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Structure Preserving Order Reduction of Large Scale Second Order Models

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Abstract

Structure Preserving Order Reduction of Large Scale Second Order Models

by Seyed Behnam Salimbahrami

This work deals with order reduction of large scale linear time-invariant systems in second order form while preserving the second order structure. The proposed methods are based on matching some of the characteristic parameters (moments and Markov parameters) of the original and reduced systems using the Krylov subspaces, knowing that the Krylov subspaces are the main kernel of the most attractive methods to reduce large scale systems.

Two main approaches are proposed in this dissertation. In the first method, the order of the original second order model is reduced by applying a projection directly to the original system. To find the projection matrices, an extension of Krylov subspace called Second Order Krylov Subspace is defined. This generalization involves two matrices and some starting vectors. To match the desired characteristic parameters, particular Second Order Krylov Subspaces are used where the projection matrices are calculated using an extension of the Arnoldi or Lanczos algorithms. In SISO case, this method matches at most Q characteristic parameters which is less than the standard Krylov methods in state space by which at most $2Q$ characteristic parameters are matched where Q is the order of the reduced system.

In the second method, the number of matching parameters is increased up to almost double in a cost of more computational effort. This approach consists of three steps: (i) conversion of the second order model into state space representation. (ii) reduction by a Krylov subspace method, preserving the second order character inside. (iii) back conversion into a second order representation by applying a similarity transformation. In this method at most $2Q - 1$ characteristic parameters match in SISO case.

The accuracy and suitability of the proposed methods are demonstrated through different examples of different orders and the results are compared and discussed.

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Garching bei München, May 2005 Behnam Salimbahrami

DEDICATION

To my wife Nasim.

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NOTATIONS AND ABBREVIATIONS

Notations

Scalers are denoted with lowercase or uppercase alphabets.

Vectors and matrices are denoted with bold alphabets. Normally the matrices are uppercase.

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^n$	the set of all vectors of dimension n with real entries
$\mathbb{R}^{n\times m}$	the set of all $n \times m$ matrices with real entries
$\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$	state space matrices of the original MIMO state space system
$\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}^T$	state space matrices of the original SISO state space system
$\mathbf{x}$	the vector of state variables of the original state space system
N	order of the original state space model
$\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{G}, \mathbf{L}$	matrices of the original second order MIMO system
$\mathbf{M}, \mathbf{D}, \mathbf{K}, \mathbf{g}, \mathbf{l}^T$	matrices of the original second order SISO system
$\mathbf{z}$	the vector of state variables of the original second order system
n	dimension of the original second order model
$\mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r$	state space matrices of the reduced MIMO state space system
$\mathbf{E}_r, \mathbf{A}_r, \mathbf{b}_r, \mathbf{c}_r^T$	state space matrices of the reduced SISO state space system
$\mathbf{x}_r$	the vector of state variables of the reduced state space system
Q	order of a reduced state space model
$\mathbf{M}_r, \mathbf{D}_r, \mathbf{K}_r, \mathbf{G}_r, \mathbf{L}_r$	matrices of the reduced second order MIMO system
$\mathbf{M}_r, \mathbf{D}_r, \mathbf{K}_r, \mathbf{g}_r, \mathbf{l}_r^T$	matrices of the reduced second order SISO system
$\mathbf{z}_r$	the vector of state variables of the reduced second order system
q	dimension of the reduced second order model

u	the input function
$\mathbf{u}$	the vector of input functions
y	the output function
$\mathbf{y}$	the vector of output functions
m_i	i-th moment of a SISO system
$\mathbf{m}_i$	i-th moment of a MIMO system
M_i	i-th Markov parameter of a SISO system
$\mathbf{M}_i$	i-th Markov parameter of a MIMO system
$\mathcal{K}_Q(.,.)$	the Krylov subspace
$\mathcal{K}_q(.,.,.)$	the second order Krylov subspace
$\mathbf{V}, \mathbf{W}$	projection matrices for the reduction in state space
$\bar{\mathbf{V}}, \bar{\mathbf{W}}$	projection matrices for the reduction in second order form

Abbreviations

LTI	Linear Time Invariant
TBR	Truncated Balanced Realization
MEMS	Micro-Electro-Mechanical System
SISO	Single input Single Output
MIMO	Multiple input Multiple Output
FEM	Finite Element Method

Part I

Preliminaries

Chapter 1

INTRODUCTION

Accurate modelling is a necessary part in all fields of engineering dealing with physical systems. To achieve an accurate model, powerful computers, newly developed methods and algorithms are necessary and modelling typically leads to or can be approximated by an ordinary differential equation. The more complex system is and the more accurate the model is, the higher the order of the corresponding differential equation should be. To model a physical behaviour with a good accuracy, a high order differential equation is unavoidable.

High order models occur in integrated circuits and micro systems [68], civil engineering, aerospace engineering [86], earthquake engineering [58], vibration problems [33] and mechanical and Micro-Electro-Mechanical Systems (MEMS) [7, 14, 69] (see also [2, 5, 21]) which are used for different purposes such as analysis, design, optimization, prediction and controller design.

1.1 Why we need order reduction

With the restrictions in numerical algorithms and digital computers, using a high order model to solve an engineering problem is a time consuming task or may lead to an incorrect result. In the field of control engineering, designing a controller for a high order system is difficult and if we are able to find a solution using a modern control technique like robust control, the controller would have an order the same as the original system posing difficulties in controller implementation. Simulation and analysis of a high order model is also difficult or even impossible.

In aerospace engineering, a good example is mechanical modelling of the International Space Station (ISS). It is composed of a complex structure containing several parts. Each

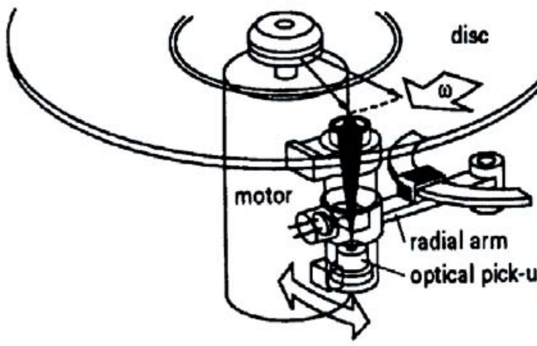

Figure 1.1: CD Player (figure from [21]).

part of this system was modelled with a system of order of several hundreds. For instance, the structural part (part 1R of the Russian Service Module) of the international space station has been modelled with a system of order 270 with 3 inputs and 3 outputs [21, 39]. Because of high complexity of this system, designing a controller without reducing the order of the original model seems to be difficult or even impossible.

One well-known example in electrical engineering is the CD player [21, 36] as shown in Figure 1.1. The most important part of this system is the optical unit (lenses, laser diode, and photo detectors) and its actuators. The main task in this system is to control the arm holding the optical unit to read the required track on the disk and to adjust the position of the focusing lens to adjust the depth of the laser beam penetrating the disc. In order to achieve this task, the system should be modelled by finite element method (FEM) leading to a differential equation of order of few hundreds which is not very easy to handle for the purpose of control.

In the field of MEMS, order reduction is a demanding task. In Figures 1.2 and 1.3 a high energy MEMS actuators is shown [13, 80]. It delivers either an impulse-bit thrust or pressure waves within a sub-millimeter volume of silicon, by producing a high amount of energy from an ignitable substance contained within the micro-system. The micro-thruster fuel is ignited by passing an electric current through a poly-silicon resistor embedded in a dielectric membrane, as shown in Figure 1.3. After the ignition phase, sustained combustion takes place and forms a high-pressure, high temperature ruptures and an impulse is imparted to the carrier frame as the gas escapes from the tank. A two-dimensional axi-symmetric model is used for the ignition phase, which after finite element based spatial discretization of the governing equations results in a linear system of 1071 ordinary

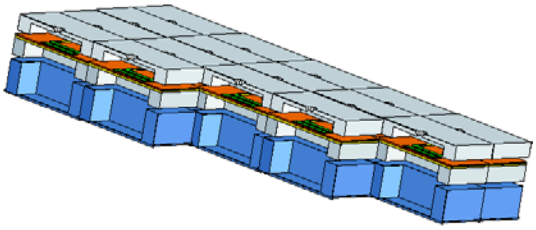

Figure 1.2: An array of Microthrusters (produced by T. Bechtold, university of Freiburg).

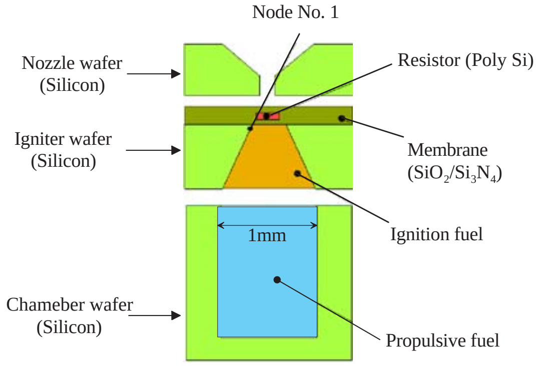

Figure 1.3: Microthruster structure (produced by T. Bechtold, university of Freiburg).

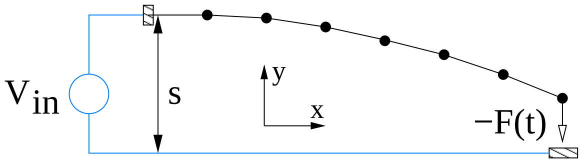

Figure 1.4: A conducting beam supported at one end with counter electrode below (produced by J. Lienemann, university of Freiburg).

differential equations.

Another example in MEMS is an electrostatically actuated beam model; see Figure 12.8. This system is used in RF switches or filters [1, 53]. Given a simple shape, they often can be modelled as one-dimensional beams embedded in two or three dimensional space. This model describes a slender beam which is actuated by a voltage between the beam and the ground electrode below. The dynamical model of this system is extracted using FEM with an order around 16000 which is quite high, making the system hard to analyze.

In Figure 1.5, another MEMS is shown. This is the Butterfly gyro which is a vibrating micro-mechanical gyro for use in inertial navigation applications [1, 52]. The sensor consists of two wing pairs that are connected to a common frame by a set of beam elements. In the excitation mode, DC-biased AC-voltages are applied to the four pairs of small electrodes and the wings are forced to vibrate in anti-phase in the wafer plane. As the structure rotates about the axis of sensitivity, each of the masses will be affected by a Coriolis acceleration. This acceleration can be represented as an inertial force that

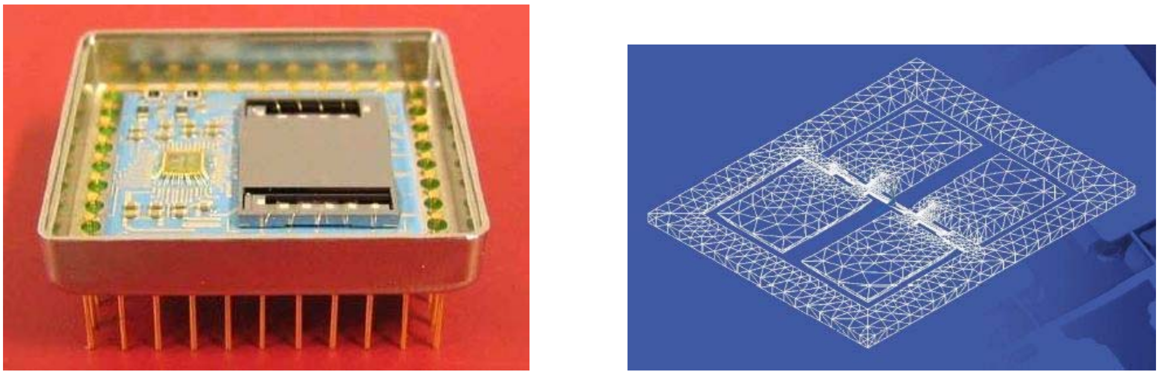

Figure 1.5: The Butterfly Gyroscope and its finite element mesh (produced by D. Billger, The Imego Institute, Sweden).

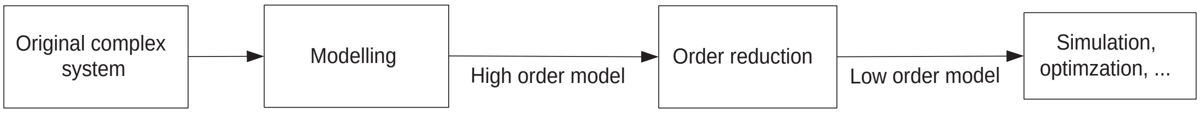

Figure 1.6: Order reduction for the purpose of simulation.

is applied at right angles with the external angular velocity and the direction of motion of the mass. The Coriolis force induces an anti-phase motion of the wings out of the wafer plane. This is the detection mode. The external angular velocity can be related to the amplitude of the detection mode, which is measured via the large electrodes. The structural model of the gyroscope has been done in ANSYS using quadratic tetrahedral elements (see Figure 1.5) leading to a model of order around 35000. When planning for and making decisions on future improvements of the Butterfly, it is of importance to improve the efficiency of the gyro simulations. The use of model order reduction indeed decreases runtime for repeated simulations.

A solution to simplify the preceding tasks in both fields of simulation [2] and controller design [41, 62] is to find a low order approximation of the original high order model. The procedure of order reduction is shown in Figure 1.6. The reduced order modelling for simulation is done directly after the step for modelling. In controller design, one can reduce the original system before designing the controller or simultaneously with controller design where the reduction is done on the controller by considering the performance of the closed loop system; see for instance [62] and the references therein.

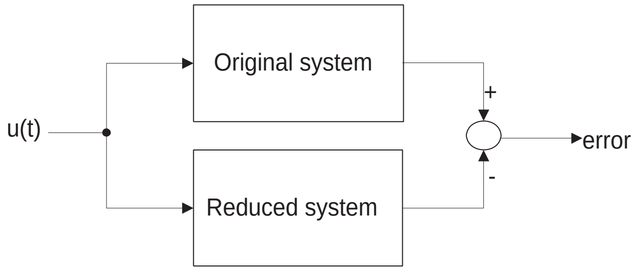

Figure 1.7: Order reduction by minimization the difference of the outputs.

1.2 Existing methods of order reduction

The main goal of the reduction is to find the best possible approximation of the output of the original system as shown in Figure 1.7. Several methods have been proposed for reduction of LTI systems in different fields like control engineering, micro-systems and applied mathematics. These methods are mostly based on minimization of some predefined error functions, deleting the less important states or matching some of the parameters of the original and reduced systems.

1.2.1 Balancing and truncation

A well-accepted method in order reduction of LTI systems is truncated balanced realization (TBR) which was first proposed by Moore [60]. TBR tries to delete the states that contribute smaller amount of energy to the outputs and needs a lot of input energy to change. Basic definitions in TBR are controllability and observability gramians and Hankel singular value. Controllability gramian represents the amount of input energy to change the states from zero to another point in the state space and observability gramian represents the amount of energy in the output produced by an initial condition when the input is set to zero. Hankel singular values are parameters that are defined as the square roots of the eigenvalues of the product of two gramians.

To delete the appropriate states, a similarity transformation is applied to the original system to transform it to a balanced form in which the controllability and observability gramians are equal and diagonal. If the diagonal entries of the gramians are sorted, a balanced system can be easily reduced by truncating the state space model which deletes the states corresponding to small entries of gramians. To calculate the gramians of a

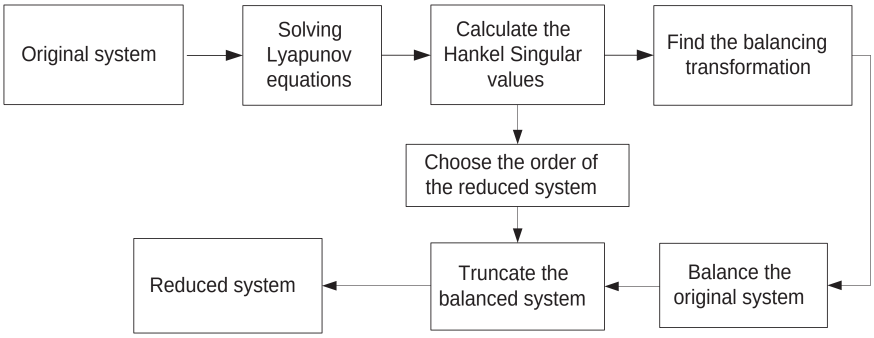

Figure 1.8: The main steps of TBR.

system, two corresponding Lyapunov equations in the size of original model are to be solved. Figure 1.8 illustrates the steps of reduction using TBR.

An important property of TBR is preserving stability of the original system, that is if the original model is stable, then the reduced order model is also stable. Glover [34] proved that there exists an error bound in order reduction using TBR. He also showed how to find the reduced order system in a different way such that the reduction procedure is optimal in the sense of Hankel norm which is defined as the largest Hankel singular value of a system.

Although TBR is not optimal in any norm, it is theoretically attractive and yields to interesting reduced order models in practice. There exist different alternatives to find a reduced system based on TBR [16, 49, 71]; see also [72] and the references therein. Balancing can also be done using a cross gramian approach where solving a Sylvester equation is involved [84]. In all methods to find a TBR reduced system, solving the Lyapunov equations (or a Sylvester equation) is a key step which is computationally expensive and restricts the use of TBR for model reduction of large scale systems.

Several methods have been proposed to extend the range of applicability of TBR to higher order models. In [15, 16], the parallel computation is used for an efficient calculation. Another option is to implement some algorithms that can solve large Lyapunov or Sylvester equations approximately, leading to the so called low rank gramians. Low rank gramians are then used for approximate balancing and truncation [40, 42, 51, 64, 65, 84]. The result of the reduction procedure by approximate TBR, depending on the iterations to produce the approximate gramians, can be close to the exact TBR.

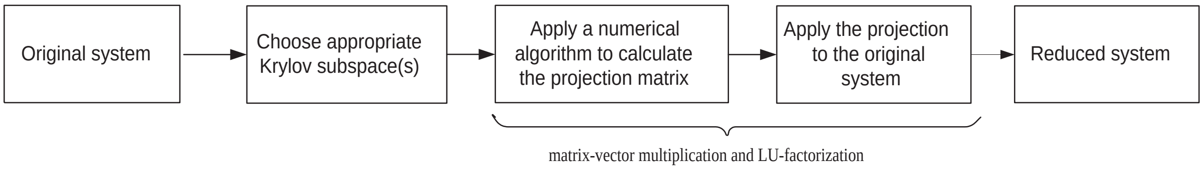

Figure 1.9: The main steps of Krylov subspace methods.

1.2.2 Krylov subspace methods

Today, one of the best choices in order reduction of large scale systems is *moment matching using Krylov subspaces* that was first proposed in [88] and then by using numerically reliable algorithms has been applied for the reduction of large scale systems [26, 29, 30, 37, 47, 54, 55, 73, 76].

In this approach, the lower order model is obtained such that some of the first moments (and/or Markov parameters) of the original and reduced systems are matched where the moments are the coefficients of the Taylor series expansion of the transfer function about a suitable point. When the expansion point tends to infinity, the coefficients are called Markov parameters. In fact, the moments about zero represent the behaviour of the system at low frequencies and the Markov parameters represent the high frequency behaviour.

Such a reduced model can be calculated using Krylov subspaces by means of well-established algorithms, Arnoldi [6], Lanczos [48] or two-sided Arnoldi [24, 78, 80], and by applying a projection to the original system. The projection matrices are calculated through an iterative procedure which is a great advantage and the reduced system is found in a relatively short time compared to TBR with more numerical reliability when dealing with high order systems. In Figure 1.9, the steps of reduction for moment matching are shown that can be compared to the similar one of TBR, in Figure 1.8 where the Lyapunov equations are to be solved.

Krylov subspace methods are actually rational Hermite interpolation as it finds a rational approximation of a function by matching the coefficients of the Taylor series expansion around a desired point. The maximum number of parameters that can be matched by these methods are double the order of the reduced system which is in fact the number of coefficients in a rational transfer function. In the case of reaching this maximum the reduced model is called a Padé approximation. Such an interpolation problem can also be solved by just solving an algebraic equation which is not in general well-conditioned compared to Krylov subspace method [2, 66, 67]. It is also possible to interpolate the transfer function over a set of points in the complex plane [2, 22].

In Table 1.1, TBR is compared to Krylov subspace methods. As it can be seen in the table, the Krylov subspace method is superior in numerical efficiency with cheaper calculations but the stability of the original system may be lost and there is no general error bound similar to TBR except under some special conditions [9, 11]. In practice, for the medium scale systems, a Krylov subspace method does not lead to reduced systems with better accuracy than TBR. For moment matching, computational cost of an *LU* factorization of the matrix $\mathbf{A}$ should be added to the table which is $\mathcal{O}(N^3)$. However, the *LU* factorization is eliminated if $\mathbf{A}$ is triangular or becomes cheaper if $\mathbf{A}$ is sparse and has some special structure (block diagonal or close to diagonal).

Table 1.1: Comparison of Krylov subspace method and TBR.

	TBR	Krylov subspace method
Numerical cost (flops)	$\mathcal{O}(N^3)$	$\mathcal{O}(Q^2N)$
Numerical reliability for large N	No	Yes
Accuracy of the reduced system	more accurate	less accurate
range of applicability	up to order few thousands	up to several ten thousands
preserving stability	Yes	No
Iterative method	No	Yes
Reliable stopping criterion	Yes	No

1.3 Reduction of systems in second order form

In some fields like electrical circuits and mechanical systems, modelling (for instance by FEM) leads to a large number of *second* order differential equations [7, 68, 69, 82]. For instance the CD player model, electrostatic beam and the butterfly Gyro introduced at the beginning of this chapter are in second order form. It is then advisable to construct a reduced order model that approximates the behavior of the original system while preserving its second-order structure [12, 55, 77, 86].

An extension of balancing and truncation to reduce second order system was first introduced by Meyer and Srinivasan in [59]. They have defined the so called *free velocity* and *zero velocity* gramians for second order systems that can be calculated using the gramians of an equivalent state space model. A balanced second order model in which the second order gramians are diagonal and equal, is then found by applying a transformation to the second order system.

Balancing based on free velocity and zero velocity gramians was then improved and extended by other authors to the so called second-order balanced truncation (SOBT) [19, 20]. The main cost of calculation in SOBT is also in calculating the gramians of an equivalent state space model.

It is not recommended to use balancing and truncation of second order models for the reduction of large scale systems for numerical reasons. To reduce the order of large scale second order systems, it is required to implement more reliable and faster algorithms and preferably iterative procedures. The first idea is of course extending the numerically efficient algorithms like Arnoldi and Lanczos, which are used in Krylov subspace methods as well-accepted approaches for the reduction of large scale state space models.

One of the oldest extensions of moment matching method for second order model was proposed by Su and Craig [86] which is equivalent to a recent work in [12] where the reduced system is found in a way different from [86]. In both papers, the Krylov subspaces were used and the structure of the original system is preserved. However, there are some disadvantages like matching smaller number of moments (up to 1/4) compared to the standard Krylov subspace methods or difficulty to match the moments about nonzero points.

Recently, in several works, it is tried to extend the Krylov subspace approach for reduced order modeling of second order systems. In [31, 32, 50, 87], it is proposed to reduce the equivalent state space system by applying a projection such that the structure of the state space matrices does not change and an algorithm is given to find the desired projection matrices. This method has difficulties to match the moments around zero as the projection matrix becomes rank deficient because of a zero column. It is also numerically more expensive than the methods proposed in this dissertation and matches half number of parameters compared to the back conversion method.

1.4 The Purpose of this thesis

In this dissertation, two different approaches are proposed for the reduction of second order systems. In the first method, we generalize the Krylov subspace method for second order systems using a *Second Order Krylov Subspace* which was first introduced in [75] for single input-single output (SISO) systems and more investigated and generalized in [57]. Related results were also found by other researchers independently [10].

By using this kind of Krylov subspaces, the original system is reduced by applying a projection directly to the second order model and the method presented in [86] is modified in different directions:

i. The projection matrix can be independent of the output equation.

i. The number of matching moments is doubled when using two-sided methods.

ii. The method is generalized to match the moments about different points.

iii. It is shown how to match the Markov parameters.

To calculate the projection matrices, modifications of the Arnoldi and Lanczos algorithms are proposed which find orthogonal or bi-orthogonal bases for given Second Order Krylov Subspaces and deflate the repeated vectors in the case of multiple starting vectors.

In the second approach, we try to match more moments compared to the first method by reducing the equivalent state space equation and then recovering the second order structure from the reduced state space system. The steps of the reduction procedure are:

i. Conversion of the second order model into a state space representation.

i. Reduction by a Krylov subspaces method, preserving the second order character inside by matching the first Markov parameter.

ii. Back conversion into a second order representation by applying a similarity transformation.

It is shown that the first Markov parameter of a second order system is zero which is a key point for the back conversion procedure. By the second method, maximum number of parameters can be matched which is almost double compared to second order Krylov methods however more calculation is needed because of reduction in a double dimension and the back conversion procedure.

1.5 Thesis outline

This dissertation consists of 4 different parts. In the upcoming chapters of the current part some preliminary information is given. We first introduce Krylov subspace methods to reduce state space systems in the next chapter, then some properties of second order models are introduced and discussed.

The second part deals with order reduction of second order systems by applying a projection directly to the original second order model. We will introduce the Second Order Krylov Subspace and present the theoretical background and proofs to match the moments about different points and Markov parameters. In Chapter 5, the necessary numerical algorithms to calculate the projection matrices are given. We discuss the invariance properties of the proposed approach in Chapter 6 and generalize it to reduce systems of higher order differential equations in Chapter 7.

The third part is about the reduction procedure by reducing the equivalent state space and back conversion into second order form. The main idea of the procedure including a simple method for back conversion in given in Chapter 8. In Chapter 9, we prove that under some weak conditions, such a back conversion is possible in SISO case and some general necessary conditions for multi-input multi-output (MIMO) systems are also discussed. In Chapter 10, we focus on undamped second order systems and the sufficient conditions to find the back conversion transformation in MIMO case is given. In Chapter 11, the steps of state space reduction and back conversion in SISO case are integrated into a single algorithm to reduce the computational effort and increase the efficiency of the method.

At the end of the report, the proposed methods are applied to different technical systems. We consider the model of a building of order 48, the structural part of the international space station of order 270 and model of a beam of order 15992.

Chapter 2

ORDER REDUCTION USING KRYLOV SUBSPACES

For the reduction of very high order systems, the methods based on Krylov subspace are among the best choices today. They define a projection from the high dimensional space of the original model to a lower dimensional space and vice versa and thereby define the reduced order model with application to circuit simulation, micro-electro-mechanical systems and more. This method was first proposed in [88] and more investigated and modified with several others [26, 29, 30, 37, 47, 54, 55, 73, 76].

In this chapter, we discuss the reduction method by applying a projection using bases of some particular Krylov subspaces. One aim of this chapter is reviewing the Krylov subspace methods including the famous algorithms used to find the reduced order matrices matching the moments, Markov parameters or both of them.

2.1 System representation and moments

We consider the dynamical MIMO system of the form

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{2.1}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{B} \in \mathbb{R}^{N\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times N}$ are given matrices and the components of the vector valued functions $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ and $\mathbf{x} \in \mathbb{R}^N$ are the inputs, outputs and states of the system, respectively. For SISO systems, $p = m = 1$, the matrices $\mathbf{B}$ and $\mathbf{C}$ change to vectors $\mathbf{b}$ and $\mathbf{c}^T$.

The transfer matrix of the system (2.1) is

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}. \tag{2.2}$$

By assuming that $\mathbf{A}$ is nonsingular, the Taylor series expansion of the transfer matrix (2.2) about zero is,

$$\mathbf{H}(s) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})\mathbf{A}^{-1}\mathbf{B}s - \cdots - \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^{i}\mathbf{A}^{-1}\mathbf{B}s^{i} - \cdots \; . \tag{2.3}$$

The coefficients of this series, without negative sign, are called moments according to the following:

Definition 2.1 *In system (2.1), suppose that* $\mathbf{A}$ *is nonsingular, then the i-th moment (about zero) of this system is*

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^{i}\mathbf{A}^{-1}\mathbf{B} \quad , \quad i = 0, 1, \cdots \; . \tag{2.4}$$

$\mathbf{m}_i$ *is a* $p \times m$ *matrix in MIMO case and a scalar* m_i*, in SISO case.*

Moments can be defined about points $s_0 \neq 0$ by rewriting the transfer matrix as

$$\mathbf{H}(s) = \mathbf{C}[(s - s_0)\mathbf{E} - (\mathbf{A} - s_0\mathbf{E})]^{-1}\mathbf{B} \; . \tag{2.5}$$

By comparing the equations (2.2) and (2.5) the moments about s_0 can be computed by substituting $\mathbf{A}$ by $\mathbf{A} - s_0\mathbf{E}$ in definition 2.1, assuming that $\mathbf{A} - s_0\mathbf{E}$ is nonsingular. In fact, the moments of $\mathbf{H}(s)$ about s_0 are the moments of $\mathbf{H}(s + s_0)$ about zero and s_0 should not be a generalized eigenvalue of the pair $(\mathbf{E}, \mathbf{A})$.

A different series results when $s_0 \longrightarrow \infty$. By putting $s = 1/\zeta$ in (2.2) and developing the Taylor series about $\zeta = 0$, the series is

$$\mathbf{H}(s) = \mathbf{C}\mathbf{E}^{-1}\mathbf{B}s^{-1} + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})\mathbf{E}^{-1}\mathbf{B}s^{-2} + \ldots + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{i}\mathbf{E}^{-1}\mathbf{B}s^{-i} + \ldots \; , \tag{2.6}$$

and its coefficients are called Markov parameters. The i-th Markov parameter is also the value of the i-th derivative of the impulse response at time zero [46]. So, the first Markov parameter, $\mathbf{M}_0$, is the impulse response at $t = 0$.

Definition 2.2 *In system (2.1), suppose that* $\mathbf{E}$ *is nonsingular, then the i-th Markov parameter is defined as*

$$\mathbf{M}_i = \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{i}\mathbf{E}^{-1}\mathbf{B} \quad , \quad i = 0, 1, \cdots \; . \tag{2.7}$$

Moments and Markov parameters will be used to describe *similarity* of original and reduced order models. In other words, moments about zero reflect the behaviour of a system at low frequencies. When the point s_0 increases, it more reflects the behaviour at higher frequencies. This can be explained by the behaviour of the method to approximate the poles with larger real parts as s_0 increases [37] or by using orthogonal polynomials where a real-valued transfer function is used [27, 45].

2.2 Order reduction using Krylov subspaces

In this section, order reduction by applying projections to system (2.1) is introduced. Suitable projections are calculated from Krylov subspaces, defined in the following.

2.2.1 Krylov subspace

Definition 2.3 *The Krylov subspace is defined as*

$$\mathcal{K}_Q(\breve{\mathbf{A}}, \breve{\mathbf{b}}) = span\{\breve{\mathbf{b}}, \breve{\mathbf{A}}\breve{\mathbf{b}}, \cdots, \breve{\mathbf{A}}^{Q-1}\breve{\mathbf{b}}\}, \tag{2.8}$$

where $\breve{\mathbf{A}} \in \mathbb{R}^{N\times N}$ *and* $\breve{\mathbf{b}} \in \mathbb{R}^{N}$ *is called the starting vector. The vectors* $\breve{\mathbf{b}}, \breve{\mathbf{A}}\breve{\mathbf{b}}, \cdots,$ $\breve{\mathbf{A}}^{Q-1}\breve{\mathbf{b}}$ *that construct the subspace are called basic vectors.*

If the i-th basic vector in Krylov subspace (2.8) is a linear combination of the previous vectors, then the next basic vectors can be written as linear combinations of the first $i-1$ vectors (this can easily be proved by pre-multiplying with $\breve{A}$). Therefore, the first independent basic vectors can be considered as a basis of a Krylov subspace.

When there exist more than one starting vector, definition 2.3 is generalized to the following form.

Definition 2.4 *The block Krylov subspace is defined as*

$$\mathcal{K}_Q(\breve{\mathbf{A}}, \breve{\mathbf{B}}) = colspan\{\breve{\mathbf{B}}, \breve{\mathbf{A}}\breve{\mathbf{B}}, \cdots, \breve{\mathbf{A}}^{Q-1}\breve{\mathbf{B}}\}, \tag{2.9}$$

where $\breve{\mathbf{A}} \in \mathbb{R}^{N\times N}$ *and the columns of* $\breve{\mathbf{B}} \in \mathbb{R}^{N\times m}$ *are the starting vectors.*

The block Krylov subspace with m starting vectors can be considered as a union of m Krylov subspaces defined for each starting vector.

2.2.2 Moment matching (SISO)

Consider a projection as follows:

$$\begin{aligned} &\mathbf{x} = \mathbf{V}\mathbf{x}_r, \\ &\mathbf{V} \in \mathbb{R}^{N\times Q}, \mathbf{x} \in \mathbb{R}^N, \mathbf{x}_r \in \mathbb{R}^Q, \end{aligned} \tag{2.10}$$

where $Q < N$. By applying this projection to the system (2.1) in SISO case and then multiplying the state equation by transpose of a matrix $\mathbf{W} \in \mathbb{R}^{N\times Q}$, a reduced model of order Q can be found,

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r + \mathbf{W}^T\mathbf{b}u, \\ y = \mathbf{c}^T\mathbf{V}\mathbf{x}_r. \end{cases} \tag{2.11}$$

The reduced order system in state space is identified by the following matrices:

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}\ ,\ \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}\ ,\ \mathbf{b}_r = \mathbf{W}^T\mathbf{b}\ ,\ \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}. \tag{2.12}$$

Now, the question is how to choose the projection matrices to find a reduced system with a behaviour similar to the original one. In the following theorems, it is shown that bases of suitable Krylov subspaces can be used in order reduction by projection. We consider that Q is small enough such that all corresponding basic vectors are linearly independent.

