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A direct method for the numerical time reversal of waves in a heterogeneous medium

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Abstract

In this work, the problem of time reversal in a heterogeneous medium is considered as the inverse problem of determining the solution of a wave equation with spatially varying coefficients from lateral Cauchy data. This problem occurs in several applications in the area of medical imaging and non-destructive testing, for example in thermoacoustic tomography.

Using the method of quasi-reversibility, the original ill-posed problem is replaced with a boundary value problem for a fourth order partial differential equation. We find a weak H^2 solution of this problem and show that it is a well-posed elliptic problem. Error estimates and convergence of the approximation follow from exact observability estimates for the wave equation, which are proven using a Carleman estimate. We derive a numerical scheme for the solution of the quasi-reversibility problem by a B-spline Galerkin method, for which we give error estimates. Finally, we present numerical results supporting the robustness of this method for the reconstruction of the wave field from lateral Cauchy data, where we also consider the case of data given only on a part of the boundary.

Zusammenfassung

In dieser Arbeit wird das Problem betrachtet, aus transienten Messungen des Schalldrucks am Rand eines beschränkten Gebiets das Schallfeld im Inneren dieses Gebiets zu einer früheren Zeit zu bestimmen. Dieses Problem, welches in der Physik als Zeitumkehrproblem bezeichnet wird, tritt zum Beispiel in der medizinischen Bildgebungstechnik der thermoakustischen Tomographie auf, die auch als Motivation für die Arbeit vorgestellt wird. Eine zentrale Bedeutung hat dabei die Berücksichtigung einer räumlich variierenden Schallgeschwindigkeit, welches auch eine wesentliche Neuerung darstellt.

Das mathematische Modell ist ein schlecht gestelltes laterales Cauchy-Problem für die Wellengleichung, für das eine stabile Approximation hergeleitet wird, die auf der Methode der Quasi-Reversibilität beruht. Für diese Approximation werden Existenz, Eindeutigkeit und Regularität der Lösung bewiesen.

Die Konvergenz der Approximationslösungen gegen die Lösung des ursprünglichen Problems, auch bei gestörten Randdaten und Messungen nur an einem Teilrand, wird mit Hilfe einer Beobachtbarkeitsungleichung für die Wellengleichung gezeigt. Zentrales Hilfsmittel ist eine Carleman-Abschätzung für hyperbolische Differentialoperatoren zweiter Ordnung.

Für die numerische Lösung der approximierenden Probleme wird ein B-Spline-Galerkin-Verfahren hergeleitet. Die Effektivität und Robustheit des Verfahrens, auch für den Fall von nur auf einem Teilrand gegebener Daten, wird schliesslich anhand numerischer Tests belegt.

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1. Introduction

Many practical problems are concerned with determining the strength and location of sources of disturbances in a medium, when only boundary measurements are available. Examples include medical imaging, seismic observations, geodynamics, or tracing electromagnetic pulses. If the sources can be temporally localized, this problem is equivalent to the determination of the initial conditions in a wave equation. If it would be possible to completely characterize a 'final' state, then the time reversibility of the wave equation can be employed to calculate the wave field backwards in time to the moment of interest. This fact is exploited in experimental time reversal (cf. [16] and references therein), where a signal is recorded by an array of transducers, time-reversed, and then re-transmitted into the medium. The re-transmitted signal propagates back through the same medium and refocuses on the source. Ideally, the array completely surrounds the source and thus the time-reversed signals go through all the scatterings, reflections and refractions that they underwent in the forward direction. If the time-reversal operation is performed on a limited angular area, only a small part of the field radiated by the source is captured and time reversed, thus limiting reversal and focusing quality. Computational time reversal, as first described in [3, 5, 59], is concerned with the mathematical analysis and numerical reproduction of this phenomenon, as well as its applications in imaging.

In an experimental setup, where the re-transmission is performed directly following the recording, the state of the medium at the start of transmission is close to that at the end of recording, hence there is no need to explicitly know the final state, which would be necessary for computational time reversal. However, in practical applications of computational time reversal, it is usually either not possible to measure a wave field in a complete region, or dissipative terms break the time reversibility of the wave equation. In both cases, the problem is then the reconstruction of initial conditions from boundary measurements only, which can be understood as a lateral Cauchy problem, where both Dirichlet and Neumann boundary conditions, but no initial conditions, are prescribed. This relation between time reversal and lateral Cauchy problems was first noticed in [3]. Such a

problem is known to be generally ill-posed in the sense of Hadamard. This means that neither existence, uniqueness, nor continuous dependence on the boundary conditions[1] must hold for its solution. However, under certain conditions, exact observability estimates can be proved, which warrant a unique and stable solution. Existence, on the other hand, can be guaranteed for a related minimization problem. By means of the observability estimate, it is possible to show that the solution of the latter are close to solutions of the original Cauchy problem.

We are thus concerned with the reconstruction of initial conditions in a scalar wave equation, which is motivated by the problem of thermoacoustic tomography, a novel method in medical imaging that uses different modalities for illumination of the target and measurement of its response. Specifically, we are interested in the case where a spatially varying absorption of radio waves induces a pressure gradient in the target, which propagates as an acoustic wave in the target and the surrounding medium. This pressure field, if measured at the boundary and played backwards into the medium, should refocus on the site where absorption is strongest. In order to investigate this problem numerically, we will compute the solution of the corresponding lateral Cauchy problem.

In contrast with the works cited above, where computational time reversal is employed in the localization of small, well-separated sources (in [59]) or scatterers (in [5]) in a random medium of small variation, we try to reconstruct arbitrary spatially varying fields in a macroscopically heterogeneous medium. Previous works on the numerical determination the initial condition in a hyperbolic equation from lateral Cauchy data were [37], [29], and [34], which only applied to constant coefficients in the principal part of the operator. The variable coefficient case was briefly considered in [40], but no numerical studies were done. Our main contribution in this dissertation is therefore the presentation and justification of a robust numerical method for the solution of the time reversal problem for the wave equation in a heterogeneous medium.

This thesis is organized as follows: In Chapter 2, after introducing some notations, we present the problem of thermoacoustic tomography as a model for a time reversal problem for the variable coefficient wave equation (Section 2.2) and give its mathematical formulation as a lateral Cauchy problem (Section 2.3). Then we introduce the method of direct quasi-reversibility for the solution of such a problem in Chapter 3. This involves minimizing a quadratic functional over a suitable Hilbert space. We show that this problem is well-posed and briefly discuss the regularity of its solution. Error estimates and convergence results for the method of quasi-reversibility can be derived by Carleman estimates, which is done in Chapter 4. The computational method used for the numerical solution of

[1] The last property will also be referred to as *stability* in the following.

the time reversal problem is discussed in Chapter 5. Here, we consider a direct approach based on a B-spline Galerkin method, which we derive for the problem of quasi-reversibility and for which we give an error estimate. We illustrate the robustness of the method with several numerical examples, which are presented in Chapter 6, with additional figures in the appendix A. Specifically, we address the problem of reconstruction from partial boundary data. Finally we discuss the results and give concluding remarks in Chapter 7.

In this work, we have chosen not to strive for the most general results, instead concentrating on formulations directly useful for the considered applications and indicating where straightforward generalizations are possible. Similarly, we have tried to keep the presentation concise, without omitting key steps. We assume the reader to be familiar with the basic theory of functional analysis, Sobolev spaces, partial differential equations, inverse problems, and the finite element method. Good introductory works on these topics are (in the order listed above) those of Yosida [65], Adams [1], Wloka [62], Engl [14], and Ciarlet [12]. A particularly nice introduction to hyperbolic partial differential equations is the short book by Ikawa [25]. Instead of restating definitions and theorems from these areas, we have opted to give precise citations from these sources.

2. Statement of the problem

2.1. Notations

For a vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we write $|x|$ for the standard Euclidean norm $|x|^2 = \sum_{i=1}^n x_i^2$.

Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with boundary $\partial\Omega$. For $T > 0$, consider the space-time cylinder $Q_T := \Omega \times [0, T]$ with lateral boundary $S_T := \partial\Omega \times [0, T]$. For a part of the boundary $\Gamma \subset \partial\Omega$, we correspondingly set $\Gamma_T := \Gamma \times [0, T]$. To simplify notations, we write $q := (x, t) \in Q_T$ for the integration variable over Q_T. We say that Q_T is of class C^k if its boundary ∂Q_T can be locally rectified by a C^k-diffeomorphism (see, e.g., [1, 4.10] for a precise definition). If all local diffeomorphisms are Lipschitz continuous, we call Q_T a Lipschitz domain. The latter will be the main class of domains we are concerned with.

We let ∂_i stand for the partial derivative with respect to x_i, with $\partial_{n+1} := \partial_t$, the derivative with respect to time. Higher order derivatives are represented by repeated indices, e.g., ∂_{tt}. The spatial gradient is written as $\nabla := (\partial_1, \dots, \partial_n)^T$, while $\Delta := \sum_{i=1}^n \partial_{ii}$ stands for the Laplacian in $\mathbb{R}^n$. For convenience, we also use the notation $\nabla^2 := (\partial_{11}, \dots, \partial_{nn})^T$ for the vector of second spatial derivatives. Finally, ∂_ν denotes the normal derivative at $\partial\Omega$ with the outer normal ν.

Consider the space $L^2(Q_T)$ of measurable real-valued functions on Q_T for which

$$\|f\|^2_{L^2(Q_T)} := \int_{Q_T} |f|^2 \, dq < \infty . \tag{2.1}$$

holds, and for $m \in \mathbb{N} \cup \{\infty\}$ the spaces $C^m(Q_T)$ of m times continuously differentiable functions on Q_T. Correspondingly, the space of smooth functions with compact support in Q_T is denoted by $C_0^\infty(Q_T)$. We also use the standard Sobolev spaces $H^m(Q_T)$ of functions in $L^2(Q_T)$ possessing weak derivatives in $L^2(Q_T)$ of order up to m. Equipped with the following inner products, these are Hilbert

spaces (cf. [1, Th. 3.6]):

$$\langle f, g\rangle_{H^1(Q_T)} := \int_{Q_T} fg\,dq + \sum_{i=1}^{n+1}\int_{Q_T} \partial_i f \partial_i g\,dq\,, \tag{2.2}$$

$$\langle f, g\rangle^2_{H^2(Q_T)} := \langle f, g\rangle_{H^1(Q_T)} + \sum_{i,j=1}^{n+1}\int_{Q_T} \partial_{ij} f \partial_{ij} g\,dq\,, \tag{2.3}$$

where the derivatives are taken in the weak sense. The corresponding norms are:

$$\|f\|^2_{H^1(Q_T)} := \|f\|^2_{L^2(Q_T)} + \sum_{i=1}^{n+1} \|\partial_i f\|^2_{L^2(Q_T)}\,, \tag{2.4}$$

$$\|f\|^2_{H^2(Q_T)} := \|f\|^2_{H^1(Q_T)} + \sum_{i,j=1}^{n+1} \|\partial_{ij} f\|^2_{L^2(Q_T)}\,, \tag{2.5}$$

In the same manner, we can define inner products $\langle\cdot,\cdot\rangle_{H^m(Q_T)}$ and norms $\|\cdot\|_{H^m(Q_T)}$ for all $m \in \mathbb{N}$, while $H^0(Q_T) = L^2(Q_T)$. Where not stated otherwise, we denote the inner product in the Hilbert space X by $\langle\cdot,\cdot\rangle_X$. For brevity, the inner product in $\mathbb{R}^n$ is simply written as $\langle\cdot,\cdot\rangle_n$. Finally, $B_r(x) := \{x \in \mathbb{R}^n : |x| < r\}$ stands for the open ball around $x \in \mathbb{R}^n$ with radius $r > 0$.

2.2. Thermoacoustic tomography

Thermoacoustic computed tomography (cf. Figure 2.1) is a new imaging method that uses different modalities for illumination of the target and measurement of its response (see, e.g., [42, 43, 44, 45, 63, 64] and references therein). It combines the advantages of purely optical imaging (high contrast) and ultrasound imaging (high resolution). Specifically, the target is subjected to a short electromagnetic impulse, which is absorbed, leading to a temperature increase and hence to expansion. This induces a pressure wave in the target, which can be measured as a change in the acoustic field outside the sample. If the absorption of the electromagnetic energy is spatially varying, the resulting wave field will carry the signature of this inhomogeneity. The premise for medical applications is that cancerous tissue absorbs more energy per volume than healthy tissue. The problem is hence to calculate this absorption density of the target from time dependent acoustic measurements outside it.

In order to make this precise, we introduce the radiation intensity of the illuminating pulse, $I(x,t)$, and the spatially varying absorption coefficient of the target and the surrounding medium, $\alpha(x)$. The generated acoustic pressure wave $u(x,t)$

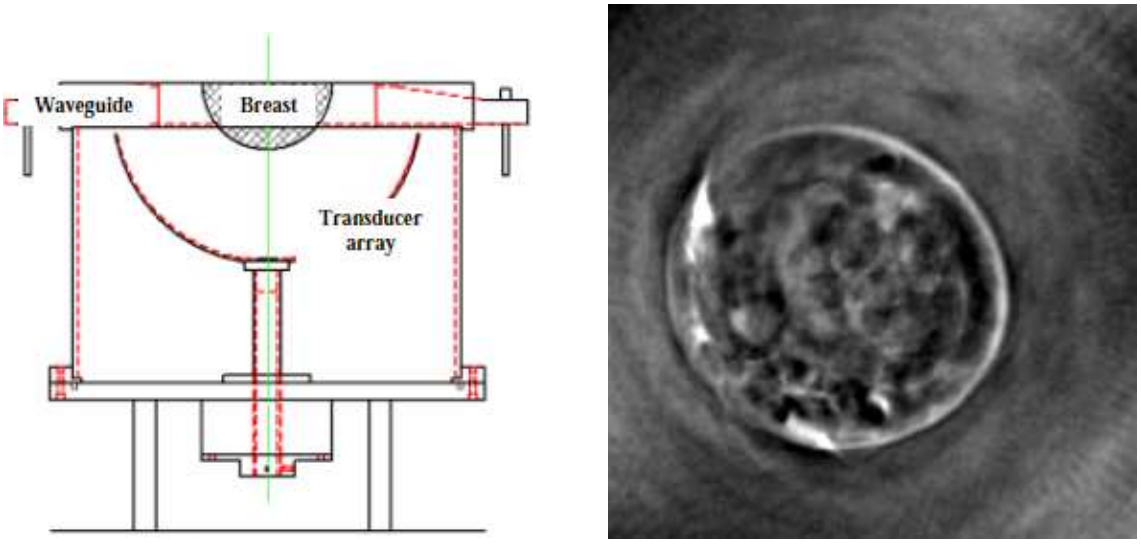


Figure 2.1: Thermoacoustic computed tomography (TCT) applied to detection of breast cancer. Left: Prototype of TCT scanner. Right: TCT scan of breast. (taken from [44]).

propagating in $\mathbb{R}^3$ can then be described by the inviscid liquid model, ignoring thermal diffusion [42, 51]:

$$\frac{1}{c^2(x)}\partial_{tt}u(x,t) - \Delta u(x,t) = \alpha(x)\frac{\beta}{c_p}\partial_t I(x,t), \qquad (x,t) \in \mathbb{R}^3 \times [0,\infty), \tag{2.6}$$

where $c(x)$ is the sound speed, β the thermal expansion coefficient, and c_p the specific heat capacity of the medium. We take the medium to be at rest prior to the irradiation, i.e.

$$u(x,t) = \partial_t u(x,t) \equiv 0, \qquad x \in \mathbb{R}^3\,,\ t < 0. \tag{2.7}$$

If we assume the pulse to be of very short duration (cf. [42]) and spatially homogeneous, we can approximate its temporal shape by the delta distribution:

$$I(x,t) \approx I_0\delta(t). \tag{2.8}$$

In this case, equation (2.6) is assumed to hold in the sense of distributions. We denote the generalized derivative with respect to time by D_t and let $\delta'(t) := D_t\delta(t)$.

We set $f(x,t) := \alpha(x)\beta c_p^{-1} I_0\delta'(t)$. If we extend the solution $u(x,t)$ and the right hand side $f(x,t)$ as zero for $t < 0$, and denote the continued solution and right hand side as $\tilde{u}$ and $\tilde{f}$, respectively, we can say that $\tilde{u}$ solves the *generalized*

Cauchy problem (cf. [61, § 12.2]):

$$\frac{1}{c^2(x)} D_{tt}\tilde{u}(x,t) - \sum_{i=1}^{3} D_{ii}\tilde{u}(x,t) = \tilde{f}(x,t). \tag{2.9}$$

We wish to formulate this as a classical Cauchy problem:

$$\begin{cases} \frac{1}{c^2(x)}\partial_{tt}u(x,t) - \Delta u(x,t) &= f(x,t) & (x,t) \in \mathbb{R}^3 \times [0,T] \ , \\ u(x,t)|_{t=0} &= u_0(x) & x \in \mathbb{R}^3 \ , \\ \partial_t u(x,t)|_{t=0} &= u_1(x) & x \in \mathbb{R}^3 \ . \end{cases} \tag{2.10}$$

Extending similarly the solution and right hand side of (2.10) as zero for $t < 0$ (and denoting again this extension by $\tilde{u}$, $\tilde{f}$), we see that for all $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, the following equalities hold:

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^3} \left(c^{-2}(x)\partial_{tt}u(x,t) - \Delta u(x,t)\right) \varphi(x,t)\,dxdt \\ &\quad = \int_0^\infty \int_{\mathbb{R}^3} u(x,t)\left(c^{-2}(x)\partial_{tt}\varphi(x,t) - \Delta\varphi(x,t)\right)\,dxdt \\ &\quad = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^3} u(x,t)\left(c^{-2}(x)\partial_{tt}\varphi(x,t) - \Delta\varphi(x,t)\right)\,dxdt \\ &\quad = \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty \int_{\mathbb{R}^3} \left(c^{-2}(x)\partial_{tt}u(x,t) - \Delta u(x,t)\right)\varphi(x,t)\,dxdt \right. \\ &\qquad \left. + \int_{\mathbb{R}^3} \partial_t u(x,\varepsilon)\varphi(x,\varepsilon)\,dx - \int_{\mathbb{R}^3} u(x,\varepsilon)\partial_t\varphi(x,\varepsilon)\,dx\right] \\ &\quad = \int_0^\infty \int_{\mathbb{R}^3} f(x,t)\varphi(x,t)\,dxdt + \int_{\mathbb{R}^3} \partial_t u(x,0)\varphi(x,0)\,dx \\ &\qquad - \int_{\mathbb{R}^3} u(x,0)\partial_t\varphi(x,0)\,dx \\ &\quad = \int_0^\infty \int_{\mathbb{R}^3} f(x,t)\varphi(x,t)\,dxdt + \int_{\mathbb{R}^3} u_1(x)\varphi(x,0)\,dx \\ &\qquad - \int_{\mathbb{R}^3} u_0(x)\partial_t\varphi(x,0)dx \\ &\quad = \int_{-\infty}^\infty \int_{\mathbb{R}^3} \left(\tilde{f}(x,t) + u_0(x)\delta'(t) + u_1(x)\delta(t)\right)\varphi(x,t)dxdt\,, \end{aligned} \tag{2.11}$$

where, in the last step, we have used the definitions of the generalized derivative and of the delta distribution.

Now, if we compare equations (2.9) and (2.10) with the help of (2.11) and the definition of $f(x,t)$, we see that the generalized Cauchy problem (2.9) for $\tilde{u}$ is equivalent to the following classical Cauchy problem for u:

$$
\begin{cases}
\frac{1}{c^2(x)}\partial_{tt}u(x,t) - \Delta u(x,t) &= 0 & (x,t) \in \mathbb{R}^3 \times [0,T] \,, \\
u(x,t)|_{t=0} &= \alpha(x) I_0 \frac{\beta}{c_p} & x \in \mathbb{R}^3 \,, \\
\partial_t u(x,t)|_{t=0} &= 0 & x \in \mathbb{R}^3 \,,
\end{cases} \tag{2.12}
$$

in the sense that a solution u of (2.12) is a solution of (2.9), and a solution $\tilde{u}$ of (2.9) for which $\tilde{u}(x,t) \equiv 0$ for $t < 0$ is (if sufficiently regular) a solution of (2.12). As a standard result (e.g. [61, § 12.3]), we know that a solution $\tilde{u}$ of (2.9) exists, is unique, and depends continuously on $\alpha(x)$. The same holds for the classical Cauchy problem (2.12) (see, e.g., [62, Th. 29.1]).

The problem of thermoacoustic tomography is the inverse problem of determining the initial conditions of (2.12), and so the absorption coefficient $\alpha(x)$, from time dependent measurement of the pressure field u on a surface $\partial\Omega$ outside the target, when c, I_0, β and c_p are known. For specific geometries of $\partial\Omega$ such as planes, cylinders and spheres, analytic expressions for the absorption density can be directly calculated (cf. [42, 43, 44, 63, 64]). This problem was also solved in [15] and [18] by inversion of the spherical mean operator, in [19] by a filtered back projection algorithm derived by the method of approximate inverse, and in [29] using Fourier expansion. These methods, however, assume a constant sound speed, while we consider thermoacoustic tomography in a heterogeneous medium. For this reason, we pose the problem of thermoacoustic tomography as a lateral Cauchy problem for a wave equation with spatially varying coefficients, which we discuss in the context of time reversal. Another advantage of our approach is the applicability to thermoacoustic tomography with arbitrary scanning geometries.

2.3. Mathematical time reversal

Time reversal is concerned with the refocussing of an acoustic wave field on point sources in a domain, when the generated acoustic pressure is recorded on the boundary and is later, quite literally, "played backwards". If we want to study this phenomenon numerically, the problem can be modeled by a scalar wave equation. However, in order for the problem to be well-posed, we have to prescribe initial conditions, as well as the boundary condition resulting from the measurement of the acoustic pressure. Except in very specific circumstances, it is not possible to measure the complete wave field in the domain prior to initiating the acoustic sources. A similar consideration holds for the field at the final time, before the reintroduction of the measured sound starts (which would be sufficient since the wave equation is time reversible, i.e. stable both for t and for $-t$ as the

time variable). Hence, as in section 2.2, we need to consider the problem of recovering the solution of the wave equation from boundary measurements only. In this case, since we deal with a second order differential equation, we need to prescribe two boundary conditions (e.g., of Dirichlet and Neumann type), called the Cauchy data, on the spatial (i.e., time-like) boundary. Problems of such type are therefore called *lateral Cauchy problems*. With this in mind, we can formulate the mathematical problem of time reversal as follows:

Problem 2.1 (Time reversal in bounded domains). Given c, φ_0, and φ_1, find $u(x,t)$ in Q_T, which solves the lateral Cauchy problem:

$$\begin{cases} \frac{1}{c(x)^2}\partial_{tt}u(x,t) - \Delta u(x,t) &= 0 & (x,t) \in Q_T\,, \\ u(x,t) &= \varphi_0(x,t) & (x,t) \in S_T\,, \\ \partial_\nu u(x,t) &= \varphi_1(x,t) & (x,t) \in S_T\,. \end{cases} \tag{2.13}$$

This problem is connected with the inverse problem considered in Section 2.2 in the following way: Given measurements of the acoustic pressure at the boundary, we can calculate the Neumann conditions by solving the classical (well-posed) exterior initial boundary value problem for the wave equation, by taking zero initial conditions and prescribing the Dirichlet conditions on the boundary S_T. From the computed solution, it is straightforward to calculate the normal derivative at S_T and thus complete the set of Cauchy data. For this reason, the inverse problem for the initial condition is also equivalent to the problem of time reversal as formulated in [59, 5] and observed experimentally ([16]), although their mathematical formulation is different from ours. (Note that we do not assume the medium to be at rest at a specified final time T.) Indeed, the high degree of refocusing of acoustic point sources described in these references is indicated by the Lipschitz stability which can be proved for this problem (cf. [35]). Of course, for thermoacoustic tomography and similar applications, we are primarily interested in recovering $u(x,0)$ in Ω, and from this, e.g., $\alpha(x)$.

We are furthermore interested in time reversal from limited boundary measurements, hence we also consider:

Problem 2.2 (Time reversal from limited data). Let $\Gamma \subset \partial\Omega$ be a subset of positive $(n-1)$-dimensional measure. Given c, φ_0, and φ_1, find $u(x,t)$ in Q_T, which solves the lateral Cauchy problem:

$$\begin{cases} \frac{1}{c(x)^2}\partial_{tt}u(x,t) - \Delta u(x,t) &= 0 & (x,t) \in Q_T\,, \\ u(x,t) &= \varphi_0(x,t) & (x,t) \in \Gamma_T\,, \\ \partial_\nu u(x,t) &= \varphi_1(x,t) & (x,t) \in \Gamma_T\,. \end{cases} \tag{2.14}$$

Finally we consider a special case of time reversal in an unbounded domain, namely a quadrant of $\mathbb{R}^2$.

Problem 2.3 (Time reversal in a quadrant). Let $D := \{x \in \mathbb{R}^2 |\ x_1 > 0,\ x_2 > 0\}$ be the first quadrant and $\Gamma := \{x \in \mathbb{R}^2 |\ x_1 = 0 \text{ or } x_2 = 0\}$. Assume the support of $u(x,0)$ is contained in $D_R = D \cap \{(x,y) \in \mathbb{R}^2 :\ |x|^2 < R\}$ for some $R > 0$. Given c, φ_0, and φ_1, find $u(x,t)$ in $D_T := D \times [0,T]$, which solves the lateral Cauchy problem:

$$\begin{cases} \frac{1}{c(x)^2}\partial_{tt}u(x,t) - \Delta u(x,t) &= 0 & (x,t) \in D_T\,, \\ u(x,t) &= \varphi_0(x,t) & (x,t) \in \Gamma_T\,, \\ \partial_\nu u(x,t) &= \varphi_1(x,t) & (x,t) \in \Gamma_T\,. \end{cases} \tag{2.15}$$

We will show that, under certain assumptions connecting c, R and T, that a stable solution of the time reversal problem in the whole of D_T is possible.

The lateral Cauchy problem is ill-posed in general — neither existence, uniqueness, nor stability of its solution is warranted. It is clear that for arbitrary pairs φ_0, φ_1, no such solution must exist. In his classical example [17], Hadamard showed that the Cauchy problem

$$\begin{cases} \partial_{11}u(x) + \partial_{22}u(x) &= 0 & x \in \Omega \subset \mathbb{R}^2\,, \\ u(x) &= 0 & x \in \partial\Omega\,, \\ \partial_\nu u(x) &= f(x) & x \in \partial\Omega\,, \end{cases} \tag{2.16}$$

has no solution unless $f(x)$ is analytic. Even if the boundary values form a compatible pair (i.e. they are known to be the traces of a function u which solves the corresponding well-posed boundary value problem), this is usually no longer the case for small perturbations $\varphi_0 + \varepsilon_0, \varphi_1 + \varepsilon_1$. This problem is one of stability, where we demand that the solution u has continuous dependence on the data f. For instance, for $f_n(x) = n^{-2}\cos(nx_1)$, Hadamard's problem has the unique solution $u(x) = n^{-3}\cos(nx_1)\sinh(nx_2)$. Although f_n (along with its first and second derivatives) tends to zero for $n \to \infty$, the corresponding solution blows up for $x_1 = 0, x_2 \neq 0$.

For practical purposes, mere continuous dependence is not sufficient, if the continuity is too weak. If we wish to compute solutions numerically, we should require at least *Hölder continuity*, i.e. $\|u - u'\| \leqslant C\varepsilon^\alpha$ for an $\alpha > 0$ and $\|f - f'\| \leqslant \varepsilon$. This means only a fixed percentage of significant digits needs be lost in recovering u from f (such problems are sometimes called *well-behaved*, cf. [28]). If we can take $\alpha = 1$, we have *Lipschitz continuity*, which guarantees a stable solution and thus well-posedness. Unfortunately, for ill-posed Cauchy problems, even if there is a continuous dependence on the data, *logarithmic continuity*

(i.e. $\|u - u'\| \leqslant C \log(1/\varepsilon)^{-\alpha}$) is the best continuity possible in general. For example (cf. [28]), there are solutions of the wave equation $\partial_{tt} u(x,t) = \Delta u(x,t)$ in $\mathbb{R}^2 \times [-\infty, \infty]$, which have the form:

$$u_n(x,t) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{in(\vartheta - x_1 \sin\vartheta + x_2 \cos\vartheta + t)} \, d\vartheta \,. \tag{2.17}$$

These functions satisfy the following bounds for $\rho < 1$ and $q < 1$:

$$\sup_{|x|<\rho} u_n(x,t) \leqslant c_1 q^n \,, \tag{2.18}$$

$$\sup_{|x|<1} u_n(x,t) \geqslant c_2 n^{-1/3} \,, \tag{2.19}$$

$$\sup_{x\in\mathbb{R}^2} (|\nabla u_n(x,t)| + u(x,t)) \leqslant c_3 n^{2/3} \,. \tag{2.20}$$

Consider the Cauchy problem for the cylinder with radius 1 with $u_n(x,t)$ and $\partial_\nu u_n(x,t)$ given on a cylinder of radius less than ρ. The solution $u(x,t) = u_n(x,t)$ for large n will satisfy the following logarithmic continuous dependence, provided $u(x,t)$ and $|\nabla u_n(x,t)|$ are bounded in $\mathbb{R}^3$ by M:

$$\sup_{|x|<1} u_n(x,t) \leqslant c_1 M (\log(M/\varepsilon)^{-1} \,, \tag{2.21}$$

where $\varepsilon := \sup_{|x|<\rho} u_n(x,t)$.

Using this construction, one can show that a solution of the Cauchy problem, if it exists, does not have to be uniquely determined. For example ([46]), the wave equation $\partial_{tt} u(x,t) - \Delta u(x,t) + b\partial_t u(x,t) + cu(x,t) = 0$ in $\mathbb{R}^2 \times [-\infty, \infty]$ with real, time-dependent, smooth coefficients b, c has a smooth solution which is zero for $|x| < 1$, yet not identically zero.

Therefore, we cannot hope to solve the lateral Cauchy problems posed above directly. For numerical calculations, we have to consider approximations of these problems that have solutions which are unique and depend continuously on the data, but are still close to that of the original problem. Previous contributions for lateral Cauchy problems with constant coefficients include [37, 39, 31, 35, 41, 29] for hyperbolic equations, and [30] for hyperbolic inequalities. We discuss one possible approach allowing variable coefficients, which is based on the minimization of a functional in a suitable Hilbert space, in the next chapter.

3. Method of quasi-reversibility

The method of quasi-reversibility was introduced in the book by Lattès and Lions [48] as an approach for the numerical solution of ill-posed boundary value problems for partial differential equations, especially overdetermined boundary value problems and backwards solutions of time-irreversible equations. This method consists in replacing the ill-posed second order problem with a well-posed fourth order problem, and was previously applied to ill-posed Cauchy problems for elliptic [48, 6, 7, 38], parabolic [48, 56, 32] and hyperbolic [37] equations as well as coefficient inverse problems [31, 33, 36, 40]. We employ an hyperbolic variant of the method described in [48, Ch. 4.8]. In contrast with the indirect approach considered there (which entailed the derivation of a partial differential equation equivalent to a variational problem), we use a direct method for calculating the solution of this weak formulation. This allows us to make weaker requirements on the regularity of the solution (H^2 instead of H^4). Furthermore, this derivation (which was first described in [33]) has the advantage of avoiding the introduction of additional boundary conditions in the Cauchy problem. The central result is a full derivation of this formulation, as well as the complete proof of the well-posedness of the quasi-reversibility problem.

3.1. Approximation by direct quasi-reversibility

Let L be a linear second order hyperbolic differential operator in a bounded domain $\Omega \subset \mathbb{R}^n$ with a boundary $\partial\Omega$ that satisfies the uniform cone condition (cf. [1, Def. 4.8] for a precise definition[1]). Specifically, for $c \in C^1(\overline{\Omega})$ and $f \in L^2(Q_T)$, we consider the wave equation:

$$Lu := \frac{1}{c(x)^2}\partial_{tt}u - \Delta u = f\,. \tag{3.1}$$

The results of this and the next chapter remain valid in the case of general linear hyperbolic differential operators of second order. The method also generalizes

[1] For our purposes, it suffices to mention that all Lipschitz domains, especially cylinders and cubes, have this property.

easily to operators of higher order.

The key idea of the method of direct quasi-reversibility is to replace the bilinear form in the weak formulation of the ill-posed problem with one that is positive definite (which implies well-posedness) and induces an equivalent norm on a suitable function space (which is necessary for the proof of convergence). For that reason, we introduce the function space

$$H_0^2(Q_T) := \left\{ u \in H^2(Q_T) : \ u|_{S_T} = \partial_\nu u|_{S_T} = 0 \right\} , \tag{3.2}$$

and for $\Gamma \subset \partial\Omega$ with positive $(n-1)$-dimensional measure the space

$$H_\Gamma^2(Q_T) := \left\{ u \in H^2(Q_T) : \ u|_{\Gamma_T} = \partial_\nu u|_{\Gamma_T} = 0 \right\} , \tag{3.3}$$

where the equalities are defined in the sense of traces (see, e.g., [52, Th. 4.12]). Both are closed subspaces of $H^2(Q_T)$. Furthermore, on $H^2(Q_T)$, consider the symmetric bilinear form

$$\langle u, v \rangle_{QR} := \int_{Q_T} \partial_{tt} u \, \partial_{tt} v \, dq + \int_{Q_T} \left\langle \nabla^2 u, \nabla^2 v \right\rangle_n dq + \int_{Q_T} u \, v \, dq . \tag{3.4}$$

Due to the presence of the $L^2(Q_T)$ norm of u and v, $\langle \cdot, \cdot \rangle_{QR}$ is positive definite and hence an inner product. Consequently, $\|u\|_{QR}^2 := \langle u, u \rangle_{QR}$ is a norm on $H^2(Q_T)$. The inclusion of all second derivatives guarantees that this norm is equivalent to the standard norm $\|\cdot\|_{H^2(Q_T)}$:

Lemma 3.1.1. *There are constants $c_1, c_2 > 0$ so that for all $u \in H_0^2(Q_T)$ the following holds:*

$$c_1 \|u\|_{H^2(Q_T)} \geqslant \|u\|_{QR} \geqslant c_2 \|u\|_{H^2(Q_T)} . \tag{3.5}$$

Proof. The second inequality follows directly from the definition of the $H^2(Q_T)$ norm. For the first inequality, we first notice that the mixed derivatives of $u \in H_0^2(Q_T)$ can be bounded from above by $\|u\|_{QR}$. We start by proving this for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Applying the Fourier transform and then Young's inequality on the Fourier variable, we see that for $i, j \in 1 \ldots, n+1$:

$$\begin{aligned} |\nabla^2 \varphi|^2 + |\partial_{tt} \varphi|^2 &= \sum_{k=1}^{n+1} \left| \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} \xi_k^2 \hat{\varphi}(x) dx \right|^2 \\ &\geqslant \frac{1}{2} \left| \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} (\xi_i^2 + \xi_j^2) \hat{\varphi}(x) dx \right|^2 \geqslant \left| \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} \xi_i \xi_j \hat{\varphi}(x) dx \right|^2 = |\partial_{ij} \varphi|^2 , \end{aligned} \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^{n+1}$ and $\hat{\varphi}$ denotes the Fourier transform of φ. Since Q_T satisfies the uniform cone condition if Ω does, the restriction

of $C_0^\infty(\mathbb{R}^{n+1})$ to $\overline{Q}_T$ is dense in $H^2(Q_T)$ (cf. [62, Th. 3.6]). Therefore we can extend this inequality to $u \in H_0^2(Q_T)$ by the Calderón extension theorem (cf. [1, Th. 5.28]). Let E denote the linear extension operator from $L^2(Q_T)$ to $L^2(\mathbb{R}^{n+1})$. Then we can apply (3.6):

$$\begin{aligned}(3.7)\quad \left\|\nabla^2 u\right\|^2_{L^2(Q_T)} + \|\partial_{tt} u\|^2_{L^2(Q_T)} &\geqslant K\left(\left\|\nabla^2(Eu)\right\|^2_{L^2(\mathbb{R}^{n+1})} + \|\partial_{tt}(Eu)\|^2_{L^2(\mathbb{R}^{n+1})}\right)\\ &\geqslant K\,\|\partial_{ij}(Eu)\|^2_{L^2(\mathbb{R}^{n+1})} \geqslant K\,\|\partial_{ij}(Eu)\|^2_{L^2(Q_T)} = K\,\|\partial_{ij} u\|^2_{L^2(Q_T)}\,,\end{aligned}$$

where K is a positive constant independent of Q, T and u, given by the definition of the extension operator. Because Q_T is bounded and both u and $\partial_\nu u$ vanish on a subset of ∂Q_T with positive measure, we can apply the Poincaré inequality [47, p. 46] to estimate $\|\partial_{ij} u\|_{L^2(Q_T)}$ by $\|\partial_i u\|_{L^2(Q_T)}$ from below for all $i, j \in 1\ldots, n+1$. Adding up all these lower bounds, we arrive at the standard norm on $H^2(Q_T)$. □

This norm will now be used as a regularization term in the derivation of the quasi-reversibility formulation. First, assume that $\varphi_0 \equiv \varphi_1 \equiv 0$ in (2.13). Now consider for a small $\varepsilon > 0$ and $f \in L^2(Q_T)$ the Tikhonov functional

$$(3.8)\qquad J_\varepsilon(u) := \frac{1}{2}\,\|Lu - f\|^2_{L^2(Q_T)} + \frac{\varepsilon}{2}\,\|u\|^2_{QR}\,.$$

Instead of the solution u of the ill-posed Problem 2.1, we search for a quasi-solution of Problem 2.1 (cf. [58]), which is defined as a minimizer u_ε of J_ε in $H_0^2(Q_T)$ (for a suitable choice of f, discussed below). In Section 3.2, we will show that this modified problem is indeed well-posed. The purpose of Chapter 4 is to prove that under certain conditions, the quasi-solutions u_ε will converge to the true solution u as ε tends to 0. We first derive a variational formulation of this minimization problem.

