Markov Processes Relat. Fields **20**, 287–328 (2014)

Markov
Processes
and
Related Fields

A Conditional Quenched CLT for Random Walks Among Random Conductances on $\mathbb{Z}^d$

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Received February 22, 2013, revised October 22, 2013

Abstract. Consider a random walk among random conductances on $\mathbb{Z}^d$ with $d \geq 2$. We study the quenched limit law under the usual diffusive scaling of the random walk conditioned to have its first coordinate positive. We show that the conditional limit law is a linear transformation of the product law of a Brownian meander and a $(d-1)$-dimensional Brownian motion.

Keywords: random conductance model, uniform heat kernel bounds, Brownian meander, reversibility

AMS Subject Classification: 60J10, 60K37

1. Introduction and results

In this paper we study random walks on a d-dimensional integer lattice with *random conductances*. One can briefly describe the model in the following way: initially, weights (i.e., some nonnegative numbers) are attached to the edges of the lattice at random. The transition probabilities are then defined to be proportional to the weights, thus obtaining a reversible Markov chain; due to a well-known correspondence between reversible Markov chains and electric networks, the weights are also called conductances. We refer to the collection of all conductances as “environment”. This model attracted considerable attention recently, and, in particular, quenched (i.e., for fixed environment) functional central limit theorems and heat kernel estimates were obtained in rather general situations, see e.g. [2, 3, 6, 16] and references therein. We also refer to the survey paper [5]. To prove the quenched functional CLT, one usually uses the so-called corrector approach, described in the following way. First, one constructs an auxiliary random field (which depends only on the environment), with the following property: the sum of the corrector and the random walk is a martingale, for which it is not difficult to show the CLT. Then, using the Ergodic Theorem, one shows that the corrector is likely to be small in comparison to the random walk itself.

While this approach has been quite fruitful, it also has its limitations, mainly due to the fact that the construction of the corrector is not very explicit. For example, it was understood only quite recently how to prove the quenched CLT for the random walk with i.i.d. conductances in *half-space*, see [7, 19]. It is therefore important to go beyond the usual setup, proving other types of limit laws. In this paper, we continue the line of research of [11] and [12] (which were, by their turn, mainly motivated by [8, 9]), where a one-dimensional model with random conductances (but with unbounded jumps) was considered.

We now define the model formally. For $x,\ y \in \mathbb{Z}^d$ with $d \geq 2$, we write $x \sim y$ if x and y are neighbors in the lattice $\mathbb{Z}^d$ and we let $\mathbb{B}_d$ be the set of unordered nearest-neighbor pairs (x, y) of $\mathbb{Z}^d$. Let $(\omega_b)_{b\in\mathbb{B}_d}$ be non-negative random variables; $\mathbb{P}$ stands for the law of this family. We assume that $\mathbb{P}$ is stationary and ergodic with respect to the family of shifts $(\theta_x, x \in \mathbb{Z}^d)$. The quantity ω_b is usually called the *conductance* of the edge b. The collection of all conductances $\omega = (\omega_b)_{b\in\mathbb{B}_d}$ is called the *environment*. If $x \sim y$, we will also write $\omega_{x,y}$ to refer to the conductance between x and y. For a particular realization ω of our environment, we define $\pi_x = \sum_{y\sim x} \omega_{x,y}$. Given that $\pi_x \in (0, \infty)$ for all $x \in \mathbb{Z}^d$ (which is $\mathbb{P}$-a.s. the case by Condition UE below), the random walk X in the environment ω is defined through its transition probabilities

$$p_\omega(x, y) = \begin{cases} \dfrac{\omega_{x,y}}{\pi_x}, & \text{if } y \sim x, \\ 0, & \text{otherwise,} \end{cases}$$

that is, if $\mathtt{P}^x_\omega$ is the quenched law of the random walk starting from x, we have

$$\mathtt{P}^x_\omega[X(0) = x] = 1, \quad \mathtt{P}^x_\omega[X(k + 1) = z \mid X(k) = y] = p_\omega(y, z).$$

Clearly, this random walk is $\mathbb{P}$-a.s. reversible with the reversible measure $(\pi_x, x \in \mathbb{Z}^d)$. Also, we denote by $\mathtt{E}^x_\omega$ the quenched expectation for the process starting from x. When the random walk starts from 0, we use the shorter notations $\mathtt{P}_\omega, \mathtt{E}_\omega$.

In order to prove our results, we need to make the *uniform ellipticity* assumption on the environment:

Condition UE. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\kappa < \omega_{0,x} < \kappa^{-1}$ for $x \sim 0$.

For all $n \geq 1$, we define the continuous map $(Z^n(t), t \in [0,1])$ as the natural polygonal interpolation of the map $k/n \mapsto n^{-1/2}X(k)$. In other words

$$\sqrt{n}Z^n(t) = X(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor)X(\lfloor nt \rfloor + 1)$$

with $\lfloor \cdot \rfloor$ the integer part. Also, we denote by $W^{(d)} = (W_1, \ldots, W_d)$ the d-dimensional standard Brownian motion. Now, let us embed the graph $\mathbb{Z}^d$ in $\mathbb{R}^d$. Denote by $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ the canonical basis of $\mathbb{R}^d$ and by $x_1, \ldots, x_d$ the vector coordinates in $\mathbb{R}^d$. By Condition UE and as our environment is stationary and ergodic there exists an invertible linear transformation $D : \mathbb{R}^d \to \mathbb{R}^d$ letting the hyperplane $\{x_1 = 0\}$ invariant and such that the sequence $(DZ^n)_{n\geq 1}$ tends weakly to $W^{(d)}$. Indeed, by Condition UE and ergodicity of the environment, it is well known (cf. [5]) that $(Z^n)_{n\geq 1}$ tends weakly to a d-dimensional Brownian motion with a positive definite covariance matrix Σ. This implies that Σ has positive eigenvalues λ_i and is diagonalizable in an orthonormal basis. If the law of the environment is also invariant under the symmetries of $\mathbb{Z}^d$, it is known that $\Sigma = \sigma^2 I$ for some constant σ, where I is the identity matrix. Thus, there exists a rotation T such that $(TZ^n)_{n\geq 1}$ tends weakly to Brownian motion with diagonal covariance matrix $\Sigma' = (\lambda_i)_{1\leq i \leq d}$ in the basis $\mathcal{B}$. This implies that $((\Sigma')^{-1/2}TZ^n)_{n\geq 1}$ tends weakly to $W^{(d)}$. Finally, by some unitary transformation R, we can rotate the hyperplane $(\Sigma')^{-1/2}T\{x_1 = 0\}$ to make it coincide with the hyperplane $\{x_1 = 0\}$. Now, using the isotropy of $W^{(d)}$ we obtain that $(R(\Sigma')^{-1/2}TZ^n)_{n\geq 1}$ tends weakly to $W^{(d)}$. For convenience, in the rest of the paper, we will choose R such that $D\mathbf{e}_1 \cdot \mathbf{e}_1 > 0$. (R can also involve a reflection). In the case that the law of the environment is also invariant under the symmetries of $\mathbb{Z}^d$, then the last statement is true with $D = \sigma^{-1}I$ (where σ is from the quenched CLT).

Denoting $X = (X_1, \ldots, X_d)$ in the basis $\mathcal{B}$, we define

$$\hat{\tau} = \inf\{k \geq 1 : X_1(k) = 0\}$$

and

$$\Lambda_n = \{\hat{\tau} > n\} = \{X_1(k) > 0 \text{ for all } k = 1, \ldots, n\}.$$

Consider the conditional quenched probability measure $Q^n_\omega[\,\cdot\,] := \mathtt{P}_\omega[\,\cdot \mid \Lambda_n]$, for all $n \geq 1$. Denote by $C([0,1])$ the space of continuous functions from $[0,1]$ into $\mathbb{R}^d$ and by $\mathcal{B}_1$ the Borel σ-field on $C([0,1])$. For each n, the random map DZ^n induces a probability measure μ^n_ω on $(C[0,1], \mathcal{B}_1)$: for any $A \in \mathcal{B}_1$,

$$\mu^n_\omega(A) := Q^n_\omega[DZ^n \in A].$$

Let us next recall the formal definition of the Brownian meander W^+. For this, define $\tau_1 = \sup\{s \in [0,1] : W_1(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$. Then,

$$W^+(s) := \Delta_1^{-1/2}|W_1(\tau_1 + s\Delta_1)|, \qquad 0 \leq s \leq 1.$$

We denote by $P_{W^+} \otimes P_{W^{(d-1)}}$ the product law of Brownian meander and $(d-1)$-dimensional standard Brownian motion on the time interval $[0,1]$. Now, we are ready to formulate the quenched invariance principle for the random walk conditioned to stay positive, which is the main result of this paper:

Theorem 1.1. *Under Condition UE, we have that, $\mathbb{P}$-a.s., μ_ω^n tends weakly to $P_{W^+} \otimes P_{W^{(d-1)}}$ as $n \to \infty$ (as probability measures on $C[0,1]$).*

The next result, referred as Uniform Central Limit Theorem (UCLT), will be useful in order to prove Theorem 1.1. Let W_Σ be a d-dimensonal Brownian motion with covariance matrix Σ defined above. Denoting by $\mathfrak{C}_b(C([0,1]),\mathbb{R})$ (respectively, $\mathfrak{C}_b^u(C([0,1]),\mathbb{R})$) the space of bounded continuous (respectively, bounded uniformly continuous) functionals from $C([0,1])$ into $\mathbb{R}$, we have the following result:

Theorem 1.2. *Under Condition UE, the following statements hold and are equivalent:*

(i) *we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in \mathfrak{C}_b(C([0,1]),\mathbb{R})$,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W_\Sigma)] \Big| = 0;$$

(ii) *we have $\mathbb{P}$-a.s., for all $H > 0$ and any $F \in \mathfrak{C}_b^u(C([0,1]),\mathbb{R})$,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W_\Sigma)] \Big| = 0;$$

(iii) *we have $\mathbb{P}$-a.s., for all $H > 0$ and any closed set B,*

$$\limsup_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \mathtt{P}_{\theta_x\omega}[Z^n \in B] \le P[W_\Sigma \in B];$$

(iv) *we have $\mathbb{P}$-a.s., for all $H > 0$ and any open set G,*

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \ge P[W_\Sigma \in G];$$

(v) *we have $\mathbb{P}$-a.s., for all $H > 0$ and any $A \in \mathcal{B}$ such that $P[W_\Sigma \in \partial A] = 0$,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big| \mathtt{P}_{\theta_x\omega}[Z^n \in A] - P[W_\Sigma \in A] \Big| = 0.$$

As mentioned, our approach does not involve the corrector in a direct way (although, of course, we use the “classical” invariance principle which relies on the corrector approach). Instead, the key ingredients are the following. We use the uniform heat kernel bounds of [10] to prove a uniform CLT (see Theorem 1.2.) In addition, we rely on several auxiliary results giving uniform bounds for exit times and hitting times of hyperplanes, leading to statements which say, roughly speaking, that the conditioning does not change the size of fluctuations (see the beginning of Section 2 for a more detailed description). These auxiliary results are shown in Section 2. They are of independent interest and might be applied to study other "fine" questions for the random conductance model than the conditional quenched CLT considered here. To prove that the conditioning does not change the size of fluctuations, we use an iteration method relying on the Markov property and the uniform bounds shown before. In Section 3, we give the proof of Theorem 1.2. Finally, in Section 4, we give the proof of Theorem 1.1.

We will denote by C_1, C_2, $\dots$ the "global" constants, that is, those that are used all along the paper and by γ, γ_1, γ_2, $\dots$ the "local" constants, that is, those that are used only in the subsection in which they appear for the first time. For the local constants, we restart the numeration in the beginning of each subsection.

Also, whenever the context is clear, to avoid heavy notations, we will not put the integer part symbol $\lfloor\cdot\rfloor$. For example, for $\delta \in (0,1)$ we will write $X(\delta n)$ instead of $X(\lfloor \delta n \rfloor)$.

2. Auxiliary results

In this section, we will prove several auxiliary results that will be needed later to prove Theorem 1.1. Before going to the technical side, let us give a short description. Lemma 2.1 gives a uniform bound on the upper tail of the exit time of a strip as well as on the lower tail of the hitting time of a set (sufficiently far away from the starting point). Lemma 2.2 provides a uniform lower bound for the probability of progressing in the direction $\mathbf{e}_1$ before backstepping to the hyperplane of the origin. Lemma 2.3 says that the probability that the hitting time of a hyperplane is larger than it should be, conditioned on the first coordinate being positive, decays fast enough. Lemma 2.4 says that the probability that the "transversal fluctuations" are larger than they should be, conditioned on the first coordinate being positive, decays fast enough.

Instead of considering the process X in the canonical basis $\mathcal{B}$ of $\mathbb{R}^d$ it is also convenient to introduce the embedded graph $\tilde{\mathbb{Z}}^d := D\mathbb{Z}^d$ with the basis $\mathcal{B}' = \{\mathbf{e}'_1, \dots, \mathbf{e}'_d\} := D\mathcal{B}$ and consider the process DX in this new basis. All the results obtained in this section concern the original random walk X expressed in $\mathcal{B}$ but they remain valid for DX expressed in $\mathcal{B}'$ with the $\|\cdot\|_1$-norm replaced by the graph distance in $\tilde{\mathbb{Z}}^d$.

Let us introduce the following notations. First, for a, $b \in \mathbb{Z}$, $a < b$, we denote by $[\![a, b]\!]$ the set $[a, b] \cap \mathbb{Z}$. Vectors of $\mathbb{Z}^d$ will be denoted by x, y or z.

For $x \in \mathbb{Z}^d$ we denote by $x_1, \dots, x_d$ its coordinates in $\mathcal{B}$. For $l \in \mathbb{R}$, we denote

$$\{l\}_j = \begin{cases} \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_j = \lfloor l \rfloor\}, & \text{if } l \geq 0, \\ \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_j = -\lfloor l \rfloor\}, & \text{if } l < 0, \end{cases}$$

for $j \in [\![1, d]\!]$. If $F \subset \mathbb{Z}^d$, let us define

$$\tau_F = \inf\{n \geq 0 : X(n) \in F\} \quad \text{and} \quad \tau_F^+ = \inf\{n \geq 1 : X(n) \in F\}.$$

At this point we mention that under Condition UE, we can apply Theorem 1.7 of [10] to the random walks $Y(n) := X(2n)$ and $Y'(n) := X(2n+1)$, to obtain that uniform heat kernel lower and upper bounds are available for this model. That is, there exist absolute constants C_1, C_2, C_3 and C_4 such that $\mathbb{P}$-a.s., for $n \in \mathbb{N}$,

$$p_\omega^n(x, y) \leq \frac{C_1}{n^{d/2}} \exp\Big\{ - C_2 \frac{\|x - y\|_1^2}{n} \Big\} \tag{2.1}$$

and if $\|x - y\|_1 \leq n$ (with $\|\cdot\|_1$ the 1-norm on $\mathbb{Z}^d$) and has the same parity as n,

$$p_\omega^n(x, y) \geq \frac{C_3}{n^{d/2}} \exp\Big\{ - C_4 \frac{\|x - y\|_1^2}{n} \Big\}. \tag{2.2}$$

We denote by d_1 the distance induced by the 1-norm. The heat kernel upper bound (2.1) has two simple consequences gathered in the following

Lemma 2.1. *Estimate* (2.1) *implies that there exist positive constants C_5 and C_6 such that $\mathbb{P}$-a.s., for $h > 0$ and $\delta > 0$, the following holds.*

(i) *Let H_1 and H_2 be two parallel hyperplanes in $\mathbb{Z}^d$ orthogonal to $\mathbf{e}_i$ for some $i \in [\![1, d]\!]$ and let us denote by $\mathcal{S}$ the strip delimited by H_1 and H_2. If $2 \leq d_1(H_1, H_2) \leq hn^{1/2}$ then there exists $n_0 = n_0(\delta, h)$ such that*

$$\sup_{x \in \mathcal{S}} \mathtt{P}_\omega^x[\tau_{H_1 \cup H_2} > \delta^2 n] \leq C_5 \frac{h}{\delta}$$

for all $n \geq n_0$;

(ii) *Let $x \in \mathbb{Z}^d$. If $A \subset \mathbb{Z}^d$ is such that $d_1(x, A) > hn^{1/2} \geq 1$ then there exists $n_1 = n_1(\delta, h)$ such that*

$$\mathtt{P}_\omega^x[\tau_A \leq \delta^2 n] \leq C_6 \frac{\delta}{h}$$

for all $n \geq n_1$.

Proof. Let us denote by $\mathcal{S}$ the strip delimited by H_1 and H_2. To prove (i), we just notice that $\mathtt{P}_\omega^x[\tau_{H_1 \cup H_2} > \delta^2 n] \leq \mathtt{P}_\omega^x[X(\delta^2 n) \in \mathcal{S}]$ and apply (2.1). More precisely, suppose that H_1 and H_2 are orthogonal to $\mathbf{e}_1$. With a slight abuse of notation, we also denote by H_1 and H_2 the coordinates where the hyperplanes H_1 and H_2 cross the first axis. We have

$$\begin{aligned}\mathtt{P}_\omega^x[X(\delta^2 n) \in \mathcal{S}] &\le \sum_{y\in\mathcal{S}} \frac{C_1}{\lfloor \delta^2 n\rfloor^{d/2}} \exp\Big\{-C_2\frac{\|x-y\|_1^2}{\lfloor \delta^2 n\rfloor}\Big\} \\ &\le \frac{C_1}{\lfloor \delta^2 n\rfloor^{d/2}} \sum_{y_1\in[H_1,H_2]} \exp\Big\{-C_2\frac{(y_1-x_1)^2}{\lfloor \delta^2 n\rfloor}\Big\} \\ &\quad\times \prod_{i=2}^{d}\sum_{y_i\in\mathbb{Z}} \exp\Big\{-C_2\frac{(y_i-x_i)^2}{\lfloor \delta^2 n\rfloor}\Big\}. \end{aligned} \tag{2.3}$$

Using (2.3), we can see that there exist positive constants γ_1, γ_2 and $n_0 = n_0(\delta, h)$ such that

$$\mathtt{P}_\omega^x[X(\delta^2 n) \in \mathcal{S}] \le \gamma_1 \int_0^{\gamma_2(h/\delta)} \exp\{-C_2 t^2\}dt$$

for all $n \ge n_0$. We deduce that there exists a constant $\gamma_3 > 0$ such that

$$\mathtt{P}_\omega^x[X(\delta^2 n) \in \mathcal{S}] \le \gamma_3 \frac{h}{\delta}$$

for all $n \ge n_0$.

