect to the quantum Bruhat graph, namely containing shortest paths for any pair of elements (Lemma 5.16) and the existence of tilted Bruhat maxima. One may ask the question which subsets of W occur as the set $\mathrm{LP}(x)$ for some $x \in \widetilde{W}$.

Corollary 5.22. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $u_1, u_2 \in W$. Then the function*

$$\varphi : W \to X_*(T)_{\Gamma_0},\ v \mapsto v^{-1}\mu - \mathrm{wt}(u_1 \Rightarrow wv) - \mathrm{wt}(v \Rightarrow u_2)$$

has a global maximum at $\rho_x(u_1)$, and another global maximum at $\rho_x^\vee(u_2)$.

Proof. If $v \in W$ is not length positive for x, and vs_α is an adjustment, it is easy to see that $\varphi(v) \leqslant \varphi(vs_\alpha)$. So we may focus on $\varphi|_{\mathrm{LP}(x)}$.

Let $v \in \mathrm{LP}(x)$ and $v' = \rho_x(u_1)$, so that

$$\begin{aligned}\varphi(v) &= v^{-1}\mu - \mathrm{wt}(u_1 \Rightarrow wv) - \mathrm{wt}(v \Rightarrow u_2)\\ &= v^{-1}\mu - \mathrm{wt}(u_1 \Rightarrow wv') - \mathrm{wt}(wv' \Rightarrow wv) - \mathrm{wt}(v \Rightarrow u_2)\\ &\underset{\mathrm{C5.17}}{=} (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow v) - \mathrm{wt}(u_1 \Rightarrow wv') - \mathrm{wt}(v \Rightarrow u_2)\\ &\leqslant (v')^{-1}\mu - \mathrm{wt}(u_1 \Rightarrow wv') - \mathrm{wt}(v' \Rightarrow u_2) = \varphi(v').\end{aligned}$$

This shows the first maximality claim. The second one follows from the duality of Lemma 5.3. ☐

Remark 5.23. Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$ and $v_1 \in \mathrm{LP}(x_1)$. Theorem 4.2 states that $x_1 \leqslant x_2$ in the Bruhat order if and only if there is some $v_2 \in W$ with

$$v_1^{-1}\mu_1 + \mathrm{wt}(v_2 \Rightarrow v_1) + \mathrm{wt}(w_1v_1 \Rightarrow w_2v_2) \leqslant v_2^{-1}\mu_2.$$

By the above corollary, it is equivalent to require this inequality for $v_2 = \rho_{x_2}(w_1v_1)$. One can alternatively require it for $v_2 = \rho^\vee_{x_2}(v_1)$.

Lemma 5.24. *Let $x_1 = w_1\varepsilon^{\mu_1}, x_2 = w_2\varepsilon^{\mu_2} \in \widetilde{W}$ and $v_1 \in \mathrm{LP}(x_1), v_2 \in \mathrm{LP}(x_2)$. The following are equivalent:*

(i) The distance $d(v_1 \Rightarrow w_2v_2)$ is minimal for all pairs in $\mathrm{LP}(x_1) \times \mathrm{LP}(x_2)$, i.e. $(v_1, v_2) \in M(x_1, x_2)$.

(ii) $v_1 = \rho^\vee_{x_1}(w_2v_2)$ and $v_2 = \rho_{x_2}(v_1)$.

Proof. (i) ⇒ (ii): Certainly, v_1 minimizes the function $d(\cdot \Rightarrow w_2v_2)$ on $\mathrm{LP}(x_1)$, showing the first claim. The second claim is analogous.

(ii) ⇒ (i): Consider Situation 5.2. By Lemmas 5.5 and 5.8, we conclude that w_2v_2 must be length positive for x_1'. It follows that $x_* \leqslant x_1 * x_2$ and

$$\ell(x_*) = \ell(x_1') + \ell(x_2) = \ell(x_1) + \ell(x_2) - d(v_1 \Rightarrow w_2v_2).$$

By Lemma 5.7, v_2 is length positive for x_*. Write $x_1 * x_2$ as $\tilde{w}\varepsilon^{\tilde{\mu}}$. Using Lemma 4.17 with Lemma 4.3, the condition $x_* \leqslant x_1 * x_2$ yields some $v_2' \in \mathrm{LP}(x_1 * x_2)$ with

$$v_1^{-1}\mu_1 + v_2^{-1}\mu_2 - \mathrm{wt}(v_1 \Rightarrow w_2v_2) + \mathrm{wt}(v_2' \Rightarrow v_2) + \mathrm{wt}(w_1v_1 \Rightarrow \tilde{w}v_2') \leqslant (v_2')^{-1}\tilde{\mu}.$$

By Proposition 5.12, we find v_1' such that $(v_1', v_2') \in M(x_1, x_2)$. By Theorem 5.11, we can express $x_1 * x_2$ in terms of (v_1', v_2'). Then the above inequality becomes

$$\begin{aligned}&v_1^{-1}\mu_1 + v_2^{-1}\mu_2 - \mathrm{wt}(v_1 \Rightarrow w_2v_2) + \mathrm{wt}(v_2' \Rightarrow v_2) + \mathrm{wt}(w_1v_1 \Rightarrow w_1v_1')\\ &\leqslant (v_1')^{-1}\mu_1 + (v_2')^{-1}\mu_2 - \mathrm{wt}(v_1' \Rightarrow w_2v_2').\end{aligned}$$

Since $v_1, v_1' \in \mathrm{LP}(x_1)$ and $v_2, v_2' \in \mathrm{LP}(x_2)$, we can apply Corollary 5.17 twice to obtain

$$\mathrm{wt}(v_1 \Rightarrow v_1') + \mathrm{wt}(w_2v_2' \Rightarrow w_2v_2) - \mathrm{wt}(v_1 \Rightarrow w_2v_2) \leqslant -\mathrm{wt}(v_1' \Rightarrow w_2v_2').$$

Rewriting, we get

$$\mathrm{wt}(v_1 \Rightarrow v_1') + \mathrm{wt}(v_1' \Rightarrow w_2v_2') + \mathrm{wt}(w_2v_2' \Rightarrow w_2v_2) \leqslant \mathrm{wt}(v_1 \Rightarrow w_2v_2).$$

In other words, there is a shortest path from v_1 to w_2v_2 that passes through v_1' and w_2v_2'. By condition (ii), this is only possible if $v_1 = v_1'$ and $v_2 = v_2'$, showing (i). $\square$

Corollary 5.25. *Consider Situation 5.2 with $v_1 \in \mathrm{LP}(x_1), v_2 \in \mathrm{LP}(x_2)$. There exists $(v_1', v_2') \in M(x_1, x_2)$ such that*

$$d(v_1 \Rightarrow w_2v_2) = d(v_1 \Rightarrow v_1') + d(v_1' \Rightarrow w_2v_2') + d(w_2v_2' \Rightarrow w_2v_2).$$

Proof. For convenience, we define a set of *admissible pairs* by

$$\begin{aligned} A := \{&(v_1', v_2') \in \mathrm{LP}(x_1) \times \mathrm{LP}(x_2) \mid \\ &d(v_1 \Rightarrow w_2v_2) = d(v_1 \Rightarrow v_1') + d(v_1' \Rightarrow w_2v_2') + d(w_2v_2' \Rightarrow w_2v_2)\}.\end{aligned}$$

Then $(v_1, v_2) \in A$, so that A is non-empty. Choose $(v_1', v_2') \in A$ such that $d(v_1' \Rightarrow w_2v_2')$ becomes minimal among all pairs in A. We claim that $(v_1', v_2') \in M(x_1, x_2)$. For this, we use Lemma 5.24. It remains to show that $v_1' = \rho^\vee_{x_1}(w_2v_2')$ and $v_2' = \rho_{x_2}(v_1)$. By Proposition 5.19 and Corollary 5.20, we obtain

$$\begin{aligned} d(v_1' \Rightarrow w_2v_2') &= d(v_1' \Rightarrow \rho^\vee_{x_1}(w_2v_2')) + d(\rho^\vee_{x_1}(w_2v_2') \Rightarrow w_2v_2'),\\ d(v_1' \Rightarrow w_2v_2') &= d(v_1' \Rightarrow w_2\rho_{x_2}(v_1)) + d(w_2\rho_{x_2}(v_1) \Rightarrow w_2v_2').\end{aligned}$$

It follows that $(\rho^\vee_{x_1}(w_2v_2'), v_2') \in A$ and $(v_1', \rho_{x_2}(v_1')) \in A$. By choice of (v_1', v_2') and the above computation, we get that $v_1' = \rho^\vee_{x_1}(w_2v_2')$ and $v_2' = \rho_{x_2}(v_1')$. This finishes the proof. $\square$

Corollary 5.26. *For $x_1, x_2 \in \widetilde{W}$, we have $\mathrm{LP}(x_1 * x_2) = \rho_{x_2}(\mathrm{LP}(x_1)) = \rho^\vee_{x_1}(w_2\,\mathrm{LP}(x_2))$, where $w_2 \in W$ is the classical part of x_2.*

Proof. We only show $\mathrm{LP}(x_1 * x_2) = \rho_{x_2}(\mathrm{LP}(x_1))$, the other claim is completely dual.

If $v_2 \in \mathrm{LP}(x_1 * x_2)$, we find $v_1 \in \mathrm{LP}(x_1)$ such that $(v_1, v_2) \in M(x_1, x_2)$. By Lemma 5.24, $v_2 = \rho_{x_2}(v_1) \in \rho_{x_2}(\mathrm{LP}(x_1))$.

Now let $v_2 \in \rho_{x_2}(\mathrm{LP}(x_1))$ and write $v_2 = \rho_{x_2}(v_1)$ for some $\widetilde{v_1} \in \mathrm{LP}(x_1)$. By Corollary 5.25, we find $(v_1', v_2') \in M(x_1, x_2)$ such that

$$d(v_1 \Rightarrow w_2v_2) = d(v_1 \Rightarrow w_2v_2') + d(w_2v_2' \Rightarrow w_2v_2).$$

Since $v_2 = \rho_{x_2}(v_1)$, we use Proposition 5.19 to obtain

$$d(v_1 \Rightarrow w_2v_2') = d(v_1 \Rightarrow w_2v_2) + d(w_2v_2 \Rightarrow w_2v_2').$$

This is only possible if $v_2 = v_2'$. Since $v_2' \in \mathrm{LP}(x_1 * x_2)$ by Proposition 5.12, we obtain the desired claim $v_2 \in \mathrm{LP}(x_1 * x_2)$. $\square$

6. σ-conjugacy classes

In this section, we review various descriptions of the set $B(G)$ of σ-conjugacy classes in $G(L)$. This serves mostly as a preparation for the next section, which discusses the *generic* σ-conjugacy class of an element $x \in \widetilde{W}$. Throughout this section, we assume that G is quasi-split.

We begin with the classical result of Kottwitz [Kot85; Kot97] that describes the σ-conjugacy class of an element $g \in G(L)$ by two invariants. These are called *Kottwitz point* $\kappa(g) \in \pi_1(G)_\Gamma = (X_*(T)/\mathbb{Z}\Phi^\vee)_\Gamma$ and *(dominant) Newton point* $\nu(g) \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.

If g lies in the normalizer of the maximal torus, $g \in N_G(T)(L)$, then it corresponds to an element in $w\varepsilon^\mu \in \widetilde{W}$. In this case, $\kappa(g)$ is the image of μ in $\pi_1(G)_\Gamma$.

Viewing both w and σ as automorphisms of $X_*(T)_{\Gamma_0}$, we write $\sigma \circ w$ for their composition. Let $N \geqslant 1$ such that the $(\sigma \circ w)^N$ is the identity map. Then $\nu(g) \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ is the unique dominant element in the W-orbit of

$$\frac{1}{N}\sum_{k=1}^{N}(\sigma \circ w)^k \mu.$$

It is true, e.g. by [He14, Section 3.3], that each σ-conjugcacy class $[b] \in B(G)$ contains an element of $N_G(T)(L)$, so that the above descriptions of $\kappa(g)$ and $\nu(g)$ actually cover all σ-conjugacy classes.

In this section, we review a few important results related to these invariants. Our main concern is to bridge the gap between the unramified case, which is often studied in the relevant literature, and the quasi-split case, which we need for our final generalization.

6.1. Parabolic averages and convex hull

We start by formally defining some averaging functions and proving their basic properties. Neither our results nor our proofs in this section should be too surprising for the educated reader, especially if one keeps the example of GL_n and its Newton polygons in mind.

Let $N \geqslant 1$ be an integer such that the action of σ^N on $X_*(T)$ becomes trivial. Then we define the *σ-average* of an element $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ by

$$\operatorname{avg}_\sigma(\mu) := \frac{1}{N}\sum_{k=1}^{N}\sigma^k(\mu) \in (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\langle\sigma\rangle}.$$

Since $\operatorname{avg}_\sigma$ vanishes on terms of the form $\mu - \sigma(\mu)$, it follows that we get a well-defined map $\operatorname{avg}_\sigma : X_*(T)_\Gamma \to (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\langle\sigma\rangle}$.

A similar notion of average is the following: For $J \subseteq \Delta$, denote by W_J the Coxeter subgroup of W generated by the reflections $\{s_\alpha \mid \alpha \in J\}$. For $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$, we define

$$\operatorname{avg}_J(\mu) := \frac{1}{\#W_J}\sum_{w \in W_J} w(\mu) \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}.$$

Finally, if $J = \sigma(J)$, we define the function π_J by

$$\pi_J := \operatorname{avg}_J \circ \operatorname{avg}_\sigma = \operatorname{avg}_\sigma \circ \operatorname{avg}_J : X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \to (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\langle\sigma\rangle}.$$

This map was introduced by Chai [Cha00, Definition 3.2]. Again, we get an induced map $\pi_J : X_*(T)_\Gamma \to (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\langle\sigma\rangle}$. If G is split, it can be identified with the *slope map* as introduced by Schieder [Sch15, Section 2.1.3].

We start with a collection of easy facts on these averages.

Lemma 6.1. *Let $\beta \in X_*(T)_\Gamma$ and $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$. Let $J \subseteq \Delta$ be any subset.*

(a) For any preimage $\beta' \in X_*(T)_{\Gamma_0}$ of β, we have

$$\langle \beta', 2\rho\rangle = \langle \operatorname{avg}_\sigma(\beta), 2\rho\rangle.$$

In particular, it makes sense to write $\langle\beta, 2\rho\rangle$.

(b) If $\langle\mu,\alpha\rangle = 0$ for all $\alpha \in J$, then $\operatorname{avg}_J(\mu) = \mu$.

(c) For all $\alpha \in J$, we have $\langle \operatorname{avg}_J(\mu), \alpha\rangle = 0$.

(d) If $\mu \geqslant 0$, then $\operatorname{avg}_J(\mu) \geqslant 0$.