By Lemma 3.1.1, the functional J_ε is coercive on $H_0^2(Q_T)$, since

$$(3.9)\qquad J_\varepsilon(u) \geqslant \varepsilon\,\|u\|^2_{QR} \geqslant c\,\|u\|^2_{H^2(Q_T)}\,.$$

As L is linear and all norms are convex, J_ε is convex as well. Hence, by the main theorem of monotone potential operators (see, e.g., [66, Th. 25.F]), if $J_\varepsilon(u)$ is Gâteaux-differentiable with derivative $J'_\varepsilon(u)$, the minimum problem (3.8) is equivalent to the abstract Euler equation $J'_\varepsilon(u) = 0$, i.e. every minimizer u_ε of (3.8) (whose existence is warranted by the above cited theorem) must satisfy

$$(3.10)\qquad J'_\varepsilon(u_\varepsilon)(v) = 0 \ \text{ for all } v \in H_0^2(Q_T)\,.$$

It remains to calculate the Gâteaux derivative of J_ε at $u \in H_0^2(Q_T)$. We begin by

considering for all $v \in H_0^2(Q_T)$ the first variation of $J_\varepsilon(u)$:

(3.11)
$$\begin{aligned}
\delta J_\varepsilon(u)(v) &= \lim_{h\to 0} \frac{J_\varepsilon(u+hv) - J_\varepsilon(u)}{h} \\
&= \lim_{h\to 0} \frac{\|(Lu+hLv)-f\|^2_{L^2(Q_T)} + \varepsilon\|u+hv\|^2_{Q_R} - \|Lu-f\|^2_{L^2(Q_T)} - \varepsilon\|u\|^2_{Q_R}}{2h} \\
&= \lim_{h\to 0} \frac{1}{2h}\Big(\|Lu\|^2_{L^2(Q_T)} + 2h\langle Lu, Lv\rangle_{L^2(Q_T)} - 2\langle Lu, f\rangle_{L^2(Q_T)} - 2h\langle Lv, f\rangle_{L^2(Q_T)} \\
&\qquad + h^2\left\|Lv^2\right\|_{L^2(Q_T)} + \|f\|^2_{L^2(Q_T)} + \varepsilon\|u\|^2_{Q_R} + 2h\varepsilon\langle u, v\rangle_{Q_R} + \varepsilon h^2\|v\|^2_{Q_R} \\
&\qquad - \|Lu\|^2_{L^2(Q_T)} + 2\langle Lu, f\rangle_{L^2(Q_T)} - \|f\|^2_{L^2(Q_T)} - \varepsilon\|u\|^2_{Q_R}\Big) \\
&= \lim_{h\to 0}\Big(\langle Lu, Lv\rangle_{L^2(Q_T)} - \langle Lv, f\rangle_{L^2(Q_T)} + h\left\|Lv^2\right\|_{L^2(Q_T)} + \varepsilon\langle u, v\rangle_{Q_R} + h\|v\|^2_{Q_R}\Big) \\
&= \langle Lu, Lv\rangle_{L^2(Q_T)} + \varepsilon\langle u, v\rangle_{Q_R} - \langle Lv, f\rangle_{L^2(Q_T)} \,.
\end{aligned}$$

Consequently, $\delta J_\varepsilon(u)$ defines a linear functional on $H_0^2(Q_T)$, which is precisely the sought Gâteux derivative $J'_\varepsilon(u)$. The abstract Euler equation for a minimizer $u_\varepsilon \in H_0^2(Q_T)$ of J_ε therefore has the form:

$$\int_{Q_T} Lu_\varepsilon\, Lv\, dq + \varepsilon\langle u_\varepsilon, v\rangle_{Q_R} - \int_{Q_T} Lv\, f\, dq = 0 \text{ for all } v \in H_0^2(Q_T)\,. \tag{3.12}$$

In order to apply this to the problems described in Section 2.3, we first have to arrange for homogeneous boundary conditions on u. The standard method is to introduce a function $\Phi \in H^2(Q_T)$ satisfying:

$$\begin{cases} \Phi(x,t) &= \varphi_0(x,t) \quad (x,t) \in S_T\,, \\ \partial_\nu\Phi(x,t) &= \varphi_1(x,t) \quad (x,t) \in S_T\,. \end{cases} \tag{3.13}$$

There exist infinitely many of these functions for a given pair φ_0, φ_1. The main difficulty in the practical application of the method of quasi-reversibility is of course the construction of such a function. We discuss one method in Section 5.2.

Now, if u is a solution of Problem 2.1, the function $u^* := u - \Phi$ satisfies

$$\begin{cases} Lu^* &= -L\Phi \quad (x,t) \in Q_T\,, \\ u^*(x,t) &= 0 \quad (x,t) \in S_T\,, \\ \partial_\nu u^*(x,t) &= 0 \quad (x,t) \in S_T\,. \end{cases} \tag{3.14}$$

Therefore, setting $f := -L\Phi$ in (3.12), we arrive at the quasi-reversibility approximation of Problem 2.1:

Problem 3.1 (Problem of direct quasi-reversibility). Set

$$M_\varepsilon(u,v) := \int_{Q_T} Lu\, Lv \, dq + \varepsilon \langle u, v\rangle_{QR} \,. \tag{3.15}$$

Given $\Phi \in H^2(Q_T)$, $c \in C^1(\overline{\Omega})$, $\varepsilon > 0$, find $u_\varepsilon \in H_0^2(Q_T)$ so that for all $v \in H_0^2(Q_T)$

$$M_\varepsilon(u_\varepsilon, v) = -\int_{Q_T} L\Phi\, Lv \, dq \,. \tag{3.16}$$

3.2. Well-posedness of the quasi-reversibility approximation

The existence, uniqueness and stability of the solution of Problem 3.1 can be established from Riesz' representation theorem.

Theorem 3.2.1. *Given $\Phi \in H^2(Q_T)$, $c \in C^1(\overline{\Omega})$, $\varepsilon > 0$, there exists a unique solution u_ε of Problem 3.1 in $H_0^2(Q_T)$. Furthermore, there exists a constant $C > 0$, depending only on Q_T and the $L^2(\Omega)$ norm of c, such that*

$$\|u_\varepsilon\|_{H^2(Q_T)} \leqslant \frac{C}{\sqrt{\varepsilon}} \|\Phi\|_{H^2(Q_T)} \,. \tag{3.17}$$

Proof. We introduce a linear functional F on $H^2(Q_T)$ by setting:

$$Fv := -\int_{Q_T} L\Phi\, Lv \, dq. \tag{3.18}$$

Since we have fixed $\Phi \in H^2(Q_T)$, this linear functional is bounded:

$$|Fv| \leqslant \left(\int_{Q_T} |L\Phi|^2 \, dq \int_{Q_T} |Lv|^2 \, dq\right)^{\frac{1}{2}} \leqslant c_1 \|\Phi\|_{H^2(Q_T)} \|v\|_{H^2(Q_T)} \,, \tag{3.19}$$

where the constant c_1 depends only on the $L^2(Q)$ norm of the coefficient c.

For each $\varepsilon > 0$, the bilinear form $M_\varepsilon(u,v)$ is symmetric and positive definite, hence an inner product on $H^2(Q_T)$. Riesz' representation theorem (see, e.g., [65, III.6]) then warrants the existence of a unique u_ε in $H_0^2(Q_T)$ which satisfies

$$M_\varepsilon(u_\varepsilon, v) = Fv \qquad \text{for all } v \in H_0^2(Q_T) \,. \tag{3.20}$$

Furthermore, by the definition of the norms on the Hilbert space $(H_0^2(Q_T), M_\varepsilon)$ and its dual, we have that for this $u_\varepsilon \in H_0^2(Q_T)$:

$$\begin{aligned} M_\varepsilon(u_\varepsilon, u_\varepsilon) = \sup_{\substack{w \in H_0^2(Q_T) \\ M_\varepsilon(w,w) \leqslant 1}} |Fw|^2 &\leqslant \sup_{\substack{w \in H_0^2(Q_T) \\ M_\varepsilon(w,w) \leqslant 1}} \left(c_1^2 \|\Phi\|^2_{H^2(Q_T)} \|Lw\|^2_{L^2(Q_T)}\right) \\ &\leqslant \sup_{\substack{w \in H_0^2(Q_T) \\ M_\varepsilon(w,w) \leqslant 1}} \left(c_1^2 \|\Phi\|^2_{H^2(Q_T)} M_\varepsilon(w,w)\right) = c_1^2 \|\Phi\|^2_{H^2(Q_T)} \,. \end{aligned} \tag{3.21}$$

Using the fact that $M_\varepsilon(u_\varepsilon, u_\varepsilon) \geqslant \varepsilon \|u_\varepsilon\|^2_{QR}$, employing Lemma 3.1.1, and setting $C := \frac{c_1}{c_2}$, we arrive at the desired estimate. □

The theorem guarantees the existence of a unique solution which depends continuously on the data Φ, hence Problem 3.1 is well-posed in the sense of Hadamard.

Note that the approximating problems of quasi-reversibility are now elliptic[2], which will be useful for proving error estimates for the numerical solution of the quasi-reversibility problem:

Lemma 3.2.2. *The bilinear form $M_\varepsilon(u,v)$ is $H^2_0(Q_T)$-elliptic, i.e. there exist constants $c_1, c_2 > 0$, depending only on the $L^2(\Omega)$ norm of c and on Q_T, respectively, such that*

$$|M_\varepsilon(u,v)| \leqslant (c_1+\varepsilon)\|u\|_{H^2(Q_T)}\|v\|_{H^2(Q_T)} \quad \text{for all } u,v \in H^2_0(Q_T), \tag{3.22}$$

$$|M_\varepsilon(u,u)| \geqslant c_2\varepsilon\|u\|^2_{H^2(Q_T)} \quad \text{for all } u \in H^2_0(Q_T)\,. \tag{3.23}$$

Proof. M_ε is an inner product on $H^2(Q_T)$, so the first inequality follows directly from Schwarz's inequality and the definition of the $H^2(Q_T)$ norm:

$$\begin{aligned} |M_\varepsilon(u,v)|^2 &\leqslant |M_\varepsilon(u,u)||M_\varepsilon(v,v)| \\ &= \left(\int_{Q_T} |Lu|^2\,dq + \varepsilon\|u\|_{QR}\right)\left(\int_{Q_T} |Lv|^2\,dq + \varepsilon\|v\|_{QR}\right) \\ &\leqslant \left(c_1\|u\|_{H^2(Q_T)} + \varepsilon\|u\|_{H^2(Q_T)}\right)\left(c_1\|v\|_{H^2(Q_T)} + \varepsilon\|v\|_{H^2(Q_T)}\right) \\ &\leqslant (c_1+\varepsilon)^2\|u\|^2_{H^2(Q_T)}\|v\|^2_{H^2(Q_T)}\,, \end{aligned} \tag{3.24}$$

where the constant c_1 again depends only on the $L^2(\Omega)$ norm of the coefficient c.

The second inequality is an immediate consequence of Lemma 3.1.1, the equivalence of the norms $\|u\|_{QR}$ and $\|u\|_{H^2(Q_T)}$ on $H^2_0(Q_T)$:

$$|M_\varepsilon(u,u)| = \int_{Q_T} |Lu|^2\,dq + \varepsilon\|u\|^2_{QR} \geqslant \varepsilon\|u\|^2_{QR} \geqslant c_2\varepsilon\|u\|^2_{H^2(Q_T)}\,. \tag{3.25}$$

□

Remark 3.1. The existence of a unique solution, together with a weaker stability estimate than (3.17) (involving ε instead of $\sqrt{\varepsilon}$), can also be derived from the Lax-Milgram lemma. The crucial properties of the direct quasi-reversibility formulation is therefore the coercivity of the bilinear form $\langle\cdot,\cdot\rangle_{QR}$. Consequently, any norm equivalent to the standard norm on $H^2_0(Q_T)$ (or even $H^2(Q_T)$) can be used as a regularization term in the Tikhonov functional (3.8). This choice can be

[2] For that reason, this approach is also known as *elliptic regularization*.

exploited to ease the numerical calculations, for instance to reduce the condition of the resulting system matrix. We have opted for the norm involving the minimal number of derivatives, in order to keep the calculations simple. Another choice would be to use different regularization parameters ε_i for the regularization terms of different orders. However, preliminary numerical studies did not show much improvement over the formulation presented above, and so we will, for the sake of presentation, keep to the specific choice (3.4) in the succeeding chapters.

As $M_\varepsilon(u,v)$ is $H_0^2(Q_T)$-elliptic and *a fortiori* $H_0^2(Q_T)$-coercive, by Gårding's theorem (see, e.g., [62, Th. 19.2]), Problem 3.1 is strongly elliptic. We can therefore apply the theory of elliptic partial differential operators to determine the regularity of solutions to Problem 3.1:

Theorem 3.2.3. *A solution u_ε of Problem 3.1 satisfies $u_\varepsilon \in H^3(\tilde{Q}_T)$ for every compact subset $\tilde{Q}_T \subset Q_T$. If ∂Q_T is of class C^4, this also holds for Q_T itself.*

Proof. This result follows from [62, Th. 20.1], since we know from Theorem 3.2.1 that a solution u_ε exists in $H_0^2(Q_T)$, and from Lemma 3.2.2 that $M_\varepsilon(u,v)$ is $H_0^2(Q_T)$-coercive. Thus, if the coefficients of $M_\varepsilon(u,v)$ are in $C^k(\overline{Q_T})$ and $L\Phi \in H^{k-2}$, we have that u_ε is also in $H^{k+2}(\tilde{Q}_T)$. The regularity up to the boundary follows from [62, Th. 20.4]. □

Remark 3.2. The results of this chapter remain valid if we replace $H_0^2(Q_T)$ by $H_\Gamma^2(Q_T)$ and set $\Phi|_{S_T\setminus\Gamma_T} = \partial_\nu\Phi|_{S_T\setminus\Gamma_T} = 0$, as long as Γ is not an $(n-1)$-dimensional set of measure zero. Hence we can also investigate a quasi-reversibility approximation of Problem 2.2. If u is a solution of (2.14), then $u^* := u - \Phi$ satisfies:

$$\begin{cases} Lu^* &= -L\Phi & (x,t) \in Q_T\,, \\ u^*(x,t) &= 0 & (x,t) \in \Gamma_T\,, \\ \partial_\nu u^*(x,t) &= 0 & (x,t) \in \Gamma_T\,. \end{cases} \tag{3.26}$$

The quasi-reversibility approximation for this problem is then to find $u_\varepsilon \in H_\Gamma^2(Q_T')$ so that for all $v \in H_\Gamma^2(Q_T')$

$$M_\varepsilon(u_\varepsilon, v) = -\int_{Q_T} L\Phi\, Lv \,dq\,. \tag{3.27}$$

It will be shown in Chapter 4 that under certain assumptions, Problem 2.3 can be treated as a special case of Problem 2.1.

The question of convergence of u_ε to the solution u of the lateral Cauchy problem as $\varepsilon \to 0$ will be the subject of Chapter 4.

4. Rates of convergence and error estimates

We now discuss error estimates for the lateral Cauchy problem, which will also yield convergence rates and error estimates for the approximation of the lateral Cauchy problems 2.1, 2.2 and 2.3 by the quasi-reversibility problem 3.1. Such considerations were absent in the book of Lions and Lattès. The proof makes use of a one-parameter family of weighted L^2 estimates called Carleman estimates. This method has been employed previously (cf. [30, 36, 40, 31]). The central new ingredient is the derivation of Theorems 4.3.3 and 4.3.6 for variable coefficients in the principal part of the operator, which extends the results of [30].

First we will introduce the main tool, a Carleman estimate for the wave equation, which can be derived from microlocal analysis [22, 27, 57] or directly by partial integration [49, 40], the latter admitting less regular boundaries and thus will be used here. We nevertheless begin by giving a short, informal overview of the former to explain the key ideas of Carleman estimates. Using these results, we first show Hölder-type estimates for the case of Cauchy data given on an arbitrary part of the boundary. In the next section, we show that under certain conditions on this boundary part, we can strengthen these results to yield Lipschitz-type estimates.

To facilitate the proof of the Lipschitz stability estimate, we consider in this chapter $Q'_T := \Omega \times [-T, T]$ instead of Q_T. Similarly, we write $S'_T := \partial\Omega \times [-T, T]$ and $\Gamma'_T := \Gamma \times [-T, T]$. Since by (2.12) we know that $\partial_t u|_{t=0} \equiv 0$, we can take the even extension of u^*, u_ε and Φ in $\Omega \times [-T, 0]$ (which we will also denote as u^*, u_ε and Φ, respectively).

4.1. Carleman estimate

Carleman estimates can be thought of as generalizations of the multiplier methods applied in the 1980's to derive observability estimates for the wave equation with constant coefficients. Here the standard multipliers are replaced by exponential

or pseudodifferential multipliers depending on the coefficients and on the domain. They were first introduced by Carleman in [10] to show unique continuation of solutions to an elliptic problem in two dimensions. Since then, they have been applied to various unique continuation and lateral Cauchy problems, as well as problems of exact controllability. By the device of Bukhgeim and Klibanov [9], they have also been employed in the study of inverse problems for source terms or coefficients in partial differential equations, by reduction to lateral Cauchy problems.

As stated above, there are two possible approaches for the derivation. One makes use of pseudodifferential calculus, with all the smoothness assumptions this entails. The other approach, which is a direct proof based on a pointwise estimate, trades the smoothness requirements for tedious calculations. Since a proof should explain as well as convince, we will first give a short sketch of the derivation via microlocal analysis in the general case, and then give a direct proof of the specific estimate needed for the problem under consideration. In this way, we hope to give an idea of the key concepts, which otherwise might get lost in the lengthy calculations.

4.1.1. Derivation by microlocal analysis

The purpose of this section is to give an informal introduction to the theory and applications of Carleman estimates, while avoiding the details of the pseudodifferential calculus. For an exhaustive and rigorous derivation in this framework, we refer to [23, 24].

For the purpose of this section, we consider a general linear partial differential operator $P(x, D)$ on $\mathbb{R}^n$ of order m:

$$P(x, D) := \sum_{|\alpha| \leqslant m} a_\alpha(x) D_x^\alpha , \tag{4.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a positive multi-index of magnitude[1] $|\alpha| := \sum_{i=1}^n \alpha_i$ and $D_x := -i\partial_x$. Denote by

$$p(x, \xi) := \sum_{|\alpha| \leqslant m} a_\alpha(x) \xi^\alpha \tag{4.2}$$

the symbol of the operator P. We call x the space variable, and ξ the Fourier variable. The symbol of the principal part P_m of the operator P is

$$p_m(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha . \tag{4.3}$$

[1] Here we consider only isotropic operators to keep the presentation clear; the results carry over to anisotropic operators (where this sum is weighted) like the heat and Schrödinger operator by modification of the Poisson bracket introduced below, cf. [26].

We assume the coefficients a_α of the principle part of P to be smooth, while for the lower order terms, which we allow to have merely bounded and measurable coefficients. We furthermore restrict ourselves here to the most common case of operators of real principal type[2], for which $p_m(x,\xi)$ is real and $\partial_x p_m(x,\xi) \neq 0$, $\partial_\xi p_m(x,\xi) \neq 0$ on the set $\{(x,\xi) \in \mathbb{R}^{2n} : \xi \neq 0,\ p_m(x,\xi) = 0\}$. Finally, let Σ be an oriented hypersurface in $\mathbb{R}^n$, which can be represented as a nondegenerate zero level surface of a smooth function φ vanishing to first order on Σ, thus $\Sigma = \{\varphi = 0\}$. We call Σ *characteristic* if $p(x, \nabla\varphi(x)) = 0$ holds for all $x \in \Sigma$. In the converse case, we differentiate between *time-like* $(p(x,\nabla\varphi(x)) > 0)$ and *space-like* $(p(x,\nabla\varphi(x)) < 0)$ surfaces. The orientation of Σ therefore determines the sign of φ away from Σ. We define the two sides as $\Sigma^+ := \{\varphi > 0\}$ and $\Sigma^- := \{\varphi < 0\}$. To motivate Carleman estimates, we consider the unique continuation property across Σ for solutions of $Pu = 0$:

Definition 4.1.1 (Unique continuation property). *The unique continuation property across Σ is said to hold for the operator P, if for each $x_0 \in \Sigma$ there exists a neighborhood V of x_0, such that the following holds: A solution u of $Pu = 0$ in V which satisfies $u = 0$ in $\Sigma^+ \cap V$ vanishes in a neighborhood of x_0.*

This property obviously depends on the definition of Σ, and hence on φ. Particularly, the orientation of Σ is crucial (indeed, in most cases, if unique continuation holds in one direction, it does not hold in the opposite direction). If Σ is noncharacteristic, this property is equivalent to the uniqueness of the Cauchy problem

$$\begin{cases} P(x,D)u = 0 & x \in \Sigma^-, \\ u = \partial_\nu u = \cdots = \partial_\nu^{m-1} = 0 & x \in \Sigma. \end{cases} \tag{4.4}$$

Roughly, the unique continuation property means that information about a solution u of $Pu = 0$ in Σ^- can be retrieved from information about u in Σ^+. Thus, we try to characterize Σ by looking at the information flow across it, in the form of special solutions u. If P is not elliptic, we can consider solutions which are highly localized in both the space and frequency domain near null bicharacteristics of P. These are given by curves of the form $(x(t),\xi(t))$ which are solutions of the Hamiltonian system

$$\frac{d}{dt}x(t) = \partial_\xi p_m(x,\xi), \qquad \frac{d}{dt}\xi(t) = -\partial_x p_m(x,\xi) \tag{4.5}$$

on $\{p_m(x,\xi) = 0\}$. The projections of these bicharacteristics on the space variable x are called *bicharacteristic rays* of p, which are the multi-dimensional analogues of the characteristics of linear hyperbolic partial differential equations of

[2] For elliptic operators, this restriction can be dropped.

two independent variables. Correspondingly, discontinuities of solutions of $Pu = 0$ are transported along bicharacteristic rays[3]. Hence, it is sensible to require that all such rays passing near Σ in Σ^- must cross into Σ^+, which implies that the bicharacteristic has non-glancing contact at Σ (cf. also [4]). This condition can be expressed using the derivative of φ along the bicharacteristic flow of p, defined as[4] $\{p, \varphi\} := \partial_\xi p \partial_x \varphi - \partial_x p \partial_\xi \varphi$:

$$\{p, \{p, \varphi\}\} > 0 \quad \text{whenever} \quad p = \{p, \varphi\} = 0,\ \xi \neq 0\,. \tag{4.6}$$

Therefore φ has to be strictly convex on null bicharacteristic rays of P near their critical points. For elliptic operators P, this condition is void, since in this case $p_m(x, \xi) \neq 0$ for all $\xi \neq 0$. The geometrical interpretation is that all null bicharacteristics tangent to Σ must curve away from Σ in the direction of Σ^+ (cf. Figure 4.1). In essence, (4.6) is a convexity condition[5] for Σ^-, with straight lines replaced with the bicharacteristic flow of P.

Even if all bicharacteristic rays cross Σ, information could fail to be transported across Σ if there exist solutions in Σ^- which decay exponentially toward Σ. To study such solutions, we complexify the symbol of P in the direction of the inner normal to Σ, i.e. we look at $p_\varphi(x, \xi) := p_m(x, \xi + i\lambda\nabla\varphi)$ for $\lambda > 0$. Since this new symbol is complex valued, the geometry of the bicharacteristic rays is no longer helpful. Instead, we look at the necessary local solvability condition for p_φ, which is that[6] $\{\Re p_\varphi, \Im p_\varphi\} \leqslant 0$ if $p_\varphi = 0$. A sufficient condition for the absence of such exponentially decreasing solutions is therefore:

$$\{\Re p_\varphi, \Im p_\varphi\} > 0 \quad \text{whenever} \quad p_\varphi = 0,\ \xi \neq 0,\ \lambda > 0\,, \tag{4.7}$$

where $\Re p_\varphi$ denotes the real part and $\Im p_\varphi$ the imaginary part of p_φ. A function φ which satisfies conditions (4.6) and (4.7) is called *strongly pseudo-convex* (with respect to P). A non-degenerate level surface of a strongly pseudo-convex function is called a *strongly pseudo-convex surface*. In fact, since for a given surface, the generating function is determined only up to multiplication by a smooth positive function, we can relax condition (4.7) for $\Sigma = \{\varphi = 0\}$ to be a strongly pseudo-

[3] This can be made explicit using the tools of microlocal analysis, e.g. Hörmander's theorem on the propagation of singularities [24, Th. 26.1.5]. For a second order hyperbolic equation, the projection of their null bicharacteristics into $\mathbb{R}^n$ are the light rays of geometric optics. See also [4], where this approach was used to prove sharp geometric conditions for the boundary control of waves.

[4] The expression $\{p, q\}$ is called the *Poisson bracket* of the two symbols p and q.

[5] This is the reason why the conditions for the unique continuation property are referred to as *pseudo-convexity* conditions.

[6] Specifically, the condition prohibits $\Im(qp_\varphi)$ to change signs along bicharacteristics of $\Re(qp_\varphi)$ for any smooth complex symbol $q \neq 0$ (cf. [24, 26.11.1]).

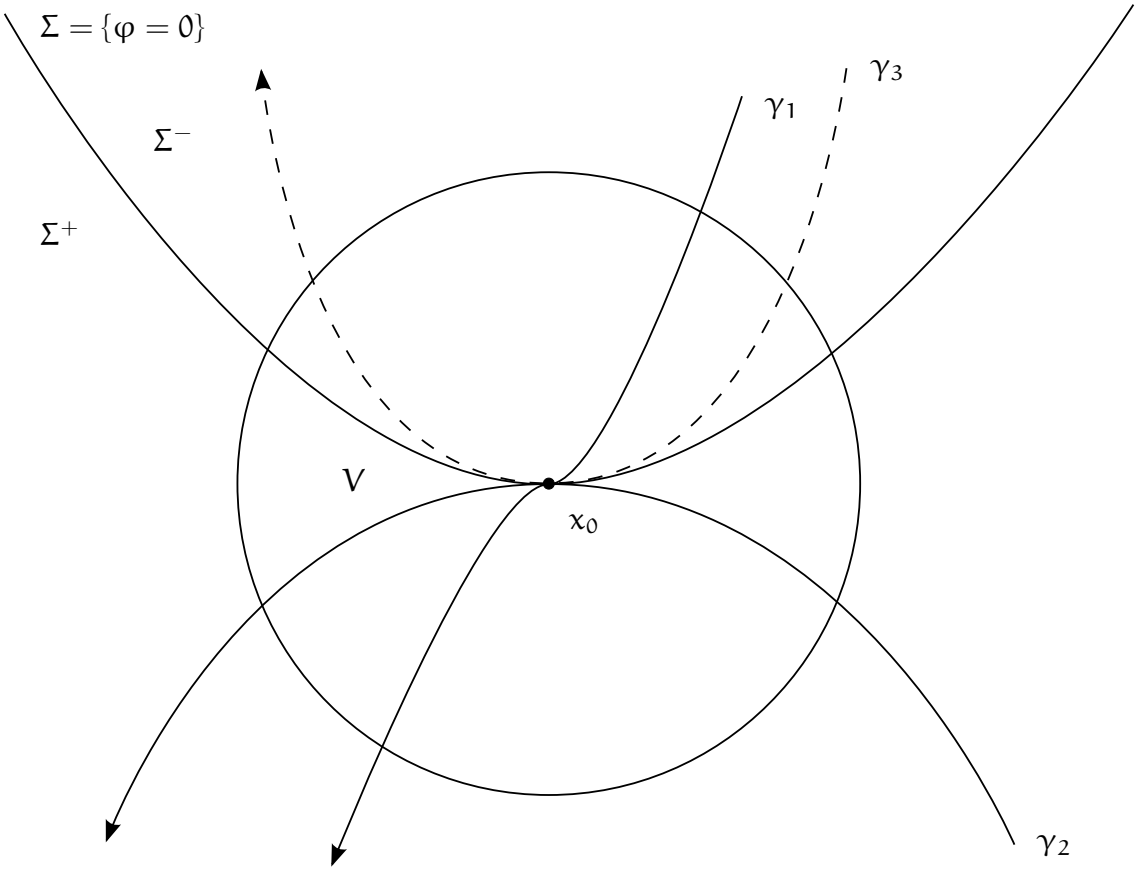


Figure 4.1: Pseudo-convexity condition on Σ in x_0. A necessary condition for the unique continuation property to hold in V is the absence of bicharacteristic rays of type γ_3. Rays of type γ_1, γ_2 are allowed.

convex surface:

$$\{\Re p_\varphi, \Im p_\varphi\} > 0 \quad \text{whenever} \quad p_\varphi = \{p_\varphi, \varphi\} = 0,\ \xi \neq 0,\ \lambda > 0\,. \tag{4.8}$$

If Σ is a strongly pseudo-convex surface (in the sense that (4.6) and (4.8) hold), then, for sufficiently large λ, $e^{\lambda\varphi}$ is a strongly pseudo-convex function in the vicinity of Σ.

Hörmander's theorem (cf. [24, Th. 28.3.4]) then states that unique continuation for solutions of $Pu = 0$ holds for every surface that is strongly pseudo-convex with respect to P. On the other hand, under the assumption that (4.6) strictly fails at some point (x_0, ξ), it was shown in [2] that unique continuation does not hold for a smooth zero-order perturbation of a wave equation. This indicates that strong pseudo-convexity is indeed the correct concept for the investigation of Cauchy problems.

Needless to say, the strong pseudo-convexity conditions are very technical and hard to check in practice. A more explicit (but no less technical) version of these

conditions can be found in [22, Th.s 8.4.2 and 8.4.3]. For elliptic operators of second order, it can be shown that condition (4.8) is always satisfied, hence any smooth surface is strongly pseudo-convex. Similarly, for second order hyperbolic operators with constant real coefficients, (4.8) holds for all non-characteristic surfaces. In the case of the wave equation $\frac{1}{c^2}\partial_{tt} - \Delta$ with constant coefficient c, condition (4.6) is fulfilled for all convex as well as for some non-convex surfaces, e.g. positive level sets of $\varphi = x^2 - \beta t^2$ for $\beta < c^2$ (which is a sharp bound, cf. [27]). The characterization in the case of non-constant coefficients is much harder (cf. [27, Th. 3.4.1]).

The central idea of the proof of Hörmander's theorem is the use of a one-parameter family of weighted L^2 estimates, where the weights are strongly pseudo-convex functions tied to the strongly pseudo-convex surface Σ, as is the parameter. These are exactly the Carleman estimates, which are applied to smooth functions with compact support:

Theorem 4.1.2 (Carleman estimate (local)). *Suppose that φ is a strongly pseudo-convex function with respect to the operator P of order m in some bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then there exists constants $C > 0$, $\lambda_0 > 0$, such that for all functions $u \in C_0^\infty(\Omega)$ and all $\lambda > \lambda_0$ the following inequality holds:*

$$\sum_{|\alpha|<m} \lambda^{2(m-|\alpha|)-1} \left\| e^{\lambda\varphi} D^\alpha u \right\|^2_{L^2(\Omega)} \leqslant C \left\| e^{\lambda\varphi} P u \right\|^2_{L^2(\Omega)} . \tag{4.9}$$

Idea of proof. We first eliminate the weight function by setting $v = e^{\lambda\varphi} u$. Then

$$e^{\lambda\varphi} P(x, D) u = e^{\lambda\varphi} P(x, D) e^{-\lambda\varphi} v =: P_\varphi(x, D, \lambda) v \,, \tag{4.10}$$

where P_φ is the conjugated operator with respect to the exponential weight, which is given by

$$P_\varphi(x, D, \lambda) = e^{\lambda\varphi} P(x, D) e^{-\lambda\varphi} = P(x, D + i\lambda\nabla\varphi). \tag{4.11}$$

If we insert the definition of v and P_φ in (4.9), we arrive at an estimate for v, where the parameter λ no longer appears explicitly on the right hand side. However, this estimate must still be uniform in λ, so the parameter λ has to treated as having the same weight as a full derivative. For this reason, we have to consider the weighted Sobolev norms

$$\|u\|^2_{k,\lambda} := \left\| (|\xi|^2 + \lambda^2)^{k/2} \hat{u} \right\|_{L^2(\Omega)} , \tag{4.12}$$

defined via the Fourier transform $\hat{u}$ of u. We have thus reduced (4.9) to the subelliptic estimate $\lambda \|v\|^2_{m-1,\lambda} \leqslant C \|P_\varphi v\|^2_{L^2(\Omega)}$ for P_φ.

