Technische Universität München
Zentrum Mathematik

Numerical Discretization of Static Hamilton-Jacobi Equations on Triangular Meshes

Christian Mark Anton Rasch

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender:		Univ.-Prof. Dr. Bernd Simeon
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Folkmar Bornemann
	2.	Univ.-Prof. Dr. Martin Brokate
	3.	Prof. Dr. Andreas Veeser (Università degli Studi di Milano, Italien) (schriftliche Beurteilung)

Die Dissertation wurde am 05.12.2006 bei der Technischen Universität eingereicht und durch die Fakultät für Mathematik am 30.01.2007 angenommen.

Contents

Preface iii

Chapter 1. Static Hamilton-Jacobi Equations 1
1.1. Statement of the Problem 1
1.1.1. Properties of the Hamilton Function 1
1.1.2. Examples of Static Hamilton-Jacobi Equations 2
1.1.3. Lipschitz Domains 3
1.2. Weak Solutions and Viscosity Solutions 3
1.2.1. Weak Solutions 4
1.2.2. Definition of Viscosity Solutions 4
1.2.3. Consistency of Viscosity Solutions 5
1.2.4. Lipschitz Continuity 7
1.2.5. Uniqueness and Stability 8
1.3. The Hopf-Lax Formula 9
1.3.1. The Support Function of the Zero Level-Set 10
1.3.2. The Optical Distance 12
1.3.3. A Variational Inequality 14
1.3.4. The Maximal Sub-Solution 16
1.3.5. The Existence Theorem 17
1.3.6. Compatibility of the Boundary Data 17
1.4. Examples, Computation of the Support Function 17
1.4.1. Equations of Eikonal Type 18
1.4.2. The Eikonal Equation 19
1.4.3. Exit-Time Optimal Control Problems 20
1.4.4. The Legendre Transformation 22

Chapter 2. Discretization by Linear Finite Elements 24
2.1. Linear Finite Elements 24
2.1.1. Triangulation 24
2.1.2. Linear Interpolation 25
2.2. Discretization by Linear Finite Elements 26
2.2.1. Motivation 26
2.2.2. The Hopf-Lax Discretization 27
2.2.3. Solvability of the Discrete System 28
2.2.4. Uniqueness of the Discrete Solution 29
2.3. Convergence 29
2.3.1. Discrete Compatibility 30
2.3.2. Uniform Boundedness and Equi-Continuity 30
2.3.3. Consistency 33
2.3.4. Convergence Theorem 34
2.4. Local Error and Error Propagation 34
2.4.1. Lipschitz Continuity of the Support Function 35
2.4.2. Local Error 35
2.4.3. Error Propagation 37

2.5. Convergence Rate 38
2.5.1. The Kružkov-Transformation 38
2.5.2. Proof of the Theorem 40
2.5.3. Example 43

Chapter 3. Computation of the Discrete Solution 46
3.1. Computation of Hopf-Lax Updates 47
3.1.1. Convex Programs 47
3.1.2. Simple Iterative Methods 47
3.1.3. The Update Formula for the Eikonal Equation 48
3.2. Iterative Methods for the Fixed-Point Problem 51
3.2.1. Jacobi Iteration 51
3.2.2. Total Complexity 51
3.2.3. Gauss-Seidel Iteration 53
3.2.4. Adaptive Gauss-Seidel Iteration 54
3.2.5. Choice of the Tolerance 56
3.3. The Fast Marching Method 56
3.3.1. Discrete Causality 57
3.3.2. The Fast Marching Method for Acute Triangulations 58
3.3.3. Extension on Non-Acute Triangulations 59
3.3.4. Implementation, Complexity 61
3.3.5. Sequencing by Untidy Priority Queues 62
3.4. The Ordered Upwind Method 65
3.4.1. Anisotropy of Hamilton-Jacobi Equations 65
3.4.2. Generalized Hopf-Lax Updates 67
3.4.3. The Algorithm 68
3.4.4. Convergence of the OUM 69
3.4.5. Implementation and Possible Extensions 72

Chapter 4. Applications and Extensions 75
4.1. Some Applications 75
4.1.1. Problems with Obstacles 75
4.1.2. Seismic Traveltimes 77
4.1.3. Exit-Time Optimal Control 78
4.2. Distance Maps on Manifolds 79
4.2.1. Differentiable Submanifolds 80
4.2.2. The Distance Function on Submanifolds 81
4.2.3. The Hopf-Lax Discretization on Submanifolds 84
4.2.4. Computation in Local Coordinates 86
4.2.5. The Fast Marching Method on Manifolds 86
4.2.6. Examples 88
4.3. A Higher Order Scheme for the Fast Marching Method 89
4.3.1. Idea: Second Order Finite Differences 89
4.3.2. The Local Error for Smooth Solutions 91
4.3.3. Connection with the Second Order Scheme in [**SV03**] 92
4.3.4. Causality and the Fast Marching Method 93
4.3.5. Numerical Experiments 95

Bibliography 96

Preface

The theory and the numerical treatment of the static Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in \Omega \tag{0.1}$$

are subjects of the doctoral thesis. Here Ω denotes a bounded Lipschitz domain, and the Hamilton function $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(x, p) \mapsto H(x, p)$ is assumed to be continuous, convex with respect to p, coercive ($H(x, p) \to \infty$ uniformly, as $\|p\| \to \infty$), and compatible ($H(x, 0) < 0$ on Ω). Examples for static Hamilton-Jacobi equations are the Eikonal equation $\|Du(x)\| f(x) - 1 = 0$ from geometric optics, describing the evolution of a wave front, given as the level-sets of u, with an underlying velocity $f(x)$, the generalized Eikonal equation $\|Du(x)\| F(x, Du(x)/\|Du(x)\|) - 1 = 0$, where the velocity depends additionally on the direction of the front normal, or the Hamilton-Jacobi-Bellman equation $\sup_{a \in A} \{\langle -f(x, a), Du(x)\rangle - l(x, a)\} = 0$, which appears in optimal control problems with restricted state spaces, involving exit-times.

A suitable notion of weak solutions for (0.1) is introduced in the first chapter, based on which existence and uniqueness results (for the Dirichlet problem) are established. In this concept of *viscosity solutions*, introduced 1981 by Crandall and Lions in [**CL81**], solutions are obtained as limits of smooth solutions of the viscous equation $H(x, Du(x)) = \epsilon\Delta u(x)$ with vanishing viscosity $\epsilon \to 0$. Viscosity solutions can be ingeniously characterized by smooth test functions, as shown in [**CL83**], [**CEL84**], which forms meanwhile the usual approach to this concept. I also follow this approach in definition 1.5, and recall basic properties of viscosity solutions, such as the consistency with the concept of classical solutions $u \in \mathcal{C}^1(\Omega)$ of (0.1). Moreover, Lemma 1.12 shows, that every viscosity solution u of (0.1) is Lipschitz-continuous, and fulfills the Hamilton-Jacobi equation, where u is differentiable (that is, almost everywhere, by Rademacher's theorem).

The comparison principle (theorem 1.13) from [**Ish87**] shows, that the maximal difference between two viscosity solutions $|u(x) - v(x)|$ is attained on the boundary $\partial\Omega$. A simple consequence is, that the Dirichlet problem

$$H(x, Du(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g, \tag{0.2}$$

has *at most* one solution.

Like in physics, where the behavior of a physical system follows the principle of the least action, which provides a variational formulation of the equations of motion, while the action can also be characterized as a solution of the Hamilton-Jacobi equation, I derive in section 1.3 a variational formulation for the viscosity solution of (0.2), which I refer to as the Hopf-Lax formula. This formula was already given in [**Lio82**] in the context of static Hamilton-Jacobi equations, and provides also an existence result for the Dirichlet problem (theorem 1.25).

A local application of the variational principle leads to a discretization of the Hamilton-Jacobi equation, which is subject of the second chapter. We endow Ω with a triangulation, and obtain a fixed point equation

$$\Lambda_h u_h = u_h, \quad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h} \tag{0.3}$$

for the linear finite-element approximation u_h to the viscosity solution of (0.2). It is shown, that this finite-element solution is well-defined, that is, (0.3) admits a unique solution in the space of linear finite-elements (theorems 2.6 and 2.7). Additionally, the convergence of the sequence of finite-element solutions $(u_h)_h$ to the viscosity solution, as the grid-diameter h vanishes, is proved (theorem 2.12), and the approximation error $\|u - u_h\|_\infty$ is analyzed (theorems 2.17 and 2.18).

Several direct and iterative methods for solving the discrete system (0.3) are considered in the third chapter. The simplest (but not the most efficient) approach is a fixed-point iteration for (0.3), where $\mathcal{O}(h^{-1})$ iterations are necessary to reach a user-defined tolerance (theorem 3.4). A competitive iterative method is the adaptive Gauss-Seidel method, originally used as a relaxation method for the multilevel solution of elliptic boundary value problems (see [**PR93**]), which was transfered for the solution of the non-linear system (0.3). I will further discuss the Fast Marching Method ([**Set96**], [**Tsi95**]), which provides an $\mathcal{O}(N \log N)$ solver for the Eikonal equation (N denotes the number of grid-points), and its generalization to anisotropic equations, the Ordered Upwind Method ([**SV01**]). The utilization of an *untidy priority queue* within the Fast Marching Method, or the Ordered Upwind Method, suggested in [**YBS06**], reduces the total complexity of those methods to $\mathcal{O}(N)$. For the obtained algorithms, I contributed a rigorous estimate on the introduced error due to the inexact minimization (lemma 3.17).

In the fourth chapter, computational examples are given. An extension of the Fast Marching Method to the solution of the Eikonal equation on Riemannian manifolds, introduced in [**KS98**], is discussed and a convergence result for the obtained discretization is supplied (theorem 4.5). Based on an idea in [**SV00**], a second order discretization of the Eikonal equation is investigated. Reflections on the applicability even in the case of non-smooth solutions lead to an adaptive scheme, which chooses the first-order variant, where the solution forms shocks (subsection 4.3.4) .

Acknowledgments. I would like to thank Prof. Bornemann for his contributions to our publication [**BR06**] (especially the simplifications in my proof of lemma 1.18, and the proof of lemma 3.1), and Thomas Satzger for the help on the complexity analysis of the modified Fast Marching Method (lemma 3.16).

CHAPTER 1

Static Hamilton-Jacobi Equations

In this chapter the theory of static Hamilton-Jacobi equation is established, along with the concept of viscosity solutions. The first section contains the statement of the problem, and the discussion of a few assumptions, that will be imposed on the Hamilton-Jacobi equation. We consider equations of the form

$$H(x, Du(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

where H is a continuous, real valued function, which is convex with respect to the second variable, and fulfills a certain growth condition (coercivity).

In section 1.2, viscosity solutions are considered, special weak solutions of Hamilton-Jacobi equations, based on which existence and uniqueness results can be proved.

The Hopf-Lax formula, derived in section 1.3 gives an explicit expression for the viscosity solution of the Dirichlet problem. Also the role of a compatibility condition for the boundary data is highlighted.

As a special case, the Hamilton-Jacobi equation

$$F(x, Du(x)) = 1, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

where F is continuous, convex and homogeneous with respect to the second variable, will be treated in subsection 1.4.1. We refer to this equation as the Hamilton-Jacobi equation of *Eikonal type*. It is further shown, how the methods developed herein may be applied to exit-time optimal control problems.

1.1. Statement of the Problem

Let $\Omega \subseteq \mathbb{R}^d$ denote a bounded domain (open and connected set). We consider partial differential equations with Dirichlet boundary conditions of the form

$$H(x, Du(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g, \tag{1.1}$$

with *Hamilton function* $H : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$, $(x, p) \mapsto H(x, p)$ and *boundary value function* $g : \partial\Omega \to \mathbb{R}$. In the following three subsections, I will precisely characterize the problem, we are concerned with herein. First, four important assumptions on the Hamilton function are collected, that underlie the most results throughout the discussion. Then, I will briefly introduce examples for Hamilton-Jacobi equations. Finally, a result on Lipschitz domains is quoted, which will enable us to prove the Lipschitz continuity of solutions of (1.1).

1.1.1. Properties of the Hamilton Function. Most results will require H to fulfill one or more of the following properties:

(H1) $H \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d)$ *(continuity)*

(H2) $p \mapsto H(x, p)$ is convex for every $x \in \overline{\Omega}$ *(convexity)*

(H3) $H(x, p) \to \infty$ for $\|p\| \to \infty$ uniformly in $x \in \overline{\Omega}$ *(coercivity)*

(H4) $H(x, 0) < 0$ for all $x \in \Omega$ *(compatibility)*

Furthermore, a compatibility condition for the boundary data is necessary for the solvability of the Dirichlet problem (1.1), which will precisely be discussed in section 1.3. For the moment, let g be at least continuous.

The following lemma gives an equivalent characterization of the coercivity of the Hamiltonian H due to its convexity with respect to p. The quantities α, β therein will appear in many estimates concerning the solution of equation (1.1).

LEMMA 1.1: *Let Ω be a bounded domain and let H fulfill (H1),(H2). Then (H3) holds, if and only if there exist real numbers $\alpha, \beta > 0$ with*

$$H(x, p) \geq \alpha \|p\| - \beta \tag{1.2}$$

for all $x \in \overline{\Omega}$ and $p \in \mathbb{R}^d$.

PROOF. Property (H3) follows directly from equation (1.2). Let (H3) be fulfilled, and let $M > 0$. As $H(x, 0)$ is bounded on $\overline{\Omega}$, there is, by (H3), a real number $m > 0$ such that $H(x, p) - H(x, 0) \geq M$ for all $x \in \overline{\Omega}$, $p \in \mathbb{R}^d$ with $\|p\| \geq m$. With $0 < t \leq 1$ we obtain from the convexity of H, that

$$H(x, tp) \leq tH(x, p) + (1 - t)H(x, 0)$$

and therefore

$$H(x, p) \geq \frac{H(x, tp) - H(x, 0)}{t} + H(x, 0)$$

For $p \in \mathbb{R}^d$ with $\|p\| \geq m$ it is $t := \frac{m}{\|p\|} \leq 1$ and it holds, that

$$H(x, p) \geq \frac{M}{m} \|p\| + H(x, 0)$$

where the last addend is bounded. Defining $\alpha := \frac{M}{m} > 0$ we can choose $\beta > 0$ such that

$$H(x, p) \geq M - \beta \geq \frac{M}{m} \|p\| - \beta = \alpha \|p\| - \beta$$

holds additionally for $p \in \mathbb{R}^d$ with $\|p\| \leq m$. □

Let Ω be a bounded domain, and assume (H1), (H2) and (H4). Then (H3) holds, if and only if the convex zero level-sets $\mathcal{Z}(x) = \{p \in \mathbb{R}^d\,;\ H(x, p) \leq 0\}$ are bounded. Loosely speaking, the property (H3) ensures the boundedness of the gradient $Du(x)$ in (1.1), and yields the Lipschitz-continuity of u (lemma 1.12).

The condition (H4) becomes important in the uniqueness result theorem 1.13 for solutions of (1.1). Of course, the trivial equation, where $H(x, p) \equiv 0$ has infinitely many solutions, while an equation, where $H(x, p) > 0$ for some x and all $p \in \mathbb{R}^d$, would not admit any solution.

Condition (H4) could be weakened. It is sufficient, if there exists a smooth sub-solution of the Hamilton-Jacobi equation, that is to say, a function $\varphi \in \mathcal{C}^1(\overline{\Omega})$ with $H(x, D\varphi(x)) < 0$ on Ω. Provided that (H1)-(H3) are satisfied, the Hamilton function, defined by $\tilde{H}(x, p) = H(x, D\varphi(x) + p)$, fulfills assumptions (H1)-(H4). If $\tilde{u}$ is a solution of $\tilde{H}(x, D\tilde{u}) = 0$, with $\tilde{u} = g - \varphi$ on the boundary, then $u = \tilde{u} + \varphi$ is a solution of the Dirichlet problem (1.1).

1.1.2. Examples of Static Hamilton-Jacobi Equations.

A famous example is, with $H(x, p) = \|p\| - f(x)$, where $f \in \mathcal{C}(\overline{\Omega})$, and $f > 0$ on Ω, the *Eikonal equation*

$$\|Du(x)\| = f(x), \quad x \in \Omega, \qquad u|_{\partial\Omega} = g.$$

Physically the solution $u(x_0)$ at $x_0 \in \Omega$ can be interpreted as the shortest time needed to reach the boundary $\partial\Omega$ from x_0, with an underlying speed $1/f(x)$ depending only on the position x (here typically $g \equiv 0$). If $f \equiv 1$, the (viscosity) solution is the distance function from $\partial\Omega$, $u(x) = \operatorname{dist}(x, \partial\Omega)$.

The computation of shortest traveltimes in an anisotropic medium, or on manifolds leads to the *generalized Eikonal equation*

$$\langle Du(x), M(x)Du(x)\rangle = f(x), \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

where $M(x)$ is a symmetric, positive definite matrix for every $x \in \Omega$. In some applications, initial values will be provided on a closed (and non-empty) subset $\Gamma \subset \Omega$ (which can be a curve, or a finite set of points, etc.) and the boundary condition will be dropped, in order to, for example, compute the distance function from a single point or from a curve on a manifold.

A further important example is the *Hamilton-Jacobi-Bellman equation*

$$\sup_{a\in A} \left\{\langle -f(x,a), Du(x)\rangle - l(x,a)\right\}, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

which arises in optimal control problems involving restricted state spaces and exit-times. Here $f(x,a)$ is the speed function of the controlled dynamical system, and $l(x,a)$ describes a running cost. The dynamical system is to be controlled in such way, that an associated cost functional takes its minimal value. The given examples of Hamilton-Jacobi equations will be discussed in more detail in section 4.1.

1.1.3. Lipschitz Domains. For our main results, we will always require Ω to be a Lipschitz domain, i.e. $\partial\Omega$ to be locally the graph of Lipschitz-continuous mappings. This is means no considerable restriction on the computational domain, as we even need to triangulate Ω for our discretization.

Definition 1.2 ([**Alt99**, A 6.2]): *A bounded domain $\Omega \subset \mathbb{R}^d$ is called a* Lipschitz domain*, if $\partial\Omega$ can be covered by finitely many open sets $U^1, \ldots, U^k$, such that $\partial\Omega \cap U^j$ is the graph of some Lipschitz-continuous mapping for $j = 1, \ldots, k$ and that $\Omega \cap U^j$ lies on one side of this graph.*

A more formal definition can be found in [**Alt99**, A 6.2]. We need the following property of Lipschitz domains.

Lemma 1.3 ([**Alt99**, Lemma 8.4]): *Let $\Omega \subset \mathbb{R}^d$ denote a Lipschitz domain. Then there is a constant $C_\Omega > 0$ depending only on Ω, such that for each two points $x, y \in \Omega$ there exists a curve $\xi \in \mathcal{C}^\infty([0,1];\Omega)$ with $\xi(0) = x$, $\xi(1) = y$, such that*

$$\ell(\xi) = \int_0^1 \left\|\dot{\xi}(t)\right\| \, dt \le \left\|\dot{\xi}\right\|_\infty \le C_\Omega \cdot \|y - x\|$$

Especially convex domains Ω are Lipschitz domains, where every two points $x, y \in \Omega$ can be connected by the straight line $\xi(t) = x + t(y - x)$ for $t \in [0,1]$. Thus, for convex domains, the lemma holds with $C_\Omega = 1$.

1.2. Weak Solutions and Viscosity Solutions

Partial differential equations, in general, require an adequate concept of weak solutions, based on which existence and uniqueness results can be shown. The problem (1.1) (with additional compatibility assumptions on the boundary values) admits a locally Lipschitz-continuous weak solution u, which satisfies the Hamilton-Jacobi equation in all points, where u is differentiable. However, this notion of weak solutions lacks in uniqueness, as an example in the next subsection teaches.

The adequate concept is the concept of viscosity solutions, where solutions are obtained by considering limits of solutions of the diffusive equation $H(x, Du_\epsilon(x)) = \epsilon \cdot \triangle u_\epsilon(x)$, as ϵ vanishes. In 1981 Crandall and Lions proposed a definition of viscosity solutions, where the differentiation is done by means of smooth test functions through partial differentiation. We follow this approach, and discuss consistency, uniqueness and stability of viscosity solutions. An existence result is subject of section 1.3.

1.2.1. Weak Solutions. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain and let $H : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be at least continuous. Let $u : \overline{\Omega} \to \mathbb{R}$ be locally Lipschitz-continuous. According to Rademacher's theorem u is differentiable almost everywhere in Ω. Such a function u, which fulfills the Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in \Omega \tag{1.3}$$

where it is differentiable, is called a *weak* or *generalized solution* of (1.3). It is called a *weak solution of the Dirichlet problem* (1.1), if it additionally fulfills

$$u(y) = g(y) \quad \text{for} \quad y \in \partial\Omega. \tag{1.4}$$

EXAMPLE 1.4: Consider the scalar Eikonal equation

$$|u'(x)| - 1 = 0 \quad \text{in} \quad]-1, 1[\ , \quad u(-1) = u(1) = 0 \tag{1.5}$$

This problem has no classical solution $u \in \mathcal{C}^1([-1,1])$ (by the theorem of Rolle, such a function u had a critical point $\xi \in]-1,1[$, in contradiction to $|u'| \equiv 1$ on $]-1,1[$). However, the functions u_n defined by

$$u_0(x) = 1 - |x|\ , \quad u_j(x) = \frac{1}{2^j} - \left| \frac{1}{2^j} - u_{j-1}(x) \right| \quad (j \in \mathbb{N})$$

are all weak solutions of (1.5).

The fact, that even in this simple example there are infinitely many weak solutions, seems unsatisfactory. Interpreting equation (1.5) as Eikonal equation for the first arrival time at x of a signal emanating from the boundary with constant speed 1, the function $u_0(x) = 1 - |x|$ would be the solution of choice. As it turns out later on, u_0 is characterized as the unique viscosity solution of (1.5).

1.2.2. Definition of Viscosity Solutions. We will now introduce the notion of viscosity solutions for static Hamilton-Jacobi equations and collect some of their fundamental properties. This notion originates from the paper [CL81] of Crandall and Lions in 1981, where in particular existence and uniqueness of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations were proved. Based on the following definition, we will be able to derive existence and uniqueness results for viscosity solutions of the Dirichlet problem (1.1).

DEFINITION 1.5 ([CEL84]): *On an open domain $\Omega \subseteq \mathbb{R}^d$ let $u \in \mathcal{C}(\Omega)$. u is called a* viscosity sub-solution *of (1.3) if, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$,*

$$H(x_0, D\varphi(x_0)) \leq 0$$

provided that $u - \varphi$ attains a local maximum in $x_0 \in \Omega$.

u is called a viscosity super-solution *of (1.3) if, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$,*

$$H(x_0, D\varphi(x_0)) \geq 0$$

provided that $u - \varphi$ attains a local minimum in $x_0 \in \Omega$.

Finally u is called a viscosity solution *of (1.3), if u is a viscosity sub-solution and a viscosity super-solution of (1.3) in union.*

As stated by theorem 3.1 in [CEL84], viscosity solutions may be obtained as uniform limits of smooth solutions of

$$H(x, Du_\epsilon(x)) = \epsilon \triangle u_\epsilon(x) \quad \text{in} \quad \Omega$$

as $\epsilon \to 0$. Similar equations appear in fluid mechanics, where $\epsilon > 0$ quantifies the fluid viscosity. Indeed the diffusive term on the right-hand side has a smoothing influence on the solution u. The method of gaining viscosity solutions of Hamilton-Jacobi equations by adding a small diffusive term is also referred to as the method of vanishing viscosity (see [CL84]).

REMARK 1.6: A function $u \in \mathcal{C}(\Omega)$ is already a viscosity sub-solution (super-solution) of (1.3), if for all $x_0 \in \Omega$ and test functions $\varphi \in \mathcal{C}_0^\infty(\Omega)$ the inequality $H(x_0, D\varphi(x_0)) \le 0 (\ge 0)$ holds true, provided that x_0 is a *strict* local maximum (minimum) point of $u - \varphi$.

To show this, let, for example, $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be such, that $u - \varphi$ has a local maximum in $x_0 \in \Omega$. Then $u - \tilde{\varphi}$ attains a strict local maximum in x_0, where

$$\tilde{\varphi}(x) = \varphi(x) + \delta \left\| x - x_0 \right\|^2$$

with some $\delta > 0$, and moreover $D\varphi(x_0) = D\tilde{\varphi}(x_0)$.

In the definition of viscosity solutions, often $\mathcal{C}^1$ test functions are considered. Before we investigate this approach in another remark, let me quote an useful proposition, borrowed from [BCD97].

