$\mathcal{H}_2$ and $\mathcal{H}_\infty$ Suboptimal Distributed Filter Design for Linear Systems

Junjie Jiao, *Member, IEEE*, Harry L. Trentelman, *Life Fellow, IEEE,* and M. Kanat Camlibel, *Senior Member, IEEE*

***Abstract*—This paper investigates the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems for continuous time linear systems. We consider a linear system monitored by a number of filters, where each of the filters receives only part of the measured output of the system. Each filter can communicate with the other filters according to an a priori given strongly connected weighted directed graph. The aim is to design filter gains that guarantee the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance input to the output estimation error to be smaller than an a priori given upper bound, while all local filters reconstruct the full system state asymptotically. We provide a centralized design method for obtaining such $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters. The proposed design method is illustrated by a simulation example.**

***Index Terms*—Distributed estimation, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering, linear time-invariant systems, suboptimality.**

I. Introduction

RECENT years have witnessed an increasing interest in problems of state estimation for spatially constrained large-scale systems. Such problems are relevant in applications, such as power grids [1], industrial plants [2] and wireless sensor networks [3]. The aim of the present paper is to deal with the situation that *no single sensor is able to estimate the system state by itself*. This can be caused by physical and/or geographical restrictions: a single sensor might be able to 'observe' only part of the overall system output. However, the individual sensors might be able to communicate (e.g. by wireless communication) and exchange their local estimates. The infrastructure of this communication is represented by a given communication graph. In this way, all of these sensors together are able to estimate the state of the system asymptotically. In this problem setting, one of the main challenges is that none of the local sensors by itself is able to estimate the system state by using its own local measurements. Consequently, standard estimation methods do not directly apply anymore.

The distributed estimation problem has been mainly studied in two research directions, namely, distributed observer design and distributed Kalman filtering. In [4], an augmented state observer was proposed to cast the distributed observer design problem into a decentralized control problem for linear systems, using the notion of 'fixed modes' [5]. Later on, in [6], the results in [4] were extended and a more general form of distributed observers was provided, allowing the rate of convergence of the observer to be freely assignable. In [7], for time-varying communication graphs, a hybrid observer was introduced to distributedly estimate the state of a linear system. Based on observability decompositions, the problem of distributed observer design was also investigated in [8]–[10]. Distributed estimation in the presence of time delays was studied in [11]–[13], while in [14]–[16], finite-time distributed observers were proposed. In [17], an attack resilient algorithm was introduced to address distributed estimation in the situation that certain nodes are compromised by adversaries.

On the other hand, in the literature much attention has also been devoted to distributed filtering problems. A Kalman-filter-based distributed filter was proposed in [18]–[20]. There, the proposed methods employ a two-step strategy: a state update rule based on a Kalman-filter and a data fusion step based on consensus. In [21], distributed fusion robust H_2 and H_∞ filters were proposed for uncertain *stable* linear systems. There, robust suboptimal fusion filters were obtained by solving large-scale LMI's. In [22], a distributed robust filtering problem was addressed using dissipativity theory. Later on in [23], the results of [22] were generalized to the case that the communication graph is allowed to randomly change. Recently, in [24], a distributed Kalman-Bucy filtering problem was studied, using the idea of averaging the dynamics of heterogeneous multi-agent systems [25].

Different from the existing work, in the present paper, we will consider two *deterministic versions* of the distributed optimal filtering problem for *general* linear systems, i.e., the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ distributed filtering problems. Given a linear system and a network of local filters, each local filter receives a portion of the measured output of the system and then exchanges its state with that of its neighboring local filters. Together, these local filters form a distributed filter. The distributed optimal filtering problem is to find filter gain matrices such that the associated $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance (as measured by the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the external disturbance to the output estimation error) is minimized, while the states of all local filters asymptotically track the system state.

Due to *non-convexity*, this problem is difficult to solve in general. Therefore, in this paper we will address a *suboptimality* version of this problem: the objective of the present paper is to design suitable filter gain matrices such that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance is *smaller than an a priori given tolerance.* To the authors' best knowledge, this paper is the first work that deals with the problems of designing distributed filters that guarantee an upper bound on the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the external disturbances to the output estimation error, while the states of all local filters asymptotically track the system state.

The main contributions of this paper are the following:

1) We establish conditions for the existence of suitable filter gains in terms of solvability of LMI's for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problem. For the $\mathcal{H}_2$ filtering problem, all except one of these LMI's will always turn out to be solvable.
2) We provide conceptual algorithms for obtaining suitable $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively.

This paper is organized as follows. In Section II, we review some basic results on graph theory, detectability properties of linear systems, and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of linear systems. Subsequently, in Section III we formulate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems. We then provide design methods for obtaining such distributed filters in Section IV. In Section V we

The work of J. Jiao was supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement no. 899987.

J. Jiao is with the Chair of Information-oriented Control, TUM School of Computation, Information and Technology, Technical University of Munich, 80333, Munich, Germany. Email: junjie.jiao@tum.de

H. L. Trentelman and M. K. Camlibel are with the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, Groningen, 9700 AV, The Netherlands. Email: h.l.trentelman@rug.nl; m.k.camlibel@rug.nl

provide a simulation example to illustrate our design method. Finally, in Section VI we formulate our conclusions.

Notation

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{R}^n$ the space of n dimensional vectors over $\mathbb{R}$. We write $\mathbf{1}_N$ for the n dimensional column vector with all its entries equal to 1. For a given matrix A, we write $A^\top$ to denote its transpose and A^{-1} its inverse (if it exists). For a symmetric matrix P, we denote $P > 0$ if it is positive definite and $P < 0$ if it is negative definite. We denote the identity matrix of dimension $n \times n$ by I_n. The trace of a square matrix A is denoted by $\mathrm{tr}(A)$. We denote by $\mathrm{diag}(d_1, d_2, \dots, d_n)$ the $n \times n$ diagonal matrix with $d_1, d_2, \dots, d_n$ on the diagonal. Given matrices $R_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, \dots, n$, we denote by $\mathrm{blockdiag}(R_i)$ the $nm \times nm$ block diagonal matrix with $R_1, R_2, \dots, R_n$ on the diagonal and we denote by $\mathrm{col}(R_i)$ the $nm\times m$ column block matrix $\left(R_1^\top, R_2^\top, \dots, R_n^\top\right)^\top$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. For a linear map $A : \mathcal{X} \to \mathcal{Y}$, the kernel and image of A are denoted by $\ker(A) := \{x \in \mathcal{X} \mid Ax = 0\}$ and $\mathrm{im}(A) := \{Ax \mid x \in \mathcal{X}\}$, respectively.