Decomposing P_φ into its real and imaginary part, $P_\varphi = \Re P_\varphi + i\Im P_\varphi$, we calculate the right hand side of this estimate:

$$\|P_\varphi v\|^2_{L^2(\Omega)} = \|\Re P_\varphi v\|^2_{L^2(\Omega)} + \|\Im P_\varphi v\|^2_{L^2(\Omega)} + 2\Im \langle \Re P_\varphi v, \Im P_\varphi v\rangle_{L^2(\Omega)} \,. \tag{4.13}$$

The third term is a differential quadratic form with a principal symbol which is a homogeneous polynomial of degree $2m-1$ in ξ and λ. The final (and most difficult) step is then obtaining an estimate of this symbol from below in order to apply Gårding's inequality, which completes the proof of the subelliptic estimate. □

The key feature of the Carleman estimates is the presence of the large parameter λ on the left hand side, which can be used to eliminate unwanted lower order terms with inconvenient signs in estimates by absorption into the corresponding terms with λ. This is used in the proof above, for instance, for the observation that the Carleman estimate depends only on the principal part of the operator P. Since this fact is of independent interest, we prove this as a lemma:

Lemma 4.1.3. *Suppose the Carleman estimate 4.9 holds for a linear differential operator P of order m and $\lambda > \lambda_0$. If r is a linear differential operator of order $m-1$, the Carleman estimate also holds for $P+r$ for all $\lambda > \lambda_1 > \lambda_0$.*

Proof. We write Pu as $(P+r)u - ru$ and estimate

$$\begin{aligned}(Pu)^2 &= [(P+r)u]^2 - 2[(P+r)u][ru] + (ru)^2 \\ &\leqslant [(P+r)u]^2 + 2\,|[(P+r)u][ru]| + (ru)^2 \leqslant 2[(P+r)u]^2 + 2(ru)^2 \\ &\leqslant 2[(P+r)u]^2 + C\sum_{|\alpha|<m} (D^\alpha u)^2 \,.\end{aligned} \tag{4.14}$$

Multiplication by $e^{2\lambda\varphi}$ and integration over Ω yields

$$\left\|e^{\lambda\varphi}Pu\right\|^2_{L^2(\Omega)} \leqslant \left\|e^{\lambda\varphi}(P+r)u\right\|^2_{L^2(\Omega)} + C\sum_{|\alpha|<m} \left\|e^{\lambda\varphi}D^\alpha u\right\|^2_{L^2(\Omega)} \,. \tag{4.15}$$

Inserting this inequality in the right hand side of the Carleman estimate (4.9), we can absorb the second term on the right hand side into the left hand side of (4.9) if $\lambda =: \lambda_1$ is sufficiently large. □

For the study of boundary value problems, where u does not have compact support, the estimate 4.1.2 is not sufficient, since the points where Σ intersects $\partial\Omega$ must be taken into account. Here, the boundary conditions play a critical role. Consider the general boundary value problem

$$\begin{cases} Pu = 0 & x \in \Omega \,, \\ Bu = 0 & x \in \partial\Omega \,, \end{cases} \tag{4.16}$$

where $B = \{B_k\}_{k=1,\dots,k_0}$ with $k_0 \leqslant m-1$ is a set of boundary operators with orders $m_k \leqslant m-1$ defined on $\partial\Omega$. These boundary operators must satisfy additional conditions for a Carleman estimate to hold up to the boundary. The following results are due to Tataru [57]: A sufficient condition is the so-called *strong Lopatinskii* condition, which is an algebraic condition connecting the symbols of P and B, the conormal ν on $\partial\Omega$ and the conormal $d\varphi$ on Σ. The precise definition is lengthy and technical, and we refer to [57] for details. For second order elliptic equations, the Dirichlet ($Bu = u$) and Neumann ($Bu = \partial_\nu u$) operator satisfy the strong Lopatinskii condition if (and only if) $\partial_\nu \varphi < 0$ on $\partial\Omega$. For the second order wave equation and the Dirichlet conditions, the strong Lopatinskii condition holds if either Σ is space-like or Σ is time-like and $\partial_\nu \varphi < 0$. For the Neumann operator, under the same assumptions, a weaker condition (which still implies unique continuation) is satisfied.

If we define trace norms $\|u\|_{k,\partial,\lambda}$ on the boundary $\partial\Omega$ similarly to (4.12), we can then extend the Carleman estimate up to the boundary by including the boundary terms Bu on the right hand side:

Theorem 4.1.4 (Carleman estimate (with boundary terms)). *Suppose that φ is a strongly pseudo-convex function with respect to P in some bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ and the strong Lopatinskii condition holds for the pair (B, P) in the direction $d\varphi$. Then there exist constants $C > 0$ and $\lambda_0 > 0$, such that for all $u \in C^\infty(\overline{\Omega})$ and all $\lambda > \lambda_0$ the following holds:*

$$\lambda \left\| e^{\lambda\varphi} u \right\|^2_{m-1,\lambda} \leqslant C \left(\left\| e^{\lambda\varphi} P u \right\|^2_{L^2(\Omega)} + \lambda \sum_{k=1}^{k_0} \left\| e^{\lambda\varphi} B_k u \right\|^2_{m-m_k-1,\partial,\lambda} \right). \tag{4.17}$$

Besides the difficulty of checking the general pseudo-convexity and Lopatinksii conditions, the main drawback of this approach are the smoothness assumptions on the function u and the boundaries. The former can be relaxed to the usual Sobolev spaces by using regularization arguments. Likewise, the regularity of the coefficients of the principal part of P can be weakened to $C^1(\Omega)$. However, the results above require Ω and Σ to be at least of class C^2. For this reason, we derive our Carleman estimate for the wave equation directly from a pointwise estimate, avoiding microlocal analysis. It should be noted that the pseudo-convexity conditions above have the advantage of yielding sharper bounds on the parameters of the pseudo-convex function φ, which play a critical role in the derivation of observability estimates from Carleman estimates[7].

[7] Essentially, they determine the minimal observation time T.

4.1.2. Derivation by a pointwise estimate

In order to allow the Carleman estimate to hold in Lipschitz domains such as cubes and cylinders, we therefore employ a direct derivation of a specific Carleman estimate for the wave equation. This derivation is based on a pointwise estimate, which was first shown in [49, Ch. 4, § 4, Lemma 1] for the wave equation with constant principal coefficient $c \equiv 1$. We extend this to our situation following [40, 11].

For a point $x_0 \in \mathbb{R}^n$, we set $r := \max_{x\in\overline{\Omega}} |x - x_0|$ and define the function

$$\varphi(x,t) := |x - x_0|^2 - \beta t^2 \,, \tag{4.18}$$

and, for $\sigma \geqslant 0$, the domain bounded by their level sets:

$$Q_\sigma := \{(x,t) \in Q_T' : \ \varphi(x,t) > \sigma\} \,. \tag{4.19}$$

These definitions will be used throughout the rest of this chapter. Note that for $\sigma_2 > \sigma_1 > 0$, it holds that $Q_{\sigma_2} \subset Q_{\sigma_1} \subset Q_0$.

The choice of x_0 and especially β is crucial for the Carleman estimate to hold. Additionally, we must impose requirements on the principal coefficient $c(x)$ in order for the bicharacteristic rays to allow pseudo-convex level sets of φ. We will show in this section that the following conditions are sufficient: For all $x \in \overline{\Omega}$, the coefficient $c(x)$ satisfies

$$0 < c_{min} \leqslant c(x) \leqslant c_{max} \,, \tag{4.20a}$$

$$2c^{-2}(x) + \left\langle \nabla(c^{-2})(x), x - x_0 \right\rangle_n > 0 \,, \tag{4.20b}$$

and the parameter $\beta \in (0,1)$ fulfills for all $x \in \overline{\Omega}$:

$$\sqrt{\beta} < \frac{2c^{-2}(x) + \left\langle \nabla(c^{-2})(x), x - x_0 \right\rangle_n}{2c_{min}^{-4} + (1 + c_{min}^{-2}) r \left(\max_{x\in\overline{\Omega}} |\nabla c^{-2}|\right)} , \tag{4.21a}$$

$$\sqrt{\beta} < \frac{4}{3c^{-4}(x) + 2c^{-2}(x) + 2r|\nabla(c^{-2})(x)|} \,. \tag{4.21b}$$

If $c(x) \equiv c$ is constant, the above conditions are reduced to $\beta < c^2$.

Under these assumptions, we can show a pointwise estimate of Carleman-type:

Lemma 4.1.5 (Pointwise Carleman estimate). *Choose $\sigma > 0$ so that $Q_\sigma \neq \emptyset$ and $\overline{Q}_\sigma \cap \{|t| = T\} = \emptyset$. Then there exist constants $C_1 > 0$, $C_2 > 0$ and $\lambda_0 > 0$, such that for $u \in C^2(\overline{Q}_T')$, the following estimate holds for all $x \in Q_\sigma$ and $\lambda > \lambda_0$:*

$$\lambda^3 |u|^2 e^{2\lambda\varphi} + \lambda \left(|\nabla u|^2 + |\partial_t u|^2\right) e^{2\lambda\varphi} + \sum_{i=1}^{n+1} \partial_i U_i \leqslant C_1 |Lu|^2 e^{2\lambda\varphi} \,, \tag{4.22}$$

where the vector function $U := (U_i)_{i=1,\dots,n+1}$ *satisfies the following estimate:*

$$|U| \leqslant C_2 \left(\lambda^3 |u|^2 + \lambda \left(|\nabla u|^2 + |\partial_t u|^2\right)\right) e^{2\lambda\varphi} . \tag{4.23}$$

Proof. We begin by observing that for $\sigma \neq 0$, we have $(x_0, 0) \notin Q_\sigma$, and so $\nabla\varphi \neq 0$ if $Q_\sigma \neq \emptyset$. The proof now roughly proceeds along the same steps as in the sketch of Theorem 4.1.2.

We first set $v = e^{\lambda\varphi} u$ and compute the conjugated operator L_φ:

$$\begin{aligned} e^{\lambda\varphi} L u = e^{\lambda\varphi} L(e^{-\lambda\varphi} v) &= (e^{\lambda\varphi}) \left(\frac{1}{c^2}\partial_{tt}(e^{-\lambda\varphi} v) - \Delta(e^{-\lambda\varphi} v)\right) \\ &= (e^{\lambda\varphi})(e^{-\lambda\varphi}) \left[\frac{1}{c^2}\left(2\lambda\beta v + 4\lambda\beta^2 t^2 v + 4\lambda\beta t \partial_t v + \partial_{tt} v\right)\right. \\ &\qquad \left. - \left(-2\lambda v + 4\lambda^2 |x - x_0|^2 v - 4\lambda \langle \nabla v, x - x_0\rangle_n + \Delta v\right)\right] . \end{aligned} \tag{4.24}$$

We therefore have:

$$\begin{aligned} |Lu|^2 e^{2\lambda\varphi} = \left[\left(Lv - 4\lambda^2 \left(|x - x_0|^2 - \frac{1}{c^2}\beta^2 t^2 + \frac{\beta - 1}{2\lambda}\right) v\right)\right. \\ \left. + \left(4\lambda \frac{1}{c^2}\beta t \partial_t v + 4\lambda \langle \nabla v, x - x_0\rangle_n\right)\right]^2 . \end{aligned} \tag{4.25}$$

The next step is the decomposition of this expression into pure quadratic and mixed terms. To get rid of terms of the form $\partial_t v \partial_i v$ from the beginning, we write the right hand side of (4.25) as $(z_1 + (z_2 + z_3))^2$ and estimate:

$$|Lu|^2 e^{2\lambda\varphi} = z_1^2 + (z_2 + z_3)^2 + 2z_1(z_2 + z_3) \geqslant z_1^2 + 2z_1 z_2 + 2z_1 z_3 . \tag{4.26}$$

Now it remains (in analogy to applying the sharp Gårding inequality) to bound this expression from below by terms involving v, $\partial_i v$, and $\partial_t v$ having positive sign (in addition to the terms containing $\partial_i U_i$). We estimate the three terms in (4.26) separately.

Step 1. We begin by computing $2z_1 z_2$. We try to collect quadratic terms by making liberal use of the Leibniz rule in the form of the identity $2uv\partial_i v =$

$\partial_i(uv^2) - (\partial_i u)v^2$ and variations thereof (e.g., for multiple factors u_i):

(4.27)
$$\begin{aligned}
2z_1z_2 &= 8\lambda c^{-2}\beta t\partial_t v\left(c^{-2}\partial_{tt}v - \Delta v - 4\lambda^2\left(|x-x_0|^2 - c^{-2}\beta^2t^2 + \frac{\beta-1}{2\lambda}\right)v\right)\\
&= \partial_t\left[4\lambda c^{-4}\beta t(\partial_t v)^2\right] - 4\lambda c^{-4}\beta(\partial_t v)^2\\
&+ \sum_{i=1}^n \partial_i[-8\lambda c^{-2}\beta t\partial_t v\partial_i v] + \sum_{i=1}^n 8\lambda c^{-2}(x)\beta t\partial_{it}v\partial_i v + 8\lambda\beta t\partial_t v\sum_{i=1}^n \partial_i c^{-2}\partial_i v\\
&+ \partial_t\left[-16\lambda^3c^{-2}\beta\left(t|x-x_0|^2 - c^{-2}\beta^2t^3 + \frac{\beta t - t}{2\lambda}\right)v^2\right]\\
&+ 16\lambda^3c^{-2}\beta\left(|x-x_0|^2 - 3c^{-2}\beta^2t^2 + \frac{\beta-1}{2\lambda}\right)v^2.
\end{aligned}$$

Similarly, we apply this identity to the fourth term on the right hand side:

$$8\lambda c^{-2}\beta t\partial_{it}v\partial_i v = \partial_t\left[4\lambda c^{-2}\beta t(\partial_i v)^2\right] - 4\lambda c^{-2}\beta(\partial_i v)^2. \tag{4.28}$$

Collecting the derivatives with respect to t and x_i, we have:

$$\begin{aligned}
2z_1z_2 &= 16\lambda^3c^{-2}\beta\left(|x-x_0|^2 - 3c^{-2}\beta^2t^2 + \frac{\beta-1}{2\lambda}\right)v^2\\
&- 4\lambda c^{-4}\beta(\partial_t v)^2 - 4\lambda c^{-2}\beta|\nabla v|^2 + 8\lambda\beta t\partial_t v\left\langle\nabla c^{-2}, \nabla v\right\rangle_n + \sum_{i=1}^{n+1}\partial_i U_i^a
\end{aligned} \tag{4.29}$$

where for $i = 1, \dots, n$,

$$U_i^a := -8\lambda c^{-2}\beta t\partial_t v\partial_i v, \tag{4.30a}$$

$$\begin{aligned}
U_{n+1}^a &:= 4\lambda c^{-4}t(\partial_t v)^2 + 4\lambda c^{-2}\beta t|\nabla v|^2\\
&- 16\lambda^3c^{-2}\beta\left(t|x-x_0|^2 - c^{-2}\beta^2t^3 + \frac{\beta t - t}{2\lambda}\right)v^2.
\end{aligned} \tag{4.30b}$$

Step 2. We repeat this procedure for $2z_1z_3$:

$$\begin{aligned}
&2z_1z_3 =\\
&8\lambda\left\langle\nabla v, x-x_0\right\rangle_n\left(c^{-2}\partial_{tt}v - \Delta v - 4\lambda^2\left(|x-x_0|^2 - c^{-2}\beta^2t^2 + \frac{\beta-1}{2\lambda}\right)v\right)\\
&= \partial_t\left[8\lambda c^{-2}\left\langle\nabla v, x-x_0\right\rangle_n v_t\right] - 8\lambda c^{-2}\left\langle\nabla v_t, x-x_0\right\rangle_n v_t - 8\lambda\left\langle\nabla v, x-x_0\right\rangle_n\Delta v\\
&- \sum_{i=1}^n\partial_i\left[16\lambda^3(x_i - x_{0i}\left(|x-x_0|^2 - c^{-2}\beta^2t^2 + \frac{\beta-1}{2\lambda}\right)v^2\right]\\
&+ 16\lambda^3\left((n+2)|x-x_0|^2 - \beta^2t^2\left(nc^{-2} + \left\langle\nabla c^{-2}, x-x_0\right\rangle_n\right) + \frac{\beta-1}{2\lambda}\right)v^2.
\end{aligned} \tag{4.31}$$

We now rearrange the second and third term above:

$$-8\lambda c^{-2}\left\langle \nabla v_t, x-x_0\right\rangle_n v_t = \sum_{i=1}^{n} \partial_i\left[-4\lambda(x_i-x_{0i})c^{-2}(\partial_t v)^2\right] + 4\lambda\left(nc^{-2}+\left\langle \nabla c^{-2}, x-x_0\right\rangle_n\right)(\partial_t v)^2\,, \tag{4.32a}$$

$$-8\lambda\left\langle \nabla v, x-x_0\right\rangle_n \Delta v = \sum_{i=1}^{n}\partial_i\left[-8\lambda\left\langle \nabla v, x-x_0\right\rangle_n \partial_i v\right] + 8\lambda|\nabla v|^2 + \sum_{i=1}^{n} 8\lambda\left\langle \nabla(\partial_i v), x-x_0\right\rangle_n \partial_i v\,. \tag{4.32b}$$

The last term in (4.32b) can be expressed as follows:

$$\sum_{i=1}^{n} 8\lambda\left\langle \nabla(\partial_i v), x-x_0\right\rangle_n \partial_i v = \sum_{i=1}^{n}\partial_i\left[4\lambda(x_i-x_{0i})|\nabla v|^2\right] - 4\lambda n|\nabla v|^2\,. \tag{4.32c}$$

By collecting the expressions in (4.31) and (4.32), we obtain:

$$\begin{aligned} 2z_1z_3 = 16\lambda^3\left((n+2)|x-x_0|^2-\beta^2t^2\left(nc^{-2}+\left\langle \nabla c^{-2}, x-x_0\right\rangle_n\right)+\frac{\beta-1}{2\lambda}\right)v^2 \\ -4\lambda(n-2)|\nabla v|^2+4\lambda\left(nc^{-2}+\left\langle \nabla c^{-2}, x-x_0\right\rangle_n\right)(\partial_t v)^2+\sum_{i=1}^{n+1}\partial_i U_i^b\,, \end{aligned} \tag{4.33}$$

where for $i=1,\dots,n$,

$$U_i^b := -16\lambda^3(x_i-x_{0i})\left(|x-x_0|^2-c^{-2}\beta^2t^2+\frac{\beta-1}{2\lambda}\right)v^2 - 8\lambda\left\langle \nabla v, x-x_0\right\rangle_n\partial_i v+4\lambda(x_i-x_{0i})|\nabla v|^2-4\lambda(x_i-x_{0i})c^{-2}(\partial_t v)^2\,, \tag{4.34a}$$

$$U_{n+1}^b := 8\lambda c^{-2}\left\langle \nabla v, x-x_0\right\rangle_n v_t\,. \tag{4.34b}$$

Step 3. Now we estimate $2z_1z_2+2z_2z_3$ from below. Noting that $\varphi(x,t)=|x-x_0|^2-\beta t^2>0$ in Q_σ, and hence $\sqrt{\beta}|t|<|x-x_0|\leqslant r$, we calculate:

$$\begin{aligned} 8\lambda\beta t\partial_t v\left\langle \nabla c^{-2}, \nabla v\right\rangle_n &\geqslant -\left|8\lambda\beta t\partial_t v\left\langle \nabla c^{-2}, \nabla v\right\rangle_n\right| \\ &\geqslant -8\lambda\beta|t|\,|\partial_t v|\,|\nabla c^{-2}(x)|\,|\nabla v| \geqslant -4\lambda\beta|t||\nabla c^{-2}(x)|\left(|\partial_t v|^2+|\nabla v|^2\right) \\ &\geqslant -4\lambda\sqrt{\beta}r|\nabla c^{-2}(x)|\left(|\partial_t v|^2+|\nabla v|^2\right). \end{aligned} \tag{4.35}$$

We apply this estimate to (4.29) and add (4.33):

$$
\begin{aligned}
(4.36)\quad 2z_1z_2 + 2z_1z_3 \geqslant 16\lambda^3 \Bigg(& (c^{-2}\beta + n + 2)|x - x_0|^2 \\
& - \left(3c^{-4}\beta + nc^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n\right)\beta^2 t^2 + \frac{2\beta - 2}{2\lambda}\Bigg) v^2 \\
& - 4\lambda \left(n - 2 + c^{-2}\beta + \sqrt{\beta} r |\nabla c^{-2}(x)|\right) |\nabla v|^2 \\
& + 4\lambda \left(nc^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n - c^{-4}\beta - \sqrt{\beta} r |\nabla c^{-2}(x)|\right) (\partial_t v)^2 \\
& + \sum_{i=1}^{n+1} \partial_i \left(U_i^a + U_i^b\right).
\end{aligned}
$$

Step 4. In order to arrange for positive coefficients in the terms containing ∇v, $\partial_t v$ and v, we make use of the terms of z_1^2. Specifically, we need to obtain a positive sign for ∇v in (4.36) (since β is small compared to n). For this reason, we introduce a new parameter $b \neq 0$ in z_1^2, which we will later use to balance n:

(4.37)
$$
z_1^2 = \left[\left(\frac{1}{c^2}\partial_{tt} v - \Delta v - 4\lambda^2 \left(|x - x_0|^2 - \frac{1}{c^2}\beta^2 t^2 + \frac{\beta - 1 - \frac{b}{2}}{2\lambda} \right) v \right) - \lambda b v \right]^2 .
$$

Dropping the quadratic terms, we obtain:

$$
(4.38)\quad z_1^2 \geqslant -2\lambda b v \left(\frac{1}{c^2}\partial_{tt} v - \Delta v - 4\lambda^2 \left(|x - x_0|^2 - \frac{1}{c^2}\beta^2 t^2 + \frac{\beta - 1 - \frac{b}{2}}{2\lambda} \right) v \right).
$$

Using the Leibniz identity for the first two terms of the product, we calculate:

$$
\begin{aligned}
(4.39)\quad z_1^2 \geqslant{}& \partial_t \left[-2\lambda c^{-2} b v \partial_t v \right] + 2\lambda b c^{-2} (\partial_t v)^2 + \sum_{i=1}^{n} \partial_i \left[2\lambda b v \partial_i v \right] - 2\lambda b |\nabla v|^2 \\
& + 8\lambda^3 b \left(|x - x_0|^2 - c^{-2}\beta^2 t^2 + \frac{\beta - 1 - \frac{b}{2}}{4\lambda} \right) v^2 .
\end{aligned}
$$

We therefore have the estimate:

$$
\begin{aligned}
(4.40)\quad z_1^2 \geqslant{}& 8\lambda^3 b \left(|x - x_0|^2 - c^{-2}\beta^2 t^2 + \frac{\beta - 1 - \frac{b}{2}}{4\lambda} \right) v^2 - 2\lambda b |\nabla v|^2 \\
& + 2\lambda b c^{-2} (\partial_t v)^2 + \sum_{i=1}^{n+1} \partial_i U_i^c ,
\end{aligned}
$$

where for $i = 1, \ldots, n$,

$$
(4.41a)\qquad U_i^c := 2\lambda b v \partial_i v ,
$$

$$
(4.41b)\qquad U_{n+1}^c := -2\lambda c^{-2} b v \partial_t v .
$$

Step 5. We finally choose b so that all coefficients are positive. First, we collect the estimates above by adding (4.36) and (4.40):

$$
\begin{aligned}
z_1^2 + 2z_1z_2 + 2z_1z_3 \geqslant 16\lambda^3 \Bigg[& \left(c^{-2}\beta + n + 2 + \frac{b}{2} \right) |x - x_0|^2 \\
& - \left(3c^{-4}\beta + nc^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n + \frac{b}{2} c^{-2} \right) \beta^2 t^2 \\
& + \frac{16\beta - 16 + 2b\beta - 2b - b^2}{16\lambda} \Bigg] v^2 + 4\lambda \left[-\frac{b}{2} - n + 2 - c^{-2}\beta - \sqrt{\beta} r |\nabla c^{-2}(x)| \right] |\nabla v|^2 \\
& + 4\lambda \left[nc^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n - c^{-4}\beta - \sqrt{\beta} r |\nabla c^{-2}(x)| + \frac{b}{2} c^{-2} \right] (\partial_t v)^2 \\
& + \sum_{i=1}^{n+1} \partial_i \left(U_i^a + U_i^b + U_i^c \right) .
\end{aligned} \tag{4.42}
$$

Now for a constant $c(x) \equiv c > 0$, the terms containing ∇c vanish, and we can set $b := -2(n-1)$. Then, the conditions that the coefficients of v^2, $(\partial_t v)^2$ and $|\nabla v|^2$ are positive are reduced to the following inequalities:

$$
(3 + c^{-2}\beta)|x - x_0|^2 - (3c^{-4}\beta + c^{-2})\beta^2 t^2 > 0 , \tag{4.43a}
$$

$$
3 - c^{-2}\beta > 0 , \tag{4.43b}
$$

$$
c^{-2} - c^{-4}\beta > 0 , \tag{4.43c}
$$

The inequalities (4.43b) and (4.43c) are obviously satisfied for $\beta < c^2$. In this case, we also have that $3c^{-4}\beta^2 < 3$. We therefore can rewrite (4.43a) as follows:

$$
\begin{aligned}
(3 + c^{-2}\beta)|x - x_0|^2 - (3c^{-4}\beta + c^{-2})\beta^2 t^2 & \\
> (3 + c^{-2}\beta) \left(|x - x_0|^2 - \beta t^2 \right) > (3 + c^{-2}\beta)\sigma > 0 &
\end{aligned} \tag{4.44}
$$

by the definition (4.19) of Q_σ. For a $\lambda =: \lambda_0$ large enough, the absolute value of the term containing $\frac{1}{\lambda}$ can be made smaller than 3σ, and so the term involving v^2 is greater than zero. Hence for constant coefficients, the condition $\beta < c^2$ is sufficient for the Carleman estimate (4.22) to hold. Note that this condition is identical to the sharp bound derived by the strong pseudo-convexity conditions, cf. above and [27].

For variable coefficients $c(x)$, we have to estimate more carefully. First, we balance n with b in the coefficient of $|\nabla v|^2$ by writing $\hat{c} := \sup_{x \in \Omega} |\nabla c^{-2}|$ and setting:

$$
b := -2 \left(n - 2 + c_{\min}^{-2}\beta + \sqrt{\beta} r \hat{c} \right) . \tag{4.45}
$$

Inserting this in (4.42), we arrive at

$$
\begin{aligned}
(4.46)\quad z_1^2 + 2z_1z_2 + 2z_1z_3 \geqslant 16\lambda^3 \Bigg[& \left((c^{-2} - c_{min}^{-2})\beta + 4 - \sqrt{\beta} r\hat{c} \right) |x - x_0|^2 \\
& - \left((3c^{-2} - c_{min}^{-2})c^{-2}\beta + 2c^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n - \sqrt{\beta} r\hat{c} \right) \beta^2 t^2 + \frac{C}{\lambda} \Bigg] v^2 \\
& + 4\lambda \left[(c_{min}^{-2} - c^{-2})\beta \right] |\nabla v|^2 \\
& + 4\lambda \left[2c^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n - (c_{min}^{-2} + c^{-2})c^{-2}\beta - (1 + c^{-2})\sqrt{\beta} r\hat{c} \right] (\partial_t v)^2 \\
& + \sum_{i=1}^{n+1} \partial_i \left(U_i^a + U_i^b + U_i^c \right),
\end{aligned}
$$

where we have used the constant $C := C(\beta, n, r, c, T)$ (which may be positive or negative) for brevity.

Now it remains to show that the terms in (4.46) can be made positive. By (4.20a), this is immediately evident for the coefficient of $|\nabla v|^2$. For the coefficient of $(\partial_t v)^2$, we first notice that for $\beta < 1$, we have $\beta < \sqrt{\beta}$. Hence, if we estimate c^{-2} from above by c_{min}^{-2}, conditions (4.20b) and (4.21a) guarantee that

$$
(4.47)\qquad 2c^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n > 2c_{min}^{-4}\beta + (1 + c_{min}^{-2})\sqrt{\beta} r\hat{c} > 0 .
$$

Last, we consider the factor of v^2. Again, we try to balance the coefficient of $|x - x_0|^2$ with that of βt. After rearrangement, the condition to be met is:

$$
(4.48)\quad (3c^{-2} - c_{min}^{-2})c^{-2}\beta^2 + c^{-2}\beta + \beta \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n + (1 - \beta)\sqrt{\beta} r\hat{c} + c_{min}^{-2}\beta < 4 .
$$

Dropping negative terms and estimating c^{-2} by its maximum in $\overline{\Omega}$, we see that the left hand side is less than:

$$
(4.49)\quad 3c_{min}^{-4}\beta^2 + 2c_{min}^{-2}\beta + \beta r\hat{c} + \sqrt{\beta} r\hat{c} \leqslant \left(3c_{min}^{-4} + 2c_{min}^{-2} \right) \sqrt{\beta} + 2\sqrt{\beta} r\hat{c} < 4 ,
$$

by assumption (4.21b) and $\beta < 1$. Hence, as above, we can estimate

$$
\begin{aligned}
(4.50)\quad & \left((c^{-2} - c_{min}^{-2})\beta + 4 - \sqrt{\beta} r\hat{c} \right) |x - x_0|^2 \\
& - \left((3c^{-2} - c_{min}^{-2})c^{-2}\beta + 2c^{-2} + \left\langle \nabla c^{-2}, x - x_0 \right\rangle_n - \sqrt{\beta} r\hat{c} \right) \beta^2 t^2 \\
& > \left(4 - c_{min}^{-2}\beta - \sqrt{\beta} r\hat{c} \right) \left(|x - x_0|^2 - \beta t^2 \right) \\
& > \left(4 - \frac{4c_{min}^{-2} + 4r\hat{c}}{3c_{min}^{-4} + 2c_{min}^{-2} + 2r\hat{c}} \right) \sigma > 0 .
\end{aligned}
$$

Taking again $\lambda =: \lambda_0$ large enough, $\frac{C}{\lambda}$ is smaller in value than this last term, and so the coefficient of v^2 is greater than zero as well.

Observing that for all $i = 1, \dots, n+1$:

$$\begin{aligned}(4.51)\quad \lambda^3 u^2 e^{2\lambda\varphi} + \lambda|\partial_i u|^2 e^{2\lambda\varphi} &= \lambda^3 (v e^{-\lambda\varphi})^2 e^{2\lambda\varphi} + \lambda|\partial_i (v e^{-\lambda\varphi})|^2 e^{2\lambda\varphi} \\ &= \lambda^3 v^2 + \lambda|\partial_i v e^{-\lambda\varphi} - \lambda\partial_i\varphi v e^{-\lambda\varphi}|^2 e^{2\lambda\varphi} \\ &\leqslant \lambda^3 v^2 + 2\lambda|\partial_i v|^2 + 2\lambda^3|\partial_i\varphi|^2 v^2 \leqslant C\left(\lambda^3 v^2 + \lambda|\partial_i v|^2\right),\end{aligned}$$

where we have estimated $\partial_i\varphi$ in Q_σ from above, we can revert to $u = v e^{-\lambda\varphi}$ in (4.46). If we collect all these positive coefficients on the right hand side, we have shown the pointwise Carleman estimate (4.22):

$$(4.52)\qquad \lambda^3 u^2 e^{2\lambda\varphi} + \lambda\left(|\nabla u|^2 + (\partial_t u)^2\right) e^{2\lambda\varphi} + \sum_{i=1}^{n+1} \partial_i U_i \leqslant C|Lu|^2 e^{2\lambda\varphi}.$$

Finally, we see from (4.30), (4.34) and (4.41), that — with coefficients a_k, $k = 1, \dots, 7$ depending on c, Q_T, β — the terms $U := U^a + U^b + U^c$ for $i = 1, \dots, n$ have the following form:

$$\begin{aligned}(4.53a)\quad U_i &= -a_1\lambda\partial_t v\partial_i v - a_2\lambda^3(x_i - x_{0i})v^2 + 4\lambda(x_i - x_{0i})\left(|\nabla v^2| - (\partial_t v)^2\right) \\ &\quad - 8\lambda\left\langle x - x_0, \nabla v\right\rangle_n \partial_i v + a_3\lambda v\partial_i v, \\ (4.53b)\quad U_{n+1} &= a_4\lambda\left(|\nabla v|^2 + (\partial_t v)^2\right) - a_5\lambda^3 v^2 + a_6\lambda\left\langle x - x_0, \nabla v\right\rangle_n \partial_t v \\ &\quad - a_7\lambda v\partial_t v.\end{aligned}$$

Using now the Cauchy-Schwarz inequality for the inner products and then Young's inequality, estimating $|x - x_0|$, and noting that $\lambda^3 > \lambda$, we have with positive constants C_1, C_2:

$$(4.54a)\qquad |U_i| \leqslant C_1\left[\lambda^3 v^2 + \lambda\left(|\nabla v^2| + (\partial_t v)^2\right) + \lambda(\partial_i v)^2\right],$$

$$(4.54b)\qquad |U_{n+1}| \leqslant C_2\left[\lambda^3 v^2 + \lambda\left(|\nabla v|^2 + (\partial_t v)^2\right)\right].$$

Summing over $i = 1, \dots, n+1$, and using an upper bound C_3 for the constants above, we see that:

$$(4.55)\qquad |U| \leqslant C_3\left(\lambda^3|v|^2 + \lambda(|\nabla v|^2 + |\partial_t v|^2)\right),$$

where the constant c_3 depends on r, T, c_{max}, c_{min}, β, and b. Inserting $v = u e^{\lambda\varphi}$, and estimating again $\nabla\varphi, \partial_t\varphi$ in Q_σ from above, yields the desired estimate (4.23). □

By integrating this pointwise estimate and applying the divergence theorem, we can prove a global (i.e., up to the boundary) Carleman estimate for the wave equation with variable coefficients.

Theorem 4.1.6 (Carleman estimate). *Under the assumptions of Lemma 4.1.5, there exist constants $C > 0$ and $\lambda_0 > 0$, such that for all $\lambda > \lambda_0$, the inequality*

$$\lambda^3 \int_{Q_\sigma} |u|^2 e^{2\lambda\varphi}\, dq + \lambda \int_{Q_\sigma} \left(|\nabla u|^2 + |\partial_t u|^2\right) e^{2\lambda\varphi}\, dq \leqslant C \int_{Q_\sigma} |Lu|^2 e^{2\lambda\varphi}\, dq + C\lambda \int_{\partial Q_\sigma \cap S'_T} |\partial_\nu u|^2 e^{2\lambda\varphi}\, ds \tag{4.56}$$

holds for all $u \in H^2(Q_\sigma)$ with $u = 0$ on ∂Q_σ and $u \equiv 0$ in a neighborhood of $\partial Q_\sigma \setminus S'_T$.