(e) If $\langle\mu,\alpha\rangle \leqslant 0$ for all $\alpha \in J$, then $\mu \leqslant w\mu$ for all $w \in W_J$. In particular, $\mu \leqslant \operatorname{avg}_J(\mu)$.

Proof. (a) follows since $\sigma(2\rho) = 2\rho$ and $\operatorname{avg}_\sigma(b) = \operatorname{avg}_\sigma(b')$.

For (b) and (c), note that the following are equivalent:

- $\langle\mu,\alpha\rangle = 0$ for all $\alpha \in J$,
- $w(\mu) = \mu$ for all $w \in W_J$.

Then both statements follow easily.

For (d), it suffices to only consider the case where μ is a simple coroot $\mu = \alpha^\vee$. If $\alpha \in J$, then $\operatorname{avg}_J(\mu) = 0$. Otherwise $w(\alpha) \in \Phi^+$ for all $w \in W_J$, such that $\operatorname{avg}_J(\mu) > 0$.

We prove (e) via induction on $\ell(w)$, the inductive start being clear. If now $\ell(w) \geqslant 1$ and $w\alpha \in \Phi^-$ for some $\alpha \in J$, then

$$w\mu = (ws_\alpha)(\mu - \langle\mu,\alpha\rangle\alpha^\vee) = (ws_\alpha)\mu + \langle\mu,\alpha\rangle w\alpha^\vee \geqslant (ws_\alpha)\mu \underset{\text{ind.}}{\geqslant} \mu.$$

This finishes the induction and the proof. ☐

Definition 6.2. Let $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ and $J \subseteq \Delta$ be any subset.

(a) We say that J is *μ-improving* if we can write $J = \{\alpha_1, \ldots, \alpha_k\}$ such that

$$\langle \operatorname{avg}_{\{\alpha_1,\ldots,\alpha_{i-1}\}}(\mu), \alpha_i\rangle \leqslant 0$$

for $i = 1, \ldots, k$.

(b) We say that J is *maximally μ-improving* if it is μ-improving, and any μ-improving superset $J' \supseteq J$ satisfies $\operatorname{avg}_J(\mu) = \operatorname{avg}_{J'}(\mu)$.

E.g. any μ-improving subset of maximal cardinality will be maximally μ-improving. Since the empty set is μ-improving, it follows that maximally μ-improving subsets always exist. We make the following immediate observations:

Lemma 6.3. *Let $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ and $J \subseteq \Delta$.*

(a) If J is μ-improving, then $\mu \leqslant \operatorname{avg}_J(\mu)$.

(b) If J is maximally μ-improving, then $\operatorname{avg}_J(\mu)$ is dominant.

(c) If $c \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ is dominant and $\mu \leqslant c$, then

$$\operatorname{avg}_J(\mu) \leqslant \operatorname{avg}_J(c) \leqslant c. \qquad \square$$

If follows that there is a uniquely determined maximum

$$\operatorname{conv}'(\mu) := \max_{J \subseteq \Delta} \operatorname{avg}_J(\mu),$$

and that $\operatorname{conv}'(\mu) = \operatorname{avg}_J(\mu)$ for every maximally μ-improving J. We define

$$\operatorname{conv}(\mu) := \operatorname{conv}'(\operatorname{avg}_\sigma(\mu)), \qquad \mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \text{ or } \mu \in X_*(T)_\Gamma.$$

Example 6.4. For the split group $G = \mathrm{GL}_n$, the operations conv and conv$'$ agree. Drawing elements of $X_*(T) \otimes \mathbb{Q}$ as polygons, the function conv corresponds to taking the upper convex hull (hence its name).

Lemma 6.5. *Let $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.*

(a) The value $\operatorname{conv}'(\mu)$ is the uniquely determined element $c \in X_*(T)_{\Gamma_0}$ satisfying the following three conditions:

- *$\mu \leqslant c$,*
- *c is dominant and*
- *$c = \operatorname{avg}_J(\mu)$ for some $J \subseteq \Delta$.*

(b) If $\mu' \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ satisfies $\mu \leqslant \mu'$, then $\operatorname{conv}'(\mu) \leqslant \operatorname{conv}'(\mu')$.

(c) Write

$$\operatorname{conv}'(\mu) - \mu = \sum_{\alpha \in \Delta} c_\alpha \alpha^\vee,$$
$$J_1 := \{\alpha \in \Delta \mid c_\alpha \neq 0\},$$
$$J_2 := \{\alpha \in \Delta \mid \langle \operatorname{conv}'(\mu), \alpha \rangle = 0\}.$$

For any subset $J \subseteq \Delta$, we have

$$\operatorname{conv}'(\mu) = \operatorname{avg}_J(\mu) \iff J_1 \subseteq J \subseteq J_2.$$

(d) *There exists* $J \subseteq \Delta$ *with* $\sigma(J) = J$ *and* $\operatorname{conv}(\mu) = \pi_J(\mu)$. *In particular,*

$$\operatorname{conv}(\mu) = \max_{\substack{J \subseteq \Delta \\ \sigma(J) = J}} \pi_J(\mu).$$

(e) *Let* $J \subseteq \Delta$ *such that*

$$\forall \alpha \in \Phi^+ \backslash \Phi_J^+ : \langle \mu, \alpha \rangle \geqslant 0.$$

Then there exists $J' \subseteq J$ *with* $\operatorname{conv}'(\mu) = \operatorname{avg}_{J'}(\mu)$. *In other words, the set* J_1 *from (c) is a subset of* J.

Proof. (a) and (b) are immediate.

(c) Let us first consider a subset $J \subseteq \Delta$ with $\operatorname{conv}'(\mu) = \operatorname{avg}_J(\mu)$. Then $\operatorname{conv}'(\mu) - \mu \in \mathbb{Q}\Phi_J^\vee$ by definition of $\operatorname{avg}_J(\mu)$. We see that $J_1 \subseteq J$ must hold. Similarly, $\langle \operatorname{conv}'(\mu), \alpha \rangle = 0$ for all $\alpha \in J$ by Lemma 6.1. Thus we must have $J_1 \subseteq J \subseteq J_2$.

We show that $\operatorname{avg}_{J_1}(\mu)$ is dominant. Let $\alpha \in \Delta$. If $\alpha \in J_1$, then $\langle \operatorname{avg}_{J_1}(\mu), \alpha \rangle = 0$ by Lemma 6.1. So let us assume that $\alpha \in \Delta \backslash J_1$. Because $\operatorname{avg}_{J_1}(\mu) \leqslant \operatorname{conv}'(\mu)$ and $\operatorname{avg}_{J_1}(\mu) \equiv \mu \equiv \operatorname{conv}'(\mu) \pmod{\mathbb{Q}\Phi_{J_1}^\vee}$, we can write

$$\operatorname{conv}'(\mu) - \operatorname{avg}_{J_1}(\mu) = \sum_{\beta \in J_1} c'_\beta \beta^\vee, \quad c'_\beta \in \mathbb{Q}_{\geqslant 0}.$$

Now we get

$$\langle \operatorname{avg}_{J_1}(\mu), \alpha \rangle = \underbrace{\langle \operatorname{conv}'(\mu), \alpha \rangle}_{\geqslant 0} + \sum_{\beta \in J_1} \underbrace{c'_\beta \langle -\beta^\vee, \alpha \rangle}_{\geqslant 0} \geqslant 0.$$

This shows that $\operatorname{avg}_{J_1}(\mu)$ is dominant.

If J is chosen such that $\operatorname{conv}'(\mu) = \operatorname{avg}_J(\mu)$, then

$$\operatorname{conv}'(\mu) \geqslant \operatorname{avg}_{J_1}(\mu) \underset{\text{L6.1}}{\geqslant} \operatorname{avg}_J \operatorname{avg}_{J_1}(\mu) \underset{J_1 \subseteq J}{=} \operatorname{avg}_J(\mu) = \operatorname{conv}'(\mu).$$

Thus $\operatorname{avg}_{J_1}(\mu) = \operatorname{conv}'(\mu)$.

So if for any intermediate set $J_1 \subseteq J \subseteq J_2$, we obtain

$$\operatorname{avg}_J(\mu) = \operatorname{avg}_J(\operatorname{avg}_{J_1}(\mu)) = \operatorname{avg}_J(\operatorname{conv}'(\mu)) \underset{J \subseteq J_2}{=} \operatorname{conv}'(\mu).$$

(d) Replacing μ by $\operatorname{avg}_\sigma(\mu)$, we may certainly assume $\mu \in (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^\sigma$. Since $\mu = \sigma(\mu)$, we conclude $\operatorname{conv}'(\mu) = \sigma(\operatorname{conv}'(\mu))$. Then we can choose J be either of the sets J_1 or J_2 from (c).

Now the "in particular" part is easy to see.

(e) Let $J' \subseteq J$ be a μ-improving subset such that there is no μ-improving subset $J' \subsetneq J'' \subseteq J$. By Lemma 6.3, $\mu \leqslant \mathrm{avg}_{J'}(\mu)$. It suffices to show that $\mathrm{avg}_{J'}(\mu)$ is dominant. Seeing μ as a coweight for the root system Φ_J, the set J' is maximally μ-improving from this perspective, so $\langle \mathrm{avg}_{J'}\mu, \alpha\rangle \geqslant 0$ for all $\alpha \in \Phi_J^+$.

If $\alpha \in \Phi^+ \backslash \Phi_J^+$, then $w\alpha \in \Phi^+\backslash\Phi_J^+$ for all $w \in W_J$, such that

$$\langle \mathrm{avg}_{J'}(\mu), \alpha\rangle = \frac{1}{\# W_{J'}} \sum_{w \in W_{J'}} \underbrace{\langle \mu, w\alpha\rangle}_{\geqslant 0} \geqslant 0.$$

Here, we use the assumption made on μ and J.

As $\mathrm{avg}_{J'}(\mu)$ is dominant, we get the desired result by (a). □

As an immediate application, let us describe Newton points of elements in $\widetilde{W}$ with this language:

Definition 6.6. For $w \in W$,we write $\mathrm{supp}(w) \subseteq \Delta$ for the set of all simple roots whose corresponding simple reflections occur in some/every reduced expression for w. Define $\mathrm{supp}_\sigma(w) := \bigcup_{n \in \mathbb{Z}} \sigma^n(\mathrm{supp}(w))$.

Lemma 6.7. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $N > 0$ such that $(\sigma \circ w)^N = \mathrm{id}$. Pick $v \in W$ such that*

$$v^{-1}\frac{1}{N}\sum_{k=1}^N (\sigma \circ w)^k(\mu) \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$$

becomes dominant. Let $J = \mathrm{supp}_\sigma(v^{-1\,\sigma}(wv))$. Then

$$\nu(x) = \pi_J(v^{-1}\mu).$$

Proof. Straightforward calculation. For an alternative proof, cf. [Cha00, Proposition 4.1].

By definition, we have

$$\begin{aligned}\nu(x) =& v^{-1}\frac{1}{N}\sum_{k=1}^N (\sigma\circ w)^k(\mu)\\ =&\frac{1}{N}\sum_{k=1}^N (v^{-1}\circ\sigma\circ wv)^k(v^{-1}\mu)\\ =&\frac{1}{N}\sum_{k=1}^N (v^{-1\,\sigma}(wv)\circ\sigma)^k(v^{-1}\mu).\end{aligned}$$

Note that

$$(v^{-1\,\sigma}(wv)\circ\sigma)(\nu(x)) = \nu(x).$$

We see that $\sigma(\nu(x))$ lies in the same W-orbit as $\nu(x)$, so $\nu(x) = \sigma(\nu(x))$ by dominance (this is well-known). Thus $v^{-1}\,{}^\sigma(wv)$ stabilizes $\nu(x)$. Write $J' = \{\alpha \in \Delta \mid \langle \nu(x), \alpha\rangle = 0\}$. Then $v^{-1}\,{}^\sigma(wv) \in W_{J'}$, so $J \subseteq J'$. Hence

$$\begin{aligned}\nu(x) =&\pi_J(\nu(x)) = \frac{1}{N}\sum_{k=1}^N \pi_J\left[(v^{-1}\,{}^\sigma(wv)\circ\sigma)^k(v^{-1}\mu)\right]\\ =&\frac{1}{N}\sum_{k=1}^N \pi_J\left[(v^{-1}\mu)\right] = \pi_J(v^{-1}\mu).\end{aligned}$$ □

6.2. λ-invariant and defect

For this section, we fix a σ-conjugacy class $[b] \in B(G)$. Following Hamacher-Viehmann [HV18, Lemma/Definition 2.1], we define its *λ-invariant* by

$$\lambda_G(b) := \max\{\tilde\lambda \in X_*(T)_\Gamma \mid \mathrm{avg}_\sigma(\tilde\lambda) \leqslant \nu(b) \text{ and } \kappa(b) = \lambda + \mathbb{Z}\Phi^\vee \text{ in } \pi_1(G)_\Gamma\}.$$

While the article of Hamacher-Viehmann assumes the group to be unramified, the construction of $\lambda_G(b)$ works without changes for quasi-split G.

Let us write

$$\begin{aligned}\nu(b) - \mathrm{avg}_\sigma(\lambda_G(b)) = \sum_{\alpha\in\Delta} c_\alpha\alpha^\vee,&\\ J_1 :=&\{\alpha\in\Delta \mid c_\alpha \neq 0\},\\ J_2 :=&\{\alpha \in \Delta \mid \langle\nu(b),\alpha\rangle = 0\}.\end{aligned}$$

We have the following simple observations:

Lemma 6.8. *(a) Pick $\mu \in X_*(T)_\Gamma$ and $J \subseteq \Delta$ with $J = \sigma(J)$ such that $\nu(b) = \pi_J(\mu)$ and $\kappa(b) = \mu + \mathbb{Z}\Phi^\vee \in \pi_1(G)_\Gamma$. Then*

$$\nu(b) = \pi_J(\lambda_G(b)) = \mathrm{conv}(\lambda_G(b)).$$

(b) We have $J_1 \subseteq J_2$. For $J \subseteq \Delta$ with $\sigma(J) = J$,

$$\nu(b) = \pi_J(\lambda_G(b)) \iff J_1 \subseteq J \subseteq J_2.$$

Proof. (a) Choose a lift $\tilde\mu \in X_*(T)_{\Gamma_0}$. Then

$$\pi_J(\mu) = \pi_J(\tilde\mu) = \mathrm{avg}_\sigma \sum_{w\in W_J} w\tilde\mu.$$

We can choose an element $w \in W_J$ such that $w\tilde\mu$ becomes anti-dominant with respect to the roots in J, i.e. $\langle w\tilde\mu, \alpha\rangle \leqslant 0$ for all $\alpha \in J$. Then $\pi_J(\tilde\mu) = \pi_J(w\tilde\mu) \geqslant w\tilde\mu$ by Lemma 6.1.