To prove (ii) we use an argument by Barlow (cf. [1] Chapter 3). First, if we denote by $B(x,r)$ the $\|\cdot\|_1$-ball of center x and radius $r := \lfloor hn^{1/2}\rfloor$ we have that

$$\mathtt{P}_\omega^x[\tau_A \le \delta^2 n] \le \mathtt{P}_\omega^x[\tau_{B^c(x,r)} \le \delta^2 n].$$

Then, we have

$$\begin{aligned}\mathtt{P}_\omega^x[\tau_{B^c(x,r)} \le \delta^2 n] &\le \mathtt{P}_\omega^x\Big[\|X(\delta^2 n) - x\|_1 > \frac{r}{2}\Big] \\ &\quad + \mathtt{P}_\omega^x\Big[\tau_{B^c(x,r)} \le \delta^2 n, \|X(\delta^2 n) - x\|_1 \le \frac{r}{2}\Big]. \end{aligned} \tag{2.4}$$

Writing $S = \tau_{B^c(x,r)}$, by the Markov property, the second term of the right-hand side of (2.4) equals

$$\begin{aligned}&\mathtt{E}_\omega^x\Big[\mathbf{1}_{\{S\le\delta^2 n\}}\mathtt{P}_\omega^{X_S}\Big[\|X(\lfloor\delta^2 n\rfloor - S) - x\|_1 \le \frac{r}{2}\Big]\Big] \\ &\quad\le \sup_{y\in\partial B(x,r+1)}\ \sup_{m\le\lfloor\delta^2 n\rfloor} \mathtt{P}_\omega^y\Big[\|X(\lfloor\delta^2 n\rfloor - m) - y\|_1 > \frac{r}{2}\Big]\end{aligned}$$

where $\partial B(x,r) := \{y \in \mathbb{Z}^d : \|y - x\|_1 = r\}$. Combining this last inequality with (2.4) we obtain,

$$\mathtt{P}_\omega^x[\tau_{B^c(x,r)} \le \delta^2 n] \le 2 \sup_{y\in\mathbb{Z}^d}\ \sup_{m\le\lfloor\delta^2 n\rfloor} \mathtt{P}_\omega^y\Big[\|X(\lfloor\delta^2 n\rfloor - m) - y\|_1 > \frac{r}{2}\Big]$$

$$\leq 2 \sup_{y\in\mathbb{Z}^d} \sup_{m\leq\lfloor\delta^2 n\rfloor} \mathtt{P}^y_\omega\Big[\|X(\lfloor\delta^2 n\rfloor - m) - y\|_\infty > \frac{r}{2d}\Big]$$

where $\|\cdot\|_\infty$ is the ∞-norm on $\mathbb{Z}^d$. Applying (2.1) to bound the last term of the above equation from above and performing the same kind of computations as in the proof of (i), we obtain (ii). □

Next, we prove the following lemma, which gives a uniform lower bound for the probability of progressing in direction $\mathbf{e}_1$ before backstepping to the hyperplane $\{0\}_1$.

Lemma 2.2. *Let $v > 0$, then there exist a constant $C_7 = C_7(v) > 0$ such that we have $\mathbb{P}$-a.s., $\inf_{y\in\{l\}_1} \mathtt{P}^y_\omega[\tau_{\{(v+1)l\}_1} < \tau_{\{0\}_1}] \geq C_7$, for all integers $l \geq 1$.*

Proof. We are going to show that we can choose $v > 0$ small enough in such a way that the statement of Lemma 2.2 is true for this v. The generalization to all $v > 0$ is then a direct consequence of the elliptic Harnack inequality, see [13].

For the moment, let $v \in (0, \frac{1}{4})$ and fix l such that $vl \geq 1$. Then, consider $w \in (v, 1]$. We start by writing

$$\begin{aligned}\mathtt{P}^y_\omega[\tau_{\{(v+1)l\}_1} < \tau_{\{0\}_1}] &\geq \mathtt{P}^y_\omega[X_1(wl^2) \geq (v+1)l, \tau_{\{0\}_1} > wl^2]\\ &\geq \mathtt{P}^y_\omega[X_1(wl^2) \geq (v+1)l] - \mathtt{P}^y_\omega[\tau_{\{0\}_1} \leq wl^2]. \qquad (2.5)\end{aligned}$$

Next, let us define $\nu := \lfloor wl^2\rfloor$ if $\lfloor wl^2\rfloor$ is even or $\nu := \lfloor wl^2\rfloor + 1$ otherwise. In the same way, we define $\rho := \lfloor vl\rfloor$ if $\lfloor vl\rfloor$ is even or $\lfloor vl\rfloor + 1$ otherwise. Observe that in any of these cases,

$$\mathtt{P}^y_\omega[X_1(wl^2) \geq (v+1)l] \geq \mathtt{P}^y_\omega[X_1(\nu) > l + \rho]. \qquad (2.6)$$

We will bound the term of the right-hand side of (2.6) from below. For $y \in \{l\}_1$, we denote by $\mathcal{P}(y)$ the (non-empty) set of vectors $z \in \mathbb{Z}^d$ that satisfy the following conditions: $z_1 - y_1 > \rho$, $\|y - z\|_1$ is even and $\|y - z\|_1 \leq \nu$. Applying (2.2), we obtain

$$\begin{aligned}&\mathtt{P}^y_\omega[X_1(\nu) > l + \rho]\\ &\quad\geq \frac{C_3}{\nu^{d/2}} \sum_{u\in\mathcal{P}(y)} \exp\Big\{-C_4\frac{\|u-y\|_1^2}{\nu}\Big\}\\ &\quad\geq \frac{C_3}{\nu^{d/2}} \sum_{v_1=(\rho+2)/2}^{\nu/2} \sum_{v_2=0}^{(\nu-2v_1)/2} \cdots \sum_{v_d=0}^{(\nu-(2v_1+\cdots+2v_{d-1}))/2} \exp\Big\{-\gamma_1\frac{v_1^2+\cdots+v_d^2}{\nu}\Big\}\\ &\quad\geq \frac{C_3}{\nu^{d/2}} \int_{(\rho+2)/2}^{\nu/2} \int_0^{(\nu-2v_1)/2} \cdots \int_0^{(\nu-(2v_1+\cdots+2v_{d-1}))/2} \exp\Big\{-\frac{\gamma_1}{\nu}\sum_{i=1}^d v_i^2\Big\} dv_d\dots dv_1\end{aligned}$$

with γ_1 a positive constant depending only on d. Now making the change of variables $u_i = (2v_i)/\sqrt{\nu}$, $i \in \{1, \dots, d\}$, in the last multiple integral, we obtain

$$\begin{aligned}
&\mathtt{P}_\omega^y[X_1(\nu) > l + \rho] \\
&\quad \geq \frac{C_3}{2^d} \int_{(\rho+2)/\sqrt{\nu}}^{\sqrt{\nu}} \int_0^{\sqrt{\nu}-u_1} \cdots \int_0^{\sqrt{\nu}-\sum_{i=1}^{d-1} u_i} \exp\Big\{ -\frac{\gamma_1}{4} \sum_{i=1}^d u_i^2 \Big\}\, du_d \dots du_1.
\end{aligned}$$

Now, by definition of ρ, ν and the way we chose v, l and w we have that

$$\frac{\rho+2}{\sqrt{\nu}} \leq 4\sqrt{2} v w^{-1/2}$$

and $\nu \geq 1$. This implies that

$$\begin{aligned}
&\mathtt{P}_\omega^y[X_1(\nu) > l + \rho] \\
&\quad \geq \frac{C_3}{2^d} \int_{1\wedge(\rho+2)/\sqrt{\nu}}^{1} \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{d-1} u_i} \exp\Big\{ -\frac{\gamma_1}{4} \sum_{i=1}^d u_i^2 \Big\} du_d \dots du_1 \\
&\quad \geq \frac{C_3}{2^d} \int_{1\wedge 4\sqrt{2} v w^{-1/2}}^{1} \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{d-1} u_i} \exp\Big\{ -\frac{\gamma_1}{4} \sum_{i=1}^d u_i^2 \Big\} du_d \dots du_1 \\
&\quad =: \frac{C_3}{2^d} J(v w^{-1/2}). \qquad (2.7)
\end{aligned}$$

By (ii) of Lemma (2.1) we obtain $\mathtt{P}_\omega^y[\tau_{\{0\}_1} \leq w l^2] \leq C_6 w^{1/2}$. Combining this last inequality with (2.5), (2.6) and (2.7) we obtain

$$\mathtt{P}_\omega^y[\tau_{\{(v+1)l\}_1} < \tau_{\{0\}_1}] \geq \frac{C_3}{2^d} J(v w^{-1/2}) - C_6 w^{1/2}. \qquad (2.8)$$

Observe that for fixed w, we have $J(vw^{-1/2}) \to J(0) > 0$ as $v \to 0$, since the integrated function is positive and the domain of integration of $J(0)$ has Lebesgue measure equal to $1/d!$. Let

$$w^* = \max\Big\{ s > 0 : C_6 s^{1/2} \leq \frac{1}{4} \frac{C_3}{2^d} J(0) \Big\}$$

that is,

$$w^* = \Big(\frac{C_3}{2^{d+2} C_6} J(0) \Big)^2.$$

Letting $v < w^* \wedge (1/4)$, we can choose a sufficiently small w in such a way that the second term of the right-hand side of (2.8) is smaller than $\frac{1}{4}\frac{C_3}{2^d} J(0)$.

Once we have chosen w, we can choose v sufficiently small in such a way that $J(vw^{-1/2}) > J(0)/2$. We obtain that

$$\mathtt{P}_\omega^y[\tau_{\{(v+1)l\}_1} < \tau_{\{0\}_1}] \geq \frac{1}{4}\frac{C_3}{2^d}J(0) > 0.$$

This shows Lemma 2.2. □

For $\varepsilon \in (0,1]$, we denote $N := \lfloor \varepsilon\sqrt{n} \rfloor$. We next prove an upper bound for the probability that the hitting time of the hyperplane $\{N\}_1$ is larger than $\varepsilon^{1/2}n$, given Λ_n.

Lemma 2.3. *There exists a function* $f = f(\varepsilon)$ *with* $\lim_{\varepsilon\to 0}\varepsilon^{-2}f(\varepsilon) = 0$ *such that we have* $\mathbb{P}$*-a.s.*

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2}n \mid \Lambda_n] \leq f(\varepsilon).$$

Proof. Let us begin the proof by sketching the main argument. Consider $\alpha \in (0,1)$, we will show that

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2}n \mid \Lambda_n] \leq \limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{2^{-1}N\}_1} > \alpha\varepsilon^{1/2}n \mid \Lambda_n] + o_1(\varepsilon)$$

when $\varepsilon \to 0$. Then, iterating the argument using hyperplanes of the form

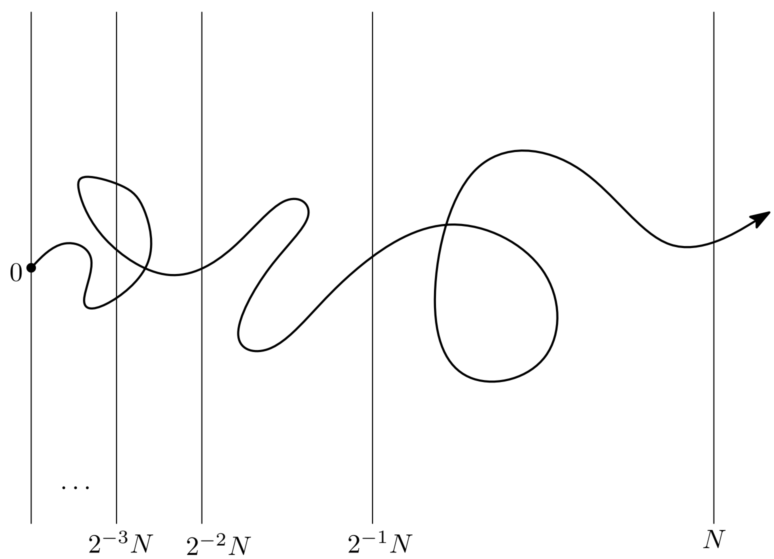

Figure 1. Iteration method.

$\{2^{-j}N\}_1$ (cf. Figure 1) we will have that for all $j \geq 0$,

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{2^{-j}N\}_1} > \alpha^j \varepsilon^{1/2} n \mid \Lambda_n] \leq \limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{2^{-(j+1)}N\}_1} > \alpha^{j+1} \varepsilon^{1/2} n \mid \Lambda_n] + o_j(\varepsilon)$$

when $\varepsilon \to 0$. Finally, restricting α to the interval $(1/4, 1)$, we will show that the $o_j(\varepsilon)$ are decreasing fast enough. Now, let us start the formal argument. Fix $\alpha \in (1/4, 1)$ and let $A_l := \{\tau_{\{l\}_1} < \tau^+_{\{0\}_1}\}$. We have

$$\begin{aligned}
&\mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n \mid \Lambda_n] \\
&\quad = \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\Big(\mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n, \tau_{\{2^{-1}N\}_1} > \alpha\varepsilon^{1/2} n, \Lambda_n] \\
&\qquad + \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n, \tau_{\{2^{-1}N\}_1} \leq \alpha\varepsilon^{1/2} n, \Lambda_n]\Big) \\
&\quad \leq \mathtt{P}_\omega[\tau_{\{2^{-1}N\}_1} > \alpha\varepsilon^{1/2} n \mid \Lambda_n] \\
&\qquad + \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n, \tau_{\{2^{-1}N\}_1} \leq \alpha\varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n].
\end{aligned} \tag{2.9}$$

Then, we have by the Markov property

$$\begin{aligned}
&\mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n, \tau_{\{2^{-1}N\}_1} \leq \alpha\varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n] \\
&\quad = \sum_{y\in\{2^{-1}N\}_1} \sum_{k\leq\lfloor\alpha\varepsilon^{1/2}n\rfloor} \mathtt{P}_\omega\Big[X(\tau_{\{2^{-1}N\}_1}) = y, \tau_{\{2^{-1}N\}_1} = k, \\
&\qquad \tau_{\{N\}_1} > \varepsilon^{1/2} n, A_{2^{-1}N}, \Lambda_n\Big] \\
&\quad \leq \max_{y\in\{2^{-1}N\}_1} \max_{k\leq\lfloor\alpha\varepsilon^{1/2}n\rfloor} \mathtt{P}^y_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n - k, \Lambda_{n-k}]\,\mathtt{P}_\omega[A_{2^{-1}N}].
\end{aligned} \tag{2.10}$$

Now, let us bound from above the term $\mathtt{P}^y_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n - k, \Lambda_{n-k}]$ uniformly in $y \in \{2^{-1}N\}_1$ and in $k \leq \lfloor\alpha\varepsilon^{1/2}n\rfloor$. Observe that, since $\varepsilon \in (0, 1]$, we have

$$\begin{aligned}
\mathtt{P}^y_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n - k, \Lambda_{n-k}] &\leq \mathtt{P}^y_\omega[\tau_{\{N\}_1} > (1-\alpha)\varepsilon^{1/2} n, \Lambda_{(1-\alpha)n}] \\
&\leq \mathtt{P}^y_\omega[\tau_{\{0\}_1\cup\{N\}_1} > (1-\alpha)\varepsilon^{1/2} n].
\end{aligned} \tag{2.11}$$

Let $\delta := \beta^{-1}\varepsilon$, where β is a positive constant to be determined later. Then, consider ε small enough in such a way that $\delta < (1-\alpha)\varepsilon^{1/2}$. Then, divide the time interval $[0, \lfloor(1-\alpha)\varepsilon^{1/2}n\rfloor]$ into intervals of size $\lfloor\delta^2 n\rfloor$. Denoting $S(0, N) = \bigcup_{i=1}^{N-1}\{i\}_1$, we obtain by the Markov property

$$\mathtt{P}^y_\omega[\tau_{\{0\}_1\cup\{N\}_1} > (1-\alpha)\varepsilon^{1/2} n]$$

$$
\begin{aligned}
&\leq \mathsf{P}_\omega^y\Big[\tau_{\{0\}_1\cup\{N\}_1} \notin \bigcup_{i=1}^{\left\lfloor \frac{\lfloor (1-\alpha)\varepsilon^{1/2}n\rfloor}{\lfloor \delta^2 n\rfloor}\right\rfloor} ((i-1)\lfloor\delta^2 n\rfloor, i\lfloor\delta^2 n\rfloor]\Big]\\
&\leq \Big(\max_{z\in S(0,N)} \mathsf{P}_\omega^z[\tau_{\{0\}_1\cup\{N\}_1} > \delta^2 n]\Big)^{(1-\alpha)\varepsilon^{1/2}\delta^{-2}-2} \qquad (2.12)
\end{aligned}
$$

for large enough n. Using (i) of Lemma 2.1, we have for all $z\in S(0,N)$,

$$
\mathsf{P}_\omega^z[\tau_{\{0\}_1\cup\{N\}_1} > \delta^2 n] \leq C_5\frac{\varepsilon}{\delta} \qquad (2.13)
$$

for sufficiently large n. Since $\varepsilon/\delta=\beta$, let us choose the constant β such that $C_5\beta\leq 1/2$. Thus, for ε sufficiently small such that $\beta^{-1}\varepsilon < (1-\alpha)\varepsilon^{1/2}$, we obtain by (2.12)

$$
\mathsf{P}_\omega^y[\tau_{\{0\}_1\cup\{N\}_1} > (1-\alpha)\varepsilon^{1/2}n] \leq 4\Big(\frac{1}{2}\Big)^{(1-\alpha)\varepsilon^{-\frac{3}{2}}\beta^2}.
$$