Theorem 2.1 *If the columns of the matrix* $\mathbf{V}$ *used in (2.11), form a basis for the Krylov subspace* $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ *and the matrix* $\mathbf{W} \in \mathbb{R}^{N\times Q}$ *is chosen such that the matrix* $\mathbf{A}_r$ *is nonsingular, then the first* Q *moments (about zero) of the original and reduced order systems match.*

Proof: Consider the vector,

$$\mathbf{V}\mathbf{A}_r^{-1}\mathbf{b}_r = \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b}.$$

The vector $\mathbf{A}^{-1}\mathbf{b}$ is in the Krylov subspace and it can be written as a linear combination of the columns of $\mathbf{V}$,

$$\exists \mathbf{r}_0 \in \mathbb{R}^Q : \mathbf{A}^{-1}\mathbf{b} = \mathbf{V}\mathbf{r}_0.$$

Therefore,

$$(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T(\mathbf{A}\mathbf{A}^{-1})\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{r}_0 = \mathbf{r}_0.$$

With this,

$$\mathbf{V}\mathbf{A}_r^{-1}\mathbf{b}_r = \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \mathbf{V}\mathbf{r}_0 = \mathbf{A}^{-1}\mathbf{b}.$$

Now, consider for $i = 0, \cdots, k$, we have,

$$\mathbf{V}\left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^i \mathbf{A}_r^{-1}\mathbf{b}_r = (\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{b}.$$

For $i = k+1$, we have,

$$\begin{aligned}
\mathbf{V}\left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^{i+1} \mathbf{A}_r^{-1}\mathbf{b}_r &= \mathbf{V}\mathbf{A}_r^{-1}\mathbf{E}_r\left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^i \mathbf{A}_r^{-1}\mathbf{b}_r \\
&= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^i \mathbf{A}_r^{-1}\mathbf{b}_r \\
&= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\left(\mathbf{A}^{-1}\mathbf{E}\right)^i \mathbf{A}^{-1}\mathbf{b} \\
&= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{A}^{-1}\mathbf{E}\left(\mathbf{A}^{-1}\mathbf{E}\right)^i \mathbf{A}^{-1}\mathbf{b} \\
&= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1} \mathbf{A}^{-1}\mathbf{b}. \qquad (2.13)
\end{aligned}$$

If $i+1 \leq Q-1$, then the vector $\left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1}\mathbf{A}^{-1}\mathbf{b}$ is in the Krylov subspace and it can be written as a linear combination of the columns of $\mathbf{V}$,

$$\exists \mathbf{r}_{i+1} \in \mathbb{R}^Q : \left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1}\mathbf{A}^{-1}\mathbf{b} = \mathbf{V}\mathbf{r}_{i+1}. \qquad (2.14)$$

We combine the equations (2.13) and (2.14),

$$\begin{aligned}
\mathbf{V}\left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^{i+1} \mathbf{A}_r^{-1}\mathbf{b}_r &= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1} \mathbf{A}^{-1}\mathbf{b} \\
&= \mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{r}_{i+1} \\
&= \mathbf{V}\mathbf{r}_{i+1} = \left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1}\mathbf{A}^{-1}\mathbf{b}. \qquad (2.15)
\end{aligned}$$

As mentioned before, the proof can be continued up to $i = Q-1$. Then, if we multiply both sides of the result in (2.15) with $\mathbf{c}^T$, then we have definitions of moments for original and reduced systems and the moments from zero to $Q-1$ match. ■

The subspace $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ is called *input Krylov subspace* and order reduction using a basis of this subspace for projection and optionally chosen the matrix $\mathbf{W}$ is called a *one-sided Krylov subspace* method.

Another important Krylov subspace used for moment matching is $\mathcal{K}_Q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$ which is called *output Krylov subspace* and is dual to the input Krylov subspace. By using this duality, it is possible to choose the matrix $\mathbf{V}$ optionally and the matrix $\mathbf{W}$ as a basis of the output Krylov subspace and the first Q moments match. It can be expressed as the following theorem.

Theorem 2.2 *If the columns of* $\mathbf{W}$ *used in (2.11) form a basis for the output Krylov subspace and the matrix* $\mathbf{V}$ *is chosen such that* $\mathbf{A}_r$ *is nonsingular, then the first* Q *moments of the original and reduced order systems match.*

Proof: The proof of this theorem is quite similar to the proof of Theorem 2.1 but here by induction, it is proved that for $i = 0, 1, \cdots, Q-1$, we have,

$$\mathbf{c}_r^T \left(\mathbf{A}_r^{-1}\mathbf{E}_r\right)^{i+1} \mathbf{A}_r^{-1}\mathbf{W}^T = \mathbf{c}^T \left(\mathbf{A}^{-1}\mathbf{E}\right)^{i+1} \mathbf{A}^{-1}. \tag{2.16}$$

We multiply the equation (2.16) from right hand side with $\mathbf{b}$, and the proof is completed. ■

Although choosing the other projection matrix in a one-sided method is optional, a typical choice is $\mathbf{W} = \mathbf{V}$ which also has some advantages like preserving stability under some additional conditions on the original state space model [29].

By combining the results of the preceding theorems and using both input and output Krylov subspaces, it is possible to match more moments as in the following theorems.

Theorem 2.3 *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (2.11), form bases for the Krylov subspaces* $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ *and* $\mathcal{K}_Q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$*, respectively, then the first* $2Q$ *moments of the original and reduced order systems match. It is assumed that* $\mathbf{A}$ *and* $\mathbf{A}_r$ *are invertible.*

Proof: According to the Theorem 2.1, the first Q moments match. Now, we write the moments of the reduced system as follows,

$$\begin{aligned} m_{ri} &= \mathbf{c}_r^T(\mathbf{A}_r^{-1}\mathbf{E}_r)^i\mathbf{A}^{-1}\mathbf{b} \\ &= \mathbf{c}_r^T(\mathbf{A}_r^{-1}\mathbf{E}_r)^{i-Q}(\mathbf{A}_r^{-1}\mathbf{E}_r)(\mathbf{A}_r^{-1}\mathbf{E}_r)^{Q}\mathbf{A}^{-1}\mathbf{b} \\ &= \mathbf{c}_r^T(\mathbf{A}_r^{-1}\mathbf{E}_r)^{i-Q}\mathbf{A}_r^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}(\mathbf{A}_r^{-1}\mathbf{E}_r)^{Q-1}\mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

By using the equations (2.15) and (2.16) for $i - Q = 0, \cdots, Q-1$, we have,

$$\begin{aligned} m_{ri} &= \mathbf{c}(\mathbf{A}^{-1}\mathbf{E})^{i-Q}\mathbf{A}^{-1}\mathbf{E}(\mathbf{A}^{-1}\mathbf{E})^{Q-1}\mathbf{A}^{-1}\mathbf{b} \\ &= \mathbf{c}(\mathbf{A}^{-1}\mathbf{E})^{i}\mathbf{b} = m_i, \end{aligned}$$

and the moments from zero to $2Q-1$ match. ■

Order reduction by using both input and output Krylov subspaces for projection is called *two-sided Krylov subspace* method.

These theorems were founded for matching the moments about zero. The results can be extended to match the moments about $s_0 \neq 0$ by substituting $\mathbf{A}$ by $\mathbf{A} - s_0\mathbf{E}$ in the definition of moments and Krylov subspaces. This means that for instance in Theorem 2.3 the subspaces $\mathcal{K}_Q((\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{b})$ and $\mathcal{K}_Q((\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{c})$ are considered. The projection is then applied to the model (2.1), as described in equations (2.10) and (2.12) (i.e. $\mathbf{A}$ in equation (2.12) is not substituted by $\mathbf{A} - s_0\mathbf{E}$). With $s_0 = 0$, the reduced and original model have the same DC gain and steady state accuracy is achieved. Small values of s_0 will also find a reduced model with good approximation of slow dynamics. An approximation of the full state vector $\mathbf{x}$ can be found from $\mathbf{x}_r$ by $\hat{\mathbf{x}} = \mathbf{V}\mathbf{x}_r$.

2.2.3 Matching the Markov parameters (SISO)

Another way to determine the similarity between LTI systems, specially at high frequencies, is comparing the Markov parameters. By suitably changing the starting vectors in input and output Krylov subspaces, not only some of the moments but also some of the Markov parameters match.

In [43] a special case for matching only the Markov parameters, called *Oblique Projection*, has been introduced. Matching the Markov parameters as discussed in [43] leads to a good approximation at high frequencies which most of the time is not desired. In the following a general case is discussed.

Theorem 2.4 *If the columns of* $\mathbf{V}$ *used in (2.11), form a basis for the Krylov subspace* $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^l\mathbf{A}^{-1}\mathbf{b})$ *where* $l \in \mathbb{Z}$ *and* $0 \leq l \leq Q$ *and* $\mathbf{W} \in \mathbb{R}^{N\times Q}$ *is chosen such that the matrices* $\mathbf{A}_r$ *and* $\mathbf{E}_r$ *are nonsingular then the first* $Q - l$ *moments and the first* l *Markov parameters of the original and reduced order systems match.*

Proof: The Krylov subspace $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^l\mathbf{A}^{-1}\mathbf{b})$ can be written in two parts as,

$$\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^l\mathbf{A}^{-1}\mathbf{b}) = \mathcal{K}_{Q-l}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b}) \cup \mathcal{K}_l(\mathbf{E}^{-1}\mathbf{A}, \mathbf{b}).$$

By considering the first subspace, the proof for moments $m_0, \cdots, m_{Q-l}$ is clear from Theorem 2.1. For the Markov parameters, the second subspace is used and in a similar way as for the moments, by using induction, it is proved that for $i = 0, 1, \cdots, l-1$

$$\mathbf{V}(\mathbf{E}_r^{-1}\mathbf{A}_r)^i\mathbf{E}_r^{-1}\mathbf{b}_r = (\mathbf{E}^{-1}\mathbf{A})^i\mathbf{E}^{-1}\mathbf{b}.$$

Then, by multiplying the left hand side with $\mathbf{c}^T$, the proof is completed. ■

By using $\mathbf{W}$ as a basis of the output Krylov subspace with a suitable starting vector, it is possible to match more parameters (moments and Markov parameters) of reduced and original models. The following theorem generalizes Theorem 2.3 to match the Markov parameters.

Theorem 2.5 *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (2.11), form bases for the Krylov subspaces* $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^{l_1}\mathbf{A}^{-1}\mathbf{b})$ *and* $\mathcal{K}_Q(\mathbf{A}^{-T}\mathbf{E}^T, (\mathbf{E}^{-T}\mathbf{A}^T)^{\mathbf{l}_2}\mathbf{A}^{-T}\mathbf{c})$ *respectively, where* $l_1, l_2 \in \mathbb{Z}$ *and* $0 \leq l_1, l_2 \leq Q$ *then the first* $2Q - l_1 - l_2$ *moments and the first* $l_1 + l_2$ *Markov parameters of the original and reduced order systems match. It is assumed that* $\mathbf{A}$, $\mathbf{E}$, $\mathbf{A}_r$ *and* $\mathbf{E}_r$ *are invertible.*

The similarity between Theorems 2.1 and 2.3 and their generalization in Theorems 2.4 and 2.5 is that in one-sided methods the number of matched characteristic parameters (moments and Markov parameter) of original and reduced order systems is Q and in two-sided methods for both theorems, it is increased $2Q$ which is the maximum that can be achieved.

2.2.4 MIMO systems

MIMO systems can also be reduced by using block Krylov subspaces to match some of the moments or Markov parameters [3, 61, 79, 89]. The generalization of reduced order model (2.11) for a system with m inputs and p outputs is

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r + \mathbf{W}^T\mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r \end{cases} \tag{2.17}$$

For this case the block Krylov subspaces $\mathcal{K}_{Q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{Q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$ are to be used. In one-sided methods, similar to Theorem 2.1, Q_1 moments and in two-sided methods $Q_1 + Q_2$ moments match.

In MIMO case, the moments are not scalars and each moment has $m \cdot p$ entries. For a system with m inputs and p outputs, each column of the matrices $\mathbf{V}$ and $\mathbf{W}$ are responsible to match one more row or column of the moment matrices. So, by choosing the first Q columns of the matrices $\mathbf{V}$ and $\mathbf{W}$, it is possible to find a reduced model of order Q. Otherwise, the order of the reduced system, chosen automatically from order of the input and output Krylov subspaces, should be a multiple of the number of inputs and outputs which can be unnecessarily high for non-square ($m \neq p$) systems.

2.3 Invariance properties

In section 2.2, it was shown that using *any* basis of input or output Krylov subspaces for order reduction results in moment matching property. It was proved by appropriately changing the starting vectors of the input and output Krylov subspaces, it is possible to match the moments and Markov parameters, simultaneously. Other degrees of freedom in the design are:

- Choice of bases of Krylov subspaces.
- Representation and the realization of the original state space model.

These items are investigated in the following two subsections; see also [73].

2.3.1 Invariance to change of Krylov bases

As we will discuss later, calculating the basis of a given Krylov subspace needs a numerically stable algorithm. Because of degrees of freedom to choose a basis, the question arises if changing the basis may affect the reduced order system.

Theorem 2.6 *The transfer matrix of the reduced order system found by a two-sided method is independent of the particular choice of the bases* $\mathbf{V}$ *and* $\mathbf{W}$ *of the input and output Krylov subspaces.*

Proof: Consider two reduced order models by using pairs of bases $\mathbf{V}_1, \mathbf{W}_1$ and $\mathbf{V}_2, \mathbf{W}_2$,

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\dot{\mathbf{x}}_{r1} = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{x}_{r1} + \mathbf{W}_1^T\mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}^T\mathbf{V}_1\mathbf{x}_{r1}, \end{cases} \tag{2.18}$$

$$\begin{cases} \mathbf{W}_2^T\mathbf{E}\mathbf{V}_2\dot{\mathbf{x}}_{r2} = \mathbf{W}_2^T\mathbf{A}\mathbf{V}_2\mathbf{x}_{r2} + \mathbf{W}_2^T\mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{V}_2\mathbf{x}_{r2}. \end{cases} \tag{2.19}$$

The columns of the matrices $\mathbf{V}_2$ and $\mathbf{W}_2$ are in the input and output Krylov subspaces, respectively. So, they can be written as a linear combination of the other bases which are columns of the matrices $\mathbf{V}_1$ and $\mathbf{W}_1$,

$$\exists \mathbf{Q}_v \in \mathbb{R}^{Q\cdot m\times Q\cdot m}, \mathbf{Q}_w \in \mathbb{R}^{Q\cdot p\times Q\cdot p} : \mathbf{V}_2 = \mathbf{V}_1\mathbf{Q}_v, \mathbf{W}_2 = \mathbf{W}_1\mathbf{Q}_w. \tag{2.20}$$

Since $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$ and $\mathbf{W}_2$ are full rank, the matrices $\mathbf{Q}_v$ and $\mathbf{Q}_w$ are invertible. By substituting equations (2.20) into the equation (2.19) we find

$$\begin{cases} \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\mathbf{Q}_v\dot{\mathbf{x}}_{r2} = \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2} + \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{B}\mathbf{u}, \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}. \end{cases}$$

$\mathbf{Q}_w$ is invertible and we can multiply both sides of the state equation by $\mathbf{Q}_w^{-T}$,

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\mathbf{Q}_v\dot{\mathbf{x}}_{r2} = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2} + \mathbf{W}_1^T\mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2} \end{cases}$$

Applying the state transformation $\mathbf{z} = \mathbf{Q}_v\mathbf{x}_{r2}$ to this system, converts it into (2.18). So, the reduced order models (2.18) and (2.19) have the same transfer matrices. ■

For one-sided methods, the invariance to change of basis is not valid in general [73]. However for the case $\mathbf{W} = \mathbf{V}$, the same result exists as in the following corollary.

Corollary 2.1 *The transfer matrix of the reduced order system found by a one-sided method with* $\mathbf{W} = \mathbf{V}$ *is independent of the particular choice of the bases* $\mathbf{V}$ *or* $\mathbf{W}$.

2.3.2 Invariance to representation and realization

Different representations and realizations of the same original model may lead to different reduced order systems which is undesired in most applications. Changing the realization means applying a similarity transformation and changing the representation means multiplying the state equation by a nonsingular matrix. As shown in [73], a two-sided method finds a reduced order model whose transfer matrix depends only on the input-output behaviour of the original model not its realization or representation.

Theorem 2.7 *In order reduction based on projection using a two-sided method, changing the representation or realization of the original system does not change the input-output behaviour of the reduced order model.*

In one-sided methods with $\mathbf{W} = \mathbf{V}$, the reduced order model and its transfer matrix changes when the representation or realization of the original model changes. In application, this can be an essential disadvantage, since it makes results depending on how the original system is modelled.

2.4 Computational aspects

In most application related models, the basic vectors used in the definition of Krylov subspaces tend to be almost linearly dependent even for moderate values of N and Q. So, they should not be used in numerical computations. Instead, there exist other suitable bases that can be applied in order reduction as explained in the following.

2.4.1 Arnoldi algorithm

In one-sided methods, the most popular algorithm is the ***Arnoldi algorithm*** which finds an orthonormal basis for a Krylov subspace [6, 28]. Consider the Krylov subspace $\mathcal{K}_Q(\breve{\mathbf{A}}, \breve{\mathbf{b}})$. The Arnoldi algorithm finds a set of normalized vectors that are orthogonal to each other,

$$\mathbf{V}_Q^T \mathbf{V}_Q = \mathbf{I}, \tag{2.21}$$

where the columns of the matrix $\mathbf{V}$ form a basis for the given Krylov subspace. In the following algorithm, in each step one more vector orthogonal to all other previous vectors is constructed and then it is normalized to have length one. In a general case, when Q is not small enough, it can happen that not all of the basic vectors are linearly independent.

In SISO case, this breaks the loop however, in MIMO case, linearly dependent vectors must be deleted during the iterations (deflation) [23, 24].

Algorithm 2.1 *Arnoldi algorithm with deflation using modified Gram-Schmidt:*

0. *Start: Delete all linearly dependent starting vectors to find* m_1 *independent starting vectors for the given Krylov subspace then set*

$$\mathbf{v}_1 = \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|_2}.$$

where $\mathbf{b}_1$ *is the first starting vector after deleting the dependent starting vectors.*

1. *For* $j = 2, 3, \cdots,$ *do,*

   (a) *Calculating the next vector: If* $j \leq m_1$ *the next vector is the* j*-th starting vector. Otherwise, the next vector is*

   $$\mathbf{r}_j = \breve{\mathbf{A}}\mathbf{v}_{j-m_1}.$$

   (b) *Orthogonalization: Set* $\hat{\mathbf{v}}_j = \mathbf{r}_j$ *then for i=1 to j-1 do:*

   $$\begin{aligned} h_{i,j-1} &= \hat{\mathbf{v}}_j^T \mathbf{v}_i \\ \hat{\mathbf{v}}_j &= \hat{\mathbf{v}}_j - h_{i,j-1}\mathbf{v}_i. \end{aligned} \tag{2.22}$$

   (c) *Normalization: If* $\hat{\mathbf{v}}_j \neq 0$[1]*, the i-th column is*

   $$h_{j,j-1} = \|\hat{\mathbf{v}}_j\|_2, \quad \mathbf{v}_j = \frac{\hat{\mathbf{v}}_j}{h_{j,j-1}}.$$

   and increase j *and go to step* (1a).

   (d) *Deflation: Reduce* m_1 *to* $m_1 - 1$ *and if* m_1 *is nonzero go to the next step and if* m_1 *is zero break the loop.*

   (e) *go to step* (1a) *without increasing* j.

In the Algorithm 2.1, if $\hat{\mathbf{v}}_i \neq 0$, then the first i basic vectors are linearly independent because $\mathbf{v}_1, \cdots, \hat{\mathbf{v}}_i$ span the same space as the first i basic vectors. If $\hat{\mathbf{v}}_i = 0$, then the

[1] In practice we check if $\|\hat{\mathbf{v}}_j\|_2 > \varepsilon$ where $\varepsilon \in \mathbb{R}^+$ is a small number

i-th basic vector is a linear combination of the previous basic vectors. Therefore, from this step on, the corresponding starting vector should be deleted. In finite precision mathematics, we check if the vectors are small.

Arnoldi algorithm, not only finds an orthonormal basis $\mathbf{V}_Q$ for the given Krylov subspace, but also if no deflation occurs, in step j, we have,

$$\breve{\mathbf{B}} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix} \underbrace{\begin{bmatrix} \|\mathbf{b}_1\|_2 & h_{11} & \cdots & h_{1m} \\ 0 & h_{21} & \cdots & h_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{mm} \end{bmatrix}}_{\mathbf{H}_s}, \tag{2.23}$$

$$\breve{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix}}_{\mathbf{V}_j} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix} \times$$

$$\underbrace{\begin{bmatrix} h_{1,m+1} & \cdots & h_{1,j-m} & \cdots & h_{1,m+j-1} \\ \vdots & \vdots & \vdots & \vdots & \\ h_{m,m+1} & \cdots & h_{m,j-m} & \cdots & h_{m,m+j-1} \\ h_{m+1,,m+1} & \cdots & h_{m+1,j-m} & \cdots & h_{m+1,m+j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & h_{j,j-m} & \cdots & h_{j,m+j-1} \end{bmatrix}}_{\mathbf{H}_j} +$$

$$\begin{bmatrix} \mathbf{v}_{j+1} & \cdots & \mathbf{v}_{j+m} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & h_{j+1,j-m+1} & h_{j+1,j-m+2} & \cdots & h_{j+1,m+j-1} \\ 0 & \cdots & 0 & 0 & h_{j+2,j-m+2} & \cdots & h_{j+2,m+j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & h_{j+m,m+j-1} \end{bmatrix}, \tag{2.24}$$

where the entries of the matrix $\mathbf{H}_j$ are produced by the algorithm. In the case of deflation in step k, the number of nonzero entries of $\mathbf{H}_j$ under the diagonal decreases. From equations (2.23) and (2.24) and by using the orthogonality of the vectors $\mathbf{v}_i$ to each other, we can conclude,

$$\mathbf{V}_j^T \breve{\mathbf{B}} = \underbrace{\begin{bmatrix} \mathbf{H}_s \\ \mathbf{0} \end{bmatrix}}_{\mathbf{H}_m}, \tag{2.25}$$

$$\mathbf{V}_j^T \breve{\mathbf{A}} \mathbf{V}_j = \mathbf{H}_j. \tag{2.26}$$

In the SISO case with a single starting vector, $\mathbf{H}_j$ is an upper Hessenberg matrix where all elements under the diagonal except for the sub-diagonal entries are zero and $\mathbf{H}_m$ is a multiple of the first unit vector.

The equation (2.25), is helpful to find the reduced order system by a one-sided method in a simpler way. In moment matching using input Krylov subspace, if we consider that the original system is multiplied by $\mathbf{A}^{-1}$ before applying the projection and the Arnoldi algorithm is run for $Q+m$ iterations, then $\mathbf{E}_r = \mathbf{H}_Q$, $\mathbf{A}_r = \mathbf{I}$ and $\mathbf{B}_r = \mathbf{H}_m$ that can directly be calculated from the Algorithm 2.1. However, the reduced order system can also be found by applying a projection to the original system (2.11) (with $\mathbf{W} = \mathbf{V}$) and the chosen characteristic parameters match but the reduced order model can be different from the one found by using $\mathbf{H}_Q$ and $\mathbf{H}_m$.

2.4.2 Lanczos algorithm

In two-sided methods, the Lanczos algorithm can be used to find the projection matrices. The classical Lanczos algorithm [26, 48] generates two sequences of basis vectors which span the Krylov subspaces $\mathcal{K}_Q(\breve{\mathbf{A}}, \breve{\mathbf{b}})$ and $\mathcal{K}_Q(\breve{\mathbf{A}}^T, \breve{\mathbf{c}})$ and are orthogonal to each other,

$$\mathbf{W}_Q^T \mathbf{V}_Q = \mathbf{I}. \tag{2.27}$$

Algorithm 2.2 *The classical Lanczos algorithm:*

1. *Set*

$$\mathbf{v}_1 = \frac{\breve{\mathbf{b}}}{\sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|}}, \quad \mathbf{w}_1 = \frac{\breve{\mathbf{c}}}{-\sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|}}$$

and set $\mathbf{v}_0 = \mathbf{w}_0 = \mathbf{0}$, $\beta_1 = 0$, $\delta_1 = 0$.

2. *For* $j = 1, 2, \cdots$ *do:*

$$\begin{aligned}
\alpha_j &= \mathbf{w}_j^T \breve{\mathbf{A}} \mathbf{v}_j, \\
\hat{\mathbf{v}} &= \breve{\mathbf{A}} \mathbf{v}_j - \alpha_j \mathbf{v}_j - \delta_j \mathbf{v}_{j-1}, \\
\hat{\mathbf{w}} &= \breve{\mathbf{A}}^T \mathbf{w}_j - \alpha_j \mathbf{w}_j - \delta_j \mathbf{w}_{j-1}, \\
\delta_{j+1} &= \sqrt{|\hat{\mathbf{v}}^T \hat{\mathbf{w}}|}, \\
\beta_{j+1} &= \frac{\hat{\mathbf{v}}^T \hat{\mathbf{w}}}{\delta_{j+1}}, \\
\mathbf{v}_{j+1} &= \frac{\hat{\mathbf{v}}}{\beta_{j+1}}, \\
\mathbf{w}_{j+1} &= \frac{\hat{\mathbf{w}}}{\delta_{j+1}}.
\end{aligned}$$

The property of the Lanczos Algorithm 2.2 is,

$$\breve{\mathbf{b}} = \mathbf{v}_1 \sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|}, \tag{2.28}$$

$$\breve{\mathbf{c}} = -\mathbf{w}_1 \sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|}, \tag{2.29}$$

$$\breve{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix}}_{\mathbf{V}_j} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 & \beta_2 & 0 & \cdots & 0 \\ \delta_2 & \alpha_2 & \beta_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \delta_{j-1} & \alpha_{j-1} & \beta_j \\ 0 & \cdots & 0 & \delta_j & \alpha_j \end{bmatrix}}_{\mathbf{T}_j} + \delta_{j+1} \mathbf{v}_{j+1} \mathbf{e}_Q^T, \tag{2.30}$$

$$\breve{\mathbf{A}}^T \underbrace{\begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_j \end{bmatrix}}_{\mathbf{W}_j} = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_j \end{bmatrix} \begin{bmatrix} \alpha_1 & \delta_2 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & \delta_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \beta_{j-1} & \alpha_{j-1} & \delta_j \\ 0 & \cdots & 0 & \beta_j & \alpha_j \end{bmatrix} + \beta_{j+1} \mathbf{w}_{j+1} \mathbf{e}_Q^T, \tag{2.31}$$

where the entries of the tridiagonal matrix $\mathbf{T}_j$ are produced by the algorithm. From equations (2.28-2.30) and by using the bi-orthogonality property, we can conclude,

$$\mathbf{W}_j^T \breve{\mathbf{b}} = \sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|} \mathbf{e}_1 \tag{2.32}$$

$$\breve{\mathbf{c}}^T \mathbf{V}_j = -\sqrt{|\breve{\mathbf{c}}^T \breve{\mathbf{b}}|} \mathbf{e}_1^T \tag{2.33}$$

$$\mathbf{W}_j^T \breve{\mathbf{A}} \mathbf{V}_j = \mathbf{T}_j. \tag{2.34}$$

The Lanczos Algorithm 2.2 can be used for moment matching of SISO systems if the state equation is considered to be multiplied by $\mathbf{A}^{-1}$ first and the Krylov subspaces $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_Q(\mathbf{E}^T\mathbf{A}^{-T}, \mathbf{c})$ are considered. The reduced system is then,

$$\begin{cases} \mathbf{T}_Q\dot{\mathbf{x}} = \mathbf{x} + \sqrt{|\breve{\mathbf{c}}^T\breve{\mathbf{b}}|}\mathbf{e}_1 u, \\ y = -\sqrt{|\breve{\mathbf{c}}^T\breve{\mathbf{b}}|}\mathbf{e}_1^T\mathbf{x}. \end{cases} \tag{2.35}$$

The standard Lanczos algorithm is not only limited to the Krylov subspaces with one starting vector but also suffers from break down when $\hat{\mathbf{v}}^T\hat{\mathbf{w}} = 0$ and loss of bi-orthogonality by increasing the iterations that can limit the usage of the algorithm. If a break down occurs, then the algorithm can not be continued. This problem makes Lanczos weaker than Arnoldi. To avoid the loss of bi-orthogonality, re-orthogonalization must be used [17]. In [3], the Lanczos algorithm is generalized to MIMO case; see also [61]. In the following, we present the Lanczos algorithm with full orthogonalization that can be applied to any pair of Krylov subspaces allowing us to use the algorithm for the original representation of systems without multiplying by $\mathbf{A}^{-1}$ [74]. In the following algorithm, we consider the Krylov subspaces, $\mathcal{K}_{Q_1}(\breve{\mathbf{A}}, \breve{\mathbf{B}})$ and $\mathcal{K}_{Q_2}(\hat{\mathbf{A}}, \hat{\mathbf{C}}^T)$ where $\breve{\mathbf{A}}, \hat{\mathbf{A}} \in \mathbb{R}^{n\times n}$, $\breve{\mathbf{B}} \in \mathbb{R}^{n\times m}$ and $\hat{\mathbf{C}} \in \mathbb{R}^{p\times n}$.

Algorithm 2.3 *Lanczos algorithm with deflation and full orthogonalization:*

0. *Start: Delete all linearly dependent starting vectors to find m_1 and p_1 independent starting vectors, $\breve{\mathbf{b}}_1, \cdots, \breve{\mathbf{b}}_{m_1}$ and $\hat{\mathbf{c}}_1, \cdots, \hat{\mathbf{c}}_{p_1}$, for input and output Krylov subspaces, respectively.*

1. *Set*

$$\mathbf{v}_1 = \frac{\breve{\mathbf{b}}_1}{\sqrt{|\hat{\mathbf{c}}_1^T\breve{\mathbf{b}}_1|}}, \quad \mathbf{w}_1 = \frac{\hat{\mathbf{c}}_1}{-\sqrt{|\hat{\mathbf{c}}_1^T\breve{\mathbf{b}}_1|}}$$

2. *For $j = 2, 3, \cdots$ do.*

   (a) *Calculating the next vector: For the input Krylov subspace, if $j \leq m_1$ then $\mathbf{r}_j = \breve{\mathbf{b}}_j$. Otherwise, the next vectors is computed as follows,*

$$\mathbf{r}_j = \breve{\mathbf{A}}\mathbf{v}_{j-m_1}.$$

For the output Krylov subspace, if $j \le p_1$ *then* $\mathbf{l}_j = \hat{\mathbf{c}}_j$. *Otherwise, the next vector is*

$$\mathbf{l}_j = \hat{\mathbf{A}}\mathbf{w}_{j-p_1}.$$

(b) *Orthogonalization: Set* $\hat{\mathbf{v}}_j = \mathbf{r}_j$ *and* $\hat{\mathbf{w}}_j = \mathbf{l}_j$ *then for* $i = 1, \cdots, j-1$ *do:*

$$\begin{aligned} \breve{h}_{i,j-1}\hat{\mathbf{v}}_j^T\mathbf{w}_i \quad &, \quad \hat{h}_{i,j-1} = \hat{\mathbf{w}}_j^T\mathbf{v}_i, \\ \hat{\mathbf{v}}_j = \hat{\mathbf{v}}_j - \breve{h}_{i,j-1}\mathbf{v}_i \quad &, \quad \hat{\mathbf{w}}_j = \hat{\mathbf{w}}_j - \hat{h}_{i,j-1}\mathbf{w}_i. \end{aligned}$$

(c) *Normalization: If* $\hat{\mathbf{v}}_i^T\hat{\mathbf{w}}_i \neq 0$ *(or not very small in practice) then*

$$\mathbf{v}_j = \frac{\hat{\mathbf{v}}_j}{\sqrt{|\hat{\mathbf{w}}_j^T\hat{\mathbf{v}}_j|}}, \quad \mathbf{w}_j = \frac{\hat{\mathbf{w}}_j}{-\sqrt{|\hat{\mathbf{w}}_j^T\hat{\mathbf{v}}_j|}}$$

increase i *and go to step* (2a).

(d) *Deflation: If* $\hat{\mathbf{v}}_j = 0$ *(or very small in practice), reduce* m_1 *to* $m_1 - 1$ *and if* m_1 *is zero break the loop.*
If $\hat{\mathbf{w}}_j = 0$ *(or very small in practice), reduce* p_1 *to* $p_1 - 1$ *and if* p_1 *is zero break the loop.*

(e) *Increasing* j *and go to step* (2a).

Similar to the Arnoldi algorithm, for the case that no deflation occurs, we have,

$$\breve{\mathbf{B}} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{|\hat{\mathbf{c}}_1^T\breve{\mathbf{b}}_1|} & \breve{h}_{11} & \cdots & \breve{h}_{1m} \\ 0 & \breve{h}_{21} & \cdots & \breve{h}_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \breve{h}_{mm} \end{bmatrix}}_{\breve{\mathbf{H}}_s}, \tag{2.36}$$

$$\breve{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix}}_{\mathbf{V}_j} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_j \end{bmatrix} \times$$

$$\underbrace{\begin{bmatrix} \breve{h}_{1,m+1} & \cdots & \breve{h}_{1,j-m} & \cdots & \breve{h}_{1,m+j-1} \\ \vdots & \vdots & \vdots & \vdots & \\ \breve{h}_{m,m+1} & \cdots & \breve{h}_{m,j-m} & \cdots & \breve{h}_{m,m+j-1} \\ \breve{h}_{m+1,,m+1} & \cdots & \breve{h}_{m+1,j-m} & \cdots & \breve{h}_{m+1,m+j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \breve{h}_{j,j-m} & \cdots & \breve{h}_{j,m+j-1} \end{bmatrix}}_{\breve{\mathbf{H}}_j} +$$

$$\begin{bmatrix} \mathbf{v}_{j+1} & \cdots & \mathbf{v}_{j+m} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \breve{h}_{j+1,j-m+1} & \breve{h}_{j+1,j-m+2} & \cdots & \breve{h}_{j+1,m+j-1} \\ 0 & \cdots & 0 & 0 & \breve{h}_{j+2,j-m+2} & \cdots & \breve{h}_{j+2,m+j-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \breve{h}_{j+m,m+j-1} \end{bmatrix}, \quad (2.37)$$

where the entries of the matrix $\breve{\mathbf{H}}_j$ are produced by the algorithm. In the case of deflation in step k, the number of nonzero entries of $\breve{\mathbf{H}}_j$ under the diagonal decreases. From equations (2.36) and (2.37) and by using the bi-orthogonality of the columns of $\mathbf{V}_j$ and $\mathbf{W}_j$ to each other, we can conclude,

$$\mathbf{W}_j^T \breve{\mathbf{B}} = \underbrace{\begin{bmatrix} \breve{\mathbf{H}}_s \\ \mathbf{0} \end{bmatrix}}_{\breve{\mathbf{H}}_m}, \quad (2.38)$$

$$\mathbf{W}_j^T \breve{\mathbf{A}} \mathbf{V}_j = \breve{\mathbf{H}}_j. \quad (2.39)$$

Another property is $\mathbf{V}_j^T \hat{\mathbf{A}} \mathbf{W}_j = \hat{\mathbf{H}}_j$ where the entries of $\hat{\mathbf{H}}_j$ are also produced in the algorithm.

In the SISO case, the Lanczos Algorithm 2.3 leads to upper Hessenberg matrices compared to the tridiagonal matrix in (2.32) as a result of the standard Lanczos algorithm. However, the Algorithm 2.3 can directly be applied to input and output Krylov subspaces and the reduced order system is then found by applying the projection as in (2.11).

If we apply the Algorithm 2.3 to the input and output Krylov subspaces to match the moments, considering that the original system is multiplied with $\mathbf{A}^{-1}$, the reduced model of order Q can be identified as, $\mathbf{E}_r = \breve{\mathbf{H}}_Q,\ \mathbf{A}_r = \mathbf{I},\ \mathbf{B}_r = \breve{\mathbf{H}}_m$ and $\mathbf{C} = \hat{\mathbf{H}}_m^T$. Such reduced system will have the same transfer function as the one found by applying a projection as in (2.17) which is a property of two-sided methods.

2.4.3 Two-sided Arnoldi algorithm

Subsequently, we introduce the two-sided Arnoldi algorithm to find the bases necessary for projection and calculating the reduced order model [24, 79, 80]. This method in comparison to Lanczos is *numerically more stable* and *easier to implement*. The algorithm comprises the following steps:

Algorithm 2.4 *Two-sided Arnoldi algorithm:*

0. *Choose the appropriate input and output Krylov subspaces for the given system, e.g.* $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ *and* $\mathcal{K}_Q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$.

1. *Apply Arnoldi Algorithm 2.1 to the input Krylov subspace to find* $\mathbf{V}_Q$.