PROPOSITION 1.7 ([BCD97, Lemma 2.2.4]): *Let $v \in \mathcal{C}(\Omega)$ and let $x_0 \in \Omega$ denote a strict maximum point of v in $\overline{B} = \overline{B(x_0, \delta)} \subseteq \Omega$. If $(v_n) \subset \mathcal{C}(\Omega)$ converges locally uniformly to v in Ω, then there exists a sequence $(x_n) \subset \overline{B}$, such that*

$$x_n \to x_0, \quad \text{and} \quad v_n(x_n) \ge v_n(x) \quad \forall x \in \overline{B}.$$

PROOF. (From [BCD97].) We take x_n to be a maximum point of v_n on $\overline{B}$. For any convergent subsequence (x_{n_k}) of (x_n), we have, by uniform convergence, $v_{n_k}(x_{n_k}) \to v(\tilde{x})$, where $\tilde{x} = \lim x_{n_k}$. As $v_{n_k}(x_{n_k}) \ge v_{n_k}(x)$ for all $x \in \overline{B}$, we deduce $v(\tilde{x}) \ge v(x_0)$, hence $\tilde{x} = x_0$, as x_0 is a strict maximum. □

REMARK 1.8: Considering test functions $\varphi \in \mathcal{C}^1(\Omega)$ leads to an equivalent characterization of viscosity solutions: Of course $\mathcal{C}_0^\infty(\Omega) \subset \mathcal{C}^1(\Omega)$. For the other direction, let u denote a viscosity solution according to definition 1.5 and let $\varphi \in \mathcal{C}^1(\Omega)$ be such, that $u - \varphi$ attains a local maximum in $x_0 \in \Omega$. Then $u - \tilde{\varphi}$ has a strict local maximum in x_0, where $\tilde{\varphi}$ is constructed as in the last remark. There is a sequence $(\varphi_n)_n \subset \mathcal{C}_0^\infty(\Omega)$, such that $\varphi_n \to \tilde{\varphi}$ and $D\varphi_n \to D\tilde{\varphi}$ uniformly in some neighborhood U of x_0 (such a sequence may be obtained by mollification of $\tilde{\varphi}$ and multiplication with a smooth cutoff function, which equals 1 on U). Then by the last proposition, there exists a sequence of points $(x_n)_n \subset U$, such that $u - \varphi_n$ has a local maximum in x_n (after passing to a subsequence, if necessary). Since u is a viscosity sub-solution, $H(x_n, D\varphi_n(x_n)) \le 0$ holds. Consequently, by uniform convergence, $H(x_0, D\tilde{\varphi}(x_0)) = H(x_0, D\varphi(x_0)) \le 0$.

1.2.3. Consistency of Viscosity Solutions. In the following, we will investigate the connection between viscosity solutions and weak solutions. If H is coercive, then every viscosity solution u is locally Lipschitz-continuous and therefore differentiable almost everywhere. A similar result for continuous and coercive Hamiltonians H on $\mathbb{R}^d \times \mathbb{R}^d$ and *bounded* continuous viscosity solutions $u : \mathbb{R}^d \to \mathbb{R}$ can be found in [BCD97, Prop. 2.4.1]. The next lemma shows, that the assumption of the boundedness in [BCD97, Prop. 2.4.1] can be dropped.

LEMMA 1.9: *Let H fulfill (H1), (H3). Then every viscosity sub-solution $u \in \mathcal{C}(\Omega)$ of (1.3) is locally Lipschitz-continuous.*

PROOF. (After [BCD97, Prop. 2.4.1]). Let $x_0 \in \Omega$ and let $R > 0$ be a real number, such that $\overline{B(x_0, 2R)} \subset \Omega$. For each $C > 0$ the function

$$\varphi(x, y) = u(y) - C \cdot \|y - x\|$$

is continuous on $\Omega \times \Omega$. If the constant C is chosen sufficiently large, then for every $x \in B(x_0, R)$ there exists some $\tilde{y} \in B(x_0, 2R)$ with

$$\varphi(x, \tilde{y}) = \sup_{y \in B(x_0, 2R)} \varphi(x, y), \tag{1.6}$$

because of the boundedness of u on the compact set $\overline{B(x_0, 2R)}$.

The point $\tilde{y}$ is then a local maximum of the function $y \mapsto \varphi(x, y)$. If $\tilde{y} \neq x$, it follows that

$$H\left(\tilde{y},\ C\frac{\tilde{y}-x}{\|\tilde{y}-x\|}\right) \leq 0$$

as u is a viscosity sub-solution of (1.3), which implies a contradiction to the coercivity (H3) of H for large C.

Therefore we can choose C, such that $\tilde{y} = x$ holds for all $x \in B(x_0, R)$ in (1.6), and as

$$\varphi(x, y) \leq \varphi(x, \tilde{y} = x)$$

we finally obtain

$$u(y) - u(x) \leq C \cdot \|y - x\| .$$

The last inequality holds for all $x, y \in B(x_0, R)$. Consequently, u is Lipschitz-continuous on $B(x_0, R)$. □

The following theorem shows the consistency of the introduced concept of viscosity solutions with the familiar notion of classical solutions or weak solutions. Similar results can be found in [CEL84] or in [BCD97] (propositions 2.1.3 and 2.1.9).

Theorem 1.10 (Consistency): *Every classical solution $u \in \mathcal{C}^1(\Omega)$ of (1.3) is also a viscosity solution. If, vice versa, $u \in \mathcal{C}(\Omega)$ is a viscosity solution of (1.3) and if u is differentiable in x_0, then $H(x_0, Du(x_0)) = 0$.*

Proof. If $u \in \mathcal{C}^1(\Omega)$ is a classical solution, and $\varphi \in \mathcal{C}_0^\infty(\Omega)$ is such, that $u - \varphi$ has a local extremum in $x_0 \in \Omega$, then $Du(x_0) - D\varphi(x_0) = 0$, and therefore $H(x_0, D\varphi(x_0)) = H(x_0, Du(x_0)) = 0$.

If $u \in \mathcal{C}(\Omega)$ is a viscosity sub-solution of (1.3), and differentiable in x_0, then there exists a function $\varphi \in \mathcal{C}^1(\Omega)$, such that $u - \varphi$ attains a local maximum in x_0 (easy exercise, or see [Eva98, Lemma, p. 544]). In the maximum point x_0, we have $Du(x_0) = D\varphi(x_0)$, and by remark 1.8, $H(x_0, Du(x_0)) = H(x_0, D\varphi(x_0)) \leq 0$. Analogously one obtains for a super-solution u, which is differentiable in x_0, that $H(x_0, Du(x_0)) \geq 0$ □

If H is coercive, i.e. if condition (H3) is fulfilled, then every viscosity solution $u \in \mathcal{C}(\Omega)$ of (1.3) is also a weak solution of (1.3), as u is locally Lipschitz-continuous by lemma 1.9, and satisfies (1.3), where it is differentiable, according to theorem 1.10. In general the converse doesn't hold true, as the following example teaches.

Example 1.11: The function $u(x) = 1 - |x|$ is a viscosity solution of

$$|u'(x)| - 1 = 0 \quad \text{in} \quad]-1, 1[\ , \quad u(-1) = u(1) = 0 \tag{1.7}$$

Anyway, u is a classical solution on the subintervals $]-1, 0[$ and $]0, 1[$, and according to the last theorem satisfies $|u'(x)| - 1 = 0$ for $x \neq 0$ in the sense of definition 1.5.

Let $\varphi \in \mathcal{C}^1(]-1, 1[)$ and assume 0 to be a local maximum point of $u - \varphi$. Then there is a $\delta > 0$, such that for all x with $|x| < \delta$ the inequality

$$(u - \varphi)(x) \leq (u - \varphi)(0)$$

holds. Therefore, it holds for all $x \in]0, \delta[$, that

$$-\frac{\varphi(x) - \varphi(0)}{x} \leq 1$$

and for all $x \in]-\delta, 0[$, that

$$\frac{\varphi(x) - \varphi(0)}{x} \leq 1$$

Thus it is $|\varphi'(0)| \leq 1$, or equivalently $|\varphi'(0)| - 1 \leq 0$, and u is a viscosity sub-solution of (1.7). One can show in an analogous manner, that u is also a viscosity super-solution of (1.7).

The function $\tilde{u}(x) = \frac{1}{2} - \left|\frac{1}{2} - u(x)\right|$ is a weak solution of (1.7) by example 1.4, but no viscosity solution. To state a reason let $\delta \leq \frac{1}{2}$ in the argumentation above, such that $\tilde{u}(x) = |x|$ for $|x| < \delta$. If for some $\varphi \in \mathcal{C}^1(]-1,1[)$ the difference function $\tilde{u} - \varphi$ had a local maximum in 0, one would conclude $|\varphi'(0)| \geq 1$.

Viscosity solutions are not preserved under the change of sign in equation (1.3). If, for example, $u \in \mathcal{C}(\Omega)$ is a viscosity solution of (1.3), then $v = -u$ is a viscosity solution of $-H(x, -Dv(x)) = 0$, but in general u is not a viscosity solution of $-H(x, Du(x)) = 0$.

1.2.4. Lipschitz Continuity. In the next lemma it is shown, that under the assumptions on H, made in section 1.1, every viscosity sub-solution u of (1.3) is even Lipschitz-continuous. We also require the Lipschitz continuity of u for our uniqueness result.

LEMMA 1.12: *Let Ω be a Lipschitz domain and let $u \in \mathcal{C}(\Omega)$ be a viscosity sub-solution of (1.3), with conditions (H1)-(H3) being satisfied. Then u is Lipschitzian with its Lipschitz constant bounded by $\frac{\beta}{\alpha}C_\Omega$ with the constants from lemma 1.1 and lemma 1.3.*

PROOF. Because of lemma 1.9 u is at least local Lipschitz-continuous and therefore we have $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ (see [EG92, 4.2, Theorem 5]). Besides, u is differentiable a.e. with its derivative Du equal to its weak derivative a.e. ([EG92, 6.2, Theorem 1]). As u is a sub-solution, it is $H(x, Du(x)) \leq 0$ a.e., and thus $\|Du(x)\| \leq \frac{\beta}{\alpha}$ a.e. in Ω by lemma 1.1. Now we can follow the argumentation in the $\mathcal{C}^{0,1}$, $W^{1,\infty}$ embedding theorem ([Alt99, Theorem 8.5]). Let $u_\epsilon = \varphi_\epsilon * u$ denote the usual mollification of u with some Dirac sequence of functions $\varphi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ (where we define $u \equiv 0$ on $\mathbb{R}^d \setminus \Omega$). Let $x_1, x_2 \in \Omega$, and let ξ denote a curve joining x_1 and x_2 as in lemma 1.3. Then:

$$\begin{aligned} |u_\epsilon(x_1) - u_\epsilon(x_2)| &= \left|\int_0^1 (u_\epsilon \circ \xi)'(t)\, dt\right| \\ &\leq \int_0^1 \|Du_\epsilon(\xi(t))\| \cdot \left\|\dot{\xi}(t)\right\| \, dt \\ &\leq \max_{0 \leq t \leq 1} \|Du_\epsilon(\xi(t))\| \cdot C_\Omega \cdot \|x_1 - x_2\| \end{aligned}$$

If ϵ is small enough, we have for all $x \in \xi([0,1])$, that

$$\|Du_\epsilon(x)\| = \|D(\varphi_\epsilon * u)(x)\| = \|(\varphi_\epsilon * Du)(x)\| \leq \|Du\|_{L^\infty}.$$

As $u_\epsilon \to u$ in $L^p(\Omega)$ for every $p < \infty$, there is a subsequence $u_{\epsilon'}$ converging to u pointwise a.e., and we obtain for almost all $x_1, x_2 \in \Omega$, that

$$|u(x_1) - u(x_2)| \leq \frac{\beta}{\alpha} C_\Omega \cdot \|x_1 - x_2\|.$$

The assertion follows, as u is continuous on Ω. □

Particularly, every viscosity sub-solution of (1.3) is bounded, and possesses a continuous extension on $\overline{\Omega}$, that is $u \in \mathcal{C}^{0,1}(\overline{\Omega})$, provided that assumptions (H1)-(H3) are fulfilled.

1.2.5. Uniqueness and Stability. Finally, the uniqueness of viscosity solutions of the static Hamilton-Jacobi equation (1.1) is a simple consequence of the following comparison principle.

THEOREM 1.13 (Comparison Principle, [Ish87]): *Assume (H1)-(H4). Let $\Omega \subset \mathbb{R}^d$ be a Lipschtiz domain and $u, v \in \mathcal{C}(\overline{\Omega})$ be a viscosity sub-solution and viscosity super-solution of equation (1.3), respectively. Then, if $u \leq v$ on the boundary $\partial\Omega$, we have $u \leq v$ on $\overline{\Omega}$.*

PROOF. According to lemma 1.12, u is Lipschitz-continuous, and the corresponding result in [Ish87] is applicable. Note that $H(x, p)$ is uniformly continuous on $\overline{\Omega} \times \overline{B(0, R)}$ for every $R > 0$, hence the continuity assumption in [Ish87] is satisfied. By assumption (H4), the function $\varphi \equiv 0$ is a strict sub-solution of 1.3.

Another proof may be found in [BCD97, page 82]. Though it does not directly apply to our case, the argumentation therein is much more elaborate, and easy to follow. It is not difficult to transfer the result to our case, if the Lipschitz continuity of u is regarded, which holds due to our coercivity assumption. With that in mind, the continuity assumption on the Hamilton function in [BCD97] can be weakened. □

As a consequence, if $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ are viscosity solutions of the Dirichlet problem (1.1), where H satisfies (H1)-(H4), then $u_1 = u_2$. This result only holds true for bounded sets Ω. Consider for example the problem

$$\|Du(x)\| = 1 \text{ on } \mathbb{R} \setminus \{0\}, \qquad u(0) = 0,$$

which has the two viscosity solutions $u_1(x) = \|x\|$ and $u_2(x) = -\|x\|$ (which are even classical solutions). Also the compatibility assumption (H4) is necessary: The functions $u_{1,2}(x) = \pm(1 - \langle x, x \rangle)$ are viscosity solutions of

$$\|Du(x)\| = 2\|x\| \text{ on } \Omega, \quad u|_{\partial\Omega} = 0,$$

with $\Omega = B(0, 1)$.

The comparison principle stated above not only shows, that, if the Dirichlet problem has a viscosity solution u, it is actually unique, it also yields the continuous dependence of the solution on the boundary values as a simple corollary:

COROLLARY 1.14: *With the assumptions from the theorem let u and v be a viscosity sub-solution and super-solution, respectively. Then*

$$\sup_{\overline{\Omega}} (u - v)^+ \leq \sup_{\partial\Omega} (u - v)^+$$

PROOF. Apply theorem 1.13 to u and $\tilde{v} = v + \sup_{\partial\Omega}(u - v)^+$. □

If both u and v are viscosity solutions, we can interchange the roles of u and v in the above argumentation, and obtain:

COROLLARY 1.15 (Continuous Dependence on the Boundary Data): *Under the same assumptions let $u_i \in \mathcal{C}(\overline{\Omega})$ be viscosity solutions of*

$$H(x, Du_i(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g_i,$$

for $i = 1, 2$. Then

$$\sup_{\overline{\Omega}} |u_1 - u_2| \leq \sup_{\partial\Omega} |g_1 - g_2|$$

The following theorem shows the stability of viscosity solutions with respect to the locally uniform convergence of continuous functions.

THEOREM 1.16 (Stability, [**BCD97**, Prop. 2.2.2]): *For* $n \in \mathbb{N}$ *let* $u_n \in \mathcal{C}(\Omega)$ *denote a viscosity solution of*

$$H_n(x, Du_n(x)) \;=\; 0, \qquad x \in \Omega.$$

If $u_n \to u$, $H_n \to H$ *uniformly on compact subsets of* Ω *and* $\Omega \times \mathbb{R}^d$*, respectively, then* u *is a viscosity solution of* $H(x, Du(x)) = 0$.

PROOF. (From [**BCD97**].) According to remark 1.6, let x_0 be a strict local maximum point of $u - \varphi$, with some test function $\varphi \in \mathcal{C}_0^\infty(\Omega)$. There exists some $N \in \mathbb{N}$ and points $x_n \in \Omega$, such that $u_n - \varphi$ has a local maximum in x_n, for $n \geq N$, and $x_n \to x_0$ (see proposition 1.7). As u_n is a viscosity solution, we have $H_n(x_n, D\varphi(x_n)) \leq 0$, and because of

$$|H_n(x_n, D\varphi(x_n)) - H(x_0, D\varphi(x_0))| \;\leq$$
$$|H_n(x_n, D\varphi(x_n)) - H(x_n, D\varphi(x_n))| \;+\; |H(x_n, D\varphi(x_n)) - H(x_0, D\varphi(x_0))|,$$

we obtain $0 \geq H_n(x_n, D\varphi(x_n)) \;\to\; H(x_0, D\varphi(x_0))$. A similar argument proves, that u is also a super-solution. □

1.3. The Hopf-Lax Formula

Making certain assumptions on the Hamiltonian H, one can explicitly state an expression for the viscosity solution u of the Dirichlet problem (1.1). Similar formulas were proposed by Lax in [**Lax63**] and by Hopf in [**Hop65**], e.g. for the Cauchy problem

$$\partial_t u(x,t) + H(\partial_x u(x,t)) = 0 \quad \text{in} \quad \mathbb{R}^d \times [0, T]$$
$$u(x, 0) = u_0(x),$$

where the viscosity solution is given by the *formula of Lax*

$$u(x,t) = \inf_{y \in \mathbb{R}^d} \left\{ u_0(y) + tH^* \left(\frac{x-y}{t} \right) \right\},$$

provided that H is convex and coercive, and u_0 is a bounded, uniformly continuous function on $\mathbb{R}^d$. Here H^* denotes the convex conjugate of H, defined by

$$H^*(q) = \sup_{p \in \mathbb{R}^d} \left\{ \langle p, q \rangle - H(p) \right\}.$$

In the following, I derive a variational formulation of the viscosity solution of

$$H(x, Du(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g, \tag{1.8}$$

which I refer to as the *Hopf-Lax formula*, and show in this way the existence of viscosity solutions of the Dirichlet problem. While the formula and the existence theorem are also contained in [**Lio82**], its proof is omitted therein, as the author considers the result a "trivial extension" of the case, where $H(x, p) = \|p\| - f(x)$ (the Eikonal equation). Contrary to the approach in [**Lio82**], the whole discussion will be based on the convex support function, which will be introduced in the next subsection, and not on the convex conjugate of H, which bears several advantages. For example, the conjugate needn't be finite on $\Omega \times \mathbb{R}^d$. Moreover, a weaker duality correspondence between the Hamilton-Jacobi equation and the characterization by means of the support function can also be obtained with the help of convex separation theory (lemma 1.20). Viscosity sub-solutions can be concisely characterized by a variational inequality (lemma 1.23). The maximal sub-solution is also a super-solution, and therefore a viscosity solution (lemma 1.24). Final consequences are the Hopf-Lax formula, the existence theorem, and the compatibility condition for the boundary values, a necessary and sufficient condition for the solvability of the Dirichlet problem (1.8).

1.3.1. The Support Function of the Zero Level-Set. By definition 1.5, $u \in \mathcal{C}(\Omega)$ is a viscosity sub-solution of (1.3), if for every $x \in \Omega$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$, such that $u - \varphi$ has a local maximum at x, it holds that

$$p = D\varphi(x) \in \left\{ p \in \mathbb{R}^d ;\ H(x,p) \le 0 \right\},$$

that is $p = D\varphi(x)$ belongs to the zero level-set $\mathcal{Z}(x)$ of H at x. The set of such vectors p forms the *super-differential* $D^+u(x)$ ([BCD97, Lemma 2.1.7]), and we could express the sub-solution property equivalently as $D^+u(x) \subseteq \mathcal{Z}(x)$. The convex support function $\rho : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$\rho(x,q) = \max_{H(x,p)\le 0} \langle p, q\rangle = \max_{\{p\in\mathbb{R}^d ;\ H(x,p)\le 0\}} \langle p, q\rangle\,, \tag{1.9}$$

provides a description of the half-spaces, which contain the zero level-set of H. Note that by assumptions (H1)-(H4), the zero level sets of $p \mapsto H(x,p)$ are non-empty compact and convex sets, with an non-empty interior for $x \in \Omega$. Hence, for every $x \in \overline{\Omega}$, $q \in \mathbb{R}^d$ there exists a $p \in \mathbb{R}^d$ with $\rho(x,q) = \langle p, q\rangle$ and $H(x,p) = 0$. Moreover, $p \in \mathbb{R}^d$ belongs to the zero level-set of H at x, if and only if

$$\langle p, q\rangle - \rho(x,q) \le 0 \text{ for all } q \in \mathbb{R}^d.$$

(which follows from a convex separation theorem, compare lemma 1.20).

But before we come to the characterization of viscosity solutions by means of the support function in lemma 1.20, we collect some important (and maybe obvious) properties of ρ in the following two lemmas. At first, the dependence on the state variable is ignored, and the partial function $q \mapsto \rho(x,q)$ is examined for every $x \in \overline{\Omega}$.

LEMMA 1.17 ([BR06, Lemma 3]): *Let (H1)-(H4) be satisfied. Then for every $x \in \overline{\Omega}$, and $q, q_1, q_2 \in \mathbb{R}^d$, the following properties hold:*

1. $\rho(x,q) \ge 0$ *(non-negativity)*
2. $\rho(x,tq) = t\rho(x,q)$ *for all* $t \ge 0$ *(positive homogeneity)*
3. $\rho(x,q_1+q_2) \le \rho(x,q_1) + \rho(x,q_2)$ *(subadditivity)*

And if $H(x,0) < 0$, we have additionally,

4. $\rho(x,q) = 0 \Rightarrow q = 0$ *for all* $q \in \mathbb{R}^d$ *(definiteness)*

For every $x \in \overline{\Omega}$ and $q \in \mathbb{R}^d$, ρ fulfills the estimate

$$\rho(x,q) \ \le\ \frac{\beta}{\alpha}\,\|q\|$$

with the coercivity constants α, β from lemma 1.1.

PROOF. (From [BR06].) As $H(x,0) \le 0$, it is $\rho(x,q) \ge \langle 0, q\rangle = 0$. If even $H(x,0) < 0$, there exists a real number $\delta > 0$, such that $H(x,p) \le 0$ for all $p \in B(0,\delta)$. Consequently,

$$\rho(x,q) \ge \sup_{p\in B(0,\delta)} \langle p, q\rangle = \delta\,\|q\|$$

The homogeneity and subadditivity are immediately obtained from the definition of ρ. By lemma 1.1,

$$\left\{ p \in \mathbb{R}^d ;\ H(x,p) \le 0 \right\} \ \subseteq\ \left\{ p \in \mathbb{R}^d ;\ \alpha\,\|p\| \le \beta \right\}$$

and therefore $\rho(x,q) \le \max_{\alpha\|p\|\le\beta} \langle p, q\rangle = \frac{\beta}{\alpha}\,\|q\|$. □

If, for $x \in \overline{\Omega}$, the Hamilton function H is symmetric with respect to p, that is

$$H(x,p) = H(x,-p) \quad \forall p \in \mathbb{R}^d$$

then $q \mapsto \rho(x,q)$ is even homogeneous, and therefore defines a semi-norm on $\mathbb{R}^d$, and a norm, if $H(x,0) < 0$.

For every $x \in \Omega$ the partial function $q \mapsto \rho(x,q)$ is convex by properties (2) and (3), hence continuous on $\mathbb{R}^d$ ([Roc97, Corollary 10.1.1]). The continuity with respect to the state variable of $\rho(x,q)$ is to be investigated in the following lemma.

LEMMA 1.18 ([BR06, Lemma 3]): *Let (H1)-(H4) be satisfied. Then $\rho \in \mathcal{C}(\Omega \times \mathbb{R}^d)$. If even $H(x,0) < 0$ on $\overline{\Omega}$, then $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d)$.*

PROOF. (From [BR06].) For $x_0 \in \overline{\Omega}$, the continuity of ρ will be proved by separately showing its upper and lower semi-continuity. As it turns out, ρ is upper-semicontinuous on $\overline{\Omega}$, and lower semi-continuous in $x_0 \in \overline{\Omega}$, if $H(x_0,0) < 0$. Let $q \in \mathbb{R}^d$, $x_0 \in \overline{\Omega}$ and $(x_n)_{n\in\mathbb{N}} \subset \Omega$ be a sequence converging to x_0.