From (2.9), we deduce

$$
\begin{aligned}
\mathsf{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2}n \mid \Lambda_n] &\leq \mathsf{P}_\omega[\tau_{\{2^{-1}N\}_1} > \alpha\varepsilon^{1/2}n \mid \Lambda_n]\\
&\quad + 4\Big(\frac{1}{2}\Big)^{(1-\alpha)\varepsilon^{-\frac{3}{2}}\beta^2}\frac{\mathsf{P}_\omega[A_{2^{-1}N}]}{\mathsf{P}_\omega[\Lambda_n]}. \qquad (2.14)
\end{aligned}
$$

Then, we will find an upper bound for the ratio in the second term of the right-hand side of (2.14). By the Markov property we have

$$
\frac{\mathsf{P}_\omega[\Lambda_n]}{\mathsf{P}_\omega[A_{2^{-1}N}]} \geq \mathsf{P}_\omega[\Lambda_n \mid A_{2^{-1}N}] \geq \min_{y\in\{2^{-1}N\}_1} \mathsf{P}_\omega^y[\tau_{\{0\}_1} > n]. \qquad (2.15)
$$

Let $K\geq 2\varepsilon$ and let $N'=\lfloor K\sqrt{n}\rfloor$. We start by noting that for any $y\in\{2^{-1}N\}_1$ we have by the Markov property

$$
\begin{aligned}
\mathsf{P}_\omega^y[\tau_{\{0\}_1} > n] &\geq \mathsf{P}_\omega^y[\tau_{\{0\}_1} > n, \tau_{\{N'\}_1} < \tau_{\{0\}_1}]\\
&\geq \min_{z\in\{N'\}_1} \mathsf{P}_\omega^z[\tau_{\{0\}_1} > n]\,\mathsf{P}_\omega^y[\tau_{\{N'\}_1} < \tau_{\{0\}_1}]. \qquad (2.16)
\end{aligned}
$$

Let us now bound from below both terms in the right-hand side of (2.16).

We first show that we can choose a sufficiently large K in such a way that $\mathsf{P}_\omega^z[\tau_{\{0\}_1} > n]\geq 1/2$ uniformly in $z\in\{N'\}_1$. Using (ii) of Lemma 2.1, we have $\mathsf{P}_\omega^z[\tau_{\{0\}_1}\leq n]\leq C_6/K$ for sufficiently large n. Choosing K sufficiently large so that $C_6/K\leq 1/2$ we obtain

$$
\mathsf{P}_\omega^z[\tau_{\{0\}_1} > n] \geq \frac{1}{2} \qquad (2.17)
$$

uniformly in $z \in \{N'\}_1$. Now going back to equation (2.16), we now show that with probability of order ε^γ with $\gamma > 0$, starting from the line $\{2^{-1}N\}_1$, the random walk reaches the line $\{N'\}_1$ before reaching the line $\{0\}_1$. By Lemma 2.2, there exists $C_7 > 0$ such that for every $l > 1$, $\mathtt{P}^u_\omega[\tau_{\{2l\}} < \tau_{\{0\}}] \geq C_7$, with $u \in \{l\}_1$. Now consider, the following sequence $(U_j)_{j\geq 1}$ of hyperplanes defined by

$$\begin{cases} U_1 & = \{2\lfloor 2^{-1}N\rfloor\}_1 \\ U_{j+1} & = \{2U_j\}_1. \end{cases}$$

Let j^* the smallest j such that $U_j \geq K\sqrt{n}$. Using the induction relation, we obtain that for some constant $\gamma_1 > 0$, $j^* \leq \gamma_1 \ln(K/\varepsilon)$ for large enough n. By convention, set $U_0 = \{2^{-1}N\}_1$. By the Markov property, we obtain that uniformly in $y \in \{2^{-1}N\}_1$,

$$\begin{aligned} \mathtt{P}^y_\omega[\tau_{\{N'\}_1} < \tau_{\{0\}_1}] &\geq \mathtt{P}^y_\omega\Big[\bigcap_{i=1}^{j^*}\{\tau_{U_i} < \tau_{\{0\}_1}\}\Big] \\ &\geq \prod_{i=1}^{j^*}\Big(\min_{u\in U_{i-1}} \mathtt{P}^u_\omega[\tau_{U_i} < \tau_{\{0\}_1}]\Big) \geq \Big(\frac{\varepsilon}{K}\Big)^{\gamma_2} \end{aligned} \tag{2.18}$$

for some constant $\gamma_2 > 0$ and large enough n. Combining (2.16), (2.17), and (2.18) we deduce

$$\min_{y\in\{2^{-1}N\}_1} \mathtt{P}^y_\omega[\tau_{\{0\}_1} > n] \geq \frac{1}{2}\Big(\frac{\varepsilon}{K}\Big)^{\gamma_2} \tag{2.19}$$

for large enough n. Then by (2.14), (2.15) and (2.19) we obtain

$$\begin{aligned} \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2}n \mid \Lambda_n] &\leq \mathtt{P}_\omega[\tau_{\{2^{-1}N\}_1} > \alpha\varepsilon^{1/2}n \mid \Lambda_n] \\ &\quad + 16K^{\gamma_2}\varepsilon^{-\gamma_2}\Big(\frac{1}{2}\Big)^{(1-\alpha)\varepsilon^{-\frac{3}{2}}\beta^2}. \end{aligned} \tag{2.20}$$

By the same argument, we can deduce that for all $j \geq 1$ we have

$$\begin{aligned} \mathtt{P}_\omega[\tau_{\{2^{-j}N\}_1} > \alpha^j\varepsilon^{1/2}n \mid \Lambda_n] &\leq \mathtt{P}_\omega[\tau_{\{2^{-(j+1)}N\}_1} > \alpha^{j+1}\varepsilon^{1/2}n \mid \Lambda_n] \\ &\quad + 16K^{\gamma_2}\Big(\frac{\varepsilon}{2^j}\Big)^{-\gamma_2}\Big(\frac{1}{2}\Big)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}(4\alpha)^j} \end{aligned} \tag{2.21}$$

for large enough n. Iterating (2.20) using (2.21), we deduce

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2}n \mid \Lambda_n] \leq 16K^{\gamma_2}\sum_{j=0}^{\infty}\Big(\frac{\varepsilon}{2^j}\Big)^{-\gamma_2}\Big(\frac{1}{2}\Big)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}(4\alpha)^j}. \tag{2.22}$$

As $\alpha \in (1/4, 1)$, the last series is convergent. Define the function f in the statement of Lemma 2.3 as

$$f(\varepsilon) := 16K^{\gamma_2} \sum_{j=0}^{\infty} \Big(\frac{\varepsilon}{2^j}\Big)^{-\gamma_2} \Big(\frac{1}{2}\Big)^{(1-\alpha)\beta^2\varepsilon^{-\frac{3}{2}}(4\alpha)^j}.$$

Using the dominated convergence theorem, it is straightforward to show that $\lim_{\varepsilon\to 0} \varepsilon^{-2} f(\varepsilon) = 0$. This proves Lemma 2.3. □

In the next lemma, N still stands for $\lfloor \varepsilon\sqrt{n} \rfloor$. However, the quantities (like α, δ, β, ...) defined in the proof of the lemma are not related to the corresponding quantities defined in the proof of Lemma 2.3. The next lemma controls the "transversal fluctuations" of $X_2, \dots, X_d$, given Λ_n.

Lemma 2.4. *We have* $\mathbb{P}$*-a.s.,*

$$\limsup_{n\to\infty} \mathtt{P}_\omega\Big[\max_{i\in[\![2,d]\!]} \sup_{j\le \tau_{\{N\}_1}} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_n\Big] \le g(\varepsilon)$$

with $\lim_{\varepsilon\to 0} \varepsilon^{-2} g(\varepsilon) = 0$.

Proof. First, observe that, by symmetry, it suffices to show that there exists $g' = g'(\varepsilon)$ such that

$$\limsup_{n\to\infty} \mathtt{P}_\omega\Big[\sup_{j\le \tau_{\{N\}_1}} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_n\Big] \le g'(\varepsilon) \tag{2.23}$$

with $\lim_{\varepsilon\to 0} \varepsilon^{-2} g'(\varepsilon) = 0$ for some $i \in [\![2,d]\!]$. For the sake of simplicity, let us take $i = 2$ in the rest of the proof. Fix $\alpha \in (1/2, 1)$ and let

$$\tilde{\varepsilon}^{-1/2} := \frac{1-\alpha}{\alpha}\varepsilon^{-1/2} > 2.$$

We introduce the following sequence of events (cf. Figure 2),

$$G_k = \Big\{\sup_{j\in(\tau_{\{2^{-k}N\}_1}, \tau_{\{2^{-k+1}N\}_1}]} |X_2(j) - X_2(\tau_{\{2^{-k}N\}_1})| \le \tilde{\varepsilon}^{-1/2}\alpha^k N\Big\}$$

for $k \ge 1$, with the convention that $\sup_{j\in\emptyset}\{\cdot\} = 0$. Then, we denote

$$B_k^\delta = \{\tau_{\{2^{-k}N\}_1} \le \delta n\} \cap \{\tau_{\{2^{-k}N\}_1} < \tau_{\{0\}_1}\}$$

for $\delta \in (0, 1]$ and $k \ge 1$.

Now, observe that on the event $B_0^\delta \cap (\cap_{k\ge 1} G_k)$ we have that

$$\sup_{j\le \tau_{\{N\}_1}} |X_2(j)| \le \varepsilon^{-1/2} N$$

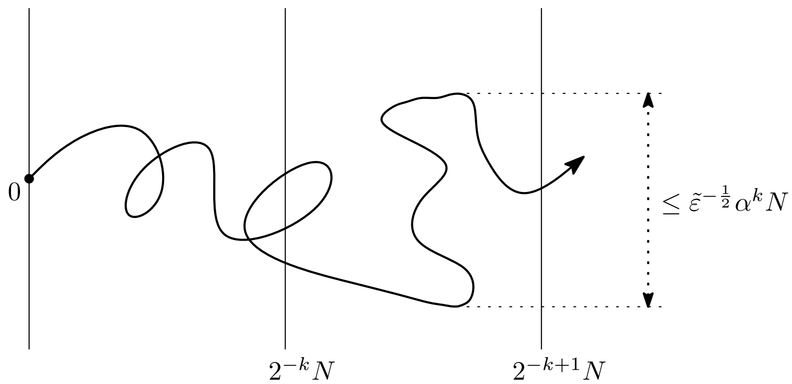

Figure 2. On the definition of G_k.

since $\alpha \in (1/2, 1)$. This implies that

$$\mathtt{P}_\omega\Big[\sup_{j \leq \tau_{\{N\}_1}} |X_2(j)| \leq \varepsilon^{-1/2} N \mid \Lambda_n\Big] \geq \mathtt{P}_\omega\Big[B_0^\delta \cap \Big(\bigcap_{k\geq 1} G_k\Big) \mid \Lambda_n\Big].$$

In order to prove Lemma 2.4, we will show that $\liminf_{n\to\infty} \mathtt{P}_\omega[B_0^\delta \cap (\cap_{k\geq 1} G_k) \mid \Lambda_n]$ tends to 1 when $\varepsilon \to 0$. We start by writing

$$\begin{aligned}\mathtt{P}_\omega\Big[B_0^\delta \cap \Big(\bigcap_{k\geq 1} G_k\Big) \mid \Lambda_n\Big] &= \mathtt{P}_\omega[B_0^\delta \mid \Lambda_n] - \mathtt{P}_\omega\Big[B_0^\delta \cap \Big(\bigcap_{k\geq 1} G_k\Big)^c \mid \Lambda_n\Big] \\ &\geq \mathtt{P}_\omega[B_0^\delta \mid \Lambda_n] - \sum_{k=1}^{\lfloor \frac{\ln N}{\ln 2}\rfloor} \mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n]. \qquad (2.24)\end{aligned}$$

From now on, we dedicate ourselves to bounding from above the terms $\mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n]$ for $k \leq \lfloor \frac{\ln N}{\ln 2}\rfloor$. We have by the Markov property,

$$\begin{aligned}&\mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n] \\ &\quad\leq \mathtt{P}_\omega[B_k^\delta \cap G_k^c \mid \Lambda_n] \\ &\quad= \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\mathtt{P}_\omega[B_k^\delta, G_k^c, \Lambda_n] \\ &\quad= \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\sum_{j\leq\lfloor\delta n\rfloor}\sum_{y\in\{2^{-k}N\}_1}\mathtt{P}_\omega\Big[B_k^\delta, G_k^c, \Lambda_n, \tau_{\{2^{-k}N\}_1} = j, X(\tau_{\{2^{-k}N\}_1}) = y\Big] \\ &\quad\leq \frac{\mathtt{P}_\omega[B_k^1]}{\mathtt{P}_\omega[\Lambda_n]}\max_{j\leq\lfloor\delta n\rfloor}\max_{y\in\{2^{-k}N\}_1}\mathtt{P}_\omega^y\Big[\sup_{i\leq\tau_{\{2^{-k+1}N\}_1}}|(X(i)-y)\cdot\mathbf{e}_2| > \tilde{\varepsilon}^{-1/2}\alpha^k N, \Lambda_{n-j}\Big]\end{aligned}$$

$$\leq \frac{\mathtt{P}_\omega[B_k^1]}{\mathtt{P}_\omega[\Lambda_n]} \max_{y\in\{2^{-k}N\}_1} \mathtt{P}_\omega^y\Big[\sup_{i\leq \tau_{\{2^{-k+1}N\}_1}} |(X(i)-y)\cdot \mathbf{e}_2| > \tilde{\varepsilon}^{-1/2}\alpha^k N, \Lambda_{(1-\delta)n}\Big].$$

Using again the Markov property, we obtain

$$\frac{\mathtt{P}_\omega[\Lambda_n]}{\mathtt{P}_\omega[B_k^1]} \geq \mathtt{P}_\omega[\Lambda_n \mid B_k^1] \geq \min_{y\in\{2^{-k}N\}_1} \mathtt{P}_\omega^y[\tau_{\{0\}_1} > n].$$

By the same argument which we used in Lemma 2.3 to treat the term

$$\min_{y\in\{2^{-1}N\}_1} \mathtt{P}_\omega^y[\tau_{\{0\}_1} > n]$$

(cf. the derivation of (2.19)), we obtain, for large enough n and all $k \leq \lfloor \frac{\ln N}{\ln 2}\rfloor$,

$$\frac{\mathtt{P}_\omega[B_k^1]}{\mathtt{P}_\omega[\Lambda_n]} \leq \gamma_1\Big(\frac{K2^k}{\varepsilon}\Big)^{\gamma_2} \tag{2.25}$$

for some positive constants γ_1, γ_2 and K from Lemma 2.3. Now, we need to bound the terms

$$\mathtt{P}_\omega^y\Big[\sup_{i\leq \tau_{\{2^{-k+1}N\}_1}} |(X(i)-y)\cdot \mathbf{e}_2| > \tilde{\varepsilon}^{-1/2}\alpha^k N, \Lambda_{(1-\delta)n}\Big]$$

from above, uniformly in $y \in \{2^{-k}N\}_1$. In order not to carry on heavy notations we treat the case $y_2 = 0$. However, as one can check, the bound we will obtain is uniform in $y \in \{2^{-k}N\}_1$. Let

$$E_k = \{(x_1,\dots,x_d) \in \mathbb{Z}^d : x_2 = \pm\lfloor \tilde{\varepsilon}^{-1/2}\alpha^k N\rfloor\}.$$

We start by writing

$$\begin{aligned}
&\mathtt{P}_\omega^y\Big[\sup_{i\leq \tau_{\{2^{-k+1}N\}_1}} |X_2(i)| > \tilde{\varepsilon}^{-1/2}\alpha^k N, \Lambda_{(1-\delta)n}\Big]\\
&= \mathtt{P}_\omega^y[\tau_{E_k} < \tau_{\{2^{-k+1}N\}_1}, \tau_{\{0\}_1} > (1-\delta)n]\\
&\leq \mathtt{P}_\omega^y[\tau_{E_k} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] + \mathtt{P}_\omega^y[\tau_{E_k} > (1-\delta)n].
\end{aligned} \tag{2.26}$$

Let us bound the first term of the right-hand side of (2.26) from above. To do so, we first write

$$\begin{aligned}
\mathtt{P}_\omega^y[\tau_{E_k} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \leq{}& \mathtt{P}_\omega^y[\tau_{\{\tilde{\varepsilon}^{-1/2}\alpha^k N\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}]\\
&+ \mathtt{P}_\omega^y[\tau_{\{-\tilde{\varepsilon}^{-1/2}\alpha^k N\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}].
\end{aligned} \tag{2.27}$$