Proof. We begin by integrating (4.22) over Q_σ. In this case, we can interpret the derivatives in (4.22) as taken in the weak sense, which justifies considering $u \in H^2(Q_T)$. We thus have:

$$\lambda^3 \int_{Q_\sigma} |u|^2 e^{2\lambda\varphi}\, dq + \lambda \int_{Q_\sigma} \left(|\nabla u|^2 + |\partial_t u|^2\right) e^{2\lambda\varphi}\, dq \leqslant C_1 \int_{Q_\sigma} |Lu|^2 e^{2\lambda\varphi}\, dq - \int_{Q_\sigma} \sum_{i=1}^{n+1} \partial_i U_i\, dq\,. \tag{4.57}$$

Since for Lipschitz domains, the exterior normal ν exists almost everywhere on Q_σ, we can apply the divergence theorem in Q_σ [52, Ch. 3, § 1, Th. 1.1] to the last integral:

$$\lambda^3 \int_{Q_\sigma} |u|^2 e^{2\lambda\varphi}\, dq + \lambda \int_{Q_\sigma} \left(|\nabla u|^2 + |\partial_t u|^2\right) e^{2\lambda\varphi}\, dq \leqslant C_1 \int_{Q_\sigma} |Lu|^2 e^{2\lambda\varphi}\, dq - \int_{\partial Q_\sigma} \langle U, \nu\rangle_{n+1}\, ds\,. \tag{4.58}$$

We split the last term into the parts on $\Gamma_0 := \partial Q_\sigma \cap S'_T$ and on $\Gamma_1 := \partial Q_\sigma \setminus \Gamma_0$:

$$-\int_{\partial Q_\sigma} \langle U, \nu\rangle_{n+1}\, ds = -\int_{\Gamma_0} \langle U, \nu\rangle_{n+1}\, ds - \int_{\Gamma_1} \langle U, \nu\rangle_{n+1}\, ds\,. \tag{4.59}$$

Using the estimate (4.23) and the fact that ν is identically zero near Γ_1, we see that the integral over Γ_1 is zero. Since $u = \partial_t u \equiv 0$ and $\nu_{n+1} = 0$ on Γ_0, the surface integrals reduce to:

$$\begin{aligned} -\int_{\Gamma_0} \langle U, \nu\rangle_{n+1}\, ds &= -4\lambda \int_{\Gamma_0} \sum_{i=1}^{n} \left((x_i - x_{0i})|\nabla\nu|^2 - 2\langle x - x_0, \nabla\nu\rangle_n\, \partial_i\nu\right)\nu_i\, ds \\ &= -4\lambda \int_{\Gamma_0} \sum_{i,j=1}^{n} \left((x_i - x_{0i})(\partial_j\nu)^2 - 2(x_j - x_{0j})\partial_i\nu\partial_j\nu\right)\nu_i\, ds\,. \end{aligned} \tag{4.60}$$

Now we decompose the derivatives with respect to the coordinates ∂_i at $\xi \in \Gamma_0$ into their normal and tangential components. Since $u \equiv 0$ on Γ_0, the tangential derivatives are zero, and thus we have for all $i = 1, \dots, n$ that $\partial_i v = \nu_i \partial_\nu v$ (since in this case $\partial_\nu v := \sum_{i=1}^{n+1} \nu_i \partial_i v = \partial_\nu v \sum_{i=1}^{n+1} \nu_i^2$, and ν is the unit normal). Hence,

$$
\begin{aligned}
(4.61)\quad & \sum_{i,j=1}^{n} \left((x_i - x_{0i})(\partial_j v)^2 - 2(x_j - x_{0j}) \partial_i v \partial_j v \right) \nu_i \\
&= \sum_{i,j=1}^{n} \left((x_i - x_{0i})(\partial_\nu v)^2 \nu_j^2 - 2(x_j - x_{0j})(\partial_\nu v)^2 \nu_i \nu_j \right) \nu_i \\
&= (\partial_\nu v)^2 \sum_{i,j=1}^{n} \left((x_i - x_{0i}) \nu_i \nu_j^2 - 2(x_j - x_{0j}) \nu_i^2 \nu_j \right) \\
&= -(\partial_\nu v)^2 \sum_{i,j=1}^{n} \left((x_i - x_{0i}) \nu_i \nu_j^2 \right) = -(\partial_\nu v)^2 \langle x - x_0, \nu \rangle_n \,,
\end{aligned}
$$

by reordering of the summation and using the fact that $\sum_{j=1}^{n} \nu_j^2 = 1$ on S_T'.

Thus, applying the Cauchy-Schwarz inequality, we estimate the divergence term:

$$
\begin{aligned}
(4.62)\quad -\int_{\Gamma_0} \langle U, \nu \rangle_{n+1} \, ds &= 4\lambda \int_{\Gamma_0} (\partial_\nu v)^2 \langle x - x_0, \nu \rangle_n \, ds \\
&\leqslant 4\lambda \int_{\Gamma_0} (\partial_\nu v)^2 |x - x_0||\nu| \, ds \leqslant 4r\lambda \int_{\Gamma_0} (\partial_\nu v)^2 \, ds \,.
\end{aligned}
$$

Reverting once more to $u = e^{-\lambda\varphi} v$, observing that $u = 0$ on Γ_0, and inserting the above inequality in (4.58), we finally arrive at the Carleman estimate (4.56). □

Remark 4.1. Since by Lemma 4.1.3 the Carleman estimate depends only on the principal part of the operator L, the results of this chapter also hold for hyperbolic operators which include absorption or potential terms. The method of quasi-reversibility consequently allows for time reversal in dissipative media, as well.

In the rest of this chapter, we always assume that the conditions (4.20) and (4.21) on c and β are met.

4.1.3. Additional estimates

We want to apply the Carleman estimates from the preceding section to the difference of solutions of the lateral Cauchy problem and of the quasi-reversibility approximation. Thus, we need estimates for this difference independent of the approximate solution.

Lemma 4.1.7. *We assume there exists a solution $u^* \in H_0^2(Q_T')$ to Problem[8] 2.1. Let $u_\varepsilon \in H_0^2(Q_T')$ be a solution of Problem 3.1. The difference $w := u^* - u_\varepsilon$ fulfills the following estimates with a constant $C > 0$ depending only on Q_T':*

$$\int_{Q_T'} (Lw)^2 \, dq \leqslant C\varepsilon \, \|u^*\|^2_{H^2(Q_T')} \,, \tag{4.63}$$

$$\|w\|^2_{H^2(Q_T')} \leqslant C \, \|u^*\|^2_{H^2(Q_T')} \,. \tag{4.64}$$

Proof. If we multiply (3.14) by Lv, integrate over Q_T, and add $\varepsilon \langle u^*, v\rangle_{QR}$ on both sides of the equation, we see that u^* satisfies for all $v \in H_0^2(Q_T)$:

$$M_\varepsilon(u^*, v) = -\int_{Q_T} L\Phi \, Lv \, dq \; + \varepsilon \langle u^*, v\rangle_{QR} \,. \tag{4.65}$$

Since $w = (u^* - u_\varepsilon) \in H_0^2(Q_T)$ as well, we can subtract (3.16), take $v = w$, and use the symmetry of w and u^* to obtain:

$$\begin{aligned} &\int_{Q_T'} (Lw)^2 \, dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \, dq + \int_{Q_T'} |\nabla^2 w|^2 \, dq + \int_{Q_T'} w^2 \, dq \right) \\ &\qquad = \varepsilon \left(\int_{Q_T'} \partial_{tt} u^* \, \partial_{tt} w \, dq + \int_{Q_T'} \left\langle \nabla^2 u^*, \nabla^2 w \right\rangle_n + \int_{Q_T'} u^* \, w \, dq \right) . \end{aligned} \tag{4.66}$$

By the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} &\int_{Q_T'} (Lw)^2 \, dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \, dq + \int_{Q_T'} |\nabla^2 w|^2 \, dq + \int_{Q_T'} w^2 \, dq \right) \\ &\quad \leqslant \varepsilon \Big(\|\partial_{tt} u^*\|_{L^2(Q_T')} \|\partial_{tt} w\|_{L^2(Q_T')} + \left\|\nabla^2 u^*\right\|_{L^2(Q_T')} \left\|\nabla^2 w\right\|_{L^2(Q_T')} \\ &\qquad\qquad + \|u^*\|_{L^2(Q_T')} \|w\|_{L^2(Q_T')} \Big) . \end{aligned} \tag{4.67}$$

Using Young's inequality $2ab \leqslant a^2 + b^2$ on the right hand side allows the subtraction of the terms containing w, which yields

$$\begin{aligned} &\int_{Q_T'} (Lw)^2 \, dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \, dq + \int_{Q_T'} |\nabla^2 w|^2 \, dq + \int_{Q_T'} w^2 \, dq \right) \\ &\qquad \leqslant \varepsilon \left(\|\partial_{tt} u^*\|^2_{L^2(Q_T')} + \left\|\nabla^2 u^*\right\|^2_{L^2(Q_T')} + \|u^*\|^2_{L^2(Q_T')} \right) . \end{aligned} \tag{4.68}$$

Including the missing derivatives on the right hand side of the inequality and using Lemma 3.1.1, the equivalence of the norms $\|u\|_{QR}$ and $\|u\|_{H^2(Q_T')}$ on $H_0^2(Q_T)$ (and

[8] by which we mean, here and below, the even extension to $[-T, 0] \times \Omega$ of a solution to the corresponding problem in Q_T.

hence on $H_0^2(Q_T')$), on the left hand side, we finally obtain:

$$\int_{Q_T'} (Lw)^2 \, dq + \varepsilon \, \|w\|^2_{H^2(Q_T')} \leqslant C\varepsilon \, \|u^*\|^2_{H^2(Q_T')} \ . \tag{4.69}$$

□

In reality, we have to assume that the Cauchy data φ_0, φ_1 are contaminated by noise, and we have, instead, φ_0^δ and φ_1^δ. In this case, the estimate above must take into account the error introduced by this:

Lemma 4.1.8. *Let $u^* \in H_0^2(Q_T')$ be a solution of (3.14). Assume there exists an even function $\Phi^\delta \in H^2(Q_T')$ which satisfies*

$$\begin{cases} \Phi^\delta(x,t) & = \varphi_0^\delta(x,t) \quad (x,t) \in S_T \ , \\ \partial_\nu \Phi^\delta(x,t) & = \varphi_1^\delta(x,t) \quad (x,t) \in S_T \ , \end{cases} \tag{4.70}$$

and

$$\left\| \Phi - \Phi^\delta \right\|_{H^2(Q_T')} \leqslant \delta. \tag{4.71}$$

Let $u_\varepsilon^\delta \in H_0^2(Q_T')$ be a solution of Problem 3.1 with Φ^δ replacing Φ.

The difference $w := u^* - u_\varepsilon^\delta$ fulfills the following estimates with a constant $C > 0$ depending only on Q_T':

$$\int_{Q_T'} (Lw)^2 \, dq \leqslant C \left(\delta^2 + \varepsilon \, \|u^*\|^2_{H^2(Q_T')} \right), \tag{4.72}$$

$$\varepsilon \, \|w\|^2_{H^2(Q_T')} \leqslant C \left(\delta^2 + \varepsilon \, \|u^*\|^2_{H^2(Q_T')} \right). \tag{4.73}$$

Proof. As above, from (3.14), we know u^* satisfies for all $v \in H_0^2(Q_T)$:

$$M_\varepsilon(u^*, v) = -\int_{Q_T} L\Phi \, Lv \, dq \ + \varepsilon \, \langle u^*, v \rangle_{QR} \, . \tag{4.74}$$

Since $(u^* - u_\varepsilon^\delta) \in H_0^2(Q_T')$ as well, we can subtract (3.16) with Φ^δ replacing Φ, take $v = w$, and again obtain by symmetry in t:

$$\begin{aligned} \int_{Q_T'} (Lw)^2 \, dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \, dq + \int_{Q_T'} |\nabla^2 w|^2 \, dq + \int_{Q_T'} w^2 \, dq \right) \\ = \int_{Q_T'} L(\Phi^\delta - \Phi) Lw \, dq \\ + \varepsilon \left(\int_{Q_T'} \partial_{tt} u^* \, \partial_{tt} w \, dq + \int_{Q_T'} \left\langle \nabla^2 u^*, \nabla^2 w \right\rangle_n \, dq + \int_{Q_T'} u^* \, w \, dq \right). \end{aligned} \tag{4.75}$$

By the Cauchy-Schwarz inequality, we have now:

$$\begin{aligned}(4.76)\quad &\int_{Q_T'} (Lw)^2 \,dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \,dq + \int_{Q_T'} |\nabla^2 w|^2 \,dq + \int_{Q_T'} w^2 \,dq \right) \\ &\leqslant \left\| L(\Phi^\delta - \Phi) \right\|_{L^2(Q_T')} \|Lw\|_{L^2(Q_T')} + \varepsilon \Big(\|\partial_{tt} u^*\|_{L^2(Q_T')} \|\partial_{tt} w\|_{L^2(Q_T')} \\ &\qquad + \left\|\nabla^2 u^*\right\|_{L^2(Q_T')} \left\|\nabla^2 w\right\|_{L^2(Q_T')} + \|u^*\|_{L^2(Q_T')} \|w\|_{L^2(Q_T')} \Big).\end{aligned}$$

Using (4.71) and again Young's inequality on the right hand side yields:

$$\begin{aligned}(4.77)\quad &\int_{Q_T'} (Lw)^2 \,dq + \varepsilon \left(\int_{Q_T'} (\partial_{tt} w)^2 \,dq + \int_{Q_T'} |\nabla^2 w|^2 \,dq + \int_{Q_T'} w^2 \,dq \right) \\ &\qquad \leqslant \delta^2 + \varepsilon \left(\|\partial_{tt} u^*\|^2_{L^2(Q_T')} + \left\|\nabla^2 u^*\right\|^2_{L^2(Q_T')} + \|u^*\|^2_{L^2(Q_T')} \right).\end{aligned}$$

Including the missing derivatives on the right hand side of the inequality, and using again Lemma 3.1.1, we finally obtain:

$$(4.78)\qquad \int_{Q_T'} (Lw)^2 \,dq + \varepsilon \|w\|^2_{H^2(Q_T')} \leqslant C_2 \left(\delta^2 + \varepsilon \|u^*\|^2_{H^2(Q_T')} \right).$$

□

Lemmata 4.1.7 and 4.1.8 remain valid if we consider solutions $u^* \in H^2_\Gamma(Q_T')$ of (3.26) and solutions $u_\varepsilon \in H^2_\Gamma(Q_T')$ of (3.16), and replace S_T' by Γ_T' (cf. Remark 3.2). We will apply these variants in the next section in deriving convergence and stability results for the case that measurements are only available on a part of the boundary.

4.2. Hölder stability for partial boundary data

If the Cauchy data φ_0, φ_1 is given only on an arbitrary part Γ of the boundary $\partial\Omega$, we cannot expect full reconstruction of the solution of the lateral Cauchy problem (cf. [4], for a sufficient and almost necessary geometric condition on the shape of this boundary part). However, it is possible to show Hölder-type convergence and stability estimates inside level sets Q_σ of pseudo-convex functions intersecting only this boundary part. For an example of such a domain, see Figure 6.9. Finding a maximal domain Q_σ given Γ is a longstanding open question.

Theorem 4.2.1 (Rate of convergence). *Assume there exists a solution $u^* \in H^2_\Gamma(Q_T')$ to problem (3.26), and (4.20) holds. Choose $x_0 \notin \overline{\Omega}$ and $\sigma \geqslant$ so that $Q_\sigma \neq \emptyset$, $\overline{Q}_\sigma \cap \{|t| = T\} = \emptyset$, and $\partial Q_\sigma \cap S_T' \subset \Gamma_T'$. Then there exists a constant $C > 0$,*

such that for sufficiently small $\varepsilon > 0$ the solution u_ε of Problem 3.1 satisfies with $\alpha \in (0,1)$:

$$\|u^* - u_\varepsilon\|^2_{H^1(Q_{3\sigma})} \leqslant C\varepsilon^\alpha \|u^*\|^2_{H^2(Q'_T)} \,. \tag{4.79}$$

Proof. We set $m := \max_{\overline{Q_0}} \varphi(x,t)$. We have:

$$\int_{Q'_T} (Lw)^2 \,dq = \int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} e^{-2\lambda\varphi} \,dq \geqslant e^{-2\lambda m} \int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} \,dq \,. \tag{4.80}$$

So from (4.63), we obtain for $w := u^* - u_\varepsilon$:

$$\int_{Q'_T} (Lw)^2 e^{2\lambda\varphi} \,dq \leqslant C_1 e^{2\lambda m} \varepsilon \|u^*\|^2_{H^2(Q'_T)} \,. \tag{4.81}$$

To apply the Carleman estimate to the left hand side, we have to arrange for zero boundary conditions for w on Q_0. Therefore we introduce a cut-off function $\chi_\sigma \in C^\infty(Q_0)$ with:

$$\chi_\sigma(x,t) \begin{cases} = 1 & \text{for } (x,t) \in Q_{2\sigma} \,, \\ = 0 & \text{for } (x,t) \in Q_0 \setminus Q_\sigma \,, \\ \in (0,1) & \text{for } (x,t) \in Q_\sigma \setminus Q_{2\sigma} \,, \end{cases} \tag{4.82}$$

and set $w_\sigma := \chi_\sigma w$. Since $\|\chi_\sigma\|_{H^2(Q_0 \setminus Q_\sigma)} = 0$, it follows that

$$(Lw_\sigma)^2 \leqslant 2(Lw)^2 + C_2(1-\chi_\sigma)(w + \partial_t w + |\nabla w|)^2 \,, \tag{4.83}$$

where the constant C_2 depends only on the $H^2(Q_\sigma)$ norm of χ_σ. Hence, multiplication with $e^{2\lambda\varphi}$ of (4.83), integration over Q'_T, and use of (4.81) yields:

$$\begin{aligned} &\int_{Q'_T} (Lw_\sigma)^2 e^{2\lambda\varphi} \,dq \\ &\qquad \leqslant C_2 \int_{Q_0 \setminus Q_{2\sigma}} (w + \partial_t w + |\nabla w|)^2 e^{2\lambda\varphi} \,dq + C_1 e^{2\lambda m} \varepsilon \|u^*\|^2_{H^2(Q'_T)} \,. \end{aligned} \tag{4.84}$$

If we restrict the domain of integration on the left hand side, we can now apply the Carleman estimate (4.56) to obtain:

$$\begin{aligned} &\lambda^3 \int_{Q_\sigma} w_\sigma^2 e^{2\lambda\varphi} \,dq + \lambda \int_{Q_\sigma} \left(|\nabla w_\sigma|^2 + |\partial_t w_\sigma|^2\right) e^{2\lambda\varphi} \,dq \\ &\qquad \leqslant C_2 e^{4\lambda\sigma} \|w\|^2_{H^1(Q_0)} + C_1 e^{2\lambda m} \varepsilon \|u^*\|^2_{H^2(Q'_T)} \,. \end{aligned} \tag{4.85}$$

Here we have used that $0 < \varphi(x,t) < 2\sigma$ on $Q_0 \setminus Q_{2\sigma}$. Replacing Q_σ with $Q_{3\sigma}$ on the left hand side and estimating $\varphi(x,t)$ by 3σ from below there, we have for $\lambda > 1$:

$$\|w\|^2_{H^1(Q_{3\sigma})} \leqslant C_2 e^{-2\lambda\sigma} \|w\|^2_{H^1(Q_0)} + C_1 e^{2\lambda m} \varepsilon \|u^*\|^2_{H^2(Q'_T)} \,. \tag{4.86}$$

Since we can bound $\|w\|^2_{H^1(Q_0)}$ by $\|w\|^2_{H^2(Q'_T)}$ and hence apply (4.64), we finally have:

$$\|w\|^2_{H^1(Q_{3\sigma})} \leqslant C_3(e^{-2\lambda\sigma} + e^{2\lambda m}\varepsilon)\,\|u^*\|^2_{H^2(Q'_T)}\ . \tag{4.87}$$

Now take

$$\lambda = \frac{\ln(1/\varepsilon)}{2(m+\sigma)}. \tag{4.88}$$

Then, $e^{-2\lambda\sigma} = e^{2\lambda m}\varepsilon$, and if ε is sufficiently small, $\lambda > \lambda_0$ holds. Setting $C := 2C_3$ and $\alpha := \frac{\sigma}{2(m+\sigma)} \in (0,1)$, estimate (4.79) follows. □

If we consider noisy Cauchy data φ_0^δ, φ_1^δ, the convergence rate will depend on the error in the data. We therefore have to modify the above proof slightly to take this into account. The following theorem consequently addresses the stability of the method of quasi-reversibility.

Theorem 4.2.2 (Error estimate). *Let $u^* \in H^2_\Gamma(Q'_T)$ be a solution of (3.26). Assume there exists an even function $\Phi^\delta \in H^2(Q'_T)$ which satisfies*

$$\begin{cases} \Phi^\delta(x,t) &= \varphi_0^\delta(x,t) \quad (x,t) \in \Gamma_T\ , \\ \partial_\nu \Phi^\delta(x,t) &= \varphi_1^\delta(x,t) \quad (x,t) \in \Gamma_T\ , \end{cases} \tag{4.89}$$

and

$$\left\|\Phi - \Phi^\delta\right\|_{H^2(Q'_T)} \leqslant \delta. \tag{4.90}$$

Denote by $u_\varepsilon^\delta \in H^2_\Gamma(Q'_T)$ the solution of (3.27) with Φ^δ replacing Φ.

Choose $x_0 \notin \overline{\Omega}$ and $\sigma \geqslant 0$ so that $Q_\sigma \neq \emptyset$, $\overline{Q}_\sigma \cap \{|t| = T\} = \emptyset$, and $\partial Q_\sigma \cap S'_T \subset \Gamma'_T$. If $\varepsilon = \delta^2$, then there exists a $C > 0$ and $\alpha \in (0,1)$, such that

$$\left\|u^* - u_\varepsilon^\delta\right\|^2_{H^1(Q_{3\sigma})} \leqslant C\delta^\alpha \left(1 + \|u^*\|^2_{H^2(Q'_T)}\right). \tag{4.91}$$

Proof. Starting from Lemma 4.1.8 instead of Lemma 4.1.7, we repeat the steps in the proof of Theorem 4.2.1, until we arrive at:

$$\|w\|^2_{H^1(Q_{3\sigma})} \leqslant C_2 e^{-2\lambda\sigma}\,\|w\|^2_{H^1(Q_0)} + C_1 e^{2\lambda m}\left(\delta^2 + \varepsilon\,\|u^*\|^2_{H^2(Q'_T)}\right). \tag{4.92}$$

Taking $\varepsilon = \delta^2$, we can use (4.73) to estimate $\varepsilon\,\|w\|^2_{H^1(Q_0)}$ by $\varepsilon\left(1 + \|w\|^2_{H^2(Q'_T)}\right)$ from above. Choosing again λ suitably, the rest of the proof follows that of Theorem 4.2.1. □

Remark 4.2. From this theorem, we see that $\varepsilon = \beta\delta^2$ for $\beta \geqslant 1$ constitutes a parameter choice rule, hence the method of quasi-reversibility is a regularization method for the lateral Cauchy problem (cf. [14]).

4.3. Lipschitz stability for full boundary data

The above proves only a Hölder rate of convergence. However, if Ω is contained in a ball of radius R, the Cauchy data is given on the whole boundary S'_T, and the final time T satisfies $T > \frac{2R}{\sqrt{\beta}}$, we can show Lipschitz stability and convergence rates for Problem 3.1. The proof consists of several steps. First, by a slight modification of the arguments of Section 4.2, we will show Hölder stability estimates for the lateral Cauchy problem 2.1. The key step is then their combination in two carefully chosen domains to obtain a Lipschitz observability estimate for the wave equation. This together with Lemma 4.1.7 yields the desired convergence and stability estimates for the quasi-reversibility approximation.

Lemma 4.3.1. *Assume that there exists an $x_0 \in \mathbb{R}^n$, such that (4.20) is satisfied. Choose $a \geqslant 0$ and $\sigma \geqslant 0$ so that $Q_{a+3\sigma} \neq \emptyset$ and $\overline{Q}_a \cap \{|t| = T\} = \emptyset$. Then there exist constants $C > 0, \lambda_0 > 0$, such that for all $w \in H_0^2(Q'_T)$ and all $\lambda > \lambda_0$ the following inequality holds:*

$$\|w\|^2_{H^1(Q_{a+3\sigma})} \leqslant C\left(e^{-2\lambda\sigma}\|w\|^2_{H^1(Q'_T)} + e^{2\lambda m}\|Lw\|^2_{L^2(Q'_T)}\right). \tag{4.93}$$

Proof. We set $m := \max_{\overline{Q_a}} \varphi(x,t)$. We have:

$$\int_{Q'_T}(Lw)^2\,dq = \int_{Q'_T}(Lw)^2 e^{2\lambda\varphi}e^{-2\lambda\varphi}\,dq \geqslant e^{-2\lambda m}\int_{Q'_T}(Lw)^2e^{2\lambda\varphi}\,dq\,, \tag{4.94}$$

and consequently:

$$\int_{Q'_T}(Lw)^2e^{2\lambda\varphi}\,dq \leqslant e^{2\lambda m}\|Lw\|^2_{L^2(Q'_T)}\,. \tag{4.95}$$

To apply the Carleman estimate (4.56), we once more have to arrange for zero boundary conditions for w on $Q_{a+\sigma}$. Therefore we introduce a cut-off function $\chi_\sigma \in C^\infty(Q'_T)$ with:

$$\chi_\sigma(x,t)\begin{cases} = 1 & \text{for } (x,t) \in Q_{a+2\sigma}\,,\\ = 0 & \text{for } (x,t) \in Q'_T \setminus Q_{a+\sigma}\,,\\ \in (0,1) & \text{for } (x,t) \in Q_{a+\sigma}\setminus Q_{a+2\sigma}\,,\end{cases} \tag{4.96}$$

and set $w_\sigma := \chi_\sigma w$. Since $\|\chi_\sigma\|_{H^2(Q_a\setminus Q_{a+\sigma})} = 0$, it follows that

$$(Lw_\sigma)^2 \leqslant 2(Lw)^2 + C(1-\chi_\sigma)(w + \partial_t w + |\nabla w|)^2\,, \tag{4.97}$$

where the constant C depends only on the $H^2(Q_{a+\sigma})$ norm of χ_σ. Hence, integration of (4.97) over $Q_{a+\sigma}$ after multiplication with $e^{2\lambda\varphi}$ and use of (4.95),

yields

$$\int_{Q_{a+\sigma}} (Lw_\sigma)^2 e^{2\lambda\varphi}\,dq \leqslant C \int_{Q_{a+\sigma}\setminus Q_{a+2\sigma}} (w + \partial_t w + |\nabla w|)^2 e^{2\lambda\varphi}\,dq + e^{2\lambda m} \|Lw\|^2_{L^2(Q'_T)}\,. \tag{4.98}$$

We can now apply the Carleman estimate (4.56) to obtain:

$$\lambda^3 \int_{Q_{a+\sigma}} w_\sigma^2 e^{2\lambda\varphi}\,dq + \lambda \int_{Q_{a+\sigma}} \left(|\nabla w_\sigma|^2 + |\partial_t w_\sigma|^2\right) e^{2\lambda\varphi}\,dq \leqslant Ce^{2\lambda(a+2\sigma)} \|w\|^2_{H^1(Q_{a+\sigma})} + Ce^{2\lambda m} \|Lw\|^2_{L^2(Q'_T)}\,. \tag{4.99}$$

Here we have used that $\varphi(x,t) < a + 2\sigma$ on $Q_a \setminus Q_{a+2\sigma}$. Replacing $Q_{a+\sigma}$ with $Q_{a+3\sigma}$ on the left hand side and estimating $\varphi(x,t)$ by $a + 3\sigma$ from below there, we have for $\lambda > 1$:

$$\|w\|^2_{H^1(Q_{a+3\sigma})} \leqslant C\left(e^{-2\lambda\sigma} \|w\|^2_{H^1(Q'_T)} + Ce^{2\lambda m} \|Lw\|_{L^2(Q'_T)}\right)\,. \tag{4.100}$$

□

We now eliminate the H^1-norm of the function on the right hand side and extend the estimate to Q'_T by applying standard energy estimates for hyperbolic operators:

Theorem 4.3.2 (Observability inequality). *Assume that $\Omega \subset \{x \in \mathbb{R}^n : |x| < R\}$ and (4.20) holds for an $x_0 \in \Omega$. If $T > \frac{2R}{\sqrt{\beta}}$, then there exists a constant $C > 0$, such that for all $w \in H^2_0(Q'_T)$, the following estimate holds:*

$$\|w\|^2_{H^1(Q'_T)} \leqslant C \|Lw\|^2_{L^2(Q'_T)}\,. \tag{4.101}$$

Proof. Since the gradient of the Carleman weight function (4.18) must not vanish inside $Q_{a+\sigma}$ in order for the Carleman estimate to hold, the point x_0 cannot lie within this domain. Hence, we need two weight functions centered on different points $x_0, x_1 \in \Omega$, chosen so that the corresponding domains overlap $\Omega \times [-\delta, \delta]$ for some $\delta > 0$, while their level sets intersect $\partial\Omega$ only on S'_T (cf. Figure 4.2).

We therefore proceed as follows: First we pick an $x_0 \in \Omega$ so that (4.20) is satisfied. By continuity of the inner product, there exists an $\varepsilon > 0$, such that $B_\varepsilon(x_0) \subset \Omega$ and for all $x_1 \in B_\varepsilon(x_0)$, (4.20) is also satisfied. Now choose $a > 0$ and$\sigma > 0$ so that the following hold:

$$6\sqrt{a + 4\sigma} < \varepsilon\,, \qquad \frac{(2R + 3\sqrt{a + 4\sigma})^2 + a}{\beta} < T^2\,. \tag{4.102}$$

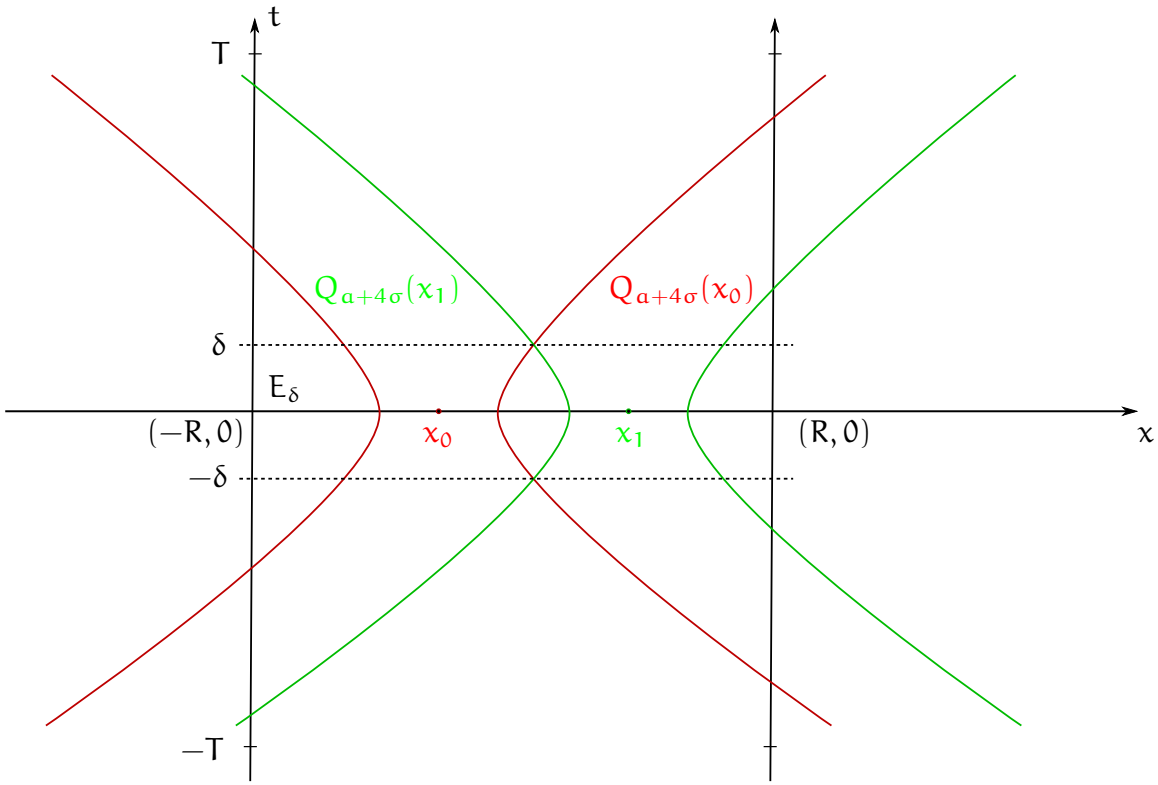


Figure 4.2: Geometric situation for the proof of the observability estimate in (x,t)-coordinates for the case $\Omega = [-R,R]$. The domains $Q_{a+4\sigma}(x_0)$, $Q_{a+4\sigma}(x_1)$ intersect $\{|x| = R\}$ for $|t| < T$, while their union covers the domain $E_\delta = [-R,R] \times [-\delta,\delta]$.

This choice is possible from the assumption that $T > \frac{2R}{\sqrt{\beta}}$. Then, fix $x_1 \in B_\varepsilon(x_0)$ with $|x_0 - x_1| = 3\sqrt{a+4\sigma}$, and set

$$Q_\sigma(x_0) := \left\{(x,t) \in Q_T' : \ |x-x_0|^2 - \beta t^2 > \sigma\right\}, \tag{4.103}$$

$$Q_\sigma(x_1) := \left\{(x,t) \in Q_T' : \ |x-x_1|^2 - \beta t^2 > \sigma\right\}. \tag{4.104}$$

Since $B_\varepsilon(x_0) \setminus B_{\sqrt{a+4\sigma}}(x_0) \neq \emptyset$ by (4.102), the same holds for the sets

$$Q_{a+4\sigma}(x_0) \subset Q_{a+3\sigma}(x_0) \subset Q_{a+2\sigma}(x_0) \subset Q_{a+\sigma}(x_0) \subset Q_a(x_0), \tag{4.105}$$

in which the former is contained. Similarly, $Q_{a+4\sigma}(x_1) \neq \emptyset$, since for all $x \in B_{\sqrt{a+4\sigma}}(x_0)$, we can show that $(x,0) \in Q_{a+4\sigma}(x_1)$:

$$\begin{aligned} |x-x_1| = |(x_1-x_0) - (x-x_0)| &\geqslant |x_0-x_1| - |x-x_0| \\ &\geqslant 3\sqrt{a+4\sigma} - \sqrt{a+4\sigma} = 2\sqrt{a+4\sigma} > \sqrt{a+4\sigma}. \end{aligned} \tag{4.106}$$

On the other hand, as $T > \frac{2R}{\sqrt{\beta}}$, we have for all $x \in \overline{\Omega}$ that $|x-x_0|^2 - \beta T^2 < 4R^2 - 4R^2 = 0$, and so for $a > 0$, it holds that $\overline{Q}_a(x_0) \cap \{|t| = T\} = \emptyset$. Because of

(4.102), for all $x \in \overline{\Omega}$,

$$|x-x_1|^2-\beta T^2 \leqslant (|x-x_0|+|x_0-x_1|)^2-\beta T^2 \leqslant (2R+3\sqrt{a+4\sigma})^2-(2R+3\sqrt{a+4\sigma})^2-a=a\,, \tag{4.107}$$

hence for $a>0$, $\overline{Q}_a(x_1)\cap\{|t|=T\}=\emptyset$ as well.