2. *Apply Arnoldi Algorithm 2.1 to the output Krylov subspace to find* $\mathbf{W}_Q$.

3. *Find the reduced order model by applying the projection as in* (2.11).

Compared to the Lanczos algorithm, two-sided Arnoldi does not suffer from break down that occurs in Lanczos. The other difference is that in Lanczos algorithm, the two set of bases are bi-orthogonal, however in the two-sided Arnoldi algorithm, each basis is orthonormal,

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \text{ and } \mathbf{W}^T\mathbf{W} = \mathbf{I}.$$

However, both algorithms lead to reduced systems with the same transfer functions.

2.4.4 Numerical remarks

In order reduction of system (2.1) based on moment matching, the matrix $\mathbf{A}^{-1}\mathbf{E}$ and $\mathbf{A}^{-T}\mathbf{E}^T$ are very important. So, the inverse of the large matrix $\mathbf{A}$ seems to be necessary. Calculating this inverse and then use it in the Algorithm 2.1 is not recommended for numerical reasons. In order to find the vector $\mathbf{t} = \mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{i-1}$ in the iterations of a numerical algorithm, it is better to solve the linear equation $\mathbf{A}\mathbf{t} = \mathbf{E}\mathbf{v}_{i-1}$ for $\mathbf{t}$ in each iteration to avoid numerical errors and to save time. In this way, a total of Q sets of linear equations are solved, whereas the calculation of $\mathbf{A}^{-1}$ would require solving a set of N such equations.

There exist many methods that can solve linear equations and find an exact or approximate solution. One of the best methods to find an exact solution is using LU-factorization [35] and then solve two triangular linear equations by Gaussian elimination. Using this method in *each* iteration leads to a slow algorithm while the most time consuming part is finding the LU factorization of the large matrix. The remedy is finding the LU factorization of the matrix $\mathbf{A}$ *at the beginning* and then solve only triangular linear equations in each iteration [2]. In this case, time is saved and the result is obtained very fast.

2.5 Passive systems and stability

An important property to be preserved in order reduction is stability or passivity of the original system. In using Krylov subspace methods to reduce the order of a stable large scale model, there is no general guarantee to find a stable reduced model. There exists a guarantee only for some types of systems which are related to passive systems [8, 28, 63]. However, stability of the reduced system can be recovered by post processing to remove the unstable poles if the reduced system is unstable, mostly by using the restarted Arnoldi and Lanczos algorithms; see for instance [36, 44, 85]. In the restarted algorithms instead of the moments matching, some of the parameters called modified moments match without any direct connection to the behavior of the system. A concept related to the passivity is positive realness as defined in the following; see also [70, 83] and the references therein.

Definition 2.5 *A square ($m = p$) transfer matrix $\mathbf{H}(s) : \mathbb{C} \mapsto (\mathbb{C}^{m\times m} \cup \infty)$ is positive real if*

1. $\mathbf{H}(s)$ *has no pole in right half complex plain.*
2. $\mathbf{H}(s^*) = (\mathbf{H}(s))^*$ *for all* $s \in \mathbb{C}$.
3. *Re* $\left(\mathbf{w}^H\mathbf{H}(s)\mathbf{w}\right) \geq 0$ *for all* $s \in \mathbb{C}$ *with Res* > 0 *and* $\mathbf{w} \in \mathbb{C}^m$.

[2] In MATLAB, instead of the command `A\V(:,i-1)`, the command `[L,U]=LU(A)` must be added to the beginning of the algorithm and in each iteration the command `U\(L\V(:,i-1))` can be used to solve triangular linear equations.

where $(.)^*$ *denotes conjugate complex and* $(.)^H$ *denotes hermitian which is complex conjugate of transpose of a complex matrix.*

In the definition of positive realness, the first condition is stability and the second condition is always true for a real transfer matrix. The third condition for SISO case means that the Nyquist diagram of the system does not go through the left half complex plain.

It can be proved that any linear dynamical system is passive if, and only if its transfer matrix is positive real; see [28] and the references therein. As mentioned in [28], the following lemma provides sufficient conditions for the system (2.1) to be passive.

Lemma 2.1 *In system (2.1), if* $\mathbf{A} + \mathbf{A}^T \preceq 0$, $\mathbf{E} = \mathbf{E}^T \succeq 0$ *and* $\mathbf{C} = \mathbf{B}^T$, *then the corresponding transfer matrix* $\mathbf{H}(s)$ *is positive real and therefore, it is passive.*

$\mathbf{F} \succeq 0$ for a symmetric matrix $\mathbf{F} \in \mathbb{R}^{N \times N}$ denotes that $\mathbf{F}$ is nonnegative definite; i.e. $\mathbf{x}^T\mathbf{F}\mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbb{R}^N$. It should be mentioned that if the conditions in Lemma 2.1 are satisfied then the system is passive but passivity is a property of the transfer matrix not its realization and if a system is passive, then the conditions of this lemma are not necessarily true. Because any passive system is necessarily stable, by using the conditions of Lemma 2.1 without considering the output equation, necessary conditions for stability can be derived.

Corollary 2.2 *The system (2.1) is stable, if* $\mathbf{A} + \mathbf{A}^T \preceq 0$ *and* $\mathbf{E} = \mathbf{E}^T \succeq 0$.

Lemma 2.2 *If* $\mathbf{F} \in \mathbb{R}^{N \times N}$ *is positive (negative) definite and* $\mathbf{V} \in \mathbb{R}^{N \times Q}$ *is a full rank matrix, then the matrix* $\mathbf{V}^T\mathbf{F}\mathbf{V}$ *is positive (negative) definite.*

Proof: Consider the vector $\mathbf{w} \in \mathbb{R}^Q$,

$$\underbrace{\mathbf{w}^T\mathbf{V}^T}_{\tilde{\mathbf{w}}^T}\mathbf{F}\underbrace{\mathbf{V}\mathbf{w}}_{\tilde{\mathbf{w}}} = \tilde{\mathbf{w}}^T\mathbf{F}\tilde{\mathbf{w}} > (<)0,$$

where $\tilde{\mathbf{w}} \in \mathbb{R}^N$. ■

Lemma 2.2 is also true when the matrix $\mathbf{F}$ is semi-definite. This lemma is helpful to keep the stability properties as in Corollary 2.2 in the reduced system calculated by a one-sided projection.

Theorem 2.8 *In the system (2.1), if* $\mathbf{A} + \mathbf{A}^T \preceq 0$ *and* $\mathbf{E} = \mathbf{E}^T \succeq 0$*, then the reduced model (2.11) using a one-sided method with the choice* $\mathbf{W} = \mathbf{V}$*, is stable and furthermore, the transfer matrix* $\mathbf{H}(s) = \mathbf{B}^T\mathbf{V}\left(s\mathbf{V}^T\mathbf{E}\mathbf{V} - \mathbf{V}^T\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{V}^T\mathbf{B}$ *is passive.*

Therefore, for certain passive (stable) systems, one-sided methods find passive (stable) reduced models.

2.6 Conclusion

In this chapter, we presented the order reduction using Krylov subspaces with an overview on corresponding numerical algorithms. We showed how a large scale systems can be reduced by applying a projection while matching some of the first moments and Markov parameters of the original and reduced order systems.

The invariance properties of the Krylov methods were also investigated. The results of our invariance properties are summarized in Table 2.1. The one-sided methods based on input Krylov subspace possess the weakest invariance properties, i.e. the transfer matrix of the resulting reduced order model depends on how the designer wrote down the equations for the original model. Reduced order models using a two-sided method not only match more moments, but also its input-output behaviour is independent of the realization and representation of the original system. In fact, the result of a two-sided method only depends on the transfer matrix of the original model (and on Q and s_0). However, the one-sided method, under certain conditions can guarantee a stable reduced model [29], whereas the two-sided method can lead to unstable reduced models.

To find such projection matrices, a numerically reliable algorithm is necessary and the Arnoldi algorithm was presented for one-sided methods and Lanczos and two-sided Arnoldi for two-sided methods. It is also presented how to simplify the procedure to find the reduced order system using the properties of the Arnoldi and Lanczos algorithms.

To stop the iterations of the Arnoldi and two-sided Arnoldi algorithms a *stopping criterion* has been presented in [79, 80] that can be used to find a suitable order for a reduced model. This measure which is based on the angle of the new vector before normalization and the hyper space spanned by all previous vectors is calculated in each iteration and there is no need to break the loop.

Table 2.1: Invariance properties of Krylov subspace methods and its effect on the reduced order model

Method	Subspace Used	Number of matching Parameters (SISO)	Change of Basis
One-sided	- Input Krylov - $\mathbf{W} = \mathbf{V}$	Q Parameters	Transfer matrix is unchanged
	- output Krylov - $\mathbf{V} = \mathbf{W}$	Q Parameters	Transfer matrix is unchanged
Two-sided	- output Krylov - Input Krylov	$2Q$ Parameters	Transfer matrix is unchanged
Method	Subspace Used	Change of Representation	Change of Realization
One-sided	- Input Krylov - $\mathbf{W} = \mathbf{V}$	Transfer matrix changes	Transfer matrix changes
	- output Krylov - $\mathbf{V} = \mathbf{W}$	Transfer matrix changes	Transfer matrix changes
Two-sided	- output Krylov - Input Krylov	Transfer matrix is unchanged	Transfer matrix is unchanged

It should also be noted that for MIMO case, where the moments and Markov parameters are matrices, there are some more degrees of freedom that can be helpful to find better results. In [79], a *selection procedure* is proposed. This selection procedure which can be used in the Arnoldi or two-sided Arnoldi algorithm, improves the approximation compared to the common block Arnoldi algorithm, by investigating dominance measures for rows and columns of the transfer matrix in each iteration. Optionally, the designer can manipulate this selection and thereby specify higher number of matching moments for certain rows and columns of the transfer matrix.

Chapter 3

SYSTEMS OF SECOND ORDER FORM

Second order systems are sets of second order differential equations. In this chapter, by introducing the second order model, we review some related preliminaries to be used later. It is explained how to find an equivalent state space equation and calculate the moments or Markov parameters of such systems.

3.1 Second order models

The high-order models considered in this paper are assumed to be given in the form,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\mathbf{z}(t), \end{cases} \tag{3.1}$$

with n second order differential equations, m inputs and p outputs. The total order of the system is $N = 2n$ and the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are called mass, damping and stiffness matrices, respectively. If $\mathbf{D} = \mathbf{0}$, then the second order system is undamped.

Equivalently, the model (3.1) can be rewritten as

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}, \end{cases}$$

with N first order differential equations where $\mathbf{F} \in \mathbb{R}^{n \times n}$ is a nonsingular matrix. The choice of $\mathbf{F}$ is optional and has no effect on the upcoming facts and results. For simplicity, one may choose $\mathbf{F} = \mathbf{I}$. However, by knowing that for most of the systems in many fields of engineering, the mass, damping and stiffness matrices are symmetric and even positive definite, for the case that $\mathbf{K}$ is nonsingular, it is recommended to transform a state space model as,

$$\begin{cases} \underbrace{\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}. \end{cases} \tag{3.2}$$

In this case, symmetry and definiteness of $\mathbf{M}$ and $\mathbf{K}$ are automatically transferred into $\mathbf{E}$ and symmetry of $\mathbf{K}$ and $\mathbf{D}$ are automatically transferred into $\mathbf{A}$; i.e. if $\mathbf{M}$, $-\mathbf{K}$ are symmetric and positive (semi-)definite and $-\mathbf{D}$ is symmetric, then $\mathbf{A}$ is symmetric and $\mathbf{E}$ is symmetric and positive (semi-)definite. One may also use $\mathbf{F} = \mathbf{K}$ which has some advantages to find a stable reduced order model although it does not lead to a symmetric $\mathbf{A}$ if $\mathbf{D}$ and $\mathbf{K}$ are symmetric.

A second order model can also be transformed into state space as,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \ddot{\mathbf{z}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{L} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}, \end{cases} \tag{3.3}$$

which is equivalent to (3.2) and $\mathbf{F} \in \mathbb{R}^{n \times n}$ is nonsingular. A common choice for $\mathbf{F}$ in this case can be identity matrix or $\mathbf{M}$ even if $\mathbf{M}$ is singular, that changes the realization into,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \ddot{\mathbf{z}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{L} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}, \end{cases} \tag{3.4}$$

3.2 Moments and Markov parameters of Second order systems

If we consider the realization (3.2), the matrix $\mathbf{A}$ is invertible if and only if $\mathbf{K}$ is nonsingular. Because,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1} & -\mathbf{K}^{-1} \\ -\mathbf{K}^{-1} & \mathbf{0} \end{bmatrix}. \tag{3.5}$$

As we saw in Chapter 2, to match the moments about zero, invertibility of $\mathbf{A}$ is necessary which changes to invertibility of $\mathbf{K}$ for second order systems. However, this condition drops when moments about a nonzero point are to be matched.

Because the transfer matrices of two systems (3.2) and (3.4) are the same, they have the same moments. Although $\mathbf{M}^{-1}$ appears in the inverse of $\mathbf{A}$ in (3.4), it will disappear in the definition of moments as we will see in the following. Therefore, the only condition to calculate the moments of a second order system is the invertibility of the stiffness matrix $\mathbf{K}$.

By considering that $\mathbf{K}$ is nonsingular, the i-th moment (about zero) of the system (3.1) is,

$$\begin{aligned}\mathbf{m}_i &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \right)^i \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1} & -\mathbf{K}^{-1} \\ -\mathbf{K}^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \right)^i \begin{bmatrix} \mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1} & -\mathbf{K}^{-1} \\ -\mathbf{K}^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{3.6}$$

By knowing that $\mathbf{C}\left(\mathbf{A}^{-1}\mathbf{E}\right)^i \mathbf{A}^{-1}\mathbf{B} = \mathbf{C}\mathbf{A}^{-1}\left(\mathbf{E}\mathbf{A}^{-1}\right)^i \mathbf{B}$ and using the realization (3.4), an equivalent equation to (3.6) is,

$$\mathbf{m}_i = \begin{bmatrix} \mathbf{0} & -\mathbf{L}\mathbf{K}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{M}\mathbf{K}^{-1} \\ \mathbf{I} & -\mathbf{D}\mathbf{K}^{-1} \end{bmatrix}^i \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}. \tag{3.7}$$

For the Markov parameters, the invertibility of the matrix $\mathbf{E}$ is necessary. If the state space system (3.2) is considered to check invertibility of $\mathbf{E}$, both matrices $\mathbf{M}$ and $\mathbf{K}$ must be invertible. But if we consider the realization (3.4) then,

$$\mathbf{E}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{D}\mathbf{M}^{-1} & \mathbf{M}^{-1} \\ \mathbf{M}^{-1} & \mathbf{0} \end{bmatrix}. \tag{3.8}$$

and the only condition is non-singularity of $\mathbf{M}$.

By considering that $\mathbf{M}$ is nonsingular, the Markov parameters of the second order system (3.1) can be calculated using state space model (3.2) as follows,

$$\begin{aligned} \mathbf{M}_i &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \right)^i \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}^i \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix}. \end{aligned} \tag{3.9}$$

Again by using the relationship $\mathbf{C}\left(\mathbf{E}^{-1}\mathbf{A}\right)^i \mathbf{E}^{-1}\mathbf{B} = \mathbf{C}\mathbf{E}^{-1}\left(\mathbf{A}\mathbf{E}^{-1}\right)^i \mathbf{B}$ and the realization (3.4), we can show that,

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{0} & \mathbf{L}\mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} -\mathbf{D}\mathbf{M}^{-1} & \mathbf{I} \\ -\mathbf{K}\mathbf{M}^{-1} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix}. \tag{3.10}$$

It should be noted that the first Markov parameter, $\mathbf{M}_0$ for second order systems is zero because of the structure of the state space matrices. In fact, for SISO systems, the Markov parameters define the relative degree. If the first k Markov parameters of a SISO system are zero, its relative degree is $k+1$ which is for second order systems at least 2 and $\mathbf{M}_0 = \mathbf{0}$ [46].

3.3 Undamped systems

A second order system is undamped if the damping matrix $\mathbf{D}$ is zero. In this case, the equivalent state space equation is,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \ddot{\mathbf{z}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{L} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \mathbf{z}(t) \end{bmatrix}. \end{cases} \tag{3.11}$$

Lemma 3.1 *Considering that* $\mathbf{M}$ *and* $\mathbf{K}$ *are nonsingular, for every undamped second order system (i.e.* $\mathbf{D} = \mathbf{0}$*), the Markov parameters* $\mathbf{M}_0, \mathbf{M}_2, \cdots$ *and the moments* $\mathbf{m}_1, \mathbf{m}_3, \cdots$ *are zero.*

Proof: Because of the structure of the matrix $\mathbf{C}$, to prove the lemma, it is enough to show that for an undamped system (3.11), for every value of $i \in \mathbb{Z}$, the lower block of the matrix $(\mathbf{E}^{-1}\mathbf{A})^{2i}\mathbf{E}^{-1}\mathbf{B}$ is zero. We can simply show that,

$$\mathbf{E}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}, \; \mathbf{A}^{-1}\mathbf{E} = \begin{bmatrix} \mathbf{0} & \mathbf{M}^{-1} \\ \mathbf{K}^{-1} & \mathbf{0} \end{bmatrix}, \; \mathbf{E}^{-1}\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

For $i = 0$, the claim is clear. Now, consider for $i = j > 0$, the upper part of the matrix $(\mathbf{E}^{-1}\mathbf{A})^{2i}\mathbf{E}^{-1}\mathbf{B}$ is zero, then,

$$\begin{aligned} (\mathbf{E}^{-1}\mathbf{A})^{2j}\mathbf{E}^{-1}\mathbf{B} &= \begin{bmatrix} \bar{\mathbf{R}} \\ \mathbf{0} \end{bmatrix} \Longrightarrow \\ (\mathbf{E}^{-1}\mathbf{A})^{2(j+1)}\mathbf{E}^{-1}\mathbf{B} &= (\mathbf{E}^{-1}\mathbf{A})^2 \begin{bmatrix} \bar{\mathbf{R}} \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{R}} \\ \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{M}^{-1}\bar{\mathbf{R}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1}\mathbf{K}\bar{\mathbf{R}} \\ \mathbf{0} \end{bmatrix}, \end{aligned}$$

and the theorem for $i = j + 1$ is also true. For the case that $i < 0$, the proof is quite similar and it should be proved that if the claim is true for $i = j \cdots, -1$, it is also true for $j = i - 1$. ■

3.4 Passivity of second order systems

A square second order system (3.1) with a full rank matrix $\mathbf{G}$ is not passive. In fact, any square positive real system has a relative degree 1 [70, 83]. To show this fact, first we mention the so called Kalmann-Yakubovich Lemma [83].

Lemma 3.2 *The system* (2.1) *is positive real if and only if, there exist a positive definite* *matrix* $\mathbf{P}$ *and a positive semi-definite matrix* $\mathbf{Q}$ *such that,*

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \tag{3.12}$$

$$\mathbf{P}\mathbf{B} = \mathbf{C}^T \tag{3.13}$$

Lemma 3.3 *If the system* (2.1) *is positive real and* $\mathbf{B}$ *is full rank, then the first Markov parameter is positive definite.*

Proof: We use Lemma 3.2 and multiply equation (3.13) from left hand side with $\mathbf{B}^T$,

$$\mathbf{B}^T\mathbf{P}\mathbf{B} = \mathbf{B}^T\mathbf{C}^T = \mathbf{M}_0^T \succ 0.$$

■

By considering the equivalent state space system (3.2) or (3.4) of the second order model (3.1), the matrix $\mathbf{B}$ is full rank if and only if $\mathbf{G}$ is full rank. By knowing that the first Markov parameter of a second order system is zero, the system (3.1) is not passive. In the SISO case, if the relative degree of the second order system (3.1) is 2, then at high frequencies, the Nyquist diagram goes to zero with an angle equal to 180 degrees which means that the Nyquist diagram does not remain in the right half plane for all values of ω that violates the third condition of definition 2.5.

Consider the shifted transfer matrix of the system (3.1), $\mathbf{H}(s+s_0) = \mathbf{L}^T\left[s^2\mathbf{M} + s(2s_0\mathbf{M}+\mathbf{D}) + (s_0^2\mathbf{M}+s_0\mathbf{D}+\mathbf{K})\right]^{-1}\mathbf{G}$, where $0 \leq s_0 \in \mathbb{R}$ and $s_0^2\mathbf{M}+s_0\mathbf{D}+\mathbf{K}$ is invertible ($s_0 = 0$ if $\mathbf{K}$ is invertible). We rewrite this system in state space as,

$$\begin{cases} \underbrace{\begin{bmatrix} s_0^2\mathbf{M}+s_0\mathbf{D}+\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \\ \qquad \underbrace{\begin{bmatrix} \mathbf{0} & s_0^2\mathbf{M}+s_0\mathbf{D}+\mathbf{K} \\ -s_0^2\mathbf{M}-s_0\mathbf{D}-\mathbf{K} & -2s_0\mathbf{M}-\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}, \\ \mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{L}^T & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}. \end{cases} \tag{3.14}$$

Theorem 3.1 *The shifted second order system* $\mathbf{H}(s+s_0)$ *is stable if* $s_0 \geq 0$, $\mathbf{D} = \mathbf{D}^T \succeq 0$, $\mathbf{M} = \mathbf{M}^T \succeq 0$ *and* $\mathbf{K} = \mathbf{K}^T \succeq 0$.

Proof: We consider the realization (3.14) and apply Corollary 2.2. It is clear that under the assumptions of the theorem $\mathbf{E} = \mathbf{E}^T \succeq 0$. For the other condition, knowing that the matrices $\mathbf{D}, \mathbf{M}$ and $\mathbf{K}$ are symmetric, we have,

$$\mathbf{A} + \mathbf{A}^T = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -4s_0\mathbf{M} - 2\mathbf{D} \end{bmatrix}.$$

This matrix is negative semi-definite if $\mathbf{M}$ and $\mathbf{D}$ are positive semi-definite. ∎

Lemma 3.4 *The second order system* (3.1),

a. has no pole in the right half complex plane, if $\mathbf{D} = \mathbf{D}^T \succeq 0$, $\mathbf{M} = \mathbf{M}^T \succeq 0$ *and* $\mathbf{K} = \mathbf{K}^T \succeq 0$.

b. is stable, if $\mathbf{D} + \mathbf{D}^T \succeq 0$, $\mathbf{M} = \mathbf{M}^T \succeq 0$ *and* $\mathbf{K} = \mathbf{K}^T \succ 0$.

Proof: In part (a), based on Theorem 3.1, for every positive number s_0, the system $\mathbf{H}(s+s_0)$ is stable. By considering that the system remains stable when s_0 tends to zero, then $\mathbf{H}(s)$ should not have any pole in the right half complex plain.

For part (b), we choose $s_0 = 0$ in the realization (3.14) and apply Corollary 2.2. The proof is similar to Theorem 3.1. ∎

The preceding results will be used in order reduction while preserving stability of the original model.

Part II

Order Reduction by Direct Projection

Chapter 4

REDUCTION OF SECOND ORDER SYSTEMS USING SECOND ORDER KRYLOV SUBSPACES

In Chapter 2, it was explained how to reduce large scale systems in state space. In this chapter, we generalize the Krylov subspace method to reduce the second order models such that the second order structure is preserved.

The idea is to reduce second order models by applying a projection. To this end, the definition of the standard Krylov subspace is extended to the so called *Second Order Krylov Subspace* which was first introduced in [75] and is used to find the projection matrices and matching the moments and more investigated and generalized in [57].

Alternatives to this method have been proposed by other authors in [32, 87] where a structured projection is applied to the equivalent state space model preserving the structure of the state space matrices. These approaches match the same number of moments as the second order Krylov methods, however all calculations and applying the projection are done in state space which is in general numerically more expensive compared to the method of this chapter. There is also a difficulty to match the moments around zero as the projection matrix for this case includes a zero column.

4.1 Second Order Krylov Subspaces

We define the Second Order Krylov Subspaces such that they are helpful to calculate the moments or Markov parameters by a recursive procedure and to find appropriate reduced order systems in the sense of moment and Markov parameter matching.

Definition 4.1 *The Second Order Krylov Subspace is defined as,*

$$\mathcal{K}_{q_1}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1) = colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{q_1-1}\}, \tag{4.1}$$

where

$$\begin{cases} \mathbf{P}_0 = \mathbf{G}_1 \ , \ \mathbf{P}_1 = \mathbf{A}_1\mathbf{P}_0 \\ \mathbf{P}_i = \mathbf{A}_1\mathbf{P}_{i-1} + \mathbf{A}_2\mathbf{P}_{i-2}, \ i = 2, 3, \cdots \end{cases} \tag{4.2}$$

and $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n\times n}, \mathbf{G}_1 \in \mathbb{R}^{n\times m}$ *are constant matrices. The columns of* $\mathbf{G}_1$ *are called the starting vectors and the matrices* $\mathbf{P}_i$ *are called basic blocks.*

Definition 4.2 *The Second Order Krylov Subspaces* $\mathcal{K}_{q_1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ *and* $\mathcal{K}_{q_2}(-\mathbf{K}^{-T}\mathbf{D}^T, -\mathbf{K}^{-T}\mathbf{M}^T, -\mathbf{K}^{-T}\mathbf{L}^T)$ *are called the input and output Second Order Krylov Subspaces for the system (3.1), respectively.*

There is a connection between the second order and standard Krylov subspaces. If we simplify the recursion (4.2) into,

$$\begin{bmatrix} \mathbf{P}_i \\ \mathbf{P}_{i-1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{i-1} \\ \mathbf{P}_{i-2} \end{bmatrix}. \tag{4.3}$$

where $\mathbf{P}_i = \mathbf{0}$ for $i < 0$, then the basic blocks of the Second Order Krylov Subspace $\mathcal{K}_q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1)$ are the upper half of the Krylov subspace $\mathcal{K}_q(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{0} \end{bmatrix}. \tag{4.4}$$

In fact, the basic blocks of the subspace $\mathcal{K}_q(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ are,

$$\begin{bmatrix} \mathbf{P}_0 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{P}_2 \\ \mathbf{P}_1 \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{P}_q \\ \mathbf{P}_{q-1} \end{bmatrix}.$$

Using this relation, the input Krylov subspace $\mathcal{K}_{q_1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ span the same space as the upper half of the standard Krylov subspace,

$$\begin{aligned}
&\mathcal{K}_{q_1}(\begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}) \\
&= \mathcal{K}_{q_1}(\begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}) \\
&= \mathcal{K}_{q_1}(\begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & -\mathbf{K} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}) \\
&= \mathcal{K}_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B}),
\end{aligned}$$

where $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{B}$ are given in (3.2). In a similar way, it is possible to show that the output Second Order Krylov Subspace span the same space as the upper half of the Krylov subspace $\mathcal{K}_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C})$. This provides us the connection between the standard and second order input and output Krylov subspaces.

To use the input and output Second Order Krylov Subspaces in order reduction by moment matching, we first prove a direct connection between these subspace and the moments of a second order system.

Lemma 4.1 *Consider the input and output Second Order Krylov Subspaces for the system (3.1) with corresponding basic blocks* $\mathbf{P}_i$ *and* $\tilde{\mathbf{P}}_i$*, respectively. Then,*

$$\mathbf{m}_i = \mathbf{L}\mathbf{P}_i = \tilde{\mathbf{P}}_i^T \mathbf{G}, \quad i = 0, 1, \cdots .$$

Proof: By using the connection between the standard and Second Order Krylov Subspace, we have

$$\begin{bmatrix} \mathbf{P}_i \\ \mathbf{P}_{i-1} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix},$$

for $i = 0, 1, \cdots$ and $\mathbf{P}_{-1} = \mathbf{0}$. By comparing this result with the equations (3.6), it can easily be concluded that $\mathbf{m}_i = \mathbf{L}\mathbf{P}_i$. To show that $\mathbf{m}_i = \tilde{\mathbf{P}}_i^T \mathbf{G}$, transpose of the equation (3.7) for the definition of the moments is used in a similar way. ■

4.2 The reduction theorems

As mentioned before, in this chapter, we find the reduced order model by applying a projection directly to the second order model. Consider a projection as follows,

$$\mathbf{z} = \bar{\mathbf{V}}\mathbf{z}_r, \; \bar{\mathbf{V}} \in \mathbb{R}^{n\times q}, \mathbf{z} \in \mathbb{R}^q, \mathbf{z}_r \in \mathbb{R}^q, \tag{4.5}$$

where $q < n$. By applying this projection to the system (3.1) and then multiplying the state equation by the transpose of a matrix $\bar{\mathbf{W}} \in \mathbb{R}^{n\times q}$, a reduced model of order $Q = 2q$ is found,

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\ddot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\dot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{z}_r = \bar{\mathbf{W}}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}_r. \end{cases} \tag{4.6}$$

Therefore, the reduced model is identified by the matrices,

$$\begin{aligned}&\mathbf{M}_r = \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\ ,\ \mathbf{D}_r = \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\ ,\ \mathbf{K}_r = \bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}},\\ &\mathbf{G}_r = \bar{\mathbf{W}}^T\mathbf{G}\ ,\ \mathbf{L}_r = \mathbf{L}\bar{\mathbf{V}}.\end{aligned}$$

Obviously, such a reduction procedure preserves the structure of the original second order model (3.1)! We transform the reduced second order system (4.6) into state space using the formulation (3.2),

$$\begin{cases}\underbrace{\begin{bmatrix}\mathbf{0} & \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\\ \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}} & \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\end{bmatrix}}_{\mathbf{E}_r}\underbrace{\begin{bmatrix}\ddot{\mathbf{z}}\\ \dot{\mathbf{z}}\end{bmatrix}}_{\dot{\mathbf{x}}_r} = \underbrace{\begin{bmatrix}\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}} & \mathbf{0}\\ \mathbf{0} & -\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\end{bmatrix}}_{\mathbf{A}_r}\underbrace{\begin{bmatrix}\dot{\mathbf{z}}\\ \mathbf{z}\end{bmatrix}}_{\mathbf{x}_r}\\ \qquad\qquad + \underbrace{\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{W}}^T\mathbf{G}\end{bmatrix}}_{\mathbf{B}_r}\mathbf{u},\\ \mathbf{y} = \underbrace{\begin{bmatrix}\mathbf{0} & \mathbf{L}\bar{\mathbf{V}}\end{bmatrix}}_{\mathbf{C}_r}\begin{bmatrix}\dot{\mathbf{z}}\\ \mathbf{z}\end{bmatrix},\end{cases} \tag{4.7}$$

Therefore, the reduced state space matrices can be calculated by applying the projection,

$$\tilde{\mathbf{V}} = \begin{bmatrix}\bar{\mathbf{V}} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{V}}\end{bmatrix},\quad \tilde{\mathbf{W}} = \begin{bmatrix}\bar{\mathbf{W}} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{W}}\end{bmatrix}, \tag{4.8}$$

to the state space model (3.2) that is equivalent to the second order model (3.1).

For the choice of $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$, the Second Order Krylov Subspaces are used, as described by the following theorems. For moment matching about zero, we consider that $\mathbf{K}^{-1}$ exists.

Theorem 4.1 *If the columns of the matrix $\bar{\mathbf{V}}$ used in (4.6) form a basis for the input Second Order Krylov Subspace $\mathcal{K}_{q_1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ and the matrix $\bar{\mathbf{W}}$ is chosen such that the matrix $\mathbf{K}_r$ is nonsingular, then the first q_1 moments (the moments from $\mathbf{m}_0$ to $\mathbf{m}_{q_1-1}$) of the original and reduced models match.*

Proof: Consider the matrices

$$\begin{cases}\mathbf{P}_{r0} = -\mathbf{K}_r^{-1}\mathbf{G}_r\ ,\ \mathbf{P}_{r1} = \mathbf{K}_r^{-1}\mathbf{D}_r\mathbf{K}_r^{-1}\mathbf{G}_r\\ \mathbf{P}_{ri} = -\mathbf{K}_r^{-1}\mathbf{D}_r\mathbf{P}_{r(i-1)} - \mathbf{K}_r^{-1}\mathbf{M}_r\mathbf{P}_{r(i-2)}\end{cases}$$

By using Lemma 4.1, we just prove that $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, q_1 - 1$ where $\mathbf{P}_i$ and $\mathbf{P}_{ri}$ are the i-th basic blocks of the input Second Order Krylov Subspace for the original and reduced order models, respectively. For the first basic block we have,

$$\begin{aligned}\bar{\mathbf{V}}\mathbf{P}_{r0} &= -\bar{\mathbf{V}}\mathbf{K}_r^{-1}\mathbf{G}_r = -\bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{G} \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}(-\mathbf{K}^{-1}\mathbf{G}) \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{P}_0.\end{aligned}$$

The matrix $\mathbf{P}_0$ is in the Second Order Krylov Subspace and there exists $\mathbf{R}_0 \in \mathbb{R}^{q\times m}$ such that $\mathbf{P}_0 = \bar{\mathbf{V}}\mathbf{R}_0$. Therefore,

$$\bar{\mathbf{V}}\mathbf{P}_{r0} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{R}_0 = \bar{\mathbf{V}}\mathbf{R}_0 = \mathbf{P}_0. \tag{4.9}$$

In the next step, the result in equation (4.9) is used and then,

$$\begin{aligned}\bar{\mathbf{V}}\mathbf{P}_{r1} &= -\bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\mathbf{P}_{r0} \\ &= -\bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\mathbf{P}_0 \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}(-\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_0) \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{P}_1.\end{aligned}$$

The matrix $\mathbf{P}_1$ is in the Second Order Krylov Subspace and can be written as $\mathbf{P}_1 = \bar{\mathbf{V}}\mathbf{R}_1$ for an $\mathbf{R}_1 \in \mathbb{R}^{q\times m}$. Thus,

$$\bar{\mathbf{V}}\mathbf{P}_{r1} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{R}_1 = \bar{\mathbf{V}}\mathbf{R}_1 = \mathbf{P}_1. \tag{4.10}$$

Now consider that the statement is true until $i = j-1$, i.e. $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, j-1$. By using the results for $i = j-2$ and $i = j-1$, for $i = j$ we have,

$$\begin{aligned}\bar{\mathbf{V}}\mathbf{P}_{rj} &= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\mathbf{P}_{r(j-1)} - (\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\mathbf{P}_{r(j-2)}\right] \\ &= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\mathbf{P}_{j-1} - (\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{P}_{j-2}\right] \\ &= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_{j-1} - (\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{j-2}\right] \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\left(-\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_{j-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{j-2}\right) \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{P}_j.\end{aligned}$$

The matrix $\mathbf{P}_j$ is in the Second Order Krylov Subspace and can be written as $\mathbf{P}_j = \bar{\mathbf{V}}\mathbf{R}_j$ for an $\mathbf{R}_j \in \mathbb{R}^{q\times m}$. Thus,

$$\bar{\mathbf{V}}\mathbf{P}_{rj} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{r}_j = \bar{\mathbf{V}}\mathbf{r}_j = \mathbf{P}_j,$$

and by induction, it is proved that $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, q_1 - 1$. For $i = q_1$, because the matrix $\mathbf{P}_{q_1}$ is not in the Second Order Krylov Subspace, the proof fails. Now, by using Lemma 4.1 the proof is completed. ■

To find the reduced order model, the matrix $\bar{\mathbf{V}}$ is calculated using a numerical algorithm as will be discussed later, and for instance $\bar{\mathbf{W}} = \bar{\mathbf{V}}$ can be chosen.