In order to prove the upper semi-continuity let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence, such that

$$\rho(x_{n_k}, q) = \langle p_{n_k}, q\rangle \to \limsup_{n\to\infty} \rho(x_n, q)$$

for $k \to \infty$, where p_{n_k} is a maximizing argument with $H(x_{n_k}, p_{n_k}) = 0$. As $(p_{n_k})_{k\in\mathbb{N}}$ is a bounded sequence by (H3), we can assume without loss of generality that there exists a $p_0 \in \mathbb{R}^d$ with $p_{n_k} \to p_0$ for $k \to \infty$. We obtain $H(x_0,p_0) = 0$ and therefore

$$\limsup_{n\to\infty} \rho(x_n, q) = \langle p_0, q\rangle \le \rho(x_0, q).$$

For the upper semi-continuity, we require $H(x_0,0) < 0$. There is a maximizing $p_0 \in \mathbb{R}^d$ with $\rho(x_0,q) = \langle p_0, q\rangle$ and $H(x_0,p_0) = 0$. We extract a subsequence, such that $\rho(x_{n_k}, q) \to \liminf_{n\to\infty} \rho(x_n,q)$ and below, construct a sequence $p_{n_k} \to p_0$ with $H(x_{n_k}, p_{n_k}) \le 0$. With it in hand we conclude

$$\liminf_{n\to\infty} \rho(x_n, q) = \lim_{k\to\infty} \rho(x_{n_k}, q) \ge \lim_{k\to\infty} \langle p_{n_k}, q\rangle = \langle p_0, q\rangle = \rho(x_0, q).$$

We can assume, that either always $H(x_{n_k}, p_0) \le 0$ or always $H(x_{n_k}, p_0) > 0$. In the first case we simply take $p_{n_k} = p_0$. In the second case, since $H(x_{n_k}, 0) < 0$, there is a $\lambda_{n_k} \in (0,1)$ with $H(x_{n_k}, \lambda_{n_k} p_0) = 0$ and we take $p_{n_k} = \lambda_{n_k} p_0$. We can assume that there exists a real number $\lambda_0 \in [0,1]$ with $\lambda_{n_k} \to \lambda_0$. Passing to the limit in

$$0 = H(x_{n_k}, \lambda_{n_k} p_0 + (1-\lambda_{n_k})0) \le \lambda_{n_k} H(x_{n_k}, p_0) + (1-\lambda_{n_k}) H(x_{n_k}, 0)$$

yields $0 \le (1-\lambda_0) H(x_0, 0)$ which, by $H(x_0,0) < 0$, implies $\lambda_0 = 1$ and $p_{n_k} \to p_0$.

The assertion follows from [Roc97, Theorem 10.7], where it is proved, that a function of two arguments, which is convex in the first, and continuous in the second argument, it jointly continuous in both arguments. ☐

Let me remark, that ρ may fail to be continuous in some boundary point $x \in \partial\Omega$, where $H(x,0) = 0$. Consider for example the function

$$H(x,p) = \max(|p| - 2, x \cdot (p-1))$$

on $\Omega \times \mathbb{R}$, where $\Omega = (0,1) \subset \mathbb{R}$. Conditions (H1)-(H4) are obviously satisfied, and denoting by $\mathcal{Z}(x) = \{p;\ H(x,p) \le 0\}$ the zero level-set of H at x, we have

$$\mathcal{Z}(x) = \begin{cases} [-2,2], & \text{for } x = 0 \\ [-2,1], & \text{for } x > 0 \end{cases}$$

Consequently, $\rho(0,1) = \max_{-2\le p\le 2} p = 2$, and $\rho(x,1) = 1$ for $x > 0$, and thus

$$1 = \liminf_{n\to\infty} \rho(1/n, 1) < \rho(0,1) = 2,$$

hence $x \mapsto \rho(x,1)$ fails to be lower semi-continuous in $x_0 = 0$.

If $H(x,0) < 0$ on $\overline{\Omega}$, then $\rho(x,q) > 0$ on $\overline{\Omega} \times S^{d-1}$ by lemma 1.17. Consequently, by homogeneity, $\rho(x,q)$ is also bounded from below by a multiple of $\|q\|$. For our convenience, let me summarize the estimates on ρ in the following corollary.

COROLLARY 1.19: *Assume (H1)-(H3) and $H(x, 0) < 0$ on $\overline{\Omega}$. Then there are numbers $\rho_*, \rho^* > 0$, such that*

$$\rho_* \|q\| \leq \rho(x, q) \leq \rho^* \|q\|$$

for all $x \in \overline{\Omega}$, $q \in \mathbb{R}^d$.

Denoting by $\mathcal{Z}(x)$ the zero level-set of H, ρ_* and ρ^* can be chosen as the radii of the largest disk with $B(0, \rho_*) \subseteq \mathcal{Z}(x)$ and the smallest disk with $\overline{B(0, \rho^*)} \supseteq \mathcal{Z}(x)$ for all $x \in \overline{\Omega}$, respectively. For the Eikonal equation, for example, where $H(x, p) = \|p\| - f(x)$, with $f_* \leq f(x) \leq f^*$ on Ω, it is $\rho(x, q) = f(x) \|q\|$, and $\rho_* = f_*$, $\rho^* = f^*$.

As initially mentioned, we use the convex support function in order to describe the zero level-sets of H. The following result makes use of a convex separation theorem. For the proof, we refer the reader to [Roc97].

LEMMA 1.20 ([BR06, Lemma 9]): *Assume (H1)-(H4). Then for all $x \in \Omega$, $p \in \mathbb{R}^d$*

$$\max_{\|q\|=1} \{\langle p, q\rangle - \rho(x, q)\} \underset{(\leq)}{\geq} 0 \quad \Leftrightarrow \quad H(x, p) \underset{(\leq)}{\geq} 0$$

PROOF. (In [BR06] the case is considered, where instead of (H4), we assume $H(x, 0) \leq 0$ on $\overline{\Omega}$. With this weaker compatibility condition only one direction "$\Rightarrow$" can be proved.) Let $x \in \Omega$, $p \in \mathbb{R}^d$ and let $\mathcal{Z}(x) = \{p \in \mathbb{R}^d\, ;\ H(x, p) \leq 0\}$ denote the zero level-set of $p \mapsto H(x, p)$. Because of [Roc97, Theorem 13.1], we have $p \in \mathcal{Z}(x)$ if and only if $\langle p, q\rangle \leq \rho(x, q)$ for all $q \in \mathbb{R}^d$ and $p \in \operatorname{int}(\mathcal{Z}(x))$ if and only if $\langle p, q\rangle < \rho(x, q)$ for every $q \in \mathbb{R}^d \setminus \{0\}$. □

As a simple consequence of lemma 1.20, u is a viscosity solution of (1.3), if and only if u is a viscosity solution of

$$\max_{\|q\|=1} \{\langle Du(x), q\rangle - \rho(x, q)\} = 0. \tag{1.10}$$

The advantage of this representation is, that we have resolved the Hamilton-Jacobi equation $H(x, Du)$ for Du, so to speak. Integrating this relation along a curve will yield the formula for u.

1.3.2. The Optical Distance.

From now on let Ω denote a Lipschitz domain. A viscosity sub-solution u of $H(x, Du(x)) \leq 0$ is Lipschitz-continuous by lemma 1.12, and therefore differentiable almost everywhere. By lemma 1.20, we have

$$\langle Du(x), q\rangle \leq \rho(x, q) \text{ for all } q \in \mathbb{R}^d$$

a.e. in Ω. Assume for a moment, that u is $\mathcal{C}^1$, such that the above inequality holds for all $x \in \Omega$. Then the integration along a curve $\xi \in \mathcal{C}^\infty([0, 1]; \Omega)$ yields

$$u(x) - u(y) = \int_0^1 \left\langle Du(\xi(t)), -\dot{\xi}(t)\right\rangle dt \leq \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt,$$

where $x = \xi(0)$ and $y = \xi(1)$. According to lemma 1.3, every two points $x, y \in \Omega$ can be joined by a smooth curve with its derivative bounded by a multiple of the distance $\|x - y\|$. The *optical distance* of x and y is defined by

$$\delta(x, y) = \inf \Big\{ \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt\, ; \\ \xi \in \mathcal{C}^\infty([0, 1]; \Omega) \text{ with } \xi(0) = x, \xi(1) = y \Big\}. \tag{1.11}$$

Based on lemma 1.20 it will be shown later on, that u is a viscosity sub-solution of (1.1), if and only if $u(x) - u(y) \leq \delta(x, y)$ for all $x, y \in \Omega$, which generalizes the above argument to not necessarily differentiable viscosity solutions. First, we collect some properties of $\delta : \Omega \times \Omega \to \mathbb{R}$ in the following lemma.

LEMMA 1.21: *Assume (H1)-(H4) and Ω a Lipschitz domain. Let $x, y, z \in \Omega$. Then*

(1) $\delta(x, y) \geq 0$ *(non-negativity)*
(2) $\delta(x, y) = 0 \Rightarrow x = y$ *(definiteness)*
(3) $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ *(triangle inequality)*

Additionally, we have

$$\delta(x, y) \leq \frac{\beta}{\alpha} C_\Omega \|x - y\|$$

with the constants from lemma 1.1 and lemma 1.3.

PROOF. As $\rho(x, q) \geq 0$ on $\Omega \times \mathbb{R}^d$, we have $\delta(x, y) \geq 0$ for all $x, y \in \Omega$. Since ρ is positively homogeneous, the value of the integral $L(\xi) = \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt$ in (1.11) is independent of the choice of parametrization of ξ, as long as the direction from x to y is preserved.

Let $x, y, z \in \Omega$ and let $\eta, \zeta \in \mathcal{C}^\infty([0, 1]; \Omega)$ be curves, which join x, z and z, y respectively. The curve $\xi(t)$ defined by $\xi(t) = \eta(t)$ for $0 \leq t \leq 1$, $\xi(t) = \zeta(t - 1)$ for $1 \leq t \leq 2$ joins x and y, and has the optical length $L(\xi) = L(\eta) + L(\zeta)$. Thus

$$\delta(x, y) \leq \delta(x, z) + \delta(z, y).$$

(In fact the compound curve $\xi = \eta \vee \zeta$ is not C^∞ on $[0, 2]$, but one could substitute ξ by the mollified curve $\xi_\epsilon = \varphi_\epsilon * \xi$, where ξ is extended to $\mathbb{R}$ by $\xi(t) = x$ for $t \leq 0$ and $\xi(t) = y$ for $t \geq 2$, and let $\epsilon \to 0$. We could also have defined δ with joining curves $\xi \in W^{1,\infty}[0, 1]$ instead of $\mathcal{C}^\infty$.)

If $x \neq y$, consider some neighborhood $B(x, \epsilon)$ of x with $y \notin B(x, \epsilon)$ and $\overline{B}(x, \epsilon) \subset \Omega$. Because of lemma 1.17 there is a real number $c > 0$ such that $\rho(x', q) \geq c$ for all $x' \in B(x, \epsilon)$ and $q \in \mathbb{R}^d$ with $\|q\| = 1$. Hence for every curve $\xi \in \mathcal{C}^\infty([0, 1]; \Omega)$ joining x and y, we have $L(\xi) \geq c\epsilon$ and thus $\delta(x, y) \geq c\epsilon$.

By lemma 1.3, there is a curve $\xi \in \mathcal{C}^\infty([0, 1]; \Omega)$ joining x and y, with its derivative bounded by $C_\Omega \|x - y\|$. From lemma 1.17, we finally get

$$\delta(x, y) \;\leq\; \frac{\beta}{\alpha} \inf_\xi \left\{ \int_0^1 \left\| \dot{\xi} \right\| dt \right\} \leq \frac{\beta}{\alpha} C_\Omega \|x - y\|$$

where the infimum is to be taken like in (1.11). Of course, if the line segment $\xi(t) = x + t(y - x)$, for $0 \leq t \leq 1$ lies entirely in Ω, we have

$$\delta(x, y) \;\leq\; \max_{0 \leq t \leq 1} \rho(\xi(t), x - y) \;\leq\; \frac{\beta}{\alpha} \cdot \|x - y\|.$$

□

In general, δ will not be a symmetric in its two arguments, unless H (and therefore ρ) is symmetric with respect to p for all $x \in \Omega$. Under the assumption of this symmetry, δ defines a metric on Ω.

Let $x, y, x', y' \in \Omega$. Then we have

$$\begin{aligned} \delta(x, y) - \delta(x', y') &= \delta(x, y) - \delta(x', y) + \delta(x', y) - \delta(x', y') \\ &\leq \delta(x, x') + \delta(y', y) \\ &\leq \frac{\beta}{\alpha} C_\Omega \left(\|x - x'\| + \|y - y'\| \right) \end{aligned}$$

Thus δ is Lipschitz-continuous, and may be extended on $\overline{\Omega} \times \overline{\Omega}$. This extension will also be denoted by δ in the following. By continuity, assertions (1), (3), and the estimate on $\delta(x, y)$ in the last lemma are valid for all $x, y, z \in \overline{\Omega}$. In view of (H4), the definiteness may fail for points x, y on the boundary $\partial\Omega$.

As the following example shows, δ is the difference metric obtained from ρ, if $H(x, p)$ does not depend on the state variable.

EXAMPLE 1.22: If $H(x,p) = H(p)$ does not depend on the state variable, then also $\rho(x,q) = \rho(q)$, and if the segment $[x,y] \subseteq \overline{\Omega}$, then $\delta(x,y) = \rho(x-y)$.

Defining $\xi(t) = x + t(y-x)$ shows, that

$$\delta(x,y) \le \int_0^1 \rho(-\dot{\xi}(t))\,dt = \int_0^1 \rho(x-y)\,dt = \rho(x-y).$$

On the other hand, if $\xi \in \mathcal{C}^\infty([0,1];\Omega)$ is some path, joining x and y in Ω, we infer from Jensen's inequality, that

$$\int_0^1 \rho(-\dot{\xi}(t))\,dt \ge \rho\Big(-\int_0^1 \dot{\xi}(t)\,dt\Big) = \rho(x-y),$$

as ρ is convex by lemma (1.17).

1.3.3. A Variational Inequality. Viscosity sub-solutions can be characterized by a variational inequality, based on the distance function, introduced in the last subsection. Factually, the inequality provides a more accurate Lipschitz bound for sub-solutions of Hamilton-Jacobi equations than the lemma 1.12. The characterization is one of the main results within this section, and will directly lead us to the Hopf-Lax formula and the existence theorem.

LEMMA 1.23: *Assume (H1)-(H4) and Ω a Lipschitz domain. Then $u \in \mathcal{C}(\Omega)$ is a viscosity sub-solution of (1.3), if and only if*

$$u(x) - u(y) \le \delta(x,y)$$

for all $x, y \in \Omega$.

PROOF. Let $u \in \mathcal{C}(\Omega)$ be a viscosity sub-solution. Then u is locally Lipschitz-continuous by lemma 1.9, and therefore $u \in W^{1,\infty}_{loc}(\Omega)$. Moreover, u is differentiable a.e., and its derivative Du equals its weak derivative (a.e.). From theorem 1.10 (consistency of viscosity solutions) and lemma 1.20, we infer

$$\langle Du(x), q\rangle \le \rho(x,q) \quad \forall q \in \mathbb{R}^d \tag{1.12}$$

almost everywhere in Ω.

Now let $x_0, y_0 \in \Omega$ and let $\alpha > 0$ be a positive real number. Then by the definition of δ, there is a curve $\xi \in \mathcal{C}^\infty([0,1];\Omega)$, joining x_0 and y_0, such that

$$\delta(x_0,y_0) \le \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\,dt \le \delta(x_0,y_0) + \alpha.$$

Next, let $(\varphi_\epsilon)_{\epsilon>0}$ be a Dirac sequence of functions $\varphi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, and let $u_\epsilon = \varphi_\epsilon * u$ denote the mollification of of u (where we set $u(x) = 0$ for $x \notin \Omega$). Let ϵ be small enough, such that $\operatorname{dist}(\xi([0,1]), \partial\Omega) > \epsilon$.

For $x \in \Omega_\epsilon$ (that is the set of all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \epsilon$), one obtains by partial integration,

$$\begin{aligned}\langle Du_\epsilon(x), q\rangle &= \int_\Omega \varphi_\epsilon(x-y)\,\langle Du(y), q\rangle\;dy \le \int_\Omega \varphi_\epsilon(x-y)\rho(y,q)\,dy \\ &\le \rho_\epsilon(x,q),\end{aligned}$$

where we used (1.12). Here ρ_ϵ denotes the mollification of ρ with respect to the state variable. It holds for $x \in \Omega_\epsilon$, that

$$\begin{aligned}\rho_\epsilon(x,q) - \rho(x,q) &= \int_\Omega \varphi_\epsilon(x-y)\,(\rho(y,q) - \rho(x,q))\;dy \\ &\le \sup_{y \in B(x,\epsilon)} (\rho(y,q) - \rho(x,q)).\end{aligned}$$

Consequently, by the continuity of ρ, we have for small enough ϵ

$$\rho_\epsilon(\xi(t), -\dot{\xi}(t)) - \rho(\xi(t), -\dot{\xi}(t)) \leq \alpha$$

for all $t \in [0,1]$. Hence,

$$\begin{aligned} u_\epsilon(x_0) - u_\epsilon(y_0) &= \int_0^1 \left\langle Du_\epsilon(\xi(t)), -\dot{\xi}(t) \right\rangle dt \\ &\leq \int_0^1 \rho_\epsilon(\xi(t), -\dot{\xi}(t))\, dt \\ &\leq \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt + \alpha \\ &\leq \delta(x_0, y_0) + 2\alpha, \end{aligned}$$

and $\epsilon \to 0$ yields $u(x_0) - u(y_0) \leq \delta(x_0, y_0) + 2\alpha$.

In order to prove the other direction, let $u \in \mathcal{C}(\Omega)$ fulfill

$$u(x) - u(y) \ \leq \ \delta(x, y)$$

for all $x, y \in \Omega$. We will now show, that u is a viscosity sub-solution of (1.3). For that purpose, let $x_0 \in \Omega$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be a function, such that $u - \varphi$ has a local maximum in x_0. Then there exists some neighborhood $B(x_0, r)$ of x_0 in Ω with

$$\varphi(x_0) - \varphi(x) \ \leq \ u(x_0) - u(x) \quad \forall x \in B(x_0, r)$$

For $q \in \mathbb{R}^d$ with $\|q\| = 1$, we consider the curve $\xi(t) = x_0 - tq$ with tangent $-q$. Since

$$u(x_0) - u(\xi(h)) \ \leq \ \delta(x_0, \xi(h)) \ \leq \ \int_0^h \rho(\xi(t), q)\, dt$$

for $h < r$ and as $\xi(h) \in B(x_0, r)$ for such h, we obtain

$$\varphi(\xi(0)) - \varphi(\xi(h)) \ \leq \ \int_0^h \rho(\xi(t), q)\, dt$$

Division by h on both sides and taking the limit $h \to 0$ yields

$$-\left\langle D\varphi(x_0), \dot{\xi}(0) \right\rangle \ \leq \ \rho(x_0, q)$$

because of the differentiability of $\varphi \circ \xi$ and because of the continuity of ρ (see lemma 1.18). Consequently, as q was chosen arbitrarily,

$$\max_{\|q\|=1} \{\langle D\varphi(x_0), q\rangle - \rho(x_0, q)\} \leq 0$$

and therefore $H(x_0, D\varphi(x_0)) \leq 0$ by lemma 1.20. □

In the proof of [**Lio82**, Theorem 5.1, part (iv)], Lions showed by a similar argument, that every sub-solution $u \in W^{1,\infty}(\Omega)$ of the Eikonal equation, that is $\|Du(x)\| \leq f(x)$ a.e., fulfills the estimate in lemma 1.23. However, Lions did not incorporate the Lipschitz continuity of u, shown in lemma 1.12. The proof of the sufficiency of the inequality was inspired by the proof of [**BCD97**, Proposition 3.2.8], where it is shown, that the value function of an infinite horizon optimal control problem is a viscosity solution of the associated Hamilton-Jacobi-Bellman equation $\lambda u + H(x, Du) = 0$.

1.3.4. The Maximal Sub-Solution. The next lemma shows, that a maximal sub-solution u of the Dirichlet problem, $H(x, Du) = 0$ on Ω, and $u = g$ on $\partial\Omega$, is already a viscosity solution.

LEMMA 1.24: *Assume (H1), and let $g \in \mathcal{C}(\partial\Omega)$. We denote by*

$$\mathcal{S} = \left\{v \in \mathcal{C}(\overline{\Omega})\,;\; H(x, Dv(x)) \leq 0 \textit{ in the viscosity sense, } v \leq g \textit{ on } \partial\Omega\right\}$$

the set of viscosity sub-solutions. If $u \in \mathcal{S}$ is such, that $u \geq v$ for all $v \in \mathcal{S}$, then u is also a viscosity super-solution, and therefore a viscosity solution of $H(x, Du) = 0$ with $u \leq g$ on the boundary.

PROOF. The argumentation in [BCD97, Proposition II.2.1 (c)] applies directly to our case. I will briefly sketch the idea. Let x_0 be a local minimum of $u - \varphi$, where $\varphi \in \mathcal{C}_0^\infty(\Omega)$, and suppose by contradiction, that $H(x_0, D\varphi(x_0)) < 0$. Then there is some $\delta_0 > 0$, such that

$$u(x_0) - \varphi(x_0) \leq u(x) - \varphi(x) \quad \text{for all } x \in \overline{B(x_0, \delta_0)} \subset \Omega.$$

Consider the function $w \in \mathcal{C}^\infty(\Omega)$, defined by

$$w(x) = \varphi(x) - \|x - x_0\|^2 + u(x_0) - \varphi(x_0) + \frac{1}{2}\delta^2,$$

where $0 < \delta < \delta_0$. One easily verifies, that $w(x_0) > u(x_0)$, and $w(x) < u(x)$ for all x, such that $\|x - x_0\| = \delta$. For small enough δ, it is shown by a local uniform continuity argument, that $H(x, Dw(x)) \leq 0$ on $B(x_0, \delta)$. Then

$$v(x) = \begin{cases} \max(u(x), w(x)), & \text{for } x \in B(x_0, \delta) \\ u(x), & \text{for } x \in \overline{\Omega} \setminus B(x_0, \delta) \end{cases}$$

defines a (continuous) sub-solution $v \in \mathcal{S}$ with $v(x_0) > u(x_0)$ in contradiction to the optimality of u. □

The last two lemmas lead to a formula for the viscosity solution of (1.1). Let $v \in \mathcal{C}(\overline{\Omega})$ denote a viscosity sub-solution of the Dirichlet problem with $v \leq g$ on the boundary. From lemma 1.23 we infer, that

$$v(x) \leq g(y) + \delta(x, y)$$

for all $y \in \partial\Omega$.

Now, let $u : \overline{\Omega} \to \mathbb{R}$ be defined by the *Hopf-Lax formula*,

$$u(x) = \min_{y \in \partial\Omega} \left\{g(y) + \delta(x, y)\right\}. \tag{1.13}$$

By construction, $u \leq g$ on $\partial\Omega$, and $v \leq u$ for every v as considered above. If u itself was a sub-solution, then, by the last lemma, it would actually be a viscosity solution of $H(x, Du) = 0$ with $u \leq g$ on $\partial\Omega$. To see this, let y_* denote a minimizing argument for some point $x \in \Omega$, that is $u(x) = g(y_*) + \delta(x, y_*)$. Then

$$u(z) - u(x) \leq \Big(g(y_*) + \delta(z, y_*)\Big) - \Big(g(y_*) + \delta(x, y_*)\Big) \leq \delta(z, x),$$

and accordingly, u fulfills the Lipschitz condition in lemma 1.23, and is indeed a sub-solution.

Moreover, the boundary condition is fulfilled, if and only if $u \geq g$ on $\partial\Omega$, that is

$$\min_{y \in \partial\Omega} \left\{g(y) + \delta(x, y)\right\} \geq g(x)$$

for all $x \in \partial\Omega$. This yields the following condition on g

$$g(x) - g(y) \leq \delta(x, y) \quad \forall x, y \in \partial\Omega, \tag{1.14}$$

which we refer to as the *compatibility condition* for the boundary data.

1.3.5. The Existence Theorem. As a final result in this section, we obtain the existence theorem for viscosity solutions of the Dirichlet problem. A similar result (without a rigorous proof) is [**Lio82**, Theorem 5.3].

THEOREM 1.25: *Assume (H1)-(H4), Ω a Lipschitz domain and $g \in \mathcal{C}(\partial\Omega)$. Then a Lipschitz-continuous viscosity solution of (1.3) with $u \leq g$ on $\partial\Omega$ is given by*

$$u(x) = \min_{y \in \partial\Omega} \{g(y) + \delta(x, y)\}, \quad x \in \overline{\Omega}.$$

The Dirichlet boundary condition $u = g$ on $\partial\Omega$ is fulfilled, if and only if the compatibility condition

$$g(x) - g(y) \leq \delta(x, y) \quad \forall x, y \in \partial\Omega$$

for g holds.