II. Preliminaries

A. Graph Theory

A weighted directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the finite nonempty node set, $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the edge set of ordered pairs (i, j) and $\mathcal{A} = [a_{ij}]$ is the associated adjacency matrix with nonnegative entries. The entry a_{ji} of the adjacency matrix $\mathcal{A}$ is the weight associated with the edge (i, j) and a_{ji} is nonzero if and only if $(i, j) \in \mathcal{E}$. Given a graph $\mathcal{G}$, a directed path from node 1 to node p is a sequence of edges $(k, k + 1)$, $k = 1, 2, \dots, p - 1$. A graph is called *strongly connected* if for any pair of distinct nodes i and j, there exists a directed path from i to j. A graph is called simple if $a_{ii} = 0$, i.e., the graph does not contain self-loops. A graph is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. A simple undirected graph is called *connected* if for each pair of nodes i and j there exists a path from i to j.

Given a graph $\mathcal{G}$, the degree matrix of $\mathcal{G}$ is denoted by $\mathcal{D} = \mathrm{diag}(d_1, d_2, \dots, d_N)$ with $d_i = \sum_{j=1}^N a_{ij}$. The Laplacian matrix of $\mathcal{G}$ is defined as $L := \mathcal{D} - \mathcal{A}$. If $\mathcal{G}$ is a weighted directed graph, the associated Laplacian matrix L has a zero eigenvalue corresponding to the eigenvector $\mathbf{1}_N$. If moreover $\mathcal{G}$ is strongly connected, then all the other eigenvalues lie in the open right half-plane.

For strongly connected weighted directed graphs, we review the following lemma [26]:

Lemma 1: Let $\mathcal{G}$ be a strongly connected weighted directed graph with Laplacian matrix L. Then there exists a unique row vector $\theta = (\theta_1, \theta_2, \dots, \theta_N)$, where $\theta_1, \theta_2, \dots, \theta_N$ are all positive real numbers, such that $\theta L = 0$ and $\theta \mathbf{1}_N = N$. Define $\Theta := \mathrm{diag}(\theta_1, \theta_2, \dots, \theta_N)$, then the matrix $\mathcal{L} := \Theta L + L^\top \Theta$ is a positive semi-definite matrix associated with a connected weighted undirected graph.

B. Detectability and Detectability Decomposition

In this subsection, we review detectability and the detectability decomposition of linear systems. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state and $y \in \mathbb{R}^p$ the measured output. The matrices A and C are of suitable dimensions.

Let $p(s)$ be the characteristic polynomial of A. Then $p(s)$ can be factorized as

$$p(s) = p_-(s)p_+(s),$$

where $p_-(s)$ and $p_+(s)$ have roots in the open left half-plane and the closed right half-plane, respectively. The undetectable subspace of the pair (C, A) is defined as

$$\mathcal{S} := \mathcal{N} \cap \ker\left(p_+(A)\right),$$

where

$$\mathcal{N} := \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

The pair (C, A) is detectable if and only if $\mathcal{S} = \{0\}$, see e.g. [27].

There exists an orthogonal matrix $T \in \mathbb{R}^{n\times n}$ such that the pair (C, A) is transformed into the detectability decomposition form

$$T^\top AT = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad CT = \begin{pmatrix} C_1 & 0 \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{v\times v}$, $A_{21} \in \mathbb{R}^{(n-v)\times v}$, $A_{22} \in \mathbb{R}^{(n-v)\times(n-v)}$, $C_1 \in \mathbb{R}^{p\times v}$ and the pair (C_1, A_{11}) is detectable. In addition, if we partition $T = (T_1\ T_2)$, where T_1 contains the first v columns, then the undetectable subspace is given by

$$\mathrm{im}(T_2) = \mathcal{S}.$$

Since T is orthogonal, we also have

$$\mathrm{im}(T_1) = \mathcal{S}^\perp.$$

C. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance of Linear Systems

In this subsection, we review the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of a linear system with external disturbances. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Ed, \\ y &= Cx, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^q$ the external disturbance and $y \in \mathbb{R}^p$ the measured output. The matrices A, C and E are of suitable dimensions.

We first review the $\mathcal{H}_2$ performance of the system (2). Let $T_d(t) = Ce^{At}E$ be the impulse response of (2). Then the associated $\mathcal{H}_2$ performance is defined to be the square of its L_2-norm, given by

$$J = \int_0^\infty \mathrm{tr}\left[T_d^\top(t)T_d(t)\right] dt. \tag{3}$$

Note that the performance (3) is finite if the system (2) is internally stable, i.e., A is Hurwitz.

The following well-known result provides a necessary and sufficient condition under which (2) is internally stable and (3) is smaller than a given upper bound (see e.g. [28], [29]):

Lemma 2: Let $\gamma > 0$. Then the system (2) is internally stable and $J < \gamma$ if and only if there exists $P > 0$ satisfying

$$\begin{aligned} A^\top P + PA + C^\top C &< 0, \\ \mathrm{tr}\left(E^\top PE\right) &< \gamma. \end{aligned}$$

Next, we review the $\mathcal{H}_\infty$ performance of the system (2). Let $T_d(s) = C(sI_n - A)^{-1}E$ be the transfer matrix of (2). If A is Hurwitz, then the $\mathcal{H}_\infty$ performance of (2) is defined as the $\mathcal{H}_\infty$ norm of $T_d(s)$, given by

$$||T_d||_\infty := \sup_{\omega\in\mathbb{R}} \sigma(T(j\omega)), \tag{4}$$

where $\sigma(T_d(j\omega))$ is the maximum singular value of the complex matrix $T_d(j\omega)$.

The well-known bounded real lemma provides a necessary and sufficient condition under which (2) is stable and (4) is smaller than a given upper bound (see e.g. [30], [31]):

Lemma 3: Let $\gamma > 0$. Then the system (2) is internally stable and $||T_d||_\infty < \gamma$ if and only if there exists $P > 0$ such that

$$A^\top P + PA + \frac{1}{\gamma^2} PEE^\top P + C^\top C < 0.$$

In the next section, we will formulate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ distributed filter design problems that will be addressed in this paper.

III. Problem Formulation

Consider the finite-dimensional linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Ed, \\ y &= Cx + Dd, \\ z &= Hx, \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^q$ the external disturbance, $y \in \mathbb{R}^r$ the measured output and $z \in \mathbb{R}^p$ the output to be estimated. The matrices A, C, D, E and H are of suitable dimensions.

The standard optimal filtering problem for the system (5) is to find a filter that takes y as input and returns an optimal estimate ζ of z, while the filter state asymptotically tracks the state x of (5). Here, 'optimal' means that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from d to the estimation error $z-\zeta$ is minimized over all such filters. In that problem setting, however, a standing assumption is that one single filter is able to acquire the complete measured output y of the system.