We treat the first term of the right-hand side of (2.27) (the method for the second term is similar). Let $L \in (2, \tilde{\varepsilon}^{-1/2})$ and divide the interval $[0, \lfloor \tilde{\varepsilon}^{-1/2}\alpha^k N\rfloor]$ into intervals of size $\lfloor L2^{-k}N\rfloor$. Furthermore, let

$$F_k = \bigcup_{j=1}^{\lfloor 2^{-k+1}N\rfloor - 1} \{j\}_1.$$

We have by the Markov property,

$$\begin{aligned}
&\mathsf{P}_\omega^y[\tau_{\{\varepsilon^{-1/2}\gamma\alpha^k N\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \\
&\quad \le \mathsf{P}_\omega^y\Big[\bigcap_{j=1}^{\left\lfloor \frac{\lfloor \tilde{\varepsilon}^{-1/2}\alpha^k N\rfloor}{\lfloor L2^{-k}N\rfloor}\right\rfloor} \big\{\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}\big\}\Big] \qquad (2.28)\\
&\quad \le \prod_{j=1}^{\lfloor L^{-1}\tilde{\varepsilon}^{-1/2}(2\alpha)^k\rfloor-2} \max_{z\in\{(j-1)\lfloor L2^{-k}N\rfloor\}_2\cap F_k} \mathsf{P}_\omega^z[\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}].
\end{aligned}$$

Let us show that

$$\max_{z\in\{(j-1)\lfloor L2^{-k}N\rfloor\}_2\cap F_k} \mathsf{P}_\omega^z[\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \le \frac{1}{2}$$

for ε sufficiently small and L sufficiently large belonging to $(2,\tilde{\varepsilon}^{-1/2})$. Consider $w\in(4,L^2)$, we have for $z\in\{(j-1)\lfloor L2^{-k}N\rfloor\}_2\cap F_k$,

$$\begin{aligned}
&\mathsf{P}_\omega^z[\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} > \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \\
&\quad \ge \mathsf{P}_\omega^z[\tau_{\{2^{-k+1}N\}_1\cup\{0\}_1} \le w2^{-2k}N^2, \tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} > w2^{-2k}N^2] \\
&\quad \ge \mathsf{P}_\omega^z[\tau_{\{2^{-k+1}N\}_1\cup\{0\}_1} \le w2^{-2k}N^2] \\
&\qquad - \mathsf{P}_\omega^z[\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} \le w2^{-2k}N^2]. \qquad (2.29)
\end{aligned}$$

Using (i) of Lemma 2.1, we deduce

$$\mathsf{P}_\omega^z[\tau_{\{2^{-k+1}N\}_1\cup\{0\}_1} \le w2^{-2k}N^2] \ge 1 - C_5 w^{-1/2}. \qquad (2.30)$$

Using (ii) of Lemma 2.1, we obtain for all $j\ge 1$,

$$\mathsf{P}_\omega^z[\tau_{\{j\lfloor L2^{-k}N\rfloor\}_2} \le w2^{-2k}N^2] \le C_6\frac{w^{1/2}}{L}. \qquad (2.31)$$

Combining (2.29), (2.30) and (2.31) we obtain for all $j\ge 1$,

$$\mathsf{P}_\omega^z[\tau_{\{jL2^{-k}N\}_2} > \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \ge 1 - C_5 w^{-1/2} - C_6\frac{w^{1/2}}{L}. \qquad (2.32)$$

First, choose w sufficiently large such that $C_5 w^{-1/2}\le 1/4$ and thus choose L sufficiently large in such a way that $C_6 w^{1/2}/L\le 1/4$. We obtain

$$\mathsf{P}_\omega^z[\tau_{\{jL2^{-k}N\}_2} > \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \ge \frac{1}{2}. \qquad (2.33)$$

Now using (2.27), (2.28) and (2.33) we have since $\tilde{\varepsilon}^{-1/2} > L$,

$$\mathsf{P}_\omega^y[\tau_{E_k} < \tau_{\{2^{-k+1}N\}_1\cup\{0\}_1}] \le 16\Big(\frac{1}{2}\Big)^{\lfloor L^{-1}\tilde{\varepsilon}^{-1/2}(2\alpha)^k\rfloor}. \qquad (2.34)$$

Next, let us treat the term $\mathtt{P}_\omega^y[\tau_{E_k} > (1-\delta)n]$. Let $\eta = \beta^{-1}\tilde{\varepsilon}$ where β is a positive constant to be chosen later. Then suppose that ε is sufficiently small such that $\eta\tilde{\varepsilon}^{-1/2}\alpha^k < 1-\delta$ and divide the time interval $[0, \lfloor (1-\delta)n\rfloor]$ into intervals of size $\lfloor \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n\rfloor$. Using the notation

$$H(E_k) = \bigcup_{j=-\lfloor \tilde{\varepsilon}^{-1/2}\alpha^k N\rfloor+1}^{\lfloor \tilde{\varepsilon}^{-1/2}\alpha^k N\rfloor-1} \{j\}_2,$$

we obtain by the Markov property

$$\begin{aligned}
&\mathtt{P}_\omega^y[\tau_{E_k} > (1-\delta)n] \\
&\quad\le \mathtt{P}_\omega^y\Big[\tau_{E_k} \notin \bigcup_{i=1}^{\left\lfloor \frac{\lfloor (1-\delta)n\rfloor}{\lfloor \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n\rfloor}\right\rfloor} ((i-1)\lfloor \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n\rfloor, i\lfloor \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n\rfloor]\Big] \\
&\quad\le \Big(\max_{z\in H(E_k)} \mathtt{P}_\omega^z[\tau_{E_k} > \eta^2\varepsilon^{-1}\alpha^{2k}n]\Big)^{(1-\delta)(\eta\tilde{\varepsilon}^{-1/2}\alpha^k)^{-2}-2} \qquad (2.35)
\end{aligned}$$

for n sufficiently large. We now bound the term $\mathtt{P}_\omega^z[\tau_{E_k} > \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n]$ from above uniformly in $z \in H(E_k)$. Using (i) of Lemma 2.1, we have

$$\mathtt{P}_\omega^z[\tau_{E_k} > \eta^2\tilde{\varepsilon}^{-1}\alpha^{2k}n] \le C_5\frac{\tilde{\varepsilon}}{\eta}.$$

Since $\tilde{\varepsilon}\eta^{-1} = \beta$, choose β small enough such that $C_5\beta \le 1/2$. For ε sufficiently small such that $\eta\tilde{\varepsilon}^{-1/2}\alpha^k < 1-\delta$, we obtain using (2.35),

$$\mathtt{P}_\omega^y[\tau_{E_k} > (1-\delta)n] \le 4\Big(\frac{1}{2}\Big)^{(1-\delta)(\beta^{-1}\tilde{\varepsilon}^{1/2}\alpha^k)^{-2}}. \qquad (2.36)$$

Combining (2.26), (2.27), (2.34) and (2.36), we deduce that, $\mathbb{P}$-a.s., for all large enough n and $k \le \lfloor \frac{\ln N}{\ln 2}\rfloor$,

$$\begin{aligned}
&\max_{y\in\{2^{-k}N\}_1} \mathtt{P}_\omega^y\Big[\sup_{i\le \tau_{\{2^{-k+1}N\}_1}} |(X(i)-y)\cdot \mathbf{e}_2| > \tilde{\varepsilon}^{-1/2}\alpha^k N, \Lambda_{(1-\delta)n}\Big] \\
&\quad\le 16\Big(\frac{1}{2}\Big)^{L^{-1}\tilde{\varepsilon}^{-1/2}(2\alpha)^k} + 4\Big(\frac{1}{2}\Big)^{(1-\delta)(\beta^2\tilde{\varepsilon}^{-1}\alpha^{-2k})}. \qquad (2.37)
\end{aligned}$$

Using (2.25) and (2.37), we obtain for all large enough n and $k \le \lfloor \frac{\ln N}{\ln 2}\rfloor$,

$$\begin{aligned}
&\mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n] \\
&\quad\le \gamma_1 K^{\gamma_2} 2^{k\gamma_2+1}\varepsilon^{-\gamma_2}\Big(16\Big(\frac{1}{2}\Big)^{L^{-1}\tilde{\varepsilon}^{-1/2}(2\alpha)^k} + 4\Big(\frac{1}{2}\Big)^{(1-\delta)(\beta^2\tilde{\varepsilon}^{-1}\alpha^{-2k})}\Big).
\end{aligned}$$

We finally deduce that, $\mathbb{P}$-a.s., for large enough n,

$$\begin{aligned}&\sum_{k=1}^{\lfloor \frac{\ln N}{\ln 2}\rfloor} \mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n] \\ &\quad\le \sum_{k=1}^{\infty} \gamma_1 K^{\gamma_2} 2^{k\gamma_2+1}\varepsilon^{-\gamma_2}\Big(16\Big(\frac{1}{2}\Big)^{L^{-1}\tilde\varepsilon^{-1/2}(2\alpha)^k} + 4\Big(\frac{1}{2}\Big)^{(1-\delta)(\beta^2\tilde\varepsilon^{-1}\alpha^{-2k})}\Big).\end{aligned}$$

Observe that since $\alpha \in (1/2,1)$, the series above converges. Let $\delta = \varepsilon^{1/2}$, we have for $\varepsilon < 1/4$,

$$\begin{aligned}\sum_{k=1}^{\lfloor \frac{\ln N}{\ln 2}\rfloor} \mathtt{P}_\omega[B_0^\delta \cap G_k^c \mid \Lambda_n] \le \sum_{k=1}^{\infty} \gamma_1 K^{\gamma_2} 2^{k\gamma_2+1}\varepsilon^{-\gamma_2}\Big(16\Big(\frac{1}{2}\Big)^{\frac{1-\alpha}{\alpha}L^{-1}\varepsilon^{-1/2}(2\alpha)^k} \\ + 4\Big(\frac{1}{2}\Big)^{1/2(\frac{1-\alpha}{\alpha})^2\beta^2\varepsilon^{-1}\alpha^{-2k})}\Big).\end{aligned}$$

Let

$$\begin{aligned}h(\varepsilon) := \sum_{k=1}^{\infty} \gamma_1 K^{\gamma_2} 2^{k\gamma_2+1}\varepsilon^{-\gamma_2}\Big(16\Big(\frac{1}{2}\Big)^{\frac{1-\alpha}{\alpha}L^{-1}\varepsilon^{-1/2}(2\alpha)^k} \\ + 4\Big(\frac{1}{2}\Big)^{1/2(\frac{1-\alpha}{\alpha})^2\beta^2\varepsilon^{-1}\alpha^{-2k})}\Big).\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have $\varepsilon^{-2}h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using (2.24) and Lemma 2.3 (since $\delta = \varepsilon^{1/2}$) we have for $\varepsilon < 1/4$,

$$\liminf_{n\to\infty} \mathtt{P}_\omega\Big[B_0^\delta \cap (\bigcap_{k\ge 1} G_k) \mid \Lambda_n\Big] \ge 1 - f(\varepsilon) - h(\varepsilon).$$

This last term tends to 1 as $\varepsilon \to 0$. Now, take $g'(\varepsilon) := f(\varepsilon) + h(\varepsilon)$ to show (2.23) and therefore Lemma 2.4. $\square$

3. Proof of the UCLT

In this section we prove Theorem 1.2. The proof is similar in spirit to the proof of Theorem 1.2 of [11], but it is greatly simplified in the present case by the use of the heat kernel upper bounds. As in [11], we will consider “good” sites (see Definition 3.1), where uniform estimates hold for the distance of Z^n and W, and then show that the random walk hits, with high probability, a good site in small distance from its starting point.

In order to take advantage of the natural left shift on the space $C(\mathbb{R}_+)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$, we will rather prove Theorem 1.2 for Z^n

assuming values in $C(\mathbb{R}_+)$ instead of $C([0,1])$. Then, the result for Z^n assuming values in $C([0,1])$ will be easily obtained by the mapping theorem (cf. [4]). Let $\mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$ be the space of bounded uniformly continuous functionals from $C(\mathbb{R}_+)$ into $\mathbb{R}$. In this section, we write W for the d-dimensional Brownian motion with covariance matrix Σ from section 1. The first step is to prove the following

Proposition 3.1. *For all $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$, we have $\mathbb{P}$-a.s., for every $H > 0$,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| = 0.$$

Fix $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+),\mathbb{R})$. We will prove that, $\mathbb{P}$-a.s., for every $\tilde{\varepsilon}, H > 0$,

$$\sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big| \mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)] \Big| \leq \tilde{\varepsilon} \tag{3.1}$$

for n large enough. Before this, we need to introduce some definitions and prove an intermediate result. Let $\mathtt{d}$ be the distance on the space $C_{\mathbb{R}_+}$ defined by

$$\mathtt{d}(f,g) = \sum_{n=1}^{\infty} 2^{-n+1} \min\Big\{1, \sup_{s\in[0,n]} \|f(s) - g(s)\|\Big\}$$

with $\|\cdot\|$ the euclidian norm on $\mathbb{R}^d$. Now, for any given $\varepsilon > 0$, let

$$h_\varepsilon := \max\Big\{h \in (0,1] : P\Big[\sup_{s\leq h} \|W(s)\| > \varepsilon\Big] + P\Big[\sup_{s\leq h} \mathtt{d}(\theta_s W, W) > \varepsilon\Big] \leq \frac{\varepsilon}{2}\Big\}. \tag{3.2}$$

Observe that $h_\varepsilon > 0$ for $\varepsilon > 0$ and $h_\varepsilon \to 0$ when $\varepsilon \to 0$. Next, adapting section 3 of [11] we introduce the following

Definition 3.1. For a given realization of the environment ω and $N \in \mathbb{N}$, we say that $x \in \mathbb{Z}^d$ is (ε, N)-good, if

- $\min\Big\{n \geq 1 : \big|\mathtt{E}_\omega[F(Z^m)] - E[F(W)]\big| \leq \varepsilon, \text{ for all } m \geq n\Big\} \leq N;$
- $\mathtt{P}_{\theta_x\omega}\Big[\sup_{s\leq h_\varepsilon} \|Z^m(s)\| \leq \varepsilon, \sup_{s\leq h_\varepsilon} \mathtt{d}(\theta_s Z^m, Z^m) \leq \varepsilon\Big] \geq 1 - \varepsilon$, for all $m \geq N$.

We now show that starting from a site $x \in [-H\sqrt{n}, H\sqrt{n}]^d$, with high probability, the random walk X will meet a (ε, n)-good site at a distance at most $h'\sqrt{n}$ before time hn (unlike as in [11], there is no need here to introduce the notion of a *nice* site since by (2.1), every point in $[-H\sqrt{n}, H\sqrt{n}]^d$ is nice). We denote by $\mathcal{G}$ the set of (ε, n)-good sites in $\mathbb{Z}^d$.

Proposition 3.2. *Fix $h' > 0$. For any $\varepsilon_1 > 0$, we can choose ε small enough in such a way that we have $\mathbb{P}$-a.s., for all sufficiently large n and all $x \in [-H\sqrt{n}, H\sqrt{n}]^d$:*

(i) $\mathsf{P}_\omega^x[\tau_{\mathcal{G}} > h_\varepsilon n] \leq \varepsilon_1$;

(ii) $\mathsf{P}_\omega^x\Big[\sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n}\Big] \leq \varepsilon_1$.

Proof. Fix ε. Then, for any $\varepsilon' > 0$ there exists N such that

$$\mathbb{P}[0 \text{ is } (\varepsilon, N)\text{-good}] > 1 - \varepsilon'.$$

By the Ergodic Theorem, we have $\mathbb{P}$-a.s. for all $n > n_1(\omega)$,

$$\Big|\{x \in [-2H\sqrt{n}, 2H\sqrt{n}]^d \text{ and } x \text{ is not } (\varepsilon, N)\text{-good}\}\Big| < 5^d \varepsilon' H^d n^{\frac{d}{2}}. \tag{3.3}$$

Let us define

$$\mathsf{Bad} := \{x \in [-2H\sqrt{n}, 2H\sqrt{n}]^d \text{ and } x \text{ is not } (\varepsilon, N)\text{-good}\}$$

and $\mathsf{Cub} := [-2H\sqrt{n}, 2H\sqrt{n}]^d$.

In order to show (i) we observe that for all $x \in [-H\sqrt{n}, H\sqrt{n}]^d$,

$$\mathsf{P}_\omega^x[\tau_{\mathcal{G}} > h_\varepsilon n] \leq \mathsf{P}_\omega^x[X(h_\varepsilon n) \in \mathsf{Bad}] + \mathsf{P}_\omega^x[\tau_{\mathsf{Cub}^c} \leq h_\varepsilon n]. \tag{3.4}$$

For the second term of the right-hand side of (3.4), we apply (ii) of Lemma 2.1 to obtain that $\mathsf{P}_\omega^x[\tau_{\mathsf{Cub}^c} \leq h_\varepsilon n] \leq \gamma_2 (h_\varepsilon)^{1/2}$. Thus, we can choose ε small enough in such a way that $\mathsf{P}_\omega^x[\tau_{\mathsf{Cub}^c} \leq h_\varepsilon n] \leq \varepsilon_1/2$. Then, using (2.1) and the fact that $|\mathsf{Bad}| < 5^d \varepsilon' H^d n^{\frac{d}{2}}$ for large n, we can show that uniformly in $x \in \mathsf{Bad} \cap [-H\sqrt{n}, H\sqrt{n}]^d$ we have $\mathsf{P}_\omega^x[X(h_\varepsilon n) \in \mathsf{Bad}] \leq \gamma_1 \varepsilon'/h_\varepsilon$ for n sufficiently large. Thus, choosing ε' sufficiently small in such a way that $\gamma_1 \varepsilon'/h_\varepsilon \leq \varepsilon_1/2$ we obtain $\mathsf{P}_\omega^x[X(h_\varepsilon n) \in \mathsf{Bad}] \leq \varepsilon_1/2$.