PROOF. Lemmas 1.23, 1.24 and the last subsection. □

REMARK 1.26: The existence of a viscosity solution can also be proved, if assumption (H4) is replaced by $H(x, 0) \leq 0$ on $\overline{\Omega}$. To see this, let u_ϵ denote the viscosity solution of $H(x, Du_\epsilon(x)) = \epsilon$ given by the Hopf-Lax formula. For some sequence $\epsilon \to 0$, the sequence of viscosity solutions (u_ϵ) is equi-continuous (by lemma 1.12), and uniformly bounded (a simple consequence of the Lipschitz-continuity). By the Arzelá-Ascoli theorem, (u_ϵ) contains a uniformly convergent subsequence. By the stability theorem 1.16, the limit function u is a viscosity solution of $H(x, Du(x)) = 0$. As one can show, also in this case, the viscosity solution u is given by the Hopf-Lax formula. However, $\rho(x, q)$ is then in general only upper semi-continuous on $\overline{\Omega} \times \mathbb{R}^d$, as the example after lemma 1.18 shows.

While we were able to state an expression for the viscosity solution u, the difficulties in solving the Dirichlet problem (1.1) seem to be only displaced, as it is not clear how $\delta(x, y)$ is to be computed. However, the Hopf-Lax formula will be used to derive a numerical approximation to the viscosity solution of (1.1) in the following chapter, by solving local, simplified problems.

1.3.6. Compatibility of the Boundary Data. From the last theorem we know, that (1.1) has a viscosity solution u, if the compatibility condition (1.14) is fulfilled. But even the converse holds true, as the following theorem shows.

THEOREM 1.27: *Assume (H1)-(H4), let Ω be a Lipschitz domain, and $g \in \mathcal{C}(\partial\Omega)$. The Dirichlet problem (1.1)*

$$H(x, Du(x)) = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega$$

has a viscosity solution $u \in \mathcal{C}(\overline{\Omega})$, if and only if g satisfies the compatibility condition (1.14).

PROOF. In view of theorem 1.25, only one direction is left to prove. Let $u \in \mathcal{C}(\overline{\Omega})$ denote a viscosity solution of (1.1). From lemma 1.23 we infer, that

$$u(x) \leq \min_{y \in \partial\Omega} \{g(y) + \delta(x, y)\}$$

for all $x \in \overline{\Omega}$, especially for $x \in \partial\Omega$, where $u(x) = g(x)$. □

A similar result is already contained in [**Lio82**, Theorem 5.3 (v)]. Lions showed the necessity and sufficiency of the compatibility condition for the existence of solutions $u \in W^{1,\infty}(\Omega)$ of the Eikonal equation, where $H(x, p) = \|p\| - f(x)$.

1.4. Examples, Computation of the Support Function

In this section, examples for Hamilton-Jacobi equations are discussed, and different ways to express the support function $\rho(x, q)$ are introduced.

1.4.1. Equations of Eikonal Type.

Consider the problem

$$H(x, Du(x)) = 0, \ x \in \Omega, \ \text{with } H(x,p) = F(x,p) - 1, \tag{1.15}$$

where $F(x,p)$ fulfills the following properties

(F1) $F \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d)$ *(Continuity)*

(F2) $p \mapsto F(x,p)$ is convex for every $x \in \overline{\Omega}$ *(Convexity)*

(F3) $F(x,tp) = tF(x,p)$ for all $x \in \overline{\Omega}$, $p \in \mathbb{R}^d$, $t > 0$ *(Homogeneity)*

(F4) $F(x,p) > 0$ for all $x \in \overline{\Omega}$, $p \neq 0$ *(Positivity)*

and $\Omega \subseteq \mathbb{R}^d$ denotes some open set. As easily seen, $H(x,p)$ defined by (1.15) fulfills the properties (H1)-(H4) and the existence and uniqueness results from the last sections are applicable.

A function $f : \mathbb{R}^d \to \mathbb{R}$, which is non-negative, positively homogeneous and convex with $f(0) = 0$ is called a *gauge*. So for fixed x in $\overline{\Omega}$ the function F is a gauge with respect to $p \in \mathbb{R}^d$. By assumptions (F2) and (F3),

$$F(x, p_1 + p_2) \ \leq \ F(x,p_1) + F(x,p_2)$$

for all $p_1, p_2 \in \mathbb{R}^d$. Consequently, provided that $F(x,p) = F(x,-p)$ for all $p \in \mathbb{R}^d$, $p \mapsto F(x,p)$ defines a norm on $\mathbb{R}^d$.

Let $\rho : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ denote the support function of the zero level-set of $H(x,p) = F(x,p) - 1$ with respect to p for every $x \in \overline{\Omega}$, as defined in equation (1.9). We have

$$\begin{aligned}\rho(x,q) \ &= \sup_{H(x,p)=0} \langle p, q \rangle \\ &= \sup_{F(x,p)=1} \langle p, q \rangle \\ &= \sup_{p \neq 0} \frac{\langle p, q \rangle}{F(x,p)}.\end{aligned}$$

So ρ is the polar function of F with respect to the second argument and thus F is the polar of ρ (see [Roc97, p. 128]):

$$F(x,p) = \sup_{q \neq 0} \frac{\langle p, q \rangle}{\rho(x,q)}, \tag{1.16}$$

and the Cauchy-Schwarz inequality

$$\langle p, q \rangle \ \leq \ F(x,p) \cdot \rho(x,q) \text{ for all } p, q \in \mathbb{R}^d$$

holds. Polar pairs of gauges qualify as the "best" pairs of functions fulfilling this inequality, in the sense, that the inequality wouldn't hold, if one function was replaced by a lesser one.

EXAMPLE 1.28: Consider the case, where $F(x,p) = \langle p, M(x)p \rangle^{1/2}$, $(x,p) \in \Omega \times \mathbb{R}^d$ and $M(x) \in \mathbb{R}^{d \times d}$ is a symmetric, positive definite matrix for every $x \in \overline{\Omega}$, that depends continuously on the state variable x. Let $R(x)^T R(x)$ denote the Cholesky factorization of $M(x)$. By the Cauchy-Schwarz inequality for the Euclidean norm, we have

$$\langle p, q \rangle = \left\langle R(x)p, R(x)^{-T} q \right\rangle \ \leq \ \|R(x)p\| \cdot \left\| R(x)^{-T} q \right\|,$$

with equality, if $p = M(x)^{-1} q$, and therefore $\rho(x,q) = \left\| R(x)^{-T} q \right\| = \left\langle q, M(x)^{-1} q \right\rangle^{1/2}$.

Given a Hamiltonian $H : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$, we can define a function $F(x,p)$ by (1.16). Under certain assumptions, the Hamilton-Jacobi equation (1.3) is then equivalent to a Hamilton-Jacobi equation of Eikonal type.

THEOREM 1.29: *Let Ω denote an open set, and $H : \overline{\Omega}\times\mathbb{R}^d \to \mathbb{R}$ fulfill the properties (H1)-(H3) and $H(x,0) < 0$ for all $x \in \overline{\Omega}$. Then $F(x,p)$ defined by (1.16) fulfills (F1)-(F4). Moreover $u \in \mathcal{C}(\Omega)$ is a viscosity solution of (1.3), if and only if it is a viscosity solution of*

$$F(x, Du(x)) = 1, \quad x \in \Omega. \tag{1.17}$$

PROOF. Let $x \in \overline{\Omega}$. By [Roc97, theorem 15.1], $p \mapsto F(x,p)$ is a gauge function, as $q \mapsto \rho(x,q)$ is gauge by lemma 1.17. As the supremum in (1.16) may be restricted to the compact set S^{d-1}, with $\rho(x,q) > 0$ for $q \in S^{d-1}$, condition (F4) is satisfied. As

$$|F(x,p) - F(y,p)| \le \|p\| \sup_{\|q\|=1} \left| \frac{1}{\rho(x,q)} - \frac{1}{\rho(y,q)} \right|$$

for all $x, y \in \overline{\Omega}$ and $p \in \mathbb{R}^d$, (F1) is fulfilled, as ρ is continuous on $\overline{\Omega}\times\mathbb{R}^d$ by lemma 1.18 with $\rho(x,q) > 0$ for all $x \in \overline{\Omega}$, $q \neq 0$.

From lemma 1.20 we infer, that $H(x,p)$ and $F(x,p)-1$ have the same sign, and the related Hamilton-Jacobi equations have thus the same viscosity solutions. □

In theorem 1.29, F may also be obtained as the gauge function generated from the convex level-set $\{p\,;\ H(x,p) \le 0\}$ (while ρ is the convex support function of this level-set). This follows from the theorems 14.5 and 15.1 in [Roc97]. The gauge function $f : \mathbb{R}^d \to \mathbb{R}$ generated from a convex set C containing the origin as an interior point, is defined to be

$$f(p) = \inf\{\lambda \ge 0\,;\ p \in \lambda C\}\,.$$

If C is a symmetric, bounded and convex set containing the origin as an interior point, then the gauge generated from C is a norm having C as its unit circle.

1.4.2. The Eikonal Equation. Let $k : \mathbb{R}^d \to \mathbb{R}$ denote some norm on $\mathbb{R}^d$. For the Eikonal equation

$$k(Du(x)) = f(x) \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{1.18}$$

where $f : \overline{\Omega} \to \mathbb{R}$ is continuous and positive, we have $\rho(x,q) = f(x)k^\circ(q)$, where k° denotes the polar of k. The polar of a norm is a norm itself, and the Cauchy-Schwarz inequality becomes

$$|\langle p, q\rangle| \ \le\ k(p)\cdot k^\circ(q),$$

by the symmetry of both k and k°. With

$$\delta(x,y) = \inf\Big\{ \int_0^T f(\xi(t))\,dt\,; T > 0 \text{ and } \xi \in \mathcal{C}^\infty([0,T];\Omega) \text{ with } \xi(0) = x,\ \xi(T) = y,\ k^\circ\left(\dot{\xi}\right) \le 1 \text{ on } [0,T]\Big\}$$

the unique viscosity solution of (1.18) is given by (1.13), provided that g is compatible.

If Ω is a convex set, and $f(x) \equiv 1$ is a constant, then $\delta(x,y) = k^\circ(x-y)$ by example 1.22. Then the viscosity solution of (1.18) is given by

$$u(x) = \min_{y\in\partial\Omega}\{g(y) + k^\circ(x-y)\}\,.$$

This relation holds as well for points x of a non-convex set Ω, where the segments joining x to an arbitrary point $y \in \partial\Omega$ are contained in $\overline{\Omega}$.

1.4.3. Exit-Time Optimal Control Problems. In the section 1.3, we proved the existence of a viscosity solution of the Dirichlet problem (1.1) by explicitly stating an expression for it. The solution u defined by

$$u(x) = \inf_{y\in\partial\Omega} \{g(y) + \delta(x, y)\}$$

may be interpreted as the value function of some optimal control problem. For that purpose, consider the set

$$\mathcal{A} = \{\alpha : [0, \infty[\to A\,;\ \alpha \text{ measurable}\}$$

of admissible controls, where $A = S^{d-1}$. For $x \in \overline{\Omega}$ and $\alpha \in \mathcal{A}$ we define $y_x(t, \alpha)$ to be the solution of the state equation

$$\dot{y}(t) = -\alpha(t) \quad \text{for } t > 0, \quad y(0) = x$$

that is $y_x(t, \alpha) = x - \int_0^t \alpha(s)\, ds$ and $y_x(\cdot, \alpha)$ fulfills the state equation almost everywhere. We denote by $t_x(\alpha)$ the first exit-time of $y_x(\cdot, \alpha)$ from Ω

$$t_x(\alpha) = \inf \{t \geq 0\,;\ y_x(t, \alpha) \in \partial\Omega\}$$

(where the infimum over the empty set is $+\infty$). One can think of g as the terminal cost and ρ as the running cost of an optimal control problem and define the cost functional by

$$J(x, \alpha) = \int_0^{t_x(\alpha)} \rho(y_x(s), \alpha(s))\, ds \quad + \quad g(y_x(t_x(\alpha)))$$

if $t_x(\alpha) < \infty$, and $J(x, \alpha) = +\infty$ otherwise. Here for simplicity of notation, y_x denotes the trajectory for the control α. We then have

$$u(x) = \inf_{\alpha\in\mathcal{A}} J(x, \alpha) \tag{1.19}$$

for every $x \in \Omega$. Additionally, if v is defined by (1.19) on $\overline{\Omega}$, we have $v = u$ in Ω and $v = g$ on the boundary $\partial\Omega$. Hence, by theorem 1.25, v is continuous on $\overline{\Omega}$, if and only if the compatibility condition (1.14) holds for the boundary data. Furthermore, u is the viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\max_{a\in A} \{\langle a, Du(x)\rangle - \rho(x, a)\} = 0$$

in Ω, as proved in theorem 1.25.

Next, I will show, that the value function of an exit-time optimal control problem coincides with the solution given by the Hopf-Lax formula of the associated Hamilton-Jacobi-Bellmann equation. For that purpose, let A denote a compact subset of $\mathbb{R}^m$, and

$$\mathcal{A} = \{\alpha : [0, \infty[\to A\,;\ \alpha \text{ measurable}\}$$

the set of admissible controls. Let $f : \mathbb{R}^d \times A \to \mathbb{R}^d$ and $l : \mathbb{R}^d \times A \to \mathbb{R}$ be continuous functions, such that, for all $x, y \in \mathbb{R}^d$, $a \in A$:

$$\|f(x, a) - f(y, a)\| \leq L_f\, \|x - y\| \tag{1.20}$$

$$|l(x, a) - l(y, a)| \leq \omega_l(\|x - y\|) \tag{1.21}$$

with a Lipschitz constant $L_f > 0$ and a modulus of continuity ω_l, and assume further, that

$$1 \leq l(x, a) \leq M \text{ with some } M > 0. \tag{1.22}$$

Let $g \in \mathcal{C}(\partial\Omega)$ with $g \geq 0$ denote the terminal cost, where Ω is a Lipschitz domain ($\mathbb{R}^d \setminus \Omega$ is our target set). Let me remark, that the rather strong assumptions were chosen in order to match the requirements in [BCD97]. It would suffice, if l and f were defined on $\overline{\Omega} \times \mathbb{R}^d$, with the above assumptions. The more general results in [BCD97] include cases, where the computational domain is unbounded.

We denote by $y_x(t, \alpha)$ the solution to the state equation

$$\dot{y}(t) = f(y(t), \alpha(t)) \quad \text{for } t > 0, \quad y(0) = x \tag{1.23}$$

for some given $\alpha \in \mathcal{A}$ and $x \in \mathbb{R}^d$, in the sense that $y_x(\cdot, \alpha)$ fulfills the integral equation

$$y(t) = x + \int_0^t f(y(s), \alpha(s))\, ds,$$

and solves (1.23) almost everywhere (by [BCD97, theorem III.5.5], the integral equation has a unique solution $y_x(t, \alpha)$ on $[0, \infty[$, which is, of course, absolutely continuous). We will further assume, that there is a number $r > 0$, such that

$$\max_{a \in A} \langle -f(x, a), p\rangle \;\geq\; r \quad \text{for all } x \in \mathbb{R}^d,\; p \in S^{d-1} \tag{1.24}$$

(a controllability assumption, which ensures, that from every point $x \in \Omega$ we can reach the boundary with some trajectory $y_x(t, \alpha)$ under an appropriate control. Of course, (f, A) is small-time controllable in the sense of [BCD97, definition IV.1.1]).

As easily seen, the Hamilton function defined by

$$H(x, p) = \max_{a \in A} \{\langle -f(x, a), p\rangle - l(x, a)\} \tag{1.25}$$

fulfills the properties (H1)-(H4) on page 1. With the cost functional

$$J(x, \alpha) \;=\; \int_0^{t_x(\alpha)} l(y_x(s, \alpha), \alpha(s))\, ds \;+\; g(y_x(t_x(\alpha), \alpha)),$$

let the value function v be defined by

$$v(x) \;=\; \inf_{\alpha \in \mathcal{A}} J(x, \alpha), \quad x \in \overline{\Omega}.$$

We obtain the following lemma.

LEMMA 1.30: *Assume (1.20)-(1.24) and $g \geq 0$. If g fulfills the compatibility condition (1.14), then the value function v is the unique viscosity solution of*

$$H(x, Dv(x)) = 0, \quad x \in \Omega, \qquad v|_{\partial\Omega} = g, \tag{1.26}$$

and hence coincides with the solution u given by the Hopf-Lax formula (1.13).

PROOF. We have $u \leq v$ by [BCD97, theorem IV.4.2] and $u \geq v$ by [BCD97, theorem IV.4.1]. For the application of the quoted results, let me remark, that u can be extended to a Lipschitz-continuous function on $\mathbb{R}^d$ by

$$\overline{u}(x) = \min_{y \in \partial\Omega} \left\{ g(y) + C_\Omega \frac{\beta}{\alpha} \cdot \|x - y\| \right\}, \quad x \in \mathbb{R}^d \setminus \Omega,$$

by the estimate on δ in lemma 1.21. The uniqueness follows from theorem 1.13. □

Particularly, all points $x \in \Omega$ are reachable by the backward system $\dot{y} = -f(y, \alpha)$ from the boundary $\partial\Omega$, and v is Lipschitz-continuous on $\overline{\Omega}$.

The Hamilton-Jacobi equation (1.26) with H being defined in (1.25) is equivalent to the equation $F(x, Dv(x)) = 1$ with

$$F(x, p) = \max_{a \in A} \left\{ \left\langle -l(x, a)^{-1} f(x, a), p \right\rangle \right\},$$

where F has the properties (F1)-(F4). Let

$$S_x = \left\{ -l(x, a)^{-1} f(x, a)\, ;\; a \in A \right\}$$

denote the *speed profile* at x. Then the following lemma holds.

LEMMA 1.31: *Under the above assumptions let $\rho(x, q)$ be defined by (1.9). Then $q \mapsto \rho(x, q)$ is the gauge function generated from $\mathrm{co} S_x$, that is*

$$\rho(x, q) = \inf \{\lambda \geq 0\, ;\; q \in \lambda\, \mathrm{co} S_x\}.$$

PROOF. The convex level-set

$$\{p \in \mathbb{R}^d;\ H(x,p) \le 0\} = \{p \in \mathbb{R}^d;\ F(x,p) \le 1\}$$

is the polar set of S_x as well as the polar set of coS_x. As S_x is compact, coS_x is convex and compact by [Roc97, theorem 17.2]. By (1.24), there is a $q \in S_x$ with $\langle q,p\rangle > 0$ for every $p \in \mathbb{R}^d \setminus \{0\}$. Consequently, $0 \in \text{int}(\text{co}S_x)$ (by convex separation theorems). The assertion follows from [Roc97, theorem 14.5]. □

So if coS_x is a symmetric set, $q \mapsto \rho(x,q)$ is the norm, that has coS_x as its unit circle. The minimal cost function v or the viscosity solution u of (1.26) depends on the shape of the speed profile S_x, rather than on the exact definition of f and l. The methods described herein for approximating viscosity solutions of Hamilton-Jacobi equations can also be used, to obtain the value function of an exit-time optimal control problem. In optimal control theory, often a different approach is chosen, where the state equation (1.23) is discretized explicitly, e.g. by Runge-Kutta methods.

The subsection 4.1.3 contains an example of an exit-time optimal control problem, where the support function is calculated with the help of lemma 1.31.

1.4.4. The Legendre Transformation. Let the Hamilton function $H(x,p)$ fulfill the properties (H1)-(H4). For every $x \in \overline{\Omega}$, we consider the convex conjugate of $p \mapsto H(x,p)$, that is, the function

$$L(x,q) = \sup_{p\in\mathbb{R}^d} \{\langle p,q\rangle - H(x,p)\}\,, \tag{1.27}$$

which provides a description of the closed half-spaces containing the epigraph of $p \mapsto H(x,p)$, and possibly takes the value $+\infty$, where the supremum is unbounded. L fulfills the following properties:

(1) $q \mapsto L(x,q)$ is a closed convex function for every x
(2) $p \mapsto H(x,p)$ is the convex conjugate of $q \mapsto L(x,q)$
(3) $L(x,q) \le \beta$, if $\|q\| \le \alpha$ with the coercivity constants from lemma 1.1
(4) $L(x,q)/\|q\| \to \infty$, as $\|q\| \to \infty$, uniformly in x
(5) $L(x,0) > 0$ for all $x \in \Omega$

PROOF. [Roc97, Theorem 12.2] shows (1) and (2). (3), (5) are trivial. (4) is shown in [Eva98, Theorem 3.3.3]. □

Moreover, the support function $\rho(x,q)$ of the zero level-set, defined in (1.9), can be obtained as the positively homogeneous function generated from L, that is,

$$\rho(x,q) = \inf_{\tau>0} \tau \cdot L(x, \tau^{-1}\cdot q),$$

by [Roc97, Theorem 13.5]. The convex conjugate of H may be obtained as the Legendre transformate of H. Let us assume, that $p \mapsto H(x,p)$ is differentiable for some $x \in \overline{\Omega}$, and let $D \subseteq \mathbb{R}^d$ denote the range of $D_pH(x,p)$. For some $q \in D$ let $p(q)$ be such, that $D_pH(x,p(q)) = q$. Then, by [Roc97, Theorem 26.4], it is

$$L(x,q) = \langle p(q), q\rangle - H(x,p(q)).$$

Let me remark, that the choice of $p(q)$, such that $D_pH(x,p(q)) = q$ does not have an influence on the value of the right hand side in the last equation (compare [Roc97, Theorem 23.5]), and hence the Legendre transformate is well-defined.

Provided, that even $H(x,p)/\|p\| \to \infty$ for some x, then $\langle p,q\rangle - H(x,p)$ is bounded from above for every $q \in \mathbb{R}^d$, and the supremum in (1.27) is attained in some $p = p(q) \in \mathbb{R}^d$. Thus, as the derivative must vanish, we obtain $q = D_pH(x,p(q))$. I summarize the results in the following lemma.

LEMMA 1.32: *Let* $H \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}^d)$ *fulfill* $H(x,0) < 0$ *on* $\overline{\Omega}$*, and let* $D_p^2 H(x,p)$ *be positive definite for all* (x,p)*. We assume additionally, that* $H(x,p) / \|p\| \to \infty$*, as* $\|p\| \to \infty$*, uniformly in* $x \in \overline{\Omega}$*. Then the Legendre transformate* $L(x,q)$ *of* H *exists on* $\overline{\Omega} \times \mathbb{R}^d$*, and is continuously differentiable. Moreover, the support function* $\rho(x,q)$*, defined by (1.9), can be obtained as the positively homogeneous function generated from* L*,*

$$\rho(x,q) = \min_{\tau > 0} \tau \cdot L(x, \tau^{-1} \cdot q),$$

with the minimum being attained at some $\tau > 0$*, where*

$$\tau \cdot L(x, \tau^{-1} \cdot q) = \left\langle D_q L(x, \tau^{-1} \cdot q), q \right\rangle .$$

PROOF. The existence of the Legendre transformate was shown above, and its differentiability is a consequence of the implicit function theorem. The minimum of $f(\tau) = \tau \cdot L(x, \tau^{-1} \cdot q)$ is attained at some $\tau > 0$, as

$$\lim_{\tau \to 0+} f(\tau) = \lim_{\tau \to \infty} f(\tau) = +\infty$$

by properties (4) and (5) of the Legendre transformate. At the minimal point, we have $f'(\tau) = 0$, which yields the last equation. $\square$

CHAPTER 2

Discretization by Linear Finite Elements

In this chapter, we develop a discretization of the Dirichlet problem (1.1)

$$H(x, Du(x)) = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = g$$

by linear finite-elements, based on the Hopf-Lax formula, which has been introduced in section 1.3. The idea to construct the finite-element solution u_h on a triangular mesh Σ_h is to locally solve simplified problems, obtained by freezing the state dependency of H. This leads to a fixed-point equation for the discrete solution. I show, that the solution is well-defined, and fulfills a discrete version of the comparison principle (theorem 1.13). Stability and consistency are considered, and a restrictive compatibility condition on the boundary data enables us to show, that the sequence of finite-element solutions (u_h) on refined triangulations is Lipschitz-continuous, uniformly with respect to the grid-spacing $h \to 0$.

In the last two sections, we prove a-priori estimates on the error $\|u - u_h\|_\infty$. First we consider the case of a classical solution $u \in \mathcal{C}^2(\overline{\Omega})$, where the estimate is obtained by tracing the propagation of the local error in the discretization. Second, in section 2.5, we follow an approach recently made by Deckelnick and Elliott in [DE04], who used a maximum principle argument in order to proof an a-priori estimate for the Eikonal equation $H(Du) = f(x)$ on Cartesian meshes. Here, their approach is adopted for the finite-element discretization of Hamilton-Jacobi equations.

2.1. Linear Finite Elements

In this section, we briefly recall the notion of linear finite elements, and introduce two necessary regularity assumptions on the triangulation. An estimate on the interpolatory error is given.