In particular, the image of $w\tilde\mu$ in $X_*(T)_\Gamma$ is $\leqslant \lambda_G(b)$ by construction of $\lambda_G(b)$. Thus

$$\nu(b) = \pi_J(w\tilde\mu) \leqslant \pi_J(\lambda_G(b)) \leqslant \mathrm{conv}(\lambda_G(b)).$$

Since $\mathrm{avg}_\sigma(\lambda_G(b)) \leqslant \nu(b)$ and $\nu(b)$ is dominant, we use Lemma 6.3 to see that $\mathrm{conv}(\lambda_G(b)) \leqslant \nu(b)$. Hence $\nu(b) = \mathrm{conv}(\lambda_G(b)) = \pi_J(\lambda_G(b))$.

(b) By [He14, Section 3.3], $b = [x]$ for some $x \in \widetilde{W}$. Applying Lemma 6.7 to x, we see that μ and J exist as in (a). In particular, $\nu(b) = \operatorname{conv}(\lambda_G(b))$.

Now all claims follow from Lemma 6.5. □

Related to the notion of the λ-invariant is the notion of *defect* of an element $[b] \in B(G)$.

Following [Kot85, Proposition 6.2], we fix an element $x = w\varepsilon^\mu$ of length zero in the extended affine Weyl group $\widetilde{W}_{J_2}$ of the Levi subgroup of G associated with J_2 such that $[b] = [x] \in B(G)$.

We denote by J_b the σ-twisted centralizer of $b \in G(L)$, i.e. the reductive group over F with F-valued points

$$J_b(F) = \{g \in G(L) \mid g^{-1} b\sigma(g) = b\}.$$

Then the defect of $[b]$ has the following equivalent descriptions:

Proposition 6.9. *The following non-negative integers all agree. The common value is called the* defect *of* $[b]$, *denoted* $\operatorname{def}(b)$.

(i) $\dim(X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^\sigma - \dim(X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\sigma w}$,

(ii) $\operatorname{rk}_F(G) - \operatorname{rk}_F(J_b)$,

(iii) $\langle \nu(b), 2\rho\rangle - \langle \lambda_G(b), 2\rho\rangle$,

(iv) $\#(J_1/\sigma)$, *the number of σ-orbits in J_1,*

(v) $\min_{v \in W} \ell(v^{-1}\,{}^\sigma(wv))$,

(vi) $\min_{v \in W_{J_1}} \ell(v^{-1}\,{}^\sigma(wv))$.

The notion of defect was originally defined in [Kot06, Equation 1.9.1] for split groups, using the expression in (i). Kottwitz shows the equality with (ii) as [Kot06, Theorem 1.10.1] and the equality with (iii) as [Kot06, Theorem 1.9.2].

If G is not split, the expression of (ii) is commonly used as definition. In the unramified case, the equality of (ii) with (iii) is then known as [Ham15, Proposition 3.8], and Hamacher's proof shows the equality with (i) and (iv).

For the remainder of this section, we sketch how to prove Proposition 6.9 for quasi-split groups G. The main idea is a reduction to the superbasic case.

Lemma 6.10. *Assume that* $[b]$ *is superbasic. Denote by* $n = \#(\Delta/\sigma)$ *the number of σ-orbits in* Δ.

(a) We have

$$(X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\sigma w} = \{\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \mid \sigma(\mu) = \mu \text{ and } \langle \mu, \Phi\rangle = \{0\}\}.$$

In particular,

$$n = \dim(X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^\sigma - (X_*(T)_{\Gamma_0} \otimes \mathbb{Q})^{\sigma w}.$$

(b) We have

$$n = \min_{v \in W} \ell(v^{-1}\,{}^{\sigma}(wv)).$$

More precisely, we find $v \in W$ and a subset $\Delta' \subseteq \Delta$ such that $\#\Delta' = n$ and $v^{-1}\,{}^{\sigma}(wv)$ is a Coxeter element for Δ'.

(c) We have

$$n = \langle \nu(b) - \mathrm{avg}_\sigma(\lambda_G(b)), 2\rho\rangle.$$

Proof. Superbasic elements only exist if each irreducible component of Φ is a root system of type A.

All claims may certainly be checked individually on each σ-connected component, so to lighten our notation, we will assume that Δ is σ-connected.

(a) If $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ is σ-stable and orthogonal to all roots, it is certainly fixed by σw. Let conversely $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ satisfy $\sigma w(\mu) = \mu$. Then we find $v \in W$ such that $v\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ is dominant. Observe that

$$\left(v\,{}^{\sigma}(wv^{-1})\right) \sigma v\mu = v\sigma w v^{-1} v\mu = v\mu.$$

Since $\sigma v\mu$ is dominant and in the W-orbit of $v\mu$, we get $\sigma v\mu = v\mu$. In particular, the dominant coweight $v\mu$ gets stabilized by $v\,{}^{\sigma}(wv^{-1}) \in W$.

Let $J := \mathrm{Stab}(v\mu)$ denote the stabilizer of the dominant coweight $v\mu$. Then $J = \sigma(J)$, so J defines a σ-stable Levi subgroup of G. Its extended affine Weyl group $\widetilde{W}_J$ contains $v^{-1}\,{}^{\sigma}(xv)$, so b comes from a σ-conjugacy class in this Levi subgroup. This is only possible if $J = \Delta$, i.e. $\langle v\mu, \Phi\rangle = \{0\}$. In particular, $v\mu = v^{-1}(v\mu) = \mu$, proving the claim.

(b) Decompose the Dynkin diagram of Δ into connected components, written as $\Delta = C_1 \sqcup \ldots \sqcup C_k$, such that $\sigma(C_i) = C_{i+1}$ for $i = 1, \ldots, k-1$ and $\sigma(C_k) = C_1$. Let $W_C := W_{C_1}$ denote the Weyl group of $C := C_1$.

Note that each C_i is of type A_n with n as given. Write C_{af} for the affine Dynkin diagram associated with $C = C_1$. Then the action of σ^k on C_{af} must fix the special node, and be either the identity or the unique involution on the complement, i.e. C. The element $x\,{}^{\sigma}x \cdots {}^{\sigma^{k-1}}x$, being an element of length zero in the affine Weyl group of C, acts on C_{af} by some cyclic permutation. The composition of these two maps, $(\sigma \circ x)^k$, should act transitively on C_{af}.

One quickly checks that this is only possible if σ^k is the identity map on C_{af}.

Now write $w = w_1\,{}^{\sigma}(w_2) \cdots {}^{\sigma^{k-1}}(w_k)$ with $w_1, \ldots, w_k \in W_C$. Let $v_1 \in W_C$ and define

$$v := v_1\,{}^{\sigma}(v_2) \cdots {}^{\sigma^{k}}(v_k) \in W, \qquad v_{i+1} = w_i v_i \text{ for } i = 1, \ldots, k-1.$$

Then

$$\begin{aligned} v^{-1}\,{}^{\sigma}(wv) =& v_1^{-1}\,{}^{\sigma}(v_2^{-1})\cdots{}^{\sigma^k}(v_k^{-1})\cdot(w_kv_k)\,{}^{\sigma}(w_1v_1)\cdot\cdots\cdot{}^{\sigma^{k-1}}(w_{k-1}v_{k-1})\\ =& v_1^{-1}w_kv_k = v_1^{-1}w_k\cdots w_1v_1\in W_C. \end{aligned}$$

We know that W_C is a Coxeter group of type A_n, so a symmetric group. It is a classical result that each element in a symmetric group is conjugate to a Coxeter element for a parabolic subgroup. In other words, we find v_1 and $\Delta'\subseteq C$ such that $v_1^{-1}w_k\cdots w_1v_1$ is a Coxeter element of Δ'.

In particular, we get

$$n = \#C \geqslant \#\Delta' = \ell(v^{-1}\,{}^{\sigma}(wv)) \geqslant \#\operatorname{supp}(v^{-1}\,{}^{\sigma}(wv)) \underset{\text{superbasic}}{\geqslant} n.$$

Thus $\#\Delta' = n$.

(c) It remains to evaluate

$$\langle\nu(b)-\operatorname{avg}_\sigma(\lambda_G(b)),2\rho\rangle = \sum_{\alpha\in\Delta}2c_\alpha.$$

This calculation is carried out by Hamacher [Ham15, Section 3], and we obtain the value n as claimed. The equality only depends on the affine root system together with the σ-action, so the fact that Hamacher only considers unramified groups is irrelevant. While his argument using characters of finite group representations is very elegant, one can also obtain the same result in a more straightforward manner with explicit calculations of Newton polygons (as we are in the A_n case). □

Proof of Proposition 6.9. The equality of (i) with (ii) is a standard Bruhat-Tits theoretic argument, cf. [Kot06, Section 4.3] or [Ham15, Proof of Prop. 3.8].

Observe that the values of (i), (iii), (iv) and (vi) do not change if we pass to the Levi subgroup of G defined by J_1. If we do so, $[b]$ becomes a superbasic σ-conjugacy class. Then the equalities of (i), (iii), (iv) and (vi) follow immediately from the preceding lemma.

It remains to show that, in the general case, (v) agrees with (vi). Suppose this was not the case. Then we would find some $v\in W$ such that

$$\ell(v^{-1}\,{}^{\sigma}(wv)) < \#(J_1/\sigma).$$

Consider the element $y = v^{-1}\,{}^{\sigma}(xv)\in\widetilde{W}$ and the subset $J\subseteq\Delta$ given by $J := \operatorname{supp}_\sigma(v^{-1}\,{}^{\sigma}(wv))$. Then J defines a σ-stable Levi subgroup $M\subseteq G$ such that $[b]$ has a preimage in $B(M)$. This is only possible if $J_1\subseteq J$, so $J=J_1$. But we must have

$$\ell(v^{-1}\,{}^{\sigma}(wv)) \geqslant \#\operatorname{supp}(v^{-1}\,{}^{\sigma}(wv)) \geqslant \#(J/\sigma) = \#(J_1/\sigma),$$

contradiction! □

6.3. Fundamental elements

Recall the equivalent characterizations of fundamental elements:

Proposition 6.11. *For $x = w\varepsilon^\mu \in \widetilde{W}$, the following are equivalent:*

(i) $\ell(x) = \langle \nu(x), 2\rho\rangle$.

(ii) For all $n \geqslant 1$, we have

$$\ell(x \cdot {}^\sigma x \cdots {}^{\sigma^{n-1}} x) = n\ell(x).$$

(iii) There exist $v \in \mathrm{LP}(x)$ and a σ-stable $J \subseteq \Delta$ such that $v^{-1}\,{}^\sigma(wv) \in W_J$ and for all $\alpha \in \Phi_J$, we have $\ell(x, v\alpha) = 0$.

(iv) For every orbit $O \subseteq \Phi$ with respect to the action of $(\sigma \circ w)$ on Φ, we have

$$(\forall \alpha \in O:\ \ell(x,\alpha) \geqslant 0) \ \text{or}\ (\forall \alpha \in O:\ \ell(x,\alpha) \leqslant 0)\,.$$

If G is defined over $\mathcal{O}_F$, this is moreover equivalent to

(v) Every element $y \in IxI$ is of the form $y = i^{-1} x\, {}^\sigma i$ for some $i \in I$.

If these equivalent conditions are satisfied, we call x fundamental*.* □

Let us first discuss the unramified case. In this case, the equivalence of (i) and (ii) is due to He [He10, Lemma 8.1]. Elements satisfying these conditions are called *good* in [He10] and *σ-straight* in more recent literature. Condition (iii) is a reformulation of the notion of *fundamental (J, w, δ)-alcoves* from Goertz-He-Nie [GHN15, Section 3.3]. Condition (v) is the notion of fundamental elements from [GHKR10]. The equivalence of (i), (iii) and (v) is a result of Nie [Nie15]. Condition (iv) is new, but we will not need it in the sequel.

If G is quasi-split but not unramified, the cited proofs fail because the map $X_*(T)_{\Gamma_0} \to X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ might no longer be injective. It is conceivable that the proofs might be generalized with a bit of work. Instead, we sketch how to prove the equivalences of (i)–(iv) using our language of length functionals, where issues with the torsion part of $X_*(T)_{\Gamma_0}$ are non-existent.

Proof of Proposition 6.11. Lemma 2.13 implies the equivalence of (ii) and (iv). Moreover, the implication (iii) $\implies$ (iv) is immediate.

Let $N > 0$ such that the action of $(\sigma \circ w)^N$ on $X_*(T)_{\Gamma_0}$ becomes trivial. For any

$v \in W$ and $\alpha \in \Phi$, we calculate

$$\begin{aligned}
&\left\langle \frac{1}{N} v^{-1} \sum_{k=1}^{N} (\sigma \circ w)^k \mu, \alpha \right\rangle \\
=&\frac{1}{N} \sum_{k=1}^{N} \langle \mu, (\sigma \circ w)^k v\alpha\rangle \\
=&\frac{1}{N} \sum_{k=1}^{N} \langle \mu, (\sigma \circ w)^k v\alpha\rangle + \Phi^+((\sigma \circ w)^k v\alpha) - \Phi^+((\sigma \circ w)^{k+1} v\alpha) \\
=&\frac{1}{N} \sum_{k=1}^{N} \ell(x, (\sigma \circ w)^k v\alpha).
\end{aligned}$$

Pick now $v \in W$ such that $v^{-1} \sum_{k=1}^{N} (\sigma \circ w)^k \mu = \nu(x)$. Then

$$\langle \nu(x), 2\rho\rangle = \sum_{\alpha \in \Phi^+} \frac{1}{N} \sum_{k=1}^{N} \ell(x, (\sigma \circ w)^k v\alpha) \geqslant \ell(x).$$

Equality holds if and only if $(\sigma \circ w)^k v \in \mathrm{LP}(x)$ for all $k \in \mathbb{Z}$. If we define $J := \mathrm{supp}_\sigma(v^{-1}\,{}^\sigma(wv))$, we see that (i) implies (iii).

It remains to show that (iv) implies (i). This follows directly from the above calculation. □

Fundamental elements play an important role for our description of generic σ-conjugacy classes. If x is fundamental, the generic σ-conjugacy class $[b_x]$ coincides with the σ-conjugacy class of x, whose Newton and Kottwitz points are easily computed. The λ-invariant and the defect of $[x]$ however are less straightforward to see. For now, we compute the defect.

Lemma 6.12. *Let x be fundamental, and choose $v \in \mathrm{LP}(x)$ and $J \subseteq \Delta$ as in Proposition 6.11 (iii).*

(a) Every $v' \in vW_J$ is length positive for x. Moreover, (x, v', J) also satisfies condition (iii) of Proposition 6.11.

(b) If $v \in W^J$, then $({}^{\sigma^{-1}}v)^{-1} x v$ coincides with an element of length zero in the extended affine Weyl group $\widetilde{W}_J = W_J \ltimes X_*(T)_{\Gamma_0}$.

