Survival of branching random walks in random environment

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Abstract

We study survival of nearest-neighbour branching random walks in random environment (BRWRE) on $\mathbb{Z}$. A priori there are three different regimes of survival: global survival, local survival, and strong local survival. We show that local and strong local survival regimes coincide for BRWRE and that they can be characterized with the spectral radius of the first moment matrix of the process. These results are generalizations of the classification of BRWRE in recurrent and transient regimes. Our main result is a characterization of global survival that is given in terms of Lyapunov exponents of an infinite product of i.i.d. 2×2 random matrices.

1 Introduction

The branching random walk in random environment (BRWRE) starts with one particle in the origin. This particle splits up in several other particles at positions $\{-1, 0, 1\}$ according to some offspring distribution. Now the process is defined inductively, at each moment each particle at x, independently of the other particles and the history of the process, splits up in particles at $\{x-1, x, x+1\}$ according to some offspring distribution that may depend on x. The collection of the offspring distributions itself is chosen randomly before starting the process and then is kept fixed during the evolution of the process. The difference with the model in [2, 10, 11] is that we start the process with one particle and not with infinitely many. In contrast to previous papers on the topic, cf. [6, 7, 8, 15, 16, 17], we allow the process to die out. A priori there are three different types of survival: global survival, local survival, and strong local survival. Global survival means that with positive probability the total number of particles is always positive. Local survival means that with positive probability every site is visited infinitely often. If the two latter probabilities are equal and positive (so that, conditioned on always having a positive number of particles, every site is visited infinitely often a.s.) we say there is strong local survival. Our first result (Theorem 2.4) says that, in fact, local and strong local survival coincide and do not depend on the realization of the environment. This is a generalization of the classification of BRWRE in recurrent and transient regimes, compare with [7] and [17]. Observe that recurrence corresponds to local survival if we assume that the process can not die out globally. Our main result is the criterion for global survival, Theorem 2.9. This criterion is given in terms of Lyapunov exponents of an infinite product of 2×2 random matrices. The main idea of the proof is to construct an embedded Galton-Watson process in an ergodic random environment that survives if and only if the BRWRE survives globally.

This model can be viewed from a different angle. Interpret the position of a particle as its type. Hence, e.g. a particle of type 0 may produce offspring particles of types $-1, 0$, and 1. Hence, our model is a particular case of a multi-type Galton-Watson process with infinitely many types. Consider a multi-type Galton-Watson process. The types are indexed by some set I that may be finite or countably infinite. Then, each type of individual $x \in I$ produces offspring according to $\mu_x = \mu_x(k_i : \ i \in I)$. This means that a particle of type x has k_i offspring of type i with probability $\mu_x(k_i)$, $i \in I$. Assume irreducibility of the process, i.e., any type of particle may, after some generations, have any other type of particle as a descendant. Let $M := (m(x,y))_{x,y\in I}$ be the first moment matrix, i.e., $m(x,y)$ is the mean number of type y offspring of one type x particle. If I is finite, it is well-known, cf. e.g. Chapter II of [12], that the multi-type Galton-Watson process survives with positive probability if and only if the the largest eigenvalue $\rho(M)$ of the matrix M, the Perron-Frobenius eigenvalue, is greater than 1. In the case when I is infinite, very little is known in general. One reason for this is the following. In the finite case, as we just mentioned, the behavior of the process is strongly connected with the Perron-Frobenius eigenvalue of the first moment matrix. In the infinite setting this matrix becomes an operator that does not necessarily have a *largest eigenvalue*. If the operator M is ergodic in the sense of [21], most results carry over from the finite case, e.g. see Chapter III, Section 10 of [12] for general results, [3] for a concrete example, and [21] for mathematical background. If the operator is not ergodic nothing is known in general and the behavior of the process becomes more subtle. There are some interesting results on an example of epidemics where a classification is obtained, see [3], and further partial results for specific models in [5] and [19], but no general explicit criterion is known. Therefore, since the first moment matrix of BRWRE is in general not ergodic, cf. Proposition 5.2.8 of [18], the classification of BRWRE constitutes a step towards the understanding of infinite-type Galton-Watson processes in general. Our model is connected with the model studied in [2, 10, 11] (where the random environment was related only to the branching mechanism). In the model studied there, the process does not start only with one particle but an infinite number of particles. While in [2, 10, 11] the authors analyse different growth rates using a variational approach, we give a description of survival and extinction using properties of branching random walks and embedded Galton-Watson processes in random environment.