In the present paper, we relax this assumption. We consider the situation that the output y of (5) is not available to one single filter, but that N different portions of the output are observed by N local filters. More specifically, each local filter acquires the portion

$$y_i = C_i x + D_i d,$$

where $y_i \in \mathbb{R}^{r_i}$, $C_i \in \mathbb{R}^{r_i \times n}$ and $D_i \in \mathbb{R}^{r_i \times q}$, for $i = 1, 2, \ldots, N$. Here, the matrices C_i and D_i are obtained by partitioning

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{pmatrix}.$$

Clearly, the original output y of (5) has then been partitioned as

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

and $\sum_{i=1}^N r_i = r$. In this paper it will be a standing assumption that the pair (C, A) is detectable. We will also assume that none of the pairs (C_i, A) is detectable itself. If, for at least one i, the pair (C_i, A) is detectable, the distributed filtering problem boils down to the standard optimal filtering problem.

In our distributed case, each local filter makes use of the portion of the measured output that it acquires and will then communicate with its neighboring local filters by exchanging filter state information. In this way, the local filters will together form a distributed filter. Following [8] and [10], we propose a distributed filter of the form

$$\begin{aligned} \dot{w}_i &= Aw_i + G_i(y_i - C_i w_i) + F_i \sum_{j=1}^N a_{ij}(w_j - w_i), \\ \zeta_i &= Hw_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{6}$$

where $w_i \in \mathbb{R}^n$ is the state of the ith local filter and $\zeta_i \in \mathbb{R}^p$ is the associated output. The matrices $G_i \in \mathbb{R}^{n \times r_i}$ and $F_i \in \mathbb{R}^{n \times n}$ are local filter gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the communication graph. In this paper, it will be a standing assumption that the communication graph between the local filters is a strongly connected weighted directed graph.

For the ith local filter, we introduce the associated local state estimation error e_i and local output estimation error η_i as

$$\begin{aligned} e_i &:= x - w_i, \\ \eta_i &:= z - \zeta_i, \quad i = 1, 2, \ldots, N. \end{aligned}$$

The dynamics of the ith local error system is then given by

$$\begin{aligned} \dot{e}_i &= (A - G_i C_i)e_i + F_i \sum_{j=1}^N a_{ij}(e_j - e_i) + (E - G_i D_i)d, \\ \eta_i &= He_i, \quad i = 1, 2, \ldots, N. \end{aligned}$$

Denote $e = (e_1^\top, e_2^\top, \ldots, e_N^\top)^\top$, $\eta = (\eta_1^\top, \eta_2^\top, \ldots, \eta_N^\top)^\top$ and

$$\begin{aligned} \bar{A} &:= \text{blockdiag}(A - G_i C_i) \in \mathbb{R}^{nN \times nN}, \\ \bar{F} &:= \text{blockdiag}(F_i) \in \mathbb{R}^{nN \times nN}, \\ \bar{E} &:= \text{col}(E - G_i D_i) \in \mathbb{R}^{nN \times q}. \end{aligned}$$

The global error system is then given by

$$\begin{aligned} \dot{e} &= \left(\bar{A} - \bar{F}(L \otimes I_n)\right) e + \bar{E}d, \\ \eta &= (I_N \otimes H)e, \end{aligned} \tag{7}$$

where $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of the communication graph. The impulse response of the system (7) from the disturbance d to the output estimation error η is equal to

$$T_d(t) = (I_N \otimes H)e^{(\bar{A} - \bar{F}(L \otimes I_n))t}\bar{E}.$$

We introduce the global $\mathcal{H}_2$ cost functional

$$J = \int_0^\infty \text{tr}\left[T_d^\top(t) T_d(t)\right] dt. \tag{8}$$

The $\mathcal{H}_2$ optimal distributed filtering problem is then the problem of minimizing the $\mathcal{H}_2$ cost functional (8) over all distributed filters (6) such that the global error system (7) is internally stable. Note that (8) is a function of the local gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$.

Unfortunately, due to the particular form of (6), this optimization problem is, in general, non-convex and it is unclear whether a closed-form solution exists. Therefore, instead of trying to find an *optimal* solution, we will address a version of this problem that only requires *suboptimality*. More concretely, we aim at designing a distributed filter such that the error system (7) is internally stable and the $\mathcal{H}_2$ performance (8) is smaller than an a priori given tolerance γ. In that case, we say that the distributed filter (6) is $\mathcal{H}_2$ γ-suboptimal:

Definition 4: Let $\gamma > 0$. The distributed filter (6) is called $\mathcal{H}_2$ γ-suboptimal if:

1) for all $i = 1, 2, \ldots, N$, whenever $d = 0$, we have that $\lim_{t\to\infty} \left(x(t) - w_i(t)\right) \to 0$ for all initial conditions on (5) and (6).
2) the associated performance (8) satisfies $J < \gamma$.

Correspondingly, the $\mathcal{H}_2$ suboptimal distributed filtering problem that we will address is the following:

Problem 1: Let $\gamma > 0$. For $i = 1, 2, \ldots, N$, find gain matrices $G_i \in \mathbb{R}^{n \times r_i}$ and $F_i \in \mathbb{R}^{n \times n}$ such that the distributed filter (6) is $\mathcal{H}_2$ γ-suboptimal.

In addition to the distributed filtering problem with $\mathcal{H}_2$ performance, in this paper we will also consider the version of this problem with $\mathcal{H}_\infty$ performance. Obviously, the transfer matrix of the system (7) from the disturbance d to the output estimation error η is equal to

$$T_d(s) = (I_N \otimes H)\left(sI_{nN} - (\bar{A} - \bar{F}(L \otimes I_n))\right)^{-1}\bar{E}.$$

The $\mathcal{H}_\infty$ performance of the distributed filter (6) is given by the $\mathcal{H}_\infty$ norm $||T_d||_\infty$ of $T_d(s)$. The problem that we will then consider is to design a distributed filter (6) such that the error system (7) is internally stable and its $\mathcal{H}_\infty$ performance is smaller than an a priori given tolerance γ. In that case, we say that the distributed filter (6) is $\mathcal{H}_\infty$ γ-suboptimal:

Definition 5: Let $\gamma > 0$. The distributed filter (6) is called $\mathcal{H}_\infty$ γ-suboptimal if:

1) for all $i = 1, 2, \ldots, N$, whenever $d = 0$, we have that $\lim_{t\to\infty}\left(x(t) - w_i(t)\right) \to 0$ for all initial conditions on (5) and (6).
2) $||T_d||_\infty < \gamma$.

Correspondingly, the $\mathcal{H}_\infty$ suboptimal distributed filtering problem that we will address is the following:

Problem 2: Let $\gamma > 0$. For $i = 1, 2, \ldots, N$, find gain matrices $G_i \in \mathbb{R}^{n\times r_i}$ and $F_i \in \mathbb{R}^{n\times n}$ such that the distributed filter (6) is $\mathcal{H}_\infty$ γ-suboptimal.

Remark 6: Note that in Problems 1 and 2, our aim is to design a distributed filter that estimates the entire system state, while guaranteeing an upper bound on the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the external disturbance d to the estimation error $z - \zeta_i$. An alternative problem would be to replace estimation of the entire state x by only estimating the output $z = Hx$. Observers achieving this are often called *functional observers*, see [32], [33].