In the SISO case, the result of Theorem 4.1 has some similarities to the results in [86] but it is independent of the output of the system. This fact is important for increasing the number of matching moments (by Theorem 4.2) compared to [86]. Also, Theorem 4.1 is more straightforward and only uses the state equations, similar to the standard Krylov subspace methods in state space in Chapter 2.

A dual formulation of Theorem 4.1 is:

Corollary 4.1 *If the columns of the matrix $\bar{\mathbf{W}}$ used in (4.6) form a basis for the output Second Order Krylov Subspace and $\bar{\mathbf{V}}$ is chosen such that $\mathbf{K}_r$ is nonsingular, then the first q_2 moments of the original and reduced order systems match.*

By using both, input and output Second Order Krylov Subspaces, it is possible to match more moments and to find better approximations of the original large scale system:

Theorem 4.2 *If the columns of the matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ used in (4.6), form bases for the second order input and output Krylov subspaces, respectively, both with the same rank, then the first $q_1 + q_2$ moments of the original and reduced order systems match. It is assumed that $\mathbf{K}$ and $\mathbf{K}_r$ are invertible.*

Proof: To prove this theorem, we use the definition (3.6) of the moments and the projection matrices (4.8) in state space. According to Theorem 4.1, independent of the definition of the output equation, the first q_1 moments match,

$$(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} = \mathbf{V}(\mathbf{A}_r^{-1}\mathbf{E}_r)^i\mathbf{A}_r^{-1}\mathbf{B}_r,\ i = 0\cdots q_1 - 1. \tag{4.11}$$

And based on corollary 4.1 and dual to equation (4.11), we have

$$\mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1} = \mathbf{C}_r(\mathbf{A}_r^{-1}\mathbf{E}_r)^i\mathbf{A}_r^{-1}\mathbf{W}^T,\ i = 0\cdots q_2 - 1. \tag{4.12}$$

The matrices $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$ are defined in system (3.2) and the matrices $\mathbf{A}_r$, $\mathbf{E}_r$ and $\mathbf{C}_r$ are defined in (4.7). We factorize the moments of the original model into two parts,

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^{i-q_1}\mathbf{A}^{-1} \times \mathbf{E} \times \left(\mathbf{A}^{-1}\mathbf{E}\right)^{q_1-1}\mathbf{A}^{-1}\mathbf{B},$$

for $i > q_1 - 1$. By using the equations (4.11) and (4.12), for $i = q_1, \cdots, q_1 + q_2 - 1$ we have,

$$\mathbf{m}_i = \mathbf{C}_r^T(\mathbf{A}_r^{-1}\mathbf{E}_r)^{i-q_1}\mathbf{A}_r^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}(\mathbf{A}_r^{-1}\mathbf{E}_r)^{q_1-1}\mathbf{A}_r^{-1}\mathbf{B}_r.$$

$\mathbf{W}^T\mathbf{E}\mathbf{V} = \mathbf{E}_r$ and then $\mathbf{m}_i = \mathbf{m}_{ri}$ where $i = 0, \cdots, q_1 + q_2 - 1$. ■

A reduction procedure using only one Second Order Krylov Subspace is called a one-sided method and when using two Second Order Krylov Subspaces, it is called a two-sided method.

4.2.1 Symmetric systems

Modelling of many systems leads to second order models with symmetric mass, damping and stiffness matrices. This special case reduces the cost of calculation in finding a reduced order approximation.

Consider that,

$$\mathbf{M}^T = \mathbf{M},\ \mathbf{D}^T = \mathbf{D},\ \mathbf{K}^T = \mathbf{K},\ \mathbf{L}^T = \mathbf{G}.$$

The input and output Second Order Krylov Subspaces for such models are equal. Therefore, if we apply a one-sided method with $\bar{\mathbf{W}} = \bar{\mathbf{V}}$, then double number of moments match.

Theorem 4.3 *If* $\mathbf{M}^T = \mathbf{M},\ \mathbf{D}^T = \mathbf{D},\ \mathbf{K}^T = \mathbf{K},\ \mathbf{L}^T = \mathbf{G}$ *and the columns of the matrix* $\bar{\mathbf{V}}$ *used in (4.6), form a basis for the input Second Order Krylov Subspace and we choose* $\bar{\mathbf{W}} = \bar{\mathbf{V}}$*, then the first* $2q_1$ *moments of the original and reduced order systems match. It is assumed that* $\mathbf{K}$ *and* $\mathbf{K}_r$ *are invertible.*

In fact, for symmetric systems, it is possible to match $2q_1$ moments with half the numerical cost compared to Theorem 4.2.

4.3 Rational interpolation

By matching the moments about zero, the behavior of the original and reduced systems are close to each other at low frequencies. To approximate the behavior at higher frequencies, the moments about $s_0 \neq 0$ can be matched, which can also be done by applying a projection to the original model (3.1).

The transfer function of the system (3.1) by direct Laplace transform is $\mathbf{H}(s) = \mathbf{L}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{G}$. The moments of $\mathbf{H}(s)$ about s_0 are equal to the moments of the following system about zero,

$$\begin{aligned}\mathbf{H}(s+s_0) &= \mathbf{L}\left((s+s_0)^2\mathbf{M} + (s+s_0)\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{G} \\ &= \mathbf{L}\left(s^2\mathbf{M} + s(\mathbf{D}+2s_0\mathbf{M}) + (\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})\right)^{-1}\mathbf{G},\end{aligned}$$

By using (3.6), the moments of $\mathbf{H}(s+s_0)$ are calculated by substituting the matrix $\mathbf{K}$ by $\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M}$ and the matrix $\mathbf{D}$ by $\mathbf{D}+2s_0\mathbf{M}$ in the definition of the moments about zero, which are the moments of the system (3.1) about s_0. Therefore, to match the moments about s_0, the same substitution as in the moments should be done in the definition of the input and output Second Order Krylov Subspaces; i.e. the subspaces $\mathcal{K}_{q_1}(-(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}(\mathbf{D}+2s_0\mathbf{M}), -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}\mathbf{M}, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}\mathbf{G})$ and $\mathcal{K}_{q_1}(-(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}(\mathbf{D}+2s_0\mathbf{M})^T, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}\mathbf{M}^T, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}\mathbf{L}^T)$ are considered and then by finding the corresponding bases as projection matrices, the reduced order system is found. In this case the condition on non-singularity of $\mathbf{K}$ is substituted by non-singularity of $\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M}$. In other words, s_0 should not be a quadratic eigenvalue of $(\mathbf{K}, \mathbf{D}, \mathbf{M})$.

This result can also be generalized to match the moments about different points $s_1, \cdots, s_k$ by considering k different Second Order Krylov Subspaces and finding a projection matrix whose columns form a bases of the union of the given subspaces.

Theorem 4.4 *If the matrix $\bar{\mathbf{V}}$ used in (4.6) is chosen such that,*

$$\bigcup_{i=1}^{l}\mathcal{K}_{q_i}\left(-(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}(\mathbf{D}+2s_i\mathbf{M}), -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}\mathbf{M}\right.$$
$$\left., -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}\mathbf{G}\right) \subseteq colspan(\bar{\mathbf{V}})$$

with an optional full rank matrix $\mathbf{W}$ (e.g. $\mathbf{W} = \mathbf{V}$), then the first q_i moments about s_i for $i = 1, \cdots, l$ of the original and reduced models match. We consider that s_i for $i = 1, \cdots, l$, is not a quadratic eigenvalue of the triples $(\mathbf{K}, \mathbf{D}, \mathbf{M})$ and $(\mathbf{K}_r, \mathbf{D}_r, \mathbf{M}_r)$.

Proof: To prove this theorem, we should consider each subspace separately and apply Theorem 4.1 for l times. ■

Theorem 4.5 *If the matrices* $\bar{\mathbf{V}}$ *and* $\bar{\mathbf{W}}$ *used in (4.6) are chosen such that,*

$$\bigcup_{i=1}^{l_1} \mathcal{K}_{q_i}\left(-(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}(\mathbf{D}+2s_i\mathbf{M}), -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}\mathbf{M}\right.$$
$$\left., -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-1}\mathbf{G}\right) \subseteq colspan(\bar{\mathbf{V}}),$$
$$\bigcup_{i=l_1+1}^{l_2} \mathcal{K}_{q_i}\left(-(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-T}(\mathbf{D}+2s_i\mathbf{M})^T, -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-T}\mathbf{M}^T\right.$$
$$\left., -(\mathbf{K}+s_i\mathbf{D}+s_i^2\mathbf{M})^{-T}\mathbf{L}^T\right) \subseteq colspan(\bar{\mathbf{W}}),$$

then the first q_i *moments about* s_i *for* $i = 1, \cdots, l_2$ *of the original and reduced models match. We consider that* s_i *for* $i = 1, \cdots, l_2$*, is not a quadratic eigenvalue of the triples* $(\mathbf{K}, \mathbf{D}, \mathbf{M})$ *and* $(\mathbf{K}_r, \mathbf{D}_r, \mathbf{M}_r)$*.*

With a two-sided method more number of characteristic parameters match. In Theorem 4.5, if $s_i = s_j$ for $i \neq j$, then $q_i + q_j$ moments about s_i match.

4.4 Matching the Markov parameters

To approximate the system behaviour at high frequencies, one can reduce the original system through matching the Markov parameters. If the matrix $\mathbf{M}$ is nonsingular then the Second Order Krylov Subspaces $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ and $\mathcal{K}_{q_2}(-\mathbf{M}^{-T}\mathbf{K}^T, -\mathbf{M}^{-T}\mathbf{D}^T, \mathbf{M}^{-T}\mathbf{L}^T)$ are used to match the Markov parameters. First, we show the relationship between these subspaces and the Markov parameters.

Lemma 4.2 *If* $\mathbf{P}_i$ *and* $\tilde{\mathbf{P}}_i$ *are basic blocks of the Second Order Krylov Subspaces* $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ *and* $\mathcal{K}_{q_2}(-\mathbf{M}^{-T}\mathbf{K}^T, -\mathbf{M}^{-T}\mathbf{D}^T, \mathbf{M}^{-T}\mathbf{L}^T)$ *for the system (3.1) then,*

$$\mathbf{M}_i = \mathbf{L}\mathbf{P}_{i-1} = \tilde{\mathbf{P}}_{i-1}^T\mathbf{G}, i = 1, 2, \cdots .$$

Proof: The basic blocks of $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ are,

$$\begin{bmatrix} \mathbf{P}_{i-1} \\ \mathbf{P}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}^i \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix},$$

for $i = 0, 1, \cdots$ and $\mathbf{P}_{-1} = \mathbf{0}$. By comparing this result with the equations (3.9), it can easily be concluded that $\mathbf{M}_i = \mathbf{L}\mathbf{P}_{i-1}$. To show that $\mathbf{M}_i = \tilde{\mathbf{P}}_{i-1}^T\mathbf{G}$, transpose of the equation (3.10) for the definition of the Markov parameters is used in a similar way. ■

A difference between lemmas 4.1 and 4.2 is the index of the basic blocks. Because for every second order system $\mathbf{M}_0 = \mathbf{0}$, there is a shift in the index of the basic vectors and the Markov parameter. In matching the Markov parameters by applying a projection to the second order model, the first Markov parameter remains zero (it automatically matches) and only other parameters are to be matched.

Theorem 4.6 *If the columns of the matrix $\bar{\mathbf{V}}$ used in (4.6) form a basis for the Second Order Krylov Subspace $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ and the matrix $\bar{\mathbf{W}}$ is chosen such that the matrix $\mathbf{M}_r$ is nonsingular, then the first q_1 Markov parameters (the Markov parameters from $\mathbf{M}_1$ to $\mathbf{M}_{q_1}$) of the original and reduced models match.*

Proof: Consider the matrices

$$\begin{cases} \mathbf{P}_{r0} = \mathbf{M}_r^{-1}\mathbf{G}_r \ , \ \mathbf{P}_{r1} = -\mathbf{M}_r^{-1}\mathbf{K}_r\mathbf{M}_r^{-1}\mathbf{G}_r \\ \mathbf{P}_{ri} = -\mathbf{M}_r^{-1}\mathbf{K}_r\mathbf{P}_{r(i-1)} - \mathbf{M}_r^{-1}\mathbf{D}_r\mathbf{P}_{r(i-2)} \end{cases}$$

By using Lemma 4.2, we just prove that $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, q_1 - 1$ where $\mathbf{P}_i$ and $\mathbf{P}_{ri}$ are the i-th basic blocks of the Second Order Krylov Subspaces $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ and $\mathcal{K}_{q_1}(-\mathbf{M}_r^{-1}\mathbf{K}_r, -\mathbf{M}_r^{-1}\mathbf{D}_r, \mathbf{M}_r^{-1}\mathbf{G}_r)$, respectively. For the first basic vector we have,

$$\begin{aligned} \bar{\mathbf{V}}\mathbf{P}_{r0} &= \bar{\mathbf{V}}\mathbf{M}_r^{-1}\mathbf{G}_r = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{G} \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}(\mathbf{M}^{-1}\mathbf{G}) \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{P}_0. \end{aligned}$$

The matrix $\mathbf{P}_0$ is in the Second Order Krylov Subspace and there exists $\mathbf{R}_0 \in \mathbb{R}^{q\times m}$ such that $\mathbf{P}_0 = \bar{\mathbf{V}}\mathbf{R}_0$. Therefore,

$$\bar{\mathbf{V}}\mathbf{P}_{r0} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\mathbf{R}_0 = \bar{\mathbf{V}}\mathbf{R}_0 = \mathbf{P}_0. \tag{4.13}$$

In the next step, the result in equation (4.13) is used and then,

$$\begin{aligned} \bar{\mathbf{V}}\mathbf{P}_{r1} &= -\bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{P}_{r0} \\ &= -\bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{P}_0 \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}(-\mathbf{M}^{-1}\mathbf{K}\mathbf{P}_0) \\ &= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{P}_1. \end{aligned}$$

The matrix $\mathbf{P}_1$ is in the Second Order Krylov Subspace and can be written as $\mathbf{P}_1 = \bar{\mathbf{V}}\mathbf{R}_1$ for an $\mathbf{R}_1 \in \mathbb{R}^{q\times m}$. Thus,

$$\bar{\mathbf{V}}\mathbf{P}_{r1} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\mathbf{R}_1 = \bar{\mathbf{V}}\mathbf{R}_1 = \mathbf{P}_1. \tag{4.14}$$

Now consider that the statement is true until $i = j-1$, i.e. $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, j-1$. By using the results for $i = j-2$ and $i = j-1$, for $i = j$ we have,

$$\begin{aligned}
\bar{\mathbf{V}}\mathbf{P}_{rj} &= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{P}_{r(j-1)} - (\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\mathbf{P}_{r(j-2)}\right] \\
&= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{K}\mathbf{P}_{j-1} - (\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{D}\mathbf{P}_{j-2}\right] \\
&= \bar{\mathbf{V}}\left[-(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{K}\mathbf{P}_{j-1} - (\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{D}\mathbf{P}_{j-2}\right] \\
&= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\left(-\mathbf{M}^{-1}\mathbf{K}\mathbf{P}_{j-1} - \mathbf{M}^{-1}\mathbf{D}\mathbf{P}_{j-2}\right) \\
&= \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\mathbf{P}_j.
\end{aligned}$$

The matrix $\mathbf{P}_j$ is in the Second Order Krylov Subspace and can be written as $\mathbf{P}_j = \bar{\mathbf{V}}\mathbf{R}_j$ for an $\mathbf{R}_j \in \mathbb{R}^{q\times m}$. Thus,

$$\bar{\mathbf{V}}\mathbf{P}_{rj} = \bar{\mathbf{V}}(\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}})^{-1}\bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\mathbf{R}_j = \bar{\mathbf{V}}\mathbf{R}_j = \mathbf{P}_j,$$

and by induction, it is proved that $\mathbf{P}_i = \bar{\mathbf{V}}\mathbf{P}_{ri}$ for $i = 0, \cdots, q_1 - 1$. For $i = q_1$, because the matrix $\mathbf{P}_{q_1}$ is not in the given order Krylov subspace, the proof fails. Now, by using Lemma 4.2 the proof is completed. ■

Similar to the moment matching, by using two Second Order Krylov Subspaces, it is possible to match more Markov parameters as in the following theorem.

Theorem 4.7 *If the columns of the matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ used in (4.6), form bases for the Second Order Krylov Subspaces $\mathcal{K}_{q_1}(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G})$ and $\mathcal{K}_{q_2}(-\mathbf{M}^{-T}\mathbf{K}^T, -\mathbf{M}^{-T}\mathbf{D}^T, \mathbf{M}^{-T}\mathbf{L}^T)$, respectively, both with the same rank, then the first $q_1 + q_2$ Markov parameters of the original and reduced order systems match. It is assumed that $\mathbf{M}$ and $\mathbf{M}_r$ are invertible.*

If we combine the Krylov subspaces used to match the moments and Markov parameters then some of the first moments and Markov parameters match, simultaneously. For the one-sided method the condition is,

$$\mathcal{K}_{q_1}\left(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}\right) \bigcup \mathcal{K}_{q_2}\left(-\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{D}, \mathbf{M}^{-1}\mathbf{G}\right) \subseteq \text{colspan}(\bar{\mathbf{V}})$$

and the first q_1 moments and q_2 Markov parameters match. The result can be extended to two-sided method in a similar way.

4.5 Guarantied stability

By considering the sufficient conditions for stability of second order systems, extracted in section 3.4, in the following theorems, we present the sufficient conditions to find a stable reduced system by applying a projection to second order systems.

Theorem 4.8 *Let the system (3.1) be reduced by a one-sided method as in theorems 4.1, 4.4 and 4.6 with* $\mathbf{W} = \mathbf{V}$*. The reduced order model,*

a. *has no pole in the right half complex plane, if* $\mathbf{D} = \mathbf{D}^T \succeq 0$, $\mathbf{M} = \mathbf{M}^T \succeq 0$ *and* $\mathbf{K} = \mathbf{K}^T \succeq 0$.

b. *is stable, if* $\mathbf{D} + \mathbf{D}^T \succeq 0$, $\mathbf{M} = \mathbf{M}^T \succeq 0$ *and* $\mathbf{K} = \mathbf{K}^T \succ 0$.

Proof: By using the assumptions of the theorem and the reduced matrices

$$\mathbf{M}_r = \mathbf{V}^T\mathbf{M}\mathbf{V},\ \mathbf{K}_r = \mathbf{V}^T\mathbf{K}\mathbf{V},\ \mathbf{D}_r = \mathbf{V}^T\mathbf{D}\mathbf{V},$$

by using Lemma 2.2, it can be verified that $\mathbf{D}_r = \mathbf{D}_r^T \succeq 0$, $\mathbf{M}_r = \mathbf{M}_r^T \succeq 0$ and $\mathbf{K}_r = \mathbf{K}_r^T \succeq 0$ (or $\succ 0$ in part (b)) and the proof is completed by applying Lemma 3.4 to the reduced system. ■

The main difference in the two parts of the theorem is in the condition on $\mathbf{K}$. In fact if $\mathbf{K}$ is invertible or equivalently if the original system has no pole at zero, then under some conditions a one-sided method preserves stability of the original system. Otherwise, we have sufficient conditions under which the reduced system has no pole in the right half complex plane but the system may have repeated poles on the imaginary axis. The symmetry and positive semi-definiteness of the matrices as in the preceding theorems are satisfied for a lot of systems in circuit simulation and MEMS.

4.6 Conclusion and comparison

In this chapter, by generalizing the definition of Krylov subspaces, the well-known method of reduction of large scale systems based on moment (or Markov parameter) matching has been applied for the reduction of second order models, resulting in reduced systems having the same structure as the original one. Preserving the structure is achieved by calculating the reduced system through a projection to the original second order model. The advantages of the proposed approach can be highlighted as follows:

- Compared to the method proposed in [86], twice the number of moments match.
- The method can easily be applied to match the moments about different points.
- Because the projection is directly applied to the second order model, some structures of the original matrices are preserved: undamped systems are approximated by undamped systems and one-sided methods preserves symmetry and definiteness of the matrices.
- Under some conditions, one-sided methods preserve stability of the original system.

Chapter 5

NUMERICAL ALGORITHMS

In the previous chapter, the conditions on the projection matrices to match the moments or Markov parameters were investigated. In this chapter, it is explained how to calculate such projection matrices. We extend the famous Arnoldi and Lanczos algorithms to calculate the desired bases.

5.1 Second order Arnoldi algorithm

Here, we extend the Arnoldi Algorithm 2.1 to find a basis for a given Second Order Krylov Subspace. Consider the Second Order Krylov Subspace $\mathcal{K}_q(\breve{\mathbf{D}}, \breve{\mathbf{M}}, \breve{\mathbf{G}})$ with m starting vectors. The algorithm given below finds an *orthonormal* basis for this subspace, i.e. $\bar{\mathbf{V}}^T\bar{\mathbf{V}} = \mathbf{I}$, and the columns of the matrix $\bar{\mathbf{V}}$ form a basis for the given subspace. The algorithm is valid for multiple starting vectors and by deflation, the vectors that can be spanned by the previous vectors are deleted.

Algorithm 5.1 *Second order Arnoldi algorithm*

0. (a) *Delete all linearly dependent starting vectors to get m_1 linearly independent vectors.*

   (b) *Set*
$$\bar{\mathbf{v}}_1 = \frac{\mathbf{g}_1}{\|\mathbf{g}_1\|_2}.$$
   where $\mathbf{g}_1$ is the first starting vector after deleting the dependent starting vectors and set $\mathbf{l}_1 = \mathbf{0}$ for $\mathbf{l}_1 \in \mathbb{R}^n$.

1. *For $i = 2, 3, \cdots,$ do,*

(a) *Calculating the next vector: if $i \leq m_1$ then set $\hat{\mathbf{v}}_i$ as the i-th starting vector and $\hat{\mathbf{l}}_i = \mathbf{0}$. Otherwise, set*

$$\hat{\mathbf{v}}_i = \breve{\mathbf{D}}\bar{\mathbf{v}}_{i-m_1} + \breve{\mathbf{M}}\mathbf{l}_{i-m_1}, \quad \hat{\mathbf{l}}_i = \bar{\mathbf{v}}_{i-m_1}.$$

(b) *Orthogonalization: For j=1 to i-1 do,*

$$h = \hat{\mathbf{v}}_i^T \bar{\mathbf{v}}_j \ , \ \hat{\mathbf{v}}_i = \hat{\mathbf{v}}_i - h\bar{\mathbf{v}}_j \ , \ \hat{\mathbf{l}}_i = \hat{\mathbf{l}}_i - h\mathbf{l}_j.$$

(c) *Deflation: If $\hat{\mathbf{v}}_i \neq \mathbf{0}$ then go to* (1d).
Else, if $\hat{\mathbf{l}}_i \neq \mathbf{0}$ then $\bar{\mathbf{v}}_i = \mathbf{0}$ and go to (1e).
Else, $m_1 = m_1 - 1$ and go to (1a) *(but go to step* (2) *if $m_1 = 0$).*

(d) *Normalization: $\bar{\mathbf{v}}_i = \frac{\hat{\mathbf{v}}_i}{\|\hat{\mathbf{v}}_i\|_2}$ and $\mathbf{l}_i = \frac{\hat{\mathbf{l}}_i}{\|\hat{\mathbf{v}}_i\|_2}$.*

(e) *Increase i and go to step* (1a).

2. *Delete the zero columns of the matrix $\bar{\mathbf{V}}$ produced by deflation process.*

To show that the Algorithm 5.1 produces the required basis for a given subspace, we simplify step (1a) to

$$\begin{bmatrix} \hat{\mathbf{v}}_i \\ \hat{\mathbf{l}}_i \end{bmatrix} = \underbrace{\begin{bmatrix} \breve{\mathbf{D}} & \breve{\mathbf{M}} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} \bar{\mathbf{v}}_{i-m_1} \\ \mathbf{l}_{i-m_1} \end{bmatrix}.$$

From the definition 4.1, the basis of $\mathcal{K}_q(\breve{\mathbf{D}}, \breve{\mathbf{M}}, \breve{\mathbf{G}})$ is the upper half of the Krylov subspace $\mathcal{K}_q(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$, where the lower half of $\tilde{\mathbf{B}}$ is zero and the upper half is $\breve{\mathbf{G}}$. Referring to the Arnoldi Algorithm 2.1, the Algorithm 5.1, is quite similar with the changes in orthogonalization and normalization to make the vectors $\bar{\mathbf{v}}_i$ orthonormal.

For deflation, it is checked if the new vector is a linear combination of the previous ones which is done in step (1c). If only $\hat{\mathbf{v}}_i$ is expanded by $\bar{\mathbf{v}}_1, \cdots, \bar{\mathbf{v}}_{i-1}$ (it is identified by $\hat{\mathbf{v}}_i = \mathbf{0}$), then the vector $\mathbf{l}_i$ should not be deleted to be used in the next iteration. In this case, $\bar{\mathbf{v}}_i$ is substituted by zero (which is deleted at the end). If both vectors $\hat{\mathbf{v}}_i$ and $\hat{\mathbf{l}}_i$ are expanded by $\bar{\mathbf{v}}_1, \cdots, \bar{\mathbf{v}}_{i-1}$ and $\mathbf{l}_1, \cdots, \mathbf{l}_{i-1}$, respectively then the algorithm deletes both vectors. In finite precision mathematics, the vectors should not be compared with zero but with a small number. In fact, in step (1c), $\hat{\mathbf{v}}_i = 0$ and $\hat{\mathbf{l}}_i = 0$ should be substituted with $\|\hat{\mathbf{v}}_i\|_2 < \epsilon$ and $\|\hat{\mathbf{l}}_i\|_2 < \epsilon$, respectively where ϵ is a small positive number.

The second order Arnoldi algorithm not only works with the original matrices of the second order model, but also it produces an orthogonal projection matrix. In moment matching by means of the Second Order Krylov Subspaces, the Algorithm 5.1, needs just an LU-factorization of the matrix $\mathbf{K}$. Compared to the standard algorithms in state space which use the LU-factorization of $\mathbf{A}$ of double dimension, the cost of computation is reduced up to a factor of 8. Furthermore, the Algorithm 5.1 takes advantage of the structures of the original matrices like symmetry, block diagonality or triangularity, produced by modeling (which happens quite often in finite element modeling) making the LU-factorization cheaper.

The moments of the MIMO system (3.1) are $p \times m$ matrices $\mathbf{m}_i$, where each column is related to an input and each row is related to an output of the system. In Algorithm 5.1, the order of the reduced system is independent of the number of starting vectors which is a great advantage specially when the system has a high number of inputs or outputs. If j columns of the matrix $\bar{\mathbf{V}}$ (or $\bar{\mathbf{W}}$) are related to the k-th input (or output), then the k-th column (or row) of the moment matrix matches up to at least the $j-1$-st moment.

To calculate the projection matrices to match the moments about several point as presented in theorems 4.4 and 4.5, an extension of the second order Arnoldi algorithm can be used. The algorithm should be run l times to match the moments about l distinct points considering all previous vectors for orthogonalization.

5.2 Two-sided methods and second order Lanczos algorithm

In this section, the Lanczos algorithm [48] is modified to find a pair of bases for a given pair of Second Order Krylov Subspaces to be used in two-sided methods. Consider the Second Order Krylov Subspaces $\mathcal{K}_{q_1}(\breve{\mathbf{D}}, \breve{\mathbf{M}}, \breve{\mathbf{G}})$ and $\mathcal{K}_{q_2}(\hat{\mathbf{D}}, \hat{\mathbf{M}}, \hat{\mathbf{L}})$ with m and p starting vectors. The algorithm given below finds *bi-orthogonal* bases for these subspaces; i.e. $\bar{\mathbf{W}}^T\bar{\mathbf{V}} = \mathbf{\Delta}$, where $\mathbf{\Delta}$ is a diagonal matrix and the columns of the matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ form bases for the given subspaces. Similar to the second order Arnoldi algorithm, the algorithm uses deflation and is valid for multiple starting vectors.

Algorithm 5.2 *Second order Lanczos algorithm*

0. Start: Delete all linearly dependent starting vectors to get m_1 *and* p_1 *independent* *starting vectors,* $\breve{\mathbf{g}}_1, \cdots, \breve{\mathbf{g}}_{m_1}$ *and* $\hat{\mathbf{l}}_1, \cdots, \hat{\mathbf{l}}_{p_1}$*, for input and output Krylov subspaces, respectively.*

1. *Set*

$$\bar{\mathbf{v}}_1 = \frac{\breve{\mathbf{g}}_1}{\|\hat{\mathbf{g}}_1\|}, \quad \bar{\mathbf{w}}_1 = \frac{\hat{\mathbf{l}}_1}{\|\hat{\mathbf{l}}_1\|}$$

2. *Set* $\mathbf{h}_i = \mathbf{0}$ *for* $\mathbf{h}_i \in \mathbb{R}^n$ *and* $i = 1, \cdots, m_1$.

3. *Set* $\mathbf{t}_i = \mathbf{0}$ *for* $\mathbf{t}_i \in \mathbb{R}^n$ *and* $i = 1, \cdots, p_1$.

4. *For* $i = 2, 3, \cdots$ *do.*

   (a) *Calculating the next vector: For the input Krylov subspace, if* $i \leq m_1$ *then* $\hat{\mathbf{v}}_i = \breve{\mathbf{g}}_i$*. Otherwise, the next vectors is computed as follows,*

$$\hat{\mathbf{v}}_i = \breve{\mathbf{D}}\bar{\mathbf{v}}_{i-m_1} + \breve{\mathbf{M}}\mathbf{h}_{i-m_1}, \quad \hat{\mathbf{h}}_i = \bar{\mathbf{v}}_{i-m_1}.$$

   For the output Krylov subspace, if $i \leq p_1$ *then* $\hat{\mathbf{w}}_i = \hat{\mathbf{l}}_i$*. Otherwise, the next vector is*

$$\hat{\mathbf{w}}_i = \hat{\mathbf{D}}\bar{\mathbf{w}}_{i-p_1} + \hat{\mathbf{M}}\mathbf{t}_{i-p_1}, \quad \hat{\mathbf{t}}_i = \bar{\mathbf{w}}_{i-p_1}.$$

   (b) *Orthogonalization: For* $j = 1, \cdots, i-1$ *do:*

$$\begin{aligned} a = \hat{\mathbf{v}}_j^T\bar{\mathbf{w}}_i \quad &, \quad b = \hat{\mathbf{w}}_j^T\bar{\mathbf{v}}_i, \\ \hat{\mathbf{v}}_j = \hat{\mathbf{v}}_j - a\bar{\mathbf{v}}_i \quad &, \quad \hat{\mathbf{w}}_j = \hat{\mathbf{w}}_j - b\bar{\mathbf{w}}_i \\ \hat{\mathbf{h}}_j = \hat{\mathbf{h}}_j - a\mathbf{h}_i \quad &, \quad \hat{\mathbf{t}}_j = \hat{\mathbf{t}}_j - b\mathbf{t}_i.. \end{aligned}$$

   (c) *Deflation on* $\bar{\mathbf{V}}$*: If* $\hat{\mathbf{v}}_i \neq \mathbf{0}$ *then* $\bar{\mathbf{v}}_i = \frac{\hat{\mathbf{v}}_i}{\|\hat{\mathbf{v}}_i\|_2}$ *and* $\mathbf{h}_i = \frac{\hat{\mathbf{h}}_i}{\|\hat{\mathbf{v}}_i\|_2}$.
   Else, if $\hat{\mathbf{h}}_i \neq \mathbf{0}$ *then* $\bar{\mathbf{v}}_i = \mathbf{0}$ *and go to step* (4d).
   Else, $m_1 = m_1 - 1$ *and go to step* (4a) *(but go to step* (5) *if* $m_1 = 0$*).*

   (d) *Deflation on* $\bar{\mathbf{W}}$*: If* $\hat{\mathbf{w}}_i \neq \mathbf{0}$ *then* $\bar{\mathbf{w}}_i = \frac{\hat{\mathbf{w}}_i}{\|\hat{\mathbf{w}}_i\|_2}$ *and* $\mathbf{t}_i = \frac{\hat{\mathbf{t}}_i}{\|\hat{\mathbf{w}}_i\|_2}$.
   Else, if $\hat{\mathbf{t}}_i \neq \mathbf{0}$ *then* $\bar{\mathbf{w}}_i = \mathbf{0}$ *and go to step* (4e).
   Else, $p_1 = p_1 - 1$ *and go to step* (4a) *(but go to step* (5) *if* $p_1 = 0$*).*

   (e) *Increase* i *and go to step* (4a).

5. *Delete the zero columns of the matrices* $\bar{\mathbf{V}}$ *and* $\bar{\mathbf{W}}$ *produced by deflation process.*

The discussion about deflation is quite similar to the second order Arnoldi algorithm. A point in the case of deflation is that if for instance both vectors $\hat{\mathbf{v}}_i, \hat{\mathbf{h}}_i$ are zero, the algorithm calculates not only the new $\hat{\mathbf{v}}_i, \hat{\mathbf{h}}_i$ but also it repeats the calculation of $\hat{\mathbf{w}}_i, \hat{\mathbf{t}}_i$ which is not necessary. In this case, we can define a flag that becomes true, if such a deflation occurs and in the next iteration, only $\hat{\mathbf{v}}_i, \hat{\mathbf{h}}_i$ are calculated. The same solution can be used if $\hat{\mathbf{w}}_i$ and $\hat{\mathbf{t}}_i$ are zero.

Similar to the second order Arnoldi algorithm, in finite precision mathematics, in steps (4c) and (4d) of the algorithm, we should check if the norm of the vectors are very small.

If we would like to have a pair of projection matrices with the property $\bar{\mathbf{W}}^T\bar{\mathbf{V}} = \mathbf{I}$, then we can easily apply the following algorithm after the Lanczos Algorithm 5.2 if $\mathbf{\Delta}$ is invertible.

Algorithm 5.3 *Bi-normalization of the projection matrices*
For $i = 1, 2, \cdots$ *do.*

1. *Normalization: Set*

$$\bar{\mathbf{v}}_i = \frac{\bar{\mathbf{v}}_i}{\sqrt{|\bar{\mathbf{w}}_i^T\bar{\mathbf{v}}_i|}}, \quad \bar{\mathbf{w}}_i = \frac{\bar{\mathbf{w}}_i}{-\sqrt{|\bar{\mathbf{w}}_i^T\bar{\mathbf{v}}_i|}}$$

2. *Increase* i *and go to step* (1).

If the part of deflation is neglected in Algorithm 5.2, for example when dealing with SISO systems, then Algorithms 5.2 and 5.3 can easily be integrated into a single algorithm. Otherwise combining the two algorithms is complicated.

The advantage of having bi-orthonormal projection matrices is that if one of the original mass, damping or stiffness matrices are identity, this property is automatically preserved in the reduced order system.

5.2.1 Two-sided second order Arnoldi

Like the two-sided Arnoldi algorithm in state space as discussed in section 2.4.3, in two-sided methods, the second order Arnoldi algorithm can be used twice to calculate the projection matrices. To do so, we apply the Algorithm 5.2 first for the input Second Order Krylov Subspace and then for the output Second Order Krylov Subspace, and the matrices $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ are found. The resulting reduction scheme can be called a *two-sided second order Arnoldi* method.