(c) The defect of x is given by

$$\mathrm{def}([x]) = \min_{v' \in vW_J} \ell((v')^{-1}\,{}^\sigma(wv')) = \min_{v' \in W} \ell((v')^{-1}\,{}^\sigma(wv)).$$

Proof. (a) This is a very straightforward calculation.

(b) By definition, $({}^{\sigma^{-1}}v)^{-1} x v \in \widetilde{W}_J$. The length calculation is straightforward using Lemma 2.12. For an alternative proof concept, cf. [HN14, Proposition 3.2].

(c) In view of (a), we may assume $v \in W^J$. Then

$$\operatorname{def}([x]) = \operatorname{def}\left([({}^{\sigma^{-1}}v)^{-1}xv]\right)$$

By (b), the element $({}^{\sigma^{-1}}v)^{-1}xv \in \widetilde{W}$ satisfies the conditions needed to compute its defect using Proposition 6.9 (v) and (vi). The claim follows. □

In order to reduce claims about arbitrary elements in $\widetilde{W}$ to fundamental ones, we need the following lemma. If G is unramified, this is a classical result of Viehmmann [Vie14, Proposition 5.5].

Lemma 6.13. *Let $x \in \widetilde{W}$ and $[b] \in B(G)_x$, i.e. $[b] \in B(G)$ with $X_x(b) \neq \emptyset$. Then there exists a fundamental element $y \in \widetilde{W}$ such that $y \leqslant x$ in the Bruhat order and $[y] = [b]$ in $B(G)$.*

Proof. Induction by $\ell(x)$. We distinguish a number of cases.

1. Suppose that x is of minimal length in its σ-conjugacy class in $\widetilde{W}$ and that $x = uy$ for some fundamental $y \in \widetilde{W}$ with $\ell(x) = \ell(u) + \ell(y)$ and $[x] = [y]$.

   By [He14, Theorem 3.5], $[b] = [x]$ so that $y \leqslant x$ satisfies the desired conditions.

2. Suppose that there exists a simple affine reflection $s \in S_{\mathrm{af}}$ such that $\ell(sx{}^{\sigma}s) < \ell(x)$. By the "Deligne-Lusztig reduction method" of Goertz-He [GH10, Corollary 2.5.3], we must have $[b] \in B(G)_{x'}$ for $x' = sx{}^{\sigma}s$ or $x' = sx$. By induction, we get an element $y \leqslant x'$ with the desired properties. Since $x' < x$, the claim follows.

3. In general, we find by [HN14, Theorem 3.4] a sequence of elements

   $$x = x_1, \dots, x_n \in \widetilde{W}$$

   such that

   - $x_{i+1} = s_i x_i {}^{\sigma}s_i$ for some simple reflection $s_i \in S$ $(i = 1, \dots, n-1)$,
   - $\ell(x_i) = \ell(x)$ for $i = 1, \dots, n$ and
   - x_n satisfies condition 1. or 2.

   In particular, we find $y' \leqslant x_n$ fundamental with $[y'] = [b]$.

   By [Nie15, Lemma 2.3], there exists $y \leqslant x$ with $\ell(y) \leqslant \ell(y')$ and y being σ-conjugate to y' in $\widetilde{W}$. While Nie's proof only covers unramified groups, this statement is purely about combinatorics of root systems and affine Weyl groups, so the generalization to quasi-split groups is immediate.

   Now observe that $[y] = [y'] = [b] \in B(G)$. In particular,

   $$\langle \nu(b), 2\rho\rangle \leqslant \ell(y) \leqslant \ell(y') = \langle \nu(b), 2\rho\rangle.$$

   We see that y must be fundamental as well.

In any case, the claim follows, finishing the induction and the proof. □

7. Generic σ-conjugacy class

For an element $x \in \widetilde{W}$, the *generic* σ-conjugacy class $[b] = [b_x] \in B(G)$ is the uniquely determined σ-conjugacy class such that $IxI \cap [b]$ is dense in IxI. For each $y \in \widetilde{W}$, we write $[y] \in B(G)$ for the σ-conjugacy class of any representative of y in $G(L)$. We have the following description due to Viehmann:

Theorem 7.1 ([Vie14, Corollary 5.6])**.** *Let $x \in \widetilde{W}$. Then $[b_x]$ is the largest σ-conjugacy class in $B(G)$ of the form $[y]$ where $y \leqslant x$ in the Bruhat order on $\widetilde{W}$.* □

Viehmann's original proof makes the assumption that the group under consideration is unramified, but it is not hard to remove this assumption. Indeed, we saw in Lemma 6.13 that [Vie14, Proposition 5.5] can be proved without this assumption, and then Viehmann's proof of [Vie14, Corollary 5.6] works without further changes.

We can now describe this generic σ-conjugacy class more explicitly:

Theorem 7.2. *Assume that G is quasi-split. Let $x = w\varepsilon^\mu \in \widetilde{W}$ and denote by $[b_x]$ is generic σ-conjugacy class. Writing $\lambda_x := \lambda_G(b_x)$, we have*

$$\lambda_x = \max_{v \in W} \left(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))\right) \in X_*(T)_\Gamma.$$

We call λ_x the *generic λ-invariant* of x. We discuss previous works and some applications of this result now, before giving its proof in the next subsection.

We begin with a more explicit way to calculate generic λ-invariants. The following lemma does not depend on the theorem, while the corollary does.

Lemma 7.3. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in W$.*

(a) If v is not length positive for x, and vs_α is an adjustment, then

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \leqslant^\sigma (vs_\alpha)^{-1} - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha)).$$

(b) We have

$$\langle v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), 2\rho\rangle \leqslant \ell(x) - d(v \Rightarrow {}^\sigma(wv)).$$

Equality holds if and only if $v \in \mathrm{LP}(x)$.

Proof. (a) We compute

$$\begin{aligned}(vs_\alpha)^{-1}\mu - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha)) \geqslant & v^{-1}\mu - \langle \mu, v\alpha\rangle\alpha^\vee - \mathrm{wt}(vs_\alpha \Rightarrow v)\\ &- \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(wvs_\alpha))\\ \geqslant^\sigma & v^{-1}\mu - \langle \mu, v\alpha\rangle\alpha^\vee - \Phi^+(v\alpha)\alpha^\vee\\ &- \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - \Phi^+(-wv\alpha)\alpha^\vee\\ =& v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - (\ell(x, v\alpha) + 1)\\ \geqslant & v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)).\end{aligned}$$

(b) Indeed, using Corollary 2.11 and Lemma 3.6, we obtain

$$\begin{aligned}\langle v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), 2\rho\rangle =&\langle v^{-1}\mu, 2\rho\rangle - \ell(v) + \ell(wv) - d(v \Rightarrow {}^\sigma(wv))\\ \leqslant&\ell(x) - d(v \Rightarrow {}^\sigma(wv)),\end{aligned}$$

with equality iff $v \in \mathrm{LP}(x)$. □

Corollary 7.4. *Let $x = w\varepsilon^\mu \in \widetilde{W}$. Among all elements $v \in \mathrm{LP}(x)$, pick one such that the distance $d(v \Rightarrow {}^\sigma(wv))$ in the quantum Bruhat graph becomes minimal. Then*

$$\lambda_x = v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \in X_*(T)_\Gamma.$$

In particular, the generic Newton point of x is given by

$$\nu_x = \mathrm{conv}(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))).$$

Proof. We know that $\lambda_x = (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv'))$ for some $v' \in W$ by the theorem. Using the above lemma, we conclude that the same equality holds for some $v' \in \mathrm{LP}(x)$.
Now $v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \leqslant (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv'))$ by the theorem, and

$$\langle v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), 2\rho\rangle \geqslant \langle (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv')), 2\rho\rangle$$

by choice of v. The claim follows. □

The following lemma might be helpful for computing ν_x.

Lemma 7.5. *Let $x = w\varepsilon^\mu \in \widetilde{W}$, $v \in \mathrm{LP}(x)$ and $J \subseteq \Delta$ such that $J = \sigma(J)$ and*

$$\forall \alpha \in \Phi^+ \backslash \Phi_J^+ : \ \ell(x, v\alpha) > 0.$$

Then there exists $J' \subseteq J$ with $\sigma(J') = J'$ and

$$\mathrm{conv}(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))) = \pi_{J'}(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))).$$

Proof. In view of Lemma 6.5 (e), it suffices to show for each $\alpha \in \Phi^+\backslash\Phi_J^+$ that

$$\langle \mathrm{avg}_\sigma(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))), \alpha\rangle \geqslant 0.$$

Let $N > 1$ such that the action of σ^N on $X_*(T)_{\Gamma_0}$ becomes trivial. Then

$$\begin{aligned}\langle \mathrm{avg}_\sigma(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))), \alpha\rangle =&\frac{1}{N}\sum_{k=1}^N \langle v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), \sigma^k(\alpha)\rangle\\ =&\frac{1}{N}\sum_{k=1}^N \langle \mu, v\sigma^k(\alpha)\rangle - \langle \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), \sigma^k(\alpha)\rangle.\end{aligned}$$

By[3] [HN21, Section 2.5], we may estimate

$$\begin{aligned}\langle \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), \sigma^k(\alpha)\rangle \leqslant & \Phi^+(-v\sigma^k(\alpha)) + \Phi^+({}^\sigma(wv)\sigma^k(\alpha))\\ =&\Phi^+(-v\sigma^k(\alpha)) + \Phi^+(wv\sigma^{k-1}(\alpha)).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{N}\sum_{k=1}^N \Big(\langle \mu, v\sigma^k(\alpha)\rangle - \langle \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), \sigma^k(\alpha)\rangle\Big)\\ \geqslant&\frac{1}{N}\sum_{k=1}^N \Big(\langle \mu, v\sigma^k(\alpha)\rangle - \Phi^+(-v\sigma^k(\alpha)) - \Phi^+(wv\sigma^{k-1}(\alpha))\Big)\\ =&\frac{1}{N}\sum_{k=1}^N \Big(\langle \mu, v\sigma^k(\alpha)\rangle - \Phi^+(-v\sigma^k(\alpha)) - \Phi^+(wv\sigma^{k}(\alpha))\Big)\\ =&\frac{1}{N}\sum_{k=1}^N \Big(\underbrace{\ell(x,\sigma^k(\alpha))}_{\geqslant 1} - 1\Big) \geqslant 0.\end{aligned}$$

This finishes the proof. □

Corollary 7.6. *If $x = w\varepsilon^\mu$ lies in a shrunken Weyl chamber and* $\mathrm{LP}(x) = \{v\}$*, then*

$$\nu_x = v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \in X_*(T)_{\Gamma_0}\otimes\mathbb{Q}.$$

Proof. Set $J := \varnothing$ in the previous lemma. □

If G is split and μ sufficiently regular, this corollary is the main result of [Mil21], which was the first paper to derive an explicit formula for ν_x from Theorem 7.1. Milićević's result since has been generalized by Sadhukhan [Sad21], who proves the statement of Corollary 7.6 if G is split and μ satisfies a regularity condition that is weaker than Milićević's. He and Nie [HN21, Proposition 3.1] proved Corollary 7.6 as stated here.

As an application of Theorem 7.2, we classify the *cordial* elements from Milićević-Viehmann [MV20].

Definition 7.7. Let $x = w\varepsilon^\mu \in \widetilde W$ and $v \in W$ be the specific length positive element constructed in Example 2.8. Then x is cordial if

$$\ell(x) - \ell(v^{-1}\,{}^\sigma(wv)) = \langle \nu_x, 2\rho\rangle - \mathrm{def}(b_x).$$

[3] The original formulation of this statement has a small typo, the version cited here is the correct one: Indeed, let $x, y \in W$ and define dominant coweights $\mu_1, \mu_2 \in X_*(T)_{\Gamma_0}$ on each simple root $\alpha \in \Delta$ as follows:

$$\langle \mu_1, \alpha\rangle := \Phi^+(-y^{-1}\alpha),\quad \langle \mu_2, \alpha\rangle := \Phi^+(x\alpha).$$

Then one checks easily that we are in the situation of [HN21, Theorem 1.1], and part (1) of this theorem yields $\langle \mathrm{wt}(y^{-1} \Rightarrow x), \alpha\rangle \leqslant \Phi^+(-y^{-1}\alpha) + \Phi^+(x\alpha)$.

Proposition 7.8. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in \mathrm{LP}(x)$. Then*

$$\ell(x) - \ell(v^{-1}\,{}^\sigma(wv)) \leqslant \langle \nu_x, 2\rho\rangle - \mathrm{def}(b_x).$$

Equality holds if and only if both conditions (a) and (b) are satisfied. Moreover, the condition (a) is always equivalent to (a').

(a) The generic λ-invariant λ_x is given by

$$\lambda_x = v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \in X_*(T)_\Gamma.$$

(a') We have

$$d(v \Rightarrow {}^\sigma(wv)) = \min_{v' \in \mathrm{LP}(x)} d(v' \Rightarrow {}^\sigma(wv')).$$

(b) We have $d(v \Rightarrow {}^\sigma(wv)) = \ell(v^{-1}\,{}^\sigma(wv))$.

Proof. By Lemma 7.3 and Theorem 7.2, (a) $\iff$ (a').

For the remaining claims, we calculate

$$\begin{aligned}\ell(x) - \ell(v^{-1}\,{}^\sigma(wv)) \leqslant& \ell(x) - d(v \Rightarrow {}^\sigma(wv))\\ =&\langle v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)), 2\rho\rangle\\ \underset{\mathrm{T7.2}}{\leqslant}& \langle \lambda_x, 2\rho\rangle \underset{\mathrm{P6.9}}{=} \langle \nu_x, 2\rho\rangle - \mathrm{def}(b_x).\end{aligned}$$ □

Corollary 7.9. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in W$ be of minimal length such that $v^{-1}\mu$ is dominant. Then x is cordial if and only if the following two conditions are both satisfied:*

(1) For each $v' \in \mathrm{LP}(x)$, $d(v \Rightarrow {}^\sigma(wv)) \leqslant d(v' \Rightarrow {}^\sigma(wv'))$.

(2) $d(v \Rightarrow {}^\sigma(wv)) = \ell(v^{-1}\,{}^\sigma(wv))$. □

This corollary generalizes the description of superregular cordial element for split G due to Milićević-Viehmann [MV20, Proposition 4.2] and the description of shrunken cordial elements due to He-Nie [HN21, Remark 3.2]. One can generalize the statement and proof of [MV20, Theorem 1.2 (b), (c)] accordingly.

The notion of cordiality depends on one specific and non-canonical length positive element for x. We conjecture that it is possible to generalize this notion to all length positive elements.