IV. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Suboptimal Distributed Filter Design

In this section, we will address Problems 1 and 2 introduced above and provide design methods for obtaining suboptimal distributed filters.

As we have explained before, the ith local filter (6) receives only a certain portion of the measured output, namely,

$$y_i = C_i x + D_i d, \quad i = 1, 2, \ldots, N.$$

In order to proceed, we first apply orthogonal transformations to the pairs (C_i, A). For $i = 1, 2, \ldots, N$, let T_i be an orthogonal matrix such that the pair (C_i, A) is transformed into the detectability decomposition form

$$T_i^\top A T_i = \begin{pmatrix} A_{i11} & 0 \\ A_{i21} & A_{i22} \end{pmatrix}, \quad C_i T_i = \begin{pmatrix} C_{i1} & 0 \end{pmatrix}, \tag{9}$$

where $A_{i11} \in \mathbb{R}^{v_i\times v_i}$, $A_{i21} \in \mathbb{R}^{(n-v_i)\times v_i}$, $A_{i22} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $C_{i1} \in \mathbb{R}^{r_i\times v_i}$ and the pair (C_{i1}, A_{i11}) is detectable. The integer v_i is equal to the dimension of the othogonal complement of the undetectable subspace of the pair (C_i, A). Accordingly, partition

$$T_i^\top E = \begin{pmatrix} E_{i1} \\ E_{i2} \end{pmatrix}, \quad HT_i = \begin{pmatrix} H_{i1} & H_{i2} \end{pmatrix}, \tag{10}$$

where $E_{i1} \in \mathbb{R}^{v_i\times q}$, $E_{i2} \in \mathbb{R}^{(n-v_i)\times q}$, $H_{i1} \in \mathbb{R}^{p\times v_i}$ and $H_{i1} \in \mathbb{R}^{p\times(n-v_i)}$.

Using the fact that (C_{i1}, A_{i11}) is detectable, let Q_{i1} be any positive definite solution to

$$A_{i11}Q_{i1} + Q_{i1}A_{i11}^\top - Q_{i1}C_{i1}^\top C_{i1}Q_{i1} < 0. \tag{11}$$

Then, by defining

$$G_{i1} := Q_{i1}C_{i1}^\top, \tag{12}$$

the matrix $A_{i11} - G_{i1}C_{i1}$ is Hurwitz.

In the sequel, we will make use of the transformed matrices (9) and (10) and the gain matrix (12) to obtain filter gains that solve Problems 1 and 2. Before presenting the main results of this paper, we will first provide a lemma that will be essential for later use. This lemma is a generalization of [8, Lemma 4], and connects the Laplacian matrix of the communication graph with detectability properties of the system (5).

Lemma 7: Assume that the pair (C, A) is detectable and that none of the pairs (C_i, A) is detectable. Suppose that the communication graph is a strongly connected directed graph with associated Laplacian matrix L. Let $\mathcal{L} := \Theta L + L^\top\Theta$, where Θ is defined as in Lemma 1. Define $T := \mathrm{blockdiag}(T_i) \in \mathbb{R}^{nN\times nN}$, where the T_i are the orthogonal matrices introduced in (9) and (10). Let $m_i > 0$ and

$$M_i := \begin{pmatrix} m_i I_{v_i} & 0 \\ 0 & 0_{n-v_i} \end{pmatrix}, \quad i = 1, 2, \ldots, N.$$

Define $M := \mathrm{blockdiag}(M_i)$. Then,

$$T^\top(\mathcal{L} \otimes I_n)T + M > 0. \tag{13}$$

The proof of Lemma 7 can be given by adapting the proof of [8, Lemma 4], replacing the observability decomposition by the detectability decomposition. We omit the details here.

In the next two subsections, we will deal with the design of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively.

A. $\mathcal{H}_2$ Suboptimal Distributed Filter Design

In this subsection, we will provide a design method for obtaining $\mathcal{H}_2$ suboptimal distributed filters. More specifically, we aim at finding a distributed filter such that the global error system (7) is stable and the associated $\mathcal{H}_2$ performance (8) is less than an a priori given tolerance.

The next lemma expresses the existence of suitable gain matrices F_i and G_i, $i = 1, 2, \ldots, N$ in terms of solvability of LMI's.

Lemma 8: Assume that the pair (C, A) is detectable and that none of the pairs (C_i, A) is detectable. Suppose that the communication graph is a strongly connected directed graph with associated Laplacian matrix L. Let $\gamma > 0$. Let the matrices T, M and $\mathcal{L}$ be as introduced in Lemma 7. Let $\epsilon > 0$ be such that

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{14}$$

Let G_{i1} be as defined in (12). For $i = 1, 2, \ldots, N$, assume there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying

$$\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} + \kappa(m_i - \epsilon)I_{v_i} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & \Psi_i \end{pmatrix} < 0 \tag{15}$$

and

$$\sum_{i=1}^{N} \mathrm{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top P_{i2}E_{i2}\right] < \gamma, \tag{16}$$

where

$$\Phi_i := A_{i11}^\top P_{i1} + P_{i1}A_{i11} - C_{i1}^\top G_{i1}^\top P_{i1} - P_{i1}G_{i1}C_{i1}, \tag{17}$$

$$\Psi_i := P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i}. \tag{18}$$

For $i = 1, 2, \ldots, N$, define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & P_{i2}^{-1} \end{pmatrix} T_i^\top \tag{19}$$

and

$$G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{20}$$

Then the corresponding distributed filter (6) is $\mathcal{H}_2$ γ-suboptimal.

Proof: First, it follows from (13) in Lemma 7 that there exists $\epsilon > 0$ such that (14) holds. Next, note that (15) is equivalent to

$$\text{blockdiag}\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} \end{pmatrix} + \kappa(M - \epsilon I_{nN}) < 0. \quad (21)$$

Using (14), it follows from (21) that

$$\text{blockdiag}\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} \end{pmatrix} - \kappa T^\top(\mathcal{L} \otimes I_n)T < 0. \quad (22)$$

Let

$$P := \text{blockdiag}(P_i), \quad P_i := T_i \begin{pmatrix} P_{i1} & 0 \\ 0 & P_{i2} \end{pmatrix} T_i^\top. \quad (23)$$

Clearly, $P > 0$. By using (19), (20), (23), (9) and (10), then (22) holds if and only if

$$\bar{A}^\top P + P\bar{A} - (L^\top \otimes I_n)\bar{F}^\top P - P\bar{F}(L \otimes I_n) + I_N \otimes H^\top H < 0 \quad (24)$$

holds, where $\bar{F} := \text{blockdiag}(F_i)$ and F_i is defined by (19). Therefore, there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ such that (15) holds for $i = 1, 2, \ldots, N$ if and only if there exists $P > 0$ of the form (23) such that (24) holds. Since the solutions of (15) also satisfy (16), we obtain

$$\text{tr}\left(\bar{E}^\top P \bar{E}\right) < \gamma. \quad (25)$$

Finally, since (24) and (25) have a solution $P > 0$, it follows from Lemma 2 that the error system (7) is internally stable and $J < \gamma$. Thus the distributed filter (6) with (20) and (19) is $\mathcal{H}_2$ γ-suboptimal. ∎

Remark 9: In Lemma 8, the choice of the parameters $m_i > 0$ is arbitrary. The parameter $\epsilon > 0$ should be chosen sufficiently small so that (14) holds. The gain G_i is defined by (20). Then, of course, the question arises: for chosen $m_i > 0$, $\epsilon > 0$ and G_i, how can we find the smallest $\gamma > 0$ such that the corresponding distributed filter (6) is $\mathcal{H}_2$ γ-suboptimal? This requires to find the smallest γ such that the LMI's (15) and (16) are solvable. It is well known that this can be done by using a standard bisection algorithm, see e.g. [30, page 115].