To show (ii), we notice that

$$\mathsf{P}_\omega^x\Big[\sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n}\Big] = \mathsf{P}_\omega^x[\tau_{B^c(x, h'\sqrt{n})} \leq h_\varepsilon n] \tag{3.5}$$

with $B(x, r)$ the euclidian ball of center x and radius r. Now, we can apply (ii) of Lemma 2.1 to the right-hand term of (3.5) to obtain that

$$\mathsf{P}_\omega^x\Big[\sup_{j \leq h_\varepsilon n} \|X(j) - X(0)\| > h'\sqrt{n}\Big] \leq \gamma_3 \frac{h_\varepsilon^{1/2}}{h'}.$$

Finally, choosing ε sufficiently small such that $\gamma_3 h^{1/2}/h' \leq \varepsilon_1$ we obtain (ii). This concludes the proof of Proposition 3.2. $\square$

Proof of Proposition 3.1. Let us prove (3.1). Consider $x \in [-H\sqrt{n}, H\sqrt{n}]^d$. We start by writing

$$\begin{aligned}\Big|\mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)]\Big| &\le \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)]\Big)\Big| \\ &\quad + \Big|\mathtt{E}_{\theta_x\omega}\Big(\mathtt{E}_{\theta_{X_{\tau_\mathcal{G}}}\omega}[F(Z^n)] - E[F(W)]\Big)\Big| \\ &:= U + V. \end{aligned} \tag{3.6}$$

First, taking $\varepsilon \le \frac{\tilde{\varepsilon}}{2}$ we obtain $V \le \tilde{\varepsilon}/2$ by definition of a (ε, n)-good site. It remains to treat the first term of the right-hand side of (3.6). Denote $X' := X - x$. Now, observe that by the Markov property

$$\begin{aligned} U &= \Big|\mathtt{E}_{\theta_x\omega}\Big(F(Z^n) - \mathtt{E}_{\theta_{X'_{\tau_\mathcal{G}}}(\theta_x\omega)}[F(Z^n)]\Big)\Big| \\ &\le \mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}(Z^n - n^{-1/2}X'_{\tau_\mathcal{G}})\Big|. \end{aligned} \tag{3.7}$$

We are going to show that for n sufficiently large we have uniformly in $x \in [-H\sqrt{n}, H\sqrt{n}]^d$,

$$\mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}(Z^n - n^{-1/2}X'_{\tau_\mathcal{G}})\Big| \le \frac{\tilde{\varepsilon}}{2}$$

for small enough ε. Let $M^n := Z^n - n^{-1/2}X'_{\tau_\mathcal{G}}$. Since F is uniformly continuous, we can choose $\eta > 0$ in such a way that if $\mathtt{d}(f,g) \le \eta$ then $|F(f) - F(g)| \le \frac{\tilde{\varepsilon}}{4}$. Then, we have

$$\begin{aligned} &\mathtt{E}_{\theta_x\omega}\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big| \\ &\quad= \mathtt{E}_{\theta_x\omega}\Big[\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big|\mathbf{1}\{\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) \le \eta\}\Big] \\ &\qquad+ \mathtt{E}_{\theta_x\omega}\Big[\Big|F\circ Z^n - F\circ\theta_{n^{-1}\tau_\mathcal{G}}M^n\Big|\mathbf{1}\{\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\}\Big] \\ &\quad\le \frac{\tilde{\varepsilon}}{4} + 2\|F\|_\infty \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\Big]. \end{aligned} \tag{3.8}$$

Since $h_\varepsilon \le 1$, we have

$$\begin{aligned} &\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta\Big] \\ &\quad\le \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n, \theta_{n^{-1}\tau_\mathcal{G}}M^n) > \eta, \tau_\mathcal{G} \le hn\Big] + \mathtt{P}_{\theta_x\omega}[\tau_\mathcal{G} > h_\varepsilon n] \\ &\quad\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_\mathcal{G}]}\|Z^n - \theta_{n^{-1}\tau_\mathcal{G}}M^n\| > \frac{\eta}{2}, \tau_\mathcal{G} \le h_\varepsilon n\Big] \\ &\qquad+ \mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_\mathcal{G}}Z^n, \theta^2_{n^{-1}\tau_\mathcal{G}}M^n) > \frac{\eta}{2}, \tau_\mathcal{G} \le h_\varepsilon n\Big] + \mathtt{P}_{\theta_x\omega}[\tau_\mathcal{G} > h_\varepsilon n]. \end{aligned} \tag{3.9}$$

Let $\mathcal{F}_{\tau_{\mathcal{G}}}$ be the σ-field generated by X until time $\tau_{\mathcal{G}}$. We first decompose the first term of the right-hand side of (3.9) in the following way:

$$\begin{aligned}
&\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n-\theta_{n^{-1}\tau_{\mathcal{G}}}M^n\|>\frac{\eta}{2},\tau_{\mathcal{G}}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]+\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,h_\varepsilon]}\|\theta_{n^{-1}\tau_{\mathcal{G}}}M^n\|>\frac{\eta}{4}\Big]\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,h_\varepsilon]}\|\theta_{n^{-1}\tau_{\mathcal{G}}}M^n\|>\frac{\eta}{4}\mid\mathcal{F}_{\tau_{\mathcal{G}}}\Big]\Big)\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}\|Z^n\|>\frac{\eta}{4}\Big]\Big). \qquad (3.10)
\end{aligned}$$

We now deal with the second term of the right-hand side of (3.9):

$$\begin{aligned}
&\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal{G}}}Z^n,\theta^2_{n^{-1}\tau_{\mathcal{G}}}M^n)>\frac{\eta}{2},\tau_{\mathcal{G}}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[\|X'_{\tau_{\mathcal{G}}}\|>\frac{\eta}{4}n\Big]+\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal{G}}}M^n,\theta^2_{n^{-1}\tau_{\mathcal{G}}}M^n)>\frac{\eta}{4},\tau_{\mathcal{G}}\le h_\varepsilon n\Big]\\
&\quad\le \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]\\
&\qquad+\mathtt{E}_{\theta_x\omega}\Big(\mathbf{1}\{\tau_{\mathcal{G}}\le h_\varepsilon n\}\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(\theta_{n^{-1}\tau_{\mathcal{G}}}M^n,\theta^2_{n^{-1}\tau_{\mathcal{G}}}M^n)>\frac{\eta}{4}\mid\mathcal{F}_{\tau_{\mathcal{G}}}\Big]\Big)\\
&\quad= \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]\\
&\qquad+\mathtt{E}_{\theta_x\omega}\Big(\mathbf{1}\{\tau_{\mathcal{G}}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal{G}}}Z^n)>\frac{\eta}{4}\Big]\Big). \qquad (3.11)
\end{aligned}$$

Combining (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}
\mathtt{P}_{\theta_x\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal{G}}}M^n)>\eta\Big]&\le \mathtt{P}_{\theta_x\omega}[\tau_{\mathcal{G}}>h_\varepsilon n]\\
&\quad+2\mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]}\|Z^n\|>\frac{\eta}{4}\Big]\\
&\quad+\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}\|Z^n\|>\frac{\eta}{4}\Big] \qquad (3.12)\\
&\quad+\mathbf{1}\{\tau_{\mathcal{G}}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal{G}}}Z^n)>\frac{\eta}{4}\Big]\Big).
\end{aligned}$$

On one hand, by definition of a (ε,n)-good point, choosing small enough $\varepsilon>0$, we have uniformly in $x\in[-H\sqrt{n},H\sqrt{n}]^d$,

$$\begin{aligned}
&\mathtt{E}_{\theta_x\omega}\Big(\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\sup_{t\in[0,h_\varepsilon]}\|Z^n\|>\frac{\eta}{4}\Big]+\mathbf{1}\{\tau_{\mathcal{G}}\le h_\varepsilon n\}\mathtt{P}_{\theta_{X_{\tau_{\mathcal{G}}}}\omega}\Big[\mathtt{d}(Z^n,\theta_{n^{-1}\tau_{\mathcal{G}}}Z^n)>\frac{\eta}{4}\Big]\Big)\\
&\quad\le\frac{\tilde{\varepsilon}}{32\|F\|_\infty} \qquad (3.13)
\end{aligned}$$

for all sufficiently large n. On the other hand, by Proposition 3.2, for sufficiently small ε, we have uniformly in $x \in [-H\sqrt{n}, H\sqrt{n}]^d$,

$$\mathtt{P}_{\theta_x\omega}[\tau_{\mathcal{G}} > h_\varepsilon n] \leq \frac{\tilde{\varepsilon}}{32\|F\|_\infty} \quad \text{and} \quad \mathtt{P}_{\theta_x\omega}\Big[\sup_{t\in[0,n^{-1}\tau_{\mathcal{G}}]} \|Z^n\| > \frac{\eta}{4}\Big] \leq \frac{\tilde{\varepsilon}}{32\|F\|_\infty} \tag{3.14}$$

for sufficiently large n. Combining (3.13), (3.14) with (3.12) and (3.7)–(3.9), we have $U \leq \tilde{\varepsilon}/2$. Together with $V \leq \tilde{\varepsilon}/2$, this leads to the desired result. □

Denote by $\mathfrak{C}_b(C(\mathbb{R}_+), \mathbb{R})$ the space of bounded continuous functionals from $C(\mathbb{R}_+)$ into $\mathbb{R}$ and by $\mathcal{B}$ the Borel σ-field on $C(\mathbb{R}_+)$. The next step is the following proposition, its proof follows essentially the proof of Theorem 2.1 of [4] (cf. also Proposition 3.7 of [11]).

Proposition 3.3. *The first statement implies the second one:*

(i) *for any $F \in \mathfrak{C}_b^u(C(\mathbb{R}_+), \mathbb{R})$, we have $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \sup_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \Big|\mathtt{E}_{\theta_x\omega}[F(Z^n)] - E[F(W)]\Big| = 0;$$

(ii) *for any open set G, we have $\mathbb{P}$-a.s.,*

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G].$$

Finally, we have Proposition 3.4, which is similar to Proposition 3.8 of [11].

Proposition 3.4. *The following statements are equivalent:*

(i) *we have $\mathbb{P}$-a.s., for every open set G,*

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G];$$

(ii) *for every open set G, we have $\mathbb{P}$-a.s.,*

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]^d} \mathtt{P}_{\theta_x\omega}[Z^n \in G] \geq P[W \in G].$$

Proof. (i) $\Rightarrow$ (ii) is trivial. Let us show that (ii) $\Rightarrow$ (i). Suppose that there exists a countable family $\mathcal{H}$ of open sets such that for every open set G there exists a sequence $(O_n)_{n=1,2,\dots} \subset \mathcal{H}$ such that $\mathbf{1}_{O_n} \uparrow \mathbf{1}_G$ pointwise as $n \to \infty$. By (ii), since the family $\mathcal{H}$ is countable we would have, $\mathbb{P}$-a.s., for all $O \in \mathcal{H}$,

$$\liminf_{n\to\infty} \inf_{x\in[-H\sqrt{n},H\sqrt{n}]} \mathtt{P}_{\theta_x\omega}[Z^n \in O] \geq P[W \in O]. \tag{3.15}$$

Then, the same kind of reasoning as that used in the proof of Proposition 3.3 to prove (i) $\Rightarrow$ (ii) would provide the desired result. The fact that $\mathcal{H}$ exists, follows from the fact that the space $C(\mathbb{R}_+)$ is second-countable. $\square$

Proof of Theorem 1.2. One can check that it is straightforward (using the same arguments as in the proof of Proposition 3.3) to deduce that (i), (ii), (iii) and (v) of Theorem 1.2 are equivalent to statement (i) of Proposition 3.4. That is, one can prove the equivalence of items (i)–(v) of Theorem 1.2. To conclude the proof of Theorem 1.2, it remains to show that (ii) of Proposition 3.4 holds. By Proposition 3.3, (ii) of Proposition 3.4 is equivalent to (i) of Proposition 3.3. Since by Proposition 3.1, (i) of Proposition 3.3 holds, the proof of Theorem 1.2 is complete. $\square$

4. Proof of Theorem 1.1

For the sake of brevity, let us denote in this section, the process DZ^n (resp. DX) by $\mathcal{Z}$ (resp. $\mathcal{X}$). We also recall that $W^{(d)} = (W_1, \dots, W_d)$ is a d-dimensional standard Brownian motion. In order to prove Theorem 1.1, we first show convergence of the finite-dimensional distributions and then, in Section 4.2, we prove the tightness of the sequence $(\mathtt{P}_\omega[\mathcal{Z}^n \in \cdot \mid \Lambda_n])_{n\geq 1}$. For $\varepsilon \in (0,1)$, we recall that $N := \lfloor \varepsilon\sqrt{n} \rfloor$. In this section for any set $F \subset \mathbb{R}^d$ we denote

$$\beta_F = \inf\{n \geq 0 : \mathcal{X}(n) \in F\} \quad \text{and} \quad \beta_F^+ = \inf\{n \geq 1 : \mathcal{X}(n) \in F\}.$$

We start by recalling the transition density function of the Brownian meander (see [14]) from $(0,0)$ to (t, x_1)

$$q(0,0;t,x_1) = t^{-3/2} x_1 \exp\Big(-\frac{{x_1}^2}{2t} \Big) \tilde{N}(x_1(1-t)^{-1/2}) \tag{4.1}$$

for $x_1 > 0$, $0 < t \leq 1$ and from (t_1, x_1) to (t_2, x_2)

$$q(t_1, x_1; t_2, x_2) = g(t_2 - t_1, x_1, x_2) \frac{\tilde{N}(x_2(1-t_2)^{-1/2})}{\tilde{N}(x_1(1-t_1)^{-1/2})}$$

for x_1, $x_2 > 0$, $0 < t_1 < t_2 \leq 1$, where

$$\tilde{N}(v) = \Big(\frac{2}{\pi}\Big)^{1/2} \int_0^v e^{-\frac{u^2}{2}} \, du$$

for $v \geq 0$ and

$$g(t, x_1, x_2) = (2\pi)^{-1/2} \Big(\exp\Big(-\frac{(x_2 - x_1)^2}{2t} \Big) - \exp\Big(-\frac{(x_1 + x_2)^2}{2t} \Big) \Big)$$

for x_1, $x_2 > 0$ and $0 < t \leq 1$.

4.1. Convergence of finite-dimensional distributions

In this subsection, we prove the convergence of the finite-dimensional distributions. A key ingredient is the decomposition according to the event $A_{0\to R}$, see (4.3), which says that the random walk progresses enough in the desired direction, without big fluctuations of the other coordinates, before it returns to the hyperplane of the origin. In order to show that this event has large probability, Lemma 2.3 and Lemma 2.4 come into play, see (4.10) and (4.29).

First, let us consider then marginal for $t = 1$.

Proposition 4.1. *We have* $\mathbb{P}$*-a.s.,*

$$\lim_{n\to\infty} \mathtt{P}_\omega[\mathcal{Z}_1^n(1) > u_1, \dots, \mathcal{Z}_d^n(1) > u_d \mid \Lambda_n] = \exp(-u_1^2/2) \prod_{i=2}^{d} \int_{u_i}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt, \quad (4.2)$$

for all $u = (u_1, \dots, u_d) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$.

Proof. First, we introduce some notations. Let

$$\mathcal{D}_u = \{x \in \mathbb{R}^d : x_1 > u_1, \dots, x_d > u_d\}$$

and

$$R_{\varepsilon,n} = \{x \in \mathbb{R}^d : x_1 = N, x_i \in [-\lfloor \varepsilon^{-1/2} N \rfloor, \lfloor \varepsilon^{-1/2} N \rfloor], i \in [\![2, d]\!]\}.$$

Let us denote $\mathcal{R}_{\varepsilon,n} = DR_{\varepsilon,n}$, and define the event

$$A_{0\to R} = \{\beta_{\mathcal{R}_{\varepsilon,n}} < \beta^+_{\{0\}_1}\}. \quad (4.3)$$

We start by bounding the term $\mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n]$ from above. Fix $\varepsilon \in (0, u_1 \wedge 1)$ and consider the following decomposition

$$\begin{aligned}
&\mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n] \\
&\quad\le \frac{1}{\mathtt{P}_\omega[\Lambda_n]} \Big(\mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u, A_{0\to R}, \Lambda_n] + \mathtt{P}_\omega[A^c_{0\to R}, \Lambda_n] \Big) \\
&\quad= \frac{1}{\mathtt{P}_\omega[\Lambda_n]} \Big(\mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} \le \varepsilon^{1/2} n] \\
&\qquad + \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n] \Big) + \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n] \\
&\quad\le (\mathtt{P}_\omega[\Lambda_n])^{-1} \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} \le \varepsilon^{1/2} n] \\
&\qquad + \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] + \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n]. \qquad (4.4)
\end{aligned}$$

Since $\varepsilon^{1/2} \in (0, 1)$, we have

$$\mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n] = \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \beta^+_{\{0\}_1} \mid \Lambda_n]$$

$$\le \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > n \mid \Lambda_n] \le \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n]. \tag{4.5}$$

Then, using the Markov property at time $\beta_{\mathcal{R}_{\varepsilon,n}}$ we deduce

$$\begin{aligned} &\frac{1}{\mathtt{P}_\omega[\Lambda_n]} \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} \le \varepsilon^{1/2} n] \\ &\qquad \le \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]} \max_{y\in R_{\varepsilon,n}} \max_{j\le \lfloor \varepsilon^{1/2} n\rfloor} \mathtt{P}_\omega^y\Big[\frac{\mathcal{X}(n-j)}{\sqrt{n}} \in \mathcal{D}_u, \Lambda_{n-j}\Big]. \end{aligned} \tag{4.6}$$

Again, using the Markov property at time $\beta_{\mathcal{R}_{\varepsilon,n}}$ we obtain

$$\frac{\mathtt{P}_\omega[\Lambda_n]}{\mathtt{P}_\omega[A_{0\to R}]} \ge \min_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y[\Lambda_n]. \tag{4.7}$$