2 Formal description of the model and main results

We now describe the model, keeping the notations of [7, 8] whenever possible. Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $\mathfrak{A} = \{-1, 0, 1\}$. Define

$$\mathcal{V} = \Big\{ v = (v_x, x \in \mathfrak{A}) : v_x \in \mathbb{Z}_+, \forall x \in \mathfrak{A} \Big\},$$

and for $v \in \mathcal{V}$ put $|v| = \sum_{x\in\mathfrak{A}} v_x$. Furthermore, let $\mathcal{M}$ be the set of all probability measures ω on $\mathcal{V}$:

$$\mathcal{M} = \Big\{\omega = (\omega(v), v \in \mathcal{V}) : \omega(v) \geq 0 \text{ for all } v \in \mathcal{V}, \sum_{v\in\mathcal{V}} \omega(v) = 1\Big\}.$$

Then, suppose that $\omega := (\omega_x \in \mathcal{M}, x \in \mathbb{Z})$ is an i.i.d. sequence with values in $\mathcal{M}$, and denote by $\mathbb{P}, \mathbb{E}$ the probability and expectation with respect to ω.

The collection $\omega = (\omega_x, x \in \mathbb{Z})$ is called *the environment*. Given the environment ω, the evolution of the process is described in the following way: start with one particle at some fixed site of $\mathbb{Z}$. At each integer time the particles branch independently in the following way: for a particle at site $x \in \mathbb{Z}$, a random element $v = (v_y, y \in \mathfrak{A})$ is chosen with probability $\omega_x(v)$, and then the particle is substituted by v_y particles in $x + y$, $y \in \mathfrak{A}$.

It is important to note that, in contrast to [7, 8], the condition $|v| \geq 1$ is dropped from the definition of $\mathcal{V}$. This means that here we allow the possibility that particles can disappear (i.e., leave no offspring), thus it can happen that the process dies out.

Denote

$$\mu_x^- = \sum_{v\in\mathcal{V}} \omega_x(v)v_{-1}, \ \mu_x^0 = \sum_{v\in\mathcal{V}} \omega_x(v)v_0, \ \text{and} \ \mu_x^+ = \sum_{v\in\mathcal{V}} \omega_x(v)v_1.$$

In words: μ_x^- is the mean number of offspring sent by a particle from x to $x-1$, μ_x^+ is the mean number of offspring sent by a particle from x to $x+1$, and μ_x^0 is the mean number of offspring which stay at x.

We always assume that the following two conditions hold:

Condition E. We have $\mathbb{P}[\min(\mu_0^-, \mu_0^+) > 0] = 1$.

Condition B. There exists $v \in \mathcal{V}$ with $|v| \geq 2$ such that $\mathbb{P}[\omega_0(v) > 0] > 0$.

Condition E is a natural ellipticity condition which ensures that the process is irreducible in the sense that for any $x, y \in \mathbb{Z}$ a particle from x can have descendants in y. Condition B says that there are places where particles are able to branch.

For the proof of Theorem 2.9 below, we will need the following stronger condition. Let $V_1 := \{v \in \mathcal{V} : v_1 \geq 1\}$ and $V_{-1} := \{v \in \mathcal{V} : v_{-1} \geq 1\}$.

Condition S. $\mathbb{E}|\ln(\omega_0(V_1))| < \infty$ and $\mathbb{E}|\ln(\omega_0(V_{-1}))| < \infty$.

Since $\mu_0^- \geq \omega_0(V_{-1})$ and $\mu_0^+ \geq \omega_0(V_1)$, Condition S implies that $\mathbb{E}(\ln \mu_0^-)^-$ and $\mathbb{E}(\ln \mu_0^+)^-$ are finite.

Let us denote by $\eta_n(y)$ the number of particles in y at time n. Define the random variable

$$\mathcal{Z}_n = \sum_{y \in \mathbb{Z}} \eta_n(y),$$

i.e., $\mathcal{Z}_n$ is the total number of particles at time n.

We denote by $\mathtt{P}_\omega^x, \mathtt{E}_\omega^x$ the probability and expectation for the process starting from x in the fixed environment ω, often denoted as "quenched" probability and expectation.

Now we define the survival regimes.

Definition 2.1 *Given ω, we say that there is* global survival *if*

$$\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0] < 1.$$

Definition 2.2 *Given ω, we say that there is* local survival *if*

$$\mathtt{P}_\omega^0[\eta_n(y) \to 0] < 1$$

for all y.

Definition 2.3 *Given ω, we say that there is* strong local survival *if*

$$\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0] = \mathtt{P}_\omega^0[\eta_n(y) \to 0] < 1$$

for all y.

We say that for a given ω there is local (respectively global) extinction, if there is no local (respectively global) survival.

In principle, in a (properly constructed) deterministic environment the definition of (strong) local survival may depend on the starting point, cf. Example 1 of [7]. Let us show, however, that in i.i.d. random environment there is no such dependence and that local survival always implies strong local survival.