Therefore, we can apply the Hölder estimate (4.93) in $Q_{a+3\sigma}(x_0)$ and $Q_{a+3\sigma}(x_1)$ to obtain for all $w\in H_0^2(Q_T')$:

$$\|w\|^2_{H^1(Q_{a+3\sigma}(x_0))} \leqslant C\left(e^{-2\lambda\sigma}\|w\|^2_{H^1(Q_T')}+e^{2\lambda m}\|Lw\|^2_{L^2(Q_T')}\right), \tag{4.108}$$

$$\|w\|^2_{H^1(Q_{a+3\sigma}(x_1))} \leqslant C\left(e^{-2\lambda\sigma}\|w\|^2_{H^1(Q_T')}+e^{2\lambda m}\|Lw\|^2_{L^2(Q_T')}\right). \tag{4.109}$$

We next show that we can combine these estimates so that the left hand side becomes an integral over a domain containing $\Omega\times[-\delta,\delta]$ for some $\delta>0$. Set

$$G_0 := \left\{(x,t)\in Q_T' :\ |x-x_0|^2>a+4\sigma\right\}\cap Q_{a+3\sigma}(x_0), \tag{4.110}$$

$$G_1 := \left\{(x,t)\in Q_T' :\ |x-x_1|^2>a+4\sigma\right\}\cap Q_{a+3\sigma}(x_1). \tag{4.111}$$

By the definition of $Q_{a+3\sigma}(x_0)$ and $Q_{a+3\sigma}(x_1)$, it is clear that

$$\left\{(x,t)\in Q_T' :\ \beta t^2<\sigma,\ |x-x_0|^2>a+4\sigma\right\}\subset G_0\,, \tag{4.112}$$

$$\left\{(x,t)\in Q_T' :\ \beta t^2<\sigma,\ |x-x_1|^2>a+4\sigma\right\}\subset G_1\,. \tag{4.113}$$

But since for $x\in Q_T'$ with $|x-x_0|^2<a+4\sigma$, we have by (4.106) that $|x-x_1|^2>a+4\sigma$, it follows that

$$\left(Q_T'\cap\{\beta t^2<\sigma\}\right)\setminus G_0=\left\{(x,t)\in Q_T' :\ \beta t^2<\sigma,\ |x-x_1|^2<a+4\sigma\right\}\subset G_1\,. \tag{4.114}$$

Hence, if we set $\delta:=\sqrt{\frac{\sigma}{\beta}}$, we have that

$$E_\delta := \left\{(x,t)\in Q_T' :\ t^2<\delta^2,\ x\in\Omega\right\}\subset G_0\cup G_1\,, \tag{4.115}$$

and so by combining the inequalities (4.108) and (4.109):

$$\|w\|^2_{H^1(E_\delta)} \leqslant 2C\left(e^{-2\lambda\sigma}\|w\|^2_{H^1(Q_T')}+e^{2\lambda m}\|Lw\|^2_{L^2(Q_T')}\right). \tag{4.116}$$

Now by writing

$$\|w\|^2_{H^1(E_\delta)}=\int_{-\delta}^{\delta}\int_\Omega w^2(x,t)+|\nabla w(x,t)|^2+(\partial_t w(x,t))^2\,dx\,dt\,, \tag{4.117}$$

we can deduce, from elementary properties of the Lebesgue integral, the existence of a $t_1 \in (-\delta, \delta)$ such that

$$\begin{aligned} \|w\|^2_{H^1(E_\delta)} &\geqslant 2\delta \int_\Omega w^2(x,t_1) + |\nabla w(x,t_1)|^2 + (\partial_t w(x,t_1))^2 \, dx \\ &= 2\delta \left(\|w(\cdot,t_1)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t_1)\|^2_{L^2(\Omega)} \right) , \end{aligned} \tag{4.118}$$

and thus, by way of (4.116),

$$\begin{aligned} &\|w(\cdot,t_1)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t_1)\|^2_{L^2(\Omega)} \\ &\qquad \leqslant \frac{C}{\delta} \left(e^{-2\lambda\sigma} \|w\|^2_{H^1(Q'_T)} + e^{2\lambda m} \|Lw\|^2_{L^2(Q'_T)} \right) . \end{aligned} \tag{4.119}$$

We complete the proof by using the above to derive the desired estimate from a standard energy inequality. Obviously, any $w \in H^2_0(Q_T)$ satisfies the following hyperbolic equation:

$$\begin{cases} L(w(x,t)) & = (Lw)(x,t) & (x,t) \in \Omega \times [t_1, T] \, , \\ w(x,t)|_{t=t_1} & = w(x,t_1) & x \in \Omega \, , \\ \partial_t w(x,t)|_{t=t_1} & = \partial_t w(x,t_1) & x \in \Omega \, , \\ w(x,t) = \partial_\nu w(x,t) & \equiv 0 & (x,t) \in \partial\Omega \times [t_1, T] \, . \end{cases} \tag{4.120}$$

Now we can apply the standard energy estimate (see, e.g. [25, Prop. 2.11]) to this equation for $t_2 \in (t_1, T)$

$$\begin{aligned} &\|w(\cdot,t_2)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t_2)\|^2_{L^2(\Omega)} \\ &\qquad \leqslant C_1 \left(e^{C_2(t_2-t_1)} \left(\|w(\cdot,t_1)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t_1)\|^2_{L^2(\Omega)} \right) \right. \\ &\qquad\qquad \left. + \int_{t_1}^{t_2} e^{C_2(t_2-t)} \|(Lw)(\cdot,t)\|^2_{L^2(\Omega)} \, dt \right) , \end{aligned} \tag{4.121}$$

where C_1, C_2 are constants independent of w, t_2 and t_1. Integrating both sides over t_2 from t_1 to T and estimating the second integral on the right hand side by its maximum at $t_2 = T$, we get:

$$\begin{aligned} &\int_{t_1}^T \|w(\cdot,t)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t)\|^2_{L^2(\Omega)} \, dt \\ &\qquad \leqslant C_3 T \left(\|w(\cdot,t_1)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot,t_1)\|^2_{L^2(\Omega)} + \int_{t_1}^T \|(Lw)(\cdot,t)\|^2_{L^2(\Omega)} \, dt \right) . \end{aligned} \tag{4.122}$$

Since L is a time reversible operator, we can repeat this procedure for $t \in (-T, t_1)$ after the transformation $\tau = -t$ and get a similar estimate on $\Omega \times [-T, t_1]$. Summing up both estimates gives:

$$\|w\|^2_{H^1(Q'_T)} \leqslant C_3 2T \left(\|w(\cdot, t_1)\|^2_{H^1(\Omega)} + \|\partial_t w(\cdot, t_1)\|^2_{L^2(\Omega)} + \|Lw\|^2_{L^2(Q'_T)} \right). \tag{4.123}$$

Finally, we insert the estimate (4.119) on the right hand side, yielding

$$\|w\|^2_{H^1(Q'_T)} \leqslant C_4 \left(e^{-2\lambda\sigma} \|w\|^2_{H^1(Q'_T)} + \|Lw\|^2_{L^2(Q'_T)} \right). \tag{4.124}$$

Taking now $\lambda > \lambda_0$ large enough so that $C_4 e^{-2\lambda\sigma} < 1$ holds, we can absorb the $H^1(Q'_T)$ norm into the left hand side and arrive at the desired estimate. □

Remark 4.3. If (4.20) holds for $|x_0| < c$, then $|x - x_0| < R + c$ for all $x \in \Omega$, and so we can replace $2R$ by R in the proof above and weaken the hypothesis on T here and below to $T > \frac{R}{\sqrt{\beta}}$. For a constant coefficient c, this becomes $T > \frac{R}{c}$, the same bound as obtained by the multiplier method (see, e.g., [20]).

Theorem 4.3.3 (Lipschitz stability). *Let $u^* \in H^2_0(Q'_T)$ be a solution of (3.14). Assume there exists an even function $\Phi^\delta \in H^2(Q'_T)$ which satisfies*

$$\begin{cases} \Phi^\delta(x,t) &= \varphi_0^\delta(x,t) \quad (x,t) \in S_T \ , \\ \partial_\nu \Phi^\delta(x,t) &= \varphi_1^\delta(x,t) \quad (x,t) \in S_T \ , \end{cases} \tag{4.125}$$

and that

$$\left\| \Phi - \Phi^\delta \right\|_{H^2(Q'_T)} \leqslant \delta. \tag{4.126}$$

Denote by $u_\varepsilon^\delta \in H^2_0(Q'_T)$ the solution of Problem 3.1 with Φ^δ replacing Φ.

Assume further that $\Omega \subset \{x \in \mathbb{R}^n : |x| < R\}$ and (4.20) holds for an $x_0 \in \Omega$. If $T > \frac{2R}{\sqrt{\beta}}$, there exists a $C > 0$ such that

$$\left\| u^* - u_\varepsilon^\delta \right\|^2_{H^1(Q'_T)} \leqslant C \left(\delta^2 + \varepsilon \|u^*\|^2_{H^2(Q'_T)} \right). \tag{4.127}$$

Proof. Set $w := u^* - u_\varepsilon^\delta$. Then $w \in H^2_0(Q'_T)$, so we can apply the observability estimate (4.101) to the difference:

$$\|w\|^2_{H^1(Q'_T)} \leqslant C_1 \|Lw\|^2_{L^2(Q'_T)} \ . \tag{4.128}$$

Estimating the right hand side by Lemma 4.1.8 yields

$$\|w\|^2_{H^1(Q'_T)} \leqslant C_2 \left(\delta^2 + \varepsilon \|u^*\|^2_{H^2(Q'_T)} \right), \tag{4.129}$$

which completes the proof. □

4.3.1. Lipschitz stability for partial boundary data

If we assume that the support of the unknown initial condition can be suitably bounded and the variations in sound speed are small, then due to the finite speed of propagation of waves, the convergence and stability of the quasi-reversibility approximation for the solution of Problem 2.3 in a quadrant can be derived from the theorems above. The following theorem is an extension of the results of [31] for variable coefficients in the principal part of the operator:

Lemma 4.3.4 (Domain of dependence). *For* $R > 0$, *let* $D_R := \{x \in D : |x| < R\}$ *be a sector of the first quadrant of* $\mathbb{R}^2$. *If* $u(x,t)$ *is a solution of* (2.12) *and* $\operatorname{supp} u(x,0) \subset D_R$, *then* $u(x,t) = 0$ *for all* $t \in [-T,T]$ *and* $|x| > c_{max}T + R$.

Proof. Consider an arbitrary point $x_0 \in D$ with $|x_0| > c_{max}T+R$. By the standard energy method, we know (see, e.g., [25, Th. 2.3]) that $u(x_0,t)$ depends only on the cone

$$\Delta(x_0,T) := \left\{(x,t) \in \mathbb{R}^2 \times [0,T] : |x-x_0| \leqslant c_{max}(T-t)\right\}. \tag{4.130}$$

Since $\Delta(x_0,T)|_{t=0} \cap D_R = \emptyset$, we have that $u(x,t) = 0$ in $\Delta(x_0,T)$ and hence $u(x_0,t) = 0$ for all $t \in [0,T]$. Since we took the even extension of u in $\Omega \times [-T,0]$, this holds for $t \in [-T,t]$ as well. □

Theorem 4.3.5 (Lipschitz stability in a quadrant). *Assume that in addition to* (4.20), $1 < c(x) < 3.5$ *and* $\left\langle \nabla c^{-2}(x), x - x_0 \right\rangle_n < \frac{1}{4}$ *are satisfied, or that* $c(x) \equiv c$ *is constant. If* $\operatorname{supp} u(x,0) \subset D_R$, $T > \frac{(2R+\eta)}{\sqrt{2\beta}-c_{max}}$ *holds for an* $\eta > 0$, *and* φ_0, φ_1 *is given on* $\Gamma_T' := \partial D \cap \{|x| < R + c_{max}T + \eta\} \times [-T,T]$, *the Lipschitz stability estimate from Theorem 4.3.3 is valid for* $D \times [-T,T]$.

Proof. Set $\tilde{R} := R + \eta + c_{max}T$ and $Q_R := D_{\tilde{R}} \times [-T,T]$. Then the boundary $\partial Q_R = \Gamma_1 \cup \Gamma_2$ consists of the two parts:

$$\Gamma_1 := \left\{x \in D : \ x_1 = 0 \text{ and } 0 \leqslant x_2 < \tilde{R}\right\} \cup \left\{x \in D : \ x_2 = 0 \text{ and } 0 \leqslant x_1 < \tilde{R}\right\}, \tag{4.131}$$

$$\Gamma_2 := \left\{x \in D : \ |x|^2 = \tilde{R}\right\}. \tag{4.132}$$

Due to Lemma 4.3.4, we have $u(x,t) = 0$ for all $x \in D$ with $|x - x_0| < \eta$ for an $x_0 \in \Gamma_2$, such that $u|_{\Gamma_2}(t) = \partial_\nu u|_{\Gamma_2}(t) \equiv 0$ for all $t \in [-T,T]$. We set

$$\psi_0(x,t) := \begin{cases} \varphi_0(x,t) & \text{on } \Gamma_1 \times [-T,T], \\ 0 & \text{on } \Gamma_2 \times [-T,T], \end{cases} \tag{4.133}$$

and

$$\psi_1(x,t) := \begin{cases} \varphi_1(x,t) & \text{on } \Gamma_1 \times [-T,T], \\ 0 & \text{on } \Gamma_2 \times [-T,T]. \end{cases} \tag{4.134}$$

Now, since $D_{\tilde{R}}$ is contained in a ball of radius $\frac{\sqrt{2}}{2}\tilde{R}$ and by the assumption $T > \frac{2\tilde{R}}{\sqrt{2\beta}}$, we can apply the stability theorem for the bounded domain Q_R with complete boundary data ψ_0, ψ_1. The conditions on c ensure that $\sqrt{2\beta} - c_{max}$ is greater than zero. □

For constant coefficients, the assumptions are always satisfied (since in that case, $\beta < c$ and so the requirements are reduced to the tautology $c < \sqrt{2}c$.) For variable coefficients, the requirements on $c(x)$ are obviously not sharp, and it is evident that the smaller ∇c^{-2} is, the larger we can take the bounds on $c(x)$ itself. In order to apply Theorem 4.3.5, the coefficient $c(x)$ must satisfy both (4.20) and the requirements above, and so this Lipschitz stability result for the case of Cauchy data given on a part of the boundary is only applicable to "almost constant" coefficients. This is clearly not satisfactory.

If only one boundary condition is given on a part of the boundary, while the other is given on the whole boundary, we can hope for a stronger result. Indeed, if the Neumann condition is prescribed only on a (specific) part of the boundary, by a slight modification of the arguments of this chapter, the observability estimate (4.101) still holds, and thus we can show Lipschitz stability for the quasi-reversibility approximation. This could be applied for time reversal in the case when the domain under consideration is bounded on one side by the measurement surface, and by a sound-soft obstacle on the other side.

Theorem 4.3.6 (Observability estimate from partial Neumann data). *Assume that* $\Omega \subset \{x \in \mathbb{R}^n : |x| < R\}$ *and* (4.20) *holds for an* $x_0 \in \mathbb{R}^n$ *with* $|x_0| > R$. *Set* $r := \max_{x\in\Omega}(x - x_0)$. *If* $T > \frac{r}{\sqrt{\beta}}$, *then there exists a constant* $C > 0$, *such that for all* $w \in H^2(Q_T')$ *with* $w = 0$ *on* S_T' *and* $\nabla w = 0$ *on*

$$\Gamma_+ := \{(x,t) \in S_T' : \langle x - x_0, \nu\rangle_n > 0\}, \tag{4.135}$$

the following estimate holds:

$$\|w\|^2_{H^1(Q_T')} \leqslant C \|Lw\|^2_{L^2(Q_T')}. \tag{4.136}$$

Proof. Choose $\sigma^2 \in (0, |x_0| - R)$. Then, for all $x \in \overline{\Omega}$, we have that

$$\varphi(x,T) = |x - x_0|^2 - \beta T^2 < r^2 - r^2 < \sigma \tag{4.137}$$

by the definition of r and the condition on T, and thus, $\overline{Q}_\sigma \cap \{|t| = T\} = \emptyset$. In addition, for $t^2 < \frac{(|x_0|-R)^2-\sigma}{\beta}$, we have for all $x \in \Omega$:

$$\varphi(x,t) = |x-x_0|^2 - \beta t^2 > (|x_0|-R)^2 - (|x_0|-R)^2 + \sigma > \sigma\,. \tag{4.138}$$

Such a choice of t is possible, since $T^2 > \frac{r^2}{\beta} > \frac{(|x_0|-R)^2-\sigma}{\beta} > 0$ by the definition of r and σ. Hence, setting $\delta^2 := \frac{(|x_0|-R)^2-\sigma}{\beta}$, we see that $E_\delta := \Omega \times (-\delta,\delta) \subset Q_\sigma$, and particularly, that $Q_\sigma \neq \emptyset$.

Now we show that the Carleman estimate (4.56) is applicable to w in Q_σ. For this, we need only to verify that we can take Γ_+ instead of S_T' for the integral of $\partial_\nu w$. Looking at (4.62), we see that the boundary integral can be decomposed into integrals over Γ_+ and $\Gamma_- := S_T' \setminus \Gamma_+$:

$$\begin{aligned} -\int_{S_T'} \langle U,\nu\rangle_{n+1}\,\mathrm{d}s \\ = 4\lambda \int_{\Gamma_+} (\partial_\nu v)^2 \langle x-x_0,\nu\rangle_n\,\mathrm{d}s + 4\lambda \int_{\Gamma_-} (\partial_\nu v)^2 \langle x-x_0,\nu\rangle_n\,\mathrm{d}s \\ \leqslant 4\lambda \int_{\Gamma_+} (\partial_\nu v)^2 \langle x-x_0,\nu\rangle_n\,\mathrm{d}s\,, \end{aligned} \tag{4.139}$$

since $\langle x-x_0,\nu\rangle_n \leqslant 0$ for $x \in \Gamma_-$ by assumption.

Consequently, we can apply Lemma 4.3.1 to w in Q_σ, which yields the following estimate:

$$\|w\|^2_{H^1(E_\delta)} \leqslant C\left(e^{-2\lambda\sigma}\|w\|^2_{H^1(Q_T')} + e^{2\lambda m}\|Lw\|^2_{L^2(Q_T')}\right). \tag{4.140}$$

From here on, the proof follows that of Theorem 4.3.3. □

The rest of this work is concerned with the numerical solution of the problem of quasi-reversibility.

5. Numerical solution

For the numerical solution of the quasi-reversibility Problem 3.1, we employ a Ritz-Galerkin approximation; that is, we look for the solution in a finite dimensional subspace S_h of $H_0^2(Q_T)$ or $H_\Gamma^2(Q_T)$. If we choose a subspace of piecewise polynomials, the approximating functions need to be at least once continuously differentiable everywhere. Of the classical finite element spaces, the Argyris and Bell triangles as well as the Bogner-Fox-Schmit rectangles (see, e.g., [12, Th. 2.2.13 and 2.2.15]) provide the requisite regularity. However, they are difficult to handle, especially in higher dimensions (to the authors' knowledge, no implementations of C^1-elements in three or more dimensions are readily available). We therefore choose a different local basis of the space of piecewise polynomials, that of B-splines, which is easier to implement. Another advantage of B-splines is the natural way to deal with the boundary conditions via (3.13). A different alternative would be the development of a mixed formulation of Problem 3.1 (cf. [6, 7] for the Laplace equation), but this approach involves the introduction of a second regularization parameter.

We first choose our finite dimensional subspace in Section 5.1 and give a result on its approximation properties, which will enable us to show error estimates for the Ritz-Galerkin approximation. In Section 5.2, we discuss the choice of a convenient basis, and with it the reduction of the approximating problem to a system of linear equations and its solution.

5.1. Ritz-Galerkin approximation

We now choose the appropriate finite dimensional subspace for our Ritz-Galerkin approximation. As it is well known in the theory of finite elements that piecewise smooth functions which are at least once continuously differentiable everywhere are also in H^2 (cf. [8, Th. II.5.2]), we consider the space of piecewise polynomials on Q_T which satisfy the necessary differentiability conditions at the break points. These are the classical spline spaces of Schoenberg [53]. To approximate our multivariate functions, we make use of tensor products of univariate splines, as is

now classical. We give a short overview of the salient facts; a thorough discussion can be found in [55].

Assume Q_T is contained in a cube $R := \prod_{i=1}^{n+1}[a_i, b_i] \subset (\mathbb{R}^n \times [0, T])$. We start by introducing a partition $\Delta := \bigotimes_{i=1}^{n+1} \Delta_i$ of R with:

$$\Delta_i := \{a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,k_i} < x_{i,k_i+1} = b_i\}, \tag{5.1}$$

where the x_i, k_i are called *break points*. The Δ_i generate the set of intervals $I_{i,j} := [x_{i,j}, x_{i,j+1})$ for $j = 0, \ldots, k_i - 1$, and $I_{i,k} := [x_{i,k_i}, x_{i,k_i+1}]$. With these partitions, we associate mesh widths

$$h_i := \max_{j\in\{0,\ldots,k_i\}}(x_{i,j+1} - x_{i,j}), \quad \overline{h} := \max_{i\in\{1,\ldots,n+1\}} h_i, \quad \underline{h} := \min_{i\in\{1,\ldots,n+1\}} h_i. \tag{5.2}$$

A set of partitions $\mathfrak{S}$ is called *quasi-uniform* if there exists a constant $K > 0$, such that for each partition $\Delta \in \mathfrak{S}$, the mesh widths satisfy $\overline{h}/\underline{h} \leqslant K$. A partition belonging to such a set will also be called quasi-uniform. The univariate spline spaces S_i^d on a partition Δ_i are then defined as:

$$S_i^d := \left\{s \in C^{d-2}([a_i, b_i]) : \ s|_{I_{i,j}} \in \mathcal{P}^d \text{ for all } j = 0, \ldots, k_i\right\}, \tag{5.3}$$

where $\mathcal{P}^d := \left\{p(x) : \ p(x) = \sum_{i=1}^{d} c_i x^{i-1}, \ c_1, \ldots, c_d \in \mathbb{R}, x \in \mathbb{R}\right\}$ is the space of polynomials of order d. Our multivariate spline space S^d is then the tensor product of these spaces:

$$S^d := \bigotimes_{i=1}^{n+1} S_i^d = \operatorname{span}\left\{\prod_{i=1}^{n+1} s_i : \ s_i \in S_i^d\right\}. \tag{5.4}$$

This is a linear space of dimension $\prod_{i=1}^{n+1}(d + k_i)$. We will give a basis with convenient properties in Section 5.2. For arbitrary domains $\Omega \subset R$, we define the spline space

$$S^d(Q_T) := \left\{s|_{Q_T} : \ s \in S^d\right\}. \tag{5.5}$$

By construction, each $s \in S^d(Q_T)$ is in $C^{d-2}(Q_T)$, and, if all but one variable are fixed, its restriction on a cube $\prod_{i=1}^{n+1} I_{i,j_i}$ is a polynomial of order d. From this, and the above cited result, we have that $S^d(Q_T) \subset H^{d-1}(Q_T)$.

It remains to give bounds for the approximation error of functions in the Sobolev spaces H^r by functions in the spline spaces $S^d(Q_T)$. The following theorem is a corollary of Theorem 4.5 in [54]:

Theorem 5.1.1 (Approximation in Sobolev spaces by splines). *Assume that $Q_T = R$ or that $Q_T \subset R$ is a Lipschitz domain. If $S^d(Q_T)$ is defined on a quasi-uniform*

partition of mesh width $\overline{h}$, *there exists a constant* $C > 0$ *such that the following estimate holds for all* $d \geqslant 2$ *and* $u \in H^{d-1}(Q_T)$*:*

$$\inf_{s \in S^d} \|u - s\|_{H^{d-2}(Q_T)} \leqslant C\overline{h} \|u\|_{H^{d-1}(Q_T)} . \tag{5.6}$$

Remark 5.1. This shows that spline spaces are optimal; no other approximation spaces of the same dimension give asymptotically smaller errors.

Thus, if we wish to construct an approximation of $u \in H^2(Q_T)$, we have to look at functions in $S^d(Q_T)$ with $d = 4$. An approximate solution u_h to (3.16) in a finite dimensional subspace of $H_0^2(Q_T)$ then has to be a function in $S^4(Q_T)$ which vanishes on $\partial\Omega$ together with its normal derivative.

For this subspace, we can give an error estimate for the Ritz-Galerkin approximation of Problem 3.1. For a given partition Δ of Q_T with mesh width $\overline{h}$, we define

$$S_h := S^4(Q_T) \cap H_0^2(Q_T). \tag{5.7}$$

Problem 5.1 (Ritz-Galerkin approximation). Given $\Phi \in H^2(Q_T)$, $c \in C^1(\overline{\Omega})$, $\varepsilon > 0$, find $u_h \in S_h$ which satisfies for all $v_h \in S_h$:

$$M_\varepsilon(u_h, v_h) = -\int_{Q_T} L\Phi \, Lv_h \, dq . \tag{5.8}$$

The existence and uniqueness of the solution to this problem follows again immediately from Riesz' representation theorem and the fact that M_ε is elliptic on $S_h \subset H_0^2(Q_T)$ as well.

Theorem 5.1.2 (Error estimate for the Ritz-Galerkin approximation). *Let* $\mathfrak{S}$ *be a quasi-uniform set of partitions of* Q_T*, and* u_ε *be a solution of Problem 3.1. For each partition* $\Delta \in \mathfrak{S}$ *with mesh width* $\overline{h}$*, let* $u_h \in S_h$ *denote the solution of Problem 5.1. Then there exists a constant* $C > 0$*, such that the following estimate holds:*

$$\|u_\varepsilon - u_h\|_{H^2(Q_T)} \leqslant C\overline{h} \|u_\varepsilon\|_{H^3(Q_T)} . \tag{5.9}$$

Proof. Since by Lemma 3.2.2 we know that M_ε is a continuous $H_0^2(Q_T)$-elliptic bilinear form and the right hand side of (5.8) is a continuous linear functional, by Céa's lemma (see, e.g., [12, Th. 2.4.1]) the approximation error is bounded by the interpolation error. The latter can be estimated in $H^2(Q_T)$ by Theorem 5.1.1, since we know from Theorem 3.2.1 that $u_\varepsilon \in H^3(Q_T)$:

$$\inf_{s_h \in S^4} \|u_\varepsilon - s_h\|_{H^2(Q_T)} \leqslant C\overline{h} \|u_\varepsilon\|_{H^3(Q_T)} . \tag{5.10}$$

Now we only have to show that this infimum is attained in $H^2_0(Q_T)$. For this, as $H^2_0(Q_T)$ is a closed subspace of $H^2(Q_T)$, we invoke the orthogonal projection theorem (see, e.g., [65, Th. III.1]). Thus, we can write $s_h \in \mathcal{S}^4$ as $s_h = s_h^0 + s_h^\perp$ with $s_h^0 \in H^2_0(Q_T)$ and $s_h^\perp \in (H^2_0(Q_T))^\perp$, which denotes the orthogonal complement of $H^2_0(Q_T)$ in $H^2(Q_T)$. Particularly, $s_h^0 \in S_h = \mathcal{S}^4(Q_T) \cap H^2_0(Q_T)$ and $s_h^\perp \in \mathcal{S}^4(Q_T) \cap (H^2_0(Q_T))^\perp$. We then have for $u_\varepsilon \in H^2_0(Q_T)$:

$$
\begin{aligned}
(5.11)\quad \inf_{s_h \in \mathcal{S}^4(Q_T)} \|u_\varepsilon - s_h\|_{H^2(Q_T)} &= \inf_{s_h \in \mathcal{S}^4(Q_T)} \left\|(u_\varepsilon - s_h^0) - s_h^\perp\right\|_{H^2(Q_T)} \\
&= \inf_{s_h \in \mathcal{S}^4(Q_T)} \left(\left\|(u_\varepsilon - s_h^0)\right\|_{H^2(Q_T)} + \left\|s_h^\perp\right\|_{H^2(Q_T)} - \left\langle (u_\varepsilon - s_h^0), s_h^\perp \right\rangle_{H^2(Q_T)} \right) \\
&= \inf_{s_h^0 \in S_h} \left\|(u_\varepsilon - s_h^0)\right\|_{H^2(Q_T)} + \inf_{s_h^\perp \in \mathcal{S}^4} \left\|s_h^\perp\right\|_{H^2(Q_T)} = \inf_{s_h \in S_h} \|u_\varepsilon - s_h\|_{H^2(Q_T)} ,
\end{aligned}
$$

because $(u_\varepsilon - s_h^0) \in H^2_0(Q_T)$ and $S_h = \mathcal{S}^4 \cap H^2_0(Q_T)$. Now Céa's lemma and the bound above give:

$$
(5.12)\quad \|u_\varepsilon - u_h\|_{H^2(Q_T)} \leqslant \frac{c_1 + \varepsilon}{c_2 \varepsilon} \inf_{s_h \in S_h} \|u_\varepsilon - s_h\|_{H^2(Q_T)} \leqslant \frac{c_3}{c_2} C \overline{h} \|u_\varepsilon\|_{H^3(Q_T)} ,
$$

where c_1, c_2 are the constants from Lemma 3.2.2 and c_3 is chosen so that $c_1 + \varepsilon \leqslant c_3 \varepsilon$. □

Again, these results carry over to the case of partial boundary data, if we consider $S_\Gamma := \mathcal{S}^4(Q_T) \cap H^2_\Gamma(Q_T)$ instead of S_h.

For general domains, the boundary conditions can be enforced by multiplication by a suitable weight function. This leads to weighted spline spaces, for which similar error bounds can be shown (cf. [21] for an overview). Since the required functions can be more easily characterized on rectangular domains $\Omega = R$ by taking an appropriate subset of $\mathcal{S}^4(Q_T)$, we restrict ourselves from now on to such domains.

5.2. Implementation

Now we turn our attention to the details of the implementation of the Galerkin approximation. For the numerical solution of the Ritz-Galerkin approximation, we transform Problem 5.1 into a system of linear equations by expressing u_h and v_h by the elements of a basis of S_h. A convenient choice are the cubic B-splines. We start by recalling the definition of univariate B-splines of order d and their relevant properties, following [13], to which we refer for proofs. To a partition[1] $\Delta = \{x_0, \ldots, x_{k+1}\}$ of the interval $[a, b] \subset \mathbb{R}$, we associate a knot vector

[1] For convenience, we drop the index i while discussing the univariate case.

$[t_0, \dots, t_{k+2d}]$ with $t_0 = t_1 = \cdots = t_{d-1} = x_0$, $t_{j+d} = x_j$ for $j \in \{0, \dots, k\}$, and $t_{k+d+1} = \cdots = t_{k+2d} = x_{k+1}$. For $j \in \{0, \dots, k+d+1\}$, the j-th normalized B-splines of order d can then be defined by the following recurrence relation:

$$B_j^d(x) := w_j^d(x) B_j^{d-1}(x) + (1 - w_{j+1}^d(x)) B_{j+1}^{d-1}(x), \tag{5.13a}$$

$$w_j^d(x) := \begin{cases} \frac{x - t_j}{t_{j+d-1} - t_j}, & \text{if } t_j \neq t_{j+d-1}, \\ 0, & \text{otherwise,} \end{cases} \tag{5.13b}$$

$$B_j^1(x) := \begin{cases} 1, & \text{if } t_j \leqslant x < t_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.13c}$$

Note that the B_j^d are identically zero for $j = 0$ and $j = k + d + 1$. The cubic B-splines $B_1^4(x), \dots, B_7^4(x)$ defined on the partition $\Delta = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ are shown in Figure 5.1, together with their first two derivatives.

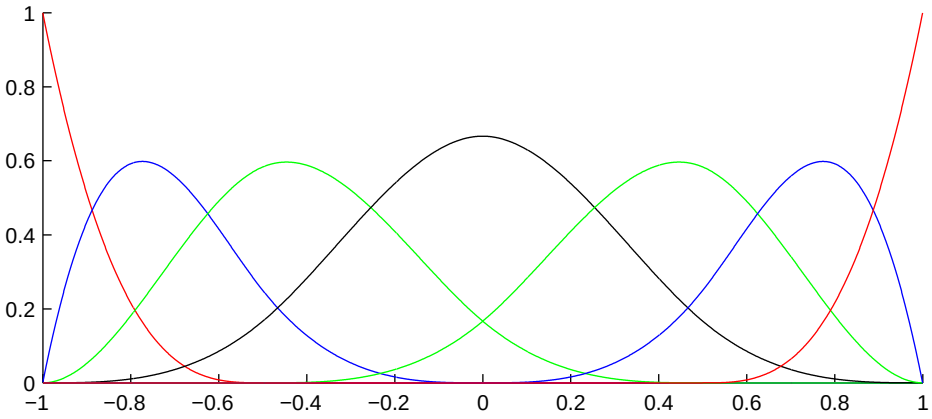
These functions have several convenient properties, which make them suitable for the purpose of Ritz-Galerkin approximation:

1. The B_j^d form a partition of unity: $B_j^d(x) \geqslant 0$ and $\sum_{j=1}^{k+d} B_j^d(x) = 1$ for all $x \in [a, b]$.
2. The B_j^d have local support: $\operatorname{supp} B_j^d = [t_j, t_{j+d})$.
3. On each interval $[x_j, x_{j+1})$, every B_j^d is a polynomial of order d.
4. The derivative of B_j^d is again a B-spline of order $d-1$, which can be calculated directly by the recurrence relation:

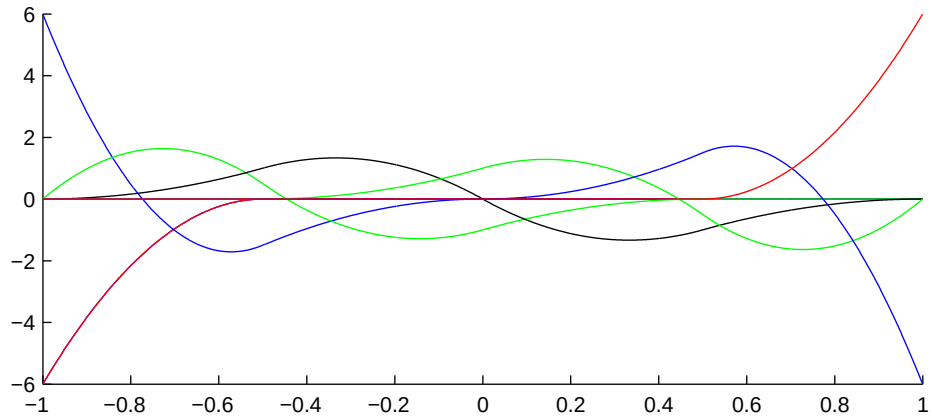
$$\frac{d}{dx} B_j^d(x) = \frac{d-1}{t_{j+d-1} - t_j} B_j^{d-1}(x) - \frac{d-1}{t_{j+d} - t_{j+1}} B_{j+1}^{d-1}(x). \tag{5.14}$$

5. The B_j^d are linear independent.

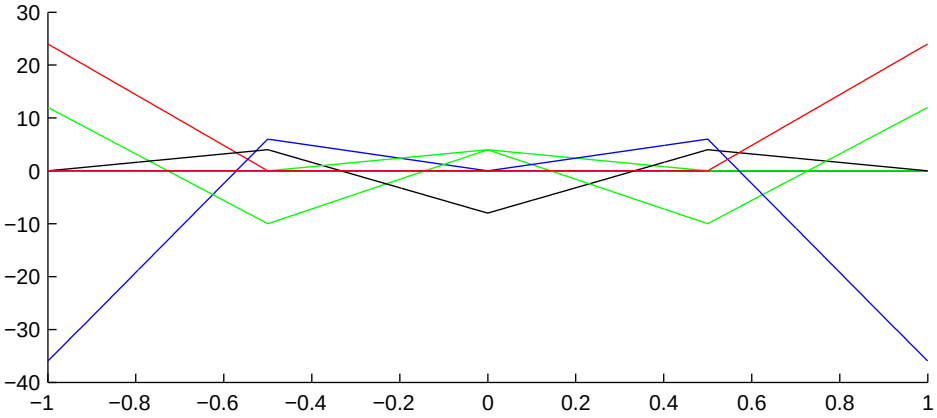
From the last three properties, it follows that the set $\mathcal{B} := \{B_1^d, \dots, B_{k+d}^d\}$ is a basis of S^d (cf. also [13, Th. IX.1]). Also, from the definition, we see that $B_j^d(x_0) \neq 0$ only for $j = 1$ and $B_j^d(x_{k+1}) \neq 0$ only for $j = k + d$. Using (5.14), a similar consideration for the derivatives of B_j^d at the boundary knots x_0 and x_{k+1} shows that apart from the derivatives of B_1^d and B_{k+d}^d, only $\frac{d}{dx} B_2^d$ and $\frac{d}{dx} B_{d+k-1}^d$ are not zero there. If we remove these, any linear combination of the remaining elements will vanish at x_0 and x_{k+1} together with its first derivative. The values at points not lying on the boundary of R, and those of higher derivatives of linear combinations, are not affected (cf. Figure 5.1).