2.1.1. Triangulation. Let $\Omega \subset \mathbb{R}^d$ be an open polytope. A *(simplicial) admissible triangulation* of Ω is a set Σ of d-dimensional closed simplices, that form a subdivision of Ω, such that any face of any simplex $\sigma \in \Sigma$ is either a subset of $\partial\Omega$, or a face of another simplex $\tau \in \Sigma$. If σ and τ share a common face, they are called *adjacent*. For every simplex $\sigma \in \Sigma$, we denote by $\mathrm{diam}(\sigma)$ its diameter, and by $\mathrm{hmin}(\sigma)$ its minimal altitude, that is the minimum of the distances of the vertices from their opposite face in σ.

For a sequence $h \to 0$, a family of admissible triangulations (Σ_h) is called *regular*, if

$$1 \leq \frac{\mathrm{diam}(\sigma_h)}{\mathrm{hmin}(\sigma_h)} \leq \theta \qquad \forall \sigma_h \in \Sigma_h \tag{2.1}$$

with some constant $\theta > 0$ and

$$\max_{\sigma_h \in \Sigma_h} \mathrm{diam}(\sigma_h) \leq h. \tag{2.2}$$

The family (Σ_h) is called *uniform*, if (2.2) holds, and, with some constant $\theta > 0$,

$$\min_{\sigma_h \in \Sigma_h} \mathrm{hmin}(\sigma_h) \geq h/\theta. \tag{2.3}$$

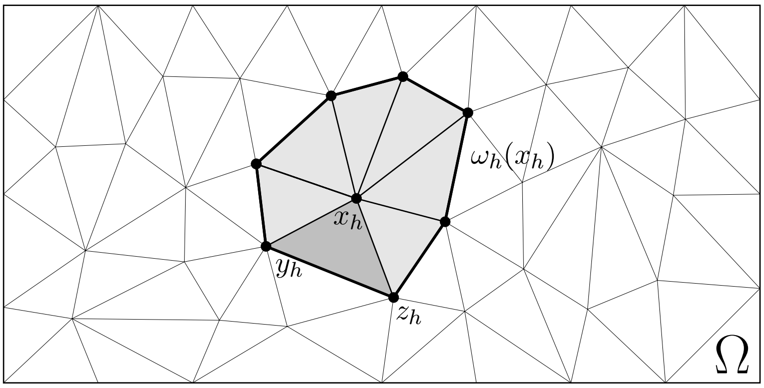

FIGURE 1. The neighborhood-patch $\omega_h(x_h)$ of some grid-point $x_h \in \Omega_h$ is the union of the adjacent triangles.

We denote the nodal points (vertices) of Σ_h, that belong to $\overline{\Omega}$, Ω, $\partial\Omega$ by $\overline{\Omega}_h$, Ω_h and $\partial\Omega_h$, respectively. For some $x_h \in \Omega_h$, we define $\omega_h(x_h)$ to be the union of all simplices $\sigma_h \in \Sigma_h$, that share x_h as a vertex. Finally, by $\mathcal{N}(x_h)$, we denote the set of all neighbors of x_h in $\overline{\Omega}_h$.

2.1.2. Linear Interpolation. The space of linear finite elements on Σ_h, that is, the continuous functions that are affine if restricted to a simplex $\sigma \in \Sigma_h$, is denoted by $\mathcal{V}_h$ and

$$I_h : \mathcal{C}(\overline{\Omega}) \to \mathcal{V}_h$$

is the corresponding nodal interpolation operator. Note that a finite-element function $u_h \in \mathcal{V}_h$ is uniquely determined by its nodal values, that is, the values $u_h(x_h)$ for all $x_h \in \overline{\Omega}_h$.

For some $q \in \mathbb{R}^d$, we denote the *directional derivative* of a (differentiable) function $f : \mathbb{R}^d \to \mathbb{R}$ at x in direction q by

$$\partial f(x;q) = \lim_{t\to 0^+} \frac{f(x+tq)-f(x)}{t}$$

Also a finite-element function $u_h \in \mathcal{V}_h$ possesses a directional derivative at every $x \in \Omega$, particularly in every grid-point $x_h \in \Omega_h$. An estimate on the interpolatory error for a smooth function φ is given in the following lemma, where also the error in the directional derivative is considered.

LEMMA 2.1: *Assume (2.1) and (2.2), $\varphi \in \mathcal{C}^2(\overline{\Omega})$ and let $\varphi_h = I_h\varphi$ denote its interpolant. Then*

$$\|\varphi - \varphi_h\|_\infty \le \frac{1}{2}\left\|D^2\varphi\right\|_\infty h^2,$$

and for every $x_h \in \Omega_h$, the error in the directional derivatives is bounded by

$$\max_{q\in S^{N-1}} |\langle D\varphi(x_h), q\rangle - \partial\varphi_h(x_h;q)| \le \frac{\theta}{2}\left\|D^2\varphi\right\|_\infty h.$$

Here $\left\|D^2\varphi\right\|_\infty$ means $\sup_{x\in\Omega}\max_{\|q\|=1}\left\|D^2\varphi(x)\cdot q\right\|$ with the Hessian matrix $D^2\varphi(x)$.

PROOF. Let $x \in \overline{\Omega}$. Then $x \in \sigma_h$ for some $\sigma_h \in \Sigma_h$, and there are numbers $0 \le \lambda_0, \dots, \lambda_d \le 1$, such that $\sum_{i=0}^d \lambda_i = 1$ and

$$x = \sum_{i=0}^{d} \lambda_i x_i, \tag{2.4}$$

where $\{x_0, \ldots, x_d\}$ is the set of vertices of σ_h. We have by Taylor expansion,

$$\begin{aligned}\varphi_h(x) - \varphi(x) &= \sum_{i=0}^{d} \lambda_i(\varphi(x_i) - \varphi(x)) \\ &= \sum_{i=0}^{d} \lambda_i \Big(D\varphi(x)(x_i - x) + \frac{1}{2} \left\langle x_i - x, D^2\varphi(\xi_i)(x_i - x) \right\rangle \Big) \\ &= \frac{1}{2} \sum_{i=0}^{d} \lambda_i \left\langle x_i - x, D^2\varphi(\xi_i)(x_i - x) \right\rangle .\end{aligned}$$

with some points $\xi_i \in \sigma_h$. Consequently, as $\operatorname{diam}(\sigma_h) \le h$ by (2.2), we obtain

$$|\varphi_h(x) - \varphi(x)| \le \frac{1}{2} \left\| D^2\varphi \right\|_\infty h^2$$

Let $q \in S^{d-1}$. As φ_h is affine on every simplex, we have

$$\frac{\varphi_h(x_h + tq) - \varphi_h(x_h)}{t} = \partial\varphi_h(x_h; q)$$

for sufficiently small t, as $x(t) = x_h + tq \in \sigma_h$ as $t \to 0$ for some $\sigma_h \in \Sigma_h$. In detail, if $t \le \operatorname{hmin}(\sigma_h)$, then $x(t) \in \sigma_h$. By (2.1), we can choose $t = \operatorname{diam}(\sigma_h)/\theta$. We obtain with $x = x(t) = x_h + tq$, expressed in barycentric coordinates as in (2.4),

$$\begin{aligned}\varphi_h(x) - \varphi_h(x_h) &= \sum_{i=0}^{d} \lambda_i(\varphi(x_i) - \varphi(x_h)) \\ &= \sum_{i=0}^{d} \lambda_i \Big(D\varphi(x_h)(x_i - x_h) + \frac{1}{2} \left\langle (x_i - x_h), D^2\varphi(\xi_i)(x_i - x_h) \right\rangle \Big)\end{aligned}$$

and thus

$$|\varphi_h(x_h + tq) - \varphi_h(x_h) - tD\varphi(x_h)q| \le \frac{1}{2} \left\| D^2\varphi \right\|_\infty \operatorname{diam}(\sigma_h)^2,$$

and division by $t = \operatorname{diam}(\sigma_h)/\theta$ yields the assertion. □

2.2. Discretization by Linear Finite Elements

In this section, I will motivate the discretization, by considering local simplified problems, which are obtained by freezing the state dependency of the Hamilton function. Then the Hopf-Lax formula is applied on every neighborhood-patch, and every grid-point x_h is assigned a value depending on the values in the neighboring grid-points, which leads to a fixed-point equation for the discrete solution. It is further shown, that this fixed-point problem admits a unique solution, the *finite-element solution* of the underlying Hamilton-Jacobi equation.

2.2.1. Motivation. The finite-element discretization which I consider is motivated by the idea of *local solutions*: At every grid-point $x_h \in \Omega_h$ the finite-element solution $u_h \in \mathcal{V}_h$ is assigned the value $v(x_h)$ of the *exact* viscosity solution $v \in C^{0,1}(\omega_h(x_h))$ that solves a simplified Hamilton-Jacobi equation on $\omega_h(x_h)$ subject to the boundary conditions $v|_{\partial\omega_h(x_h)} = u_h|_{\partial\omega_h(x_h)}$.

A good candidate for such a simplification of the Hamilton-Jacobi equation (1.1) is obtained by freezing locally the dependence of H on its first variable. This way $v \in C^{0,1}(\omega_h(x_h))$ is obtained as the viscosity solution of the *local Dirichlet problem*

(2.5) $$H(x_h, Dv(x)) = 0 \qquad x \in \operatorname{int}(\omega(x_h)), \qquad v|_{\partial\omega_h(x_h)} = u_h|_{\partial\omega_h(x_h)}.$$

Example 1.22 and theorem 1.25 show, that the viscosity solution of (2.5) is given by

$$v(x) = \min_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x - y)\}$$

if $u_h|_{\partial\omega_h(x_h)}$ is compatible. Under the assumptions (H1)-(H4), the Hopf-Lax formula is well-defined independently of the compatibility of the boundary values of the local Dirichlet problem (2.5). Thus we can define

$$u_h(x_h) = \min_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}\,. \tag{2.6}$$

The questions, that now arise, are whether there exists a finite-element function u_h, which fulfills (2.6) for all $x_h \in \Omega_h$ along with the boundary condition on $\partial\Omega_h$, and if so, how good it approximates the viscosity solution of (1.1).

2.2.2. The Hopf-Lax Discretization. In view of the last subsection, we define the discrete sub- and super-solutions of the Hamilton-Jacobi equation.

DEFINITION 2.2: *The function $\Lambda_h : \mathcal{V}_h \to \mathcal{V}_h$, defined by*

$$(\Lambda_h u_h)(x_h) = \begin{cases} \min\limits_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\} & x_h \in \Omega_h \\ u_h(x_h) & x_h \in \partial\Omega_h \end{cases} \tag{2.7}$$

for $u_h \in \mathcal{V}_h$ will be called the Hopf-Lax update function. *Every fixed-point of Λ_h, such that $u_h|_{\partial\Omega} = g$, will be called* finite-element solution *of (1.1) on Σ_h. Finite-element functions $u_h \in \mathcal{V}_h$ are called* finite-element sub-solution *(*finite-element super-solution*) if $u_h \leq \Lambda_h u_h$ ($u_h \geq \Lambda_h u_h$).*

The denomination finite-element sub- or super-solution is chosen, as they represent the discrete equivalent of viscosity sub- or super-solutions. One of the main results of this chapter states, that the finite-element solutions, defined by the fixed-point equation

$$u_h = \Lambda_h u_h, \qquad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}, \tag{2.8}$$

converge uniformly to the viscosity solution of (1.1).

In the next subsections we prove, that the fixed-point equation (2.8) has a unique solution $u_h \in \mathcal{V}_h$. Thus for a Dirichlet problem (1.1) and an admissible triangulation Σ_h, the approximate solution is well-defined.

As stated before, the finite-element solution has directional derivatives in every grid-point $x_h \in \Omega_h$. The following remark shows an alternate formulation of the Hopf-Lax scheme (2.8).

REMARK 2.3: Let (H1)-(H4) be fulfilled. Then u_h fulfills (2.8) if and only if

$$\max_{q \in S^{d-1}} \{-\partial u_h(x_h; -q) - \rho(x_h, q)\} = 0 \text{ in } \Omega_h, \quad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}.$$

Thus the Hopf-Lax approximation is an *upwind* discretization, that is, u_h is updated from the simplex where the characteristic direction $-q$ points into.

PROOF. The following equations are equivalent:

$$u_h(x_h) = (\Lambda_h u_h)(x_h)$$

$$u_h(x_h) = \min_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}$$

$$\max_{y \in \partial\omega_h(x_h)} \{u_h(x_h) - u_h(y) - \rho(x_h, x_h - y)\} = 0$$

$$\max_{q \in S^{d-1}} \{-\partial u_h(x_h; -q) - \rho(x_h, q)\} = 0$$

□

As shown in lemma 1.20, the equations

$$\max_{q\in S^{d-1}} \{\langle Du(x), q\rangle - \rho(x,q)\} = 0, \quad x \in \Omega$$

and $H(x, Du(x)) = 0$ are equivalent in the sense, that they have the same viscosity solutions. In view of definition 1.5, we can also formulate the following characterization of finite-element solutions:

REMARK 2.4: A function $u_h \in \mathcal{V}_h$ is a finite-element sub-solution, if and only if

$$v_h(x_h) \ \le \ (\Lambda_h v_h)(x_h), \tag{2.9}$$

for every $v_h \in \mathcal{V}_h$, provided that $u_h - v_h$ attains a local maximum in $x_h \in \Omega_h$. Super-solutions may be characterized analogously. To give a reason let $u_h \in \mathcal{V}_h$ denote a finite-element function, and let condition (2.9) be fulfilled. Then trivially also $v_h = u_h$ fulfills (2.9), and therefore u_h is a sub-solution. If u_h, on the other hand, is a finite-element sub-solution, and $v_h \in \mathcal{V}_h$, such that $u_h - v_h$ attains a local maximum in x_h, then

$$(u_h - v_h)(x_h) \ \ge \ (u_h - v_h)(y) \quad \text{for all } y \in \partial\omega_h(x_h),$$

and as $u_h(x_h) \le u_h(y) + \rho(x_h, x_h - y)$ for such y, we have

$$v_h(x_h) \le u_h(x_h) + (v_h(y) - u_h(y)) \le v_h(y) + \rho(x_h, x_h - y)$$

for all $y \in \partial\omega_h(x_h)$, thus $v_h(x_h) \le \Lambda_h v_h(x_h)$.

2.2.3. Solvability of the Discrete System.

The existence of a finite-element solution as implicitly defined by (2.8) is based on two simple properties of the Hopf-Lax update function Λ_h.

PROPOSITION 2.5 ([BR06, Lemma 5]): *Assume (H1)-(H4). Let $u_h, v_h \in \mathcal{V}_h$.*

(1) *Λ_h is monotonic, that is, $u_h \le v_h$ implies $\Lambda_h u_h \le \Lambda_h v_h$*
(2) *Λ_h is non-expanding, that is, $\|\Lambda_h u_h - \Lambda_h v_h\|_\infty \le \|u_h - v_h\|_\infty$*

PROOF. (From [BR06].) The first property is an immediate consequence of the definition of Λ_h. To prove the second, let the maximum be attained at a nodal point $x_h \in \overline{\Omega}_h$, without loss of generality $\|\Lambda_h u_h - \Lambda_h v_h\|_\infty = (\Lambda_h u_h)(x_h) - (\Lambda_h v_h)(x_h)$. If $x_h \in \partial\Omega_h$ there is nothing to show; so we can assume $x_h \in \Omega_h$. Let $y \in \partial\omega_h(x_h)$ be such that

$$(\Lambda_h v_h)(x_h) = v_h(y) + \rho(x_h, x_h - y).$$

Hence

$$\begin{aligned}(\Lambda_h u_h)(x_h) &- (\Lambda_h v_h)(x_h)\\ &\le \{u_h(y) + \rho(x_h, x_h - y)\} - \{v_h(y) + \rho(x_h, x_h - y)\}\\ &\le \|u_h - v_h\|_\infty\end{aligned}$$

which proves the assertion. □

The existence of solutions of the fixed point equation can be proved by considering the fixed point iteration $u_h^{n+1} = \Lambda_h u_h^n$ with an appropriate initial value. This so-called *Jacobi iteration* provides simultaneously a first method to compute the discrete solution. The Jacobi iteration, together with other numerical methods for solving the fixed point problem, will be discussed in detail in chapter 3.

THEOREM 2.6 ([BR06, Theorem 6]): *Assume (H1)-(H4) and $g : \partial\Omega \to \mathbb{R}$. Then the finite-element discretization (2.8) has a solution $u_h \in \mathcal{V}_h$. If $u_h^0 \in \mathcal{V}_h$ is such that $u_h^0|_{\partial\Omega_h} = g|_{\partial\Omega_h}$ and $\Lambda_h u_h^0 \ge u_h^0$, then the Jacobi iteration*

$$u_h^{n+1} = \Lambda_h u_h^n, \qquad n = 0, 1, 2, \ldots,$$

converges monotonically to a solution of (2.8).

PROOF. (From [BR06].) An initial iterate $u_h^0 \in \mathcal{V}_h$ with $\Lambda_h u_h^0 \geq u_h^0$ is given by

$$u_h^0|_{\partial\Omega_h} = g|_{\partial\Omega_h}, \qquad u_h|_{\Omega_h} \equiv \min_{x \in \partial\Omega_h} g(x).$$

Inductively the monotonicity of Λ_h implies $u_h^{n+1} = \Lambda_h u_h^n \geq u_h^n$. Hence, the monotonic convergence of the sequence follows if we establish a uniform bound on the iterates. Such a bound is given by

$$\|u_h^n\|_\infty \leq \max_{y_h \in \partial\Omega_h} g(y_h) + \sum_{x_h \in \Omega_h, y_h \in \overline{\Omega}_h} \rho(x_h, x_h - y_h)$$

Thus, $u_h^n \to u_h \in \mathcal{V}_h$ for some $u_h \in \mathcal{V}_h$, which by continuity must be a fixed point of Λ_h. □

2.2.4. Uniqueness of the Discrete Solution. Like in the continuous case, uniqueness of the finite-element solution is a simple corollary of the following *discrete comparison principle*, which is formulated in analogy to 1.13. In particular, the finite-element discretization is a *monotonic scheme.*

THEOREM 2.7 ([BR06, Theorem 7]): *Assume (H1)-(H4). Let $u_h, v_h \in \mathcal{V}_h$ be finite-element sub- and super-solutions, respectively. If $u_h \leq v_h$ on $\partial\Omega_h$ then we have $u_h \leq v_h$ on $\overline{\Omega}$.*

PROOF. (From [BR06].) Let be $\Delta_h = u_h - v_h \in \mathcal{V}_h$. Note that the maximum of Δ_h will be attained in a nodal point. We will show that the existence of $x_h \in \Omega_h$ with $\Delta_h(x_h) = \max_{x \in \overline{\Omega}} \Delta_h(x) = \delta > 0$ yields a contradiction. To this end we choose such a maximizing x_h with minimal value of $v_h(x_h)$. There exists a point $y \in \partial\omega_h(x_h)$, such that $(\Lambda_h v_h)(x_h) = v_h(y) + \rho(x_h, x_h - y)$, hence

$$\delta = u_h(x_h) - v_h(x_h) \leq (\Lambda_h u_h)(x_h) - (\Lambda_h v_h)(x_h) \leq u_h(y) - v_h(y).$$

On the boundary of the face that contains y in its relative interior there is, by the maximality of δ, a point $\hat{x}_h \in \Omega_h$ such that $\Delta_h(\hat{x}_h) = \delta$ and $v_h(\hat{x}_h) \leq v_h(y)$. By lemma 1.17 we have $\rho(x_h, x_h - y) > 0$ and obtain

$$v_h(\hat{x}_h) \leq v_h(y) = (\Lambda_h v_h)(x_h) - \rho(x_h, x_h - y) < (\Lambda_h v_h)(x_h) \leq v_h(x_h)$$

in contradiction to the minimality of $v_h(x_h)$. □

And we get the continuous dependence on the boundary data as a simple consequence, likewise the corollary 1.14 to the comparison principle for viscosity solutions.

COROLLARY 2.8: *Assume (H1)-(H4) and let $u_h, v_h \in \mathcal{V}_h$ denote finite-element sub- and super-solutions, respectively. Then*

$$\sup_{\overline{\Omega}_h} (u_h - v_h)^+ \leq \sup_{\partial\Omega_h} (u_h - v_h)^+.$$

PROOF. Apply theorem 2.7 to u_h and $\tilde{v}_h = v_h + \sup_{\partial\Omega_h}(u_h - v_h)^+$. □

If u_h and v_h are both finite-element solutions, then the last corollary yields the estimate $\|u_h - v_h\|_\infty \leq \max_{y_h \in \partial\Omega_h} |(u_h - v_h)(y_h)|$.

2.3. Convergence

The aim of this section is to show the convergence of (u_h), as the grid-diameter vanishes. For that purpose, we introduce a stronger compatibility condition on the boundary value function g, which enables us to show the equi-continuity and uniform boundedness of the sequence of grid-functions (u_h). Such a sequence has a convergent sub-sequence, as the theorem of Arzelà-Ascoli teaches. A notion of consistency is required in order to prove, that the limit function is indeed a viscosity solution of the underlying Hamilton-Jacobi equation.

2.3.1. Discrete Compatibility. The existence of a continuous viscosity solution of the Dirichlet problem $H(x, Du(x)) = 0$ on Ω with $u = g$ on the boundary is equivalent to the compatibility condition (1.14)

$$g(x) - g(y) \;\leq\; \delta(x, y) \quad \text{for all } x, y \in \partial\Omega$$

for the boundary values, as we have seen in subsection 1.3.6. A more restrictive bound on g will be needed in order to show the uniform Lipschitz-continuity of the sequence of grid-functions (u_h).

There exists a constant $\rho_* \geq 0$, such that

$$\rho(x, q) \;\geq\; \rho_* \cdot \|q\| \qquad \forall x \in \Omega,\; q \in \mathbb{R}^d.$$

Of course, this is trivially fulfilled with $\rho_* = 0$. If H is strictly coercive, that is

$$H(x, 0) \;<\; 0 \quad \text{for all } x \in \overline{\Omega},$$

we can choose $\rho_* > 0$, as shown in corollary 1.19. With the regularity constant $\theta \geq 1$ of the sequence of triangulations from (2.1), we consider the *discrete compatibility condition*

$$|g(x) - g(y)| \;\leq\; \frac{\rho_*}{\theta} \cdot \|x - y\| \;\text{ for all } x, y \in \partial\Omega. \tag{2.10}$$

Of course, this condition is actually stronger than (1.14), as $\delta(x, y) \geq \rho_* \|x - y\|$ and $\theta \geq 1$. Note that the *homogeneous* Dirichlet condition $g \equiv 0$ always satisfies (2.10).

2.3.2. Uniform Boundedness and Equi-Continuity. If g fulfills the discrete compatibility condition (2.10), then we can derive an uniform Lipschitz bound for the sequence of grid-functions. This is done in the following theorem. The boundedness follows as a simple consequence. A similar result has already been part of my diploma thesis [Ras02], where I have considered only homogeneous Dirichlet boundary values $g \equiv 0$.

THEOREM 2.9 ([BR06, Theorem 8]): *Assume (H1)-(H4), (2.10) and (2.1). Then the sequence (u_h) of finite-element solutions of (2.8) is equi-continuous and uniformly bounded. In detail, we have*

$$|u_h(x) - u_h(y)| \leq C_\Omega \cdot d \cdot \theta \cdot \frac{\beta}{\alpha} \cdot \|x - y\|$$

for all $x, y \in \overline{\Omega}$, and a uniform bound on the sequence is given by

$$\|u_h\|_\infty \leq \max_{y \in \partial\Omega} |g(y)| + C_\Omega \cdot d \cdot \theta \cdot \frac{\beta}{\alpha} \cdot \operatorname{diam}(\Omega).$$

Here, $\alpha, \beta > 0$ are defined in lemma 1.1 and $C_\Omega > 0$ is from lemma 1.3.

PROOF. (From [BR06].) The uniform bound on $\|u_h\|_\infty$ is a simple consequence of the Lipschitz condition. The proof of the Lipschitz condition proceeds in three steps, imposing less and less restrictions on the possible choices of $x, y \in \overline{\Omega}$.

Step 1. For *neighboring* nodal points $x_h, y_h \in \overline{\Omega}_h$ we prove

$$|u_h(x_h) - u_h(y_h)| \leq \frac{\beta}{\alpha} \cdot \|x_h - y_h\|.$$

Since $\beta/\alpha \geq \rho_* \geq \rho_*/\theta$ this is, by (2.10), obviously true for $x_h, y_h \in \partial\Omega_h$. If $x_h \in \Omega_h$ we have $y_h \in \partial\omega_h(x_h)$ and hence

$$u_h(x_h) = (\Lambda_h u_h)(x_h) \leq u_h(y_h) + \rho(x_h, x_h - y_h) \leq u_h(y_h) + \frac{\beta}{\alpha} \|x_h - y_h\|.$$

If $y_h \in \Omega_h$ we can change the roles of x_h and y_h and the Lipschitz bound follows.