Conjecture 7.10. *Assume that $\mathrm{char}(F) > 0$ and let $x = w\varepsilon^\mu \in \widetilde{W}$ and $v \in \mathrm{LP}(x)$. For all $b \in B(G)$, we expect*

$$\dim X_x(b) \leqslant \frac{1}{2}\left(\ell(x) + \ell(v^{-1}\,{}^\sigma(wv)) - \langle \nu(b), 2\rho\rangle - \mathrm{def}(b)\right).$$

Remark 7.11. (a) The conjecture has been tested for a large number of randomly generated x using the sagemath computer algebra system [Sage; SaCo]. This computer search did not find a single counter-example.

(b) For the one v constructed in Example 2.8, the right-hand side of Conjecture 7.10 is exactly the virtual dimension of [He14], and then the conjecture is proved in that paper. He's proof does not seem to be easily generalized to arbitrary $v \in \mathrm{LP}(x)$, though.

(c) If $b = b_x$ is the generic σ-conjugacy class associated with x, then

$$\begin{aligned}\dim X_x(b_x) =& \ell(x) - \langle \nu(b_x), 2\rho\rangle = \ell(x) - \langle \lambda_x, 2\rho\rangle - \mathrm{def}(b_x)\\ =& \min_{v\in\mathrm{LP}(x)} d(v \Rightarrow {}^{\sigma}(wv)) - \mathrm{def}(b_x).\end{aligned}$$

The first equality is [He15, Theorem 2.23] and the second one is Proposition 6.9.

We see that the conjecture is true whenever $b = b_x$. In particular, $X_x(b)$ is non-empty for only one element $b \in B(G)$, the conjecture is true. This is e.g. the case if x is of minimal length in its σ-conjugacy class in $\widetilde{W}$, cf. [He14, Theorem 3.5].

(d) Let $x \in \widetilde{W}$ and $v \in \mathrm{LP}(x)$ such that the following two assumptions are both satisfied:

(1) Conjecture 7.10 is satisfied for (x, v) and all $b \in B(G)$ and

(2) for $b = b_x$, the inequality in conjecture 7.10 becomes an equality:

$$\dim X_x(b_x) = \frac{1}{2}\left(\ell(x) - \ell(v^{-1}\,{}^{\sigma}(wv)) - \langle \nu_x, 2\rho\rangle - \mathrm{def}(b_x)\right).$$

One can check assumption (2) using Proposition 7.8.

Under these two assumptions, the major results and proofs of [MV20] can be generalized in a straightforward manner.

(e) In view of Corollary 7.9, we are led to ask which $(w_1, w_2) \in W^2$ satisfy the condition

$$d(w_1 \Rightarrow w_2) = \ell(w_2^{-1}w_1). \tag{$*$}$$

By [MV20, Remark 4.4], this is the case if and only if there is a shortest path $w_1 \to \cdots \to w_2$ where each arrow is of the form $u \to us_\alpha$ for some $\alpha \in \Delta$.

While it appears unreasonable to ask for a "general formula" for $d(w_1 \Rightarrow w_2)$, describing the elements for which $(*)$ holds might prove to be an easier task.

If w_1 is smaller than w_2 in the right weak Bruhat order, then $(*)$ is certainly satisfied. This applies in particular for $w_1 = 1$ (cf. [He21b, Theorem 4.2]) or $w_2 = w_0$ (cf. [MV20, Theorem 1.2 (a)]).

From [MV20, Theorem 1.2 (b), (c)], we obtain moreover the following criteria:

- If $\ell(w_2^{-1}w_1) = \#\,\mathrm{supp}(w_2^{-1}w_1)$, then $(*)$ holds true.
- If $w_2 = 1$, then $(*)$ holds true if and only if w_1 is *small height avoiding* as in [MV20, Definition 4.7]. This notion is discussed in [MV20]; yet one may still hope for a more explicit classification of those elements.

If $w_2 \neq 1$, we may of course use Lemma 3.8 to reduce to the $w_2 = 1$ case. Indeed, if $\alpha \in \Delta$ satisfies $w_2^{-1}\alpha \in \Phi^-$, one may argue as follows:

- If $w_1^{-1}\alpha \in \Phi^-$, then $d(w_1 \Rightarrow w_2) = d(s_\alpha w_1 \Rightarrow s_\alpha w_2)$. Thus (w_1, w_2) satisfies $(*)$ iff $(s_\alpha w_1, s_\alpha w_2)$ satisfies $(*)$.
- If $w_1^{-1}\alpha \in \Phi^-$, then $d(w_1 \Rightarrow w_2) = d(w_1 \Rightarrow s_\alpha w_2) + 1$. Moreover,

$$\ell((s_\alpha w_2)^{-1} w_1) = \ell(w_2^{-1} s_\alpha w_1) < \ell(w_2^{-1} w_1).$$

  Thus (w_1, w_2) satisfy $(*)$ if and only if the following two conditions are both satisfied: $(w_1, s_\alpha w_2)$ satisfies $(*)$ and $\ell(w_2^{-1} s_\alpha w_1) = \ell(w_2^{-1} w_1) - 1$.

While these partial results are somewhat promising, the question which pairs satisfy $(*)$ is still very much open.

7.1. Proof of the Theorem

Fix $x = w\varepsilon^\mu \in \widetilde{W}$. We need to show the following two claims:

- There exists some $v \in W$ such that

$$\lambda_x \leqslant v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \in X_*(T)_\Gamma.$$

- For each $v \in W$, we have

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \leqslant \lambda_x \in X_*(T)_\Gamma.$$

  By definition of $\lambda_G(x)$, this is equivalent to

$$\mathrm{avg}_\sigma(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))) \leqslant \nu_x \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}.$$

Let us use the shorthand notation $\lambda \leqslant^\sigma \lambda'$ to say that the image of λ in $X_*(T)_\Gamma$ is less than or equal to the image of λ' in $X_*(T)_\Gamma$ (λ, λ' being elements of $X_*(T), X_*(T)_{\Gamma_0}$ or $X_*(T)_\Gamma$).

We write $\lambda \equiv^\sigma \lambda'$ to denote $\lambda \leqslant^\sigma \lambda'$ and $\lambda' \leqslant^\sigma \lambda$. Similarly, we write $\lambda <^\sigma \lambda'$ to denote $\lambda \leqslant^\sigma \lambda'$ but $\lambda' \nleqslant^\sigma \lambda$.

For this section, call an element $v \in W$ *maximal* if there exists no $v' \in W$ such that

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) <^\sigma (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv')).$$

Lemma 7.12. *Let $v \in W$ be maximal. Moreover, fix a root $\alpha \in \Phi^+$ such that*

$$\mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma \alpha^\vee \Phi^+(-v\alpha) + \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wv)).$$

Then precisely one of the following conditions is satisfied:

(1) $\ell(x, v\alpha) > 0$*, and the element*

$$x' := w'\varepsilon^{\mu'} := x r_{v\alpha,\Phi^+(-v\alpha)} \in \widetilde{W}$$

satisfies $x' < x$ *and*

$$(vs_\alpha)^{-1}\mu' - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(w'vs_\alpha)) \equiv^\sigma v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)).$$

(2) $\ell(x, v\alpha) = 0$*,* $vs_\alpha \in W$ *is maximal with*

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma (vs_\alpha)^{-1}\mu - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha))$$

and

$$\mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha)) \equiv^\sigma \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wv)) + \alpha^\vee\Phi^+(-wv\alpha).$$

Remark 7.13. If $v \neq {}^\sigma(wv)$ and $v \to vs_\alpha$ is an edge in $\mathrm{QB}(W)$ that is part of a shortest path from v to ${}^\sigma(wv)$, then the root $\alpha \in \Phi^+$ will satisfy the condition of the Lemma.

Proof of Lemma 7.12. We use maximality of v by comparing to vs_α. Now calculate

$$\begin{aligned}
&(vs_\alpha)^{-1}\mu - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha))\\
\geqslant&(vs_\alpha)^{-1}\mu - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wv)) - \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(wvs_\alpha)).\\
\equiv^\sigma&(vs_\alpha)^{-1}\mu + \alpha^\vee\Phi^+(-v\alpha) - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - \underbrace{\mathrm{wt}(wv \Rightarrow wvs_\alpha)}_{\leqslant \alpha^\vee\Phi^+(wv\alpha) \text{ by C3.17}}\\
\geqslant&v^{-1}\mu - \langle\mu, v\alpha\rangle\alpha^\vee + \alpha^\vee\Phi^+(-v\alpha) - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - \alpha^\vee\Phi^+(wv\alpha)\\
=&v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - \ell(x, v\alpha)\alpha^\vee.
\end{aligned}$$

If $\ell(x, v\alpha) < 0$, we get a contradiction to the maximality of v.

Next assume that $\ell(x, v\alpha) = 0$. Then every inequality in the above computation must be an equality (up to σ-coinvariants), or we would again get a contradiction. In particular, vs_α must be maximal, as

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma (vs_\alpha)^{-1}\mu - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha)).$$

Moreover, we obtain

$$\mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wvs_\alpha)) \equiv^\sigma \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wv)) + \alpha^\vee(-wv\alpha).$$

This shows all the claims in (2).

Finally assume $\ell(x, v\alpha) > 0$. The claim $x' < x$, i.e. $x(v\alpha, \Phi^+(-v\alpha)) \in \Phi^-_{\mathrm{af}}$, follows from Lemma 2.9. Calculating explicitly, we get

$$w'\varepsilon^{\mu'} = w\varepsilon^\mu s_{v\alpha}\varepsilon^{\Phi^+(-v\alpha)v\alpha^\vee} = ws_{v\alpha}\varepsilon^{s_{v\alpha}(\mu)+\Phi^+(-v\alpha)v\alpha^\vee}.$$

So indeed,

$$\begin{aligned}
(vs_\alpha)^{-1}\mu' - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(w'vs_\alpha)) =&v^{-1}\mu - \alpha^\vee\Phi^+(-v\alpha) - \mathrm{wt}(vs_\alpha \Rightarrow {}^\sigma(wv))\\
\equiv^\sigma&v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)).
\end{aligned}$$

□

Corollary 7.14. *Let v be maximal. Then at least one of the following conditions is satisfied:*

(1) There exists $x' = w'\varepsilon^{\mu'} < x$ and $v' \in W$ such that

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma (v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow {}^\sigma(w'v')).$$

(2) The element ${}^\sigma(wv) \in W$ is maximal, and we have

$$v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma {}^\sigma(wv)^{-1}\mu - \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(w\,{}^\sigma(wv))).$$

Proof. Choose a shortest path in $\mathrm{QB}(W)$

$$p : v \to vs_{\alpha_1} \to vs_{\alpha_1}s_{\alpha_2} \to \cdots \to vs_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k} = {}^\sigma(wv).$$

Consider the roots

$$\beta_i = vs_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i) \in \Phi, \qquad i = 1, \dots, k.$$

We fix $i^* \in \{0, \dots, k\}$ maximally such that $\ell(x, \beta_i) = 0$ for $1 \leqslant i \leqslant i^*$.

We claim that each v_i for $i = 0, \dots, i^*$ satisfies the following conditions:

(a) v_i is maximal,

(b) $d(v_i \Rightarrow {}^\sigma(wv_i)) = d(v_i \Rightarrow {}^\sigma(wv)) + d({}^\sigma(wv) \Rightarrow {}^\sigma(wv_i))$.

(c) $v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma v_i^{-1}\mu - \mathrm{wt}(v_i \Rightarrow {}^\sigma(wv_i))$.

Induction on i. Since $v_0 = v$, the claim is clear for $i = 0$. Now in the inductive step, assume that $i < i^*$ and that the conditions (a)–(c) are true for v_i. We apply Lemma 7.12 to (v_i, α_i). This is possible, as $v_i \to v_{i+1}$ is part of a shortest path from v_i to ${}^\sigma(wv)$ (by choice of the path p), hence part of a shortest path from v_i to ${}^\sigma(wv_i)$ by (b).

Since $i < i^*$, we get $\ell(x, v_i\alpha_i) = 0$, so condition (2) of Lemma 7.12 must be satisfied. Now (a) and (c) follow immediately for v_{i+1}. For condition (b), use condition (2) of the lemma to compute

$$\begin{aligned}
&\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv_{i+1}))\\
\equiv^\sigma\ &\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv_i)) + \alpha_i^\vee \Phi^+(-wv_i\alpha_i)\\
\underset{\text{(b)}}{=}\ &\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv)) + \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(wv_i)) + \alpha_i^\vee \Phi^+(-wv_i\alpha_i)\\
\underset{\text{C3.17}}{\geqslant^\sigma}\ &\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv)) + \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(wv_i)) + \mathrm{wt}({}^\sigma(wv_i) \Rightarrow {}^\sigma(wv_{i+1})\\
\geqslant\ &\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv)) + \mathrm{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(wv_{i+1}))\\
\geqslant\ &\mathrm{wt}(v_{i+1} \Rightarrow {}^\sigma(wv_{i+1})).
\end{aligned}$$

We see that equality must hold in every step (up to the σ-action). In light of Lemma 3.6, condition (b) for v_{i+1} follows, finishing the induction.

With the above claim proved for all $i \in \{0, \dots, i^*\}$, we distinguish two cases:

(1) Case $i^* < k$. Then $\ell(x, \beta_{i^*+1}) = \ell(x, v_{i^*}(\alpha_{i^*+1})) > 0$ by choice of i^*. Applying Lemma 7.12 to v_{i^*} and α_{i^*+1}, we immediately get the desired x'.

(2) Case $i^* = k$. Then ${}^\sigma(wv) = v_{i^*}$ and we obtain everything claimed. □

Lemma 7.15. *Let $v \in W$. Then there exists some $x' \leqslant x$ with*

$$\nu(x') \geqslant \operatorname{avg}_\sigma(v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv))).$$

In other words, $\lambda_x \geqslant^\sigma v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv))$.

Proof. Induction on $\ell(x)$. We may certainly assume that v is maximal. If there exists $x' = w'\varepsilon^{\mu'} < x$ and $v' \in W$ with

$$v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv)) \ \equiv^\sigma \ (v')^{-1}\mu' - \operatorname{wt}(v' \Rightarrow {}^\sigma(w'v')),$$

we may apply the inductive hypothesis to x' and are done.