(a) $B_j^4(x)$ for $j = 1, \ldots, 7$



(b) $\frac{d}{dx} B_j^4(x)$ for $j = 1, \ldots, 7$



(c) $\frac{d^2}{dx^2} B_j^4(x)$ for $j = 1, \ldots, 7$

Figure 5.1: Cubic B-splines on the partition $\Delta = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$ and derivatives.

Hence, if we define a knot vector $[t_{0,i}, \dots, t_{k_i+2d,i}]$ on each partition Δ_i, the tensor product of the $\mathcal{B}_i$ constitutes a basis $\mathcal{B}$ of the multivariate splines $\mathcal{S}^d(Q_T)$. Similarly, if we set $\mathcal{B}_i^0 := \{B_3^d, \dots, B_{k_i+d-2}^d\}$, the tensor product

$$\mathcal{B}^0 := \left(\bigotimes_{i=1}^{n} \mathcal{B}_i^0 \right) \otimes \mathcal{B}_{n+1} \tag{5.15}$$

is a basis of S_h (recalling that u, $\partial_\nu u$ vanish only on the time-like part of the boundary). In order to keep the presentation simple, we restrict ourselves to cubic B-splines in three dimensions from now. In this case, we have with $(x, y, t) \in \mathbb{R}^3$ the following representation:[2]

$$\begin{aligned} \mathcal{B} = \Big\{ & B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t), \\ & i \in \{1, \dots, k_1 + 4\}, j \in \{1, \dots, k_2 + 4\}, k \in \{1, \dots, k_3 + 4\} \Big\}, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \mathcal{B}^0 = \Big\{ & B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t), \\ & i \in \{3, \dots, k_1 + 2\}, j \in \{3, \dots, k_2 + 2\}, k \in \{1, \dots, k_3 + 4\} \Big\}. \end{aligned} \tag{5.17}$$

Using this basis $\mathcal{B}$, we can construct a function $\Phi \in H^2(Q_T)$ so that (3.13) is satisfied: Since the univariate cubic B-spline basis $\mathcal{B}_i$ has $k_i + 4$ elements, prescribing the values at the $k_i + 2$ partition points leaves two degrees of freedom per dimension, which can be used to prescribe the derivatives at the end points (so-called *complete cubic spline interpolation*, cf. [13, Ch.s IV and V]). For rectangular Ω, this translates exactly to the normal derivative at the boundary $\partial\Omega$. The value of Φ at partition points in the interior of R is simply set to zero. Evaluating each element of $\mathcal{B}$ at the partition points and adding a row each for the derivatives at the boundary points, we receive the cubic spline interpolation matrix A_j, $j = 1, 2, 3$. These tridiagonal matrices can be constructed analytically from (5.13) and (5.14), and the coefficients φ_{ijk} of the cubic B-spline interpolant Φ_h of Φ,

$$\Phi_h(x, y, t) = \sum_{i=1}^{k_1+4} \sum_{j=1}^{k_2+4} \sum_{k=1}^{k_3+4} \varphi_{ijk} B_{i,1}^4(x) B_{j,2}^4(y) B_{k,3}^4(t), \tag{5.18}$$

are the solution of a series of successive linear equations (cf. [13, Th. XVII.1]). The expression of Φ as a cubic B-spline will guarantee the necessary regularity of

[2] From here on, i, j, k will always denote indices of univariate spline basis elements, while $k_i + 2$ still denotes the number of breakpoints for the univariate splines of $\mathcal{B}_i$.

u_ε for Theorem 5.1.2 via Theorem 3.2.3, as well as facilitate the assembly of the system of linear equations.

Since S_h is a linear space, we can demand that equation (5.8) holds only for all $v_h \in \mathcal{B}^0$. We express u_h as a linear combination of elements of $\mathcal{B}^0$ as well:

$$u_h(x,y,t) = \sum_{i=3}^{k_1+2} \sum_{j=3}^{k_2+2} \sum_{k=1}^{k_3+4} a_{ijk} B^4_{i,1}(x) B^4_{j,2}(y) B^4_{k,3}(t). \tag{5.19}$$

Problem 5.1 can now be reduced to a system of linear equations for the coefficients a_{ijk}. In order to write this system in matrix-vector form, we set $N := (k_1+2)(k_2+2)(k_3+4)$ and use linear indices $\imath = \imath(i,j,k)$ of the coefficient tensor a_{ijk} and of the elements of $\mathcal{B}^0$:

$$\imath(i,j,k) := (k_1+2)(k_2+2)(k-1) + (k_1+2)(j-3) + i - 2\,. \tag{5.20}$$

In the same way we define $\jmath = \jmath(m,n,o)$:

$$\jmath(m,n,o) := (k_1+2)(k_2+2)(o-1) + (k_1+2)(n-3) + m - 2\,. \tag{5.21}$$

Then the system matrix M can be written as:

$$M := (M_\varepsilon(u_\imath, u_\jmath))_{\imath\jmath}\,,\ u_\imath, u_\jmath \in \mathcal{B}^0,\ \imath,\jmath = 1,\dots,N\,, \tag{5.22}$$

As the B-splines have compact support contained in Q_T, we can extend the integrals to $\mathbb{R}^3$. Denoting the inner product of u and v on $L^2(\mathbb{R})$ by $\langle u, v\rangle$ and the inner product on $L^2(\mathbb{R}^2)$ by $\langle u, v\rangle_2$, the matrix entries decompose into sums and products of inner products (for clarity, we drop the arguments and the superscript, and write $B''_{i,i}$ for the second derivative with respect to the argument):

$$\begin{aligned}
M_\varepsilon(u_\imath, u_\jmath) = {} & \left\langle c^{-2} B_{i,1} B_{j,2}, c^{-2} B_{m,1} B_{n,2} \right\rangle_2 \langle B''_{k,3}, B''_{o,3}\rangle \\
& - \left\langle c^{-2} B_{i,1} B_{j,2}, B''_{m,1} B_{n,2} \right\rangle_2 \langle B_{k,3}, B''_{o,3}\rangle \\
& - \left\langle c^{-2} B_{i,1} B_{j,2}, B_{m,1} B''_{n,2} \right\rangle_2 \langle B''_{k,3}, B_{o,3}\rangle \\
& - \left\langle c^{-2} B''_{i,1} B_{j,2}, B_{m,1} B_{n,2} \right\rangle_2 \langle B''_{k,3}, B_{o,3}\rangle \\
& - \left\langle c^{-2} B_{i,1} B''_{j,2}, B_{m,1} B_{n,2} \right\rangle_2 \langle B_{k,3}, B''_{o,3}\rangle \\
& + \langle B''_{i,1}, B_{m,1}\rangle \langle B_{j,2}, B''_{n,2}\rangle \langle B_{k,3}, B_{o,3}\rangle \\
& + \langle B_{i,1}, B''_{m,1}\rangle \left\langle B''_{j,2}, B_{n,2}\right\rangle \langle B_{k,3}, B_{o,3}\rangle \\
& + (1+\varepsilon) \langle B''_{i,1}, B''_{m,1}\rangle \langle B_{j,2}, B_{n,2}\rangle \langle B_{k,3}, B_{o,3}\rangle \\
& + (1+\varepsilon) \langle B_{i,1}, B_{m,1}\rangle \left\langle B''_{j,2}, B''_{n,2}\right\rangle \langle B_{k,3}, B_{o,3}\rangle \\
& + \varepsilon \langle B_{i,1}, B_{m,1}\rangle \langle B_{j,2}, B_{n,2}\rangle \langle B''_{k,3}, B''_{o,3}\rangle \\
& + \varepsilon \langle B_{i,1}, B_{m,1}\rangle \langle B_{j,2}, B_{n,2}\rangle \langle B_{k,3}, B_{o,3}\rangle\,.
\end{aligned} \tag{5.23}$$

Similarly, we calculate the right hand side

$$F := (F_\iota)_\iota\,,\ F_\iota := -\langle L\Phi, Lu_\iota\rangle_{L^2(Q_T)}\,,\ u_\iota \in \mathcal{B}^0\,,\ \iota = 1,\dots,N \tag{5.24}$$

by inserting (5.18), the cubic B-spline interpolant of Φ (which does not vanish on the spatial boundary):

$$\begin{aligned} F_\iota = \sum_{m=1}^{k_1+4}\sum_{n=1}^{k_2+4}\sum_{o=1}^{k_3+4} \varphi_{ijk}\cdot\Big(&-\left\langle c^{-2}B_{i,1}B_{j,2}, c^{-2}B_{m,1}B_{n,2}\right\rangle_2 \langle B''_{k,3}, B''_{o,3}\rangle \\ &+\left\langle c^{-2}B_{i,1}B_{j,2}, B''_{m,1}B_{n,2}\right\rangle_2 \langle B_{k,3}, B''_{o,3}\rangle \\ &+\left\langle c^{-2}B_{i,1}B_{j,2}, B_{m,1}B''_{n,2}\right\rangle_2 \langle B''_{k,3}, B_{o,3}\rangle \\ &+\left\langle c^{-2}B''_{i,1}B_{j,2}, B_{m,1}B_{n,2}\right\rangle_2 \langle B''_{k,3}, B_{o,3}\rangle \\ &+\left\langle c^{-2}B_{i,1}B''_{j,2}, B_{m,1}B_{n,2}\right\rangle_2 \langle B_{k,3}, B''_{o,3}\rangle \\ &-\langle B''_{i,1}, B_{m,1}\rangle \langle B_{j,2}, B''_{n,2}\rangle \langle B_{k,3}, B_{o,3}\rangle \\ &-\langle B_{i,1}, B''_{m,1}\rangle \left\langle B''_{j,2}, B_{n,2}\right\rangle \langle B_{k,3}, B_{o,3}\rangle \\ &-\langle B''_{i,1}, B''_{m,1}\rangle \langle B_{j,2}, B_{n,2}\rangle \langle B_{k,3}, B_{o,3}\rangle \\ &-\langle B_{i,1}, B_{m,1}\rangle \left\langle B''_{j,2}, B''_{n,2}\right\rangle \langle B_{k,3}, B_{o,3}\rangle\Big)\,. \end{aligned} \tag{5.25}$$

Unfortunately, the B-splines do not constitute an orthonormal base with respect to the inner product on $L^2(\mathbb{R})$, but their inner products can still be precalculated efficiently in several ways. Since B-splines are piecewise polynomials, Gaussian quadrature is the most stable of these which is exact for B-splines and generalizes easily to products of arbitrary and multivariate functions. For low order splines, the performance is comparable to other methods based on recurrence relations or partial integration (cf. [60]).

A Gaussian quadrature rule of order m integrates polynomials of order $2m-1$ exactly. Between their break points x_i, cubic B-splines are polynomials of order 4, so their product is a polynomial of order 7. These can be exactly integrated by a quadrature rule of order 4. The products involving the second derivatives of B-splines are of order 5, so a quadrature rule of order 3 suffices. Hence we split the integrals (e.g., over x) into sums of integrals over the k_1+1 intervals defined by the break points $x_\nu \in \Delta_1$, and after scaling to the interval $[-1,1]$, apply the appropriate quadrature rule of order $d \in \{3,4\}$:

$$\begin{aligned} \langle B_{i,1}, B_{m,1}\rangle &= \int_{-\infty}^{\infty} B_{i,1}(t)B_{m,1}(t)\,dt = \sum_{\nu=0}^{k_1}\int_{x_\nu}^{x_{\nu+1}} B_{i,1}(t)B_{m,1}(t)\,dt \\ &\approx \sum_{\nu=0}^{k_i}\left(\frac{(x_{\nu+1}-x_\nu)}{2}\sum_{\mu=1}^{d}\gamma_\mu B_{i,1}(x_\mu)B_{m,1}(x_\mu)\right), \end{aligned} \tag{5.26}$$

with $x_\mu := \frac{1}{2}\left((x_{\nu+1} - x_\nu)t_\mu + (x_{\nu+1} + x_\nu)\right)$, where t_μ are the Gauss-Legendre points with the corresponding weights γ_μ.

Similarly, for the inner product on $L^2(\mathbb{R}^2)$, we use the tensor product Gauss rule

$$\begin{aligned}(5.27)\quad &\left\langle c^{-2}B_{i,1}B_{j,2}, c^{-2}B_{m,1}B_{n,2}\right\rangle_2 \\ &\qquad = \int_{\mathbb{R}^2} c^{-4}(x,y)B_{i,1}(x)B_{j,2}(y)B_{m,1}(x)B_{n,2}(y)\,d(x,y) \\ &\approx \sum_{\nu_1=0}^{k_1}\sum_{\nu_2=0}^{k_2}\left(h_{\nu_1,\nu_2}\sum_{\mu_1,\mu_2=1}^{d}\gamma_{\mu_1}\gamma_{\mu_2}\frac{B_{i,1}(x_{\mu_1})B_{j,2}(y_{\mu_2})B_{m,1}(x_{\mu_1})B_{n,2}(y_{\mu_2})}{c^4(x_{\mu_1},y_{\mu_2})}\right),\end{aligned}$$

where $h_{\nu_1,\nu_2} := \frac{1}{4}(x_{\nu_1+1} - x_{\nu_1})(y_{\nu_2+1} - y_{\nu_2})$, and x_{μ_1}, y_{μ_2} are again the one dimensional transformed Gauss-Legendre points with the weights $\gamma_{\mu_1}, \gamma_{\mu_2}$.

The inner products can be precalculated, where one can make use of the fact that the support of B_i and B_m (and *a fortiori* of their derivatives) intersect only for $i - 3 \leqslant m \leqslant i + 3$ to reduce the number of computations. This can also be employed in the assembly of the matrix M (where only entries close to the diagonal will need to be filled) as well as in the assembly of the right hand side vector F (where the sums only have to be taken from $i-3$ to $i+3$). Therefore at most $343N$ entries of M are nonzero.

By the above, we arrive at the system of linear equations $Ma = F$ for the unknown $a = (a_\iota)_\iota$ in (5.19), which is large, sparse and banded (cf. Figures 5.2 and 5.3). Since the bilinear form M_ε is symmetric and elliptic, M is also symmetric and positive definite, so a preconditioned conjugate gradient method can be used for the fast solution of this system. We employ a Jacobi prescaling (ie. solving $DMDa = DF$ with the diagonal matrix $D_{ii} = \frac{1}{\sqrt{M_{ii}}}$, $i = 1, \ldots, N$) in order to improve performance and stability. This has proved more effective than other preconditioners (e.g., SPAI or incomplete Cholesky), since the matrix is strongly diagonally dominant (cf. Figure 5.4).

Reverting then from the linear index ι to i, j, k, we have calculated the B-spline coefficient tensor a_{ijk} of u_h. After successive application of the univariate B-spline interpolation matrices A_1, A_2 and A_3 , we recover the solution u_h of the discretized quasi-reversibility Problem 5.1. In the next chapter, we illustrate the effectiveness and robustness of this method with numerical results.

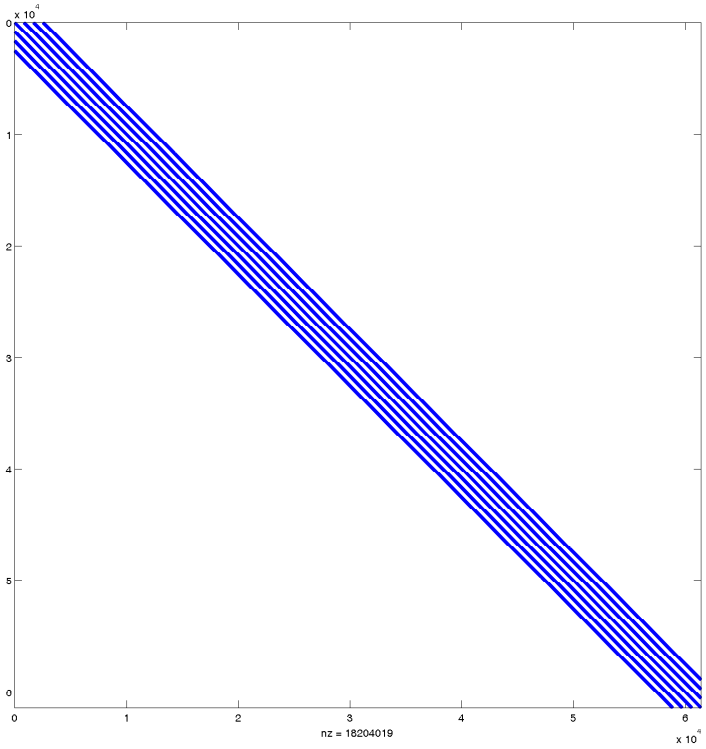
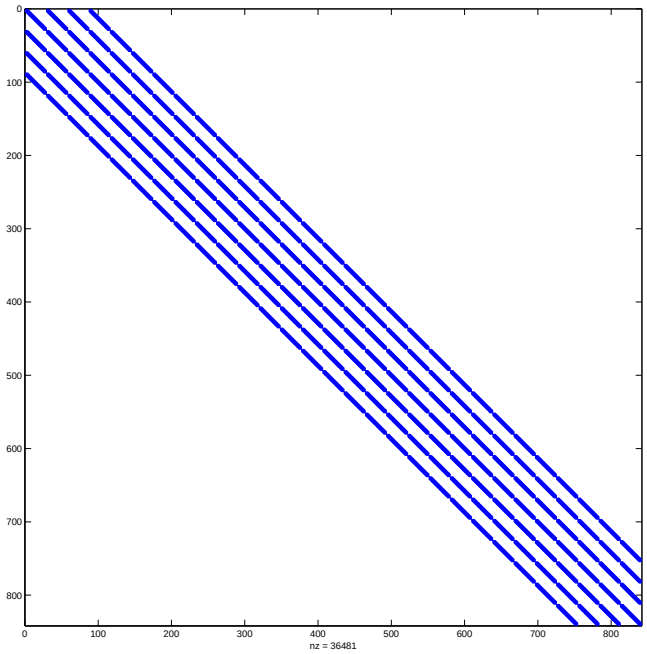
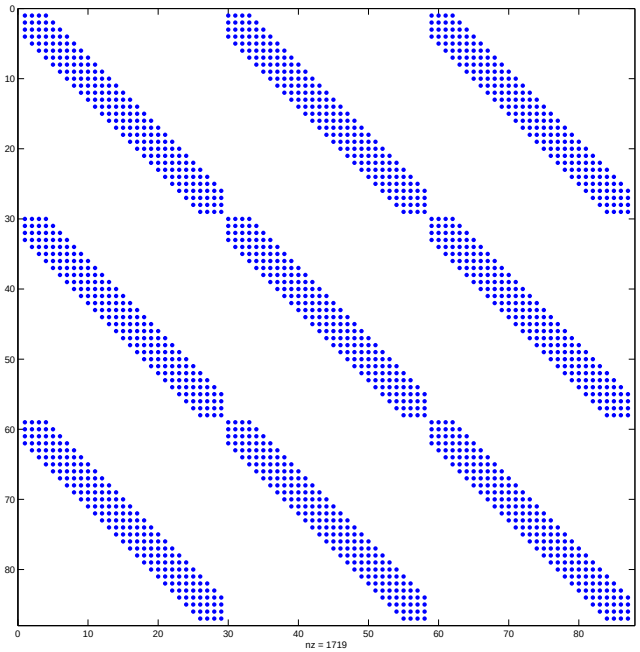


Figure 5.2: Sparsity pattern of the matrix M. In this example, $N = 61393$ and $\varepsilon = 10^{-3}$.



(a) Top left 841×841 block of M.



(b) Top left 87×87 block of M.

Figure 5.3: Details of the band structure of the matrix M. Each figure shows a magnification of a single band of the larger structure.

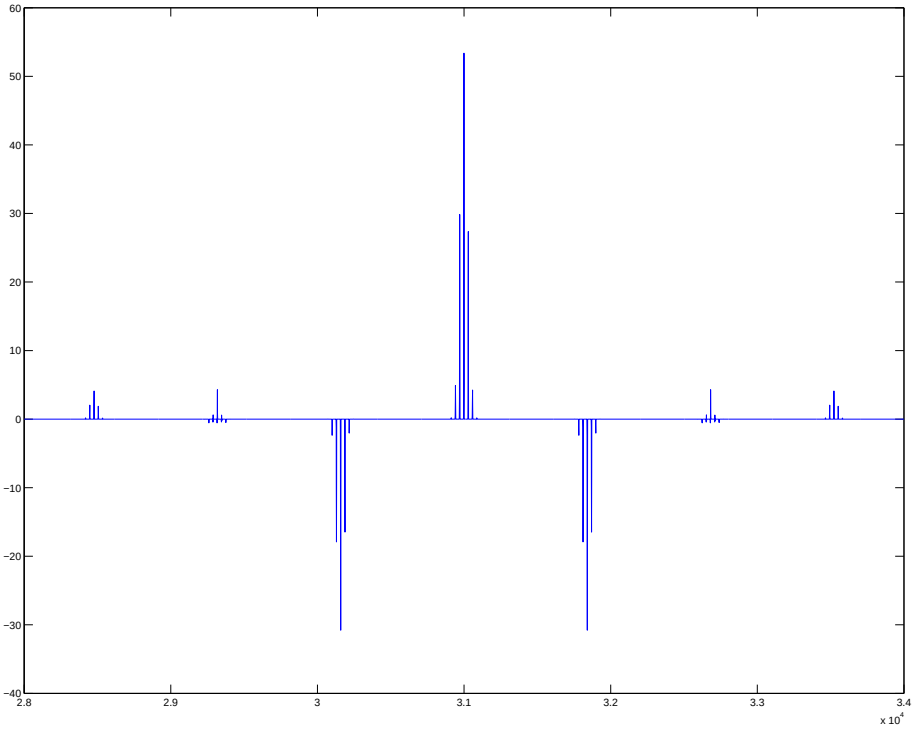


Figure 5.4: Distribution of the matrix entry values of above matrix $M = (M_{ij})_{ij}$. Shown is a section $(M_{ij})_i$, $j = 31000$.

6. Numerical results

We have implemented the Ritz-Galerkin scheme described in Chapter 5 using Matlab. In order to avoid committing an inverse crime, we generate the Cauchy data by solving the forward problem using a semidiscretized finite element method on an unstructured domain (using Comsol Multiphysics, formerly Femlab). Specifically, on the square $\Omega = [-3,3] \times [-3,3]$, we prescribe a sound speed $c(x,y)$ and an initial condition $u_0(x,y)$. We then solve equation (2.12) for $t \in [0,T]$ on the rectangular domain $[-3 - c_{max}T, 3 + c_{max}T]^2$, with T chosen according to Theorem 4.3.3, and homogeneous Dirichlet boundary conditions due to the finite speed of propagation. Both sound speed and initial condition are continued on this extended domain by linear extrapolation. The domain is discretized with an unstructured quasi-uniform triangular mesh. For each time step, the solution is approximated by quadratic elements, yielding a system of second order ordinary differential equations, which are solved by a variable order variable stepsize backward differentiation formula. The solution and its gradient is then extracted at a given partition Δ of $\Omega \times [0,T]$ (if necessary, by interpolation), and Φ_h is calculated. To simulate errors in the measurement data, we introduce noise to the calculated B-spline coefficients φ_{ijk}:

$$\varphi^{\delta}_{ijk} = (1 + \delta\xi)\varphi_{ijk}\,, \tag{6.1}$$

where $\delta \geqslant 0$ is a given noise level and ξ is a random number uniformly distributed between -1 and 1. Note that in this way, we do not include a smoothing step for the noisy data. With these coefficients, we compute the matrix M and right hand side vector F as described above. The system of linear equations $Ma = F$ for the spline coefficients of u_h is solved to a tolerance of 10^{-6} by a stabilized biconjugate gradient method (BICGSTAB, provided by Matlab). Since the B-splines are nonnegative and sum to 1 at any point, this tolerance also holds for the function u_h by way of the triangle inequality. We illustrate the effectiveness and robustness of our approach in different situations with several tests. For all our tests, we choose a uniform discretization of Ω with mesh width $h_{1,2} = 0.2$ and of $[0,T]$ width mesh width $h_3 = 0.1$. With this discretization, the calculations took around 25 minutes on a 2.2 GHz Opteron workstation, using 2 gigabytes of memory.

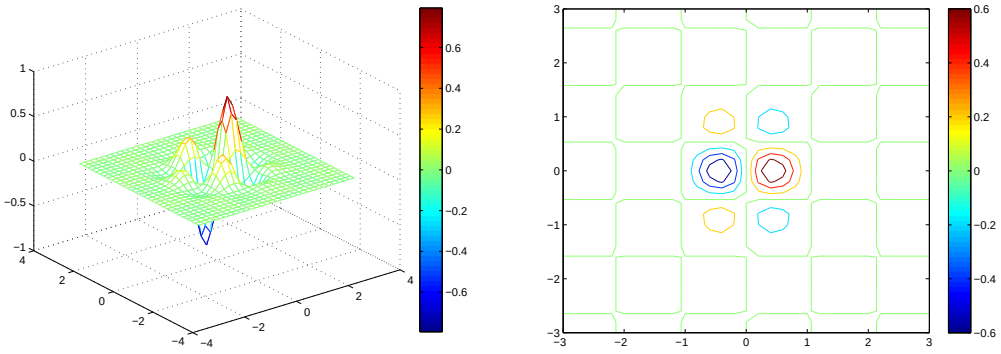


Figure 6.1: Mesh plot (left) and contour plot (right) of the initial condition to be reconstructed.

6.1. Test 1: Smooth initial conditions, constant coefficients

To evaluate the influence of a heterogeneous medium, we first test our method with a constant coefficient, which we take to be $c(x) \equiv 1$. Therefore we set $T := 7$. As our initial condition to be reconstructed, we use

$$u_0(x,y) = e^{-(x^2+y^2)} \sin(3x)\cos(3y)\,, \tag{6.2}$$

which is negligible outside Ω (cf. Figure 6.1). The reconstructions $u_\varepsilon(x,y,0)$ for various $\delta \in [0,1]$ are compared in Figure 6.2, while the relative $L^2(\Omega)$-errors for several values of δ and ε are given in Table 6.1. In Figure 6.3, we show that the time evolution of the solution $u(x,y,t)$ can also be recovered. In Figure A.1, mesh and contour plots of the reconstructed initial conditions for several noise levels are compared.

As can be seen, the reconstructions of the initial conditions are good, even with very high noise levels of up to 300%. The Lipschitz stability estimate of Theorem 4.3.3 provides an indication of this. The time evolution is also close to the reference solution. The more pronounced oscillation in the calculated solution is due to the fact that during the reconstruction method, we seek to identify both initial conditions, $u(x,y,0)$ and $\partial_t u(x,y,0)$ at the same time. Starting a forward solver with $u(x,y,0) = u_\varepsilon(x,y0)$ and $\partial_t u(x,y,0) = 0$ would give an even closer match. Of note is that the optimal parameter $\varepsilon = 10^{-3}$ is the same for all noise levels up to $\delta = 1$, and gives comparable results for the higher levels. This further emphasizes the robustness of the method.

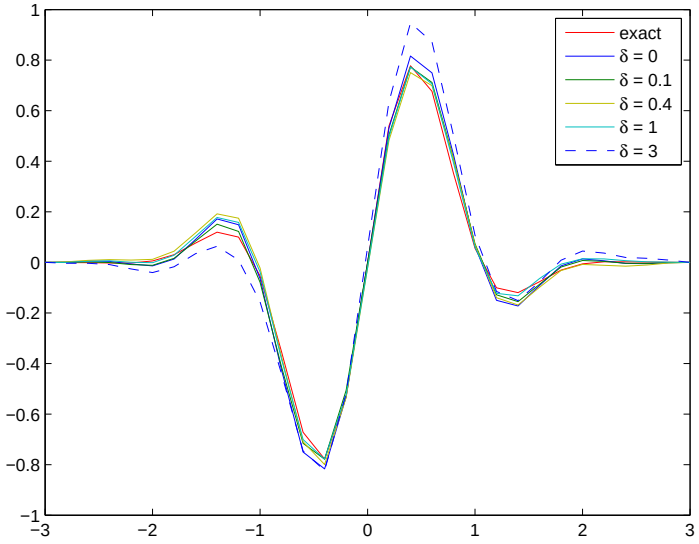


Figure 6.2: Comparisons of reconstructed initial conditions in a homogeneous medium for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are slices $u_\varepsilon(x, 0, 0)$, $x \in [-3, 3]$.

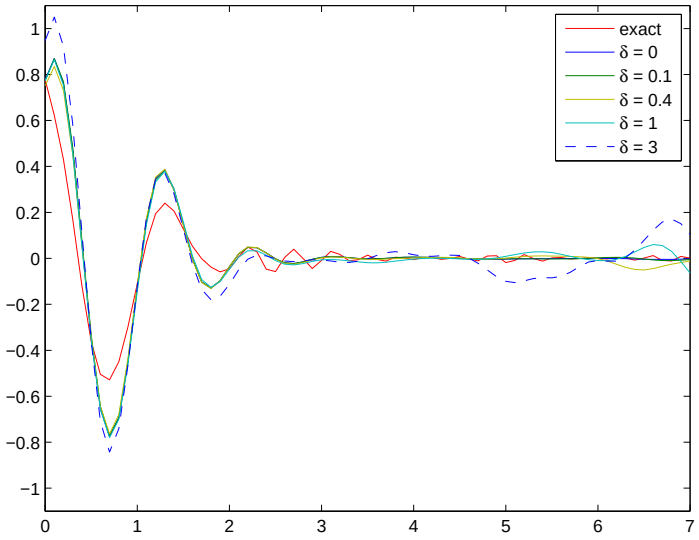


Figure 6.3: Comparisons of the time evolution of the solution u in a homogeneous medium for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are plots $u_\varepsilon(0.4, 0, t)$, $t \in [0, 7]$.

δ, ε	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-1}	1.0
0	0.15971	0.15964	0.15901	0.15281	0.09858	0.87138	0.99522
0.05	0.16029	0.16041	0.15768	0.15073	0.09941	0.87178	0.99524
0.1	0.16211	0.15708	0.15837	0.15076	0.09618	0.87057	0.99527
0.2	0.16614	0.15912	0.16017	0.15119	0.09509	0.86999	0.99539
0.4	0.16653	0.18398	0.17926	0.15146	0.11096	0.87364	0.99583
0.5	0.19039	0.14141	0.18613	0.16458	0.13000	0.86619	0.99617
1.0	0.20092	0.23337	0.18634	0.21325	0.12906	0.88003	0.99444
2.0	0.31615	0.38390	0.43442	0.27649	0.34904	0.85463	0.99199
3.0	0.54785	0.43724	0.52161	0.48232	0.55094	0.84709	0.99659
4.0	0.78441	0.69156	0.64085	0.98342	0.64522	0.86108	0.99543
6.0	0.90998	1.08680	0.84162	1.28090	1.26290	0.91392	0.99826

Table 6.1: Relative $L^2(\Omega)$-errors of reconstructions of (6.2) in a homogeneous medium for various noise levels $\delta \in [0, 6]$ and regularization parameters $\varepsilon \in [0, 1]$.

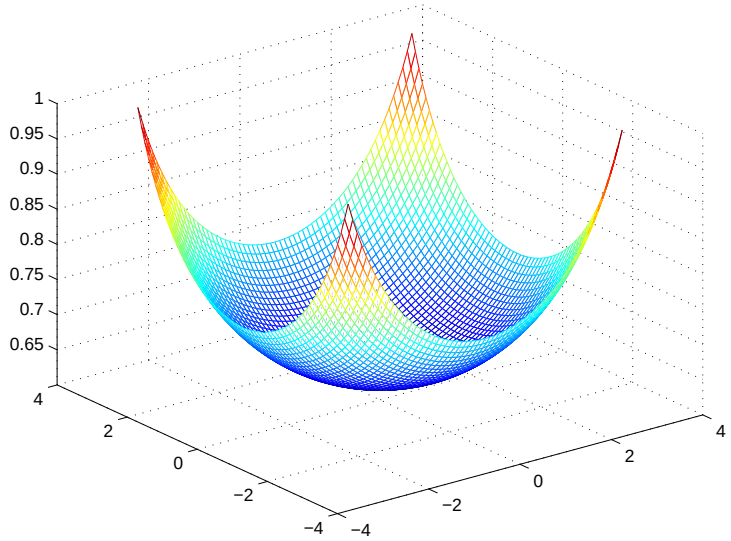


Figure 6.4: Heterogeneous medium: sound speed c from (6.3).

6.2. Test 2: Smooth initial conditions, smooth coefficients

Next we repeat these experiments for a heterogeneous medium. Taking (cf. Figure 6.4)

$$\frac{1}{c(x,y)^2} = \frac{5}{2} - \frac{1}{12}(x^2+y^2)\,, \tag{6.3}$$

we satisfy the condition (4.20) with $x_0 = 0$. According to Theorem 4.3.3, we should take $\sqrt{\beta} < \frac{16}{95}$, hence $T := 26$. However, for numerical calculations, $T := 7$ (a lower bound for the minimal time needed for a wave to propagate from any point in Ω to the closest point on $\partial\Omega$) turned out to be sufficient. Since the bounds from Carleman estimates for hyperbolic equations with variable coefficients are known not to be sharp, this was to be expected. Again, the relative $L^2(\Omega)$-errors are given in Table 6.2, while the solutions are compared in Figures 6.5 and 6.6. Contour plots of the reconstructions can be found in the appendix, Figure A.3.