Theorem 2.4 *Local survival and strong local survival do not depend on the starting point in Definitions 2.2 and 2.3. Also, either there is strong local survival for $\mathbb{P}$-a.a. ω, or there is local extinction for $\mathbb{P}$-a.a. ω.*

Similarly to [6, 15, 16, 17] we can obtain a simple and explicit criterion for local extinction. As in the above references, it turns out that local extinction does not depend on the environmental law itself, but only on its support.

Definition 2.5 *We say that the process vanishes on the right (respectively, on the left) if for any $z \in \mathbb{Z}$, the set $\{z, z+1, z+2, \dots\}$ (respectively, $\{z, z-1, z-2, \dots\}$) is visited only finitely many times a.s.*

The criterion for local extinction is then given by

Theorem 2.6 *There is local extinction iff there exists $\lambda > 0$ such that*

$$\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \le 1 \tag{2.1}$$

for $\mathbb{P}$-a.a. ω. Moreover, if $\lambda > 1$, then the process vanishes on the right, and if $\lambda < 1$, then the process vanishes on the left.

Remark 2.7 *If (2.1) holds for $\mathbb{P}$-a.a. ω with $\lambda = 1$, then there is global extinction a.s. (so that the process vanishes in both directions). This is easy to see: $\mu_0^- + \mu_0^0 + \mu_0^+ \le 1$ implies that the mean offspring in all sites is less than or equal to 1, and so the total number of particles in the process is a (nonnegative) supermartingale. This supermartingale converges a.s. to some limit, and it is straightforward to obtain (using Conditions B, E, and the fact that the environment is i.i.d.) that this limit can only be 0.*

Remark 2.8 *By Theorem 2.6, local extinction implies that $\mathbb{P}[\mu_0^0 < 1] = 1$. Then, particles cannot accumulate in any site without help from outside. Using Condition E we obtain that if $\mathbb{P}\big[\omega_0\big((0,0,0)\big) > 0\big] > 0$, then it is* possible *that the process dies out, i.e., $\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0] > 0$.*

Now, the goal is to obtain a criterion for global extinction in the case when the process becomes locally extinct. To this end we introduce the following matrices. For $k \in \{1, 2, 3, \dots\}$, denote

$$A_k = \begin{pmatrix} \frac{1-\mu_k^0}{\mu_k^+} & -\frac{\mu_k^-}{\mu_k^+} \\ 1 & 0 \end{pmatrix}, \text{ and } \tilde{A}_k = \begin{pmatrix} \frac{1-\mu_k^0}{\mu_k^-} & -\frac{\mu_k^+}{\mu_k^-} \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

So, $A_1, A_2, A_3, \dots$ and $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \dots$ are two sequences of i.i.d. random matrices. Denote by γ_1 the top Lyapunov exponent associated with the sequence $\{A_k\}$, i.e.,

$$\gamma_1 = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\ln \|A_n \cdots A_1\|),$$

where $\|\cdot\|$ is any matrix norm (this limit exists provided that $\mathbb{E} \ln^+ \|A_1\|$ is finite, cf. e.g. Section I.2 of [4]). Analogously, let $\tilde{\gamma}_1$ be the top Lyapunov exponent of the sequence $\{\tilde{A}_k\}$. The criterion for global survival is then given by

Theorem 2.9 *Suppose that Condition S holds. Assume also that there is local extinction, so, by Theorem 2.6, there is $\lambda > 0$ such that* (2.1) *holds $\mathbb{P}$-a.s. Then,*

- *if there is some $\lambda > 1$ such that $\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \leq 1$ $\mathbb{P}$-a.s., then there is global survival iff*

$$\gamma_1 < \mathbb{E} \ln \left(\frac{\mu_0^-}{\mu_0^+} \right); \tag{2.3}$$

- *if there is some $\lambda < 1$ such that $\mu_0^- \lambda^{-1} + \mu_0^0 + \mu_0^+ \lambda \leq 1$ $\mathbb{P}$-a.s., then there is global survival iff*

$$\tilde{\gamma}_1 < \mathbb{E} \ln \left(\frac{\mu_0^+}{\mu_0^-} \right). \tag{2.4}$$

3 Proofs

In this section we prove Theorems 2.4, 2.6, and 2.9.

3.1 Proof of Theorem 2.4.

An important object is the first moment matrix $M_\omega = (m_\omega(x, y))_{x,y \in \mathbb{Z}}$ of the process which is defined by

$$m_\omega(x, x-1) = \mu_x^-, \quad m_\omega(x, x) = \mu_x^0, \quad m_\omega(x, x+1) = \mu_x^+,$$

and $m_\omega(x, y) = 0$ for $y \notin \{x-1, x, x+1\}$. Let $M_\omega^n = (m_\omega^{(n)}(x, y))_{x,y \in \mathbb{Z}}$ denote the n-fold convolution of M_ω; in other words, $m^{(n)}(x, y) = \mathtt{E}_\omega^x[\eta_n(y)]$. Due to Condition E, the matrix M_ω is irreducible. We have, by a supermultiplicativity argument, that

$$\rho(M_\omega) := \limsup_{n \to \infty} \left(m_\omega^{(n)}(x, y) \right)^{1/n} \tag{3.1}$$

does not depend on x and y (cf. e.g. [9]).