Let us assume that this is not the case. By the above corollary, we see that ${}^\sigma(wv)$ is maximal and

$$v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv)) \ \equiv^\sigma \ {}^\sigma(wv)^{-1}\mu - \operatorname{wt}({}^\sigma(wv) \Rightarrow {}^\sigma(w\,{}^\sigma(wv))).$$

For $n \geqslant 0$, we define the element $v_n \in W$ by $v_0 := v$ and $v_{n+1} := {}^\sigma(wv_n) \in W$. A simple induction argument shows that each v_n is maximal and

$$v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv)) \equiv^\sigma v_n^{-1}\mu - \operatorname{wt}(v_n \Rightarrow {}^\sigma(wv_n)).$$

We calculate for $\lambda \in X_*(T)_{\Gamma_0}$:

$$v_n\lambda = {}^\sigma(wv_{n-1})\lambda = \sigma \circ wv_{n-1}\left(\sigma^{-1}\lambda\right) = (\sigma \circ w)^n v(\sigma^{-n}\lambda).$$

Thus

$$v_n^{-1}\lambda = \sigma^n v^{-1}(\sigma \circ w)^{-n}(\lambda).$$

Let $N \geqslant 1$ such that the action of $(\sigma \circ w)^N$ on $X_*(T)$ becomes trivial. We see that

$$\begin{aligned}
\operatorname{avg}_\sigma\left(v^{-1}\mu - \operatorname{wt}(v \Rightarrow {}^\sigma(wv))\right) &= \frac{1}{N}\sum_{n=1}^N \operatorname{avg}_\sigma\left(v_n^{-1}\mu - \operatorname{wt}(v_n \Rightarrow {}^\sigma(wv_n))\right)\\
&\leqslant \frac{1}{N}\sum_{n=1}^N \operatorname{avg}_\sigma\left(v_n^{-1}\mu\right)\\
&= \frac{1}{N}\sum_{n=1}^N \operatorname{avg}_\sigma\left(v^{-1}(\sigma \circ w)^{-n}\mu\right)\\
&= \operatorname{avg}_\sigma v^{-1}\frac{1}{N}\sum_{n=1}^N (\sigma \circ w)^{-n}\mu.\\
&\leqslant \operatorname{avg}_\sigma \nu(x) = \nu(x).
\end{aligned}$$

Thus we may choose $x' = x$, finishing the induction and the proof. □

Lemma 7.16. *Let $x = w\varepsilon^\mu \in \widetilde{W}$ be a fundamental element, and choose $v' \in \mathrm{LP}(x)$ with* $\mathrm{def}([x]_\sigma) = \ell((v')^{-1}\,{}^\sigma(wv'))$ *as in Lemma 6.12. Then*

$$\lambda_x \equiv^\sigma (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv')).$$

Proof. By Lemma 7.15, we have

$$\lambda_x \geqslant^\sigma (v')^{-1}\mu - \mathrm{wt}(v' \Rightarrow {}^\sigma(wv')).$$

Now we calculate

$$\begin{aligned} &\langle \lambda_x - (v')^{-1}\mu + \mathrm{wt}(v' \Rightarrow {}^\sigma(wv')), 2\rho\rangle \\ \underset{\text{L7.3}}{=}&\langle \lambda_x, 2\rho\rangle - \ell(x) + d(v' \Rightarrow {}^\sigma(wv')) \\ \underset{\text{fund.}}{=}&\langle \lambda_G(x), 2\rho\rangle - \langle \nu(x), 2\rho\rangle + d(v' \Rightarrow {}^\sigma(wv')) \\ \underset{\text{P6.9}}{=}& -\mathrm{def}([x]_\sigma) + d(v' \Rightarrow {}^\sigma(wv')) \\ \leqslant& -\mathrm{def}([x]_\sigma) + \ell((v')^{-1}\,{}^\sigma(wv')) \underset{\text{assump.}}{=} 0. \end{aligned}$$

The inequality on the last line is [MV20, Lemma 4.3]. ☐

Lemma 7.17. *There exists $v \in W$ such that*

$$\lambda_x \leqslant^\sigma v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)).$$

Proof. Induction on $\ell(x)$.

Let us first consider the case that there exists an element $x' = w'\varepsilon^{\mu'} < x$ with $[b_{x'}] = [b_x] \in B(G)$. If this is the case, we may further assume by definition of the Bruhat order that $x' = xr_a$ for some affine root $a \in \Phi^+_{\mathrm{af}}$.

Using the induction assumption, we find some $v' \in W$ such that

$$\lambda_{x'} = \lambda_x \leqslant^\sigma (v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow {}^\sigma(w'v')).$$

Write $a = (\alpha, k)$ such that $w' = ws_\alpha$ and $\mu' = s_\alpha(\mu) + k\alpha^\vee$. The condition $\ell(x') < \ell(x)$ means that $xa \in \Phi^-_{\mathrm{af}}$, which we can rewrite as

$$k - \langle \mu, \alpha\rangle < \Phi^+(w\alpha).$$

We distinguish the following cases.

- Case $(v')^{-1}\alpha \in \Phi^-$. Define $v := s_\alpha v'$ and compute

$$\begin{aligned} \lambda_x &\leqslant^\sigma (v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow {}^\sigma(w'v')) \\ &= v^{-1}(\mu - k\alpha^\vee) - \mathrm{wt}(s_\alpha v \Rightarrow {}^\sigma(wv)) \\ &\leqslant v^{-1}\mu - kv^{-1}\alpha^\vee - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) + \mathrm{wt}(v \Rightarrow s_\alpha v) \\ &\underset{\text{C3.17}}{\leqslant} v^{-1}\mu - kv^{-1}\alpha^\vee - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) + v^{-1}\alpha^\vee\Phi^+(-\alpha) \\ &= v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) + (\Phi^+(-\alpha) - k)v^{-1}\alpha^\vee \\ &\leqslant v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)). \end{aligned}$$

The inequality on the last line follows since $\Phi^+(-\alpha) - k \leqslant 0$ (as $a \in \Phi_{\mathrm{af}}^+$) and $v^{-1}\alpha \in \Phi^+$ by assumption.

- Case $(v')^{-1}\alpha \in \Phi^+$. Define $v := v'$ and compute

$$\begin{aligned}
\lambda_x \leqslant^\sigma\ & (v')^{-1}\mu' - \mathrm{wt}(v' \Rightarrow {}^\sigma(w'v')) \\
=& v^{-1}(\mu - \langle \mu, \alpha\rangle\alpha^\vee + k\alpha^\vee) - \mathrm{wt}(v \Rightarrow ws_\alpha v) \\
\leqslant^\sigma & v^{-1}\mu + (-\langle \mu, \alpha\rangle + k)v^{-1}\alpha^\vee - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) + \mathrm{wt}(ws_\alpha v \Rightarrow wv) \\
\underset{\text{C3.17}}{\leqslant} & v^{-1}\mu + (-\langle \mu, \alpha\rangle + k)v^{-1}\alpha^\vee - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) - v^{-1}\alpha^\vee\Phi^+(w\alpha) \\
=& v^{-1}\mu + (-\langle \mu, \alpha\rangle + k - \Phi^+(w\alpha))v^{-1}\alpha^\vee - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)) \\
\leqslant & v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv)).
\end{aligned}$$

The inequality on the last line follows since $-\langle \mu, \alpha\rangle + k - \Phi^+(w\alpha) \leqslant 0$ (as $xa \in \Phi_{\mathrm{af}}^-$) and $v^{-1}\alpha \in \Phi^+$ by assumption.

In any case, we find an element $v \in W$ with the desired property, proving the claim for x.

It remains to study the case where $[b_x] > [b_{x'}]$ for all $x' < x$. By Lemma 6.13, x must be fundamental. The result follows from Lemma 7.16. ☐

Proof of Theorem 7.2. The Theorem follows immediately from Lemmas 7.15 and 7.17. ☐

7.2. General groups

In this section, we drop the assumption that G should be quasi-split. We keep the notation from Section 2.1. As announced, we show how to compute generic σ-conjugacy classes and classify cordial elements in this case.

The Frobenius action on the apartment $\mathcal{A}$ preserves the base alcove $\mathfrak{a}$, but no longer the chosen special vertex $\mathfrak{x}$. We denote by $\mu_\sigma \in V$ the uniquely determined element such that $\sigma(\mathfrak{x}) = \mathfrak{x} + \mu_\sigma$.

Moreover, there is a natural Frobenius action on $X_*(T)_{\Gamma_0}$. We denote the induced linear map by $\sigma_{\mathrm{lin}} : V \to V$

Under the identification of $\mathcal{A}$ with V by $\mathfrak{x} \mapsto 0$, the map σ_{lin} is given by

$$\sigma_{\mathrm{lin}} : V \to V, \quad v \mapsto \sigma(v) - \mu_\sigma.$$

Since σ_{lin} permutes the alcoves in $\mathcal{A}$, it permutes the Weyl chambers in V. We hence find a uniquely determined element $\sigma_1 \in W$ with $\sigma_{\mathrm{lin}}(C) = \sigma_1(C)$. Define $\sigma_2 := \sigma_1^{-1} \circ \sigma_{\mathrm{lin}}$ such that $\sigma_2(C) = C$. Then the action of σ on V is given by the composed action

$$\sigma = t_{\mu_\sigma} \circ \sigma_1 \circ \sigma_2,$$

where t_{μ_σ} is the translation by μ_σ. Note that σ_2 fixes both 0 and C, hence it fixes $\mathfrak{a}$ being the only alcove in C adjacent to 0. It follows that also $t_{\mu_\sigma} \circ \sigma_1$ fixes $\mathfrak{a}$. So the map

$t_{\mu_\sigma} \circ \sigma_1 : V \to V$ "looks like" the action of an element in $\Omega \subseteq \widetilde{W}$, except that a lift of $\mu_\sigma \in V$ to $X_*(T)_{\Gamma_0}$ might not exist; and if it exists, it might not be unique.

For each $w_1, w_2 \in W$, the difference $w_1\mu_\sigma - w_2\mu_\sigma$ lies in $\mathbb{Z}\Phi^\vee$, so we may consider $w_1\mu_\sigma - w_2\mu_\sigma$ as a well-defined element of $X_*(T)_{\Gamma_0}$ even if neither $w_1\mu_\sigma$ nor $w_2\mu_\sigma$ lies in $X_*(T)_{\Gamma_0}$.

We define maps

$$\begin{aligned}\operatorname{avg}_{\sigma_2} &: X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \to X_*(T)_{\Gamma_0} \otimes \mathbb{Q},\\ \operatorname{avg}_J &: X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \to X_*(T)_{\Gamma_0} \otimes \mathbb{Q} \quad (J \subseteq \Delta)\end{aligned}$$

as in Section 6.1. If $J = \sigma_2(J)$, we define $\pi_J := \operatorname{avg}_J \circ \operatorname{avg}_{\sigma_2}$. For an element $\mu \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$ or $\mu \in X_*(T)_\Gamma$, we define

$$\operatorname{conv}(\mu) := \max_{\substack{J \subseteq \Delta\\ J = \sigma_2(J)}} \operatorname{avg}_J \operatorname{avg}_{\sigma_2}(\mu) \in X_*(T)_{\Gamma_0} \otimes \mathbb{Q}.$$

Then we can describe generic Newton points as follows:

Theorem 7.18. *Assume that* $\operatorname{char}(F)$ *does not divide the order of* $\pi_1(G_{\mathrm{ad}})$*, the Borovoi fundamental group of the adjoint quotient*[4]*.*

Let $x = w\varepsilon^\mu \in \widetilde{W}$*. The generic Newton point of* x *is given by*

$$\nu_x = \max_{v \in W} \operatorname{conv}\left(v^{-1}\mu - \operatorname{wt}(\sigma_1^{-1}v \Rightarrow {}^{\sigma_2}(wv)) + \frac{1}{\#W}\sum_{u \in W}(v^{-1}\mu_\sigma - u^{-1}\mu_\sigma)\right).$$

In fact, the maximum is attained for some $v \in \operatorname{LP}(x)$*.*

We prove this theorem by reduction to the previously established results for quasi-split groups, following Goertz-He-Nie [GHN15, Section 2].

By [GHN15, Corollary 2.2.2], it suffices to prove the Theorem for adjoint groups, by comparing $B(G)_x$ with $B(G_{\mathrm{ad}})_x$.

Let us now assume that G is adjoint. Then $\gamma := \varepsilon^{\mu_\sigma} \circ \sigma_1$ is a well-defined element of $\widetilde{W}$, hence of Ω. Following [GHN15, Proposition 2.5.1], we can identify $B(G)_x$ with $B(\tilde{G})_{x\gamma} \cdot \gamma^{-1}$. Here, $\tilde{G}$ is a quasi-split inner form of G with maximal torus T and Frobenius given by σ_2. We see that

$$\nu_x = \nu^G\Big(\max_{[b] \in B(G)_x}[b]\Big) = \nu^G\Big(\max_{[b] \in B(\tilde{G})_{x\gamma}}[b\gamma^{-1}]\Big).$$

A quick calculation shows that for all $[b] \in B(G)$, we have

$$\nu^G([b]) = \nu^{\tilde{G}}([b\gamma]) - \frac{1}{\#W}\sum_{u \in W} u\mu_\sigma.$$

[4]It is conjectured in [GHN15, Section 2.2] that this assumption can be dropped; and in fact, it does not appear any more in [HN21, Section 3.2].

Thus

$$\nu_x = \nu^{\tilde{G}}([b_{x\gamma}]) - \frac{1}{\#W}\sum_{u\in W} u\mu_\sigma.$$

Calculating $\nu^{\tilde{G}}([b_{x\gamma}])$ using Corollary 7.4 shows Theorem 7.18.

Let us return to the general situation. Following Milićević-Viehmann [MV20, Remark 1.3], we define an element $x \in \widetilde{W}$ to be cordial if the corresponding element $\tilde{x}$ in the extended affine Weyl group of the quasi-split group $\tilde{G}$ under the above reduction is cordial. Then the results from [MV20] on cordial elements guarantee that the affine Deligne-Lusztig varieties associated with $\tilde{x}$ satisfy the most desirable properties as discussed earlier. By the above reduction method of [GHN15], it follows that also the affine Deligne-Lusztig varieties associated with x satisfy these properties.

Straightforward calculation shows the following:

Proposition 7.19. *Assume that* $\operatorname{char}(F)$ *does not divide the order of* $\pi_1(G_{\mathrm{ad}})$.

Let $x = w\varepsilon^\mu \in \widetilde{W}$ *and pick* $v \in W$ *of minimal length such that*

$$v^{-1}\mu + v^{-1}\mu_\sigma \in V$$

is dominant. Then $\sigma_1^{-1}v \in \mathrm{LP}(x)$. *The element* x *is cordial if and only if the following two conditions are both satisfied:*

(1) For any $\sigma_1^{-1}v' \in \mathrm{LP}(x)$, *we have*

$$d(\sigma_1^{-1}v' \Rightarrow {}^{\sigma_2}(wv')) \geqslant d(\sigma_1^{-1}v \Rightarrow {}^{\sigma_2}(wv)).$$

(2) We have

$$d(\sigma_1^{-1}v \Rightarrow {}^{\sigma_2}(wv)) = \ell\left(v^{-1}\sigma_1 {}^{\sigma_2}(wv)\right). \qquad \square$$

7.3. Connection to Demazure products

To conclude the section, we use our previous results on Demazure products to find a different description of generic Newton points. Following He [He21a], we consider twisted Demazure powers of x.