Here as well the reconstructions are close to the prescribed initial conditions, although the method is now a little less tolerant to noise.

δ, ε	0	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-1}	1.0
0	0.14984	0.14690	0.14850	0.15036	0.12667	0.76300	0.99530
0.05	0.14253	0.13900	0.15360	0.13643	0.11971	0.76330	0.99540
0.1	0.14457	0.14330	0.14200	0.14502	0.10915	0.76364	0.99513
0.2	0.13169	0.14170	0.15068	0.13298	0.10365	0.76320	0.99554
0.5	0.15338	0.17681	0.16089	0.15436	0.14229	0.75604	0.99508
1.0	0.17350	0.22785	0.22412	0.21235	0.19087	0.75570	0.99376

Table 6.2: Relative $L^2(\Omega)$-errors of reconstructions of (6.2) in a heterogeneous medium for various noise levels δ and regularization parameters ε.

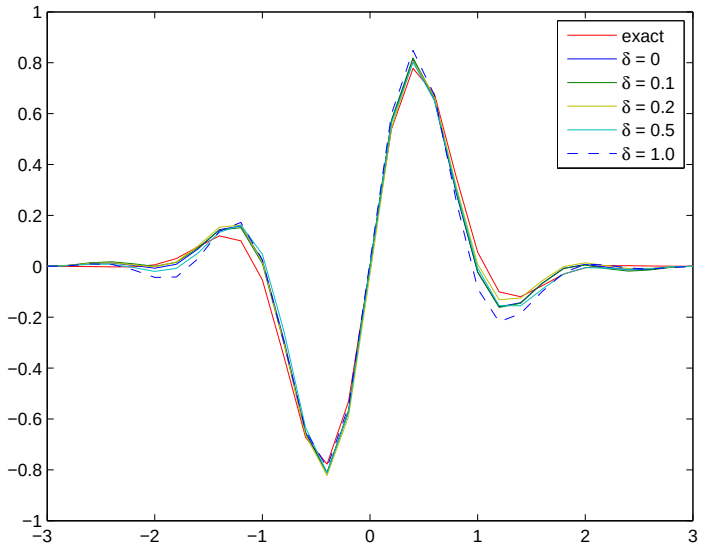


Figure 6.5: Comparisons of reconstructed initial conditions in a heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are slices $u_\varepsilon(x, 0, 0)$, $x \in [-3, 3]$.

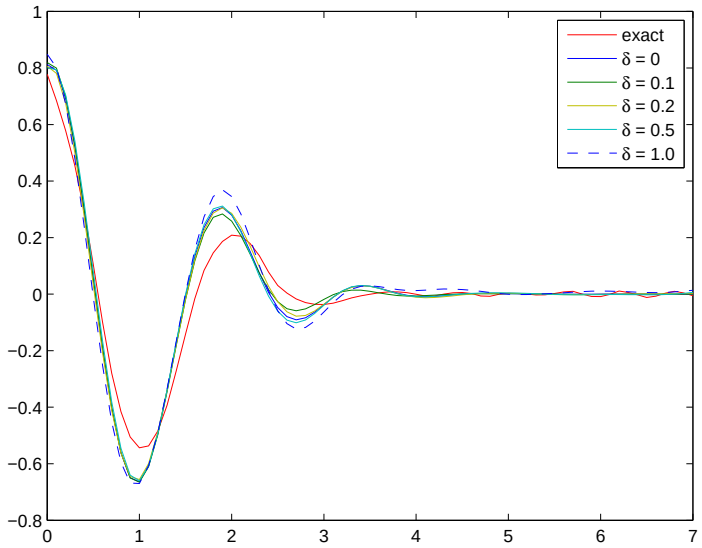


Figure 6.6: Comparisons of the time evolution of the solution u in a heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are plots $u_\varepsilon(0.4, 0, t)$, $t \in [0, 7]$.

6.3. Test 3: Smooth initial conditions, nondifferentiable coefficients

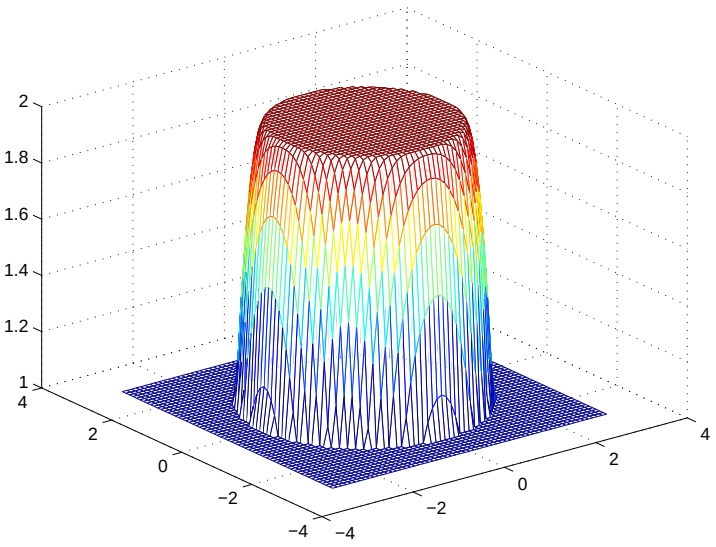


Figure 6.7: Heterogeneous medium: sound speed c from (6.4).

Since we are interested in applications in thermoacoustic tomography, we also test our method with a more realistic sound speed, which is not covered by our theoretical results, such as a bone surrounded by soft tissue or water. Since the speed of sound in bone is roughly twice of that in water, we model this situation by the following speed distribution (cf. Figure 6.7):

$$c(x,y) = \max\left(2 - \left(\frac{\max\left(2 - 5 + x^2 + y^2, 0\right)}{2}\right)^2, 1\right). \tag{6.4}$$

The results are shown in Figures 6.8 and A.4.

Since the constant in the error estimate (4.127) depends (by the Carleman estimate (4.56)) on the gradient of c^{-2}, we expect in this case the inevitable numerical errors to be larger. Still, the reconstruction is only slightly worse than for smooth coefficients.

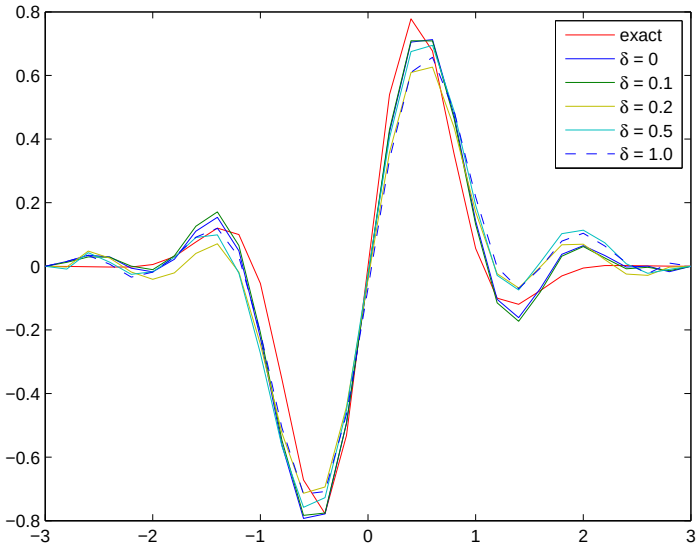


Figure 6.8: Comparisons of reconstructed initial conditions in a heterogeneous medium (6.4) for various noise levels δ and regularization parameter $\varepsilon = 10^{-4}$. Shown are slices $u_\varepsilon(x, 0, 0)$, $x \in [-3, 3]$.

6.4. Test 4: Smooth initial conditions, smooth coefficients, limited boundary data

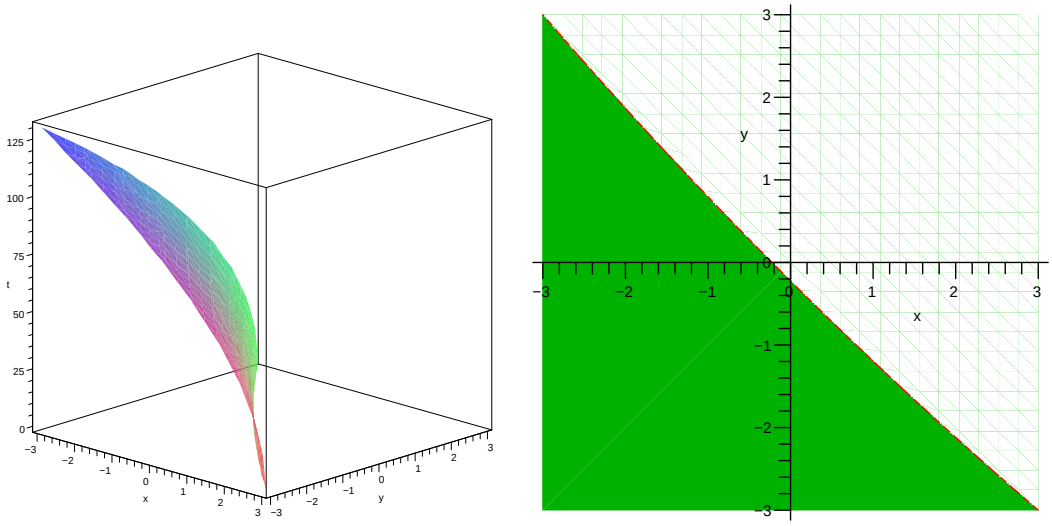


Figure 6.9: Domain $Q_\sigma = \{(x-40)^2 + (y-40)^2 - \left(\frac{16}{95}t\right)^2 > 3218\}$ (left, below the surface), $Q_\sigma \cap \{t = 0\}$ (right) for reconstruction from limited boundary data.

Motivated by applications in thermoacoustic tomography (where the measurements are available on the half-sphere, cf. Figure 2.1), we investigate the case of Cauchy data given only on half of the boundary. Note that this is more ambitious than the setting of thermoacoustic tomography, since we try reconstruction in the full domain, not only inside the convex closure of the measurement boundary.

If the boundary data is given only on a part of the boundary, we expect the reconstruction to deteriorate, due to the weaker Hölder stability estimate for this case. Here we take for our computations only the boundary data given on $\Gamma := \{(x,y) \in \partial\Omega : \ x = -3 \text{ or } y = -3\}$. As a heterogeneous medium, we again take the smooth coefficient (6.3). According to Theorem 4.2.2, we can expect stable reconstruction inside the domain (4.18) (cf. Figure 6.9, where we take $(x_0, y_0) = (40, 40)$, $\sqrt{\beta} = \frac{16}{25}$, and $\sigma = 3218$;). The theoretical bound for $T > 130$ notwithstanding, it was sufficient to take the minimal time for a wave starting in any point in Q_σ to reach Γ, which is again $T := 7$. The results are shown in Figures 6.11, 6.12, and A.5.

Due to energy loss at the boundary $\partial\Omega \setminus \Gamma$, the amplitude of the reconstructed solution is roughly half that of the true solution. The shape of the initial condition, however, is still recovered well for all noise levels. The reconstruction is only slightly worse outside $Q_\sigma \cap \{t = 0\}$. If we restrict the measurement to the

line $\{(x, y) \in \partial\Omega : \ x = -3\}$, loss of amplitude becomes even more pronounced. The accuracy on the far side of the (bottom) boundary also deteriorates (cf. Figure 6.10).

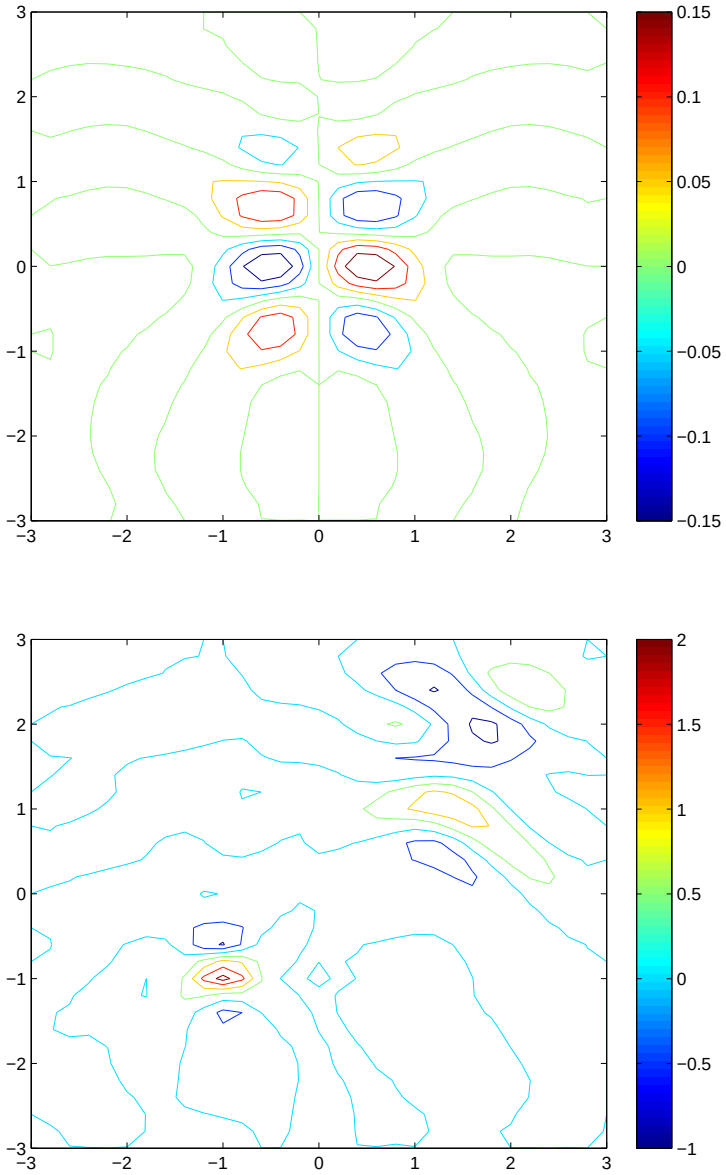


Figure 6.10: Contour plot of reconstructed initial conditions in heterogeneous medium (6.3) from boundary data given on bottom side of the square. No noise, regularization parameter $\varepsilon = 10^{-5}$. top: smooth initial condition (6.2), bottom: delta-like sources (6.5).

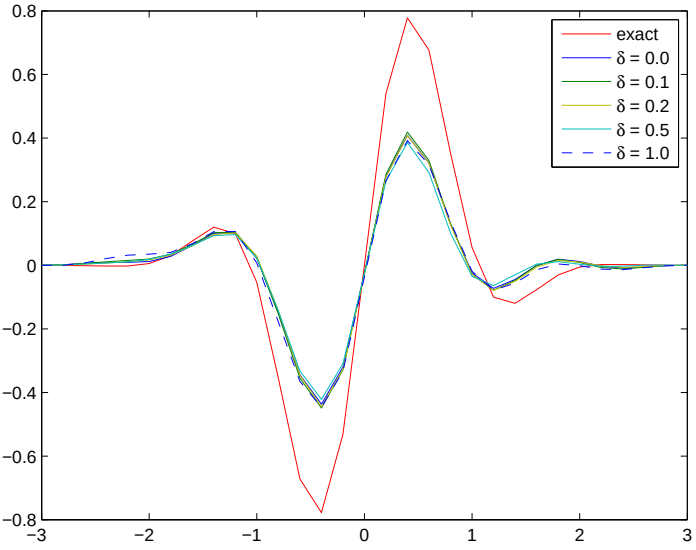


Figure 6.11: Comparisons of reconstructed initial conditions from limited boundary data in a heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are slices $u_\varepsilon(x, 0, 0)$, $x \in [-3, 3]$.

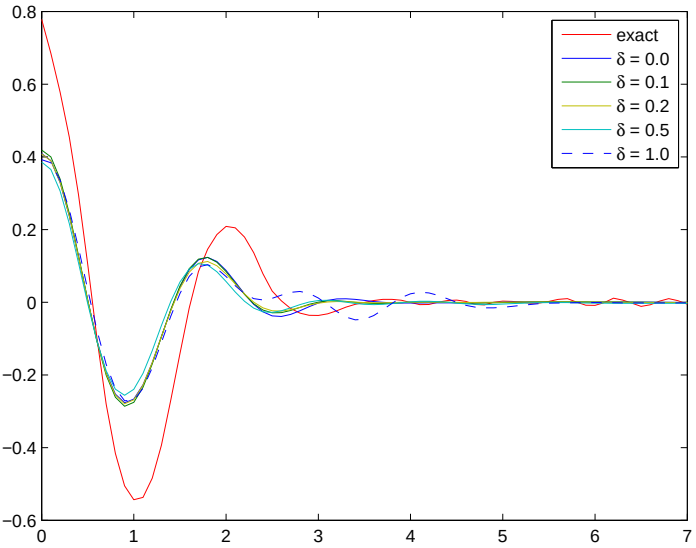


Figure 6.12: Comparisons of the time evolution of the solution u from limited boundary data in a heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$. Shown are plots $u_\varepsilon(0.4, 0, t)$, $t \in [0, 7]$.

6.5. Test 5: Delta-like sources

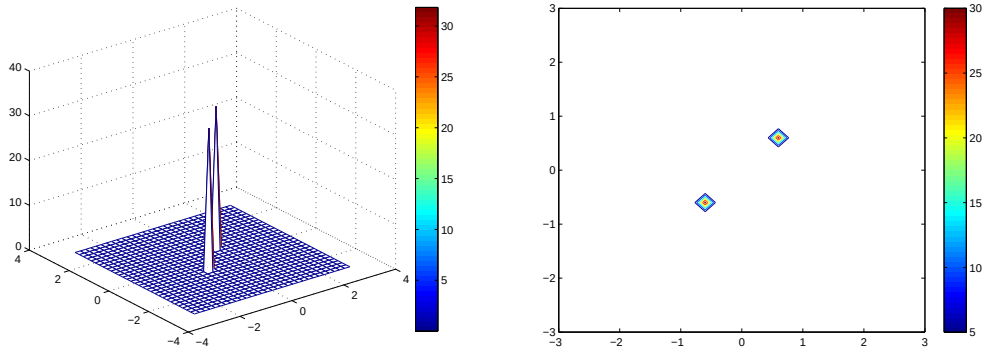


Figure 6.13: Mesh plot (left) and contour plot (right) of the delta-like sources to be reconstructed.

Of interest as well is the refocusing of delta-like sources in experimental time reversal, a fact which has been mathematically investigated in [41]. As a model for two delta sources, we use the sum of two nascent delta functions:

(6.5)
$$u_0(x,y) = \frac{1}{\pi}\frac{0.01}{(0.01^2 + (x-0.6)^2 + (y-0.6)^2} + \frac{1}{\pi}\frac{0.01}{(0.01^2 + (y+0.6)^2 + (y+0.6)^2}\,.$$

We try the reconstruction in a homogeneous ($c \equiv 1$, Test 5a) medium, in the heterogeneous medium (6.3) (Test 5b) and (6.4) (Test 5c). Test 5b is repeated with the boundary data given only on the part Γ defined in Section 6.4 (Test 5d). The results are shown in Figures A.6, A.8, A.9, and A.10, respectively. The case of data given only on one side of the square can be seen in Figure 6.10

Refocusing does indeed take place, even if the boundary data is severely contaminated by noise. For this problem, the numerical solution of the forward problem had to be stabilized by introducing damping in the time stepping algorithm. This explains the lower amplitude of the reconstructed solution, since energy is not conserved in the solution used to generate the measurement data. The large gradient of the coefficient again limits the reconstruction for the medium (6.4). In Figure A.10, the influence of the domain Q_σ can be seen. Inside Q_σ, the delta-like peak is captured much better (although not as well as in the preceding figures, due to the weaker Hölder stability).

6.6. Test 6: Shepp-Logan phantom

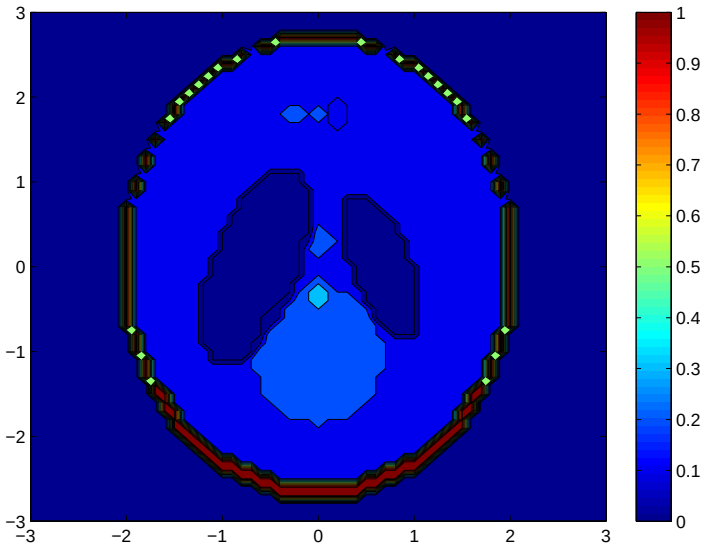


Figure 6.14: Contour plot (color coded) of the Shepp-Logan phantom.

A standard benchmark in medical imaging and tomography in particular is the Shepp-Logan phantom. This consists in a grayscale intensity image that is made up of one large ellipse (representing the brain) containing several smaller ellipses (representing features in the brain). We use a slightly modified variant of the Shepp-Logan phantom, in which the contrast is improved for better visual perception (cf. Figure 6.14, color coded).

This is an extreme test for our method, with its strong discontinuities and high discrepancies between the outer shell (of amplitude 1), the small inclusions (amplitude between 0.2 and 0.4) and the inner cavities (amplitude 0). Indeed, the forward solver used to generate the measurement data managed a stable computation of the wave field only with heavy numerical damping. We nevertheless tried our method for this target, for a homogeneous as well as the heterogeneous medium (6.3), in order to determine its limits. The results for zero noise and $\varepsilon = 0$ are shown in Figure 6.15. As can be seen, the large structures are recovered quite well, at least for constant coefficients. For variable coefficients, the reconstruction visibly suffers. On the other hand, the smaller structures in the phantom show up. The deterioration becomes more pronounced if noise is introduced in the measurements (cf. Figure A.11).

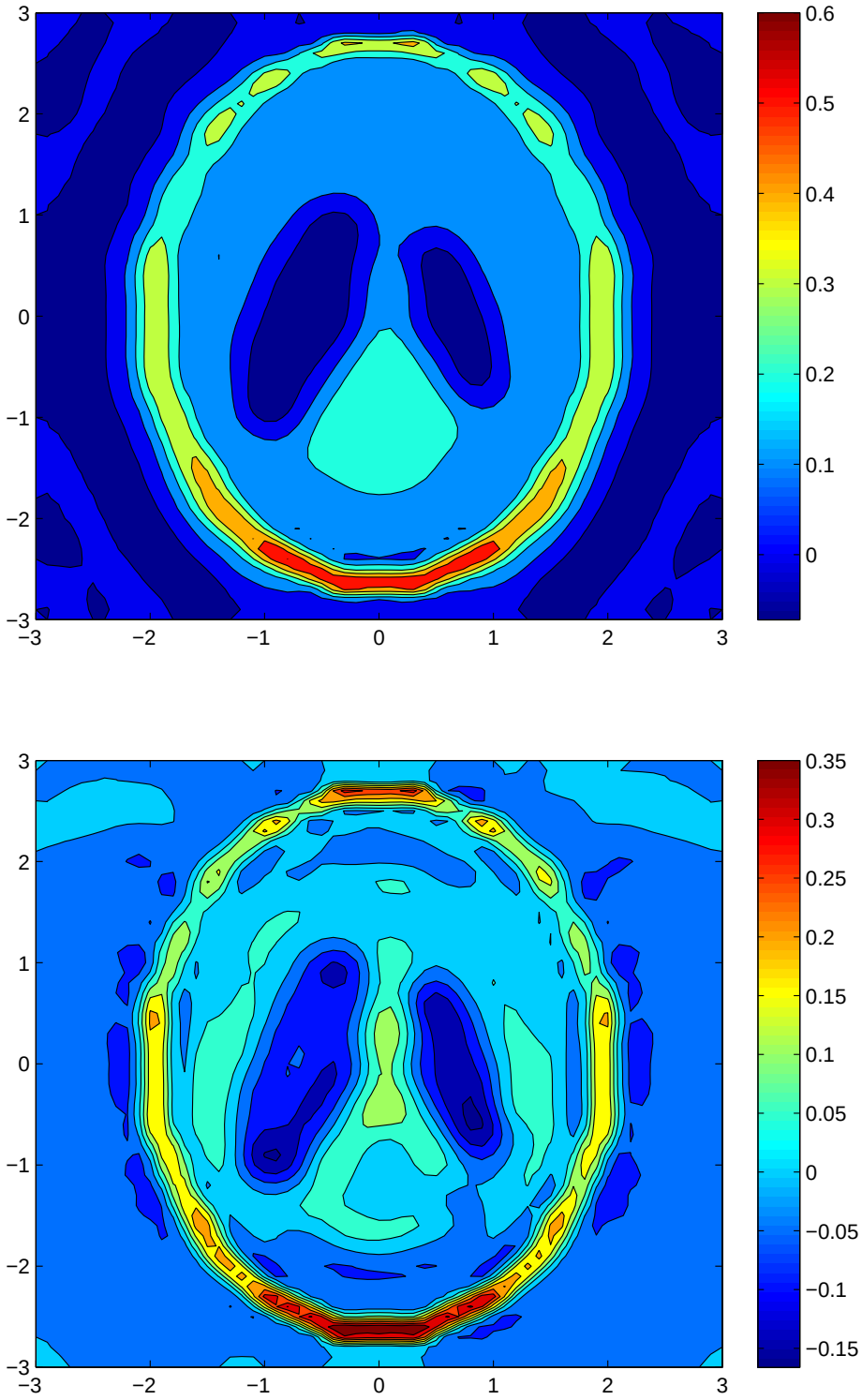


Figure 6.15: Contour plot of reconstructed Shepp-Logan phantom in homogeneous medium (top) and in heterogeneous medium (6.3) (bottom) without noise, $\varepsilon = 0$.

7. Conclusion

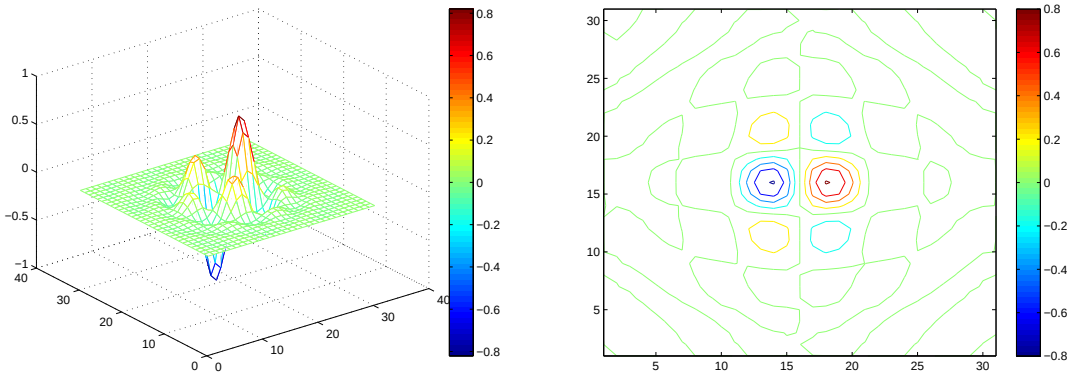
We have presented a direct and robust method for the numerical time reversal of waves in a heterogeneous medium. The main advantages are the high resistance to noise in the lateral Cauchy data and the relative independence from parameter choices (since taking, e.g., $\varepsilon = 10^{-3}$ yielded the best results in most of our calculations, and comparable results in the other cases). This, and not relying on an initial guess, is traded for the higher memory requirements compared to iterative methods (e.g., Newton-type methods). We feel that a more efficient implementation of our approach in a lower level programming language such as C or FORTRAN will be competitive with such methods.

Since the results in this work also hold for general linear hyperbolic operators of second order, our method is also applicable when lower order terms are present, for example for numerical time reversal in dissipative media. Our approach is also feasible for time-dependent coefficients, which theoretically would allow medical imaging of moving targets (e.g., beating hearts, breathing lungs, or kidney stones).

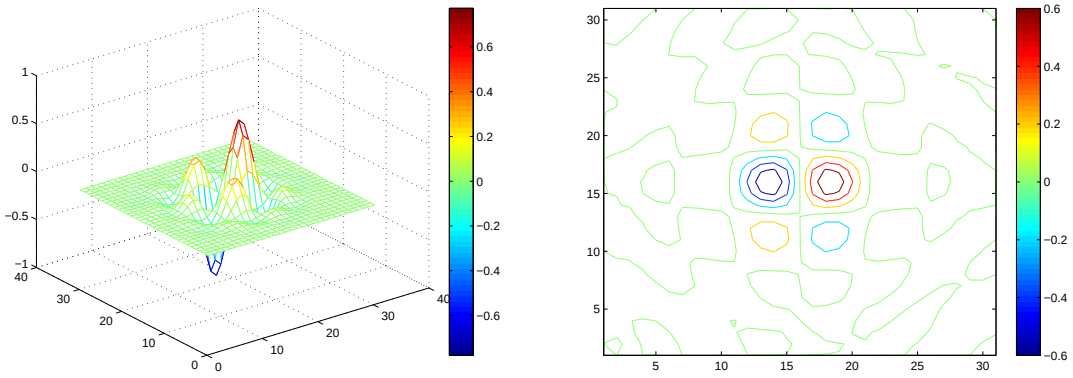
The computational method presented can also be applied to arbitrary (Lipschitz) domains by using weighted B-splines which vanish to second order on the boundary (cf. [21]). The only difficulty is then the construction of the function Φ, which could be handled by a transformation to a rectangular domain. The restriction to uniform knot vectors can also be lifted, which together with the efficient knot insertion algorithms for B-splines opens the way to adaptive refinement strategies based on *a posteriori* error estimates.

Finally, it would be interesting to derive quasi-reversibility approximations for systems of linear equations and investigate their numerical solutions. Since exact observability estimates have been proven for the systems of elasticity and electromagnetism (cf. [11], [50]), it is expected that stability and convergence estimates similar to the ones derived here can be obtained.

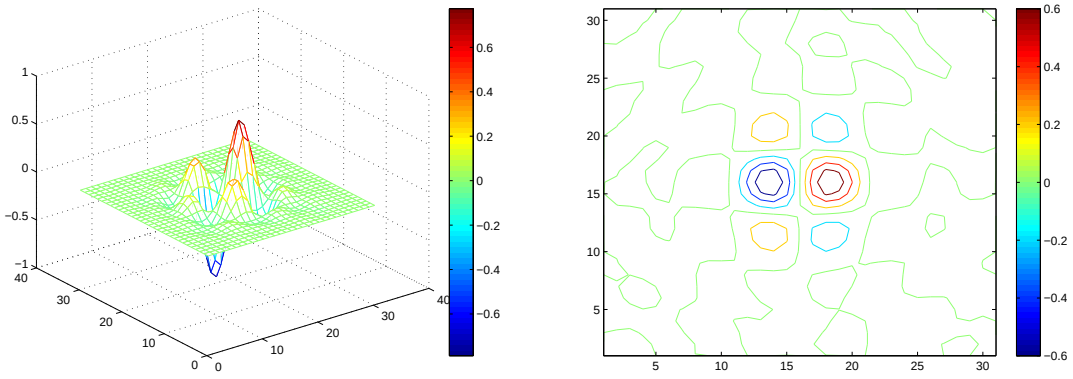
A. Comparisons of reconstructions for different noise levels



(a) $\delta = 0$, $\varepsilon = 0$

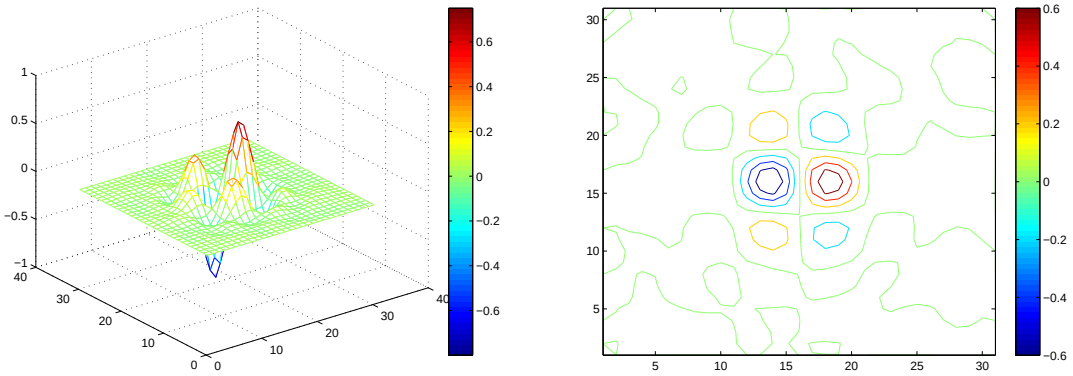


(b) $\delta = 0.1$, $\varepsilon = 10^{-3}$

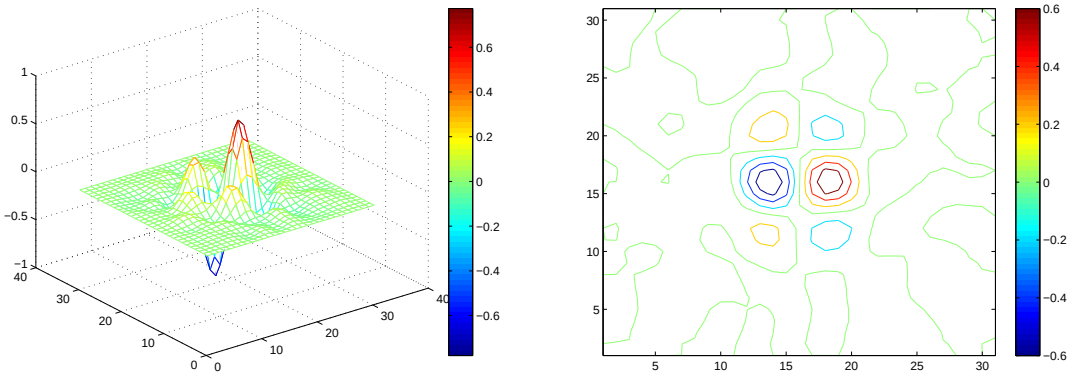


(c) $\delta = 0.2$, $\varepsilon = 10^{-3}$

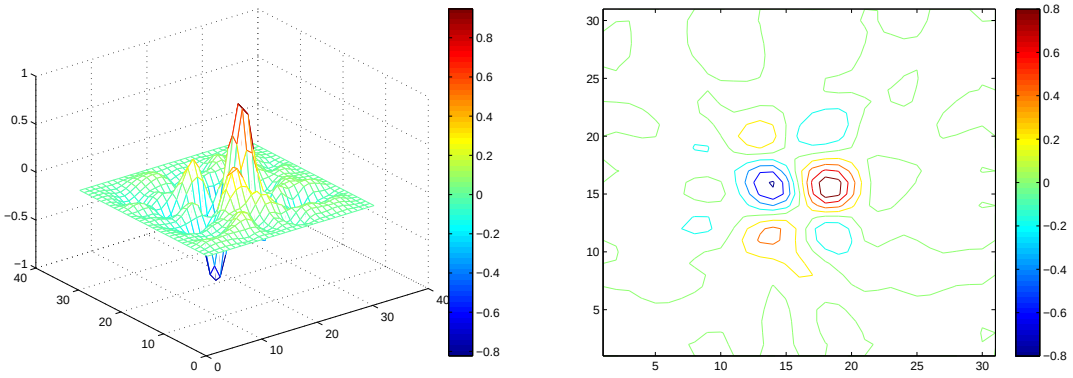
Figure A.1: (Test 1) Reconstructed initial conditions in a homogeneous medium for various noise levels δ and regularization parameters ε.