The difference of the two-sided second order Arnoldi algorithm to the second order Lanczos algorithm is that the projection matrices are orthogonal $\bar{\mathbf{V}}^T\bar{\mathbf{V}} = \mathbf{I}, \bar{\mathbf{W}}^T\bar{\mathbf{W}} = \mathbf{I}$ while in the Lanczos algorithm the projection matrices are bi-orthogonal $\bar{\mathbf{W}}^T\bar{\mathbf{V}} = \boldsymbol{\Delta}$.

In the next chapter, it is shown that the transfer function of the reduced systems found by the two-sided second order Arnoldi and the second order Lanczos algorithms are exactly the same.

5.3 Conclusion and comparison

In this chapter, the standard algorithms of Krylov subspace methods, Arnoldi, Lanczos and two-sided Arnoldi algorithms were extended to calculate bases of Second Order Krylov Subspaces to calculate the projection matrices to reduce the order of second order models.

Compared to the state space methods, half number of iterations is needed to reduce to the same order and to match the moments about s_0, the proposed algorithms uses a cheaper calculation via doing the calculation in the size of the second order model and using only an LU-factorization of the matrix $\mathbf{K} + s_0\mathbf{D} + s_0^2\mathbf{M}$ (compared to the LU-factorization of $\mathbf{A} - s_0\mathbf{E}$).

Chapter 6

INVARIANCE PROPERTIES

In Chapter 4, the conditions on the projection matrices were discussed in order to match some of the characteristic parameters of the original and reduced system. It was shown that using *any* basis of Second Order Krylov Subspaces as projection matrices leads us to the reduced systems with desired properties. In this chapter, first the effect of the choice of bases on input-output behaviour of the reduced order system is investigated. We also discuss if changing the realization or representation of the original model affects the transfer function of the reduced system.

6.1 Invariance to change of bases

There can be several ways to calculate the basis for a given subspace. In order reduction using Krylov subspaces, the main question is what are the effects of changing the basis on input-output behaviour of the reduced order system. This question is answered by the following theorems.

Theorem 6.1 *The transfer function of the reduced order system found by a one-sided reduction method in theorems 4.1, 4.4 and 4.6 with* $\bar{\mathbf{W}} = \bar{\mathbf{V}}$*, is independent of the particular choice of the bases* $\bar{\mathbf{V}}$*.*

Proof: Consider two reduced order models by using bases $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_2$ of the same subspace. The reduced order models are

$$\begin{cases} \bar{\mathbf{V}}_1^T\mathbf{M}\bar{\mathbf{V}}_1\ddot{\mathbf{z}}_{r1} + \bar{\mathbf{V}}_1^T\mathbf{D}\bar{\mathbf{V}}_1\dot{\mathbf{z}}_{r1} + \bar{\mathbf{V}}_1^T\mathbf{K}\bar{\mathbf{V}}_1\mathbf{z}_{r1} = \bar{\mathbf{V}}_1^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_1\mathbf{z}_{r1}, \end{cases} \tag{6.1}$$

$$\begin{cases} \bar{\mathbf{V}}_2^T\mathbf{M}\bar{\mathbf{V}}_2\ddot{\mathbf{z}}_{r2} + \bar{\mathbf{V}}_2^T\mathbf{D}\bar{\mathbf{V}}_2\dot{\mathbf{z}}_{r2} + \bar{\mathbf{V}}_2^T\mathbf{K}\bar{\mathbf{V}}_2\mathbf{z}_{r2} = \bar{\mathbf{V}}_2^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_2\mathbf{z}_{r2}, \end{cases} \tag{6.2}$$

Because the columns of the matrix $\bar{\mathbf{V}}_2$ are in the given subspace, they can be written as a linear combination of the other bases which are columns of the matrix $\bar{\mathbf{V}}_1$,

$$\exists \mathbf{Q} \in \mathbb{R}^{q\times q} : \bar{\mathbf{V}}_2 = \bar{\mathbf{V}}_1\mathbf{Q}. \tag{6.3}$$

Since $\bar{\mathbf{V}}_1$ and $\bar{\mathbf{V}}_2$ are full rank, the matrix $\mathbf{Q}$ is invertible. By substituting the equation (6.3) into the equation (6.2), we find

$$\begin{cases} \mathbf{Q}^T\bar{\mathbf{V}}_1^T\mathbf{M}\bar{\mathbf{V}}_1\mathbf{Q}\ddot{\mathbf{z}}_{r2} + \mathbf{Q}^T\bar{\mathbf{V}}_1^T\mathbf{D}\bar{\mathbf{V}}_1\mathbf{Q}\dot{\mathbf{z}}_{r2} + \mathbf{Q}^T\bar{\mathbf{V}}_1^T\mathbf{K}\bar{\mathbf{V}}_1\mathbf{Q}\mathbf{z}_{r2} = \mathbf{Q}^T\bar{\mathbf{V}}_1^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_1\mathbf{Q}\mathbf{z}_{r2}, \end{cases}$$

Now, if we multiply the state equation with $\mathbf{Q}^{-T}$ and apply the state transformation $\mathbf{z}_{r1} = \mathbf{Q}\mathbf{z}_{r2}$, the reduced system (6.1) is found. This says that the transfer function of reduced systems (6.1) and (6.2) are the same. ∎

Based on the result of Theorem 6.1, the transfer function of the reduced system does not depend on the numerical algorithm used to calculate the projection matrix. In fact, any easy to calculate basis of the Second Order Krylov Subspace can freely be chosen to calculate the reduce order system. Similar result can be proved for the two-sided methods:

Theorem 6.2 *The transfer function of the reduced order system found by a two-sided reduction method as in theorems 4.2, 4.5 and 4.7, is independent of the particular choice of the bases* $\bar{\mathbf{V}}$ *and* $\bar{\mathbf{W}}$.

Proof: Consider two reduced order models by using bases $\bar{\mathbf{V}}_1, \bar{\mathbf{W}}_1$ and $\bar{\mathbf{V}}_2, \bar{\mathbf{W}}_2$. The reduced order models are

$$\begin{cases} \bar{\mathbf{W}}_1^T\mathbf{M}\bar{\mathbf{V}}_1\ddot{\mathbf{z}}_{r1} + \bar{\mathbf{W}}_1^T\mathbf{D}\bar{\mathbf{V}}_1\dot{\mathbf{z}}_{r1} + \bar{\mathbf{W}}_1^T\mathbf{K}\bar{\mathbf{V}}_1\mathbf{z}_{r1} = \bar{\mathbf{W}}_1^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_1\mathbf{z}_{r1}, \end{cases} \tag{6.4}$$

$$\begin{cases} \bar{\mathbf{W}}_2^T\mathbf{M}\bar{\mathbf{V}}_2\ddot{\mathbf{z}}_{r2} + \bar{\mathbf{W}}_2^T\mathbf{D}\bar{\mathbf{V}}_2\dot{\mathbf{z}}_{r2} + \bar{\mathbf{W}}_2^T\mathbf{K}\bar{\mathbf{V}}_2\mathbf{z}_{r2} = \bar{\mathbf{W}}_2^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_2\mathbf{z}_{r2}, \end{cases} \tag{6.5}$$

Similar to the proof of Theorem 6.1, we have,

$$\exists \mathbf{Q}_v, \mathbf{Q}_w \in \mathbb{R}^{q\times q} : \bar{\mathbf{V}}_2 = \bar{\mathbf{V}}_1\mathbf{Q}_v, \bar{\mathbf{W}}_2 = \bar{\mathbf{W}}_1\mathbf{Q}_w, \tag{6.6}$$

where the matrices $\mathbf{Q}_v$ and $\mathbf{Q}_w$ are invertible. By substituting equations (6.6) into equation (6.5) we find

$$\begin{cases} \mathbf{Q}_w^T\bar{\mathbf{W}}_1^T\mathbf{M}\bar{\mathbf{V}}_1\mathbf{Q}_v\ddot{\mathbf{z}}_{r2} + \mathbf{Q}_w^T\bar{\mathbf{W}}_1^T\mathbf{D}\bar{\mathbf{V}}_1\mathbf{Q}_v\dot{\mathbf{z}}_{r2} + \mathbf{Q}_w^T\bar{\mathbf{W}}_1^T\mathbf{K}\bar{\mathbf{V}}_1\mathbf{Q}_v\mathbf{z}_{r2} = \mathbf{Q}_w^T\bar{\mathbf{W}}_1^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}_1\mathbf{Q}_v\mathbf{z}_{r2}, \end{cases}$$

Now, if we multiply the state equation with $\mathbf{Q}_w^{-T}$ and apply the state transformation $\mathbf{z}_{r1} = \mathbf{Q}_v \mathbf{z}_{r2}$, the reduced system (6.4) is found. This says that the transfer function of reduced systems (6.4) and (6.5) are the same. ■

The main result of Theorem 6.2 is that any two-sided method like second order Lanczos and two-sided second order Arnoldi algorithms as explained in section 5.2 leads to the reduced systems with the same transfer functions.

6.2 Invariance to representation and realization

The effect of different modelling of a single system on the input-output behaviour of the reduced order model is important as it is desired to achieve reduced systems with the same transfer function, if the transfer function of the original system does not change. First, we check if the transfer function of the reduced system changes when the state equation of the original model is multiplied with a constant invertible matrix (change of representation).

Theorem 6.3 *In order reduction based on projection using a two-sided method as mentioned in theorems 4.2, 4.5 and 4.7, changing the representation of the original system does not change the input-output behaviour of the reduced order model.*

Proof: Consider two different representations of an original system

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}} + \mathbf{D}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{z}, \end{cases} \tag{6.7}$$

$$\begin{cases} \mathbf{T}\mathbf{M}\ddot{\mathbf{z}} + \mathbf{T}\mathbf{D}\dot{\mathbf{z}} + \mathbf{T}\mathbf{K}\mathbf{z} = \mathbf{T}\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{z}, \end{cases} \tag{6.8}$$

where $\mathbf{T}$ is an invertible matrix. For reducing the representation (6.8), the input Second Order Krylov Subspace is

$$\mathcal{K}_{q_1}(-(\mathbf{TK})^{-1}\mathbf{TD}, -(\mathbf{TK})^{-1}\mathbf{TM}, -(\mathbf{TK})^{-1}\mathbf{TG}) = \mathcal{K}_{q_1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}),$$

which is equal to the input Second Order Krylov Subspace of the representation (6.7). The output Second Order Krylov Subspace for the representation (6.8) is,

$$\mathcal{K}_{q_2}(-(\mathbf{TK})^{-T}(\mathbf{TD})^T, -(\mathbf{TK})^{-T}(\mathbf{TM})^T, -(\mathbf{TK})^{-T}\mathbf{L}^T) =$$
$$\mathcal{K}_{q_2}(-\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{T}^T, -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{M}^T\mathbf{T}^T, -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{L}^T).$$

The basic blocks of this subspace are,

$$\begin{cases} \tilde{\mathbf{P}}_0 = -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{L}^T \ , \ \tilde{\mathbf{P}}_1 = -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{T}^T\mathbf{P}_0 = \mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{K}^{-T}\mathbf{L}^T \\ \tilde{\mathbf{P}}_i = -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{T}^T\tilde{\mathbf{P}}_{i-1} - \mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{M}^T\mathbf{T}^T\tilde{\mathbf{P}}_{i-2}, \ i = 2, 3, \cdots \end{cases} \tag{6.9}$$

The basic blocks of the output Second Order Krylov Subspace of representation (6.7) are,

$$\begin{cases} \mathbf{P}_0 = -\mathbf{K}^{-T}\mathbf{L}^T \ , \ \mathbf{P}_1 = \mathbf{K}^{-T}\mathbf{D}^T\mathbf{P}_0 \\ \mathbf{P}_i = -\mathbf{K}^{-T}\mathbf{D}^T\mathbf{P}_{i-1} - \mathbf{K}^{-T}\mathbf{M}^T\mathbf{P}_{i-2}, \ i = 2, 3, \cdots \end{cases} \tag{6.10}$$

By induction, we prove that $\tilde{\mathbf{P}}_i = \mathbf{T}^{-T}\mathbf{P}_i$ for $i = 1, 2, \cdots$. By comparing equations (6.9) and (6.10), it is clear that $\tilde{\mathbf{P}}_0 = \mathbf{T}^{-T}\mathbf{P}_0$ and $\tilde{\mathbf{P}}_1 = \mathbf{T}^{-T}\mathbf{P}_1$. Now, consider this relation is true for $i = 1, 2, \cdots, j-1$. For $i = j$, we have,

$$\begin{aligned} \tilde{\mathbf{P}}_j &= -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{T}^T\tilde{\mathbf{P}}_{j-1} - \mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{M}^T\mathbf{T}^T\tilde{\mathbf{P}}_{j-2} \\ &= -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{P}_{j-1} - \mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{M}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{P}_{j-2} \\ &= -\mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{D}^T\mathbf{P}_{j-1} - \mathbf{T}^{-T}\mathbf{K}^{-T}\mathbf{M}^T\mathbf{P}_{j-2} \\ &= \mathbf{T}^{-T}\left(-\mathbf{K}^{-T}\mathbf{D}^T\mathbf{P}_{j-1} - \mathbf{K}^{-T}\mathbf{M}^T\mathbf{P}_{j-2}\right) = \mathbf{T}^{-T}\mathbf{P}_j. \end{aligned}$$

According to Theorem 6.2, we can choose any bases of the Krylov subspaces. So, we choose the basic blocks for projection. Therefore for both representations, $\bar{\mathbf{V}}$ is the same because the input Krylov subspace is independent of representation. If the other matrix for representation (6.7) is $\bar{\mathbf{W}}$, for representation (6.8), it is $\mathbf{T}^{-T}\bar{\mathbf{W}}$. Thus, the reduced systems of the representations (6.7) and (6.8) are,

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\ddot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\dot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{z}_r = \bar{\mathbf{W}}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}_r, \end{cases}$$

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{T}^{-1}\mathbf{T}\mathbf{M}\bar{\mathbf{V}}\ddot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{T}^{-1}\mathbf{T}\mathbf{D}\bar{\mathbf{V}}\dot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{T}^{-1}\mathbf{T}\mathbf{K}\bar{\mathbf{V}}\mathbf{z}_r = \bar{\mathbf{W}}^T\mathbf{T}^{-1}\mathbf{T}\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}_r, \end{cases}$$

and it is clear that the two realizations are exactly the same. ■

Therefore, before applying a two-sided reduction methods, we can multiply the state equation with an invertible matrix without any change on the transfer function of the reduced system. In one-sided methods using input Second Order Krylov Subspace, a related theorem is not valid, although the corresponding subspace is independent of the representation. In this case the reduced order system after changing the representation is,

$$\begin{cases} \bar{\mathbf{V}}^T\mathbf{T}\mathbf{M}\bar{\mathbf{V}}\ddot{\mathbf{z}} + \bar{\mathbf{W}}^T\mathbf{T}\mathbf{D}\bar{\mathbf{V}}\dot{\mathbf{z}} + \bar{\mathbf{W}}^T\mathbf{T}\mathbf{K}\bar{\mathbf{V}}\mathbf{z} = \bar{\mathbf{W}}^T\mathbf{T}\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}, \end{cases}$$

where the effect of $\mathbf{T}$ is not removed after reduction and reduced systems with different transfer function are achieved. In application, this can be an essential disadvantage, since it makes results depending on the representation.

Another important property to be investigated is the effect of realization of the original system on the reduced order system.

Theorem 6.4 *In two-sided second order Krylov method as in theorems 4.2, 4.5 and 4.7, changing the realization of the original system does not change the input-output behaviour of the reduced order model.*

Proof: Consider two realizations of the original system using an invertible matrix $\mathbf{T}$,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}} + \mathbf{D}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{z}, \end{cases} \tag{6.11}$$

$$\begin{cases} \mathbf{M}\mathbf{T}\ddot{\tilde{\mathbf{z}}} + \mathbf{D}\mathbf{T}\dot{\tilde{\mathbf{z}}} + \mathbf{K}\mathbf{T}\tilde{\mathbf{z}} = \mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{T}\tilde{\mathbf{z}}, \end{cases} \tag{6.12}$$

Because the two-sided method is independent of the representation, we can use the representation in equation (6.12). For the realization (6.12), the output Second Order Krylov Subspace is

$$\mathcal{K}_{q_2}(-(\mathbf{KT})^{-T}(\mathbf{DT})^T, -(\mathbf{KT})^{-T}(\mathbf{MT})^T, -(\mathbf{KT})^{-T}(\mathbf{LT})^T) = \\ \mathcal{K}_{q_2}(-\mathbf{K}^{-T}\mathbf{D}^T, -\mathbf{K}^{-T}\mathbf{M}^T, -\mathbf{K}^{-T}\mathbf{L}^T),$$

which is equal to the output Second Order Krylov Subspace of the realization (6.11). The input Second Order Krylov Subspace for the realization (6.12) is,

$$\mathcal{K}_{q_1}(-(\mathbf{KT})^{-1}\mathbf{DT}, -(\mathbf{KT})^{-1}\mathbf{MT}, -(\mathbf{KT})^{-1}\mathbf{G}) = \\ \mathcal{K}_{q_1}(-\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{DT}, -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{MT}, -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{G}),$$

The basic blocks of this subspace are,

$$\begin{cases} \tilde{\mathbf{P}}_0 = -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{G} \ , \ \tilde{\mathbf{P}}_1 = -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{DTP}_0 = -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1}\mathbf{G} \\ \tilde{\mathbf{P}}_i = -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{DT}\tilde{\mathbf{P}}_{i-1} - \mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{MT}\tilde{\mathbf{P}}_{i-2}, \ i = 2, 3, \cdots \end{cases} \tag{6.13}$$

The basic blocks of the input Second Order Krylov Subspace of realization (6.11) is,

$$\begin{cases} \mathbf{P}_0 = -\mathbf{K}^{-1}\mathbf{G}\ ,\ \mathbf{P}_1 = -\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_0 \\ \mathbf{P}_i = -\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_{i-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{i-2},\ i = 2, 3, \cdots \end{cases} \tag{6.14}$$

By induction, we prove that $\tilde{\mathbf{P}}_i = \mathbf{T}^{-1}\mathbf{P}_i$ for $i = 1, 2, \cdots$. By comparing equations (6.13) and (6.14), it is clear that $\tilde{\mathbf{P}}_0 = \mathbf{T}^{-1}\mathbf{P}_0$ and $\tilde{\mathbf{P}}_1 = \mathbf{T}^{-1}\mathbf{P}_1$. Now, consider this relation is true for $i = 1, 2, \cdots, j-1$. For $i = j$, we have,

$$\begin{aligned} \tilde{\mathbf{P}}_j &= -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{D}\mathbf{T}\tilde{\mathbf{P}}_{i-1} - \mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{M}\mathbf{T}\tilde{\mathbf{P}}_{i-2} \\ &= -\mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{D}\mathbf{T}\mathbf{T}^{-1}\mathbf{P}_{i-1} - \mathbf{T}^{-1}\mathbf{K}^{-1}\mathbf{M}\mathbf{T}\mathbf{T}^{-1}\mathbf{P}_{i-2} \\ &= \mathbf{T}^{-1}\left(-\mathbf{K}^{-1}\mathbf{D}^{-1}\mathbf{P}_{i-1} - \mathbf{K}^{-1}\mathbf{M}^{-1}\mathbf{P}_{i-2}\right) = \mathbf{T}^{-1}\mathbf{P}_j. \end{aligned}$$

According to Theorem 6.2, we can choose any bases of the Krylov subspaces. So, we choose the basic blocks for projection. With this choice, for both realizations, $\bar{\mathbf{W}}$ is the same because the output Krylov subspace is independent of realization. If the other matrix for the realization (6.11) is $\bar{\mathbf{V}}$, for the realization (6.12), it is $\mathbf{T}^{-1}\bar{\mathbf{V}}$. Thus, the reduced systems of the realizations (6.11) and (6.12) are,

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\ddot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\dot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{z}_r = \bar{\mathbf{W}}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}_r, \end{cases}$$

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{M}\bar{\mathbf{V}}\mathbf{T}^{-1}\mathbf{T}\ddot{\tilde{\mathbf{z}}}_r + \bar{\mathbf{W}}^T\mathbf{D}\bar{\mathbf{V}}\mathbf{T}^{-1}\mathbf{T}\dot{\tilde{\mathbf{z}}}_r + \bar{\mathbf{W}}^T\mathbf{K}\bar{\mathbf{V}}\mathbf{T}^{-1}\mathbf{T}\tilde{\mathbf{z}}_r = \bar{\mathbf{W}}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\mathbf{T}^{-1}\mathbf{T}\tilde{\mathbf{z}}_r, \end{cases}$$

and it is clear that the two realizations are exactly the same. ■

In both theorems 6.3 and 6.4, a particular pair of bases is used for the proof leading to the same realization. In practice, by using the same numerical algorithm, reduced order systems with different realizations but the same transfer function are found.

If a one-sided method is applied, changing the realization of the original model changes the transfer function of the reduced order model. By a one-sided method, the reduced order model of the realization (6.12) is,

$$\begin{cases} \bar{\mathbf{V}}^T\mathbf{T}^{-T}\mathbf{M}\bar{\mathbf{V}}\ddot{\tilde{\mathbf{z}}}_r + \bar{\mathbf{V}}^T\mathbf{T}^{-T}\mathbf{D}\bar{\mathbf{V}}\dot{\tilde{\mathbf{z}}}_r + \bar{\mathbf{V}}^T\mathbf{T}^{-T}\mathbf{K}\bar{\mathbf{V}}\tilde{\mathbf{z}}_r = \bar{\mathbf{V}}^T\mathbf{T}^{-T}\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\bar{\mathbf{V}}\tilde{\mathbf{z}}_r, \end{cases}$$

which is in general different from the reduced order model of the realization (6.11).

Table 6.1: Invariance properties of Second Order Krylov subspace methods and its effect on the reduced order model

Method	Subspace Used	Number of matching Parameters (SISO)	Change of Basis
One-sided	- Input Krylov - $\bar{\mathbf{W}} = \bar{\mathbf{V}}$	q Parameters	Transfer function is unchanged
	- output Krylov - $\bar{\mathbf{V}} = \bar{\mathbf{W}}$	q Parameters	Transfer function is unchanged
Two-sided	- output Krylov - Input Krylov	$2q$ Parameters	Transfer function is unchanged
Method	**Subspace Used**	**Change of Representation**	**Change of Realization**
One-sided	- Input Krylov - $\bar{\mathbf{W}} = \bar{\mathbf{V}}$	Transfer function changes	Transfer function changes
	- output Krylov - $\bar{\mathbf{V}} = \bar{\mathbf{W}}$	Transfer function changes	Transfer function changes
Two-sided	- output Krylov - Input Krylov	Transfer function is unchanged	Transfer function is unchanged

6.3 Conclusion

In this chapter, the invariance properties of the Second Order Krylov methods were investigated. The results of our invariance properties are summarized in Table 6.1. Similar to the state space Krylov methods, the one-sided methods possess the weakest invariance properties. Reduced order models using two-sided methods not only match more moments, but also their input-output behaviour is independent of the realization and representation of the original system.

Chapter 7

GENERALIZATION TO R-TH ORDER MODELS

The Krylov subspace methods can be generalized to reduce sets of differential equations of orders higher than 2 [32]. We generalize second order Krylov subspace methods to reduce a large set of R-th order differential equations by applying the projection directly to the original system and extend the definition of Krylov subspaces to higher orders; see also [38] for more discussions and alternatives.

Consider the system of the form,

$$\begin{cases} \mathbf{A}_R \mathbf{z}^{(R)}(t) + \mathbf{A}_{R-1}\mathbf{z}^{(R-1)}(t) + \cdots + \mathbf{A}_0 \mathbf{z} = \mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\mathbf{z}(t), \end{cases} \tag{7.1}$$

with r, R-th order differential equations, m inputs and p outputs. The order of the system (7.1) is $N = R \cdot r$.

Equivalently, the model (7.1) can be rewritten in state space as

$$\left\{\begin{array}{l} \underbrace{\begin{bmatrix} \mathbf{F}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}_{R-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_R \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \\ \vdots \\ \mathbf{z}^{(R-1)} \\ \mathbf{z}^{R} \end{bmatrix}}_{\dot{\mathbf{x}}} = \\ \qquad \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{F}_{R-1} \\ -\mathbf{A}_0 & -\mathbf{A}_1 & \cdots & -\mathbf{A}_{R-2} & -\mathbf{A}_{R-1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \\ \vdots \\ \mathbf{z}^{(R-2)} \\ \mathbf{z}^{(R-1)} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}, \\ \mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \\ \vdots \\ \mathbf{z}^{(R-2)} \\ \mathbf{z}^{(R-1)} \end{bmatrix}, \end{array}\right.$$

where $\mathbf{F}_i \in \mathbb{R}^{R\times R}$ are invertible matrices for $i = 1, \cdots R-1$.

The i-th moment (about zero) of the system (7.1) is,

$$\mathbf{m}_i = \begin{bmatrix} \mathbf{L}^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}^T \begin{bmatrix} -\mathbf{A}_0^{-1}\mathbf{A}_1 & \cdots & -\mathbf{A}_0^{-1}\mathbf{A}_{R-1} & -\mathbf{A}_0^{-1}\mathbf{A}_R \\ \mathbf{I} & \vdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{A}_0^{-1}\mathbf{G} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \tag{7.2}$$

Definition 7.1 *The R-th order Krylov subspace is defined as,*

$$\mathcal{K}_{q_1}(\tilde{\mathbf{A}}_1, \cdots, \tilde{\mathbf{A}}_R, \mathbf{B}_1) = colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{q_1-1}\}, \tag{7.3}$$

where

$$\left\{\begin{array}{l} \mathbf{P}_0 = \mathbf{B}_1\ ,\ \ \mathbf{P}_i = \mathbf{0}\ \textit{for}\ i < 0 \\ \mathbf{P}_i = \mathbf{A}_1\mathbf{P}_{i-1} + \cdots + \mathbf{A}_R\mathbf{P}_{i-R},\ \ i = 1, 2, \cdots \end{array}\right. \tag{7.4}$$

and $\mathbf{A}_i \in \mathbb{R}^{r\times r}, \mathbf{B}_1 \in \mathbb{R}^{r\times m}$ *are constant matrices.*

To match the moments of an R-th order model, the matrix $\mathbf{A}_0$ should be invertible. By this assumption, the subspaces $\mathcal{K}_{q_1}(-\mathbf{A}_0^{-1}\mathbf{A}_1, \cdots, -\mathbf{A}_0^{-1}\mathbf{A}_R, -\mathbf{A}_0^{-1}\mathbf{G})$ and $\mathcal{K}_{q_2}(-\mathbf{A}_0^{-T}\mathbf{A}_1^T, \cdots, -\mathbf{A}_0^{-T}\mathbf{A}_R^T, -\mathbf{A}_0^{-T}\mathbf{L}^T)$ are used for moment matching that are called the input and output R-th order Krylov subspaces for the system (7.1).

Lemma 7.1 *Consider the input and output R-th order Krylov subspaces for the system (7.1) with corresponding basic blocks $\mathbf{P}_i$ and $\tilde{\mathbf{P}}_i$, respectively. Then,*

$$\mathbf{m}_i = \mathbf{L}\mathbf{P}_i = \tilde{\mathbf{P}}_i^T\mathbf{G}, i = 0, 1, \cdots .$$

By applying a projection directly to the system (7.1), a reduced model with the same structure can be found,

$$\begin{cases} \bar{\mathbf{W}}^T\mathbf{A}_R\bar{\mathbf{V}}\mathbf{z}_r^{(R)} + \cdots + \bar{\mathbf{W}}^T\mathbf{A}_1\bar{\mathbf{V}}\dot{\mathbf{z}}_r + \bar{\mathbf{W}}^T\mathbf{A}_0\bar{\mathbf{V}}\mathbf{z}_r = \bar{\mathbf{W}}^T\mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\bar{\mathbf{V}}\mathbf{z}_r(t). \end{cases} \tag{7.5}$$

For choosing $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ the R-th order Krylov-subspaces are used, as described by the following theorems. The proofs are quite similar to the second order case.

Theorem 7.1 *If the columns of the matrix $\bar{\mathbf{V}}$ in (7.5), form a basis for the input R-th order Krylov subspace and $\bar{\mathbf{W}}$ is chosen such that $\bar{\mathbf{W}}^T\mathbf{A}_0\bar{\mathbf{V}}$ is nonsingular, then the first q_1 moments of the original and reduced order models match.*

Theorem 7.2 *If the columns of the matrix $\bar{\mathbf{V}}$ and $\bar{\mathbf{W}}$ used in (7.5), form bases for the input and output R-th order Krylov subspaces, respectively, both with the same rank, then the first $q_1 + q_2$ moments of the original and reduced order systems match. It is assumed that $\mathbf{A}_0$ and $\bar{\mathbf{W}}^T\mathbf{A}_0\bar{\mathbf{V}}$ are invertible.*

Part III

Order Reduction by Back Conversion

Chapter 8

REDUCTION OF SECOND ORDER SYSTEMS BY BACK CONVERSION

A disadvantage of Second Order Krylov methods is that they match smaller number of moments compared to the state space methods. In order to match more characteristic parameters, in this chapter, we propose a method based on reducing in state space and back conversion into second order form where a maximum number of parameters match; up to double compared to the Second Order Krylov Subspace method.

In this method, the second order system is first converted into a state space model (i.e. a set of first order differential equations) and then its order is reduced by applying Krylov-subspace methods as described in chapter 2. However, in doing so, the reduced-order system will be of the first-order type as well, making a physical interpretation difficult. The idea is to convert the reduced state space model back into the second order form to recover the original structure.

One of the methods proposed in [59], was based on applying the standard balancing and truncation to the equivalent state space model and then convert it back to the second order structure using a similarity transformation. The disadvantage of using the approach proposed in [59] is that the output equation of the final second order model is in general a linear combination of the second order state vector and its derivative, which is different from the structure of the original second order system (3.1).

As we showed in Chapter 3, the first Markov parameter of the second order system (3.1) is zero. This can be used as a key point to convert a state space model into second order form such that the output equation has the same structure as the original system.

In [55, 56, 75], we showed that by matching the first Markov parameter and a number of moments, the reduced order system can be transformed into second order form. This method has been extended to MIMO case in [77] and a different way to calculate the back transforming procedure has been proposed in [18].

8.1 Reduction by matching the moments and the first Markov parameter

As mentioned before, when the method of Chapter 2 is applied to the state space model (2.1), it destroys the second order structure but can match the maximum number of moments using a two-sided method. The idea to is to calculate a reduced model from the state space model (2.1) and then convert this model to a second order representation. This conversion requires the characteristic property of the second order type system which is $\mathbf{M}_0 = \mathbf{0}$.

Consider the state space system,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}, \\ \mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}. \end{cases} \tag{8.1}$$

which is equivalent to the second order system (3.1). As mentioned in Chapter 2, a Krylov subspace method can be applied to reduce the system (8.1) to match the first Markov parameter together with some of the moments [76, 88] resulting in,

$$\begin{cases} \underbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}_{\mathbf{E}_r} \dot{\mathbf{x}}_r = \underbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}_{\mathbf{A}_r} \mathbf{x}_r + \underbrace{\mathbf{W}^T\mathbf{B}}_{\mathbf{B}_r} \mathbf{u}, \\ \mathbf{y} = \underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r} \mathbf{x}_r. \end{cases} \tag{8.2}$$

In the following, we remind two related theorems in one and two-sided methods.

Theorem 8.1 *If the columns of* $\mathbf{V}$ *used in (8.2), form a basis for the Krylov subspace* $\mathcal{K}_{Q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{E}^{-1}\mathbf{B})$ *and the matrix* $\mathbf{W}$ *is chosen such that the matrices* $\mathbf{A}_r$ *and* $\mathbf{E}_r$ *are nonsingular, then the first* $Q_1 - 1$ *moments and the first Markov parameter of the original and reduced order systems match.*

We typically choose $\mathbf{W} = \mathbf{V}$.

Theorem 8.2 *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (8.2), form bases for Krylov subspaces* $\mathcal{K}_{Q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{E}^{-1}\mathbf{B})$ *and* $\mathcal{K}_{Q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$*, respectively, both with the same rank, then the first* $Q_1 + Q_2 - 1$ *moments and the first Markov parameter of the original and reduced order systems match. It is assumed that* $\mathbf{A}$*,* $\mathbf{E}$*,* $\mathbf{A}_r$ *and* $\mathbf{E}_r$ *are nonsingular.*

An alternative to Theorem 8.2 is using the Krylov subspaces $\mathcal{K}_{Q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{Q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{E}^{-T}\mathbf{C}^T)$. To match the moments about s_0, we should simply substitute the matrix $\mathbf{A}$ with $\mathbf{A} - s_0\mathbf{E}$ in the Krylov subspaces. To find the projection matrices based on Theorems 8.1 and 8.2, the standard algorithms in section 2.4 can be applied.

Because the first Markov parameter of a second order model is zero, by matching the first Markov parameter, it remains zero in the reduced order model.

8.2 Conversion into second order type model

In the following, we show how a Q-th order (Q is even) state space model with the property $\mathbf{M}_0 = \mathbf{0}$ can be converted into a second order representation (3.1) with $q = \frac{Q}{2}$ second order differential equations.

Without loss of generality, we consider that the matrix $\mathbf{E}_r$ in (8.1) is the identity matrix (if not, we can multiply the state equation by $\mathbf{E}_r^{-1}$), and the state space model to be converted to second order form is,

$$\begin{cases} \dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}, \\ \mathbf{y} = \mathbf{C}_r\mathbf{x}. \end{cases} \tag{8.3}$$

The first Markov-parameter is assumed to be zero,

$$\mathbf{M}_0 = \mathbf{C}_r\mathbf{B}_r = 0. \tag{8.4}$$

Now, we introduce a vector $\mathbf{z}_r$ defined as

$$\mathbf{z}_r = \mathbf{C}_z\mathbf{x}_r, \tag{8.5}$$

$$\text{with } \mathbf{C}_z = \begin{bmatrix} \mathbf{C}_r \\ \mathbf{R} \end{bmatrix}, \tag{8.6}$$

where the matrix $\mathbf{R} \in \mathbb{R}^{(q-p)\times Q}$ is chosen such that the matrix $\mathbf{C}_z$ is full row rank and $\mathbf{R}\mathbf{B}_r = \mathbf{0}$. This can be achieved by constructing a sequence of linear independent vectors in the Kernel of $\mathbf{B}_r$. In section 8.2.1, there is an option how to find $\mathbf{R}$.

In fact, the variable $\mathbf{z}_r$ is an extended output such that the resulting system with output $\mathbf{z}_r$ is observable and the first Markov parameter remains zero. The vector $\mathbf{z}_r$ is intended to become the vector of variables of the reduced second order system, we are looking for.