Definition 7.20. Let $n \geqslant 1$. We define the n-th σ-twisted Demazure power of x as

$$x^{*,\sigma,n} := x * ({}^\sigma x) * \cdots * \left({}^{\sigma^{n-1}}x\right) \in \widetilde{W}.$$

For $n \geqslant 2$, let us write

$$x_n := {}^{\sigma^{1-n}}\left(\left(x^{*,\sigma,n-1}\right)^{-1} x^{*,\sigma,n}\right),$$

such that

$$x^{*,\sigma,n} = x^{*,\sigma,n-1} * \left({}^{\sigma^{n-1}}x\right) = x^{*,\sigma,n-1}\cdot\left({}^{\sigma^{n-1}}x_n\right).$$

We can calculate x_n in terms of x and ${}^{\sigma^{1-n}}\mathrm{LP}(x^{*,\sigma,n-1})$ using Theorem 5.11. By Corollary 5.26, we have

$$\mathrm{LP}(x^{*,\sigma,n}) = \rho_{\sigma^{n-1}x}\left(\mathrm{LP}(x^{*,\sigma,n-1})\right) = \cdots = \rho_{\sigma^{n-1}x} \circ \cdots \circ \rho_{\sigma x}\left(\mathrm{LP}(x)\right).$$

Observe that by definition of the generic action ρ_x, we may write

$$\rho_{\sigma^n x}({}^{\sigma^n}(u)) = {}^{\sigma^n}(\rho_x(u)).$$

Let us define the map $\rho_{x,\sigma} := \rho_x \circ {}^{\sigma^{-1}}(\cdot) : W \to W$ by

$$\rho_{x,\sigma}(u) := \rho_x({}^{\sigma^{-1}}(u)).$$

Then

$$\begin{aligned}
\mathrm{LP}(x^{*,\sigma,n}) =& \rho_{\sigma^{n-1}x} \circ \cdots \circ \rho_{\sigma x}\left(\mathrm{LP}(x)\right).\\
=& \left({}^{\sigma^{n-1}}(\cdot) \circ \rho_x \circ {}^{\sigma^{1-n}}(\cdot)\right) \circ \cdots \circ \left({}^{\sigma^{1}}(\cdot) \circ \rho_x \circ {}^{\sigma^{-1}}(\cdot)\right)(\mathrm{LP}(x))\\
=& {}^{\sigma^{n-1}}(\cdot) \circ \rho_{x,\sigma} \circ \cdots \circ \rho_{x,\sigma}(\mathrm{LP}(x))\\
=& {}^{\sigma^{n-1}}\left(\rho_{x,\sigma}^{n-1}(\mathrm{LP}(x))\right).
\end{aligned}$$

Lemma 7.21. *(a) There exists an integer $N > 1$ such that for each $n \geqslant N$,*

$$x_N = x_n \text{ and } \rho_{x,\sigma}^N(\mathrm{LP}(x)) = \rho_{x,\sigma}^n(\mathrm{LP}(x)).$$

Denote the eventual values by $x_\infty := x_N$ resp. $\rho_{x,\sigma}^\infty(\mathrm{LP}(x)) := \rho_{x,\sigma}^N(\mathrm{LP}(x))$.

(b) We have

$$\begin{aligned}
\rho_{x,\sigma}^\infty(\mathrm{LP}(x)) =& \{v \in \mathrm{LP}(x) \mid \exists n \geqslant 1 : \ v = \rho_{x,\sigma}^n(v)\}.\\
\lim_{n\to\infty} \frac{\ell(x^{*,\sigma,n})}{n} =& \ell(x_\infty).
\end{aligned}$$

(c) The element x_∞ is fundamental. For each $v \in \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$, it can be written as

$$x_\infty = ({}^{\sigma^{-1}}v)\rho_{x,\sigma}(v)^{-1}\varepsilon^{\mu - \rho_{x,\sigma}(v)\,\mathrm{wt}\left({}^{\sigma^{-1}}v \Rightarrow w\rho_{x,\sigma}(v)\right)}.$$

Proof. (a) Observe that $\rho_{x,\sigma}^n$ induces an endomorphism $\mathrm{LP}(x) \to \mathrm{LP}(x)$. We obtain a weakly decreasing sequence of subsets of W

$$\mathrm{LP}(x) \supseteq \rho_{x,\sigma}(\mathrm{LP}(x)) \supseteq \rho_{x,\sigma}^2(\mathrm{LP}(x)) \supseteq \cdots.$$

Since W is finite, this sequence must stabilize eventually.

Because x_n only depends on the values of $\rho_{x,\sigma}^{n-1}(\mathrm{LP}(x))$ and x, the result follows.

(b) Both claims follow immediately from (a).

(c) Let N be as in (a), and let $n \geqslant 1$. Then

$$x^{*,\sigma,N+n} = x^{*,\sigma,N} \cdot {}^{\sigma^N}x_\infty \cdots {}^{\sigma^{N+n-1}}x_\infty$$

is a length additive product. In particular,

$$\ell(x_\infty \cdots {}^{\sigma^{n-1}}x_\infty) = n\ell(x_\infty).$$

By [Nie15, Theorem 1.3] or Proposition 6.11, x_∞ is fundamental.

Next let $v \in \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$. Then also $\rho_{x,\sigma}(v) \in \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$, and we get

$${}^{\sigma^N}\rho_{x,\sigma}(v) \in \mathrm{LP}(x^{*,\sigma,N+1}) = \mathrm{LP}(x^{*,\sigma,N} * {}^{\sigma^N}x) = \mathrm{LP}(x^{*,\sigma,N} \cdot {}^{\sigma^N}(x_\infty)).$$

In view of Proposition 5.12, we find a uniquely determined element ${}^{\sigma^N}v' \in \mathrm{LP}(x^{*,\sigma,N})$ such that

$$({}^{\sigma^N}v', {}^{\sigma^N}\rho_{x,\sigma}(v)) \in M(x^{*,\sigma,N}, {}^{\sigma^N}x).$$

Then by Theorem 5.11,

$$x_\infty = v'\rho_{x,\sigma}(v)^{-1}\varepsilon^{\mu - \rho_{x,\sigma}(v)\,\mathrm{wt}(v' \Rightarrow w\rho_{x,\sigma}(v))}.$$

Note that ${}^\sigma v' \in {}^{\sigma^{1-N}}\mathrm{LP}(x^{*,\sigma,N}) = \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$. The minimality condition on the tuple $({}^{\sigma^N}v', {}^{\sigma^N}\rho_{x,\sigma}(v))$ moreover implies that $\rho_x(v') = \rho_{x,\sigma}({}^\sigma v') = \rho_{x,\sigma}(v)$ (Lemma 5.24). The map $\rho_{x,\sigma} : \rho_{x,\sigma}^\infty(\mathrm{LP}(x)) \to \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$ is a surjective, and the set $\rho_{x,\sigma}^\infty(\mathrm{LP}(x))$ is finite. It follows that the restriction of $\rho_{x,\sigma}$ to $\rho_{x,\sigma}^\infty(\mathrm{LP}(x))$ is bijective. Recall that v and ${}^\sigma v'$ are two elements of $\rho_{x,\sigma}^\infty(\mathrm{LP}(x))$ whose images under $\rho_{x,\sigma}$ coincide. Thus $v = {}^\sigma v'$, finishing the proof. □

Theorem 7.22. *(a) The σ-conjugacy class $[x_\infty] \in B(G)$ is the generic σ-conjugacy class of x.*

(b) For any $v \in \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$, we have $\ell(x_\infty) = \ell(x) - d(v \Rightarrow {}^\sigma(w\rho_{x,\sigma}(v)))$.

(c) Fix $v \in \rho_{x,\sigma}^\infty(\mathrm{LP}(x))$ and define $J = \mathrm{supp}_\sigma(\rho_{x,\sigma}(v)^{-1}v)$, so $J \subseteq \Delta$ consists of all σ-orbits of simple roots whose corresponding simple reflections occur in some reduced decomposition of $\rho_{x,\sigma}(v)^{-1}v \in W$.

We can express the generic Newton point of x as

$$\nu_x = \pi_J\left(v^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^\sigma(wv))\right).$$

Proof. (a) By Theorem 7.1, we can express the generic σ-conjugacy class of x as

$$[b_x] = \max\{[y] \mid y \leqslant x\} = \max\{[y] \mid y \leqslant x \text{ and } y \text{ is fundamental}\}.$$

In particular, $[b_x] \geqslant [x_\infty]$. For the converse inequality, pick some $y \leqslant x$ fundamental with $[b_x] = [y] \in B(G)$.

By definition of the Demazure product, we get

$$x^{*,\sigma,n} = x * ({}^{\sigma}x) \cdots * \left({}^{\sigma^{n-1}}x\right) \geqslant y\,({}^{\sigma}y) \cdots \left({}^{\sigma^{n-1}}y\right).$$

Thus, using the fact that y and x_∞ are fundamental, we get

$$\begin{aligned}\langle \nu(x_\infty), 2\rho\rangle =&\ell(x_\infty) = \lim_{n\to\infty} \frac{\ell(x^{*,\sigma,n})}{n}\\ &\geqslant \lim_{n\to\infty} \frac{\ell(y\,{}^{\sigma}y \cdots {}^{\sigma^{n-1}}y)}{n} = \lim_{n\to\infty} \ell(y) = \langle \nu(y), 2\rho\rangle = \langle \nu(b_x), 2\rho\rangle.\end{aligned}$$

This estimate shows that $[x_\infty] = [b_x]$.

(b) This follows from the explicit description of x_∞ in Lemma 7.21 together with Corollary 2.11 and the simple observation $\rho_{x,\sigma}(v) \in \mathrm{LP}(x_\infty)$.

(c) Let us write $x_\infty = w_\infty \varepsilon^{\mu_\infty}$. The generic Newton point of x is the Newton point of x_∞, which we express using Lemma 6.7.

Let $N \geqslant 1$ such that the action of $(\sigma \circ w_\infty)$ on X_* becomes trivial. We want to show for each $v \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$ that

$$v^{-1} \sum_{k=1}^{N} (\sigma \circ w_\infty)^k \mu_\infty \in X_* \otimes \mathbb{Q}$$

is dominant.

Note each $v \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$ may be written as $v = \rho_{x,\sigma}(u)$ for some $u \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$. By Lemma 7.21, it follows that $w_\infty = ({}^{\sigma^{-1}}u)v^{-1}$. Thus $u = {}^{\sigma}(w_\infty v) \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$. This shows ${}^{\sigma}(w_\infty v) \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$ for each $v \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$. It follows for each $\alpha \in \Phi^+$ that

$$\begin{aligned}&\left\langle v^{-1} \sum_{k=1}^{N} (\sigma \circ w_\infty)^k \mu_\infty, \alpha \right\rangle = \sum_{k=1}^{N} \langle \mu_\infty, (\sigma \circ w_\infty)^k v\alpha\rangle\\ &= \sum_{k=1}^{N} \Big(\langle \mu_\infty, (\sigma \circ w_\infty)^k v\alpha\rangle + \Phi^+((\sigma \circ w_\infty)^k v\alpha) - \Phi^+((\sigma \circ w_\infty)^{k+1} v\alpha)\Big)\\ &= \sum_{k=1}^{N} \ell(x_\infty, (\sigma \circ w_\infty)^k v\alpha) \geqslant 0.\end{aligned}$$

This shows the above dominance claim. As $v \in \rho_{x,\sigma}^{\infty}(\mathrm{LP}(x))$ was arbitrary, the same claim holds for $\rho_{x,\sigma}(v)$. With

$$J := \mathrm{supp}_\sigma(\rho_{x,\sigma}(v)^{-1}\,{}^{\sigma}(w_\infty \rho_{x,\sigma}(v))) = \mathrm{supp}_\sigma(\rho_{x,\sigma}(v)^{-1} v),$$

Lemma 6.7 proves that

$$\begin{aligned}\nu(x_\infty) = \pi_J(\rho_{x,\sigma}(v)^{-1}\mu_\infty) &\underset{\text{L7.21}}{=} \pi_J(\rho_{x,\sigma}(v)^{-1}\mu - \mathrm{wt}({}^{\sigma^{-1}}v \Rightarrow w\rho_{x,\sigma}(v)))\\ &\underset{\text{Def. }\pi_J}{=} \pi_J(\rho_{x,\sigma}(v)^{-1}\mu - \mathrm{wt}(v \Rightarrow {}^{\sigma}(w\rho_{x,\sigma}(v)))).\end{aligned}$$

Now observe that

$$\begin{aligned}\rho_{x,\sigma}(v)^{-1}\mu &\equiv v^{-1}\mu \pmod{\mathbb{Q}\Phi_J^\vee},\\ \mathrm{wt}(v \Rightarrow {}^{\sigma}(w\rho_{x,\sigma}(v))) &\equiv \mathrm{wt}(v \Rightarrow {}^{\sigma}(wv)) \pmod{\mathbb{Q}\Phi_J^\vee}.\end{aligned}$$ □

Part (a) of the above Theorem readily implies [He21a, Theorem 0.1]. Our previous result Corollary 7.4 expresses the generic Newton point ν_x as a formula similar to part (c) of the above theorem, but the allowed elements $v \in \mathrm{LP}(x)$ are usually different ones.

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A. Some quantum Bruhat graphs

In this appendix, we show pictures of a couple of quantum Bruhat graphs and parabolic quantum Bruhat graphs. For size reasons, only the root systems A_2, B_2, G_2 and A_3 are covered.

The simple roots are numbered α_1, α_2 (and α_3 for A_3). For types B_2 and G_2, we use the convention that α_1 is long and α_2 is short[5]. We write s_i as a shorthand for s_{α_i} ($i = 1, 2, 3$).

Elements of the Weyl group W are represented by lexicographically minimal reduced words. In each ot the diagrams, the elements of the same length form a row of the diagram, with the neutral element on the bottom and the longest element on the top. Within each row, the elements are sorted lexicographically.

It follows that Bruhat edges always go upwards and quantum edges always go downwards.

For parabolic quantum Bruhat graphs, the quantum edges are drawn in a dark blue shade and are labelled by their respective weight (in the parabolic case, a representative of the coset in $\mathbb{Z}\Phi^\vee/\mathbb{Z}\Phi_J^\vee$). The Bruhat edges are drawn in black and are unlabelled.

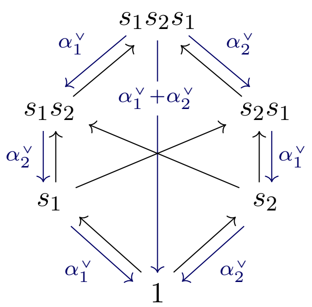
The pictures are rendered using the LaTeX package `tikz-cd`. The LaTeX-code was generated using the computer algebra system `sage-combinat` ([Sage], [SaCo]), with manual tweaking of the arrows to improve readability.