Due to the irreducibility Condition E, we obtain for all $x, z \in \mathbb{Z}$ that $\mathtt{P}_\omega^x[z \text{ is visited by some particle}] = \mathtt{P}_\omega^x[\eta_n(z) > 0 \text{ for some } n] > 0$. Since for $y \in \mathbb{Z}$

$$\mathtt{P}_\omega^x[\eta_n(y) \not\to 0] \geq \mathtt{P}_\omega^x[z \text{ is visited by some particle}] \times \mathtt{P}_\omega^z[\eta_n(y) \not\to 0],$$

local survival does not depend on the choice of the starting point in its definition. To see that the same holds true for strong local survival, observe first that for all $x, y \in \mathbb{Z}$

$$\mathtt{P}_\omega^x[\mathcal{Z}_n \to 0] \leq \mathtt{P}_\omega^x[\eta_n(y) \to 0]. \tag{3.2}$$

We denote by $\eta_n = (\eta_n(x))_{x\in\mathbb{Z}}$ the "global" configuration of particles at time n. Let Ξ_n be the set of all possible particle configurations at time n; observe that, since we start with one particle at 0, this set is finite or countably infinite. Now, assume that

$$\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0] = \mathtt{P}_\omega^0[\eta_n(y) \to 0].$$

Conditioning on the first time step we obtain

$$\begin{aligned}
\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0] &= \sum_{\eta\in\Xi_1} \mathtt{P}_\omega^0[\eta_1 = \eta]\mathtt{P}_\omega^0[\mathcal{Z}_n \to 0 \mid \eta_1 = \eta] \\
&= \sum_{\eta\in\Xi_1} \mathtt{P}_\omega^0[\eta_1 = \eta] \prod_{x\in\{-1,0,1\}} \big(\mathtt{P}_\omega^x[\mathcal{Z}_n \to 0]\big)^{\eta(x)}
\end{aligned}$$

and

$$\begin{aligned}
\mathtt{P}_\omega^0[\eta_n(y) \to 0] &= \sum_{\eta\in\Xi_1} \mathtt{P}_\omega^0[\eta_1 = \eta]\mathtt{P}_\omega^0[\eta_n(y) \to 0 \mid \eta_1 = \eta] \\
&= \sum_{\eta\in\Xi_1} \mathtt{P}_\omega^0[\eta_1 = \eta] \prod_{x\in\{-1,0,1\}} \big(\mathtt{P}_\omega^x[\eta_n(y) \to 0]\big)^{\eta(x)}
\end{aligned}$$

Therefore, using (3.2),

$$\mathtt{P}_\omega^x[\mathcal{Z}_n \to 0] = \mathtt{P}_\omega^x[\eta_n(y) \to 0]$$

for $x \in \{-1, 1\}$. Now, the statement follows from an induction argument due to Condition E.

The remaining part of the proof splits into three steps (a similar reasoning can be found in [9]).

Step 1: *Local survival is equivalent to* $\rho(M_\omega) > 1$. Let $\rho(M_\omega) > 1$ and use the following well-known approximation property of the spectral radius, namely

$$\rho(M_\omega) = \sup_{|F|<\infty} \rho(M_{\omega,F}). \tag{3.3}$$

Here $M_{\omega,F}$ is the finite matrix over the set F defined as $m_{\omega,F}(x,y) = m_\omega(x,y)$ for all $x, y \in F$. Due to (3.3) there exists a finite set F such that $\rho(M_{\omega,F}) > 1$. Since $\rho(M_{\omega,F}) \leq \rho(M_{\omega,G})$ for $F \subseteq G$, we can choose F to be connected and such that $0 \in F$. Observe that $M_{\omega,F}$ is the first moment matrix of the multi-type Galton-Watson process that lives on F. This process can also be interpreted as the embedded process where particles live only on the set F and die if they leave this set. Since $\rho(M_{\omega,F}) > 1$ this embedded process is supercritical. This implies the local survival of the BRWRE.