Remark 10: Lemma 8 states that if there exist solutions $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying (15) and (16), then the distributed filter (6) with gain matrices (19) and (20) is $\mathcal{H}_2$ γ-suboptimal. There, the inequality (16) is a global condition for checking suboptimality. In fact, such suboptimality condition can also be checked locally. Indeed, if for $i = 1, 2, \ldots, N$ there exist solutions satisfying (15) and

$$\text{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top P_{i2}E_{i2}\right] < \frac{\gamma}{N},$$

then the corresponding distributed filter (6) with (19) and (20) is $\mathcal{H}_2$ γ-suboptimal.

Lemma 8 provides a condition for the existence of suitable gain matrices F_i and G_i in terms of solvability of LMI's. In the next theorem, we show that, in fact, the LMI's (15) in Lemma 8 always have solutions. In fact, we can take P_{i2} to be the identity matrix of dimension $n - v_i$ and P_{i1} to be the unique solution of a given Lyapunov equation. In this way we obtain the following conceptual algorithm for computing suitable gain matrices.

Theorem 11: Assume that the pair (C, A) is detectable and that none of the pairs (C_i, A) is detectable. Suppose that the communication graph is a strongly connected directed graph with associated Laplacian matrix L. Let $\gamma > 0$. Then an $\mathcal{H}_2$ γ-suboptimal distributed filter of the form (6) is obtained as follows:

1) Compute $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ with $\theta_i > 0$ such that $\theta L = 0$ and $\theta \mathbf{1}_N = N$.

Then, for $i = 1, 2, \ldots, N$:

2) Compute orthogonal matrices T_i that put A, C_i, E and H into the form (9) and (10).
3) Take $m_i = 1$ and compute $\epsilon > 0$ such that

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \quad (26)$$

4) Compute $Q_{i1} > 0$ satisfying (11). Define $G_{i1} := Q_{i1}C_{i1}^\top$.
5) Take $\kappa > 0$ sufficiently large such that

$$A_{i22} + A_{i22}^\top + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i} + \frac{1}{\kappa\epsilon}(A_{i21} + H_{i2}^\top H_{i1})(A_{i21} + H_{i2}^\top H_{i1})^\top < 0. \quad (27)$$

6) Compute $P_{i1} > 0$ satisfying the Lyapunov equation

$$(A_{i11} - G_{i1}C_{i1})^\top P_{i1} + P_{i1}(A_{i11} - G_{i1}C_{i1}) + H_{i1}^\top H_{i1} + \kappa I_{v_i} = 0. \quad (28)$$

7) Define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & I_{n-v_i} \end{pmatrix} T_i^\top, \quad G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \quad (29)$$

Then for all $\gamma > 0$ satisfying

$$\sum_{i=1}^N \text{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top E_{i2}\right] < \gamma, \quad (30)$$

the corresponding distributed filter (6) with gain matrices (29) is $\mathcal{H}_2$ γ-suboptimal.

Proof: Using Lemma 7, by choosing $m_i = 1$ for $i = 1, 2, \ldots, N$, there exists $\epsilon > 0$ such that (26) holds. Next, for $i = 1, 2, \ldots, N$, we choose $\kappa > 0$ sufficiently large such that (27) holds . Since Q_{i1} is a positive definite solution of (11) and $G_{i1} := Q_{i1}C_{i1}^\top$, then the matrix $A_{i11} - G_{i1}C_{i1}$ is Hurwitz. Consequently, for $i = 1, 2, \ldots, N$, the Lyapunov equation (28) has unique solution $P_{i1} > 0$. Since (27) holds and $-\kappa\epsilon I_{v_i} < 0$, by using the Schur complement, we obtain

$$\begin{pmatrix} -\kappa\epsilon I_{v_i} & A_{i21}^\top + H_{i1}^\top H_{i2} \\ A_{i21} + H_{i2}^\top H_{i1} & \tilde{\Psi}_i \end{pmatrix} < 0, \quad i = 1, 2, \ldots, N, \quad (31)$$

where $\tilde{\Psi}_i := A_{i22} + A_{i22}^\top + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i}$. Using (28) and $P_{i2} = I_{n-v_i}$, it then follows that (15) holds.

On the other hand, by taking $P_{i2} = I_{n-v_i}$ in (16), we obtain (30). It then follows from Lemma 8 that the corresponding distributed filter is $\mathcal{H}_2$ γ-suboptimal. ∎

Remark 12: Note that, in step 1) of Theorem 11, we need to compute the left eigenvector θ of the Laplacian matrix corresponding to the eigenvalue 0. This requires so-called global information on the communication graph. This dependency on global information can be removed using algorithms that compute left eigenvectors of the Laplacian matrix in a distributed fashion, see e.g. [34] or [35]. On the other hand, in step 3) we need to compute ϵ. To do so, we need knowledge of the orthogonal matrices T_i, the matrix M and the Laplacian matrix $\mathcal{L}$, which is global information. Also in step 5), we need to find one κ that satisfy (27) for $i = 1, 2, \ldots, N$. Note that, however, we can always take $\epsilon > 0$ sufficiently small and $\kappa > 0$ sufficiently large such that (26) and (27) hold, respectively. This might however lead to an achievable tolerance γ that is very large, giving poor suboptimality of the corresponding distributed filter.

In general, the computation of our suboptimal filters requires global information, so cannot be performed in a decentralized fashion. This is in contrast with the decentralized computation of distributed state observers as described in [36].

Remark 13: In case of large N, in Theorem 11, in step 1) it might be difficult to directly compute a left eigenvector. However, as mentioned in Remark 12, there exist algorithms for computing left eigenvectors of Laplacian matrices in a distributed fashion. In step 3), it might be difficult to compute a suitable $\epsilon > 0$ that satisfies (26) . However, it obviously suffices to take $\epsilon > 0$ sufficiently small. The remaining matrix operations need to be performed N times.