Combining (4.4), (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned} \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n] \le{}& \frac{\max_{y\in R_{\varepsilon,n}} \max_{j\le\lfloor\varepsilon^{1/2}n\rfloor} \mathtt{P}_\omega^y[\mathcal{X}(n-j) \in \mathcal{D}_u\sqrt{n}, \Lambda_{n-j}]}{\min_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y[\Lambda_n]} \\ &+ 2\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n]. \end{aligned} \tag{4.8}$$

Now, to bound the term $\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n]$ from above we notice that

$$\begin{aligned} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] &= \mathtt{P}_\omega[\tau_{R_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] \\ &\le \mathtt{P}_\omega\Big[\max_{i\in[\![2,d]\!]} \sup_{j\le\tau_{\{N\}_1}} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_n\Big] \\ &\quad + \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n \mid \Lambda_n]. \end{aligned} \tag{4.9}$$

By Lemmas 2.3 and 2.4 we have

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] \le f(\varepsilon) + g(\varepsilon). \tag{4.10}$$

By definition of $\mathcal{Z}^n$, we have $\mathtt{P}_\omega^y[\Lambda_n] = \mathtt{P}_\omega^y\Big[\mathcal{Z}_1^n(1) > 0, t \in [0,1]\Big]$. Thus, from Theorem 1.2 we obtain, recalling that W_1 is the first component of $W^{(d)}$,

$$\begin{aligned} \lim_{n\to\infty} \min_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y\Big[\mathcal{Z}_1^n(t) > 0, t\in[0,1]\Big] &= P^{\varepsilon\sigma_1}\Big[\min_{0\le t\le 1} W_1(t) > 0\Big] \\ &= P\Big[|W_1(1)| < \varepsilon\sigma_1\Big] = \frac{2\varepsilon\sigma_1}{\sqrt{2\pi}} + o(\varepsilon) \end{aligned} \tag{4.11}$$

as $\varepsilon \to 0$, where P^x is law of $W^{(d)}$ starting at x and $\sigma_1 := D\mathbf{e}_1 \cdot \mathbf{e}_1 > 0$ (cf. Section 1). Now, let us treat the term

$$\max_{y\in R_{\varepsilon,n}} \max_{j\le\lfloor\varepsilon^{1/2}n\rfloor} \mathtt{P}_\omega^y[\mathcal{X}(n-j) \in \mathcal{D}_u\sqrt{n}, \Lambda_{n-j}].$$

Fix $\delta' > 0$ and let $\mathrm{sgn}(x) = -1$ if $x \le 0$ and 1 if $x > 0$. Denote

$$U_i := \Big\{\mathcal{X}_i(n - \lfloor \varepsilon^{1/2} n\rfloor) > (u_i - \mathrm{sgn}(u_i)\delta')\sqrt{n}\Big\} \tag{4.12}$$

and

$$V_i := \Big\{\max_{j \le \lfloor \varepsilon^{1/2} n\rfloor} |\mathcal{X}_i(n - \lfloor \varepsilon^{1/2} n\rfloor) - \mathcal{X}_i(n-j)| \ge \delta'\sqrt{n}\Big\} \tag{4.13}$$

for $i = 1, \dots, d$. Observe that we have for $y \in R_{\varepsilon,n}$ and $j \le \lfloor \varepsilon^{1/2} n\rfloor$

$$\mathtt{P}^y_\omega[\mathcal{X}(n-j) \in \mathcal{D}_u\sqrt{n}, \Lambda_{n-j}] \le \mathtt{P}^y_\omega\Big[\bigcap_{i=1}^d (U_i \cup V_i) \cap \Lambda_{n - \lfloor \varepsilon^{1/2} n\rfloor}\Big].$$

Let us consider the set $\mathcal{I} = \{U_1, \dots, U_d, V_1, \dots, V_d\}$ and denote by $\mathcal{J}$ the set formed by all intersections of d distinct elements of $\mathcal{I}$: $\mathcal{J}$ contains $\binom{2d}{d}$ elements. Let us denote by $J_1, \dots, J_{\binom{2d}{d}}$ all the elements of $\mathcal{J}$. Therefore, we obtain

$$\max_{j \le \lfloor \varepsilon^{1/2} n\rfloor} \mathtt{P}^y_\omega[\mathcal{X}(n-j) \in \mathcal{D}_u\sqrt{n}, \Lambda_{n-j}] \le \sum_{i \le \binom{2d}{d}} \mathtt{P}^y_\omega\Big[J_i, \Lambda_{n - \lfloor \varepsilon^{1/2} n\rfloor}\Big]. \tag{4.14}$$

Let us treat the term $\mathtt{P}^y_\omega[\cap_{i=1}^d U_i, \Lambda_{n - \lfloor \varepsilon^{1/2} n\rfloor}]$. We have by definition of $\mathcal{Z}^n$

$$\begin{aligned}
&\mathtt{P}^y_\omega\Big[\bigcap_{i=1}^d U_i, \Lambda_{n - \lfloor \varepsilon^{1/2} n\rfloor}\Big] \\
&\le \mathtt{P}^y_\omega\Big[\bigcap_{i=1}^d \Big\{\mathcal{Z}_i^{n - \lfloor \varepsilon^{1/2} n\rfloor}(1) > (u_i - \mathrm{sgn}(u_i)\delta')\Big\}, \mathcal{Z}_1^{n - \lfloor \varepsilon^{1/2} n\rfloor}(t) > 0, t \in [0,1]\Big].
\end{aligned}$$

By Theorem 1.2 we deduce

$$\begin{aligned}
\limsup_{n\to\infty} \max_{y \in R_{\varepsilon,n}} &\mathtt{P}^y_\omega\Big[\bigcap_{i=1}^d U_i, \Lambda_{n - \lfloor \varepsilon^{1/2} n\rfloor}\Big] \\
&\le P^{\frac{\varepsilon\sigma_1}{\sqrt{1-\varepsilon^{1/2}}}}\Big[W_1(1) > (u_1 - \mathrm{sgn}(u_1)\delta'), \min_{0\le t\le 1} W_1(t) > 0\Big] \\
&\quad \times \prod_{i=2}^d P^{\frac{\gamma_1\varepsilon^{1/2}}{\sqrt{1-\varepsilon^{1/2}}}}[W_i(1) > (u_i - \mathrm{sgn}(u_i)\delta')]
\end{aligned} \tag{4.15}$$

for some constant γ_1. Abbreviate $\varepsilon' := \sigma_1\varepsilon(1-\varepsilon^{1/2})^{-1/2}$ and let us compute the first term of the right-hand side of (4.15) for sufficiently small ε. By the reflection principle for the Brownian motion, we have

$$P^{\varepsilon'}\Big[W_1(1) > (u_1 - \mathrm{sgn}(u_1)\delta'), \min_{0\le t\le 1} W_1(t) > 0\Big]$$

$$
\begin{aligned}
&= P^{\varepsilon'}\Big[W_1(1) > (u_1 - \operatorname{sgn}(u_1)\delta')\Big] \\
&\quad - P^{\varepsilon'}\Big[W_1(1) < -(u_1 - \operatorname{sgn}(u_1)\delta')\Big] \\
&= P\Big[W_1(1) > (u_1 - \operatorname{sgn}(u_1)\delta') - \varepsilon'\Big] \\
&\quad - P\Big[W_1(1) < -(u_1 - \operatorname{sgn}(u_1)\delta') - \varepsilon')\Big] \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{(u_1 - \operatorname{sgn}(u_1)\delta') - \varepsilon'}^{(u_1 - \operatorname{sgn}(u_1)\delta') + \varepsilon'} e^{-\frac{x^2}{2}}\, dx.
\end{aligned}
$$

Therefore, we obtain, as $\varepsilon \to 0$

$$
\begin{aligned}
&\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y\Big[\bigcap_{i=1}^d U_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}\Big] \\
&\quad \le \Big(\frac{2\varepsilon\sigma_1 e^{-\frac{(u_1-\operatorname{sgn}(u_1)\delta')^2}{2}}}{\sqrt{2\pi(1-\varepsilon^{1/2})}} + o(\varepsilon)\Big) \prod_{i=2}^d \int\limits_{(u_i - \operatorname{sgn}(u_i)\delta') - \frac{\gamma_1 \varepsilon^{1/2}}{\sqrt{1-\varepsilon^{1/2}}}}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt. \qquad (4.16)
\end{aligned}
$$

The other terms $\mathtt{P}_\omega^y[J_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}]$ necessarily contain a term V_j for some $j \in [\![1, d]\!]$. Thus, we have for $J_i \neq \cap_{i=1}^d U_i$,

$$
\mathtt{P}_\omega^y[J_i, \Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}] \le \sum_{j=1}^d \mathtt{P}_\omega^y[V_j]. \qquad (4.17)
$$

Let us bound the terms $\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y[V_j]$ for $j \in [\![1, d]\!]$. We start by writing

$$
\begin{aligned}
\mathtt{P}_\omega^y[V_j] &= \mathtt{P}_\omega^y\Big[\max_{i\le\lfloor\varepsilon^{1/2}n\rfloor} |\mathcal{X}_j(n-\lfloor\varepsilon^{1/2}n\rfloor) - \mathcal{X}_j(n-i)| \ge \delta'\sqrt{n}\Big] \\
&= \mathtt{P}_\omega^y\Big[\max_{n-\lfloor\varepsilon^{1/2}n\rfloor \le k\le n} \Big|\mathcal{X}_j(k) - \mathcal{X}_j(n-\lfloor\varepsilon^{1/2}n\rfloor)\Big| \ge \delta'\sqrt{n}\Big] \\
&\le \mathtt{P}_\omega^y\Big[\max_{1-\varepsilon^{1/2}\le t\le 1} \Big(\mathcal{Z}_j^n(t) - \min_{1-\varepsilon^{1/2}\le s\le t} \mathcal{Z}_j^n(s)\Big) \ge \delta'\Big] \\
&\quad + \mathtt{P}_\omega^y\Big[\min_{1-\varepsilon^{1/2}\le t\le 1} \Big(\mathcal{Z}_j^n(t) - \max_{1-\varepsilon^{1/2}\le s\le t} \mathcal{Z}_j^n(s)\Big) \le -\delta'\Big].
\end{aligned}
$$

By Theorem 1.2, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}_\omega^y\Big[\max_{1-\varepsilon^{1/2}\le t\le 1} \Big(\mathcal{Z}_j^n(t) - \min_{1-\varepsilon^{1/2}\le s\le t} \mathcal{Z}_j^n(s)\Big) \ge \delta'\Big] \\
&\quad = P\Big[\max_{1-\varepsilon^{1/2}\le t\le 1} \Big(W_j(t) - \min_{1-\varepsilon^{1/2}\le s\le t} W_j(s)\Big) \ge \delta'\Big] \qquad (4.18)
\end{aligned}
$$

and

$$\lim_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega\Big[\min_{1-\varepsilon^{1/2}\le t\le 1}\Big(\mathcal{Z}^n_j(t) - \max_{1-\varepsilon^{1/2}\le s\le t} \mathcal{Z}^n_j(s)\Big) \le -\delta'\Big]$$
$$= P\Big[\min_{1-\varepsilon^{1/2}\le t\le 1}\Big(W_j(t) - \max_{1-\varepsilon^{1/2}\le s\le t} W_j(s)\Big) \le -\delta'\Big]. \tag{4.19}$$

Observe that the right-hand sides of (4.18) and (4.19) are equal since $(-W_j)$ is a Brownian motion. Thus, let us compute for example the right-hand side term of (4.18). By Lévy's Theorem (cf. [18], Chapter VI, Theorem 2.3), we have

$$P\Big[\max_{0\le t\le\varepsilon^{1/2}}\Big(W_j(t) - \min_{0\le s\le t} W_j(s)\Big) \ge \delta'\Big] = P\Big[\max_{0\le t\le\varepsilon^{1/2}} |W_j(t)| \ge \delta'\Big].$$

Then,

$$P\Big[\max_{0\le t\le\varepsilon^{1/2}} |W_j(t)| \ge \delta'\Big] \le 2P\Big[\max_{0\le t\le\varepsilon^{1/2}} W_j(t) \ge \delta'\Big] = 4P[W_j(\varepsilon^{1/2}) \ge \delta'].$$

Using an estimate on the tail of the Gaussian law ([17], Appendix B, Lemma 12.9) we obtain

$$P\Big[\max_{0\le t\le\varepsilon^{1/2}} |W_j(t)| \ge \delta'\Big] \le \frac{4\varepsilon^{1/4}}{\delta'\sqrt{2\pi}} \exp\Big\{-\frac{(\delta')^2}{2\varepsilon^{1/2}}\Big\}.$$

We finally obtain

$$\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \sum_{i=1}^d \mathtt{P}^y_\omega[V_i] \le \frac{8d\varepsilon^{1/4}}{\delta'\sqrt{2\pi}} \exp\Big\{-\frac{(\delta')^2}{2\varepsilon^{1/2}}\Big\}. \tag{4.20}$$

To sum up, combining (4.11), (4.14), (4.16), (4.17), and (4.20), we have $\mathbb{P}$-a.s.

$$\begin{aligned}
&\limsup_{n\to\infty} \mathtt{P}_\omega[\mathcal{Z}^n(1)\in\mathcal{D}_u \mid \Lambda_n] \\
&\quad\le \Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi}} + o(\varepsilon)\Big)^{-1}\Big(\frac{2\varepsilon\sigma_1 e^{-\frac{(u_1-\mathrm{sgn}(u_1)\delta')^2}{2}}}{\sqrt{2\pi(1-\varepsilon^{1/2})}} + o(\varepsilon)\Big) \\
&\qquad\times \prod_{i=2}^d \int_{(u_i-\mathrm{sgn}(u_i)\delta')-\frac{\gamma_1\varepsilon^{1/2}}{\sqrt{1-\varepsilon^{1/2}}}}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt \\
&\qquad + \binom{2d}{d}\frac{8d\varepsilon^{1/4}}{\delta'\sqrt{2\pi}} \exp\Big\{-\frac{(\delta')^2}{2\varepsilon^{1/2}}\Big\} + 2(f(\varepsilon)+g(\varepsilon)).
\end{aligned} \tag{4.21}$$

Let us now bound the term $\mathtt{P}_\omega[\mathcal{Z}^n(1)\in\mathcal{D}_u \mid \Lambda_n]$ from below. We have by the Markov property

$$\mathtt{P}_\omega[\mathcal{Z}^n(1)\in\mathcal{D}_u \mid \Lambda_n]$$

$$\geq \frac{\mathtt{P}_\omega[A_{0\to R}, \beta_{\mathcal{R}_{\varepsilon,n}} \leq \varepsilon^{1/2} n]}{\mathtt{P}_\omega[\Lambda_n]} \min_{y \in R_{\varepsilon,n}} \min_{j \leq \lfloor \varepsilon^{1/2} n \rfloor} \mathtt{P}_\omega^y[\mathcal{X}(n-j) \in \mathcal{D}_u \sqrt{n}, \Lambda_{n-j}]. \tag{4.22}$$

We first decompose the term $(\mathtt{P}_\omega[\Lambda_n])^{-1}\mathtt{P}_\omega[A_{0\to R}, \beta_{\mathcal{R}_{\varepsilon,n}} \leq \varepsilon^{1/2} n]$ in the following way

$$\begin{aligned} \frac{\mathtt{P}_\omega[A_{0\to R}, \beta_{\mathcal{R}_{\varepsilon,n}} \leq \varepsilon^{1/2} n]}{\mathtt{P}_\omega[\Lambda_n]} &= \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]} - \frac{\mathtt{P}_\omega[A_{0\to R}, \beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n]}{\mathtt{P}_\omega[\Lambda_n]} \\ &= \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]}(1 - \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid A_{0\to R}]). \end{aligned} \tag{4.23}$$

Then, we write

$$\begin{aligned} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid A_{0\to R}] &= \frac{\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n, A_{0\to R}]}{\mathtt{P}_\omega[A_{0\to R}]} \\ &\leq \frac{\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n, \Lambda_{\varepsilon^{1/2} n}]}{\mathtt{P}_\omega[A_{0\to R}, \Lambda_{\varepsilon^{1/2} n}]} \\ &= \frac{\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}]}{1 - \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_{\varepsilon^{1/2} n}]}. \end{aligned} \tag{4.24}$$

For the term $\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}]$, we have, recalling that $N = \lfloor \varepsilon \sqrt{n} \rfloor$,

$$\begin{aligned} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] &= \mathtt{P}_\omega[\tau_{R_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] \\ &\leq \mathtt{P}_\omega\Big[\max_{i \in [\![2,d]\!]} \sup_{j \leq \tau_{\{N\}_1}} |X_i(j)| > \varepsilon^{-1/2} N \mid \Lambda_{\varepsilon^{1/2} n}\Big] \\ &\quad + \mathtt{P}_\omega[\tau_{\{N\}_1} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}]. \end{aligned} \tag{4.25}$$

By Lemmas 2.3 and 2.4 we deduce

$$\limsup_{n\to\infty} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}] \leq g(\varepsilon^{3/4}) + f(\varepsilon^{3/4}). \tag{4.26}$$

For the term $\mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_{\varepsilon^{1/2} n}]$, we write

$$\mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_{\varepsilon^{1/2} n}] = \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \beta^+_{\{0\}_1} \mid \Lambda_{\varepsilon^{1/2} n}] \leq \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_{\varepsilon^{1/2} n}].$$

Hence, by (4.26) we obtain

$$\limsup_{n\to\infty} \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_{\varepsilon^{1/2} n}] \leq f(\varepsilon^{3/4}) + g(\varepsilon^{3/4}). \tag{4.27}$$