(a) $\delta = 0.4$, $\varepsilon = 10^{-3}$

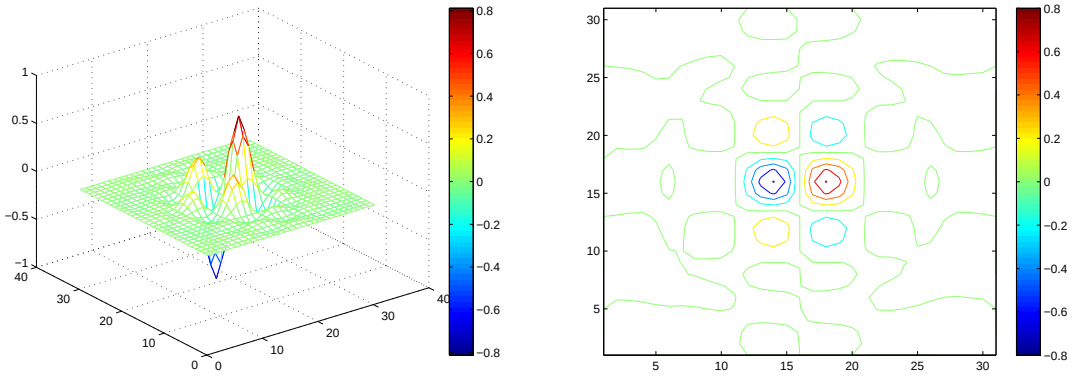


(b) $\delta = 1$, $\varepsilon = 10^{-3}$

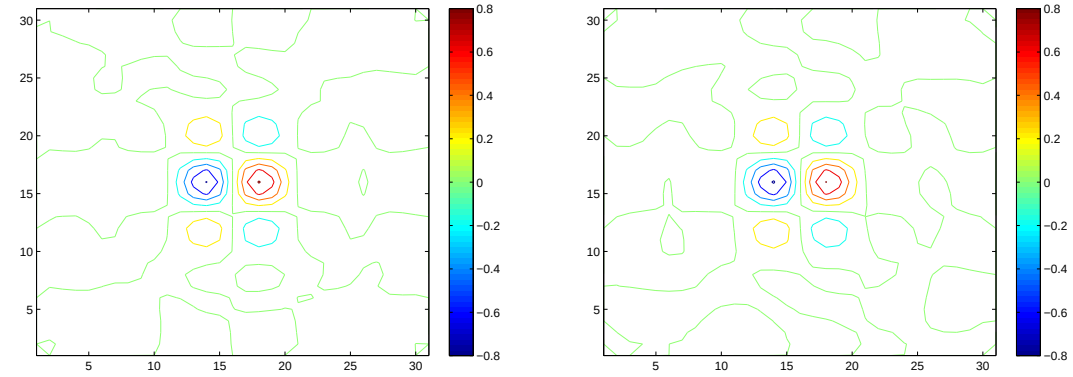


(c) $\delta = 3$, $\varepsilon = 10^{-3}$

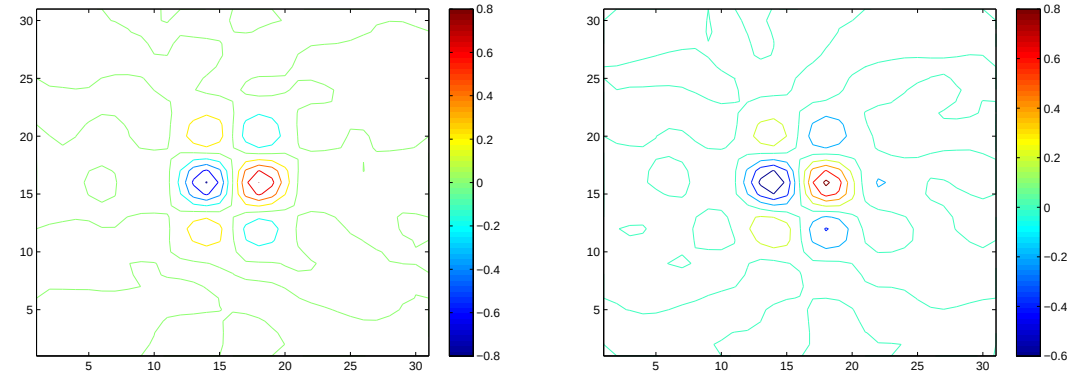
Figure A.2: (Test 1) Reconstructed initial conditions in a homogeneous medium for various noise levels δ and regularization parameters ε (cont'd).



(a) $\delta = 0$

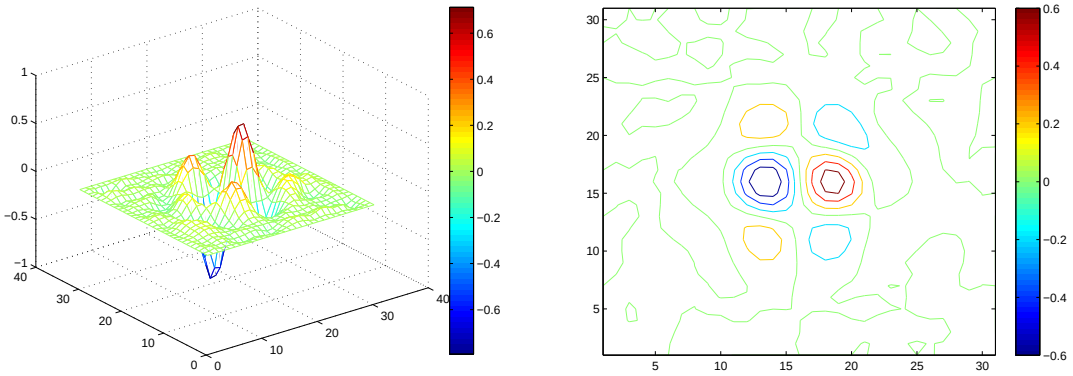


(b) $\delta = 0.1$ (c) $\delta = 0.2$

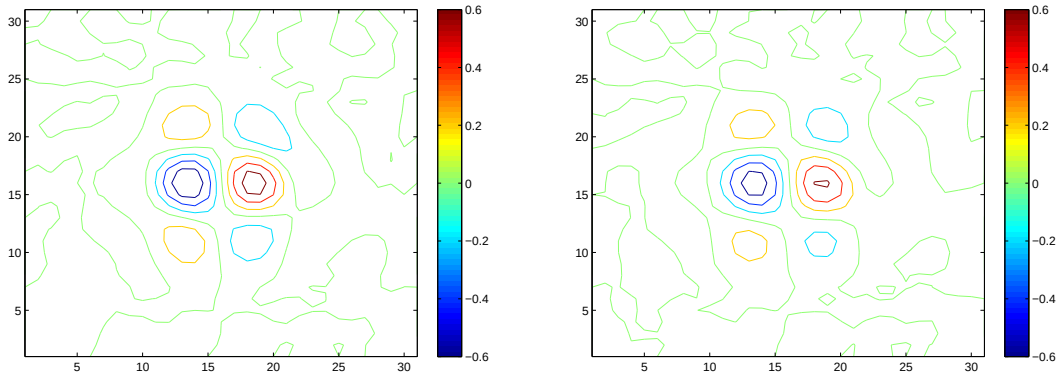


(d) $\delta = 0.5$ (e) $\delta = 1$

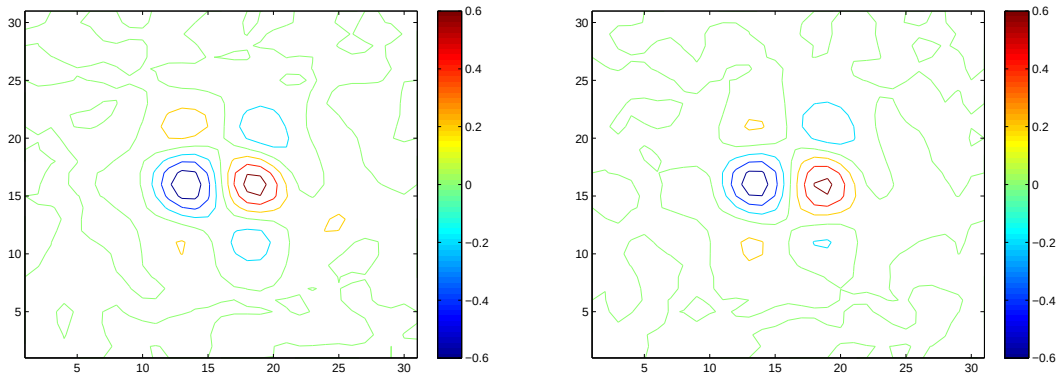
Figure A.3: (Test 2) Reconstructed initial conditions in a heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-3}$.



(a) $\delta = 0$

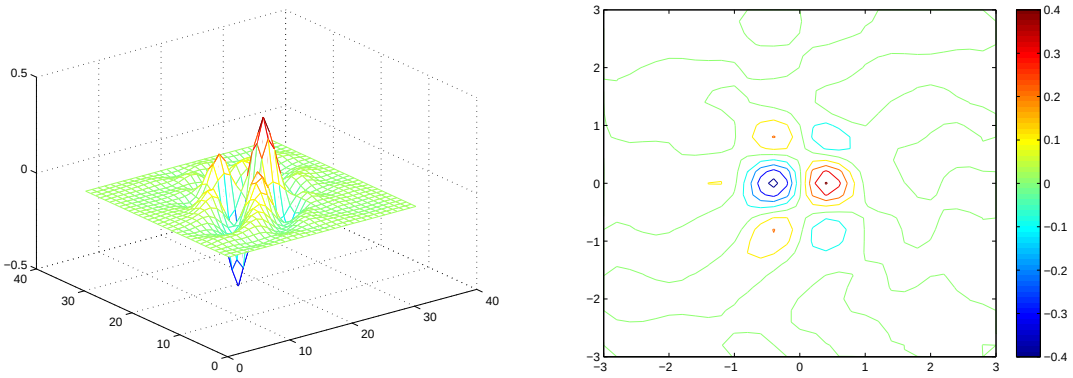


(b) $\delta = 0.1$ (c) $\delta = 0.2$

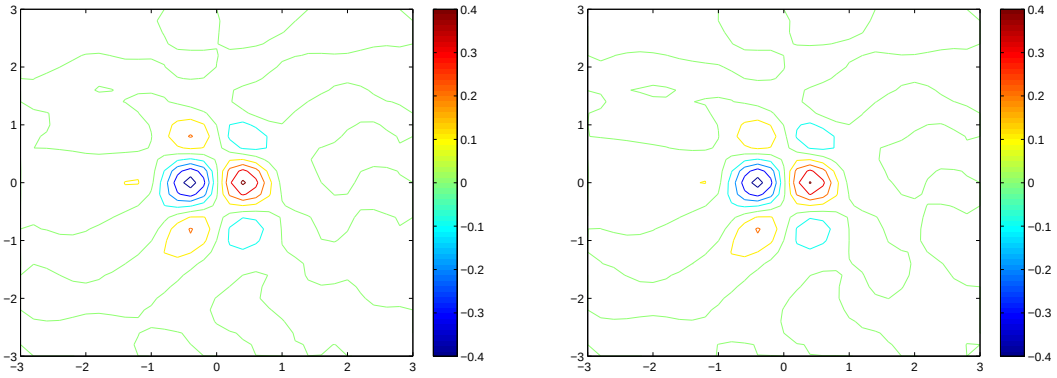


(d) $\delta = 0.5$ (e) $\delta = 1$

Figure A.4: (Test 3) Reconstructed initial conditions in heterogeneous medium (6.4) for various noise levels δ and regularization parameter $\varepsilon = 10^{-4}$.

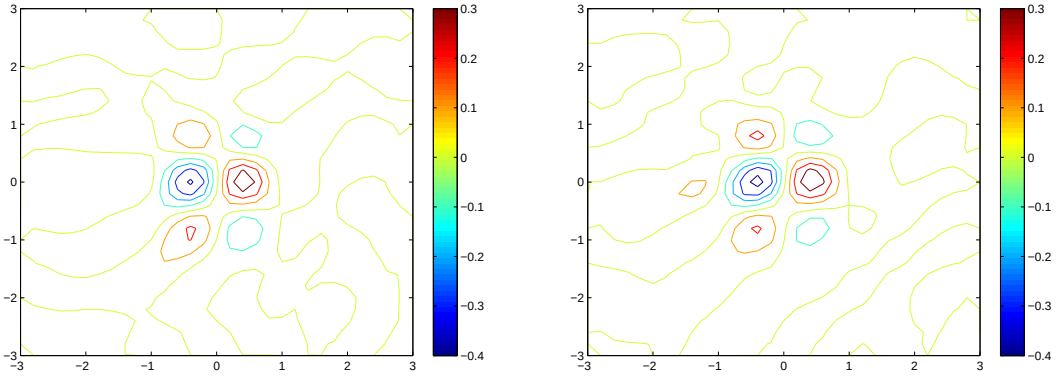


(a) $\delta = 0$



(b) $\delta = 0.1$

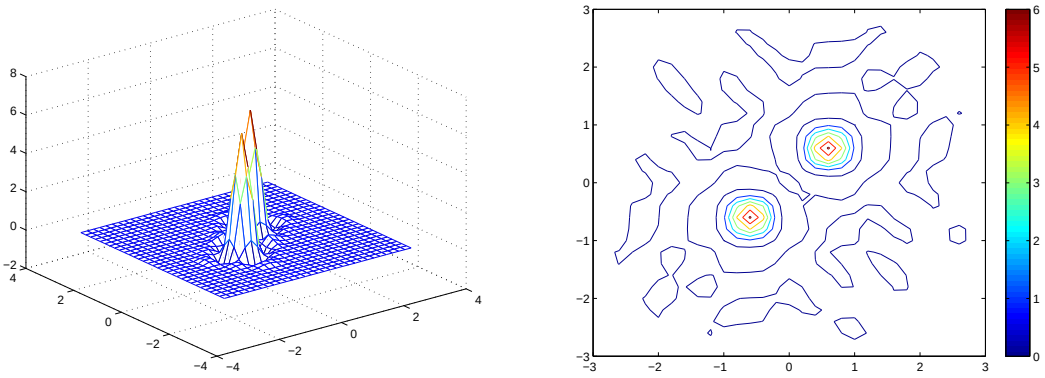
(c) $\delta = 0.2$



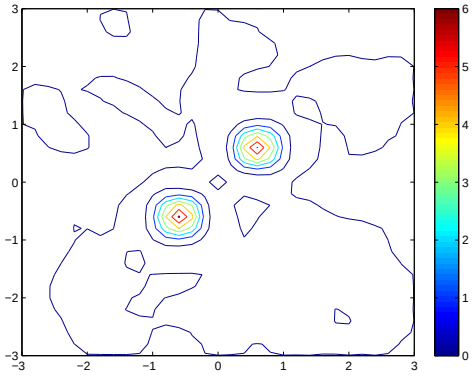
(d) $\delta = 0.5$

(e) $\delta = 1$

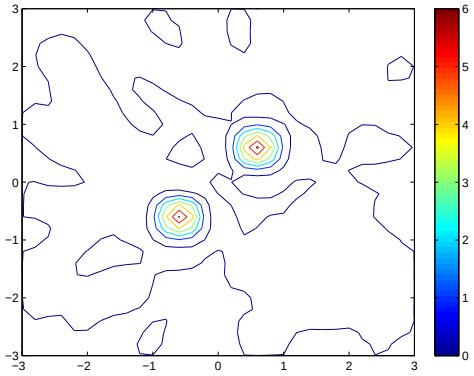
Figure A.5: (Test 4) Reconstructed initial conditions in heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-4}$ from boundary data given only on bottom and left side.



(a) $\delta = 0$, $\varepsilon = 10^{-5}$

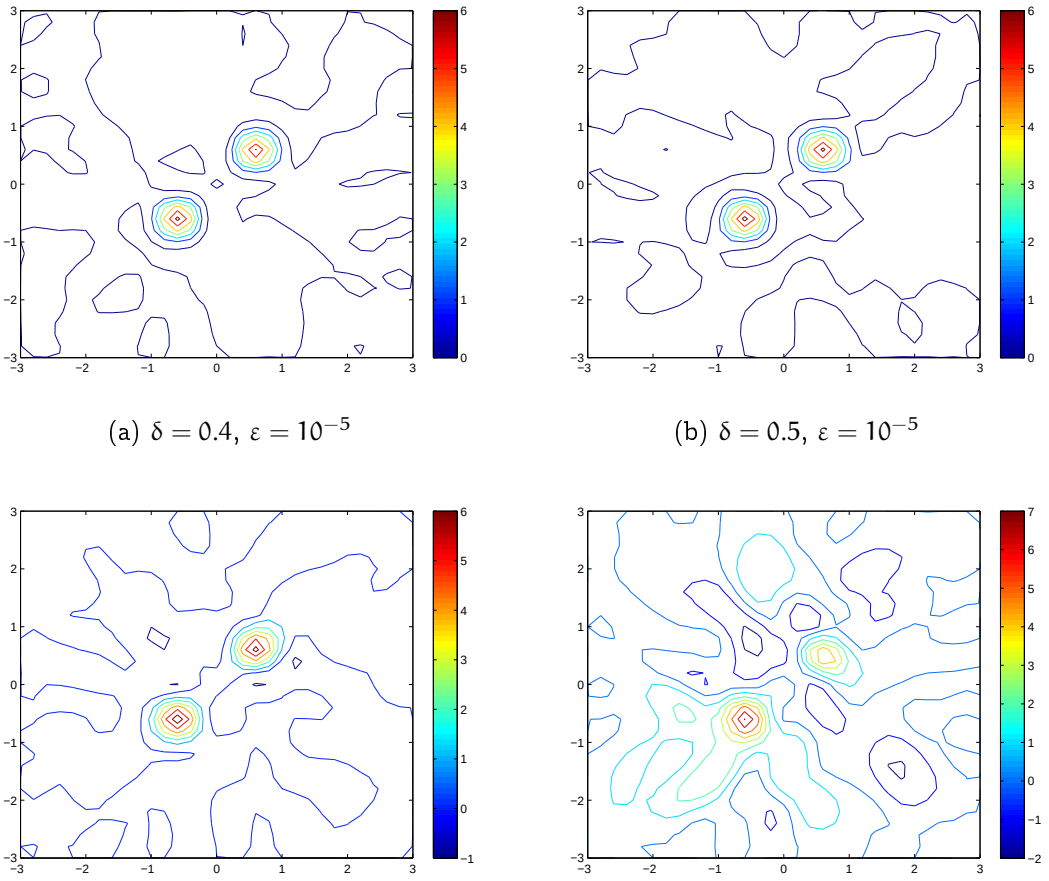


(b) $\delta = 0.1$, $\varepsilon = 10^{-5}$



(c) $\delta = 0.2$, $\varepsilon = 10^{-5}$

Figure A.6: (Test 5a) Reconstructed delta-like sources in homogeneous medium for various noise levels δ and regularization parameters ε.



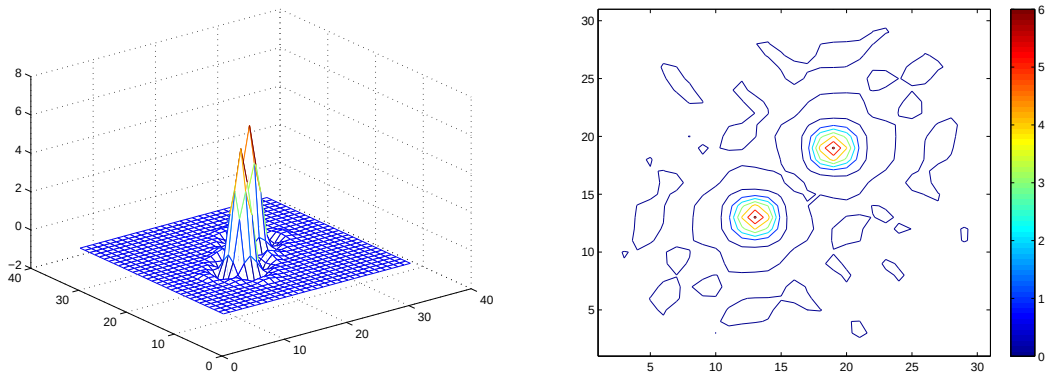
(a) $\delta = 0.4$, $\varepsilon = 10^{-5}$

(b) $\delta = 0.5$, $\varepsilon = 10^{-5}$

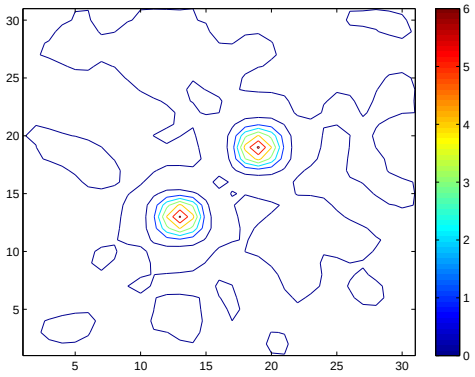
(c) $\delta = 1$, $\varepsilon = 10^{-3}$

(d) $\delta = 3$, $\varepsilon = 10^{-3}$

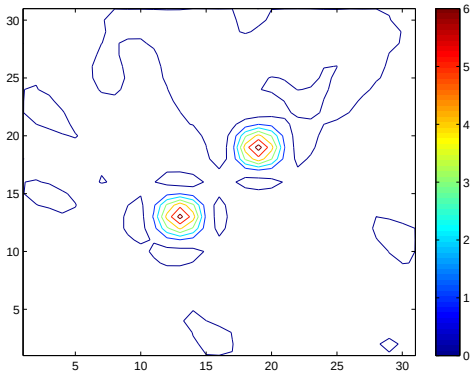
Figure A.7: (Test 5a) Reconstructed delta-like sources in homogeneous medium for various noise levels δ and regularization parameters ε. (cont'd)



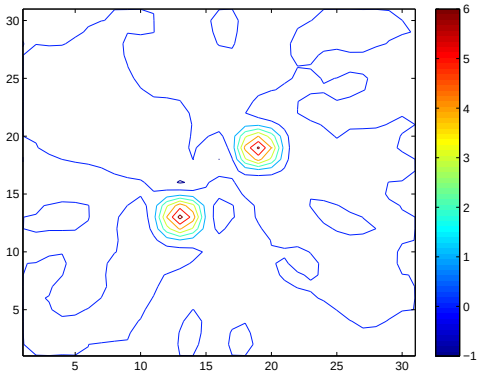
(a) $\delta = 0$



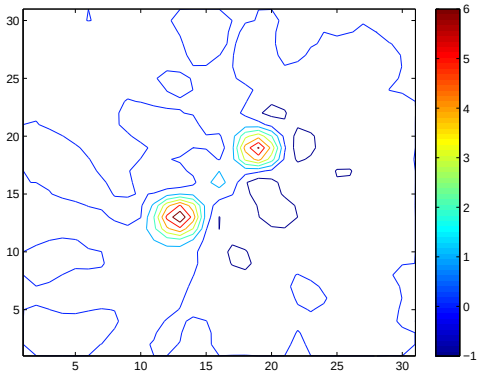
(b) $\delta = 0.1$



(c) $\delta = 0.2$

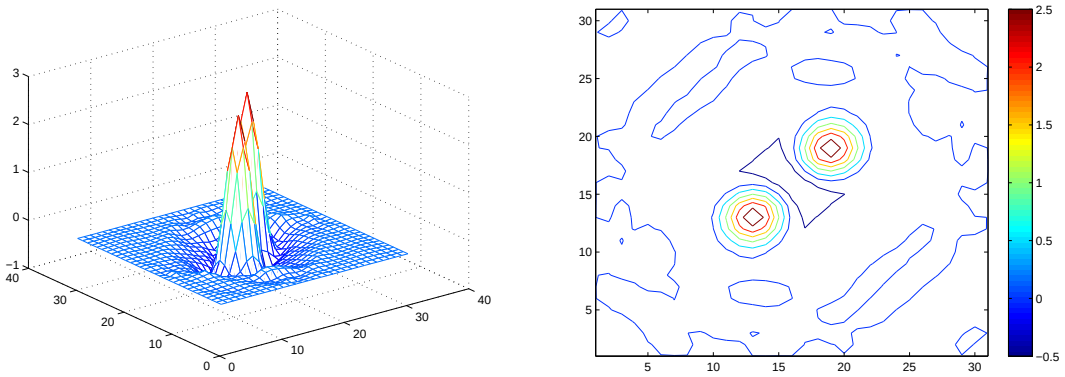


(d) $\delta = 0.5$

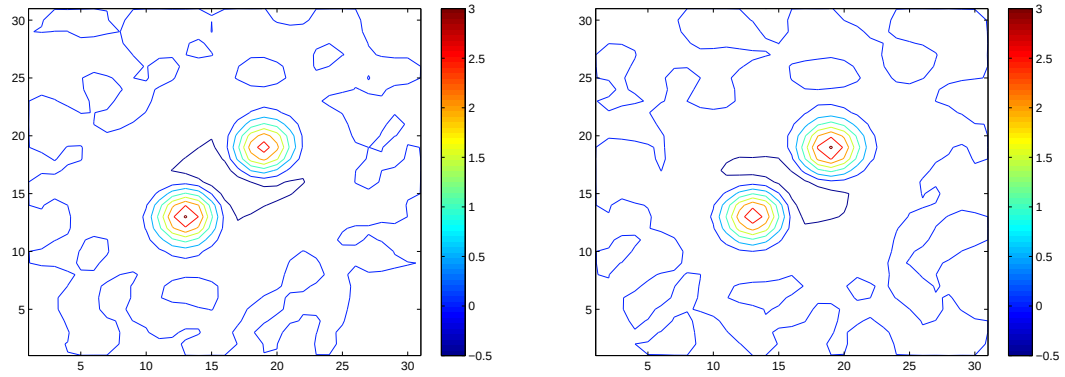


(e) $\delta = 1$

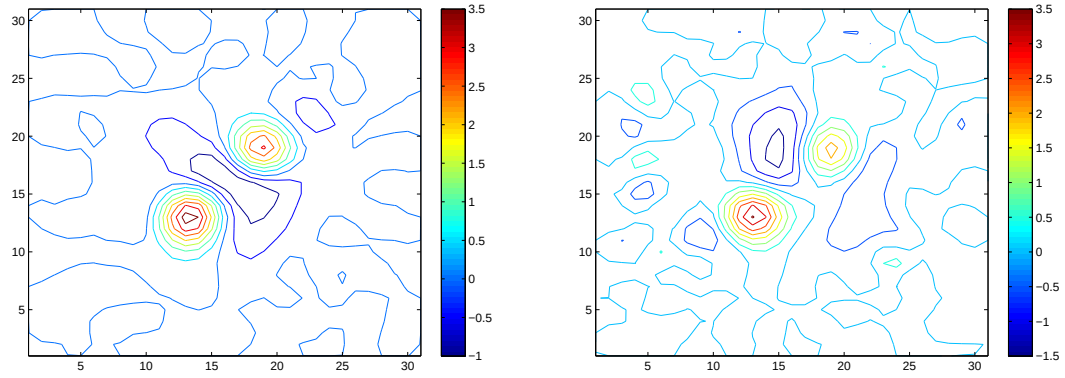
Figure A.8: (Test 5b) Reconstructed delta-like sources in heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-6}$.



(a) $\delta = 0$

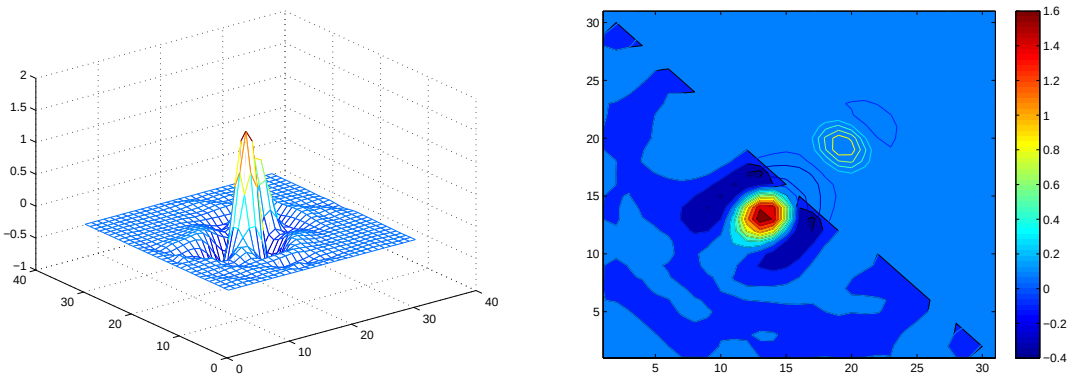


(b) $\delta = 0.1$ (c) $\delta = 0.2$

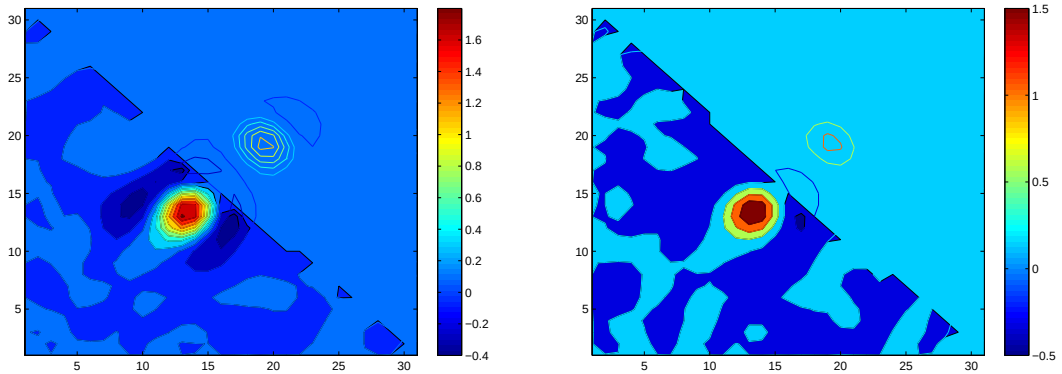


(d) $\delta = 0.5$ (e) $\delta = 1$

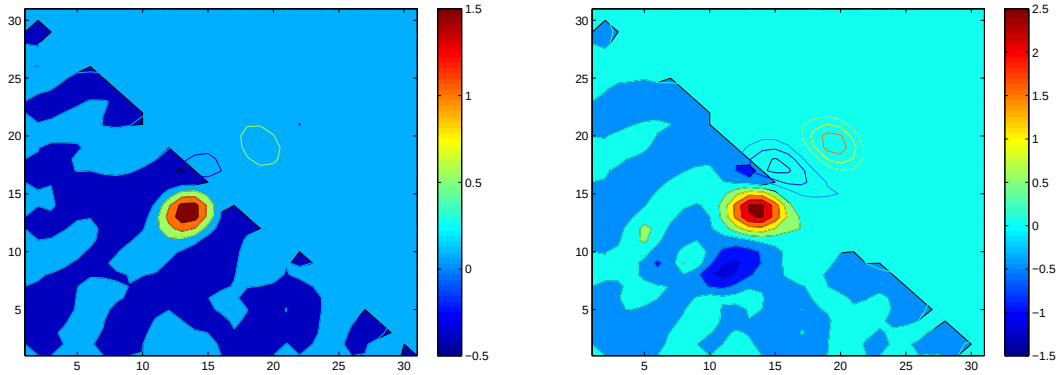
Figure A.9: (Test 5c) Reconstructed delta-like sources in heterogeneous medium (6.4) for various noise levels δ and regularization parameter $\varepsilon = 10^{-4}$.



(a) $\delta = 0$

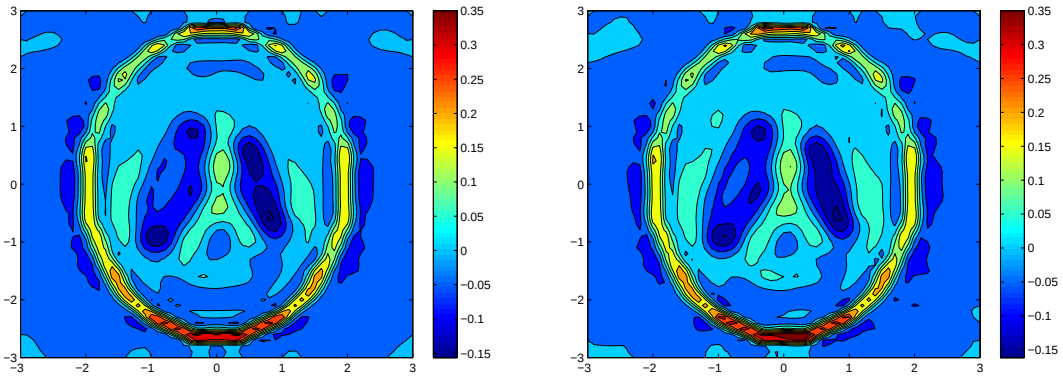


(b) $\delta = 0.1$ (c) $\delta = 0.2$

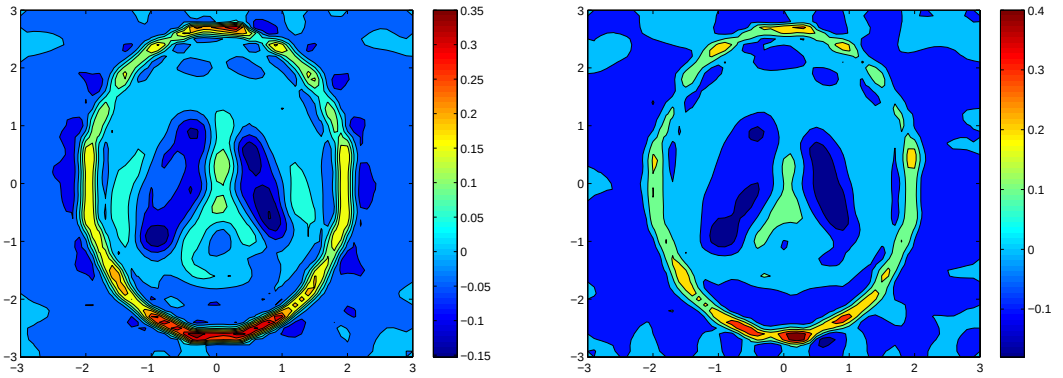


(d) $\delta = 0.5$ (e) $\delta = 1$

Figure A.10: (Test 5d) Reconstructed delta-like sources in heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-5}$ from boundary data given at bottom and left half only. The solution inside domain Q_σ is filled.



(a) $\delta = 0$ (b) $\delta = 0.05$



(c) $\delta = 0.1$ (d) $\delta = 0.2$

Figure A.11: (Test 6) Reconstructed Shepp-Logan phantom in heterogeneous medium (6.3) for various noise levels δ and regularization parameter $\varepsilon = 10^{-5}$.

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