Now, assume local survival of the process. We proceed by constructing an embedded Galton-Watson process counting the number of particles in the origin. Let the particles that are the first particles in their ancestry line (of the BRWRE) to return to 0 form the first generation of the new process. The process is defined inductively: the i-th generation consists of particles that are the i-th particle in their ancestry line to return to 0. Denote by ψ_i the size of i-th generation. Observe that $\psi_i \in \mathbb{N} \cup \{\infty\}$ is a Galton-Watson process with mean $\mathtt{E}_\omega \psi_1 > 1$ (in fact, one can even show that $\mathtt{E}_\omega \psi_1 = \infty$) since the process survives locally. Now, we define an embedded process of the above Galton-Watson process, which is formed by particles that do not go too far away from the origin. Let the restricted first generation consist of particles that are the first particles in their ancestry line to return to 0 before time N. Inductively, the restricted i-th generation is formed by the particles having an ancestor in the restricted $(i-1)$th generation and being the first in their ancestry line of this ancestor to return to 0 in at most N time steps. Let $\psi_i^{(N)}$ be the size of the restricted i-th generation and let us choose N such that $\mathtt{E}_\omega \psi_1^{(N)} > 1$. Setting $\mathcal{B}_N := [-N, \ldots, N]$ we obtain $\rho(M_\omega) > \rho(M_{\omega,\mathcal{B}_N}) > 1$.

Step 2: *Either there is local survival for* $\mathbb{P}$*-a.a.* ω*, or there is local extinction for* $\mathbb{P}$*-a.a.* ω*.* The spectral radius $\rho(M_\omega)$ is deterministic. To see this observe that $\rho(M_\omega) = \limsup_n (m^{(n)}(x,y))^{1/n}$ does not depend on x and y and is constant $\mathbb{P}$-a.s. by ergodicity of the environment as discussed in [17].

Step 3: *Local survival implies strong local survival.* We assume local survival. Recall that a set F with $\rho(M_{\omega,F}) > 1$ gives rise to a supercritical multi-type Galton-Watson process. In analogy to [7] we call these sets *(recurrent) seeds.* We make the following observation that is obvious if $\mathbb{P}$ is discrete and easy to check otherwise: There exists some $N \in \mathbb{N}$ and some $\varepsilon > 0$

$$\mathbb{P}\left[\mathtt{P}^x_{\omega|\mathcal{B}_N}[\mathcal{Z}_n \not\to 0] > \varepsilon\right] > 0, \tag{3.4}$$

here $\mathtt{P}^x_{\omega|\mathcal{B}_N}$ denotes the probability measure of the embedded process that starts in $x \in \mathcal{B}_N$ and lives on $\mathcal{B}_N$ (i.e. particles leaving $\mathcal{B}_N$ die). We proceed similarly to the proof of Lemma 2.6 of [7] and partition $\mathbb{Z}$ into translates of $\mathcal{B}_N$. Then, by the Borel-Cantelli Lemma, infinitely many of the translates of $\mathcal{B}_N$ contain a seed with survival probability at least ε, cf. (3.4). Now, if the process survives globally infinitely many such seeds will be visited and it is straightforward to construct an independent sequence of embedded supercritical multi-type Galton-Watson processes whose survival probability is greater than ε. Eventually, one of those processes survives and strong local survival follows. □

3.2 Proof of Theorem 2.6.

First let us observe that Lemma 3.5 in [17] implies that

$$\rho(M_\omega) = \sup_{\mu\in\hat{\mathcal{K}}} \inf_{\lambda\in\mathbb{R}} \Big(\lambda\mu^+ + \mu^0 + \lambda^{-1}\mu^-\Big), \tag{3.5}$$

where $\mu = (\mu^+, \mu^0, \mu^-)$ and $\hat{\mathcal{K}}$ is the convex hull of the support of the one-dimensional marginal of $\mathbb{P}$. Observing that sup and inf are attained with say $\hat{\mu}$ and $\hat{\lambda}$, we obtain that $\hat{\lambda}\hat{\mu}^+ + \hat{\mu}^0 + \hat{\lambda}^{-1}\hat{\mu}^- \le 1$ implies (2.1). Clearly, (2.1) implies that $\inf_{\lambda\in\mathbb{R}} \sup_{\mu\in\hat{\mathcal{K}}}(\lambda\mu^+ + \mu^0 + \lambda^{-1}\mu^-) \le 1$. Since, by a minimax argument, one can exchange inf and sup, we obtain that $\rho(M_\omega) \le 1$ a.s.

Now, let us suppose that $\lambda > 1$ and let us prove that the process vanishes on the right. Note that, by (2.1), the function

$$h(n) = \sum_{x\in\mathbb{Z}} \eta_n(x)\lambda^x$$

is a positive supermartingale for $\mathbb{P}$-a.a. ω (see, for example, the proof of Theorem 1.6 in [7]). Therefore, as $n \to \infty$, it converges a.s. to some random variable h_∞. Using Fatou's Lemma, for the process starting at the origin we obtain that, $\mathbb{P}$-a.s.,

$$\mathtt{E}_\omega h_\infty \le \mathtt{E}_\omega h(0) = 1$$

On the event that the set $\{1, 2, 3, \ldots\}$ is visited infinitely often we have that every $k \ge 1$ is visited at least once. Hence, $\limsup h(n) \ge \lambda^k$ for all k. Since this contradicts the fact that $h(n)$ converges to a finite random variable, we obtain that $\{1, 2, 3, \ldots\}$ is only visited finitely often a.s. Using the irreducibility we obtain that the set $\{z, z+1, z+2, \ldots\}$ is visited finitely often a.s. for any $z \in \mathbb{Z}$. The case $\lambda < 1$ can be treated analogously. $\square$