Going back to the term $(\mathtt{P}_\omega[\Lambda_n])^{-1}\mathtt{P}_\omega[A_{0\to R}]$ in (4.23), we write

$$\frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]} = \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n, A_{0\to R}] + \mathtt{P}_\omega[\Lambda_n, A^c_{0\to R}]}$$

$$
\begin{aligned}
&= \Big(\mathtt{P}_\omega[\Lambda_n \mid A_{0\to R}] + \mathtt{P}_\omega[\Lambda_n, A^c_{0\to R}](\mathtt{P}_\omega[A_{0\to R}])^{-1}\Big)^{-1}\\
&\geq \Big(\mathtt{P}_\omega[\Lambda_n \mid A_{0\to R}] + \mathtt{P}_\omega[\Lambda_n, A^c_{0\to R}](\mathtt{P}_\omega[\Lambda_n, A_{0\to R}])^{-1}\Big)^{-1}\\
&= \Big(\mathtt{P}_\omega[\Lambda_n \mid A_{0\to R}] + \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n](1-\mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n])^{-1}\Big)^{-1}.
\end{aligned}
\tag{4.28}
$$

By (4.10), we have

$$
\limsup_{n\to\infty} \mathtt{P}_\omega[A^c_{0\to R} \mid \Lambda_n] \leq \limsup_{n\to\infty} \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] \leq f(\varepsilon) + g(\varepsilon). \tag{4.29}
$$

Then, we have by the Markov property

$$
\mathtt{P}_\omega[\Lambda_n \mid A_{0\to R}] \leq \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega[\Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}] + \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid A_{0\to R}]. \tag{4.30}
$$

Thus, by (4.28), (4.30), (4.24), (4.26), (4.27), and (4.29), we deduce

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]} \qquad (4.31)\\
&\geq \Big(\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega[\Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}] + \frac{f(\varepsilon^{3/4})+g(\varepsilon^{3/4})}{1-f(\varepsilon^{3/4})-g(\varepsilon^{3/4})} + \frac{f(\varepsilon)+g(\varepsilon)}{1-f(\varepsilon)-g(\varepsilon)}\Big)^{-1}.
\end{aligned}
$$

Combining (4.22), (4.23), (4.26), (4.27), and (4.31), we obtain $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\liminf_{n\to\infty} \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n]\\
&\geq \Big(\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega[\Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}] + \frac{f(\varepsilon^{3/4})+g(\varepsilon^{3/4})}{1-f(\varepsilon^{3/4})-g(\varepsilon^{3/4})} + \frac{f(\varepsilon)+g(\varepsilon)}{1-f(\varepsilon)-g(\varepsilon)}\Big)^{-1}\\
&\quad \times \Big(1 - \frac{f(\varepsilon^{3/4})+g(\varepsilon^{3/4})}{1-f(\varepsilon^{3/4})-g(\varepsilon^{3/4})}\Big)\\
&\quad \times \liminf_{n\to\infty} \min_{y\in R_{\varepsilon,n}} \min_{j\leq \varepsilon^{1/2} n} \mathtt{P}^y_\omega[\mathcal{X}(n-j) \in \mathcal{D}_u\sqrt{n}, \Lambda_{n-j}].
\end{aligned}
\tag{4.32}
$$

Analogously to (4.11) we have

$$
\lim_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega[\Lambda_{n-\lfloor \varepsilon^{1/2} n\rfloor}] = \frac{2\varepsilon\sigma_1}{\sqrt{2\pi(1-\varepsilon^{1/2})}} + o(\varepsilon). \tag{4.33}
$$

At this point, let us introduce more notations. Let $\delta' > 0$ be the constant used in the definitions of V_i and U_i (cf. (4.12) and (4.13)) and introduce

$$
E_i = \Big\{\mathcal{X}_i(n) > (u_i + \operatorname{sgn}(u_i)\delta')\sqrt{n}\Big\}
$$

and

$$F_i = \Big\{ \max_{j \le \lfloor \varepsilon^{1/2} n \rfloor} |\mathcal{X}_i(n) - \mathcal{X}_i(n-j)| \le \delta' \sqrt{n} \Big\}$$

for $i \in [\![1, d]\!]$. Observe that for all $y \in R_{\varepsilon,n}$ and $j \le \lfloor \varepsilon^{1/2} n \rfloor$ we have

$$\begin{aligned} \mathtt{P}_\omega^y[\mathcal{X}(n-j) \in \mathcal{D}_u \sqrt{n}, \Lambda_{n-j}] &\ge \mathtt{P}_\omega^y \Big[\bigcap_{i=1}^d (E_i \cap F_i), \Lambda_n \Big] \\ &\ge \mathtt{P}_\omega^y \Big[\bigcap_{i=1}^d E_i, \Lambda_n \Big] - \sum_{i=1}^d \mathtt{P}_\omega^y [F_i^c]. \end{aligned} \tag{4.34}$$

By Theorem 1.2 and similar computations as those to derive equations (4.16) and (4.20), we obtain for some constant γ_2,

$$\begin{aligned} \lim_{n\to\infty} \min_{y \in R_{\varepsilon,n}} \mathtt{P}_\omega^y \Big[\bigcap_{i=1}^d E_i, \Lambda_n \Big] = \Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi}} \exp\Big\{ -\frac{(u_1 + \operatorname{sgn}(u_1)\delta')^2}{2} \Big\} + o(\varepsilon) \Big) \\ \times \prod_{i=2}^d \int_{(u_i + \operatorname{sgn}(u_i)\delta') - \gamma_2 \varepsilon^{1/2}}^{\infty} \frac{\exp\Big\{ -\frac{t^2}{2} \Big\}}{\sqrt{2\pi}} \, dt \end{aligned} \tag{4.35}$$

as $\varepsilon \to 0$ and

$$\limsup_{n\to\infty} \max_{y \in R_{\varepsilon,n}} \sum_{i=1}^d \mathtt{P}_\omega^y [F_i^c] \le \frac{8d\varepsilon^{1/4}}{\delta' \sqrt{2\pi}} \exp\Big\{ -\frac{(\delta')^2}{2\varepsilon^{1/2}} \Big\}. \tag{4.36}$$

Combining (4.32), (4.33), (4.35), and (4.36), we obtain $\mathbb{P}$-a.s.

$$\begin{aligned} &\liminf_{n\to\infty} \mathtt{P}_\omega[\mathcal{Z}^n(1) \in \mathcal{D}_u \mid \Lambda_n] \\ &\quad \ge \Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi(1-\varepsilon^{1/2})}} + o(\varepsilon) + \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - g(\varepsilon^{3/4})} + \frac{f(\varepsilon) + g(\varepsilon)}{1 - f(\varepsilon) - g(\varepsilon)} \Big)^{-1} \\ &\quad \times \Big(1 - \frac{f(\varepsilon^{3/4}) + g(\varepsilon^{3/4})}{1 - f(\varepsilon^{3/4}) - g(\varepsilon^{3/4})} \Big) \\ &\quad \times \Big(\Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi}} e^{-\frac{(u_1 + \operatorname{sgn}(u_1)\delta')^2}{2}} + o(\varepsilon) \Big) \prod_{i=2}^d \int_{(u_i + \operatorname{sgn}(u_i)\delta') - \gamma_2\varepsilon^{1/2}}^{\infty} \frac{\exp\Big\{ -\frac{t^2}{2} \Big\}}{\sqrt{2\pi}} \, dt \\ &\quad - \frac{8d\varepsilon^{1/4}}{\delta'\sqrt{2\pi}} \exp\Big\{ -\frac{(\delta')^2}{2\varepsilon^{1/2}} \Big\} \Big). \end{aligned} \tag{4.37}$$

Finally, take $\delta' = \varepsilon^{1/8}$ and let $\varepsilon \to 0$ in (4.21) and (4.37) to prove (4.2). $\square$

The next steps in showing that the f.d.d.'s converge are standard and we follow [14] and [11].
First, we will prove the following

Proposition 4.2. *We have* $\mathbb{P}$*-a.s., for* $u_1 > 0$, $-\infty < a_i < b_i < \infty$, $i \in [\![2, d]\!]$ *and* $0 < t < 1$,

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[\mathcal{Z}_1^n(t) \le u_1, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(t) \in (a_i, b_i]\Big\} \mid \Lambda_n\Big]$$
$$= \int_0^{u_1} q(0,0;t,v)dv \prod_{i=2}^{d}\int_{a_i}^{b_i} \frac{\exp\Big\{-\frac{v^2}{2t}\Big\}}{\sqrt{2\pi t}}dv. \tag{4.38}$$

Proof. For $\varepsilon > 0$ we have

$$\mathtt{P}_\omega\Big[\mathcal{Z}_1^n(n^{-1}\lfloor nt\rfloor) \le u_1 - \varepsilon, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(n^{-1}\lfloor nt\rfloor) \in (a_i - \varepsilon, b_i + \varepsilon]\Big\} \mid \Lambda_n\Big]$$
$$\le \mathtt{P}_\omega\Big[\mathcal{Z}_1^n(t) \le u_1, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(t) \in (a_i, b_i]\Big\} \mid \Lambda_n\Big] \tag{4.39}$$
$$\le \mathtt{P}_\omega\Big[\mathcal{Z}_1^n(n^{-1}\lfloor nt\rfloor) \le u_1 + \varepsilon, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(n^{-1}\lfloor nt\rfloor) \in (a_i + \varepsilon, b_i - \varepsilon]\Big\} \mid \Lambda_n\Big].$$

for all sufficiently large n. Now, suppose that we have for all $u_1 \ge 0$, $a_i < b_i$ and $0 < t < 1$,

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[\mathcal{Z}_1^n(n^{-1}\lfloor nt\rfloor) \le u_1, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(n^{-1}\lfloor nt\rfloor) \in (a_i, b_i]\Big\} \mid \Lambda_n\Big]$$
$$= \int_0^{u_1} q(0,0;t,v)\,dv \prod_{i=2}^{d}\int_{a_i}^{b_i} \frac{\exp\Big\{-\frac{v^2}{2t}\Big\}}{\sqrt{2\pi t}}\,dv. \tag{4.40}$$

Combining (4.39) and (4.40) yields (4.38) since the limit distribution $q(0,0;t,x_1)$ is absolutely continuous. Let us denote by $l = l(t,n)$ the quantity $(n\lfloor nt\rfloor^{-1})^{1/2}$. We recall that x is the vector of coordinates $(x_1, \dots, x_d)$. Then, observe that

$$\mathtt{P}_\omega\Big[\mathcal{Z}_1^n(n^{-1}\lfloor nt\rfloor) \le u_1, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(n^{-1}\lfloor nt\rfloor) \in (a_i, b_i]\Big\} \mid \Lambda_n\Big]$$
$$= \frac{1}{\mathtt{P}_\omega[\Lambda_n]}$$

$$
\begin{aligned}
&\times \mathtt{P}_\omega\Big[\mathcal{Z}_1^{\lfloor nt\rfloor}(1)\le lu_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\in(la_i,lb_i]\Big\}, \Lambda_{nt}, \mathcal{X}_1(k)>0, \lfloor nt\rfloor<k\le n\Big]\\
&=\frac{1}{\mathtt{P}_\omega[\Lambda_n]}\int_0^{lu_1}\int_{la_2}^{lb_2}\cdots\int_{la_d}^{lb_d}\\
&\qquad \mathtt{P}_\omega\Big[\mathcal{Z}_1^{\lfloor nt\rfloor}(1)\in dx_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\in dx_i\Big\}, \Lambda_{nt}, \mathcal{X}_1(k)>0, \lfloor nt\rfloor<k\le n\Big]\\
&=\frac{\mathtt{P}_\omega[\Lambda_{nt}]}{\mathtt{P}_\omega[\Lambda_n]}\int_0^{lu_1}\int_{la_2}^{lb_2}\cdots\int_{la_d}^{lb_d}\\
&\qquad \mathtt{P}_\omega\Big[\mathcal{X}_1(k)>0, \lfloor nt\rfloor<k\le n \mid \mathcal{Z}_1^{\lfloor nt\rfloor}(1)\in dx_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\in dx_i\Big\}\Big]\\
&\qquad \times \mathtt{P}_\omega\Big[\mathcal{Z}_1^{\lfloor nt\rfloor}(1)\in dx_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\in dx_i\Big\} \mid \Lambda_{nt}\Big]\\
&=\frac{\mathtt{P}_\omega[\Lambda_{nt}]}{\mathtt{P}_\omega[\Lambda_n]}\int_0^{lu_1}\int_{la_2}^{lb_2}\cdots\int_{la_d}^{lb_d} \mathtt{P}_\omega^{x\sqrt{\lfloor nt\rfloor}}\Big[\mathcal{Z}_1^n(s)>0, 0\le s\le 1-n^{-1}\lfloor nt\rfloor\Big]\\
&\qquad \times \mathtt{P}_\omega\Big[\mathcal{Z}_1^{\lfloor nt\rfloor}(1)\in dx_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\in dx_i\Big\} \mid \Lambda_{nt}\Big].
\end{aligned}
\tag{4.41}
$$

By (4.7), (4.11), (4.31), and (4.33) we have $\mathbb{P}$-a.s.

$$
\lim_{n\to\infty}\frac{\mathtt{P}_\omega[\Lambda_{nt}]}{\mathtt{P}_\omega[\Lambda_n]} = t^{-1/2}. \tag{4.42}
$$

Using Theorem 1.2 and Dini's theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain

$$
\lim_{n\to\infty}\mathtt{P}_\omega^{z\sqrt{\lfloor nt\rfloor}}\Big[\mathcal{Z}_1^n(s)>0, 0\le s\le 1-n^{-1}\lfloor nt\rfloor\Big] = \tilde{N}\Big(z_1\Big(\frac{t}{1-t}\Big)^{1/2}\Big)
$$

uniformly in z on every compact set of the form $[0,K]\times[-K,K]^{d-1}$. By Proposition 4.1, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\mathtt{P}_\omega\Big[\mathcal{Z}_1^{\lfloor nt\rfloor}(1)\le x_1, \bigcap_{i=2}^d \Big\{\mathcal{Z}_i^{\lfloor nt\rfloor}(1)\le x_i\Big\} \mid \Lambda_{nt}\Big]\\
&\quad = \exp\Big(-\frac{x_1^2}{2}\Big)\prod_{i=2}^d\int_{-\infty}^{x_i}\frac{\exp\big\{-\frac{v^2}{2}\big\}}{\sqrt{2\pi}}\,dv.
\end{aligned}
$$

Now, applying Lemma 2.18 of [14] to (4.41), we obtain

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[\mathcal{Z}_1^n(n^{-1}\lfloor nt\rfloor) \le u, \bigcap_{i=2}^{d}\Big\{\mathcal{Z}_i^n(n^{-1}\lfloor nt\rfloor) \in (a_i, b_i]\Big\} \mid \Lambda_n\Big]$$
$$= \int_0^{u_1 t^{-1/2}} \int_{a_2 t^{-1/2}}^{b_2 t^{-1/2}} \dots \int_{a_d t^{-1/2}}^{b_d t^{-1/2}} t^{-1/2} \tilde{N}\Big(x_1\Big(\frac{t}{1-t}\Big)^{1/2}\Big)$$
$$\times\, x_1 \exp\Big\{-\frac{x_1^2}{2}\Big\} \prod_{i=2}^{d} \frac{\exp\Big\{-\frac{x_i^2}{2}\Big\}}{\sqrt{2\pi}}\, dx_1 \dots dx_d.$$

Finally, make the change of variables $y = t^{1/2}x$ to obtain the desired result. $\square$

The final step in showing convergence of the f.d.d.'s is

Proposition 4.3. *We have $\mathbb{P}$-a.s., for all $k \ge 1$, $u_i > 0$, $-\infty < a_j^i < b_j^i < \infty$, $i \in [\![1,k]\!]$, $j \in [\![2,d]\!]$ and $0 < t_1 < t_2 < \dots < t_k \le 1$,*

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[\bigcap_{i=1}^{k}\Big\{\mathcal{Z}_1^n(t_i) \le u_i, \mathcal{Z}_2^n(t_i) \in (a_2^i, b_2^i], \dots, \mathcal{Z}_d^n(t_i) \in (a_d^i, b_d^i]\Big\} \mid \Lambda_n\Big]$$
$$= \prod_{j=2}^{d} \int_{a_j^1}^{b_j^1} \dots \int_{a_j^k}^{b_j^k} \frac{\exp\Big\{-\frac{x_1^2}{2t_1}\Big\}}{\sqrt{2\pi t_1}} \frac{\exp\Big\{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\Big\}}{\sqrt{2\pi(t_2-t_1)}} \dots \frac{\exp\Big\{-\frac{(x_k-x_{k-1})^2}{2(t_k-t_{k-1})}\Big\}}{\sqrt{2\pi(t_k-t_{k-1})}}\, dx_k \dots dx_1$$
$$\times \int_0^{u_1} \dots \int_0^{u_k} q(0,0;t_1,x_1) q(t_1,x_1;t_2,y_2) \dots q(t_{k-1},x_{k-1};t_k,x_k)\, dx_k \dots dx_1. \tag{4.43}$$

Proof. The proof is by induction in k. This result holds for $k = 1$ by virtue of (4.38). Suppose (4.43) is true for $k = m-1$, we show that it can be extended to $k = m$. Let $t_i' = n^{-1}\lfloor t_i n\rfloor$ and let

$$\mathcal{D}_i = \{x \in \mathbb{R}^d : x_1 \le u_1, a_j^i < x_i \le b_j^i, j \in [\![2,d]\!]\}$$

for $i \in [\![1,m]\!]$. We mention here that in this proof, y^i for $i \in [\![1,m]\!]$ are all elements of $\mathbb{R}^d$ while y_i for $i \in [\![1,m]\!]$ belong to $\mathbb{R}$. By the same argument as in the beginning of the proof of Proposition 4.2, observe that