Assume on the other hand that $y_h \in \partial\Omega_h$. There is a minimizing $y \in \partial\omega_h(x_h)$ such that

$$u_h(x_h) = (\Lambda_h u_h)(x_h) = u_h(y) + \rho(x_h, x_h - y) > u_h(y),$$

where the last inequality follows from Lemma 1.17. The boundary of the face that contains y in its relative interior has a point $x_h^1 \in \overline{\Omega}_h$ with $u_h(x_h^1) \le u_h(y) < u_h(x_h)$. By the definition of ρ_* and θ we obtain

$$\rho(x_h, x_h - y) \ge \rho_* \|x_h - y\| \ge \frac{\rho_*}{\theta} \left\|x_h - x_h^1\right\|.$$

Continuing this construction, we obtain a sequence $x_h = x_h^0, x_h^1, \dots, x_h^m$ of nodal points with strictly decreasing u_h-values that necessarily reaches the boundary at some index m: $x_h^m \in \partial\Omega_h$. Thus, by construction and (2.10),

$$\begin{aligned} u_h(x_h) &\ge g(x_h^m) + \frac{\rho_*}{\theta} \sum_{i=0}^{m-1} \|x_h^i - x_h^{i+1}\| \\ &\ge g(y_h) + \frac{\rho_*}{\theta} \left(\sum_{i=0}^{m-1} \|x_h^i - x_h^{i+1}\| - \|x_h^m - y_h\| \right) \\ &\ge u_h(y_h) - \frac{\rho_*}{\theta} \|x_h - y_h\| \ge u_h(y_h) - \frac{\beta}{\alpha} \|x_h - y_h\|, \end{aligned}$$

which concludes the proof of Step 1.

Step 2. Let $\sigma_h \in \Sigma_h$ be a simplex of the triangulation. For $x, y \in \sigma_h$, $x \ne y$, we prove that

$$|u_h(x) - u_h(y)| \le \theta \cdot d \cdot \frac{\beta}{\alpha} \cdot \|x - y\|$$

Let $x_0, \dots, x_d$ denote the vertices of σ_h and let

$$x = \sum_{i=0}^{d} \lambda_i x_i \quad \text{and} \quad y = \sum_{i=0}^{d} \mu_i x_i.$$

be represented as convex combinations.

Let $I = \{i = 1, \dots, d;\ \lambda_i \ne \mu_i\}$ and $i \in I$. Then we have

$$|\lambda_i - \mu_i| = \frac{\|x - y\|}{\left\| \sum_{j=0}^{d} \frac{\lambda_j}{\lambda_i - \mu_i} x_j - \sum_{j=0}^{n} \frac{\mu_j}{\lambda_i - \mu_i} x_j \right\|} = \frac{\|x - y\|}{\|x_i - z_i\|}$$

where

$$z_i = \sum_{j=0,\, j\ne i}^{d} \frac{\mu_j - \lambda_j}{\lambda_i - \mu_i} x_j$$

is contained in the affine hull of the x_j for $j \ne i$. Consequently,

$$\begin{aligned} |u_h(x) - u_h(y)| &\le \sum_{i \in I} |\lambda_i - \mu_i| \cdot |u_h(x_i) - u_h(x_0)| \\ &= \sum_{i \in I} \frac{\|x - y\|}{\|x_i - z_i\|} \cdot |u_h(x_i) - u_h(x_0)| \\ &\le \frac{\beta}{\alpha} \cdot \|x - y\| \cdot \sum_{i \in I} \frac{\|x_i - x_0\|}{\|x_i - z_i\|} \\ &\le \frac{\beta}{\alpha} \cdot \|x - y\| \cdot d \cdot \frac{\operatorname{diam}(\sigma_h)}{\operatorname{hmin}(\sigma_h)} \end{aligned}$$

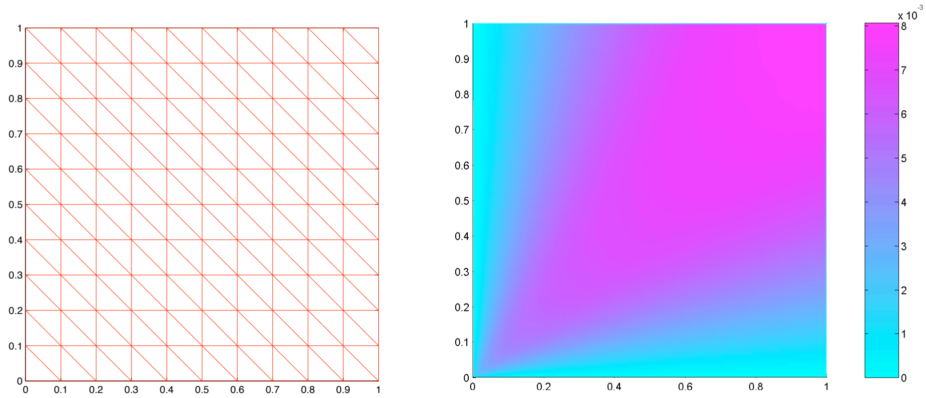

FIGURE 2. Example 2.10: Triangulation (left) and approximation error on the finest mesh (right).

Step 3. For $x, y \in \Omega$, there is a curve $\xi \in \mathcal{C}^\infty([0,1];\Omega)$ joining x and y, with its derivative bounded by $C_\Omega \|x - y\|$ (lemma 1.3). Let $0 = t_0 < t_1 < \ldots < t_m = 1$ be a subdivision of $[0,1]$ such that $\xi(t_{i-1})$ and $\xi(t_i)$ are elements of a common simplex. By step 2 we obtain

$$\begin{aligned} |u_h(x) - u_h(y)| &\leq \sum_{i=0}^{m-1} |u_h(\xi(t_i)) - u_h(\xi(t_{i+1}))| \\ &\leq \theta \cdot d \cdot \frac{\beta}{\alpha} \sum_{i=0}^{m-1} \|\xi(t_i) - \xi(t_{i+1})\| \\ &\leq \theta \cdot d \cdot \frac{\beta}{\alpha} \cdot C_\Omega \cdot \|x - y\| \sum_{i=0}^{m-1} |t_i - t_{i+1}| \end{aligned}$$

which concludes the proof of the asserted Lipschitz bound. □

The crucial point in the proof, where the discrete compatibility condition (2.10) is needed, is the estimate on $u_h(y_h) - u_h(x_h)$, where $y_h \in \partial\Omega_h$ and x_h is a neighbor of y_h in Ω_h. Let me remark, that a necessary and sufficient condition for the uniform Lipschitz continuity of (u_h) is: There is some $M > 0$, such that

$$g(y_h) - u_h(x_h) \leq M \cdot \|y_h - x_h\| \tag{2.11}$$

for al $y_h \in \partial\Omega_h$ and neighbors $x_h \in \overline{\Omega}_h$ of y_h,uniformly in $h \to 0$.

The following example shows, why in the discrete case a stricter compatibility condition is actually necessary, in order to obtain the uniform Lipschitz continuity.

EXAMPLE 2.10: With $\Omega = (0,1) \times (0,1) \subseteq \mathbb{R}^2$ let $g : \partial\Omega \to \mathbb{R}$ be defined by $g(x) = \|x\|$. We consider the Eikonal equation

$$\|Du(x)\| = 1, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

which possesses the unique viscosity solution $u : \overline{\Omega} \to \mathbb{R}$, $u(x) = \|x\|$, by theorems 1.13, 1.25. As easily seen, $\delta(x, y) = \|x - y\|$ for $x, y \in \overline{\Omega}$ and g fulfills (1.14), whereas the discrete compatibility condition (2.10) is violated, as $\theta > 1$ in the two dimensional case.

We computed the associated finite-element solution u_h on a sequence of refined triangulations, constructed as in figure 2. The mesh spacing h and the corresponding approximation error $\|u_h - u\|_\infty$ are given in the following table:

grid-spacing h	error $\|u_h - u\|_\infty$
0.2000	0.1161
0.1000	0.0627
0.0500	0.0388
0.0250	0.0233
0.0125	0.0143
0.0063	0.0081

Following Zhao in [Zha05] one can show, that the approximation error is of order $\mathcal{O}(h\log(h))$ for $h \to 0$, which is done by estimating the interpolatory error in every triangle and by carefully tracing the error propagation in the mesh. As it can be seen in figure 2, the maximal error is attained in the grid-point $x_h = (1-h, 1-h)$. With $y_h = (1,1)$, we obtain

$$\frac{u_h(y_h) - u_h(x_h)}{\|y_h - x_h\|} \approx \frac{g(y_h) - u(x_h) + \mathcal{O}(h\log(h))}{\sqrt{2}h} = 1 + \mathcal{O}(\log(h)),$$

and $|\log(h)| \to \infty$, as h approaches 0. In this case, the uniform Lipschitz continuity asserted in theorem 2.9 is violated.

2.3.3. Consistency. Loosely speaking, in the framework of viscosity solutions the notion of *consistency* of a discretization means that a smooth function is already a sub-solution (super-solution) of the differential equation if it is a sub-solution (super-solution) of the discrete scheme.

THEOREM 2.11 ([BR06, Theorem 10]): *Assume (H1)-(H4), (2.1) and (2.2). Let $\varphi \in C_0^\infty(\Omega)$, $x \in \Omega$, and $x_h \in \Omega_h$ be a sequence of nodal points that converges to x as $h \to 0$. Then*

$$\varphi_h(x_h) \le (\Lambda_h\varphi_h)(x_h) \text{ for all } h \quad \Rightarrow \quad H(x, D\varphi(x)) \le 0,$$
$$\varphi_h(x_h) \ge (\Lambda_h\varphi_h)(x_h) \text{ for all } h \quad \Rightarrow \quad H(x, D\varphi(x)) \ge 0,$$

where $\varphi_h = I_h\varphi$ denotes the interpolant of φ.

PROOF. (From [BR06].) First, let $\varphi_h(x_h) \le (\Lambda_h\varphi_h)(x_h)$ for all h of the sequence, that is,

$$\varphi_h(x_h) - \varphi_h(y) - \rho(x_h, x_h - y) \le 0, \qquad y \in \partial\omega_h(x_h).$$

After division by $\|x_h - y\|$ we get, by lemma 2.1, a constant $C > 0$ such that

$$\langle D\varphi(x_h), q\rangle - \rho(x_h, q) \le Ch, \qquad \|q\| = 1.$$

If we pass to the limit $h \to 0$ (note the continuity of ρ at $x \in \Omega$ as stated in lemma 1.18) and take thereafter the maximum over all $q \in S^1$, we obtain

$$\max_{\|q\|=1} \{\langle D\varphi(x), q\rangle - \rho(x,q)\} \le 0.$$

From lemma 1.20 we infer the assertion $H(x, D\varphi(x)) \le 0$.

On the other hand, let $\varphi_h(x_h) \ge (\Lambda_h\varphi_h)(x_h)$ for all h of the sequence, that is,

$$\varphi_h(x_h) - \varphi_h(y_h) - \rho(x_h, x_h - y_h) \ge 0$$

for some $y_h \in \partial\omega_h(x_h)$. After division by $\|x_h - y_h\|$ we get, by lemma 2.1, a constant $C > 0$ such that

$$\langle D\varphi(x_h), q_h\rangle - \rho(x_h, q_h) \ge -Ch, \qquad q_h = (x_h - y_h)/\|x_h - y_h\|$$

By compactness, we can assume that there is a convergent sub-sequence $q_h \to q$ with $\|q\| = 1$. Passing to the limit yields

$$\langle D\varphi(x), q\rangle - \rho(x,q) \ge 0,$$

from which we infer the assertion $H(x, D\varphi(x)) \geq 0$ by lemma 1.20. □

2.3.4. Convergence Theorem. Under the assumptions of theorem (2.9), the sequence $(u_h)_h$ of grid functions is equi-continuous and uniformly bounded. Thus, by the Arzelà-Ascoli theorem, it has a convergent sub-sequence in $C(\overline{\Omega})$. By the consistency theorem 2.11, the limit function is a viscosity solution of (1.1). The details are given in the following theorem, and the subsequent proof.

Theorem 2.12 ([BR06, Theorem 11]): *Assume (H1)-(H4), (2.10) or (2.11), and (2.1), (2.2). Then, as $h \to 0$, the sequence of finite-element solutions $u_h \in \mathcal{V}_h$ defined by*

$$u_h = \Lambda_h u_h, \qquad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h},$$

converges uniformly to the unique viscosity solution u of the Dirichlet problem

$$H(x, Du(x)) = 0, \qquad u|_{\partial\Omega} = g.$$

Proof. (From [BR06].) Theorems 2.6 and 2.7 show the existence and uniqueness of the finite-element solutions $u_h \in \mathcal{V}_h$. Theorem 2.9 shows that $(u_h) \subseteq \mathcal{C}(\overline{\Omega})$ is equi-continuous and uniformly bounded. By the theorem of Arzelà-Ascoli there is a sub-sequence $(u_{h'})$ that converges uniformly to a function $u \in C(\overline{\Omega})$. Let $y \in \partial\Omega$. As $u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h}$, this limit satisfies the boundary condition $u|_{\partial\Omega} = g$.

To show that u is a viscosity *sub-solution* of $H(x, Du(x)) = 0$ let $\varphi \in C_0^\infty(\Omega)$ and $x_0 \in \Omega$ be such that $u - \varphi$ attains a strict local maximum in x_0 (see remark 1.6). By lemma 2.1, $u_{h'} - \varphi_{h'}$ converges uniformly to $u - \varphi$, where $\varphi_{h'} = I_{h'}\varphi$ denotes the interpolant of φ. Hence, after passing to a sub-sequence again, if necessary, there is a sequence of nodal points $x_{h'} \in \Omega_{h'}$ such that $x_{h'} \to x_0$ and

$$(u_{h'} - \varphi_{h'})(x_{h'}) \geq (u_{h'} - \varphi_{h'})(y), \qquad y \in \partial\omega_{h'}(x_{h'}),$$

that is, $(u_{h'} - \varphi_{h'})$ attains a local maximum in $x_{h'}$ (see proposition 1.7). By remark 2.4, we obtain

$$\varphi(x_{h'}) \leq (\Lambda_{h'}\varphi_{h'})(x_{h'}).$$

The consistency of the discretization, stated in Theorem 2.11, yields that

$$H(x_0, D\varphi(x_0)) \leq 0,$$

which concludes the proof that u is a viscosity sub-solution.

In the same way we prove that u is a viscosity *super-solution* of $H(x, Du(x)) = 0$. Therefore, u is a viscosity solution, which, by the comparison principle (theorem 1.13), is actually *unique*. Hence, the full sequence $(u_h)_h$ cannot have limit points different from u. □

2.4. Local Error and Error Propagation

Let u denote a viscosity solution of $H(x, Du(x)) = 0$. In this section we will see, that the local error $|u - \Lambda_h u|$ is of order h^2, where h denotes the grid-spacing. Together with the investigation of the propagation of errors in the Hopf-Lax approximation, I will show, that the rate of convergence is $\mathcal{O}(h)$, if the Hamilton-Jacobi equation admits a classical solution. The general case will be treated in section 2.5.

In order to prove the assertions made in this section, we have to impose stronger assumptions on the Hamiltonian than in theorem 2.12. The following two assumptions will be needed, in addition to (H2),(H3).

(H1)′ *(Lipschitz-Continuity)*
$$\begin{cases} \text{For every } R > 0 \text{ there is a constant } L_R > 0, \text{ such that} \\ \qquad |H(x,p) - H(y,p)| \leq L_R \, \|x - y\| \\ \text{for all } x, y \in \overline{\Omega}, \|p\| \leq R. \end{cases}$$

$$\text{(H4)}' \quad \begin{array}{l}\textit{(Strict}\\ \textit{Compatibility)}\end{array}\begin{cases}\text{There is a number } m > 0, \text{ such that} \\ \qquad H(x, 0) \leq -m \\ \text{for all } x \in \overline{\Omega}\end{cases}$$

2.4.1. Lipschitz Continuity of the Support Function. In section 1.3 we showed the continuity of $\rho(x, q)$, defined by (1.9) as the support function of the zero level-sets of $p \mapsto H(x, p)$. Under the assumptions of (H1)′, (H4)′ we can prove, that $\rho(x, q)$ is Lipschitz-continuous with respect to the first argument.

PROPOSITION 2.13: *Assume (H1)′,(H2),(H3) and (H4)′. Then there is a number* $L_\rho > 0$, *such that*

$$|\rho(x, q) - \rho(y, q)| \leq L_\rho \cdot \|x - y\| \cdot \|q\| \quad \textit{for all } x, y \in \overline{\Omega},\ q \in \mathbb{R}^d.$$

PROOF. Let $\mathcal{Z}(x) = \{p \in \mathbb{R}^d\,;\ H(x, p) \leq 0\}$ for $x \in \overline{\Omega}$ denote the zero level-set of $p \mapsto H(x, p)$. We observe, that

$$\begin{aligned}\rho(x, q) - \rho(y, q) &= \max_{p_1 \in \mathcal{Z}(x)} \langle p_1, q\rangle - \max_{p_2 \in \mathcal{Z}(y)} \langle p_2, q\rangle \\ &\leq \|q\| \cdot \max_{p_1 \in \mathcal{Z}(x)} \min_{p_2 \in \mathcal{Z}(y)} \|p_1 - p_2\|,\end{aligned}$$

for all $x, y \in \overline{\Omega}$, $q \in \mathbb{R}^d$, and thus

$$|\rho(x, q) - \rho(y, q)| \leq \|q\| \cdot d_H(\mathcal{Z}(x), \mathcal{Z}(y)), \tag{2.12}$$

where d_H denotes the Hausdorff distance. From (H3) we infer, that the level-sets $\mathcal{Z}(x)$ are bounded independently of x. Thus there is a radius $R > 0$, such that $\mathcal{Z}(x) \subseteq B(0, R)$ for all $x \in \overline{\Omega}$. Let $x, y \in \overline{\Omega}$. By (H1)′, we have for $p \in \mathcal{Z}(x)$,

$$H(y, p) \leq H(x, p) + L_R \cdot \|x - y\| \leq L_R \cdot \|x - y\|.$$

Consequently, by assumptions (H2) and (H4)′, it holds for $0 \leq t \leq 1$, that

$$\begin{aligned}H(y, tp) &\leq tH(y, p) + (1 - t)H(y, 0) \\ &\leq tL_R \cdot \|x - y\| - (1 - t)m\end{aligned}$$

And thus $H(y, tp) \leq 0$ for

$$t = \frac{m}{m + L_R \cdot \|x - y\|}.$$

So with $\tilde{p} = tp$ we have $\tilde{p} \in \mathcal{Z}(y)$, and therefore

$$\begin{aligned}\operatorname{dist}(p, \mathcal{Z}(y)) &\leq \|p - \tilde{p}\| = (1 - t)\|p\| \\ &\leq \frac{L_R}{m} \cdot \|x - y\| \cdot \|p\| \leq R \cdot \frac{L_R}{m} \cdot \|x - y\|\end{aligned}$$

As $p \in \mathcal{Z}(x)$ was chosen arbitrarily and by changing the roles of x and y, we get

$$d_H(\mathcal{Z}(x), \mathcal{Z}(y)) \leq R \cdot \frac{L_R}{m} \cdot \|x - y\|,$$

which, together with (2.12), yields the asserted Lipschitz continuity. □

2.4.2. Local Error. The finite-element solution $u_h \in \mathcal{V}_h$ of the Hamilton-Jacobi equation $H(x, Du) = 0$ on a triangulation Σ_h takes the value

$$u_h(x_h) = \min_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}$$

in every point $x_h \in \Omega_h$. As easily seen, the viscosity solution of (1.3), given by the Hopf-Lax formula (1.13), fulfills

$$u(x_h) = \min_{y \in \partial\omega_h(x_h)} \{u(y) + \delta(x_h, y)\}.$$

The local error in x_h, that is $u(x_h) - \Lambda_h(I_h u)(x_h)$, consists of two components:

$$u(x_h) - \Lambda_h(I_h u)(x_h) \leq \max_{y\in\partial\omega_h} \{u(y) - I_h u(y)\} + \max_{y\in\partial\omega_h} \{\delta(x_h, y) - \rho(x_h, x_h - y)\}. \tag{2.13}$$

The first addend is the interpolatory error of u on $\partial\omega_h(x_h)$, and the second addend is the approximation error in the optical distance. While we have dealt with the first addend in lemma 2.1, the approximation error is subject of the following lemma.

Lemma 2.14: *Assume (H1)′,(H2),(H3),(H4)′. Then there is some $C > 0$ with*

$$|\delta(x, y) - \rho(x, x - y)| \leq C \|x - y\|^2$$

for all $x, y \in \overline{\Omega}$, for which the joining segment $[x, y] \subseteq \overline{\Omega}$.

Proof. For $x, y \in \Omega$, such that $[x, y] \subseteq \Omega$, we have with $\xi(t) = x + t(y - x)$

$$\delta(x, y) \leq \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt \leq \rho(x, x - y) + L_\rho \|x - y\|^2, \tag{2.14}$$

by the Lipschitz-continuity of ρ from proposition 2.13. For every $\eta > 0$, there exists a curve $\xi \in C^\infty([0, 1]; \Omega)$ joining x and y, with

$$\delta(x, y) \geq \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt - \eta. \tag{2.15}$$

By the Lipschitz continuity of ρ, it holds that

$$\int_0^1 \left|\rho(x, -\dot{\xi}(t)) - \rho(\xi(t), -\dot{\xi}(t))\right| dt \leq \int_0^1 L_\rho \cdot \|x - \xi(t)\| \cdot \left\|\dot{\xi}(t)\right\| dt \leq L_\rho \cdot l(\xi)^2.$$

So by Jensen's inequality, we get from (2.15)

$$\begin{aligned} \delta(x, y) \quad &\geq \quad \rho\left(x, -\int_0^1 \dot{\xi}(t)\, dt\right) - L_\rho \cdot l(\xi)^2 - \eta \\ &= \quad \rho(x, x - y) - L_\rho \cdot l(\xi)^2 - \eta \end{aligned} \tag{2.16}$$

Using (2.15) in order to estimate the length of ξ, we obtain

$$C_\Omega \cdot \rho^* \cdot \|x - y\| + \eta \;\geq\; \delta(x, y) + \eta \;\geq\; \rho_* \cdot l(\xi).$$

Here $\rho^*, \rho_* > 0$ are the bounds for ρ from corollary 1.19. (2.14) and (2.16) yield

$$|\delta(x, y) - \rho(x, x - y)| \leq L_\rho \left(C_\Omega \frac{\rho^*}{\rho_*}\right)^2 \cdot \|x - y\|^2,$$

by letting $\eta \to 0$. □

We obtain the following result on the local error, in regions $U \subset \Omega$, where the viscosity solution is smooth.

Theorem 2.15: *Assume (H1)′,(H2),(H3),(H4)′, (2.2). Let $u \in \mathcal{C}(\Omega)$ denote a viscosity solution of $H(x, Du(x)) = 0$, such that the restriction of u to some open subset $U \subset \Omega$ is $\mathcal{C}^2(\overline{U})$. Then*

$$\max_{x_h\in U_h} |u(x_h) - (\Lambda_h I_h u)(x_h)| = \mathcal{O}(h^2),$$

where $U_h = \{x_h \in \Omega_h\, ;\ \omega_h(x_h) \subset \overline{U}\}$.

Proof. Follows from (2.13) with lemmas 2.1 and 2.14. □

2.4.3. Error Propagation. Next, we discuss the propagation of errors in the Hopf-Lax scheme, in order to obtain an error estimation in the smooth case. For the sake of simplicity, we consider uniform triangulations Σ_h, where $\mathrm{hmin}(\sigma_h) \geq h/\theta$ for all $\sigma_h \in \Sigma_h$.

LEMMA 2.16: *Assume (H1)′,(H2),(H3),(H4)′, (2.2),(2.3). Let u_h denote the solution of (2.8) and (v_h) a uniformly bounded sequence of functions $v_h \in \mathcal{V}_h$ with*

$$\max_{x_h \in \Omega_h} |(v_h - \Lambda_h v_h)(x_h)| \leq \mu(h)h, \qquad \max_{y_h \in \partial\Omega_h} |u_h(y_h) - v_h(y_h)| \leq \eta,$$

where μ is some modulus of continuity and $\eta > 0$. Then there is some $C > 0$ with

$$\|u_h - v_h\|_\infty \leq \eta + C\mu(h)$$

for sufficiently small $h \to 0$.