3.3 Proof of Theorem 2.9.

Let us assume that there exists $\lambda > 1$ which satisfies (2.1) for $\mathbb{P}$-a.a. ω (so, the process vanishes on the right). The proof for the case $\lambda < 1$ follows then by exchanging λ and λ^{-1} and μ^- and μ^+ (that is, using $\tilde{A}_k$ instead of A_k).

Since the matrices $\{A_k\}$ are not nonnegative, we introduce the following sequences of nonnegative matrices. For $k \in \{1, 2, 3, \ldots\}$, denote

$$A_k^{(\lambda)} = \begin{pmatrix} \frac{\mu_k^-}{\lambda^2 \mu_k^+} & \frac{1-\mu_k^0-\lambda^{-1}\mu_k^- - \lambda\mu_k^+}{\lambda\mu_k^+} \\ \frac{\mu_k^-}{\lambda^2 \mu_k^+} & 1 + \frac{1-\mu_k^0-\lambda^{-1}\mu_k^- - \lambda\mu_k^+}{\lambda\mu_k^+} \end{pmatrix}. \tag{3.6}$$

That is, $A_1^{(\lambda)}, A_2^{(\lambda)}, A_3^{(\lambda)}, \ldots$ is a sequence of i.i.d. random matrices, which are nonnegative by (2.1). Denote by $\gamma_1^{(\lambda)}$ the top Lyapunov exponent associated with the sequence $\{A_k^{(\lambda)}\}$ and by $\gamma_2^{(\lambda)}$ the second Lyapunov exponent. It holds that (cf., for example, Corollary 1.3 of [14])

$$\gamma_1^{(\lambda)} + \gamma_2^{(\lambda)} = \mathbb{E}\Big[\ln \det A_1^{(\lambda)}\Big] = \mathbb{E} \ln \Big(\frac{\mu_k^-}{\lambda^2 \mu_k^+}\Big) = \mathbb{E} \ln \Big(\frac{\mu_k^-}{\mu_k^+}\Big) - 2 \ln \lambda. \tag{3.7}$$

The proof splits into two parts. First, we prove that there is global survival iff

$$\gamma_1^{(\lambda)} < \mathbb{E} \ln \Big(\frac{\mu_0^-}{\mu_0^+}\Big) - \ln \lambda. \tag{3.8}$$

We conclude then by comparing the Lyapunov spectra of $\{A_k\}$ and $\{A_k^{(\lambda)}\}$.

We will consider two modifications of our BRWRE. The first modification is the following. Start the original BRWRE with one particle at 0. When a particle hits -1, it is frozen and remains at -1 until all the existing particles hit -1 (this will happen in finite time, as our process vanishes on the right). Let ξ_1 be the total number of frozen particles at -1. Then, release the frozen particles, let them perform a BRW in random environment ω, and freeze all particles that hit -2. When all the existing particles are frozen at -2, let ξ_2 be the number of particles at -2. We repeat the above construction in this way to obtain a branching process $\{\xi_n\}_{n=1,2,\ldots}$ in stationary ergodic random environment. By Theorem 5.5 and Corollary 6.3 of [20] (taking into account Condition S), the above process survives with positive probability iff $\mathbb{E}\ln \mathtt{E}_\omega\xi_1 > 0$. But survival of the process $\{\xi_n\}$ is equivalent to survival of our original process $\{\mathcal{Z}_n\}$.

We are going to calculate $\mathbb{E}\ln \mathtt{E}_\omega\xi_1$ by means of constructing another sequence of random variables $\{\zeta_n\}$ in such a way that $\mathbb{E}\ln \mathtt{E}_\omega\xi_1 = \mathbb{E}\ln \mathtt{E}_\omega\zeta_1$.

Now, our second modification of the original BRWRE is defined in the following way. We start with one particle in $k \geq 0$. When a particle hits 0, it is frozen and remains at 0 forever, i.e., we modify the environment by putting $\omega'_0(v') = 1$, where $v' = (0,1,0)$. Denote by ζ_k the total number of frozen particles at 0 starting with one particle at k. As the environment ω is stationary, ζ_k and ξ_k have the same annealed law, $k = 1,2,3,\dots$.