B. $\mathcal{H}_\infty$ Suboptimal Distributed Filter Design

In this subsection, we will provide a method for obtaining $\mathcal{H}_\infty$ suboptimal distributed filters. More concretely, we aim at finding, for a given tolerance $\gamma > 0$, a distributed filter such that the global error system (7) is stable and $||T_d||_\infty < \gamma$.

The next lemma expresses the existence of suitable gain matrices F_i and G_i, $i = 1, 2, \ldots, N$ in terms of solvability of N nonlinear matrix inequalities.

Lemma 14: Assume that the pair (C, A) is detectable and that none of the pairs (C_i, A) is detectable. Suppose that the communication graph is a strongly connected directed graph with associated Laplacian matrix L. Let $\gamma > 0$. Let the matrices T, M and $\mathcal{L}$ be as introduced in Lemma 7. Let $\epsilon > 0$ be such that

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{33}$$

Let G_{i1} be as defined in (12). For $i = 1, 2, \ldots, N$, assume there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying

$$\begin{pmatrix} \Phi_i + \kappa(m_i - \epsilon)I_{v_i} & \Omega_i \\ \Omega_i^\top & \Psi_i - \kappa\epsilon I_{n-v_i} \end{pmatrix} < 0, \tag{34}$$

where

$$\begin{aligned} \Phi_i = {} & (A_{i11} - G_{i1}^\top C_{i1})^\top P_{i1} + P_{i1}(A_{i11} - G_{i1}^\top C_{i1}) \\ & + \frac{1}{\gamma^2} P_{i1}(E_{i1} - G_{i1}D_i)(E_{i1} - G_{i1}D_i)^\top P_{i1} + H_{i1}^\top H_{i1}, \end{aligned} \tag{35}$$

$$\Omega_i = A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} + \frac{1}{\gamma^2} P_{i1}(E_{i1} - G_{i1}D_i)E_{i2}^\top P_{i2}, \tag{36}$$

$$\Psi_i = P_{i2}A_{i22} + A_{i22}^\top P_{i2} + \frac{1}{\gamma^2} P_{i2}E_{i2}E_{i2}^\top P_{i2} + H_{i2}^\top H_{i2}.$$

For $i = 1, 2, \ldots, N$, define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & P_{i2}^{-1} \end{pmatrix} T_i^\top \tag{37}$$

and

$$G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{38}$$

Then the corresponding distributed filter (6) is $\mathcal{H}_\infty$ γ-suboptimal.

Proof: First, it follows from (13) in Lemma 7 that there exists $\epsilon > 0$ such that (14) holds. Next, note that (34) is equivalent to

$$\text{blockdiag}\begin{pmatrix} \Phi_i & \Omega_i \\ \Omega_i^\top & \Psi_i \end{pmatrix} + \kappa(M - \epsilon I_{nN}) < 0. \tag{39}$$

Using (14), it then follows from (39) that

$$\text{blockdiag}\begin{pmatrix} \Phi_i & \Omega_i \\ \Omega_i^\top & \Psi_i \end{pmatrix} - \kappa T^\top(\mathcal{L} \otimes I_n)T < 0. \tag{40}$$

Let

$$P := \text{blockdiag}(P_i), \quad P_i := T_i \begin{pmatrix} P_{i1} & 0 \\ 0 & P_{i2} \end{pmatrix} T_i^\top. \tag{41}$$

Clearly, $P > 0$. By using (37), (38), (41), (9) and (10), then (40) holds if and only if

$$\begin{aligned} & \bar{A}^\top P + P\bar{A} - (L^\top \otimes I_n)\bar{F}^\top P - P\bar{F}(L \otimes I_n) \\ & + \frac{1}{\gamma^2} P\bar{E}\bar{E}^\top P + I_N \otimes H^\top H < 0 \end{aligned} \tag{42}$$

holds, where $\bar{F} := \text{blockdiag}(F_i)$ and F_i is defined by (37). Therefore, there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ such that (34) holds for $i = 1, 2, \ldots, N$ if and only if there exists $P > 0$ of the form (23) such that (42) holds. Finally, since (42) has a solution $P > 0$, it follows from Lemma 3 that the error system (7) is internally stable and $||T_d||_\infty < \gamma$. Thus the distributed filter (6) with (38) and (37) is $\mathcal{H}_\infty$ γ-suboptimal. ■

Lemma 14 provides a condition for the existence of suitable gain matrices F_i and G_i in terms of solvability of the nonlinear matrix inequalities (34). However, these inequalities are not LMI's. However, by using suitable Schur complements, we can transform the inequalities (34) into LMI's. In this way we obtain the following conceptual algorithm for computing suitable gain matrices.

Theorem 15: Assume that the pair (C, A) is detectable and that none of the pairs (C_i, A) is detectable. Suppose that the communication graph is a strongly connected directed graph with associated Laplacian matrix L. Let $\gamma > 0$. Then an $\mathcal{H}_\infty$ suboptimal distributed filter of the form (6) is obtained as follows:

1) Compute $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ with $\theta_i > 0$ such that $\theta L = 0$ and $\theta \mathbf{1}_N = N$.

For $i = 1, 2, \ldots, N$:

2) Compute an orthogonal matrix T_i that puts A, C_i, E and H into the form (9) and (10).
3) Take arbitrary $m_i > 0$ and compute $\epsilon > 0$ such that

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{43}$$

4) Compute $Q_{i1} > 0$ satisfying (11). Define $G_{i1} := Q_{i1}C_{i1}^\top$.
5) Compute $P_{i1} > 0$, $P_{i2} > 0$ and $\kappa > 0$ such that the inequality (32) (see next page) holds.
6) Define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & I_{n-v_i} \end{pmatrix} T_i^\top, \quad G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{44}$$

Then the corresponding distributed filter (6) is $\mathcal{H}_\infty$ γ-suboptimal.

Proof: By taking the appropriate Schur complements in (32), it follows that (32) hold if and only if (34) hold. The rest follows from Lemma 14. ■

We conclude this section by noting that remarks similar to Remark 9, Remark 12 and Remark 13 hold in the $\mathcal{H}_\infty$ case.

V. Simulation Example

In this section, we will use a simulation example borrowed from [8] to illustrate the conceptual algorithm in Theorem 11 for designing $\mathcal{H}_2$ suboptimal distributed filters. Consider the linear system (5) with

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -8 & 1 & -1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^\top,$$

$$C = \begin{pmatrix} C_1 \\ \hline C_2 \\ \hline C_3 \\ \hline C_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ \hline 2 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 2 & 0 & 4 & 0 & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} D_1 \\ \hline D_2 \\ \hline D_3 \\ \hline D_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 \end{pmatrix}.$$

$$\begin{pmatrix} (A_{i11}-G_{i1}^{\top}C_{i1})^{\top}P_{i1}+P_{i1}(A_{i11}-G_{i1}^{\top}C_{i1})+\kappa(m_i-\epsilon)I_{v_i}+H_{i1}^{\top}H_{i1} & A_{i21}^{\top}P_{i2}+H_{i1}^{\top}H_{i2} & P_{i1}(E_{i1}-G_{i1}D_i) \\ P_{i2}A_{i21}+H_{i2}^{\top}H_{i1} & P_{i2}A_{i22}+A_{i22}^{\top}P_{i2} & P_{i2}E_{i2} \\ (E_{i1}-G_{i1}D_i)^{\top}P_{i1} & E_{i2}^{\top}P_{i2} & -\gamma^2 I_q \end{pmatrix} < 0. \tag{32}$$

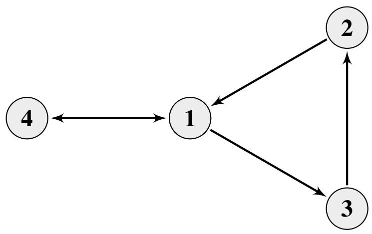

Fig. 1: The communication graph between the local filters.