From the definition (8.5), we find the time derivative

$$\dot{\mathbf{z}}_r = \mathbf{C}_z\dot{\mathbf{x}}_r = \mathbf{C}_z\mathbf{A}_r\mathbf{x}_r + \mathbf{C}_z\mathbf{B}_r\mathbf{u} = \mathbf{C}_z\mathbf{A}_r\mathbf{x}_r. \tag{8.7}$$

Therefore, (with (8.5) and (8.7)), the relation between the state vector $\mathbf{x}_r$ and $\mathbf{z}_r, \dot{\mathbf{z}}_r$ is,

$$\begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A}_r \end{bmatrix} \mathbf{x}_r. \tag{8.8}$$

This defines the similarity transformation $\mathbf{x}_r = \mathbf{T}\mathbf{x}_t$ where,

$$\mathbf{T} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A}_r \end{bmatrix}^{-1}, \mathbf{x}_t = \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix}, \tag{8.9}$$

assuming that the matrix $\mathbf{T}$ exists. A necessary condition is that the matrix $\begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r \end{bmatrix}$ is full rank which is true for observable systems. A sufficient condition for the SISO case is that the system (8.3) is controllable that will be discussed later, but for the MIMO case, a sufficient condition is not known, yet. Examples show that probably almost always such a matrix $\mathbf{T}$ exists.

Applying this transformation to the system (8.3) leads to,

$$\begin{cases} \dot{\mathbf{x}}_t = \mathbf{T}^{-1}\mathbf{A}_r\mathbf{T}\mathbf{x}_t + \mathbf{T}^{-1}\mathbf{B}_r\mathbf{u}, \\ \mathbf{y} = \mathbf{C}_r\mathbf{T}\mathbf{x}_t. \end{cases} \tag{8.10}$$

Now, we show that this model is in the form (8.1) and by comparison can directly be converted into the representation (3.1), and the matrices $\mathbf{M}_r$, $\mathbf{D}_r$, $\mathbf{K}_r$, $\mathbf{G}_r$ and $\mathbf{L}_r$ are found. Considering the facts that,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A}_r \end{bmatrix}, \quad \mathbf{C}_z\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{8.11}$$

$$\mathbf{C}_z\mathbf{A}_r\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{C}_z\mathbf{B}_r = \mathbf{0}, \tag{8.12}$$

the system (8.10) can be rewritten as follows,

$$\begin{cases} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}_r \\ \ddot{\mathbf{z}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_z\mathbf{A}_r\mathbf{T} \\ \mathbf{C}_z\mathbf{A}_r^2\mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{C}_z\mathbf{B}_r \\ \mathbf{C}_z\mathbf{A}_r\mathbf{B}_r \end{bmatrix} \mathbf{u}, \\ \mathbf{y} = \mathbf{C}_r\mathbf{T} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix}. \end{cases}$$

By using equations (8.12), we have,

$$\begin{cases} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_r \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}_r \\ \ddot{\mathbf{z}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} \mathbf{u}, \\ \mathbf{y} = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix}. \end{cases}$$

where,

$$\mathbf{M}_r = \mathbf{I}\ ,\ \mathbf{G}_r = \mathbf{C}_z\mathbf{A}_r\mathbf{B}_r\ ,\ \begin{bmatrix} -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} = \mathbf{C}_z\mathbf{A}_r^2\mathbf{T}. \tag{8.13}$$

The output equation in (8.10), $\mathbf{y} = \mathbf{C}_r\mathbf{T}\mathbf{x}_t$, simplifies to $\mathbf{y} = \begin{bmatrix} \mathbf{I} & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}_t$, because $\mathbf{C}_r$ is the upper block of $\mathbf{T}^{-1}$. Thereby, we conclude

$$\mathbf{L}_r = \begin{bmatrix} \mathbf{I} & 0 & \cdots & 0 \end{bmatrix}. \tag{8.14}$$

which determines all parameters of the reduced model of second order type,

$$\begin{cases} \mathbf{M}_r\ddot{\mathbf{z}}_r + \mathbf{D}_r\dot{\mathbf{z}}_r + \mathbf{K}_r\mathbf{z}_r = \mathbf{G}_r\mathbf{u}, \\ \mathbf{y} = \mathbf{L}_r\mathbf{z}_r. \end{cases}$$

So, the sufficient conditions for a state space model to be converted to a second order type model are

- The first Markov parameter is zero.
- The order Q of the system is even.
- The (fictitious) output vector, $\mathbf{z}_r$, can be defined such that the matrix $\mathbf{T}$ from (8.9) exists.

8.2.1 Finding R in equation (8.6)

One way to find the matrix $\mathbf{R}$ in equation (8.6) is to find the vectors that are orthogonal to the columns of the matrices $\mathbf{B}_r$ and $\mathbf{C}_r^T$. Here we use the QR-factorization [35] to find the matrix $\mathbf{R}$.

Consider the QR-factorization of the matrix,

$$\begin{bmatrix} \mathbf{B}_r & \mathbf{C}_r^T \end{bmatrix}_{Q\times(m+p)} = \mathbf{Q} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Based on the properties of the QR-factorization, the matrix $\mathbf{Q}$ is orthogonal. Now, we have,

$$\mathbf{0} = \begin{bmatrix} \mathbf{0}_{(n-(m+p))\times(m+p)} & \mathbf{I}_{n-(m+p)} \end{bmatrix} \mathbf{Q}^T\mathbf{Q} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{R}_1 \begin{bmatrix} \mathbf{B}_r & \mathbf{C}_r^T \end{bmatrix},$$

where $\mathbf{R}_1$ is the $Q-(m+p)$ last columns of the matrix $\mathbf{Q}$. Therefore $Q-(m+p)$ vectors are found that are orthogonal to the columns of $\mathbf{B}_r$. Because of orthogonality of the matrix $\mathbf{R}_1$ to $\mathbf{C}_r^T$, all columns of $\mathbf{R}_1$ are linearly independent of the rows of $\mathbf{C}_r$. Any $q-p$ rows of the matrix $\mathbf{R}_1$ can be chosen as the rows of the matrix $\mathbf{R}$.

So, ***the steps of reducing second order type models*** are:

1. Convert the original second order model into the state space representation.

2. Apply an order reduction method as described in Chapter 2, to match the first Markov parameter (which equals zero) and some of the moments; choose an even order Q. The reduced model is then converted to the state space representation (8.3) by multiplying by $\mathbf{E}_r^{-1}$.

3. Convert the reduced order state space model into a second order type model by first constructing a matrix $\mathbf{C}_z$ as in equation (8.6), and then calculating the transformation matrix $\mathbf{T}$ from (8.9). Finally, the matrices $\mathbf{M}_r$, $\mathbf{D}_r$, $\mathbf{K}_r$, $\mathbf{G}_r$ and $\mathbf{L}_r$ of the reduced model of type (3.1) are computed from equations (8.13) and (8.14).

Different from the approach in [59], in the second order reduced model, the same as in the original system, the output does not depend on derivative of the second order state vector. This can be done by keeping the first Markov parameter equal to zero.

8.3 Numerical issues

In order reduction of the state space system equivalent to the second order system (3.1) as described in section 2, it seems to be a need of calculating the LU-factorization of the matrices $\mathbf{A}$ and $\mathbf{E}$ of dimension $N = 2n$. Calculating these factorizations and then using them in the Arnoldi or Lanczos algorithm to find the projection matrices is not recommended for numerical reasons but, it is better to consider the structure of these matrices as described subsequently.

From (11.8) we observe that,

$$\begin{aligned}
\mathbf{E}^{-1} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} \end{bmatrix} \\
\mathbf{A}^{-1} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.
\end{aligned} \tag{8.15}$$

Now, the starting vectors of the input Krylov subspace are,

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix}.$$

Therefore, the set of linear equations $\mathbf{M\Gamma} = \mathbf{G}$ must be solved. For the other vectors to be calculated in each iteration, we have,

$$\begin{aligned}
\hat{\mathbf{v}}_j &= \begin{bmatrix} \hat{\mathbf{v}}_{1j} \\ \hat{\mathbf{v}}_{2j} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{1(j-m)} \\ \mathbf{v}_{2(j-m)} \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{K}^{-1}\left(\mathbf{D}\mathbf{v}_{1(j-m)} + \mathbf{M}\mathbf{v}_{2(j-m)}\right) \\ \mathbf{v}_{1(j-m)} \end{bmatrix},
\end{aligned}$$

and only the linear equation $\mathbf{K}\hat{\mathbf{v}}_{1j} = -\left(\mathbf{D}\mathbf{v}_{1(j-m)} + \mathbf{M}\mathbf{v}_{2(j-m)}\right)$ must be solved in each iteration.

In this way finding the LU-factorizations of the matrices $\mathbf{K}$ and $\mathbf{M}$ is necessary. This is done *only once* and *before* starting the iterations so that within the iterations, only triangular linear equations are to be solved (which is very fast and accurate).

The steps of finding the matrix $\mathbf{V}$ as a basis of the Krylov subspace $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{E}^{-1}\mathbf{B})$ for the system (3.2) by Arnoldi algorithm are shown in the following algorithm:

Algorithm 8.1 *Arnoldi algorithm for second order systems*

0. *Start: Delete all linearly dependent columns of* $\mathbf{G}$ *to find* $\mathbf{G}_1$ *with* m_1 *independent columns.*

1. *Calculate the LU factorizations* $\mathbf{M} = \mathbf{L}_M\mathbf{U}_M$ *and* $\mathbf{K} = \mathbf{L}_K\mathbf{U}_K$.

2. *Solve two sets of triangular equations* $\mathbf{L}_M(\mathbf{U}_M\boldsymbol{\Gamma}) = \mathbf{G}_1$.

3. *Set* $\mathbf{v}_1 = \frac{1}{\|\gamma_1\|_2}\begin{bmatrix} \mathbf{0} \\ \gamma_1 \end{bmatrix}$ *where* γ_1 *is the first column of* $\boldsymbol{\Gamma}$.

4. *For* $j = 2, 3, \cdots$, *do,*

   (a) *Calculating the next vector: If* $j \leq m_1$ *the next vector is* $\hat{\mathbf{v}}_j = \begin{bmatrix} \mathbf{0} \\ \gamma_j \end{bmatrix}$.
   Else, solve two triangular equations $\mathbf{L}_K(\mathbf{U}_K\hat{\mathbf{v}}_{1j}) = -\left(\mathbf{D}\mathbf{v}_{1(j-m_1)} + \mathbf{M}\mathbf{v}_{2(j-m_1)}\right)$ *and set* $\hat{\mathbf{v}}_j = \begin{bmatrix} \hat{\mathbf{v}}_{1j} \\ \mathbf{v}_{1(j-m_1)} \end{bmatrix}$.

   (b) *Orthogonalization: For i=1 to j-1 do,*

   $$h_{i,j-1} = \hat{\mathbf{v}}_j^T\mathbf{v}_i \ , \ \hat{\mathbf{v}}_j = \hat{\mathbf{v}}_j - h_{i,j-1}\mathbf{v}_i.$$

   (c) *Normalization: If* $\hat{\mathbf{v}}_j$ *is zero, reduce* m_1 *to* $m_1 - 1$ *and if* m_1 *is nonzero go to step* (4a) *and if* m_1 *is zero break the loop. Else, if* $\hat{\mathbf{v}}_j \neq 0$ *the* j*-th column of* $\mathbf{V}$ *is*

   $$h_{j,j-1} = \|\hat{\mathbf{v}}_j\|_2 \,, \quad \mathbf{v}_j = \frac{\hat{\mathbf{v}}_j}{h_{j,j-1}}.$$

   (d) *Increase* j *and go to step* (4a).

In a two-sided method using the Lanczos algorithm, similar changes can be helpful to reduce the numerical effort. By considering the Krylov subspaces $\mathcal{K}_{Q_1}(\mathbf{A}_1^{-1}\mathbf{E}_1, \mathbf{A}_1^{-1}\mathbf{B})$ and $\mathcal{K}_{Q_2}(\mathbf{A}_1^{-T}\mathbf{E}_1^T, \mathbf{E}_1^{-T}\mathbf{C})$, the *LU*-factorization of $\mathbf{M}$ is not necessary making reduction procedure cheaper.

8.4 Matching the moments about $s_0 \neq 0$

As discussed in Chapter 2, the moments of a state space model about a point $s_0 \neq 0$, can also be matched by applying a projection to the original state space model. This is normally done by substituting the matrix $\mathbf{A}$ with $\mathbf{A} - s_0\mathbf{E}$ in the definition of the Krylov subspaces. Such a substitution changes the structure of the matrices of a second order model and the Algorithm 2.1 can not be helpful anymore.

As discussed in section 4.3, by substituting the matrix $\mathbf{K}$ by $\mathbf{K} + s_0\mathbf{D} + s_0^2\mathbf{M}$ and the matrix $\mathbf{D}$ by $\mathbf{D} + 2s_0\mathbf{M}$ in the definition of the moments about zero, the moments about s_0 is found. To match the moments about s_0, the same substitution as in the moments should be done in the definition of input and output Krylov subspaces as in the following algorithm:

Algorithm 8.2 *Algorithm to match the moments about s_0*

1. *Set*

$$\begin{aligned} \mathbf{E} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} - s_0\mathbf{D} - s_0^2\mathbf{M} & -\mathbf{D} - 2s_0\mathbf{M} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \end{aligned}$$

2. *Apply the algorithm 2.1 to find the matrix* $\mathbf{V}$ *using the structure of the state space matrices in step 1.*

3. *Find the state space reduced system as follows,*

$$\mathbf{E}_r = \mathbf{V}^T\mathbf{E}\mathbf{V}, \mathbf{A}_r = \mathbf{V}^T\mathbf{A}\mathbf{V} + s_0\mathbf{E}_r, \mathbf{b}_r = \mathbf{V}^T\mathbf{b}, \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}$$

The advantage of the Algorithm 8.2 is that the structure of the matrices $\mathbf{E}, \mathbf{A}$ is preserved and the Algorithm 2.1 can be applied by cheaper calculation compared to the standard algorithms without considering the structure of the original system.

8.5 Conclusion

In this chapter, a new method in order reduction of large scale second order models was introduced, resulting in reduced systems having the same structure as the original model. We reduced the equivalent state space system to match the first Markov parameter (which is zero for second order models) and some of the first moments. The second order structure is then recovered by applying a similarity transformation.

This method matches a maximum number of parameters (with the first Markov parameter among them) which is doubled compared to the method of part II. By knowing that the most expensive part of the numerical algorithms to calculate the projection matrices is the LU-factorization, the numerical effort of second order Krylov subspace and back conversion by a two-sided method is very close to each other where only the LU-factorization of the matrix $\mathbf{K}$ is necessary while in the back conversion method based on only input Krylov subspace the LU-factorization of the matrices $\mathbf{K}$ and $\mathbf{M}$ should be calculated. However, to reduce the original model to the same order, the number of iterations in the back conversion method is double the second order Krylov method.

Because the back conversion procedure does not have any influence on stability of the system, the conditions to preserve stability of a second order system using the method proposed in this chapter is the same as the conditions in section 4.5 when a one-sided method with the choice $\mathbf{W} = \mathbf{V}$ is applied.

Chapter 9

EXISTENCE OF THE TRANSFORMATION MATRIX

In this chapter we investigate the conditions to find a similarity transformation matrix $\mathbf{T}$ to transform the system,

$$\begin{cases} \mathbf{E}_r \dot{\mathbf{x}}_r = \mathbf{A}_r \mathbf{x}_r + \mathbf{B}_r \mathbf{u}, \\ \mathbf{y} = \mathbf{C}_r \mathbf{x}_r, \end{cases} \tag{9.1}$$

of an even order, into second order form,

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{z}}_r \\ \ddot{\mathbf{z}}_r \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} \mathbf{u}, \\ \mathbf{y} = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \dot{\mathbf{z}}_r \end{bmatrix}. \end{cases} \tag{9.2}$$

First, we discuss some of the necessary conditions and then we find the sufficient conditions for SISO case.

9.1 The necessary conditions

In the following, some necessary conditions for the existence of the transformation matrix is given. The most important difference between a general state space model and the one models a second order behaviour is the notion of Markov parameter as formulated in the following Lemma.

Lemma 9.1 *For every MIMO system* (9.1)*, if* $\mathbf{C}_r\mathbf{B}_r \neq 0$ *then, there is no nonsingular matrix* $\mathbf{T}$ *to transform it into* (9.2).

Proof: If such a transformation exists by considering system (9.2) there is the following contradiction,

$$\mathbf{0} = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} (\mathbf{T}^{-1}\mathbf{T}) \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix}$$
$$= \left(\begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \mathbf{T}^{-1} \right) \left(\mathbf{T} \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} \right) = \mathbf{C}_r \mathbf{B}_r \neq \mathbf{0}.$$

■

In Lemma 9.1, as we know $\mathbf{C}_r\mathbf{B}_r$ is the first Markov parameter of the system (9.1). In fact, the first Markov parameter of every second order system of the form (9.2) is zero and this is a necessary condition for the existence of $\mathbf{T}$. If the first Markov parameter is nonzero, then the output of the transformed system in second order form depends on the derivative of the states and the structure of $\mathbf{C}_r\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{L}_r \end{bmatrix}$ can not be achieved.

Theorem 9.1 *For every MIMO system* (9.1), *if* $\mathbf{C}_r$ *is full rank and* $\begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r \end{bmatrix}$ *is singular, then there is no nonsingular matrix* $\mathbf{T}$ *to transform the system into* (9.2).

Proof: Consider that $\begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r \end{bmatrix}$ is singular and such a transformation $\mathbf{T}$ exists. We have

$$\mathbf{C}_r\mathbf{T} = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \Longrightarrow \mathbf{C}_r = \begin{bmatrix} \mathbf{L}_r & \mathbf{0} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix}}_{\mathbf{T}^{-1}} = \mathbf{L}_r\mathbf{T}_1.$$

From the other side we have the relation,

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{A}_r\mathbf{T} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \Longrightarrow \mathbf{T}^{-1}\mathbf{A}_r = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \mathbf{T}^{-1} \Longrightarrow \\
\begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \mathbf{A}_r &= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \Longrightarrow \\
\begin{bmatrix} \mathbf{T}_1\mathbf{A}_r \\ \mathbf{T}_2\mathbf{A}_r \end{bmatrix} &= \begin{bmatrix} \mathbf{T}_2 \\ -\mathbf{K}_r\mathbf{T}_1 - \mathbf{D}_r\mathbf{T}_2 \end{bmatrix} \Longrightarrow \\
\mathbf{T}_1\mathbf{A}_r &= \mathbf{T}_2 \Longrightarrow \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_1\mathbf{A}_r \end{bmatrix}
\end{aligned}$$

Because $\mathbf{T}^{-1}$ is nonsingular, its multiplication with the full rank matrix $\begin{bmatrix} \mathbf{L}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_r \end{bmatrix}$ should be full rank but,

$$\begin{bmatrix} \mathbf{L}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_r \end{bmatrix} \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{L}_r\mathbf{T}_1 \\ \mathbf{L}_r\mathbf{T}_1\mathbf{A}_r \end{bmatrix} = \begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r \end{bmatrix}$$

which is not full rank by the assumption and such a nonsingular matrix $\mathbf{T}$ does not exist. ■

The assumption of Theorem 9.1 for MIMO case can be checked easily and for SISO systems, observability guaranties this necessary condition.

Corollary 9.1 *For every SISO system,*

$$\begin{cases} \dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{b}_r u, \\ y = \mathbf{c}_r^T\mathbf{x}_r, \end{cases} \tag{9.3}$$

if $\begin{bmatrix} \mathbf{c}_r^T \\ \mathbf{c}_r^T\mathbf{A}_r \end{bmatrix}$ *is singular and* $\mathbf{c}_r \neq 0$ *then there is no nonsingular matrix* $\mathbf{T}$ *to transform it into second order form.*

The following theorem is dual to Theorem 9.2, related to the matrix $\mathbf{B}_r$ and controllability.

Theorem 9.2 *For every MIMO system* (9.1)*, if* $\mathbf{B}_r$ *is full rank and* $\begin{bmatrix} \mathbf{B}_r & \mathbf{A}_r\mathbf{B}_r \end{bmatrix}$ *is singular, then there is no nonsingular matrix* $\mathbf{T}$ *to transform it into* (9.2).

Proof: Consider that $\begin{bmatrix} \mathbf{B}_r & \mathbf{A}_r\mathbf{B}_r \end{bmatrix}$ is singular and such a transformation $\mathbf{T}$ exists. We have

$$\mathbf{T}^{-1}\mathbf{B}_r = \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} \Longrightarrow \mathbf{B}_r = \underbrace{\begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{0} \\ \mathbf{G}_r \end{bmatrix} \Longrightarrow \mathbf{B}_r = \mathbf{T}_2\mathbf{G}_r.$$

From the other side, we have,

$$\begin{aligned} \mathbf{A}_r\mathbf{B}_r &= \mathbf{T}(\mathbf{T}^{-1}\mathbf{A}_r\mathbf{T})\mathbf{T}^{-1}\mathbf{B}_r = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_r & -\mathbf{D}_r \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{P} \end{bmatrix} \\ &= \mathbf{T}_1\mathbf{G}_r - \mathbf{T}_2\mathbf{D}_r\mathbf{G}_r \end{aligned}$$

Now, multiplication of the full rank matrix $\begin{bmatrix} \mathbf{0} & \mathbf{G}_r \\ \mathbf{G}_r & \mathbf{0} \end{bmatrix}$ with every invertible matrix should be full rank,

$$\begin{aligned} \mathbf{T}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}_r & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{G}_r \\ \mathbf{G}_r & \mathbf{0} \end{bmatrix} &= \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{G}_r \\ \mathbf{G}_r & -\mathbf{D}_r\mathbf{G}_r \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{T}_2\mathbf{G}_r & \mathbf{T}_1\mathbf{G}_r - \mathbf{T}_2\mathbf{D}_r\mathbf{G}_r \end{bmatrix} = \begin{bmatrix} \mathbf{B}_r & \mathbf{A}_r\mathbf{B}_r \end{bmatrix} \end{aligned}$$

which is not full rank by the assumption and such a nonsingular matrix $\mathbf{T}$ does not exist. ■

Corollary 9.2 *For every SISO system* (9.3)*, if* $\begin{bmatrix} \mathbf{b}_r & \mathbf{A}_r\mathbf{b}_r \end{bmatrix}$ *is singular and* $\mathbf{b}_r \neq 0$ *then there is no nonsingular matrix* $\mathbf{T}$ *to achieve the structure in* (9.2)*.*

For the SISO systems, if the state space model is minimal, then the conditions of the Theorems 9.1 and 9.2 are automatically satisfied.

9.2 Sufficient conditions for SISO case

In this section, we investigate the sufficient conditions for the existence of a similarity transformation by which a state space SISO model can be transformed to a set of second order differential equations. We consider the SISO system, (9.3) of even order Q where the first Markov parameter is zero. We study the possibility of transforming this system to a second order system.

Theorem 9.3 *For every controllable SISO system* (9.3) *with* $\mathbf{c}_r^T\mathbf{b}_r = 0$*, there exists a nonsingular matrix of the form,*

$$\mathbf{T} = \begin{bmatrix} \mathbf{c}_r^T \\ \mathbf{R} \\ \mathbf{c}_r^T\mathbf{A}_r \\ \mathbf{R}\mathbf{A}_r \end{bmatrix},$$

such that $\mathbf{R}\mathbf{b}_r = 0$*.*

Proof: Because the system is controllable, without loss of generality, we just consider that the system is in controller canonical form [46],

$$\begin{cases} \begin{bmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \\ \vdots \\ \dot{x}_{rQ-1} \\ \dot{x}_{rQ} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{r0} & -a_{r1} & a_{r2} & \cdots & -a_{rQ-1} \end{bmatrix} \begin{bmatrix} x_{r1} \\ x_{r2} \\ \vdots \\ x_{rQ-1} \\ x_{rQ} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \\ y = \begin{bmatrix} c_{r1} & c_{r2} & \cdots & c_{rQ-1} & c_{rQ} \end{bmatrix} \mathbf{x}_r, \end{cases} \tag{9.4}$$

Now, consider the last nonzero element of the vector $\mathbf{c}_r$ is c_{rk}; i.e.

$$\mathbf{c}_r^T = \begin{bmatrix} c_{r1} & \cdots & c_{rk} & 0 & \cdots & 0 \end{bmatrix} \text{ where } c_{rk} \neq 0. \tag{9.5}$$

Because $\mathbf{c}_r^T \mathbf{b}_r = 0$, with the structure of $\mathbf{b}_r$ we have $c_{rQ} = 0$ and therefore $k < Q$. We search for the vectors $\mathbf{r}_i$ for $i = 1, \cdots, q-1$ that are orthogonal to $\mathbf{b}_r$ such that the matrix,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{c}_r^T \\ \mathbf{c}_r^T \mathbf{A}_r \\ \mathbf{r}_1^T \\ \mathbf{r}_1^T \mathbf{A}_r \\ \vdots \\ \mathbf{r}_{n-1}^T \\ \mathbf{r}_{n-1}^T \mathbf{A}_r \end{bmatrix},$$

is full rank, where $q = \frac{Q}{2}$. By interchanging the rows of the matrix $\bar{\mathbf{T}}$, the matrix $\mathbf{T}$ is found. To be orthogonal to $\mathbf{b}_r$, it is sufficient to have zero in the last entry and by using the structure of the matrix $\mathbf{A}_r$, the entries of $\mathbf{r}^T$ are shifted to right when multiplied by $\mathbf{A}_r$. In the following, we construct the matrix $\bar{\mathbf{T}}$ in two cases:

First, consider the value of k is odd. The matrix $\bar{\mathbf{T}}$ can be chosen as,

$$\bar{\mathbf{T}} = \begin{bmatrix} c_{r1} & c_{r2} & \cdots & c_{rk-1} & c_{rk} & 0 & 0 & \cdots & 0 \\ 0 & c_{r1} & \cdots & c_{rk-2} & c_{rk-1} & c_{rk} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{9.6}$$

The matrix $\bar{\mathbf{T}}$ is square and by knowing that $c_{rk} \neq 0$, it is obvious that the rows are linearly independent and $\bar{\mathbf{T}}$ is full rank.

If the value of k is even, the matrix $\bar{\mathbf{T}}$ can be chosen as,

$$\bar{\mathbf{T}} = \begin{bmatrix} c_{r1} & c_{r2} & \cdots & c_{rk-2} & c_{rk-1} & c_{rk} & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_{r1} & \cdots & c_{rk-3} & c_{rk-2} & c_{rk-1} & c_{rk} & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \alpha & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \alpha & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{9.7}$$

The value of α should be chosen such that $\alpha c_{rk}^2 \neq c_{rk-2}c_{rk} - c_{rk-1}^2$. Again, the matrix $\bar{\mathbf{T}}$ is square and by knowing that $c_{rk} \neq 0$, it is obvious that the rows are linearly independent and $\bar{\mathbf{T}}$ is full rank. ■

Consider the similarity transformation (8.9). According to Theorem 9.3, such a transformation exists if the system is controllable. This condition is satisfied for all reduced system because the first step of reduction is finding a minimal system and a reduced system found by a Krylov subspace method using input Krylov subspace is controllable. So, the sufficient conditions for a SISO state space model to be converted to a second order type model are:

- The system is controllable.
- The first Markov parameter is zero.
- The order of the system is even.

An alternative proof can be found in [38] where instead of a controller canonical form an upper Hessenberg matrix is considered leading to the same result.

Chapter 10

UNDAMPED SECOND ORDER MODEL

A second order system is undamped if the damping matrix $\mathbf{D}$ is zero. This point is interesting because, some technical systems of second order form are undamped and it is worthy to discuss this issue in order reduction.

10.1 Reducing undamped systems

From the result of Lemma 3.1, if an undamped system is reduced in state space by moment (or Markov parameter) matching, then half of the matched moments (or Markov parameters) of the reduced order model are zero. With this property back transforming the reduced order model into second order form becomes simpler as stated in the following theorem.

Theorem 10.1 *Consider the reduced system,*

$$\begin{cases} \dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}, \\ \mathbf{y} = \mathbf{C}_r\mathbf{x}_r. \end{cases}$$

of an even order $Q = 2q$. If this system is observable and the parameters $\mathbf{M}_l, \mathbf{M}_{l-2}, \cdots, \mathbf{M}_0, \mathbf{m}_1, \mathbf{m}_3, \cdots, \mathbf{m}_{Q-l-3}$ for an even value of l are zero, then the matrix $\mathbf{C}_z$ in *equation* (8.9) *can be chosen as,*

$$\mathbf{C}_z = \mathbf{Q} \begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r^{l} \\ \mathbf{C}_r\mathbf{A}_r^{l-2} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{2} \\ \mathbf{C}_r\mathbf{A}_r^{-2} \\ \mathbf{C}_r\mathbf{A}_r^{-4} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{l-Q+2} \end{bmatrix}, \tag{10.1}$$

where $\mathbf{Q} \in \mathbb{R}^{q\times q}$ *is any nonsingular matrix.*

Proof: The first property is orthogonality to $\mathbf{B}_r$,

$$\mathbf{C}_z\mathbf{B}_r = \mathbf{Q} \begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r^{l} \\ \mathbf{C}_r\mathbf{A}_r^{l-2} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{2} \\ \mathbf{C}_r\mathbf{A}_r^{-2} \\ \mathbf{C}_r\mathbf{A}_r^{-4} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{l-Q+2} \end{bmatrix} \mathbf{B}_r = \mathbf{Q} \begin{bmatrix} \mathbf{C}_r\mathbf{B}_r \\ \mathbf{C}_r\mathbf{A}_r^{l}\mathbf{B}_r \\ \mathbf{C}_r\mathbf{A}_r^{l-2}\mathbf{B}_r \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{2}\mathbf{B}_r \\ \mathbf{C}_r\mathbf{A}_r^{-2}\mathbf{B}_r \\ \mathbf{C}_r\mathbf{A}_r^{-4}\mathbf{B}_r \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{l-Q+2}\mathbf{B}_r \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \mathbf{M}_0 \\ \mathbf{M}_l \\ \mathbf{M}_{l-2} \\ \vdots \\ \mathbf{M}_2 \\ \mathbf{m}_1 \\ \mathbf{m}_3 \\ \vdots \\ \mathbf{m}_{Q-l-3} \end{bmatrix} = \mathbf{0}.$$

Therefore, the rows of $\mathbf{C}_z$ are in the null space of $\mathbf{B}_r$. The only condition is that the matrix $\mathbf{T}$ is full rank,

$$\mathbf{T} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A}_r \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{C}_r \\ \mathbf{C}_r\mathbf{A}_r^{l} \\ \mathbf{C}_r\mathbf{A}_r^{l-2} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{2} \\ \mathbf{C}_r\mathbf{A}_r^{-2} \\ \mathbf{C}_r\mathbf{A}_r^{-4} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{l-Q+2} \\ \mathbf{C}_r\mathbf{A}_r \\ \mathbf{C}_r\mathbf{A}_r^{l+1} \\ \mathbf{C}_r\mathbf{A}_r^{l-1} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{3} \\ \mathbf{C}_r\mathbf{A}_r^{-1} \\ \mathbf{C}_r\mathbf{A}_r^{-3} \\ \vdots \\ \mathbf{C}_r\mathbf{A}_r^{l-Q+3} \end{bmatrix}$$

Because $\mathbf{Q}$ is invertible, the rank of $\mathbf{T}$, depends only on the second part which by rearranging the rows and multiplying with the nonsingular matrix $\mathbf{A}_r^{Q-l-2}$ can be transformed to the observability matrix. Therefore, if the system is observable, the matrix $\mathbf{T}$ is full rank and the proof is completed. ■

Corollary 10.1 *If an undamped system is reduced to an observable system of order $Q = 2q$, by matching some of the first moments and Markov parameters such that the first Markov parameter is matched and the total number of matched parameters is at least Q then, there exists a nonsingular matrix which transforms the reduced system into the second order form* (3.1).

10.2 Calculating the transformation matrix

According to Theorem 10.1, the rows of the matrix $\mathbf{C}_z$ should form a basis for the subspace,

$$\begin{aligned}\mathcal{S} = \Big\{ \ \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^2 \mathbf{C}_r^T, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{l-2} \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^{l} \mathbf{C}_r^T, \\ \left(\mathbf{A}_r^T\right)^{-2} \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^{-4} \mathbf{C}_r^T, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{l-Q+2} \mathbf{C}_r^T \ \Big\}.\end{aligned} \tag{10.2}$$

The subspace $\mathcal{S}$ can be rewritten as,

$$\begin{aligned}\mathcal{S} = & \Big\{ \ \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^2 \mathbf{C}_r^T, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{l-2} \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^{l} \mathbf{C}_r^T \ \Big\} \\ & \bigcup \Big\{ \ \left(\mathbf{A}_r^T\right)^{-2} \mathbf{C}_r^T, \ \left(\mathbf{A}_r^T\right)^{-4}, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{l-Q+2} \mathbf{C}_r^T \ \Big\} \\ = & \mathcal{K}_{\frac{l}{2}+1} \left(\left(\mathbf{A}_r^T\right)^2, \mathbf{C}_r^T \right) \bigcup \mathcal{K}_{\frac{Q-l}{2}-1} \left(\left(\mathbf{A}_r^T\right)^{-2}, \left(\mathbf{A}_r^T\right)^{-2} \mathbf{C}_r^T \right).\end{aligned} \tag{10.3}$$

To calculate the corresponding basis, the Arnoldi Algorithm 2.1 can be applied to two Krylov subspaces and calculate the matrix $\mathbf{C}_z$.

If the value of l is small as it is common in order reduction, then two subspaces in (10.3) can be integrated into a single Krylov subspace by rewriting the equation (10.2) as

$$\begin{aligned}\mathcal{S} = & \Big\{ \ \left(\mathbf{A}_r^T\right)^{l} \mathbf{C}_r^T \ \left(\mathbf{A}_r^T\right)^{l-2} \mathbf{C}_r^T, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{2} \mathbf{C}_r^T, \ \mathbf{C}_r^T, \ \cdots, \ \left(\mathbf{A}_r^T\right)^{l-Q+2} \mathbf{C}_r^T \ \Big\} \\ = & \mathcal{K}_{\frac{Q}{2}} \left(\left(\mathbf{A}_r^T\right)^{-2}, \left(\mathbf{A}_r^T\right)^{l} \mathbf{C}_r^T \right).\end{aligned} \tag{10.4}$$

For instance if we match only the first Markov parameter to reduce an undamped system, the Arnoldi Algorithm 2.1 should be applied to the Krylov subspace $\mathcal{K}_{\frac{Q}{2}} \left(\left(\mathbf{A}_r^T\right)^{-2}, \mathbf{C}_r^T \right)$, to find the matrix $\mathbf{C}_z$.

If the value of $Q - l$ is small which happens when more number Markov parameters are matched, then the Krylov subspace $\mathcal{K}_{\frac{Q}{2}} \left(\left(\mathbf{A}_r^T\right)^{2}, \left(\mathbf{A}_r^T\right)^{l-Q+2} \mathbf{C}_r^T \right)$ is to be considered.

10.3 Conclusion

Order reduction of undamped second order system by back conversion has been investigated. We showed that for undamped systems, if the number of matching Markov parameters is larger than or equal to the order of the reduced system then there is a guarantee for the existence of the transformation matrix.

By knowing that, in state space Krylov subspace method, the number of matching parameters is at least Q, this fact not only proves the existence of the transformation matrix, but also a numerical reliable way using the Arnoldi algorithm is concluded to calculate the matrix $\mathbf{C}_z$ which is the only unknown part of the transformation matrix.