A.1. Root System A_2

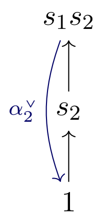
The root system has three positive roots, namely α_1, α_2 and $\alpha_1 + \alpha_2$. The Weyl group consists of six elements. These are the two reflection orderings:

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2,$$
$$\alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1.$$

Quantum Bruhat graph QB(W)



Parabolic quantum Bruhat graph QB(W^J) with $J = \{\alpha_1\}$



The parabolic quantum Bruhat graph QB($W^{\{\alpha_2\}}$) is isomorphic to the one printed above,

[5] For type G_2, the opposite labelling is used in `sage`.

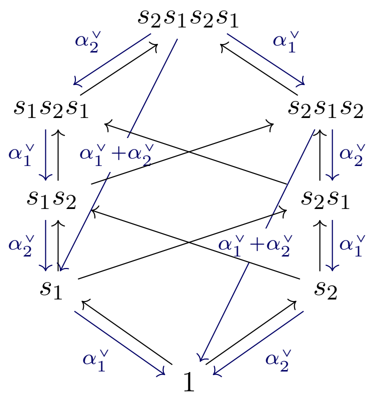
by interchanging α_1 and α_2. The parabolic quantum Bruhat graph $\mathrm{QB}(W^{\{\alpha_1,\alpha_2\}})$ consists of only one point.

A.2. Root System B_2

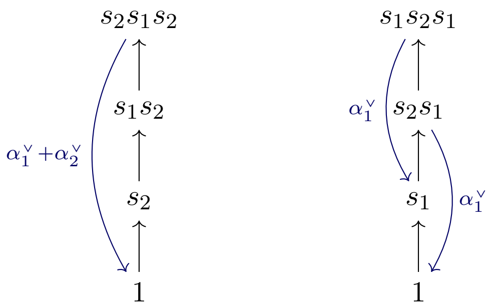
The root system has four positive roots, namely the long roots $\alpha_1, \alpha_1 + 2\alpha_2$ and the short roots $\alpha_2, \alpha_1 + \alpha_2$. The Weyl group consists of eight elements. These are the two reflection orderings:

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_2,$$
$$\alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1.$$

The quantum Bruhat graph and the double Bruhat graph are given as follows:



These are the two non-trivial parabolic quantum Bruhat graphs, with $\mathrm{QB}(W^{\{\alpha_1\}})$ on the left and $\mathrm{QB}(W^{\{\alpha_2\}})$ on the right:

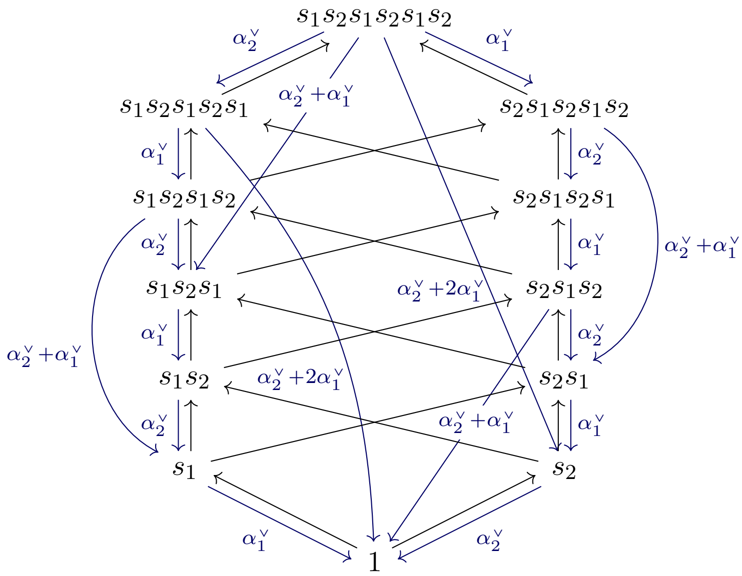


A.3. Root System G_2

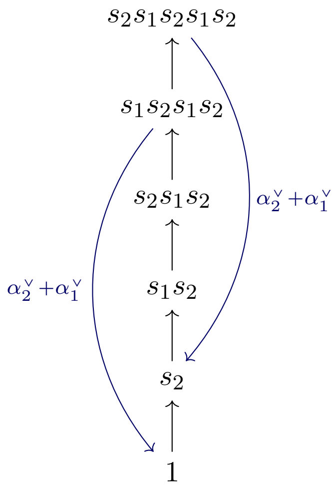
The root system has six positive roots, namely the three long roots $\alpha_1, \alpha_1 + 3\alpha_2$ and $2\alpha_1 + 3\alpha_2$ as well as the three short roots $\alpha_2, \alpha_1 + \alpha_2$ and $\alpha_1 + 2\alpha_2$. The Weyl group consists of twelve elements. These are the two reflection orderings:

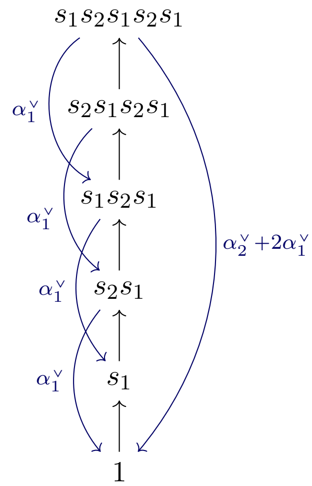
$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec 2\alpha_1 + 3\alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_1 + 3\alpha_2 \prec \alpha_2,$$
$$\alpha_2 \prec \alpha_1 + 3\alpha_2 \prec \alpha_1 + 2\alpha_2 \prec 2\alpha_1 + 3\alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1.$$

Now the quantum Bruhat graph is given as follows:



The two non-trivial parabolic quantum Bruhat graphs are given as follows, with $\mathrm{QB}(W^{\{\alpha_1\}})$ on the left and $\mathrm{QB}(W^{\{\alpha_2\}})$ on the right:





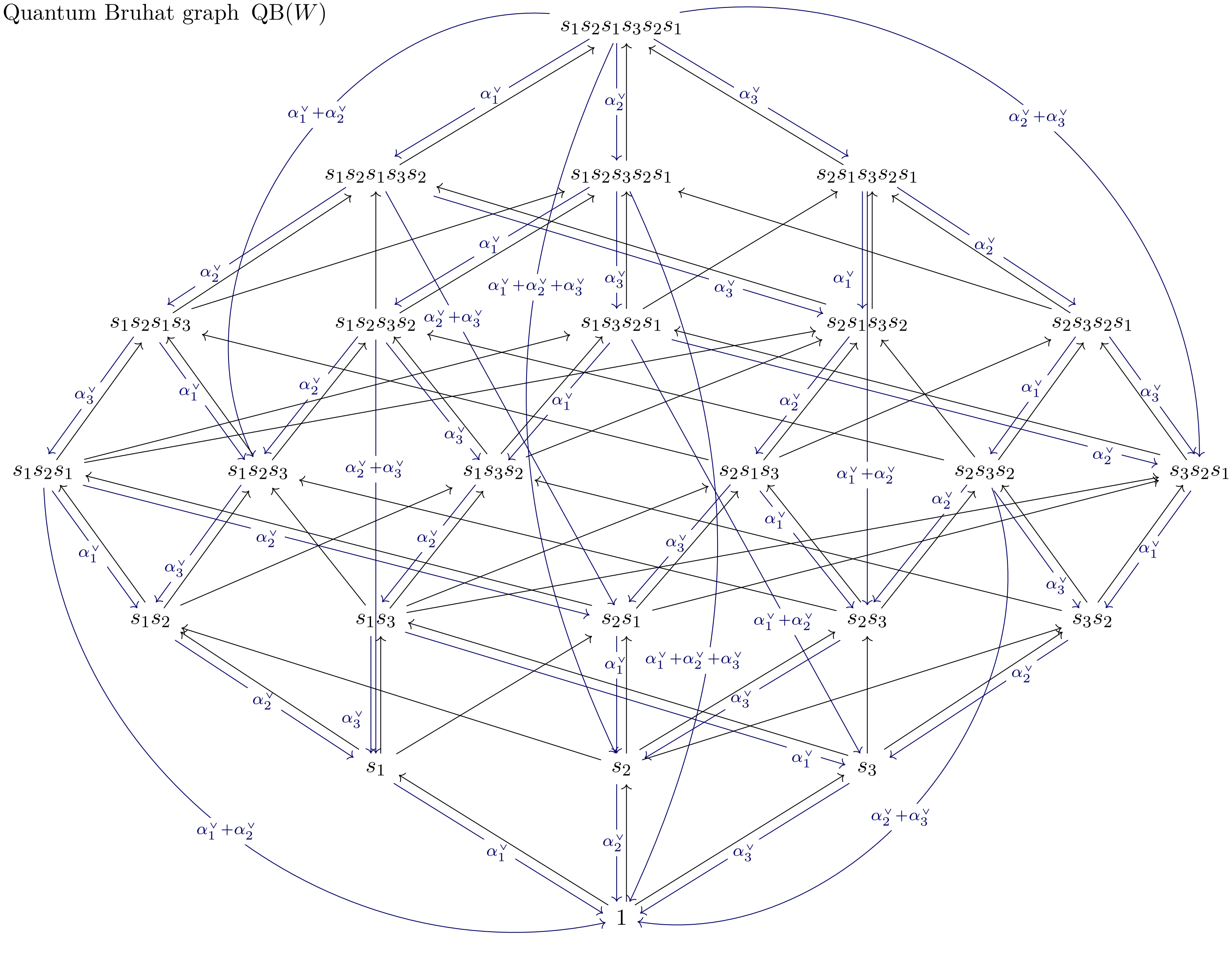
A.4. Root System A_3

The root system has six positive roots, namely $\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_2+\alpha_3$ and $\alpha_1+\alpha_2+\alpha_3$. The Weyl group consists of 24 elements. These are the 16 reflection orderings:

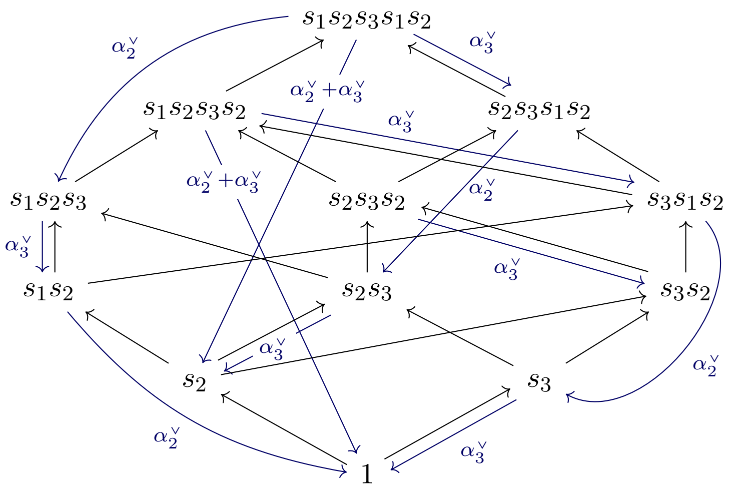
$$
\begin{gathered}
\alpha_3 \prec \alpha_1 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_2, \\
\alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_3 \prec \alpha_1, \\
\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1 \prec \alpha_1+\alpha_2 \prec \alpha_2, \\
\alpha_3 \prec \alpha_1 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_2, \\
\alpha_1 \prec \alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_2, \\
\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_1, \\
\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_3 \prec \alpha_1, \\
\alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_1 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_3, \\
\alpha_1 \prec \alpha_1+\alpha_2 \prec \alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_3, \\
\alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_1, \\
\alpha_1 \prec \alpha_1+\alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_3 \prec \alpha_2+\alpha_3 \prec \alpha_2, \\
\alpha_1 \prec \alpha_1+\alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_3, \\
\alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1 \prec \alpha_3, \\
\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1 \prec \alpha_3, \\
\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_1, \\
\alpha_1 \prec \alpha_3 \prec \alpha_1+\alpha_2+\alpha_3 \prec \alpha_1+\alpha_2 \prec \alpha_2+\alpha_3 \prec \alpha_2.
\end{gathered}
$$

The quantum Bruhat graph is printed on the next page.

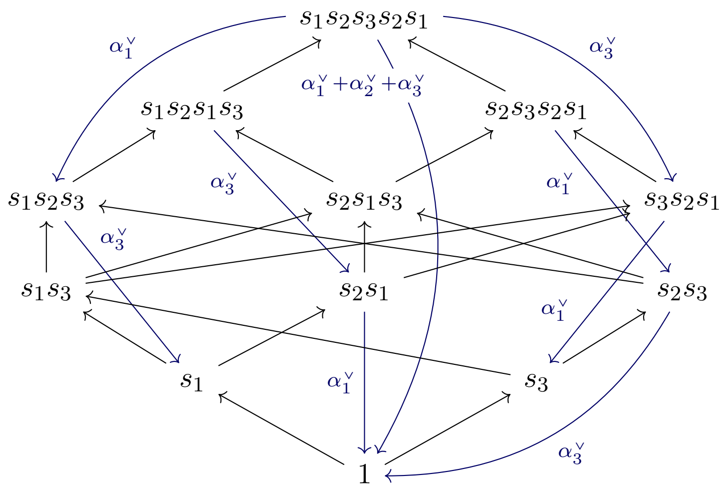
Quantum Bruhat graph $\mathrm{QB}(W)$



Below is the parabolic quantum Bruhat graph $\mathrm{QB}(W^{\{\alpha_1\}})$. It is isomorphic to $\mathrm{QB}(W^{\{\alpha_3\}})$ after interchanging α_1 and α_3.



This is $\mathrm{QB}(W^{\{\alpha_2\}})$:



Finally, we have $\mathrm{QB}(W^{\{\alpha_1,\alpha_2\}})$ on the left and $\mathrm{QB}(W^{\{\alpha_1,\alpha_3\}})$ on the right. Note that

$\mathrm{QB}(W^{\{\alpha_1,\alpha_2\}}) \cong \mathrm{QB}(W^{\{\alpha_2,\alpha_3\}})$ after interchanging α_1 and α_3.

$$\begin{tikzcd}
s_1s_2s_3 \arrow[ddd, bend right, "\alpha_3^\vee"'] \\
s_2s_3 \arrow[u] \\
s_3 \arrow[u] \\
1 \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & s_2s_1s_3s_2 \arrow[ddd, bend right, "\alpha_2^\vee"'] & \\
 & s_1s_3s_2 \arrow[u] \arrow[ddd, bend left, "\alpha_2^\vee"] & \\
s_1s_2 \arrow[ur] & & s_3s_2 \arrow[ul] \\
 & s_2 \arrow[ul] \arrow[ur] & \\
 & 1 \arrow[u] &
\end{tikzcd}$$