The system (5) is monitored by four local filters, and each local filter acquires a portion of the measured output y, namely, $y_i = C_i x + D_i d$, for $i = 1,2,3,4$. The pair (C, A) is detectable, but none of the pairs (C_i, A) is detectable.

We assume the four local filters to be of the form (6). The communication graph between the four local filters is depicted in Figure 1. The graph is a strongly connected unweighted directed graph with associated Laplacian matrix L. The normalized left eigenvector θ of L associated with eigenvalue 0 is computed to be $\theta = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$.

Next, for $i = 1,2,3,4$, we compute an orthogonal matrix T_i such that the matrices A, C_i, E and H are transformed into the form (9) and (10). For $i = 1,2,3,4$, we take $m_i = 1$. We also compute that for $\epsilon = 0.0398$, the inequality (26) holds. Subsequently, for $i = 1,2,3,4$, we solve (27) and compute $\kappa = 198.2$. Following the steps in Theorem 11, we also compute gain matrices F_i and G_i. We then compute the upper bound

$$\sum_{i=1}^{4} \mathrm{tr}\left[(E_{i1}^{\top} - D_i^{\top}G_{i1}^{\top})P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^{\top}E_{i2}\right] = 71.8728.$$

Thus, for all $\gamma > 71.8728$, the distributed filter (6) with gain matrices F_i and G_i is $\mathcal{H}_2$ γ-suboptimal, and the $\mathcal{H}_2$ norm of the associated global error system (7) is guaranteed to satisfy

$$||T_{\mathrm{df}}||_{\mathcal{H}_2} < \sqrt{71.8728} = 8.4778.$$

Using Matlab, the actual $\mathcal{H}_2$ norm of the associated global error system (7) is computed to be

$$||T_{\mathrm{df}}||_{\mathcal{H}_2} = 1.5212.$$

As an illustration, we have compared the performance of our distributed filter with that of the centralized optimal filter (which can be computed using [31, page 256]) and those of the distributed observers proposed in [8], [36] (with a decay rate at least $\alpha = 0.5$). The computed H_2 performances of the associated error systems can be found in TABLE I.

TABLE I

	Centralized filter	Distributed filter	Distributed observer 1 ([8])	Distributed observer 2 ([36])
$\mathcal{H}_2$−norm	$\lVert T_{\mathrm{cf}}\rVert_{\mathcal{H}_2} = 0.5443$	$\lVert T_{\mathrm{df}}\rVert_{\mathcal{H}_2} = 1.5212$	$\lVert T_{\mathrm{do1}}\rVert_{\mathcal{H}_2} = 5.4517$	$\lVert T_{\mathrm{do2}}\rVert_{\mathcal{H}_2} = 5.5774$

It can be seen that, as expected, the $\mathcal{H}_2$ norm of the global error system using our distributed filter ($||T_{\mathrm{df}}||_{\mathcal{H}_2} = 1.5212$) is larger than the one using the (unconstrained) centralized optimal filter ($||T_{\mathrm{cf}}||_{\mathcal{H}_2} = 0.5443$). Interestingly however, the performance of the distributed filter (consisting of four local filters) is smaller than four times the performance of the centralized filter, in the sense that $||T_{\mathrm{df}}||_{\mathcal{H}_2} < 4 \times ||T_{\mathrm{cf}}||_{\mathcal{H}_2}$. Next, note that the performances of the distributed observers in [8], [36] are not comparable to that of our distributed filter, in the sense that their associated $\mathcal{H}_2$ norms are much bigger than that of our distributed filter. This shows that our proposed distributed filter indeed outperforms these two distributed observers.

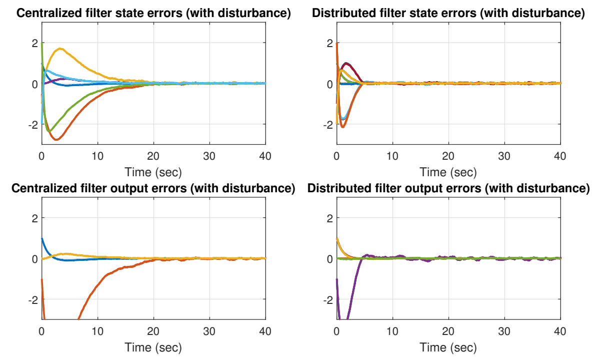

Fig. 2: Left two plots: trajectories of the state and output error components using the centralized filter (with disturbance). Right two plots: trajectories of the state and output error components using the proposed distributed filter (with disturbance).

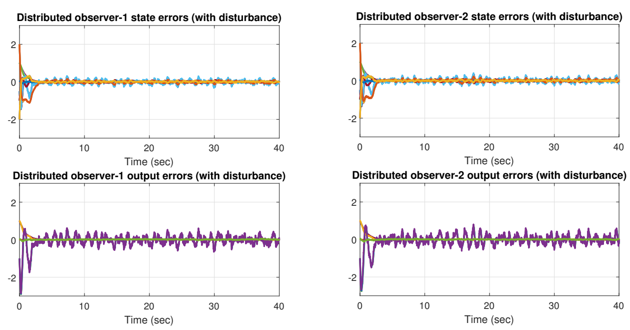

Fig. 3: Left two plots: trajectories of the state and output error components using the distributed observer in [8] (with disturbance). Right two plots: trajectories of the state and output error components using the distributed observer in [36] (with disturbance).

To further illustrate the performance of our distributed suboptimal filter, we have also plotted the state and output error trajectories of the centralized optimal filter, our distributed filter and two other distributed observers, respectively, in presence of external disturbances. In particular, we use matlab command `20*rand()` to generate a random disturbance signal d. As an example, we take the initial state of the system (5) to be $x_0 = \begin{pmatrix} 1 & -0.5 & -1 & 0 & 2 & -2 \end{pmatrix}^{\top}$ and the initial state of the distributed filter to be zero. The state and output error trajectories of the centralized filter and our proposed distributed filter are plotted in Figure 2. The state and output error trajectories of two other distributed observers are plotted in Figure 3. It can be seen that our proposed distributed filter has a comparable performance comparing with the centralized optimal one, in the sense of disturbance tolerance. It also validates that our proposed distributed filter indeed outperforms two other distributed observers.