Chapter 11

AN INTEGRATED STATE SPACE AND BACK CONVERSION PROCEDURE, (SISO CASE)

The state space reduced system by moment matching using Arnoldi (or Lanczos) algorithm has a special structure, including an upper Hessenberg (or a tridiagonal matrix) as a coefficient of $\dot{\mathbf{x}}_r$ directly calculated from the algorithm. However, in the first step of the proposed approach by back conversion, because of matching the first Markov parameter, the reduced state space model can only be calculated by applying a projection and the special structure of the reduced system as in equations (2.25), (2.26) and (2.35) is destroyed. Furthermore, there was a lack of good numerical algorithms for back conversion into second order form.

In this chapter, by considering SISO systems, first we modify the Arnoldi algorithm such that the first Markov parameter is matched and the structure in the matrices of the reduced system is preserved; see also [81]. The proposed algorithms directly calculate the matrices of the reduced state equation. Then, by using this structure in the reduced system, a numerical procedure is proposed to find a transformation matrix which transforms the state space system into a second order form. This procedure not only suggests a numerically reliable procedure to compute the transformation matrix, but also extracts sufficient conditions for the possibility of the back conversion into second order form.

11.1 One-sided Krylov subspace methods

Consider the Arnoldi Algorithm 2.1 applied to reduce the SISO system,

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\ y = \mathbf{c}^T\mathbf{x}, \end{cases} \tag{11.1}$$

based on moment matching. As discussed in section 2.4.1, the reduced order system in SISO case can be written in the special form,

$$\begin{cases} \mathbf{H}_Q\dot{\mathbf{x}}_r = \mathbf{x}_r + \|\mathbf{A}^{-1}\mathbf{b}\|_2\mathbf{e}_1 u, \\ y = \mathbf{c}^T\mathbf{V}_Q\mathbf{x}_r, \end{cases} \tag{11.2}$$

where $\mathbf{e}_1$ is the first unit vector and $\mathbf{H}_Q$ is calculated directly from the Arnoldi algorithm.

To match the first Markov parameter and maximum number of moments, we consider the Krylov subspace $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{E}^{-1}\mathbf{b})$ for the reduction of system (11.1). In this case, the Arnoldi Algorithm 2.1 finds the orthonormal matrix $\mathbf{V}_Q$ and the reduced system can be found only by applying the projection $\mathbf{x}_r = \mathbf{V}_Q\mathbf{x}$ and the first Markov parameter and $Q-1$ moments match. In fact, because of changing the starting vector of the Krylov subspace, the reduced order model can not be written in the form (11.2) using the Hessenberg matrix $\mathbf{H}_Q$ calculated directly from the Arnoldi algorithm and $\mathbf{b}_r$ is not a multiple of the first unit vector $\mathbf{e}_1$. To achieve the reduced system (11.2), we modify the Arnoldi algorithm into Algorithm 11.1.

Algorithm 11.1 *Modified Arnoldi algorithm*

1. *Apply the standard Arnoldi Algorithm 8.1 to the Krylov subspace* $\mathcal{K}_{Q-1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ *to produce* $\mathbf{V}_{Q-1}$, $\mathbf{H}_{Q-2}$ *and* $h_{Q-1,Q-2}$.

2. *Find the normalized vector* $\mathbf{v}_Q = \alpha_0\mathbf{E}^{-1}\mathbf{b} + \sum_{i=1}^{Q-1}\alpha_i\mathbf{v}_i$ *using modified Gram-Schmidt procedure [35] such that* $\mathbf{v}_Q^T\mathbf{V}_{Q-1} = \mathbf{0}$.

3. *Find the normalized vector* $\mathbf{v}_{Q+1}$ *using modified Gram-Schmidt procedure such that* $\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{Q-1} = \sum_{i=1}^{Q+1} h_{i,Q-1}\mathbf{v}_i$ *and* $\mathbf{v}_{Q+1}^T\mathbf{V}_Q = \mathbf{0}$.

4. Calculate the Q-th column of the matrix $\mathbf{H}_Q$ *as follows,*

$$\begin{bmatrix} h_{1,Q} \\ h_{2,Q} \\ \vdots \\ h_{Q-2,Q} \\ h_{Q-1,Q} \\ h_{Q,Q} \end{bmatrix} = \begin{bmatrix} \alpha_0\|\mathbf{A}^{-1}\mathbf{b}\|_2 + \alpha_{Q-1}h_{1,Q-1} \\ \alpha_{Q-1}h_{2,Q-1} \\ \vdots \\ \alpha_{Q-1}h_{Q-2,Q-1} \\ \alpha_{Q-1}h_{Q-1,Q-1} \\ \alpha_{Q-1}h_{Q,Q-1} \end{bmatrix} + \begin{bmatrix} & \mathbf{H}_{Q-2} & & \\ 0 & \cdots & 0 & h_{Q-1,Q-2} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{Q-2} \end{bmatrix}. \tag{11.3}$$

In this algorithm, because $\mathbf{v}_1 = \frac{1}{\|\mathbf{A}^{-1}\mathbf{b}\|_2}\mathbf{A}^{-1}\mathbf{b}$ and $\mathbf{V}_Q$ is orthonormal, we have,

$$\mathbf{V}_Q^T\mathbf{A}^{-1}\mathbf{b} = \mathbf{V}_Q^T\mathbf{v}_1\|\mathbf{A}^{-1}\mathbf{b}\|_2 = \|\mathbf{A}^{-1}\mathbf{b}\|_2\mathbf{e}_1.$$

From the other side, because the Arnoldi algorithm is applied to calculate the first $Q-1$ vectors, we have,

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{Q-2} = \mathbf{V}_{Q-2}\mathbf{H}_{Q-2} + h_{Q-1,Q-2}\mathbf{v}_{Q-1}\mathbf{e}_{Q-2}^T.$$

By calculating $\mathbf{v}_Q$ and $\mathbf{v}_{Q+1}$ in steps 2 and 3, we have,

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_Q = \alpha_0\mathbf{A}^{-1}\mathbf{E}\mathbf{E}^{-1}\mathbf{b} + \mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{Q-2}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{Q-2} \end{bmatrix} + \alpha_{Q-1}\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{Q-1}$$

$$= \alpha_0\mathbf{A}^{-1}\mathbf{b} + \mathbf{V}_{Q-2}(\mathbf{H}_{Q-2} + h_{Q-1,Q-2}\mathbf{v}_{Q-1}\mathbf{e}_{Q-2}^T)\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{Q-2} \end{bmatrix} + \alpha_{Q-1}\sum_{i=1}^{Q+1} h_{i,Q-1}\mathbf{v}_i. \tag{11.4}$$

By knowing that $\mathbf{A}^{-1}\mathbf{b} = \mathbf{v}_1\|\mathbf{A}^{-1}\mathbf{b}\|_2$, equation (11.4) can be rewritten as,

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_Q = \mathbf{V}_Q\left(\begin{bmatrix} \alpha_0\|\mathbf{A}^{-1}\mathbf{b}\|_2 + \alpha_{Q-1}h_{1,Q-1} \\ \alpha_{Q-1}h_{2,Q-1} \\ \vdots \\ \alpha_{Q-1}h_{Q,Q-1} \end{bmatrix} + \begin{bmatrix} & \mathbf{H}_{Q-2} & & \\ 0 & \cdots & 0 & h_{Q-1,Q-2} \\ 0 & \cdots & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{Q-2} \end{bmatrix}\right) + \alpha_{Q-1}h_{Q,Q-1}\mathbf{v}_{Q+1}\mathbf{e}_Q^T. \tag{11.5}$$

The value inside the parentheses in (11.5) is the Q-th column of $\mathbf{H}_Q$ calculated in (11.3).

Now, if we combine the results of steps 1 and 3 of the Algorithm 11.1 with equation (11.5) we conclude,

$$\mathbf{A}^{-1}\mathbf{E}\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_Q \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{Q-2} & \mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{Q-1} & \mathbf{A}^{-1}\mathbf{E}\mathbf{v}_Q \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{Q+1} \end{bmatrix} \underbrace{\begin{bmatrix} & & & & h_{1,Q-1} & h_{1,Q} \\ & & \mathbf{H}_{Q-2} & & \vdots & \vdots \\ 0 & \cdots & 0 & h_{Q-1,Q-2} & h_{Q-1,Q-1} & h_{Q-1,Q} \\ 0 & \cdots & 0 & 0 & h_{Q,Q-1} & h_{Q,Q} \\ 0 & \cdots & 0 & 0 & h_{Q+1,Q-1} & 0 \end{bmatrix}}_{\bar{\mathbf{H}}}.$$

The matrix $\mathbf{H}_Q = \mathbf{V}_Q^T\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_Q$ is calculated by deleting the last row of the matrix $\bar{\mathbf{H}}$. All entries of the matrix $\mathbf{H}_Q$ can directly be calculated from the Algorithm 11.1.

After calculating the projection matrix $\mathbf{V}_Q$ by applying the Algorithm 11.1, the reduced order model is of the form (11.2) which matches the first Markov parameter and $q-1$ first moments with the original system.

11.2 Two-sided Krylov subspace methods

In two-side methods, by applying the Lanczos algorithm to the Krylov subspaces $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_Q(\mathbf{E}^T\mathbf{A}^{-T}, \mathbf{A}^T\mathbf{E}^{-T}\mathbf{c})$, it is possible to match the first Markov parameter and the first $2Q-1$ first moments. The reduced order model is of the form,

$$\begin{cases} \mathbf{T}_Q\dot{\mathbf{x}}_r = \mathbf{x}_r + sign(\mathbf{c}^T\mathbf{E}^{-1}\mathbf{b})\sqrt{|\mathbf{c}^T\mathbf{E}^{-1}\mathbf{b}|}\mathbf{e}_1 u, \\ y = \mathbf{c}^T\mathbf{V}_Q\mathbf{x}_r, \end{cases} \tag{11.6}$$

where the matrix $\mathbf{T}_Q$ is tridiagonal. The difference to the reduced system found by moment matching as in equation (2.35) is that $\mathbf{c}^T\mathbf{V}_Q$ is not a multiple of the first unit vector. To achieve exactly the same structure as the standard Lanczos algorithm to match the moments, the algorithm should be changed to first consider the basic vector $\mathbf{c}$ and then at the end add the vector $\mathbf{A}^T\mathbf{E}^{-T}\mathbf{c}$ similar to the modification in the Arnoldi algorithm however, for the upcoming procedure in reducing second order systems, the structure in (11.6) is enough which is an special case of the form found in (11.2).

11.3 Reduction of second order systems

We consider the second order system,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}} + \mathbf{D}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{g}u, \\ y = \mathbf{l}^T\mathbf{z}, . \end{cases} \tag{11.7}$$

that can be transformed into state space as,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \ddot{\mathbf{z}} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \mathbf{z} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{g} \end{bmatrix}}_{\mathbf{b}} u, \\ y = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{l}^T \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} \dot{\mathbf{z}} \\ \mathbf{z} \end{bmatrix}. \end{cases} \tag{11.8}$$

We also consider that the matrices $\mathbf{K}$ and $\mathbf{M}$ are nonsingular. By this assumption, the matrices $\mathbf{E}$ and $\mathbf{A}$ become nonsingular and therefore, the method proposed in the Chapter 8 to match the moments and Markov parameters can be applied to reduce system (11.8).

For the reduction of second order systems, first we reduce the equivalent state space equation (11.8) using Algorithm 11.1 while the first Markov parameter of the reduced system is zero by matching with the one of the original system. Then, the reduced order system (11.2) is transformed into the second order form (11.8) by applying a similarity transformation as explained in the following. Because the state space system (11.6) is a special case of (11.2), we only consider the general case which is valid for the reduced order models found by both one sided method as in Algorithm 11.1 and two-sided method as in section 11.2.

To calculate the transformation matrix, consider that the first nonzero entry of $\mathbf{r} = \mathbf{c}_r^T\mathbf{H}_Q^{-1}$ is r_k. Because $\mathbf{c}_r^T\mathbf{H}_Q^{-1}\mathbf{b}_r = 0$ (this is the first Markov parameter of reduced system) and $\mathbf{b}_r$ is a multiple of the first unit vector, $k > 1$. We construct the transformation matrix for two different cases. First consider k is an even number. Then, we construct the matrix $\mathbf{S} \in \mathbb{R}^{Q\times\frac{Q}{2}}$ as,

$$\mathbf{S} = \begin{bmatrix} \mathbf{H}_Q^{-T}\mathbf{c}_r & \mathbf{e}_2 & \mathbf{e}_4 & \cdots & \mathbf{e}_{k-2} & \mathbf{e}_{k+2} & \cdots & \mathbf{e}_Q \end{bmatrix}.$$

Because $\mathbf{c}_r^T\mathbf{H}_Q^{-1}\mathbf{b} = 0$, the first row of $\mathbf{S}$ is zero and $\mathbf{S}^T\mathbf{b}_r = \mathbf{0}$. The transformation matrix

$\mathbf{T}$ is constructed as,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{S}^T\mathbf{H}_Q \end{bmatrix}. \tag{11.9}$$

If we interchange the rows of $\mathbf{T}^{-1}$ then,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{e}_2^T\mathbf{H}_Q \\ \mathbf{e}_2^T \\ \mathbf{e}_4^T\mathbf{H}_Q \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{e}_{k-2}^T\mathbf{H}_Q \\ \mathbf{e}_{k-2}^T \\ \mathbf{c}_r^T\mathbf{H}_Q^{-1}\mathbf{H}_Q \\ \mathbf{c}_r^T\mathbf{H}_Q^{-1} \\ \mathbf{e}_{k+2}^T\mathbf{H}_Q \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{e}_Q^T\mathbf{H}_Q \\ \mathbf{e}_Q^T \end{bmatrix} = \begin{bmatrix} \mathbf{h}_2^T \\ \mathbf{e}_2^T \\ \mathbf{h}_4^T \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{h}_{k-2}^T \\ \mathbf{e}_{k-2}^T \\ \mathbf{c}_r^T \\ \mathbf{c}_r^T\mathbf{H}_Q^{-1} \\ \mathbf{h}_{k+2}^T \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{h}_Q^T \\ \mathbf{e}_Q^T \end{bmatrix},$$

where $\mathbf{h}_j^T$ is the j-th row of the matrix $\mathbf{H}_Q$. Because the matrix $\mathbf{H}_Q$ is upper Hessenberg, the first $j-2$ entries of $\mathbf{h}_j$ are zero and the matrix $\bar{\mathbf{T}}$ is upper triangular whose diagonal entries are one or the sub-diagonal entries of even rows of $\mathbf{H}_Q$, except for the $k-1$-st row with $r_k h_{k,k-1}$ and k-th row with r_k as,

$$\bar{\mathbf{T}} = \begin{bmatrix} h_{2,1} & * & \cdots & * & * & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{k,k-1}r_k & * & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & r_k & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & h_{k+2,k+1} & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Therefore, $\mathbf{T}^{-1}$ is full rank if all sub-diagonal entries of even rows of $\mathbf{H}_Q$ are nonzero.

If k is odd, then the matrix $\mathbf{S}$ is constructed as

$$\mathbf{S} = \begin{bmatrix} \mathbf{c}_r & \mathbf{e}_2 & \mathbf{e}_4 & \cdots & \mathbf{e}_{k-1} + \beta \mathbf{e}_{k+1} & \mathbf{e}_{k+3} & \cdots & \mathbf{e}_Q \end{bmatrix},$$

where β is a parameter. Again $\mathbf{S}^T \mathbf{b}_r = \mathbf{0}$ and the transformation matrix $\mathbf{T}$ is constructed using (11.9). To show that $\mathbf{T}$ is full rank, we interchange the rows as follows,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{e}_2^T \mathbf{H}_Q \\ \mathbf{e}_2^T \\ \mathbf{e}_4^T \mathbf{H}_Q \\ \mathbf{e}_4^T \\ \vdots \\ (\mathbf{e}_{k-1} + r\mathbf{e}_{k+1})^T \mathbf{H}_Q \\ \mathbf{e}_{k-1}^T + \beta \mathbf{e}_{k+1}^T \\ \mathbf{c}_r^T \mathbf{H}_Q^{-1} \mathbf{H}_Q \\ \mathbf{c}_r^T \mathbf{H}_Q \\ \mathbf{e}_{k+3}^T \mathbf{H}_Q \\ \mathbf{e}_{k+3}^T \\ \vdots \\ \mathbf{e}_Q^T \mathbf{H}_Q \\ \mathbf{e}_Q^T \end{bmatrix} = \begin{bmatrix} \mathbf{h}_2^T \\ \mathbf{e}_2^T \\ \mathbf{h}_4^T \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{h}_{k-1}^T + r\mathbf{h}_{k+1}^T \\ \mathbf{e}_{k-1}^T + \beta \mathbf{e}_{k+1}^T \\ \mathbf{c}_r^T \\ \mathbf{c}_r^T \mathbf{H}_Q \\ \mathbf{h}_{k+2}^T \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{h}_Q^T \\ \mathbf{e}_Q^T \end{bmatrix}$$

$$= \begin{bmatrix} h_{2,1} & * & \cdots & * & * & * & * & * & \cdots \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{k,k-1} & * & * & * & * & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \beta & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \alpha_3 & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & r_k & r_{k+1} & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & h_{k+2,k+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\alpha_1 = r_k h_{k,k-1}$, $\alpha_2 = r_k h_{k,k} + r_{k+1} h_{k+1,k}$, $\alpha_3 = r_k h_{k,k+1} + r_{k+1} h_{k+1,k+1} + r_{k+2} h_{k+2,k+2}$. If $\beta \neq \frac{1}{\alpha_1}\left(\alpha_3 - \frac{r_{k+1}}{r_k}\alpha_2\right)$ is chosen and the sub-diagonal entries of $\mathbf{H}_Q$ at even rows are nonzero then the matrix $\bar{\mathbf{T}}$ is full rank, knowing that $\alpha_1 \neq 0$ because $h_{k,k-1}, r_k$ are assumed to be nonzero.

Now, consider the similarity transformation $\mathbf{x}_r = \mathbf{T}\mathbf{x}_t$ is applied to the system (11.2),

$$\begin{cases} \mathbf{T}^{-1}\mathbf{H}_Q\mathbf{T}\dot{\mathbf{x}}_t = \mathbf{x}_t + \mathbf{T}^{-1}\mathbf{b}_r u, \\ y = \mathbf{c}_r^T\mathbf{T}\mathbf{x}_t. \end{cases} \tag{11.10}$$

Considering the facts that,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{S}^T\mathbf{H}_Q \end{bmatrix} \quad , \quad \mathbf{S}^T\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{11.11}$$

$$\mathbf{S}^T\mathbf{H}_Q\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix} \quad , \quad \mathbf{S}^T\mathbf{b} = \mathbf{0}, \tag{11.12}$$

the system (11.10) can be rewritten as follows,

$$\begin{cases} \begin{bmatrix} \mathbf{S}^T\mathbf{H}_Q\mathbf{T} \\ \mathbf{S}^T\mathbf{H}_Q^2\mathbf{T} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{S}^T\mathbf{b}_r \\ \mathbf{S}^T\mathbf{H}_Q\mathbf{b}_r \end{bmatrix} u, \\ y = \mathbf{c}_r^T\mathbf{T} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{cases}$$

which is equivalent to,

$$\begin{cases} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}_r & \mathbf{D}_r \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_r \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_r \end{bmatrix} u, \\ y = \begin{bmatrix} \mathbf{0} & \mathbf{l}_r^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{cases} \tag{11.13}$$

where,

$$\mathbf{K}_r = -\mathbf{I}, \ \mathbf{g}_r = \mathbf{S}^T\mathbf{H}_Q\mathbf{b}_r, \ \begin{bmatrix} \mathbf{M}_r & \mathbf{D}_r \end{bmatrix} = \mathbf{S}^T\mathbf{H}_Q^2\mathbf{T}. \tag{11.14}$$

The output equation in (11.10), $y = \mathbf{c}_r^T\mathbf{T}\mathbf{x}_t$, is simplified to $y = \mathbf{e}_{\frac{Q}{2}+1}\mathbf{x}_t$, because $\mathbf{c}_r^T$ is the first line of $\mathbf{S}^T\mathbf{H}_Q$ in $\mathbf{T}^{-1}$. Thereby, we conclude

$$\mathbf{l}_r^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{11.15}$$

Because, in system (11.13), $\dot{\mathbf{x}}_2 = \mathbf{x}_1$ by defining $\mathbf{z}_r = \mathbf{x}_1$ the state space equation (11.13) is equivalent to second order system,

$$\begin{cases} \mathbf{M}_r\ddot{\mathbf{z}}_r + \mathbf{D}_r\dot{\mathbf{z}}_r + \mathbf{K}_r\mathbf{z}_r = \mathbf{g}_r u, \\ y = \mathbf{l}_r^T\mathbf{z}_r. \end{cases} \tag{11.16}$$

We conclude the results of the back conversion procedure as the following theorem:

Theorem 11.1 *The sufficient conditions for a state space model of the form* (11.2) *to be converted to a second order type model are: the first Markov parameter is zero, the order of the system is even,* $\mathbf{H}_Q$ *is full rank and its sub-diagonal entries at even rows are nonzero.*

Lemma 11.1 *Consider the Hessenberg matrix* $\mathbf{H}_Q$ *in system* (11.2) *is full rank. The system* (11.2) *is controllable if and only if all sub-diagonal entries of* $\mathbf{H}_Q$ *are nonzero.*

Proof: The Kalman controllability matrix of system (11.2) is

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{b}_r, & \mathbf{H}_Q^{-1}\mathbf{b}_r, & \cdots, & \mathbf{H}_Q^{-Q+1}\mathbf{b}_r \end{bmatrix} \\ &= \mathbf{H}_Q^{-Q+1} \underbrace{\begin{bmatrix} \mathbf{H}_q^{Q-1}\mathbf{b}_r, & \mathbf{H}_Q^{Q-2}\mathbf{b}_r, & \cdots, & \mathbf{b}_r \end{bmatrix}}_{\mathcal{C}_H}, \end{aligned}$$

where $\mathbf{b}_r$ is a multiple of the first unit vector. Because $\mathbf{H}_Q$ is nonsingular, the system (11.2) is controllable if and only if the matrix $\mathcal{C}_H$ is full rank. If we change the sequence of the columns of $\mathcal{C}_H$ to

$$\mathcal{C}_t = \begin{bmatrix} \mathbf{b}_r, & \mathbf{H}_Q\mathbf{b}_r, & \cdots, & \mathbf{H}_Q^{Q-1}\mathbf{b}_r \end{bmatrix},$$

then using the structure of $\mathbf{H}_Q$ and $\mathbf{b}_r$, the matrix $\mathcal{C}_t$ is an upper triangular matrix whose diagonal entries are the sub-diagonal entries of the matrix $\mathbf{H}_Q$. Therefore, $\mathcal{C}_H$ is full rank (or the system is controllable) if all sub-diagonal entries of $\mathbf{H}_Q$ are nonzero and vice versa.

■

By using Lemma 11.1, if a system is controllable then the third condition of Theorem 11.1 is fulfilled which confirms the result of section 9.2. If a two-sided method using Lanczos is applied as explained in section 11.2, then the matrix $\mathbf{H}_Q$ becomes tridiagonal which is a special case of Hessenberg form and all steps of back conversion to second order form are quite similar.

So, the steps of reducing second order type models are:

1. Apply Algorithm 11.1 (or Lanczos as explained in section 11.2) to find a reduced system of the form (11.2) (using Lanczos $\mathbf{H}_Q$ is substituted by $\mathbf{T}_Q$).

2. Calculate the matrix $\mathbf{S}$ as explained in this section and $\mathbf{T}$ using (11.9).

3. calculate the state space matrices of the reduced second order system (11.16) using equations (11.14) and (11.15).

11.3.1 Matching the moments about s_0

To match the moments about $s_0 \neq 0$, it is normally sufficient to substitute $\mathbf{A}$ with $\mathbf{A} - s_0\mathbf{E}$ in the corresponding Krylov subspaces. To deal with second order systems, we do the substitution as step 1 of the Algorithm 8.2 and then apply Algorithm 11.1 or the Lanczos algorithm. Then, the moments of the reduced system (11.2) or (11.6) about zero are matched with the moments of the original system about s_0. Therefore, after transforming such a reduced order model into second order form (11.16) the reduced matrices should be modified as follows,

$$\begin{aligned} \mathbf{M}_{s_0} &= \mathbf{M}_r, \\ \mathbf{D}_{s_0} &= \mathbf{D}_r - 2s_0\mathbf{M}_r, \\ \mathbf{K}_{s_0} &= \mathbf{K}_r - s_0\mathbf{D}_r - s_0^2\mathbf{M}_r, \end{aligned}$$

and the matrices $\mathbf{M}_{s_0}$, $\mathbf{D}_{s_0}$ and $\mathbf{K}_{s_0}$ define the final reduced second order system.

11.4 Conclusion

In this chapter, a modified Arnoldi algorithm is proposed to reduce a state space equation matching the first Markov parameter which is zero for second order models and some of the first moments. The structure of the reduced state space matrices found by applying the modified Arnoldi algorithm is the same as the one by moment matching in standard Arnoldi algorithm.

The structure in the reduced state space matrices found by the proposed algorithm is used to calculate a similarity transformation to transform the reduced system into a second order form.

It was also explained how to use the method to match the moments about other points and how to increase the number of matching parameters by applying the Lanczos algorithm.

In the part for back conversion, a difference to the method in Chapter 8 is that the numerical way is straightforward without any preliminary calculation on the state space reduced system like multiplying with $\mathbf{E}^{-1}$. The other variation is that in Chapter 8, the matrix $\mathbf{E}$ is considered to be identity but here the matrix $\mathbf{A}$ is identity.

Part IV

Examples

Chapter 12

TECHNICAL EXAMPLES

In order to illustrate the suitability of the methods proposed in this dissertation, in this chapter, we apply them to different technical systems. Three different methods are considered to be applied for order reduction:

- *Method 1:* Applying a projection directly to the second order model using the Second Order Krylov Subspaces as proposed in part II.
- *Method 2:* Find an equivalent state space model and find a state space reduced model using a Krylov subspace method by matching the moments together with the first Markov parameter and then calculate the equivalent second order system by a back conversion procedure as proposed in part III.
- *Method 3:* Find an equivalent state space model and find a state space reduced model using a Krylov subspace method by matching only the moments (and not preserving the second order structure) as in Chapter 2.

All three methods are applied in two different cases: one-sided and two-sided. In all reduced order models, the best value of s_0 is chosen to reduce the error. For the systems considered in this chapter, the highest value leading to a stable reduced system is chosen in order to find the best possible result. To compare the results, we consider different types of error functions. For relatively small systems, we calculate the relative error defined as:

$$\begin{aligned}\mathcal{H}_\infty\text{error norm} &= \frac{\|\mathbf{H}(s)-\mathbf{H}_r(s)\|_\infty}{\|\mathbf{H}(s)\|_\infty},\\ \mathcal{H}_2\text{error norm} &= \frac{\|\mathbf{H}(s)-\mathbf{H}_r(s)\|_2}{\|\mathbf{H}(s)\|_2},\end{aligned}$$

where $\|.\|_\infty$, $\|.\|_2$, $\mathbf{H}(s)$ and $\mathbf{H}_r(s)$ are $\mathcal{H}_\infty$ norm, $\mathcal{H}_2$ norm, the transfer function of the original and reduced systems, respectively.

12.1 Building model

As the first example, we consider a relatively low order system which is the model of a building (the Los Angeles University Hospital) with 8 floors each having 3 degrees of freedom, namely displacement in x and y directions, and rotation[1] [4, 21]. Hence, we have 24 variables in second order form (order $N = 48$ in state space) with one input and one output. We reduce the system to order $Q = 14$ in state space and $q = 7$ in the second order form.

In Table 12.1, the relative errors of reduced systems are shown. In Figures 12.1 and 12.2, bode diagram of the original and reduced order models with one-sided and two-sided methods are compared to each other. In Figure 12.3, the step response of the original and reduced systems by two-sided methods are illustrated where all systems have very close responses.

Table 12.1: Relative errors in reducing the building model.

	Method	$\mathcal{H}_2$ relative error norm	$\mathcal{H}_\infty$ relative error norm	s_0
One-sided	Method 1	0.1196	0.0860	0
	Method 2	0.0780	0.0655	2.9
	Method 3	0.0850	0.0758	1.9
Two-sided	Method 1	0.1195	0.0861	0.1
	Method 2	0.0766	0.0625	2.3
	Method 3	0.0762	0.0604	3

From Table 12.1 and Figures 12.1 and 12.2, the best reduced order model is from method 3 which matches more moments, however destroys the second order structure. Methods 1 and 2, both preserve the second order structure. This can be seen from the bode diagram at high frequencies, where the gain of the reduced systems by methods 1 and 2 have the same slope as the original system, differing 20 dB/dec from the one by method 3. Because method 2 matches more moments than method 1, it has a better performance at medium frequencies. As expected the two-sided methods have led to better results than the one-sided methods.

[1] The model is available online at `http://www.win.tue.nl/niconet/NIC2/benchmodred.html`.

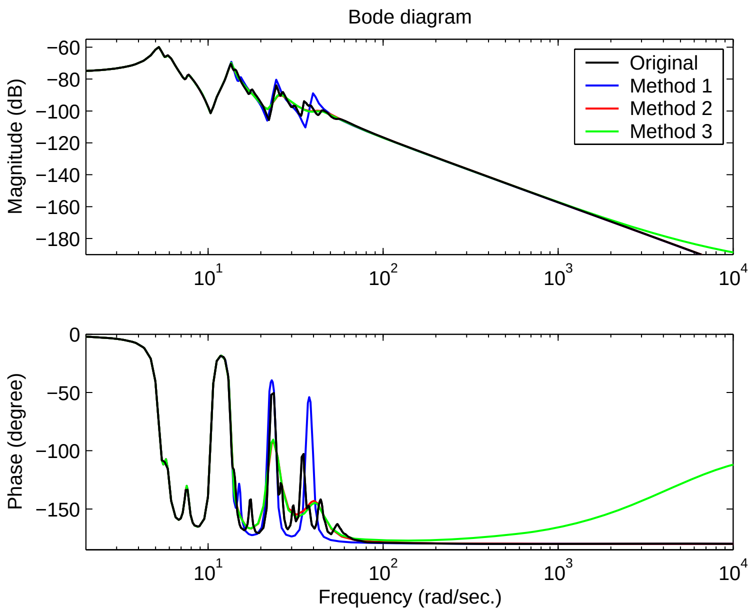

Figure 12.1: Bode diagram of the building model and reduced systems using one-sided methods.

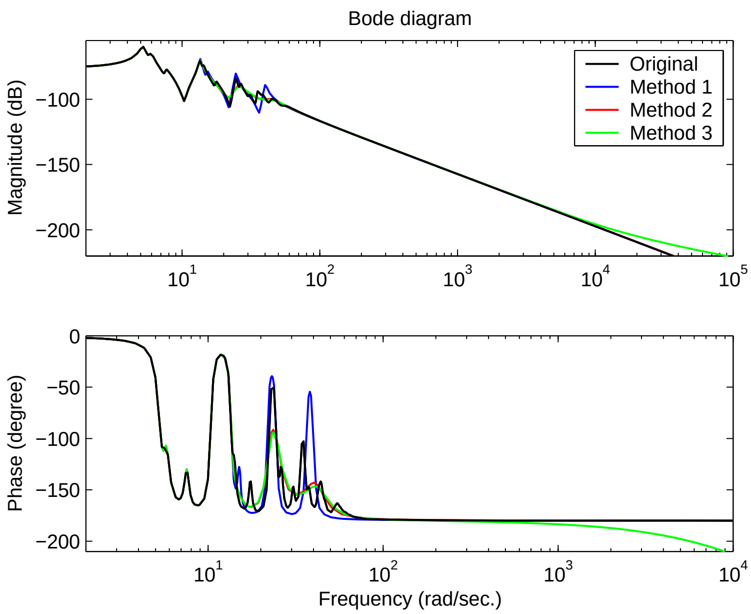

Figure 12.2: Bode diagram of the building model and reduced systems using two-sided methods.

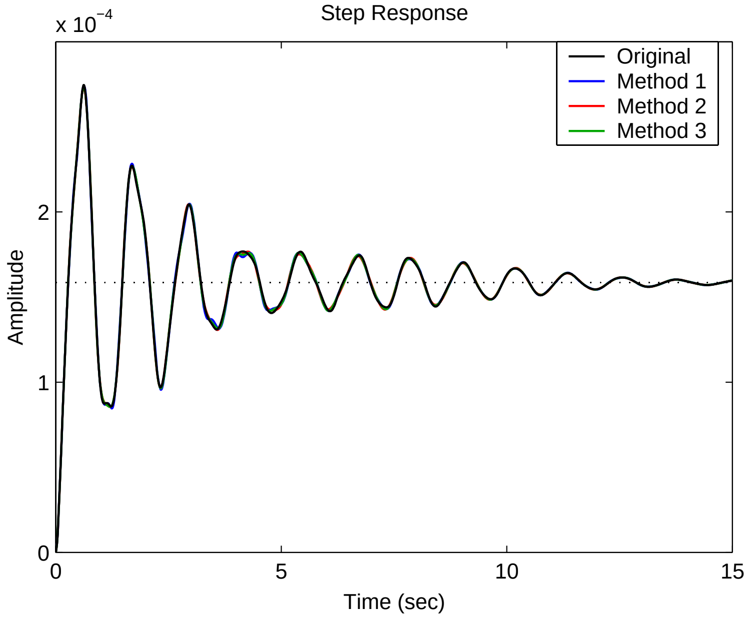

Figure 12.3: Step response of the building model and reduced systems using two-sided methods.

12.2 International space station

As the second example, we consider the International Space Station (ISS) model[2] [4, 21, 39]. It is composed of a complex structure containing several parts. Each part of this system was modelled with a system of order of several hundreds. For instance, the structural part (part 1R of the Russian Service Module) of the international space station has been modelled with a system of order 270 with 3 inputs and 3 outputs (a second order model of dimension $n = 135$). The original system is reduced to order $Q = 30$ in state space and $q = 15$ in second order form.

In Table 12.2 the norm of the error systems are shown with the frequency responses as in Figures 12.4 and 12.5 where the largest singular values of the original and reduced systems are shown.

Table 12.2: Relative errors in reducing the ISS model.

	Method	$\mathcal{H}_2$ relative error norm	$\mathcal{H}_\infty$ relative error norm	s_0
One-sided	Method 1	0.0608	0.0097	0.05
	Method 2	0.0411	0.0089	0.4
	Method 3	0.0437	0.0089	0.4
Two-sided	Method 1	0.0619	0.0095	0.05
	Method 2	0.0500	0.0093	0.051
	Method 3	0.0500	0.0094	0.0025

The same as in the previous example, the worst result is for method 1 while accuracy of methods 2 and 3 are very close to each other. Because of preserving the structure, methods 1 and 2 have led to better approximations at high frequencies that can be seen in the frequency responses. However, at medium frequencies, methods 2 and 3 are closer to the original system. For this system, the value of s_0 for one-sided and two-sided methods are different, because the same values of s_0 in two-sided methods lead to unstable reduced systems. Because of larger s_0 in one-sided methods, their reduced systems have better accuracy than two-sided methods although less number of moments match.

The same results can be concluded from the step responses, in Figures 12.6 and 12.7.