PROOF. Let $x_h \in \Omega_h$. By assumption, we have for every $x_h \in \Omega_h$

$$v_h(x_h) \geq (\Lambda_h v_h)(x_h) - \mu(h)h. \tag{2.17}$$

By the assumptions on the triangulation, it holds, that

$$\rho(x_h, x_h - y) \geq \rho_* \|x_h - y\| \geq \rho_* h/\theta. \tag{2.18}$$

Consequently, we deduce by (2.17), that

$$\max_{y \in \partial\omega_h(x_h)} \left(\frac{v_h(x_h) - v_h(y)}{\rho(x_h, x_h - y)} \right) \geq 1 - \frac{\theta\mu(h)}{\rho_*}.$$

So if h is sufficiently small, such that $\frac{\theta\mu(h)}{\rho_*} \leq \frac{1}{2}$, then the function

$$\tilde{v}_h = \frac{1}{1 - \frac{\theta\mu(h)}{\rho_*}} \cdot v_h$$

is a finite-element super-solution. Since u_h is a finite-element sub-solution, we obtain from corollary 2.8, that

$$\begin{aligned}
(u_h - v_h)(x_h) &= (u_h - \tilde{v}_h)(x_h) + (\tilde{v}_h - v_h)(x_h) \\
&\leq \max_{\partial\Omega_h} (u_h - \tilde{v}_h)^+ + \|\tilde{v}_h - v_h\|_\infty \\
&\leq \max_{\partial\Omega_h} |u_h - v_h| + 2\|\tilde{v}_h - v_h\|_\infty \\
&\leq \eta + \frac{4\theta\mu(h)}{\rho_*} \cdot \|v_h\|_\infty .
\end{aligned}$$

Analogously, we have for every $x_h \in \Omega_h$,

$$v_h(x_h) \leq (\Lambda_h v_h)(x_h) + \mu(h)h$$

or equivalently,

$$\rho(x_h, x_h - y) \cdot \left(\frac{v_h(x_h) - v_h(y)}{\rho(x_h, x_h - y)} - 1 \right) \leq \mu(h)h$$

for all $y \in \partial\omega_h(x_h)$. By (2.18),

$$\tilde{v}_h = \frac{1}{1 + \frac{\theta\mu(h)}{\rho_*}} \cdot v_h$$

defines a finite-element sub-solution. From the discrete comparison principle, we get

$$(v_h - u_h)(x_h) = (v_h - \tilde{v}_h)(x_h) + (\tilde{v}_h - u_h)(x_h) \leq \frac{2\theta\mu(h)}{\rho_*} \cdot \|v_h\|_\infty + \eta.$$

□

If Σ_h is a non-uniform (e.g. local refined) triangulation, with properties (2.1) and (2.2) being fulfilled, then the last lemma holds, provided that the sequence (v_h) satisfies

$$|v_h(x_h) - (\Lambda_h v_h)(x_h)| \leq \mu(h) \cdot \operatorname{diam}(\omega_h(x_h))$$

for all $x_h \in \Omega_h$. Combining the error propagation in the scheme with the local error, we obtain an estimation of the global error $\|u - u_h\|_\infty$, provided that (1.3) has a classical solution $u \in \mathcal{C}^2(\overline{\Omega})$. However, the differentiability assumptions on u are too strong, and will generally not be fulfilled. Consider, for example, a Hamilton-Jacobi equation (1.3) with homogeneous Dirichlet boundary condition ($g \equiv 0$). By assumption (H4), the viscosity solution cannot be constant, thus it takes a maximum or minimum in some point $x \in \Omega$. If u was differentiable in x, then $Du(x) = 0$, but then equation (1.3) would not be fulfilled, as $H(x, 0) < 0$. Anyway, the convergence is of order $\mathcal{O}(h)$ in regions, where the viscosity solution is smooth.

THEOREM 2.17: *Assume (H1)′,(H2),(H3),(H4)′, (2.2),(2.3) and let (u_h) denote the sequence of finite-element solutions. If the Dirichlet problem (1.1) admits a classical solution $u \in \mathcal{C}^2(\overline{\Omega})$, then*

$$\|u - u_h\|_\infty = \mathcal{O}(h)$$

for $h \to 0$.

PROOF. Follows from theorem 2.15 and lemma 2.16 applied to u_h, and $v_h = I_h u$. □

As $u \in \mathcal{C}^2(\overline{\Omega})$ in the last theorem, the compatibility condition

$$g(x) - g(y) \;\leq\; \delta(x, y)$$

is necessarily fulfilled by theorem 1.27, and the stronger discrete compatibility requirement for theorem 2.12 may be omitted. It will take a lot more effort to obtain the convergence rate in the non-smooth case, which will be done in the next section.

2.5. Convergence Rate

The main result of this section is the following theorem. Under the regularity assumption on the triangulation, and the discrete compatibility condition, I show, that the order of convergence of (u_h) is at least $\mathcal{O}(\sqrt{h})$.

THEOREM 2.18 (Convergence Rate): *Assume (H1)′,(H2),(H3),(H4)′, (2.1),(2.2) and (2.10). Let $u \in \mathcal{C}(\overline{\Omega})$ denote the viscosity solution of $H(x, Du(x)) = 0$ with $u = g$ on $\partial\Omega$, and (u_h) the sequence of finite-element solutions. Then there are numbers $C, h_0 > 0$, such that*

$$\|u - u_h\|_\infty \;\leq\; C\sqrt{h} \quad \text{for all } h \leq h_0.$$

The proof of the theorem requires two short propositions, and is postponed to subsection 2.5.2.

2.5.1. The Kružkov-Transformation.

In order to prove the theorem, we begin with two short propositions. The following transformation of a viscosity solution $u \to v = -e^{-u}$ leads to the Hamilton-Jacobi equation $v + F(x, Dv) = 0$, where the dependent variable v appears explicitely as an addend.

Let me remark, that $v \in \mathcal{C}(\Omega)$ is called *viscosity sub-solution* of the equation $F(x, v(x), Dv(x)) = 0$, if for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$,

$$F(x_0, v(x_0), D\varphi(x_0)) \leq 0$$

provided that $u - \varphi$ attains a local maximum in $x_0 \in \Omega$. Viscosity super-solutions are defined likewise the definition 1.5, with $F(x_0, v(x_0), D\varphi(x_0)) \geq 0$, where $u - \varphi$

has a local minimum. Of course, a viscosity solution is a sub- and super-solution in union.

PROPOSITION 2.19 (Kružkov-transformation): *Let u be a viscosity solution of (1.3). Then $v = -e^{-u}$ is a viscosity solution of*

$$F(x, v(x), Dv(x)) = 0 \quad \textit{in } \Omega, \tag{2.19}$$

where $F : \overline{\Omega} \times]-\infty, 0[\times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$F(x, r, p) = -r\, H\left(x, -\frac{1}{r}p\right).$$

PROOF. This follows directly from [BCD97, proposition II.2.5] with $\Phi(t) = -e^{-t}$. There it is shown, that $v = \Phi(u)$, with $\Phi \in \mathcal{C}^1(\mathbb{R})$ and $\Phi'(t) > 0$ for all t, is a viscosity solution of $F(x, \Psi(v), \Psi'(v)Dv) = 0$, where $\Psi = \Phi^{-1}$, provided, that u is a viscosity solution of $F(x, u, Du) = 0$. Consequently, with $\Psi(t) = -\log(-t)$, v is a viscosity solution of $H(x, -\frac{1}{v}Dv) = 0$. □

As shown in subsection 1.4.1, we can assume, that $H(x, p) = F(x, p) - 1$ with some function $F \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d)$, which fulfills the properties (F1) - (F4) on page 18, that is, F is convex, positively homogeneous with respect to p, and positive except for $p = 0$. Moreover, F will be Lipschitz-continuous as the following proposition shows.

PROPOSITION 2.20: *Assume (H1)′,(H2),(H3) and (H4)′ and let $F : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be defined by (1.16) on page 18 as the polar of the support function or equivalently, as the gauge generated from the convex zero level-sets of $p \mapsto H(x, p)$. Then F fulfills properties (F1)-(F4), F is even Lipschitz-continuous, that is,*

$$|F(x, p) - F(y, p)| \;\leq\; L_F \cdot \|p\| \cdot \|x - y\| \quad \textit{for all } x, y \in \overline{\Omega},\ p \in \mathbb{R}^d$$

with some constant $L_F > 0$. If $u \in \mathcal{C}(\Omega)$ is a viscosity solution of $H(x, Du(x)) = 0$ in Ω, then $v = \Phi(u) = -e^{-u}$ is a viscosity solution of

$$v(x) \;+\; F(x, Dv(x)) \;=\; 0, \qquad x \in \Omega.$$

PROOF. The support function $\rho : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz-continuous with respect to its first argument by proposition 2.13. As $\rho(x, q) \geq \rho_* \cdot \|q\|$ with some $\rho_* > 0$ by corollary 1.19, we obtain

$$\begin{aligned}
|F(x, p) - F(y, p)| \;&\leq\; \|p\| \sup_{\|q\|=1} \left| \frac{1}{\rho(x, q)} - \frac{1}{\rho(y, q)} \right| \\
&\leq\; \frac{1}{\rho_*^2} \|p\| \sup_{\|q\|=1} |\rho(y, q) - \rho(x, q)| \\
&\leq\; \frac{1}{\rho_*^2} \|p\| \cdot L_\rho \cdot \|x - y\|,
\end{aligned}$$

which shows the asserted Lipschitz-continuity of F. If u is a viscosity solution of $H(x, Du(x)) = 0$, then by theorem 1.29, it is also a viscosity solution of $F(x, Du(x)) - 1 = 0$, and by the last proposition 2.19, the transformed function $v = -e^{-u}$ is a viscosity solution of

$$-v(x) \cdot (F(x, -v(x)^{-1}Dv(x)) - 1) = 0 \quad \Leftrightarrow \quad F(x, Dv(x)) + v(x) = 0.$$

□

2.5.2. Proof of the Theorem. Now we come to the proof of the theorem. The technique, that is used to obtain the convergence rate, is similar to the maximum principle argument used by Crandall and Lions in [CL84] for the convergence of approximations to the viscosity solution of the Cauchy problem, which was recently adapted by Deckelnick and Elliott in [DE04] for Hamilton-Jacobi equations $H(Du(x)) = f(x)$ on Cartesian meshes. In a similar way, also uniqueness results for Hamilton-Jacobi equations are proved, see for example the proof of the uniqueness theorem II.3.1 in [BCD97] for the equation $v + F(x, Dv) = 0$.

PROOF. (of the theorem) By proposition 2.20, we may assume, that the Kružkov-transformate $v = -e^{-u}$ of u is a viscosity solution of

$$v(x) \;+\; F(x, Dv(x)) \;=\; 0, \tag{2.20}$$

where F is Lipschitz-continuous, and satisfies (F1)-(F4). By theorem 1.12 and theorem 2.9, there is a Lipschitz bound $L_u > 0$ such that

$$\|u(x) - u(y)\| \leq L_u \|x - y\| \quad \text{and} \quad \|u_h(x) - u_h(y)\| \leq L_u \|x - y\|$$

for all $x, y \in \overline{\Omega}$, and a number $M > 0$, such that

$$\|u\|_\infty, \|u_h\|_\infty \leq M$$

uniformly in h. Consequently, by the mean value theorem,

$$|v(x) - v(y)| = \left|-e^{-u(x)} + e^{-u(y)}\right| \leq e^M |u(x) - u(y)| \leq e^M L_u \|x - y\|$$

for all $x, y \in \overline{\Omega}$, so v (and accordingly $v_h = -e^{-u_h}$) is Lipschitz-continuous with Lipschitz constant $L_v = e^M L_u$. Moreover, we have

$$-e^M \leq v(x), v_h(x) \leq -e^{-M} \quad \text{for all } x \in \overline{\Omega}$$

uniformly in h. As we have

$$|v(x) - v_h(x)| \geq e^{-M} |u(x) - u_h(x)|,$$

for all $x \in \overline{\Omega}$, it suffices to show, that $\|v - v_h\|_\infty = \mathcal{O}(\sqrt{h})$.

Let $x_h^* \in \overline{\Omega}_h$ be such, that

$$|v(x_h^*) - v_h(x_h^*)| = \max_{x_h \in \overline{\Omega}_h} |v(x_h) - v_h(x_h)|. \tag{2.21}$$

The cases, where $v(x_h^*) > v_h(x_h^*)$ and $v(x_h^*) < v_h(x_h^*)$ have to be treated seperately, although the approach is very similar in both cases.

First part: $v_h(x_h^*) > v(x_h^*)$. In this case, let $\varphi : \overline{\Omega}_h \times \overline{\Omega} \to \mathbb{R}$ be defined by

$$\varphi(x_h, x) = v_h(x_h) - v(x) - \frac{\|x_h - x\|^2}{2\sqrt{h}}$$

and let $(\overline{x}_h, \overline{x}) \in \overline{\Omega}_h \times \overline{\Omega}$ be such, that

$$\varphi(\overline{x}_h, \overline{x}) = \max_{\overline{\Omega}_h \times \overline{\Omega}} \varphi(x_h, x).$$

As $\varphi(\overline{x}_h, \overline{x}) \geq \varphi(x_h^*, x_h^*)$, we have

$$\begin{aligned} \frac{\|\overline{x}_h - \overline{x}\|^2}{2\sqrt{h}} &\leq v_h(\overline{x}_h) - v(\overline{x}) + v(x_h^*) - v_h(x_h^*) \\ &= v_h(\overline{x}_h) - v(\overline{x}_h) + v(\overline{x}_h) - v(\overline{x}) - (v_h(x_h^*) - v(x_h^*)) \\ &\leq v(\overline{x}_h) - v(\overline{x}) \\ &\leq L_v \|\overline{x}_h - \overline{x}\| \end{aligned}$$

and thus

$$\|\overline{x}_h - \overline{x}\| \leq 2L_v\sqrt{h}. \tag{2.22}$$

We first consider the case, where $(\overline{x}_h, \overline{x}) \in \Omega_h \times \Omega$. Then $\overline{x}$ is a minimum point of $x \mapsto -\varphi(\overline{x}_h, x)$, and as v is a viscosity (super-)solution of (2.19), we infer

$$v(\overline{x}) + F\left(\overline{x}, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right) \geq 0. \tag{2.23}$$

The fact, that $\overline{x}_h$ is a maximum point of $x_h \mapsto \varphi(x_h, \overline{x})$ enables us to show, that

$$v_h(\overline{x}_h) + F\left(\overline{x}_h, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right) \leq \mathcal{O}(\sqrt{h}), \tag{2.24}$$

which will be proved later on. As $\varphi(\overline{x}_h, \overline{x}) \geq \varphi(x_h^*, x_h^*)$, we have

$$v_h(x_h^*) - v(x_h^*) \leq v_h(\overline{x}_h) - v(\overline{x}) - \frac{\|\overline{x}_h - \overline{x}\|^2}{2\sqrt{h}} \leq v_h(\overline{x}_h) - v(\overline{x}). \tag{2.25}$$

By the Lipschitz-continuity of F and by (2.22), we obtain

$$F\left(\overline{x}, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right) \leq F\left(\overline{x}_h, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right) + L_F \cdot 2L_v \cdot 2L_v\sqrt{h}.$$

Thus (2.25), (2.23) and (2.24) and the above estimate yield

$$\begin{aligned} v_h(x_h^*) - v(x_h^*) &\leq v_h(\overline{x}_h) - v(\overline{x}) \\ &= \left(v_h(\overline{x}_h) + F\left(\overline{x}, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right)\right) - \left(v(\overline{x}) + F\left(\overline{x}, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right)\right) \\ &\leq \mathcal{O}(\sqrt{h}). \end{aligned}$$

Now we turn to the proof of (2.24). As $\varphi(\overline{x}_h, \overline{x}) \geq \varphi(x_h, \overline{x})$ for all $x_h \in \overline{\Omega}_h$, we have

$$v_h(x_h) \leq v_h(\overline{x}_h) + \frac{\|x_h - \overline{x}\|^2 - \|\overline{x}_h - \overline{x}\|^2}{2\sqrt{h}} =: w_h(x_h). \tag{2.26}$$

It holds for all grid-points $y_h \in \overline{\Omega}_h$, that

$$w_h(\overline{x}_h) - w_h(y_h) = \left\langle \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}, \overline{x}_h - y_h \right\rangle - \frac{\|\overline{x}_h - y_h\|^2}{2\sqrt{h}}. \tag{2.27}$$

We have $w_h(\overline{x}_h) = v_h(\overline{x}_h)$, and for all neighbors $y_h \in \overline{\Omega}_h$ of $\overline{x}_h$, we deduce from (2.27), as $\|\overline{x}_h - y_h\| \leq h$ by regularity assumption (2.2), the following estimates

$$|w_h(\overline{x}_h) - w_h(y_h)| \ \leq \ (2L_v + \frac{1}{2}\sqrt{h}) \cdot \|\overline{x}_h - y_h\|, \tag{2.28}$$

where (2.22) was used, and

$$\left| w_h(\overline{x}_h) - w_h(y_h) - \left\langle \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}, \overline{x}_h - y_h \right\rangle \right| \ \leq \ \frac{1}{2}\sqrt{h} \cdot \|\overline{x}_h - y_h\|. \tag{2.29}$$

Recall, that with $\delta = e^{-M}$, where M denotes the uniform bound on u_h, we have $v_h(x_h) \leq -\delta$ for all $x_h \in \overline{\Omega}_h$. So one obtains from (2.28), that for sufficiently small h,

$$w_h(y_h) \leq -\frac{\delta}{2} \tag{2.30}$$

for all $y_h \in \mathcal{N}(\overline{x}_h)$. For such h, $\hat{w}_h = \Psi(w_h) = -\log(-w_h)$ is defined for $\overline{x}_h$ and the neighboring grid-points. By Taylor expansion, we have for $a, b < 0$, that

$$\Psi(a) - \Psi(b) \ = \ \Psi'(a) \cdot (a - b) \ + \ R \ = \ -\frac{1}{a} \cdot (a - b) \ + \ R,$$

where the remainder term R is bounded by $\frac{1}{2m^2}(a-b)^2$, where $m = \min(|a|, |b|)$. Consequently, we obtain by (2.30)

$$\hat{w}_h(\overline{x}_h) - \hat{w}_h(y_h) \geq -\frac{1}{w_h(\overline{x}_h)} \cdot (w_h(\overline{x}_h) - w_h(y_h)) - \frac{2}{\delta^2} \cdot (w_h(\overline{x}_h) - w_h(y_h))^2,$$

Thus, by (2.28) and (2.29), there is a constant $C > 0$ depending only on δ and L_v, such that

$$\hat{w}_h(\overline{x}_h) - \hat{w}_h(y_h) \geq -\frac{1}{v_h(\overline{x}_h)} \left\langle \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}, \overline{x}_h - y_h \right\rangle - C\sqrt{h}\, \|\overline{x}_h - y_h\|$$

for all neighbors y_h of $\overline{x}_h$. We can think of $\hat{w}_h$ as a finite-element function on $\omega_h(\overline{x}_h)$, where the values $\hat{w}_h(y)$ for $y \in \partial\omega_h(\overline{x}_h)$ are defined by linear interpolation. Although $\hat{w}_h$ is defined on a neighborhood of $\overline{x}_h$, the values of $\hat{w}_h$ outside of $\omega_h(\overline{x}_h)$ don't play a role, so one could assign arbitrary values. From the last equation, we infer by regularity assumption (2.1), that

$$\hat{w}_h(\overline{x}_h) - \hat{w}_h(y) \geq -\frac{1}{v_h(\overline{x}_h)} \left\langle \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}, \overline{x}_h - y \right\rangle - C\theta\sqrt{h}\, \|\overline{x}_h - y\|, \tag{2.31}$$

for all $y \in \partial\omega_h(\overline{x}_h)$. Moreover, we have $\hat{w}_h(\overline{x}_h) = \Psi(w_h(\overline{x}_h)) = \Psi(v_h(\overline{x}_h)) = u_h(\overline{x}_h)$ and, by (2.26), $\hat{w}_h(y_h) \geq u_h(y_h)$ for neighbors y_h of x_h. Thus $u_h - \hat{w}_h$ attains a local maximum in $\overline{x}_h$, and as u_h is a finite-element sub-solution, we obtain $\hat{w}_h(\overline{x}_h) \leq (\Lambda_h \hat{w}_h)(\overline{x}_h)$ by remark 2.4, and therefore, in view of (2.31),

$$\max_{q \in S^{d-1}} \left\{ -\frac{1}{v_h(\overline{x}_h)} \left\langle \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}, q \right\rangle - \rho(\overline{x}_h, q) \right\} \leq C\theta\sqrt{h}.$$

Consequently we have, as $-v_h \leq e^M$, and $\rho(x,q) \geq \rho_* \cdot \|q\|$ on $\overline{\Omega} \times \mathbb{R}^d$, that

$$v_h(\overline{x}_h) + F\left(\overline{x}_h, \frac{\overline{x}_h - \overline{x}}{\sqrt{h}}\right) \leq \frac{C\theta \cdot e^M}{\rho_*} \cdot \sqrt{h},$$

and (2.24) is proved.

The cases remain, where $\overline{x}_h \in \partial\Omega_h$ or $\overline{x} \in \partial\Omega$. In the first case we obtain by (2.22) and (2.25), as $v = v_h$ on $\partial\Omega_h$, that

$$\begin{aligned} v_h(x_h^*) - v(x_h^*) &\leq v_h(\overline{x}_h) - v(\overline{x}) \\ &\leq v_h(\overline{x}_h) - v(\overline{x}_h) + v(\overline{x}_h) - v(\overline{x}) \\ &\leq L_v \|\overline{x}_h - \overline{x}\| = \mathcal{O}(\sqrt{h}). \end{aligned}$$

In the second case, where $\overline{x} \in \partial\Omega$, we have

$$\begin{aligned} v_h(x_h^*) - v(x_h^*) &\leq v_h(\overline{x}_h) - v(\overline{x}) \\ &\leq v_h(\overline{x}_h) - v_h(\overline{x}) + v_h(\overline{x}) - v(\overline{x}) \\ &\leq L_v \|\overline{x}_h - \overline{x}\| + \mathcal{O}(h) = \mathcal{O}(\sqrt{h}), \end{aligned}$$

where $v_h(\overline{x}) - v(\overline{x}) = \mathcal{O}(h)$ is the interpolation error on the boundary.

Second part: $v_h(x_h^*) < v(x_h^*)$. Because of the analogy to the first part of the proof, I only sketch the arguments here. Let $\psi : \overline{\Omega} \times \overline{\Omega}_h \to \mathbb{R}$ be defined by

$$\psi(x, x_h) = v(x) - v_h(x_h) - \frac{\|x - x_h\|^2}{2\sqrt{h}}$$

and let $(\overline{x}, \overline{x}_h) \in \overline{\Omega} \times \overline{\Omega}_h$ be such, that

$$\psi(\overline{x}, \overline{x}_h) = \max_{\overline{\Omega} \times \overline{\Omega}_h} \psi(x, x_h).$$

As $\psi(\overline{x},\overline{x}_h) \geq \psi(x_h^*,x_h^*)$, we have by the uniform Lipschitz-continuity of (v_h),

$$\begin{aligned}\frac{\|\overline{x}-\overline{x}_h\|^2}{2\sqrt{h}} &\leq v(\overline{x}) - v_h(\overline{x}_h) + v_h(x_h^*) - v(x_h^*) \\ &= v(\overline{x}) - v_h(\overline{x}) + v_h(\overline{x}) - v_h(\overline{x}_h) - (v(x_h^*) - v_h(x_h^*)) \\ &\leq v_h(\overline{x}) - v_h(\overline{x}_h) \\ &\leq L_v \|\overline{x}-\overline{x}_h\|,\end{aligned}$$

and thus (2.22) holds.

We assume, that $(\overline{x},\overline{x}_h) \in \Omega\times\Omega_h$. Then $\overline{x}$ is a maximum point of $x \mapsto \psi(x,\overline{x}_h)$, and as v is a viscosity (sub-)solution of (2.20), we obtain

$$v(\overline{x}) + F\left(\overline{x}, \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}\right) \leq 0. \tag{2.32}$$

From the fact, that $\overline{x}_h$ is a minimum point of $x_h \mapsto -\Psi(\overline{x},x_h)$, we infer that

$$v_h(\overline{x}_h) + F\left(\overline{x}_h, \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}\right) \geq -\mathcal{O}(\sqrt{h}), \tag{2.33}$$

which will be proved below. As in the first part, (2.32) and (2.33) yield the assertion.