Denote $f(k) = \mathtt{E}^k_\omega\zeta_k$, $k = 0,1,2,\dots$. Note that $f(0) = 1$, and for $k \geq 1$ we can write the recursive equation

$$f(k) = \mu^-_k f(k-1) + \mu^0_k f(k) + \mu^+_k f(k+1). \tag{3.9}$$

Let $g(k) = \lambda^{-k} f(k)$. Then, (3.9) implies that

$$\lambda^k g(k) = \lambda^{k-1}\mu^-_k g(k-1) + \lambda^k \mu^0_k g(k) + \lambda^{k+1}\mu^+_k g(k+1). \tag{3.10}$$

Denote $\Delta(k) = g(k) - g(k-1)$. Observe that (3.10) can be rewritten as

$$\begin{pmatrix} \Delta(k+1) \\ g(k+1) \end{pmatrix} = A^{(\lambda)}_k \begin{pmatrix} \Delta(k) \\ g(k) \end{pmatrix},$$

where $A^{(\lambda)}_k$ is the matrix defined in (3.6). Recall that, by (2.1), the matrix $A^{(\lambda)}_k$ is nonnegative.

In fact, to define Lyapunov exponents and use the classical results about them, we need $\ln\|A^{(\lambda)}_1\|$ and $\ln\|(A^{(\lambda)}_1)^{-1}\|$ to be integrable. It is straightforward to check that this is the case iff $\mathbb{E}(\ln\mu^-_k)^-$ and $\mathbb{E}(\ln\mu^+_k)^-$ are finite, which is a consequence of Condition S.

Denote by $H_\omega \subset \mathbb{R}^2$ the random one-dimensional subspace of $\mathbb{R}^2$ associated with $\gamma^{(\lambda)}_2$. So, for all $e \in \mathbb{R}^2 \setminus H_\omega$, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln\|A^{(\lambda)}_n \cdots A^{(\lambda)}_1 e\| = \gamma^{(\lambda)}_1, \tag{3.11}$$

and for all $e' \in H_\omega \setminus \{0\}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \ln\|A^{(\lambda)}_n \cdots A^{(\lambda)}_1 e'\| = \gamma^{(\lambda)}_2 \tag{3.12}$$

(cf., for example, Theorem 3.1 of [14]). If $\mu_0^-\lambda^{-1}+\mu_0^0+\mu_0^+\lambda<1$ with positive probability, then Corollary 2 of [13] implies that $\gamma_2^{(\lambda)}<\gamma_1^{(\lambda)}$. If, on the other hand, $\mu_0^-\lambda^{-1}+\mu_0^0+\mu_0^+\lambda=1$ $\mathbb{P}$-a.s., after some elementary computations it is straightforward to obtain that $\{\gamma_1^{(\lambda)},\gamma_2^{(\lambda)}\}=\{0,\mathbb{E}\ln(\mu_k^-/(\lambda^2\mu_k^+))\}$ and thus we also have $\gamma_2^{(\lambda)}<\gamma_1^{(\lambda)}$ (or $\gamma_1^{(\lambda)}=\gamma_2^{(\lambda)}=0$; we treat this case later).

So, suppose that $\gamma_2^{(\lambda)}<\gamma_1^{(\lambda)}$. Now, our goal is to prove that $(\Delta(1),g(1))\in H_\omega$. We argue by contradiction. Observe that there exists some vector $e_\omega\geq(1,1)$ such that $e_\omega\notin H_\omega$ and $e_\omega\cdot(\Delta(1),g(1))\neq 0$. Suppose that $\varphi_\omega:=(\Delta(1),g(1))\notin H_\omega$. Then, there exists a number $c_\omega\neq 0$ such that $u_\omega:=\varphi_\omega+c_\omega e_\omega\in H_\omega$. Now, let us write

$$\begin{pmatrix}\Delta(k+1)\\ g(k+1)\end{pmatrix}=\Big(\prod_{i=1}^k A_i^{(\lambda)}\Big)\varphi_\omega=\Big(\prod_{i=1}^k A_i^{(\lambda)}\Big)u_\omega-c_\omega\Big(\prod_{i=1}^k A_i^{(\lambda)}\Big)e_\omega. \quad (3.13)$$

As $\gamma_1^{(\lambda)}>\gamma_2^{(\lambda)}$, using (3.11) and (3.12), and the uniform positiveness of e_ω, we see that for all k large enough

$$\operatorname{sgn}\Delta(k)=\operatorname{sgn}g(k)=-\operatorname{sgn}(c_\omega). \quad (3.14)$$

On the other hand, let us show that for all k we have $\Delta(k)\leq 0$. Then, since $g(k)>0$ by definition, we obtain a contradiction with (3.14).