[2] The model is available online at `http://www.win.tue.nl/niconet/NIC2/benchmodred.html`

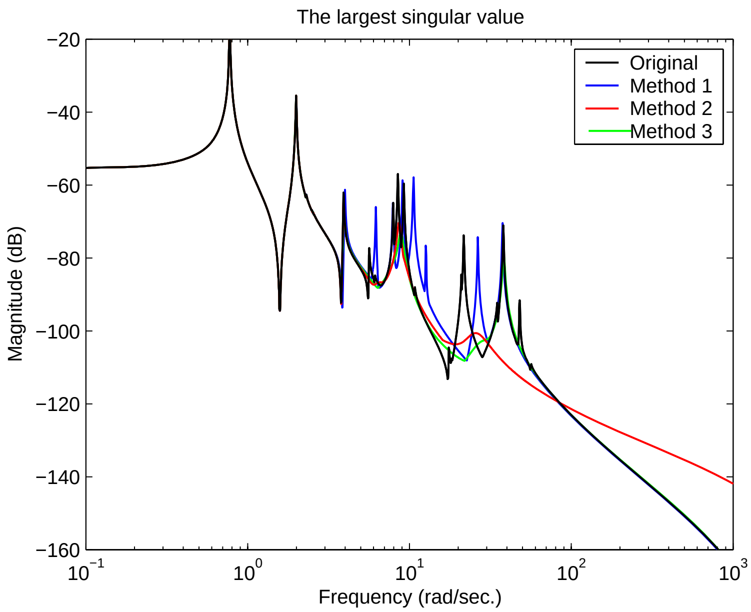

Figure 12.4: The largest singular values of the original and order 15 reduced model of ISS using one-sided methods.

However, method 2, in one-sided methods, performs a little bit worse than two other methods while in the two-sided methods method 1 is the worst as expected.

12.3 Application to a beam model

The system we consider in this section is a beam model which is a typical structure whose generic layout corresponds to atomic force microscopy tips and gas sensors as well as radio frequency switches and filters[3]. Given a simple shape, they often can be modelled as one-dimensional beams embedded in two or three dimensional space. This model describes a slender beam which is actuated by a voltage between the beam and the ground electrode below; see Figure 12.8. On the beam, at least three degrees of freedom per node have to be considered. On the ground electrode, all spatial degrees of freedom are fixed, so only charge has to be considered.

Based on the finite element discretization presented in [90], an interactive matrix genera-

[3]The model can be downloaded from Oberwolfach Model Reduction Benchmark Collection available online at http://www.imtek.uni-freiburg.de/simulation/benchmark/

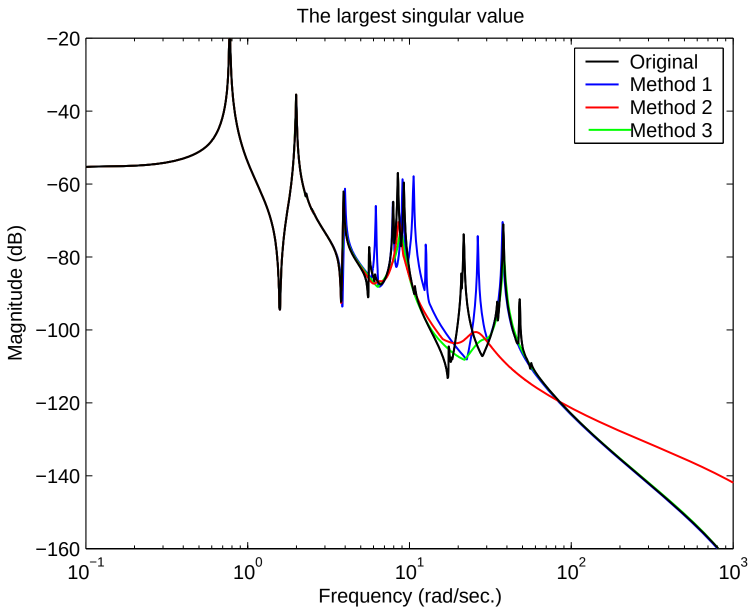

Figure 12.5: The largest singular values of the original and order 15 reduced model of ISS using two-sided methods.

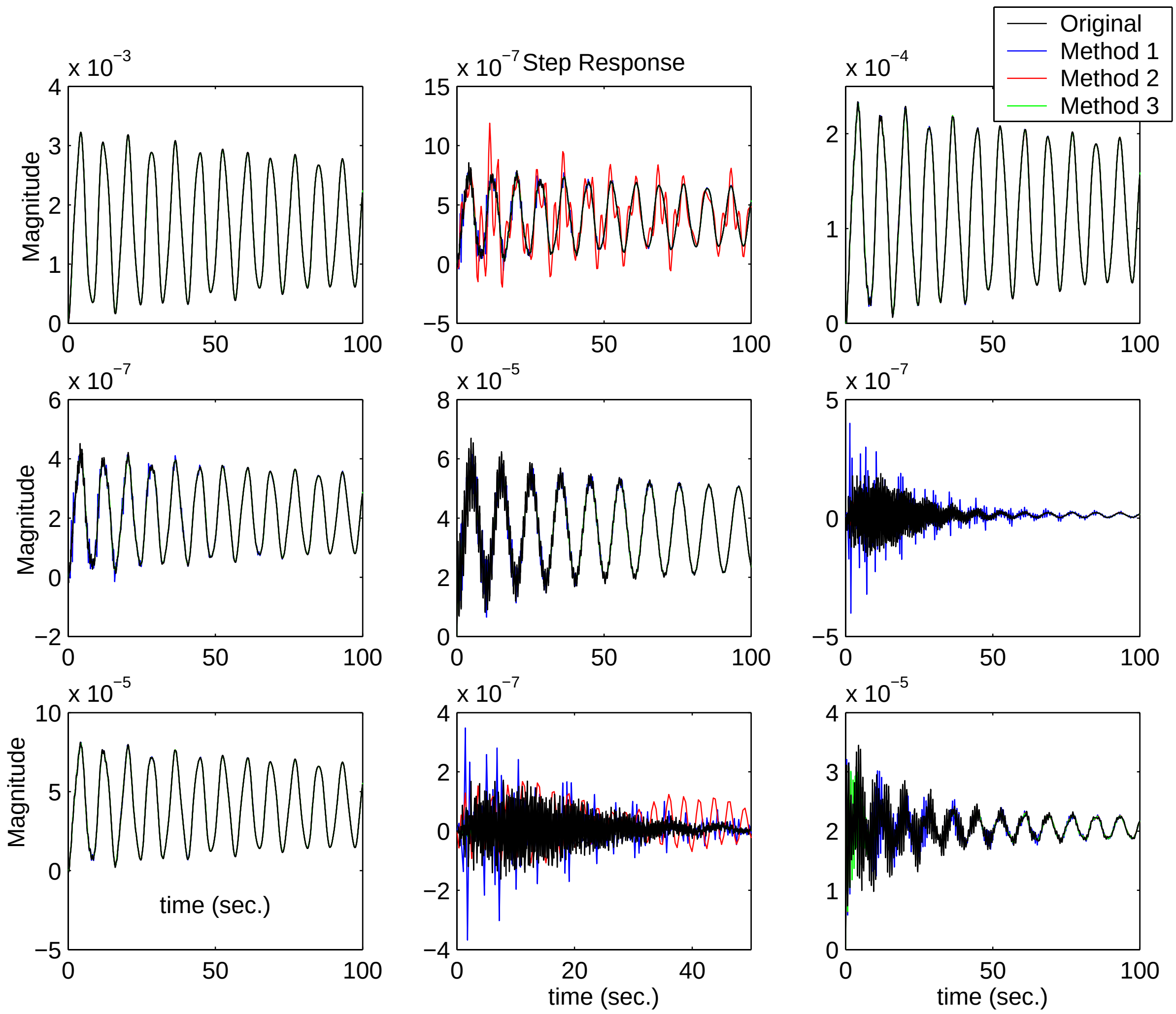

Figure 12.6: Step response of the original and order 15 reduced model of ISS using one-sided methods.

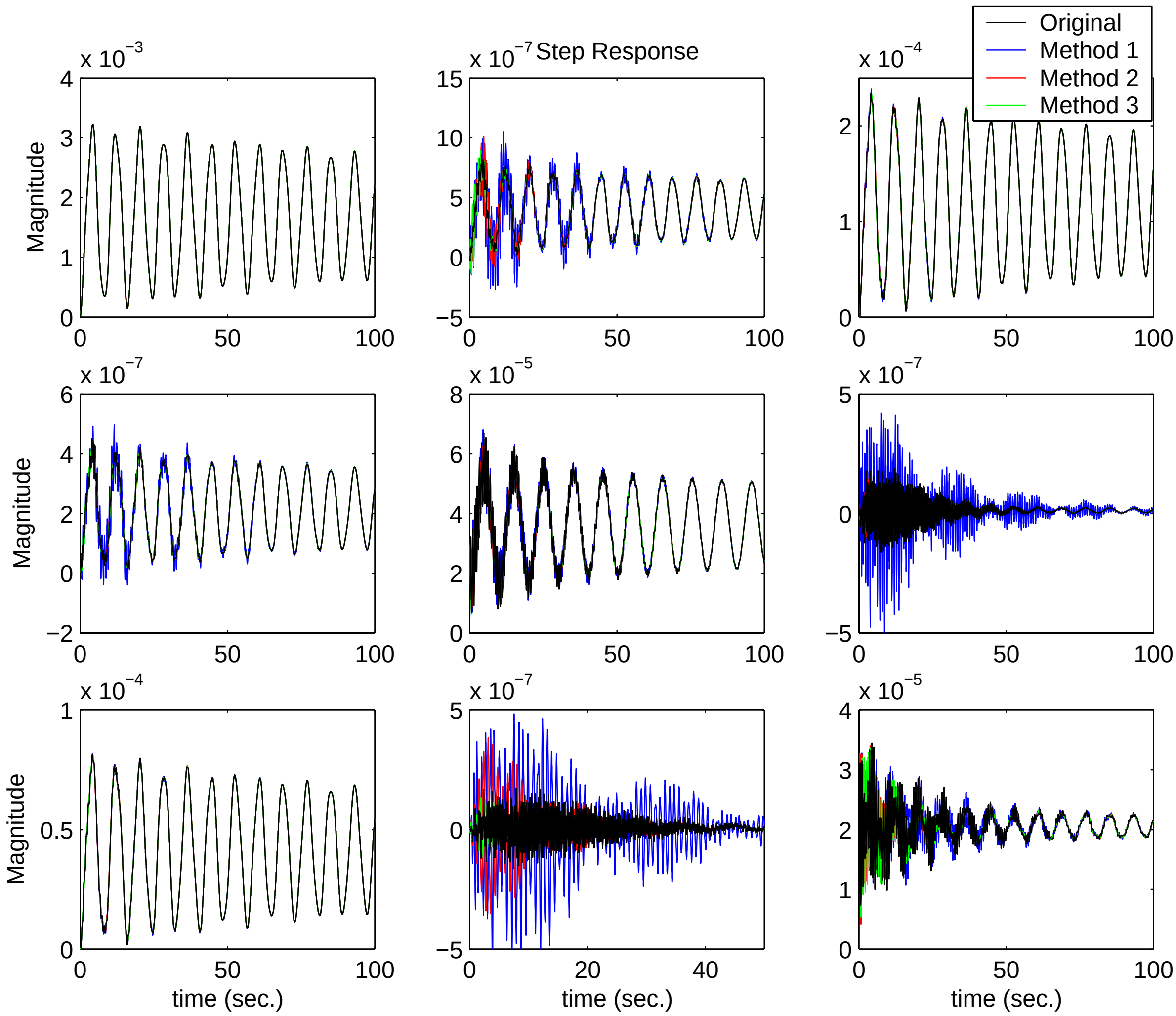

Figure 12.7: Step response of the original and order 15 reduced model of ISS using two-sided methods.

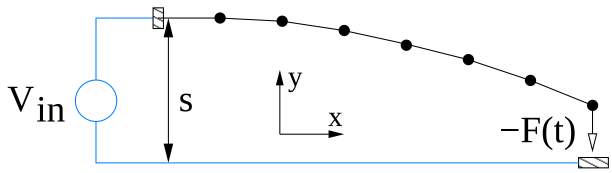

Figure 12.8: A conducting beam supported at one end with counter electrode below.

tor has been created. After modelling of the beam, a set of differential-algebraic equations in second order form is found where the damping matrix is calculated as a linear combination of the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$. The input of the system is the voltage applied between the beam and the counter electrode and the output is the displacement of the nodes typically the node at the end of the beam. A typical input to this system is a step function; periodic on/off switching is also possible. The reduced model should thus both represent the step response as well as the possible influence of higher order harmonics. Details of the implementation are available in [53].

Two types of model are considered to be reduced by the proposed methods: an undamped model ($\mathbf{D} = \mathbf{0}$) and a lightly damped model, both of order $N = 15992$ with $n = 7996$ second order differential equations. The original models are reduced to different orders. By reducing the original systems to higher orders, better approximation at higher frequency can be achieved. Because the norm of the error system and frequency response for such high order models can not be calculated by the algorithms implemented in MATLAB, we compare the frequency response of the reduced models with a higher order reduced system. For a better comparison, we also extract the highest frequency f_{max}, up to which the frequency response of the lower order model is almost the same as the one of the higher order reduced model.

The undamped model is symmetric, however the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are not positive definite. Therefore, the one-sided and two-sided methods leads to the same results. It should be noted that the reduced system of the undamped model by method 1 leads to an undamped system! For this system, the first Markov parameter of the reduced systems by method 3 is very small making the back conversion possible and the results are very close to the method 2. There fore only two sets of reduced systems are compared for the undamped beam model.

In Table 12.3, the maximum frequency that the reduced system is accurate is given. These results can be compared to the Figures 12.9 and 12.10 where the frequency response of

Table 12.3: Maximum accurate frequency f_{max} compared to the reduced system of order 100 for the undamped system.

	Method	order Q	s_0	f_{max} (Rad/sec.)
One-sided	Method 1	6	1	4.3×10^3
		10	10^3	1.7×10^4
		20	10^4	6×10^4
	Method 2,3	6	0	5.3×10^3
		10	0	1.8×10^4
		20	0	6.2×10^4

some of the reduced systems are plotted. Because of preserving the second order structure, the slope of the bode plots at high frequencies is $-40dB/dec.$. The state space method performs better (but not much better) than the second order Krylov method.

To achieve a good approximation of step response, order 6 performs well and after this order, the step response remains almost unchanged. The results can be seen in Figure 12.11 where the step responses of the reduced systems are very close to each other.

The lightly damped model is not symmetric, because we use the point at 0.75 of length of the beam to its end as the output node. In one-sided and two-sided cases, method 2 by back conversion leads to unstable reduced system if the order is less than 8 and 20, respectively. In Table 12.4, the maximum frequency that the reduced system is accurate is given. These result can be compared to the Figures 12.12, 12.13, 12.14, 12.15, 12.16 and 12.17 where the frequency response of some of the reduced systems are plotted. Similar to the undamped model, by going to higher orders better accuracy at higher frequencies can be achieved.

For this system, although the method 3 does not theoretically preserve the second order structure, the first Markov parameter of reduced systems found by this method are small and their frequency responses at high frequency has $-40dB/dec.$ slope up to very high frequencies. However, the frequency responses of the reduced system by method 2 tend the one of the system of order 48 at high frequencies, different from method 3.

Method 2 and 3 do not lead to better results compared to method 1. For some orders, method 1 has a better approximation because of using a larger value of s_0.

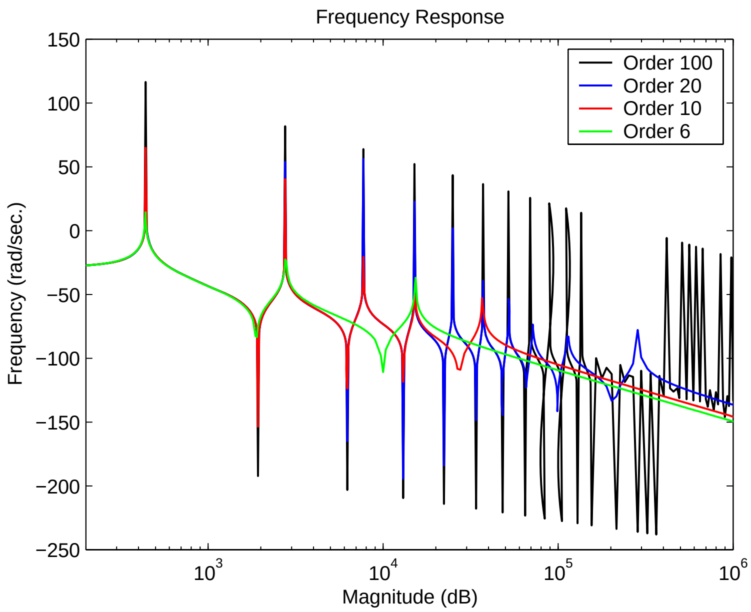

Figure 12.9: Bode diagram of the reduced systems of the undamped model using a second order Krylov method.

Figure 12.18 show the step response of the reduced systems by one-sided methods. The step response of the reduced systems of order 6 by method 1 and 3 have a good approximation while method 2 needs at least order 8 because of losing stability. However, all step responses are almost the same.

12.4 Conclusion

In this chapter, the proposed methods have been applied to three different systems of low, medium and high orders and the results are discussed. The results show that the state space method usually leads to better approximation however the structure is destroyed and the result is not much different from the back conversion approach whereas the second order Krylov methods lead to worse results because of matching less number of moments. The reduced systems with the same structure as the original model (method 1 and 2) have a better approximation at high frequencies with the slope of $-40dB/dec.$.

Another parameter that paly an important role is s_0. Different values of s_0 lead to different results and they can be important to find a stable reduced system. Therefore, finding

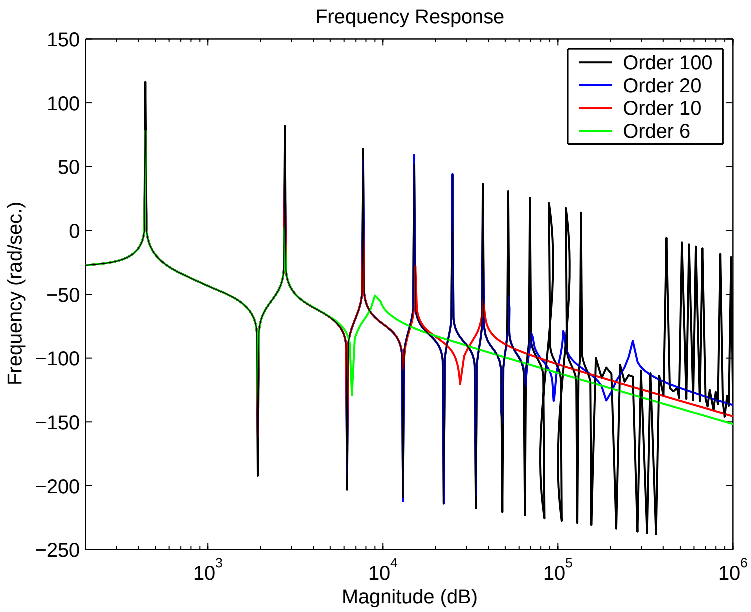

Figure 12.10: Bode diagram of the reduced systems of the undamped model using a state space method.

a stable reduced system is a restriction to choose s_0 and for some cases as experienced in the preceding examples, method 1 may lead to better results although less number of moments match.

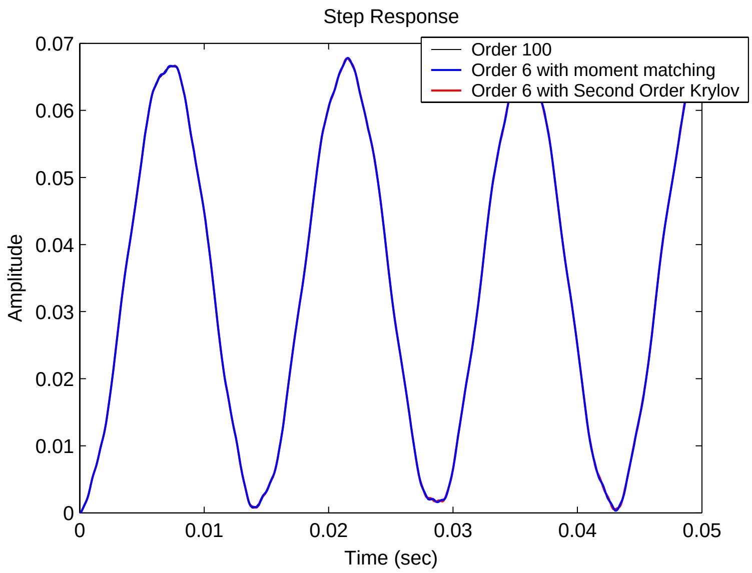

Figure 12.11: Step Response of the reduced systems of the undamped model.

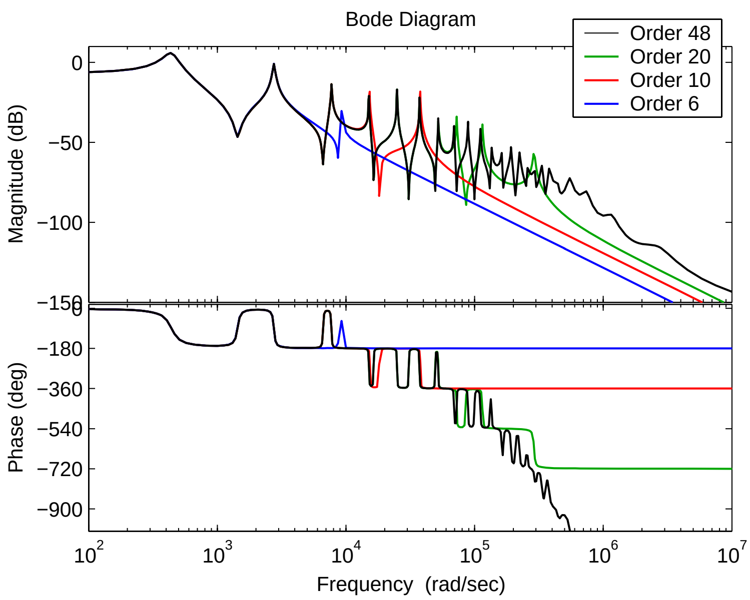

Figure 12.12: Bode diagram of the reduced systems of the damped model using a one-sided second order Krylov method.

Table 12.4: Maximum accurate frequency f_{max} compared to the reduced system of order 48 for the damped system.

	Method	order Q	s_0	f_{max} (Rad/sec.)
One-sided	Method 1	6	0	4.3×10^3
		10	2×10^3	1.3×10^4
		20	1.5×10^4	5.5×10^4
	Method 2	8	0	5.1×10^3
		10	0	1.2×10^4
		20	0	6.5×10^4
	Method 3	6	0	4.3×10^3
		10	0	1.5×10^4
		20	0	6.5×10^4
two-sided	Method 1	6	1e3	4.1×10^3
		10	4×10^3	2.6×10^4
		20	6×10^4	5.5×10^4
	Method 2	6	-	unstable
		10	-	unstable
		20	0	6.5×10^4
	Method 3	6	0	4.1×10^3
		10	0	3.0×10^4
		20	0	6.5×10^4

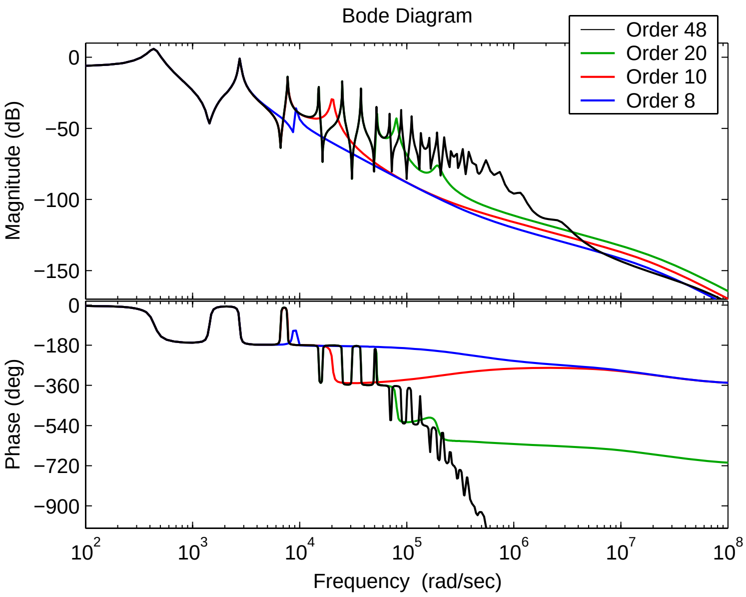

Figure 12.13: Bode diagram of the reduced systems of the damped model using a one-sided back conversion method.

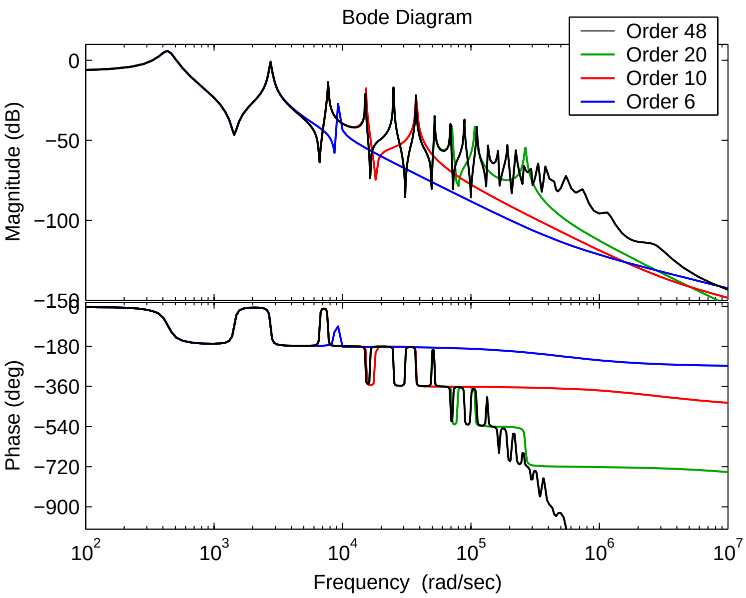

Figure 12.14: Bode diagram of the reduced systems of the damped model using a one-sided state space method.

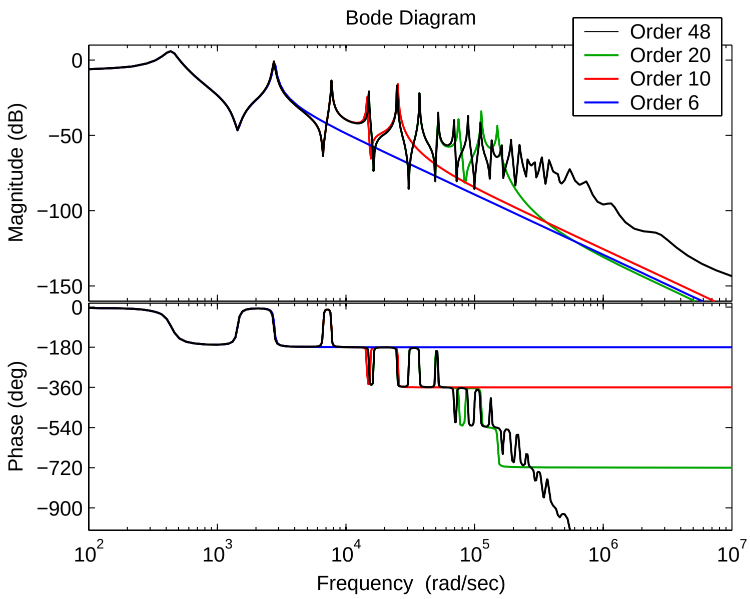

Figure 12.15: Bode diagram of the reduced systems of the damped model using a two-sided second order Krylov method.

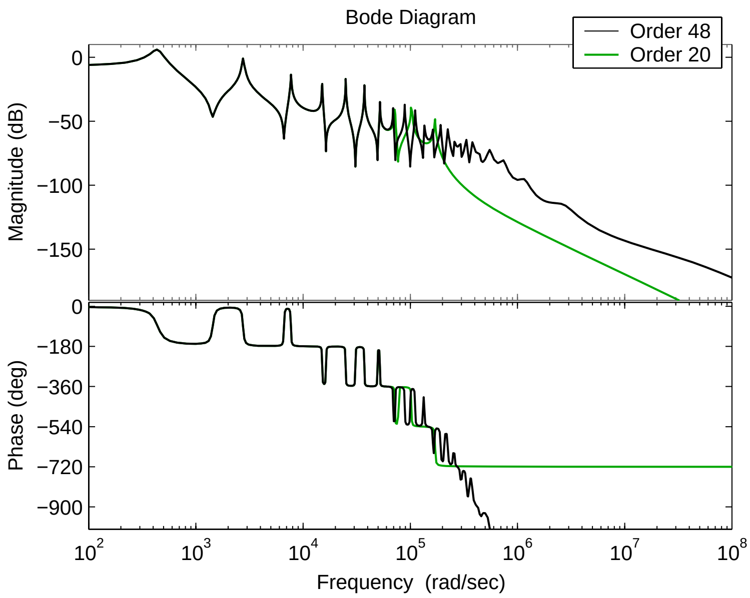

Figure 12.16: Bode diagram of the reduced systems of the damped model using a two-sided back conversion method.

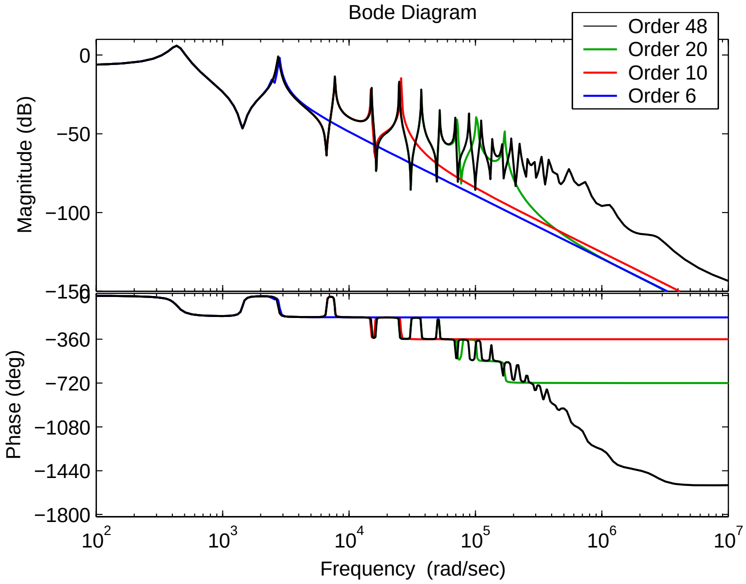

Figure 12.17: Bode diagram of the reduced systems of the damped model using a two-sided state space method.

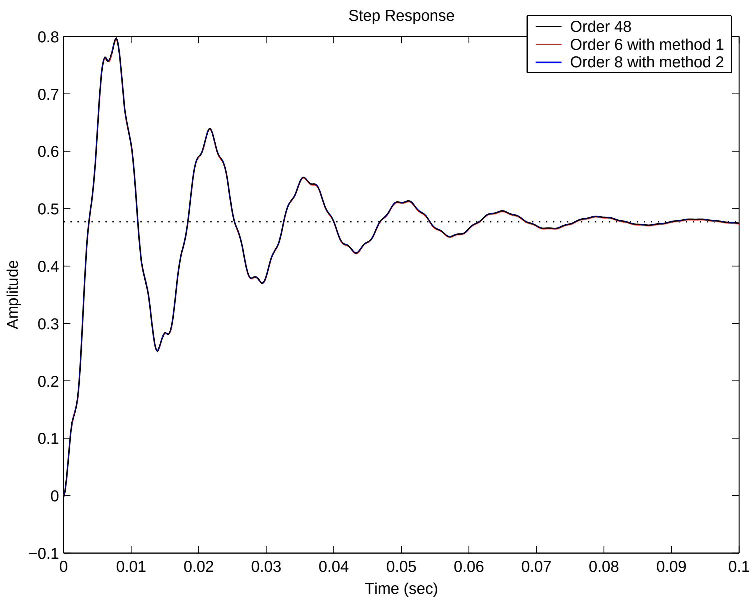

Figure 12.18: Step Response of the reduced systems of the damped model using one-sided methods.

Chapter 13

CONCLUSION AND DISCUSSION

In this dissertation, we have proposed two different methods to reduce systems of second order form such that the second order structure is preserved. To avoid numerical problems and in order to apply the reduction methods to large scale systems, the Krylov subspace reduction approach has been used.

In the first method, by generalizing the definition of Krylov subspaces to second order Krylov subspace, the well-known method of reduction of large scale systems based on moment (or Markov parameter) matching has been generalized to reduce second order models. Preserving the structure is achieved by applying a projection to the original second order model. The advantages of the proposed approach can be highlighted as follows:

- The method proposed in [86] has been modified and the number of matching parameters has been increased up to double.
- The method has been generalized to match the Markov parameters or moments about different points.
- The proposed method preserves some structures of the original matrices: undamped systems are approximated by undamped systems and one-sided methods preserves symmetry and definiteness of the mass, damping and stiffness matrices.
- Under some conditions, one-sided methods preserve stability of the original system.
- Calculating the projection matrices in the size of original second order model makes the numerical calculation cheaper and simpler.

To calculate the projection matrices, the Arnoldi and Lanczos algorithms have been generalized to the so called second order Arnoldi and second order Lanczos algorithms.

We have also generalized the second order Krylov subspaces to reduce a large number of high order differential equations.

Despite the advantages of second order Krylov subspace method, compared to the state space reduction methods, this method matches half number of parameters if we reduce the original system to the same order. To improve the results found by the first approach and in order to increase the number of matching parameters, we have proposed the second method based on reduction in state space and back conversion to second order form.

In the second approach, we showed that if we reduce the equivalent state space model while matching the first Markov parameter, then the reduced order model can be converted into second order form calculating the mass, damping and stiffness matrices. The sufficient conditions for the back conversion procedure in SISO case and for MIMO undamped models have been presented however, there is no proof for the existence of back conversion transformation in MIMO case.

For undamped models, a numerical algorithm has been presented to back convert the state space model into second order form which avoids numerical problems. For SISO systems, we have integrated the reduction and back conversion procedures by modifying the Arnoldi algorithm and taking advantage of the structure of the reduced state space system.

In comparison to the first approach, the back conversion method matches almost double number of characteristic parameters while using double number of iterations to reduce to the same order. The most expensive part of the algorithms is the LU-factorization. In the second order Krylov methods and two-sided back conversion, only one LU-factorization of $\mathbf{K}$ should be calculated in moment matching while the one-sided back conversion method needs 2 LU-factorizations of $\mathbf{M}$ and $\mathbf{K}$; see Table 13.1. By considering the structure of the state space matrices, the cost of computation of back conversion method using the Lanczos algorithm is very close to the second order Krylov method.

In practice, each method may have some advantages over the other, specially by changing the point around which the moments match or losing stability by one of the methods.

The results of this dissertation can be generalized to match the coefficients of the Laguerre

Table 13.1: Comparison of the reduction approaches to reduce to order $Q = 2q$

Method	One/two-sided	iterations	LU-factorization	matching moments
direct projection	One-sided	q	$\mathbf{K}$	q
	Two-sided	q	$\mathbf{K}$	$2q$
back conversion	One-sided	$2q$	$\mathbf{K}, \mathbf{M}$	$2q-1$
	Two-sided	$2q$	$\mathbf{K}$	$4q-1$

series expansion instead of the Taylor series which is an alternative to moment matching approach [25].

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