VI. Conclusion

In this paper, we have studied the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problem for linear systems. We have established conditions for the existence of suitable filter gains. These are expressed in terms of solvability of LMI's. Based on these conditions, we have provided conceptual algorithms for obtaining the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively. The computation of these distributed filters requires centralized computation, i.e. global information is needed.

As a possibility for future research, we mention the extension of the results in this paper to the case that the filter gains need to be computed in a decentralized fashion, see for example [36]. It would also be interesting to extend the results in this paper to linear time-varying systems, or to the case that the communication graph is no longer strongly connected, or can even be a switching graph. In this paper, we have designed distributed filters to estimate the entire system state. Another potential research direction would be to design distributed functional observers/filters that directly estimate the system output instead of the entire system state [33].

References

[1] Y. Huang, S. Werner, J. Huang, N. Kashyap, and V. Gupta, "State estimation in electric power grids: meeting new challenges presented by the requirements of the future grid," *IEEE Signal Processing Magazine*, vol. 29, no. 5, pp. 33–43, 2012.

[2] R. Vadigepalli and F. J. Doyle, "A distributed state estimation and control algorithm for plantwide processes," *IEEE Transactions on Control Systems Technology*, vol. 11, no. 1, pp. 119–127, 2003.

[3] R. Olfati-Saber and J. S. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 6698–6703.

[4] S. Park and N. C. Martins, "Design of distributed LTI observers for state omniscience," *IEEE Transactions on Automatic Control*, vol. 62, no. 2, pp. 561–576, 2017.

[5] Shih-Ho Wang and E. Davison, "On the stabilization of decentralized control systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, 1973.

[6] L. Wang and A. S. Morse, "A distributed observer for a time-invariant linear system," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2123–2130, 2018.

[7] L. Wang, A. S. Morse, D. Fullmer, and J. Liu, "A hybrid observer for a distributed linear system with a changing neighbor graph," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 2017, pp. 1024–1029.

[8] W. Han, H. L. Trentelman, Z. Wang, and Y. Shen, "A simple approach to distributed observer design for linear systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 1, pp. 329–336, 2019.

[9] A. Mitra and S. Sundaram, "Distributed observers for LTI systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3689–3704, 2018.

[10] T. Kim, H. Shim, and D. D. Cho, "Distributed Luenberger observer design," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, 2016, pp. 6928–6933.

[11] H. Silm, R. Ushirobira, D. Efimov, E. Fridman, J.-P. Richard, and W. Michiels, "Distributed observers with time-varying delays," *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.

[12] H. Basu and S. Y. Yoon, "Distributed state estimation by a network of observers under communication and measurement delays," *Systems & Control Letters*, vol. 133, p. 104554, 2019.

[13] K. Liu, J. Lü, and Z. Lin, "Design of distributed observers in the presence of arbitrarily large communication delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 9, pp. 4447–4461, 2018.

[14] H. Silm, R. Ushirobira, D. Efimov, J.-P. Richard, and W. Michiels, "A note on distributed finite-time observers," *IEEE Transactions on Automatic Control*, vol. 64, no. 2, pp. 759–766, 2019.

[15] H. Silm, D. Efimov, W. Michiels, R. Ushirobira, and J.-P. Richard, "A simple finite-time distributed observer design for linear time-invariant systems," *Systems & Control Letters*, vol. 141, p. 104707, 2020.

[16] R. Ortega, E. Nuno, and A. Bobtsov, "Distributed observers for lti systems with finite convergence time: A parameter estimation-based approach," *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.

[17] A. Mitra and S. Sundaram, "Byzantine-resilient distributed observers for LTI systems," *Automatica*, vol. 108, p. 108487, 2019.

[18] R. Olfati-Saber, "Distributed Kalman filter with embedded consensus filters," in *Proceedings of the 44th IEEE Conference on Decision and Control*, 2005, pp. 8179–8184.

[19] ——, "Kalman-consensus filter : optimality, stability, and performance," in *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, 2009, pp. 7036–7042.

[20] ——, "Distributed Kalman filtering for sensor networks," in *2007 46th IEEE Conference on Decision and Control*, 2007, pp. 5492–5498.

[21] J. Sun, C. Zhang, and B. Guo, "Distributed fusion robust H_2 and H_∞ filtering for uncertain discrete-time systems," in *2010 8th World Congress on Intelligent Control and Automation*, 2010, pp. 806–810.

[22] V. Ugrinovskii, "Distributed robust filtering with H_∞ consensus of estimates," *Automatica*, vol. 47, no. 1, pp. 1–13, 2011.

[23] ——, "Distributed robust estimation over randomly switching networks using H_∞ consensus," *Automatica*, vol. 49, no. 1, pp. 160–168, 2013.

[24] J. Kim, H. Shim, and J. Wu, "On distributed optimal Kalman-Bucy filtering by averaging dynamics of heterogeneous agents," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, 2016, pp. 6309–6314.

[25] J. Kim, J. Yang, H. Shim, J. Kim, and J. H. Seo, "Robustness of synchronization of heterogeneous agents by strong coupling and a large number of agents," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3096–3102, 2016.

[26] J. Mei, W. Ren, and J. Chen, "Distributed consensus of second-order multi-agent systems with heterogeneous unknown inertias and control gains under a directed graph," *IEEE Transactions on Automatic Control*, vol. 61, no. 8, pp. 2019–2034, 2016.

[27] W. Wonham, *Linear Multivariable Control: A Geometric Approach*. Springer-Verlag, New York, 1985.

[28] T. Iwasaki, R. Skelton, and J. Geromel, "Linear quadratic suboptimal control with static output feedback," *Systems & Control Letters*, vol. 23, no. 6, pp. 421–430, 1994.

[29] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, 1997.

[30] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey, NJ, USA: Prentice-Hall, 1996.

[31] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control Theory for Linear Systems*. Springer Verlag, 2001.

[32] M. Darouach, "Existence and design of functional observers for linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 940–943, 2000.

[33] A. Mitra and S. Sundaram, "Distributed functional observers for LTI systems," in *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, 2017, pp. 3519–3524.

[34] T. Charalambous, M. G. Rabbat, M. Johansson, and C. N. Hadjicostis, "Distributed finite-time computation of digraph parameters: left-eigenvector, out-degree and spectrum," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 2, pp. 137–148, 2016.

[35] A. Gusrialdi and Z. Qu, "Distributed estimation of all the eigenvalues and eigenvectors of matrices associated with strongly connected digraphs," *IEEE Control Systems Letters*, vol. 1, no. 2, pp. 328–333, 2017.

[36] T. Kim, C. Lee, and H. Shim, "Completely decentralized design of distributed observer for linear systems," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4664–4678, 2020.