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[\bigcap_{i=1}^{m}\{\mathcal{Z}^n(t_i) \in \mathcal{D}_i\} \mid \Lambda_n\Big]$$

$$= \lim_{n\to\infty} \mathtt{P}_\omega\Big[\bigcap_{i=1}^{m-2}\{\mathcal{Z}^n(t_i)\in\mathcal{D}_i\}, \bigcap_{i=m-1}^{m}\{\mathcal{Z}^n(t'_i)\in\mathcal{D}_i\} \mid \Lambda_n\Big] \tag{4.44}$$

provided that the limits exist. Then, we write for sufficiently large n

$$\begin{aligned}
&\mathtt{P}_\omega\Big[\bigcap_{i=1}^{m-2}\{\mathcal{Z}^n(t_i)\in\mathcal{D}_i\}, \bigcap_{i=m-1}^{m}\{\mathcal{Z}^n(t'_i)\in\mathcal{D}_i\} \mid \Lambda_n\Big]\\
&\quad= \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\int\limits_{\mathcal{D}_{m-1}}\int\limits_{\mathcal{D}_m} \mathtt{P}_\omega\Big[\mathcal{Z}^n(t_1)\in\mathcal{D}_1,\dots,\mathcal{Z}^n(t_{m-2})\in\mathcal{D}_{m-2},\\
&\qquad \mathcal{Z}^n(t'_{m-1})\in dy^{m-1}, \mathcal{Z}^n(t'_m)\in dy^m, X_1(1)>0,\dots,X_1(n)>0\Big]\\
&\quad= \frac{\mathtt{P}_\omega[\Lambda_{nt_{m-1}}]}{\mathtt{P}_\omega[\Lambda_n]}\int\limits_{\mathcal{D}_{m-1}}\int\limits_{\mathcal{D}_m}\\
&\qquad \mathtt{P}_\omega\Big[\mathcal{Z}^n(t_1)\in\mathcal{D}_1,\dots,\mathcal{Z}^n(t_{m-2})\in\mathcal{D}_{m-2}, \mathcal{Z}^n(t'_{m-1})\in dy^{m-1} \mid \Lambda_{nt_{m-1}}\Big]\\
&\qquad \times \mathtt{P}_\omega^{y^{m-1}\sqrt{n}}\Big[\mathcal{Z}_1^n(s)>0, 0\le s\le t'_m-t'_{m-1}, \mathcal{Z}^n(t'_m-t'_{m-1})\in dy^m\Big]\\
&\qquad \times \mathtt{P}_\omega^{y^m\sqrt{n}}\Big[\mathcal{Z}_1^n(s)>0, 0\le s\le 1-t'_m\Big].
\end{aligned} \tag{4.45}$$

By the induction hypothesis we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathtt{P}_\omega\Big[\mathcal{Z}^n(t_1)\in\mathcal{D}_1,\dots,\mathcal{Z}^n(t_{m-2})\in\mathcal{D}_{m-2}, \mathcal{Z}^n(t'_{m-1})\in\mathcal{D}_{m-1} \mid \Lambda_{nt_{m-1}}\Big]\\
&\quad= \prod_{j=2}^{d}\int\limits_{a_j^1}^{b_j^1}\cdots\int\limits_{a_j^{m-1}}^{b_j^{m-1}} \frac{\exp\Big\{-\frac{y_1^2}{2t_1}\Big\}}{\sqrt{2\pi t_1}}\,\frac{\exp\Big\{-\frac{(y_2-y_1)^2}{2(t_2-t_1)}\Big\}}{\sqrt{2\pi(t_2-t_1)}}\cdots\\
&\qquad \times \frac{\exp\Big\{-\frac{(y_{m-1}-y_{m-2})^2}{2(t_{m-1}-t_{m-2})}\Big\}}{\sqrt{2\pi(t_{m-1}-t_{m-2})}}\,dy_{m-1}\dots dy_1\\
&\qquad \times \int\limits_0^{u_1 t_{m-1}^{-1/2}}\cdots\int\limits_0^{u_{m-1}t_{m-1}^{-1/2}} q(0,0;t_1/t_{m-1},y_1)\,q(t_1/t_{m-1},y_1;t_2/t_{m-1},y_2)\dots\\
&\qquad \times q(t_{m-2}/t_{m-1},y_{m-2};1,y_{m-1})\,dy_{m-1}\dots dy_1.
\end{aligned} \tag{4.46}$$

On the other hand, by (4.42) we have $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\frac{\mathtt{P}_\omega[\Lambda_{nt_{m-1}}]}{\mathtt{P}_\omega[\Lambda_n]} = t_{m-1}^{1/2}. \tag{4.47}$$

Using Theorem 1.2 and Dini's theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \mathtt{P}_\omega^{y^{m-1}\sqrt{n}}\Big[\mathcal{Z}_1^n(s) > 0, 0 \le s \le t'_m - t'_{m-1}, \mathcal{Z}^n(t'_m - t'_{m-1}) \in \mathcal{D}_m\Big] \\
&\quad = \prod_{j=2}^{d} \int_{a_j^m}^{b_j^m} \frac{\exp\Big\{-\frac{(y_m - y_j^{m-1})^2}{2(t_m - t_{m-1})}\Big\}}{\sqrt{2\pi(t_m - t_{m-1})}}\, dy_m \times \int_0^{u_m} g(t_m - t_{m-1}, y_1^{m-1}, v)dv \qquad (4.48)
\end{aligned}$$

uniformly in y^{m-1} on every compact set of the form $[0,K]\times[-K,K]^{d-1}$, and

$$\lim_{n\to\infty} \mathtt{P}_\omega^{y^{m}\sqrt{n}}\Big[\mathcal{Z}_1^n(s) > 0, 0 \le s \le 1 - t'_m\Big] = \tilde{N}(y_1^m(1-t_m)^{-1/2}) \qquad (4.49)$$

uniformly in y^m on every compact set of the form $[0,K]\times[-K,K]^{d-1}$. Combining (4.44), (4.45), (4.46), (4.47), (4.48), (4.49), and using Lemma 2.18 of [14] twice, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \mathtt{P}_\omega\Big[\bigcap_{i=1}^{m}\{\mathcal{Z}^n(t_i) \in \mathcal{D}_i\} \mid \Lambda_n\Big] \\
&\quad = \prod_{j=2}^{d} \int_{a_j^1}^{b_j^1} \cdots \int_{a_j^m}^{b_j^m} \frac{\exp\Big\{-\frac{x_1^2}{2t_1}\Big\}}{\sqrt{2\pi t_1}} \frac{\exp\Big\{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}\Big\}}{\sqrt{2\pi(t_2-t_1)}} \cdots \\
&\quad \times \frac{\exp\Big\{-\frac{(x_m-x_{m-1})^2}{2(t_m-t_{m-1})}\Big\}}{\sqrt{2\pi(t_m-t_{m-1})}}\, dx_m \dots dx_1 \\
&\quad \times t_{m-1}^{-1} \int_0^{u_{m-1}} \int_0^{u_m} \int_0^{u_1 t_{m-1}^{-1/2}} \cdots \int_0^{u_{m-2} t_{m-1}^{-1/2}} q(0,0;t_1/t_{m-1}, y_1) \\
&\quad \times q(t_1/t_{m-1}, y_1; t_2/t_{m-1}, y_2) \dots \\
&\quad \times q(t_{m-2}/t_{m-1}, y_{m-2}; 1, y_{m-1} t_{m-1}^{-1/2})\, dy_{m-1} \dots dy_1 \\
&\quad \times g(t_m - t_{m-1}, y_{m-1}, y_m)\tilde{N}(y_m(1-t_m)^{-1/2})dy_m. \qquad (4.50)
\end{aligned}$$

Now, make the change of variables $t_{m-1}^{1/2} y_1 = x_1, \dots, t_{m-1}^{1/2} y_{m-2} = x_{m-2}$ in (4.50) to obtain (4.43) for $k = m$. □

4.2. Tightness

In this section, to finish the proof of Theorem 1.1, we prove that the sequence of measures $(\mathtt{P}_\omega[\mathcal{Z}^n \in \cdot \mid \Lambda_n])_{n\ge1}$ is tight $\mathbb{P}$-a.s. The proof is standard: we consider the modulus of continuity, divide the time interval into small subintervals, and have to control the probability of fluctuations of our conditioned process over these small time intervals, where we use the results of the last subsection.

First, we define the modulus of continuity for functions $f \in C[0,1]$:

$$w_f(\delta') = \sup_{|t-s|\leq\delta'} \{\|f(s)-f(t)\|_\infty\}$$

where $s,t \in [0,1]$ and $\|\cdot\|_\infty$ is the ∞-norm on $\mathbb{R}^d$. By Theorem 14.5 of [15] it suffices to show that $\mathbb{P}$-a.s., for every $\hat{\varepsilon} > 0$

$$\lim_{\delta'\downarrow 0} \limsup_{n\to\infty} \mathtt{P}_\omega[w_{\mathcal{Z}^n}(\delta') \geq \hat{\varepsilon} \mid \Lambda_n] = 0 \tag{4.51}$$

since $\mathcal{Z}^n(0) = 0$. Now observe that

$$\begin{aligned}\mathtt{P}_\omega[w_{\mathcal{Z}^n}(\delta') \geq \hat{\varepsilon} \mid \Lambda_n] &= \mathtt{P}_\omega\Big[\sup_{|t-s|\leq\delta'} \|\mathcal{Z}^n(t)-\mathcal{Z}^n(s)\|_\infty \geq \hat{\varepsilon} \mid \Lambda_n\Big]\\ &\leq \mathtt{P}_\omega\Big[\sup_{|t-s|\leq 2\delta'} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big] \end{aligned}\tag{4.52}$$

for $n \geq 2/\delta'$. Let $m := \lfloor 1/4\delta' \rfloor$ and divide the interval $[0,1]$ into intervals $I_k := [k/m,(k+1)/m]$, for $0 \leq k \leq m-1$. Additionally, consider the intervals $J_l := [(2l+1)/2m,(2l+3)/2m]$, for $0 \leq l \leq m-2$ and $J_{m-1} := \emptyset$. Observe that

$$\begin{aligned}&\mathtt{P}_\omega\Big[\sup_{|t-s|\leq 2\delta'} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big]\\ &\quad\leq \mathtt{P}_\omega\Big[\Big\{\max_{k\leq m-1}\sup_{s,t\in I_k} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big\}\\ &\qquad\cup\Big\{\max_{l\leq m-1}\sup_{s,t\in J_l} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big\} \mid \Lambda_n\Big]\\ &\quad\leq m\Big(\max_{k\leq m-1}\mathtt{P}_\omega\Big[\sup_{s,t\in I_k} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big]\\ &\qquad+\max_{l\leq m-1}\mathtt{P}_\omega\Big[\sup_{s,t\in J_l} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big]\Big)\end{aligned}\tag{4.53}$$

with the convention that $\sup_{s,t\in\emptyset}\{\cdot\} = 0$. Our next step is to bound from above the $\limsup_{n\to\infty}$ of both terms in parentheses in the right-hand side of (4.53). As an example, let us treat the terms indexed by I_k for $k \in [\![1,m-1]\!]$. The term indexed by I_0 and those indexed by J_k, $k \in [\![1,m-1]\!]$ can be treated in a similar way. To do that, we will use the same approach as in the proof of Proposition 4.1. Analogously to (4.4) we have for $\varepsilon \in (0,1)$ and $\delta \in (0,1)$,

$$\begin{aligned}&\mathtt{P}_\omega\Big[\sup_{s,t\in I_k} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big]\\ &\quad\leq (\mathtt{P}_\omega[\Lambda_n])^{-1}\mathtt{P}_\omega\Big[\sup_{s,t\in I_k} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} \leq \delta n m^{-1}\Big]\end{aligned}$$

$$+\,\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \delta n m^{-1} \mid \Lambda_n] + \mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n]. \tag{4.54}$$

Analogously to (4.6), we obtain

$$\begin{aligned}(\mathtt{P}_\omega[\Lambda_n])^{-1}\mathtt{P}_\omega\Big[\sup_{s,t\in I_k}\|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}, A_{0\to R}, \Lambda_n, \beta_{\mathcal{R}_{\varepsilon,n}} \leq \delta n m^{-1}\Big]\\ \leq \frac{\mathtt{P}_\omega[A_{0\to R}]}{\mathtt{P}_\omega[\Lambda_n]} \max_{y\in R_{\varepsilon,n}} \max_{j\leq \lfloor \frac{\delta n}{m}\rfloor} \mathtt{P}^y_\omega\Big[\sup_{s,t\in I_k}\|\mathcal{X}(nt-j)-\mathcal{X}(ns-j)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big].\end{aligned}$$

Now, observe that for all sufficiently large n

$$\begin{aligned}\max_{j\leq \lfloor \frac{\delta n}{m}\rfloor} \mathtt{P}^y_\omega\Big[\sup_{s,t\in I_k}\|\mathcal{X}(nt-j)-\mathcal{X}(ns-j)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big]\\ \leq \mathtt{P}^y_\omega\Big[\sup_{s,t\in I'_k}\|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big]\end{aligned} \tag{4.55}$$

with $I'_k = [(k-2\delta)/m, (k+1)/m]$. Now, let $I''_k = [(k-3\delta)/m, (k+1)/m]$. By Theorem 1.2 and the estimate on the tail of the Gaussian law given in [17], Appendix B, Lemma 12.9, we have

$$\begin{aligned}\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \mathtt{P}^y_\omega\Big[\sup_{s,t\in I'_k}\|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big]\\ \leq d\cdot P\Big[\sup_{s,t\in I''_k}|W_1(t)-W_1(s)| \geq \hat{\varepsilon}\Big]\\ \leq 8d\cdot P\Big[W_1\Big(\frac{1+3\delta}{m}\Big) \geq \hat{\varepsilon}\Big]\\ \leq \frac{16d}{\hat{\varepsilon}\sqrt{2\pi m}}\exp\Big\{-\frac{\hat{\varepsilon}^2 m}{8}\Big\}\end{aligned} \tag{4.56}$$

since $\delta < 1$. We obtain

$$\begin{aligned}\limsup_{n\to\infty} \max_{y\in R_{\varepsilon,n}} \max_{j\leq \lfloor \frac{\delta n}{m}\rfloor} \mathtt{P}^y_\omega\Big[\sup_{s,t\in I_k}\|\mathcal{X}(nt-j)-\mathcal{X}(ns-j)\|_\infty \geq \hat{\varepsilon}\sqrt{n}\Big]\\ \leq \frac{16d}{\hat{\varepsilon}\sqrt{2\pi m}}\exp\Big\{-\frac{\hat{\varepsilon}^2 m}{8}\Big\}.\end{aligned} \tag{4.57}$$

Thus, we have by (4.7), (4.10), (4.54), and (4.57)

$$\begin{aligned}\limsup_{n\to\infty} \mathtt{P}_\omega\Big[\sup_{s,t\in I_k}\|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \geq \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big]\\ \leq \Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi}} + o(\varepsilon)\Big)^{-1}\Big(\frac{16d}{\hat{\varepsilon}\sqrt{2\pi m}}\exp\Big\{-\frac{\hat{\varepsilon}^2 m}{8}\Big\}\Big) + f(\varepsilon) + g(\varepsilon)\\ + \limsup_{n\to\infty}\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \delta n m^{-1} \mid \Lambda_n].\end{aligned} \tag{4.58}$$

Combining (4.53) and (4.58) we find

$$\begin{aligned}\limsup_{n\to\infty} \mathtt{P}_\omega\Big[\sup_{|t-s|\le 2\delta'} \|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \ge \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big] \\ \le 2m\Big(\frac{16d}{\hat{\varepsilon}\sqrt{2\pi m}}\exp\Big\{-\frac{\hat{\varepsilon}^2 m}{8}\Big\}\Big(\frac{2\varepsilon\sigma_1}{\sqrt{2\pi}}+o(\varepsilon)\Big)^{-1} \\ + f(\varepsilon)+g(\varepsilon)+\limsup_{n\to\infty}\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \delta n m^{-1} \mid \Lambda_n]\Big).\end{aligned} \tag{4.59}$$

Then, let $\varepsilon = m^{-3}$ and $\delta = m^{-1/2}$ in (4.59). We have by (4.10)

$$\begin{aligned}\limsup_{n\to\infty}\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \delta n m^{-1} \mid \Lambda_n] \\ = \limsup_{n\to\infty}\mathtt{P}_\omega[\beta_{\mathcal{R}_{\varepsilon,n}} > \varepsilon^{1/2} n \mid \Lambda_n] \le f(m^{-3}) + g(m^{-3}).\end{aligned}$$

Therefore, we obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathtt{P}_\omega\Big[\sup_{s,t\in \hat{I}_k}\|\mathcal{X}(nt)-\mathcal{X}(ns)\|_\infty \ge \hat{\varepsilon}\sqrt{n} \mid \Lambda_n\Big] = 0.$$

As $\hat{\varepsilon}$ is arbitrary and $m = \lfloor 1/4\delta' \rfloor$, using (4.52), this last expression proves (4.51) and consequently the tightness of the sequence $\big(\mathtt{P}_\omega[\mathcal{Z}^n \in \cdot \mid \Lambda_n]\big)_{n\ge 1}$. □

Acknowledgements

This work was mainly done when N.G. was visiting Brazil in the end of 2010; this visit was supported by FAPESP (2010/16085–7). C.G. is grateful to FAPESP (grant 2009/51139–3) for financial support. S.P. and M.V. were partially supported by CNPq (grant 300886/2008–0 and 301455/2009–0). C.G., S.P., and M.V. also thank CNPq (472431/2009–9) and FAPESP (2009/52379–8) for financial support. We gratefully acknowledge the John von Neumann guest professor program of Technische Universität München which supported a visit of S.P. and M.V. to Munich.

We also thank the referee for helpful comments.

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