Indeed, we have

$$\Delta(k)=g(k)-g(k-1)=\lambda^{-k}\mathtt{E}_\omega^k\zeta_k-\lambda^{-(k-1)}\mathtt{E}_\omega^{k-1}\zeta_{k-1}.$$

Thus, we need to show that

$$\mathtt{E}_\omega^k\zeta_k\leq\lambda\mathtt{E}_\omega^{k-1}\zeta_{k-1}. \quad (3.15)$$

Note that

$$\mathtt{E}_\omega^k\zeta_k=\mathtt{E}_\omega^{k-1}\zeta_{k-1}\ \mathtt{E}_\omega\hat{\zeta}_k,$$

where $\hat{\zeta}_k$ is a random variable defined as follows: start the process with one particle at k and freeze all particles that reach $k-1$; then, $\hat{\zeta}_k$ is the number of frozen particles at $k-1$. Observe that, similarly to the proof of Theorem 2.6, the function

$$h(\eta_n)=\sum_{x\in\mathbb{Z}}\eta_n(x)\lambda^x$$

is still a supermartingale for $\mathbb{P}$-a.a. ω for this process as well. So, suppose that we start from one particle in 1 and freeze all particles that reach 0 and let τ be the moment when all particles are frozen. As we assumed that the cloud of particles vanishes on the right, τ is finite a.s. Then, using Fatou's Lemma, we obtain that, $\mathbb{P}$-a.s.,

$$\mathtt{E}_\omega \zeta_1 = \mathtt{E}_\omega \hat{\zeta}_1 = \mathtt{E}_\omega h(\eta_\tau) \le h(\eta_0) = \lambda.$$

By stationarity, $\mathtt{E}_\omega \hat{\zeta}_k \le \lambda$ for all $k = 1, 2, 3, \dots$, and this shows (3.15).

Hence, $(\Delta(1), g(1)) \in H_\omega$ and

$$\begin{pmatrix} \Delta(k+1) \\ g(k+1) \end{pmatrix} = \Big(\prod_{i=1}^{k} A_i^{(\lambda)} \Big) \begin{pmatrix} \Delta(1) \\ g(1) \end{pmatrix}. \tag{3.16}$$

Let $\|\cdot\|_1$ be the L_1-norm in $\mathbb{R}^2$. Then,

$$\|(\Delta(k+1), g(k+1))\|_1 = |g(k+1) - g(k)| + g(k+1) = g(k),$$

as $\Delta(k+1) \le 0$. Thus, (3.16) and (3.12) imply that

$$\lim_{k\to\infty} \frac{1}{k} \ln g(k) = \gamma_2^{(\lambda)} \tag{3.17}$$

and so

$$\lim_{k\to\infty} \frac{1}{k} \ln f(k) = \gamma_2^{(\lambda)} + \ln \lambda. \tag{3.18}$$

As mentioned above, it may happen also that $\gamma_1^{(\lambda)} = \gamma_2^{(\lambda)} = 0$, but in this case it is straightforward to obtain that (3.17) and (3.18) hold as well (since the limits in (3.11) and (3.12) are both equal to 0).

Now, note that, $\mathtt{E}_\omega^k \zeta_k = \mathtt{E}_\omega^k \hat{\zeta}_k \mathtt{E}_\omega^{k-1} \hat{\zeta}_{k-1} \dots \mathtt{E}_\omega^1 \hat{\zeta}_1$, and $\hat{\zeta}_1, \dots, \hat{\zeta}_k$ have the same annealed law as ζ_1. Therefore, by the Ergodic Theorem, we have

$$\lim_{k\to\infty} \frac{1}{k} \ln f(k) = \mathbb{E} \ln \mathtt{E}_\omega \zeta_1, \quad \mathbb{P}\text{-a.s.}$$

Thus, $\mathbb{E} \ln \mathtt{E}_\omega \zeta_1 = \ln \lambda + \gamma_2^{(\lambda)}$, and the process $\mathcal{Z}_n$ survives globally iff

$$0 < \ln \lambda + \gamma_2^{(\lambda)} = \mathbb{E} \ln \Big(\frac{\mu_k^-}{\mu_k^+} \Big) - \ln \lambda - \gamma_1^{(\lambda)},$$

by (3.7). Now condition (3.8) follows. It remains to prove that $\gamma_1 = \gamma_1^{(\lambda)} + \ln \lambda$. Observe that (3.9) can be written in terms of $\{A_k\}$:

$$\begin{pmatrix} f(k+1) \\ f(k) \end{pmatrix} = A_k \begin{pmatrix} f(k) \\ f(k-1) \end{pmatrix}.$$

Using this relation, it is straightforward to check that $A_k = \lambda B^{-1} A_k^{(\lambda)} B$, where

$$B := \begin{pmatrix} 1 & -\lambda \\ 1 & 0 \end{pmatrix}.$$

Now the desired statement follows since $\prod_{k=1}^n A_k = \lambda^n B^{-1} \prod_{k=1}^n A_k^{(\lambda)} B$. $\square$

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