Now we turn to the proof of (2.33). As $\psi(\overline{x},\overline{x}_h) \geq \psi(\overline{x},x_h)$ for all $x_h \in \overline{\Omega}_h$, we now have

$$v_h(x_h) \geq v_h(\overline{x}_h) + \frac{\|\overline{x}-\overline{x}_h\|^2 - \|\overline{x}-x_h\|^2}{2\sqrt{h}} =: w_h(x_h). \tag{2.34}$$

It holds for neighboring grid-points y_h of $\overline{x}_h$, that

$$w_h(\overline{x}_h) - w_h(y_h) = \left\langle \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}, \overline{x}_h - y_h \right\rangle + \frac{\|\overline{x}_h - y_h\|^2}{2\sqrt{h}}. \tag{2.35}$$

Again, (2.28) holds true, and we define by $\hat{w}_h = -\log(-w_h)$ on $\overline{\Omega}_h$ a finite-element function $\hat{w}_h$ with $\hat{w}_h(\overline{x}_h) = u_h(\overline{x}_h)$. Following equations (2.28)-(2.31), we obtain a constant $C > 0$ independent of h, such that

$$\hat{w}_h(\overline{x}_h) - \hat{w}_h(y) \leq -\frac{1}{v_h(\overline{x}_h)} \left\langle \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}, \overline{x}_h - y \right\rangle + C\theta\sqrt{h}\,\|\overline{x}_h - y\|$$

for all $y \in \partial\omega_h(\overline{x}_h)$. From (2.34) we read off, that $\hat{w}_h(\overline{x}_h) \geq (\Lambda_h\hat{w}_h)(\overline{x}_h)$, and thus

$$\max_{q\in S^{d-1}} \left\{ -\frac{1}{v_h(\overline{x}_h)} \left\langle \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}, q \right\rangle - \rho(\overline{x}_h, q) \right\} \geq -C\theta\sqrt{h}.$$

And (2.33) is proved, as we have with $-v_h \leq e^M$,

$$v_h(\overline{x}_h) + F\left(\overline{x}_h, \frac{\overline{x}-\overline{x}_h}{\sqrt{h}}\right) \geq -\frac{e^M}{\rho_*}\cdot C\theta\sqrt{h}.$$

The cases, where $\overline{x}_h \in \partial\Omega_h$ or $\overline{x} \in \partial\Omega$, can be treated as in the first part. □

2.5.3. Example. Let me remark, that the asserted convergence rate of $\mathcal{O}(\sqrt{h})$ is in most cases too pessimistic, even if the Hamilton-Jacobi equation has no classical solution $u \in \mathcal{C}^2(\overline{\Omega})$. Although the theorem 2.17 doesn't apply in the general case, the Hamilton-Jacobi equation has a smooth solution at least in a neighborhood of the boundary, where the boundary is smooth, and if the boundary value function g is appropriate, which can be shown by characteristic theory. As the shock lines in the solution then appear, where the characteristics intersect, and as the information flows into those shocks, and does not emanate from them, we also compute Hopf-Lax updates near the shock lines from a subregion, where the solution is smooth. Thus the convergence behavior is in such cases determined by the theory developed in section 2.4. Examples, where a lower rate of convergence is

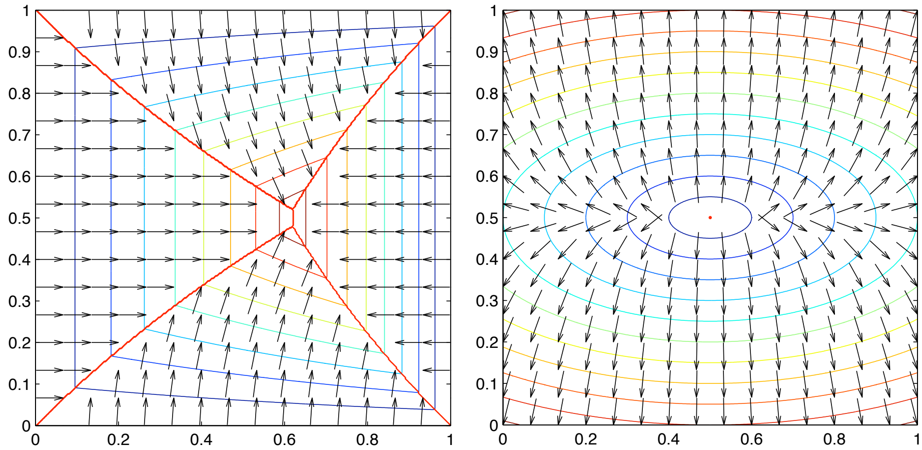

FIGURE 3. *Left.* The characteristics intersect at the discontinuities of Du (red lines). *Right.* The characteristics emante from a point, where u is not differentiable.

observed, typically include situations, where information spreads out from a singularity (a sharp corner) in the boundary, or when a distance function from a single point is computed.

To give an example, we consider first the following two dimensional problem:

$$\|Du(x,y)\| = e^x \text{ for } (x,y) \in \Omega, \quad u(x,y) = 0 \text{ on } \partial\Omega, \tag{2.36}$$

with $\Omega =]0,1[\times]0,1[$. As one easily verifies, the functions

$$\begin{aligned} u_1(x,y) &= e^x - 1 \\ u_2(x,y) &= e - e^x \\ u_3(x,y) &= e^x \cdot \sin(y) \\ u_4(x,y) &= e^x \cdot \sin(1-y) \end{aligned}$$

are (classical) solutions of $\|Du(x,y)\| = e^x$, subject to $u_1(0,y) = 0$, $u_2(1,y) = 0$, $u_3(x,0) = 0$ and $u_4(x,1) = 0$. The viscosity solution of (2.36) is given by

$$u(x,y) = \min\{u_1(x,y), u_2(x,y), u_3(x,y), u_4(x,y)\},$$

for $(x,y) \in \overline{\Omega}$. The directions of the characteristics, and the contours of u are depicted in figure 3 (left). The viscosity solution $u(x,y)$ takes the value u_1, u_2, u_3 and $u_4(x,y)$ in the left, right, upper and lower region, respectively. On the region boundaries (the red lines) we observe jumps in the derivative of u. Nevertheless, the characteristics run into the singularities, and first order convergence of the finite-element solution is observed. Results of a numerical calculation can be found in table 1. For the computation, I used the Fast Marching Method on regular meshes like the one shown in figure 2.

The second example is the (generalized) Eikonal equation

$$u_x(x,y)^2 + \frac{1}{4}u_y(x,y)^2 = 1 \text{ for } (x,y) \in \Omega, \quad u(x_0,y_0) = 0,$$

where $x_0 = y_0 = 0.5$, and $\Omega = [0,1]^2 \setminus \{(x_0,y_0)\}$is the computational domain. A viscosity solution u ($-u$ is another one) is given by the distance function

$$u(x,y) = \sqrt{(x-0.5)^2 + 4(y-0.5)^2},$$

	Example 1		Example 2	
Mesh	L^∞ error	order	L^∞ error	order
11×11	0.07163		0.07192	
21×21	0.03781	0.92169	0.04343	0.72749
41×41	0.01940	0.96239	0.02846	0.61010
81×81	0.00982	0.98158	0.01776	0.67992
161×161	0.00510	0.94533	0.01079	0.71961
321×321	0.00253	1.01367	0.00790	0.448321

TABLE 1. Numerical Results for the two examples depicted in figure 3. To obtain the convergence order, I calculated $\log(e_1/e_2)/\log(2)$, where $e_2 = \|u_h - u\|_\infty$ denotes the error on the finer mesh, and the grid-spacing is reduced by half in each step.

which is shown in figure 3 (right). The solution u has a singularity at (x_0, y_0), and the characteristics spread out from that point. Thus we expect a lower rate of convergence, as it can be confirmed by a numerical calculation, where we used the adaptive Gauss-Seidel method on regular $n \times n$ meshes. The results are summarized in table 1.

CHAPTER 3

Computation of the Discrete Solution

This chapter is devoted to the computation of the discrete solution u_h, which has been characterized as the unique fixed-point of the Hopf-Lax update function

$$u_h = \Lambda_h u_h, \qquad u_h|_{\partial\Omega_h} = g|_{\partial\Omega_h},$$

in section 2.2. For the whole chapter, we assume at least (H1)-(H4) to be fulfilled, such that the fixed-point equation has a unique solution u_h, and we assume the regularity conditions (2.1) and (2.2) for the triangulation. For some results, even (2.3) will be required.

In section 3.2, we discuss iterative methods for the fixed-point problem, motivated by similar iterative methods in linear algebra for solving systems of linear equations, and analyze their complexity. All these methods are usually terminated, when the residual $\|\hat{u}_h - \Lambda_h \hat{u}_h\|_\infty$ falls below a user-defined tolerance. So the termination error $\|u_h - \hat{u}_h\|_\infty$ has to be determined in terms of the local residual.

For isotropic Hamilton-Jacobi equations, that is equations, where $H(x,p) = f(x, \|p\|)$ does not depend on the direction of p, a direct (non-iterative) method is available to compute the finite-element solution, the so-called Fast Marching Method, described by Tsitsiklis in [Tsi95] and Sethian in [Set96]. It is based on a causality principle, which states, that on an acute triangulation the value of u_h in some grid-point x_h depends only on the smaller grid-function values in the neighboring grid-points. Then the finite-element solution can be computed, starting at the boundary, where values are provided, and following increasing values of u_h. In an implementation, a heap sort strategy will be involved, in order to arrange the values of u_h by their size, leading to a total complexity of $\mathcal{O}(N \log N)$, where N denotes the number of grid-points. An extension of the Fast Marching Method to non-acute triangulations was given in [KS98]. We briefly follow this approach, and introduce the virtual update strategy for triangles with obtuse angles. Recently Yatziv, Bartesaghi and Sapiro suggested in [YBS06] to use a bucket sort algorithm for the Fast Marching Method, reducing its overall complexity to $\mathcal{O}(N)$, but introducing a local error due to inexact sequencing. In subsection 3.3.5 I discuss this approach, and give a rigorous estimate on the introduced error.

An elaborate generalization of the Fast Marching Method for anisotropic equations, suggested in [SV03], leads to the so-called Ordered Upwind Method. With a similar technique of discretization, this method does not compute a solution of the above fixed-point problem, but it also constructs a suitable approximation to the viscosity solution. However, the complexity $\mathcal{O}(\Upsilon \cdot N \log N)$ of the method is additionally affected by the anisotropy coefficient Υ of the Hamilton-Jacobi equation, a quantity, which measures the anisotropic deformation of the zero level-set of $p \mapsto H(x,p)$. The description of the Ordered Upwind Method is subject of section 3.4.

All methods for solving the fixed-point problem require, of course, the computation of the Hopf-Lax update for single grid-points $x_h \in \Omega_h$,

$$(\Lambda_h u_h)(x_h) = \min_{y \in \partial\omega_h(x_h)} \left\{ u_h(y) + \rho(x_h, x_h - y) \right\},$$

the local approximation of the viscosity solution. Generally, this results in the minimization of a convex function, which can be done by iterative methods, as suggested in subsection 3.1.2. For the generalized Eikonal equation $\langle Du(x), M(x)Du(x)\rangle = 1$, an update formula can be derived, which allows for the computation of Hopf-Lax updates in $\mathcal{O}(1)$ time (see subsection 3.1.3). With a different approach, this formula was already given in [**KS98**] for the Eikonal equation $\|Du\| = f(x)$.

3.1. Computation of Hopf-Lax Updates

The computation of Hopf-Lax updates leads to an ordinary convex program. In the two dimensional case, a scalar function $f : \mathbb{R} \to \mathbb{R}$ has to be minimized on the interval $[0, 1]$. Two simple iterative methods for computing the minimum, direct search, and golden section search are considered in subsection 3.1.2. The error brought in because of the approximate minimization is studied, and it is shown, how the iterative minimization affects the total complexity for solving $u_h = \Lambda_h u_h$. For equations of Eikonal type, $\langle Du(x), M(x)Du(x)\rangle = 1$, an update formula is derived in subsection 3.1.3. From a different point of view, Kimmel and Sethian attained in [**KS98**] an update formula for the *standard* Eikonal equation $\|Du\| = f(x)$.

3.1.1. Convex Programs.

In order to compute the finite-element approximation to the viscosity solution of (1.1), the Hopf-Lax update

$$(\Lambda_h u_h)(x_h) = \min_{y\in\partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}$$

has to be evaluated. Let Σ_h be some triangulation of the computational domain $\overline{\Omega}$. For some simplex $\sigma_h = [x_h, x_1, \ldots, x_d] \in \Sigma_h$ adjacent to x_h (the interval brackets denote the convex hull here), we denote by

$$v^{\sigma_h}(x_h) = \min\left\{\sum_{i=1}^{d} t_i u_h(x_i) + \rho\Big(x_h, x_h - \sum_{i=1}^{d} t_i x_i\Big)\,;\quad \sum_{i=1}^{d} t_i = 1,\ t_i \geq 0\right\} \tag{3.1}$$

the *update from the simplex* σ_h. Then we get the Hopf-Lax update as the minimum

$$\Lambda_h u_h(x_h) = \min_{\sigma_h \text{ adjacent to } x_h} v^{\sigma_h}(x_h)$$

of the updates computed from the adjacent simplices.

Let me remark, that (3.1) is an ordinary convex program, where the convex function

$$f(t_1, \ldots, t_d) = \sum_{i=1}^{d} t_i u_h(x_i) + \rho\Big(x_h, \sum_{i=1}^{d} t_i(x_h - x_i)\Big)$$

is to be minimized subject to the the constraints

$$\sum_{i=1}^{d} t_i = 1, \qquad t_i \geq 0, \quad i = 1, \ldots, d.$$

3.1.2. Simple Iterative Methods.

In the two dimensional case, we have to minimize the scalar convex function

$$f(t) = t\tilde{u}_h(y_h) + (1-t)\tilde{u}_h(z_h) + \rho\big(x_h, x_h - ty_h - (1-t)z_h\big), \quad t \in [0, 1]$$

in order to compute an update in x_h from the triangle $[x_h, y_h, z_h]$. Here $\tilde{u}_h(y_h), \tilde{u}_h(z_h)$ denote boundary values, or already computed trial values in the neighboring points. In this subsection, two simple iterative methods are proposed, in order to approximate the minimum, direct search and golden section search. For notational simplicity, let f be extended to $\mathbb{R}$ by $f(0)$ to the left of $[0, 1]$, and by $f(1)$ to the right of $[0, 1]$.

For the direct search, we choose with some $n \in \mathbb{N}$ equidistant points $t_i = i/n$, $i = 0, \dots, n$ in the interval $[0, 1]$, and simply approximate the minimal value by

$$\min_{t \in [0,1]} f(t) \approx \min_{i=0,\dots,n} f(t_i).$$

Let t^* denote the point in $[0, 1]$ where f takes its minimal value, and let $i^* \in \{0, \dots, n\}$ denote the index, where the right hand side is minimal. As $f(i^*/n) \leq f((i^*-1)/n), f((i^*+1)/n)$, and by convexity of f, we have $t^* \in](i^*-1)/n, (i^*+1)/n[$ (more precisely there is a minimal point of f in this interval). Thus, with $\epsilon = 1/n$, we have $|t^* - i^*/n| < \epsilon$. Consequently,

$$|f(t^*) - f(i^*/n)| \leq \epsilon\big(|\tilde{u}_h(y_h) - \tilde{u}_h(z_h)| + \rho(x_h, y_h - z_h)\big). \tag{3.2}$$

By assumption (2.2) on the triangulation, we have $\|y_h - z_h\| \leq h$, and by theorem 2.9 the finite-element solution u_h, which is to be computed, is Lipschitz-continuous (with a Lipschitz constant L independent of the grid-spacing). In theorem 2.15 we have seen, that the local truncation error, for a smooth solution, is $\mathcal{O}(h^2)$. If $\tilde{u}_h(y_h), \tilde{u}_h(z_h)$ were good approximations to the finite-element solution, then we could deduce from inequality (3.2), that $|f(t^*) - f(i^*/n)| \leq \epsilon(L + \rho^*)h$. Thus, in view of the truncation error, we should have $\epsilon \approx \mathcal{O}(h)$, such that the local error is not worsened by the approximation of $f(t^*)$. Consequently, we would choose $n = \mathcal{O}(1/h)$ points in $[0, 1]$ for the approximate minimization of f. Let me remark, that of course the total complexity for solving the discrete system (2.8) is affected by the complexity of the local minimization. For example, with N denoting the number of grid-points, $\mathcal{O}(N)$ updates in single grid-points have to be computed within the Ordered Upwind Method, which leads to a total complexity of $\mathcal{O}(N \cdot \log N)$ for computing the finite-element solution, when it is assumed, that the local minimization requires a constant number of operations. If the local minimization in every triangle is done by means of the described direct search, then the total complexity becomes $\mathcal{O}(N^{3/2})$, as $h \propto N^{-1/2}$ for a uniform family of triangulations.

As an alternative, the golden section search method may be used. This method generates a sequence (I_k) of nested intervals beginning with $I_0 = [0, 1]$, such that the ratio of the lengths of two subsequent intervals is the golden section ratio $g = (1 + \sqrt{5})/2$. The limit point of (I_k) would be the minimal point t^* for $f(t)$. As the length of the kth interval is g^{-k}, and by the consideration above for the direct search, we obtain a suitable approximation of $f(t^*)$ after $|\log(h)/\log(g)|$ subdivisions. As $|\log h| \propto \log N$, where N denotes the number of grid-points, the total complexity of the Ordered Upwind Method with golden section search is still $\mathcal{O}(N \log(N))$. Details on the implementation of the golden section method can be found in [**PTVF02**], among other sources.

3.1.3. The Update Formula for the Eikonal Equation. For the Eikonal equation in two dimensions, we derive now a formula for (3.1). For this purpose, let $\Omega \subset \mathbb{R}^2$ and $M : \overline{\Omega} \to \mathbb{R}^{2\times 2}$ be a continuous mapping into the *symmetric positive definite* 2×2-matrices. We denote the corresponding inner product by $\langle p, q\rangle_{M(x)} = \langle p, M(x)q\rangle$, and its subordinate norm by $\|p\|_{M(x)} = \langle p, p\rangle_{M(x)}^{1/2}$.

Now, we consider the Dirichlet problem for the *generalized eikonal equation*,

$$\|Du(x)\|_{M(x)} = 1 \quad x \in \Omega, \qquad u|_{\partial\Omega} = g.$$

The associated Hamiltonian $H(x, p) = \|p\|_{M(x)} - 1$ satisfies the assumptions (H1)–(H4). The support function of the zero-level set defined in (1.9) is simply given by polar of $\|\cdot\|_{M(x)}$, namely,

$$\rho(x, q) = \max_{H(x,p)=0} \langle p, q\rangle = \max_{\|p\|_{M(x)}=1} \langle p, q\rangle = \|q\|_{M(x)^{-1}}, \tag{3.3}$$

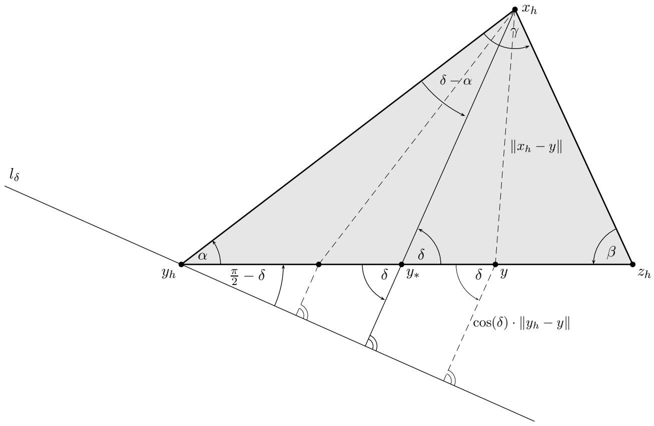

FIGURE 1. Geometry of the minimization of $\cos(\delta)\|y-y_h\|+\|x_h-y\|$ for $y \in [y_h, z_h]$. Note that for $\delta > \pi/2$ the segment through y perpendicular to l_δ has *negative* length $\cos(\delta)\cdot\|y-y_h\|$.

compare example 1.28. The Hopf–Lax update function becomes for $x_h \in \Omega_h$,

$$(\Lambda_h u_h)(x_h) = \min_{y \in \partial\omega_h(x_h)} \left\{ u_h(y) + \|x_h - y\|_{M(x_h)^{-1}} \right\}. \tag{3.4}$$

The next lemma shows, how the update $v^{\sigma_h}(x_h)$ from a single simplex adjacent to x_h can be determined by an elementary geometric argument. At first, only the classic eikonal equation (with $M(x) \equiv I$) is considered.

LEMMA 3.1 ([BR06, Lemma 12]): *Let $\sigma_h \in \Sigma_h$ be the triangle with the vertices x_h, y_h, z_h and $u_h \in \mathcal{V}_h$. Denote the angles at y_h, z_h by α, β, respectively. Defining*

$$\Delta = \frac{u_h(z_h) - u_h(y_h)}{\|z_h - y_h\|}$$

and $\cos(\delta) = \Delta$ if $|\Delta| \le 1$, we obtain

$$v^{\sigma_h}(x_h) = \begin{cases} u_h(y_h) + \|x_h - y_h\|, & \cos(\alpha) \le \Delta, \\ u_h(y_h) + \cos(\delta - \alpha)\cdot\|x_h - y_h\|, & \alpha \le \delta \le \pi - \beta, \\ u_h(z_h) + \|x_h - z_h\|, & \Delta \le \cos(\pi - \beta). \end{cases} \tag{3.5}$$

PROOF. (From [BR06].) As one figures out, (3.5) yields the same value, if the roles of y_h and z_h are interchanged in lemma 3.1. Thus it suffices to treat the case, where $\Delta \ge 0$.

By the definition of Δ, we have

$$v_h^{\sigma_h} = \min_{y \in [y_h, z_h]} \{u_h(y) + \|x_h - y\|\} = u_h(y_h) + \min_{y \in [y_h, z_h]} \{\Delta \cdot \|y - y_h\| + \|x_h - y\|\}.$$

For $\Delta \ge 1$, the first case in formula (3.5) applies, and as

$$\Delta \cdot \|y - y_h\| + \|x_h - y\| \ge \|y - y_h\| + \|x_h - y\| \ge \|x_h - y_h\|,$$

it is indeed $v^{\sigma_h}(x_h) = u_h(y_h) + \|x_h - y_h\|$.

Now, let $0 \le \Delta < 1$ so that $\cos(\delta) = \Delta$ defines an angle $0 < \delta \le \pi/2$. A look at Figure 1 shows that

$$\cos(\delta)\cdot\|y - y_h\| + \|x_h - y\| \tag{3.6}$$

attains its minimum at the unique intersection y_* of two straight lines: the first line running through y_h and z_h, the second line running through x_h perpendicular to l_δ. Here, l_δ is the straight line that encloses at y_h with $[y_h, z_h]$ the angle $\pi/2 - \delta$. We observe that the value of the minimum is simply $\cos(\delta - \alpha) \cdot \|x_h - y_h\|$. A further look at Figure 1 teaches that $y_* \in [y_h, z_h]$ if and only if

$$0 \le \delta - \alpha \le \gamma = \pi - \alpha - \beta, \qquad \text{that is,} \qquad \alpha \le \delta \le \pi - \beta.$$

If $\delta < \alpha$, or equivalently $\Delta > \cos(\alpha)$, y_* is to the left of y_h and the minimum of (3.6) in $[y_h, z_h]$ is attained at y_h. On the other hand, if $\delta > \pi - \beta$, or equivalently $\Delta < \cos(\pi - \beta)$, y_* is to the right of z_h and the minimum of (3.6) in $[y_h, z_h]$ is attained at z_h. □

Now we return to the construction of an update formula for (3.4). As for $x_h \in \Omega_h$, $M(x_h)^{-1}$ is a symmetric positive definite matrix, there is a symmetric positive definite matrix $T(x_h)$, such that $T(x_h)^2 = M(x_h)^{-1}$, and we have

$$\|q\|^2_{M(x_h)^{-1}} = \left\langle T(x_h)^2 q, q \right\rangle = \langle T(x_h)q, T(x_h)q\rangle = \|T(x_h)q\|^2 .$$

Thus the update formula for the general case is obtained from lemma 3.1 by transition to the transformed triangle $\hat{\sigma}_h$ with the corners

$$\hat{x}_h,\ \hat{y}_h,\ \hat{z}_h = T(x_h)x_h,\ T(x_h)y_h,\ T(x_h)z_h,$$

associated with values $u_h(x_h), u_h(y_h), u_h(z_h)$ of the finite-element function.

This way we immediately obtain the following update procedure, writing $\langle p, q\rangle_x = \langle p, q\rangle_{M(x)^{-1}}$, $\|p\|_x = \|p\|_{M(x)^{-1}}$, $c_\alpha = \cos(\alpha)$, and $c_\beta = \cos(\beta)$ for short (note that we used the cosine addition formula to compute $\cos(\alpha - \delta)$ for implementation purposes):

Algorithm 3.2 (Update Formula for the Eikonal Equation):
Let $x_h \in \Omega$, $u_h \in \mathcal{V}_h$, and $\sigma_h = [x_h, y_h, z_h] \in \Sigma_h$.

$$\Delta = \frac{u_h(z_h) - u_h(y_h)}{\|z_h - y_h\|_{x_h}}$$

$$c_\alpha = \frac{\langle x_h - y_h, z_h - y_h\rangle_{x_h}}{\|x_h - y_h\|_{x_h} \cdot \|z_h - y_h\|_{x_h}}, \quad c_\beta = \frac{\langle x_h - z_h, y_h - z_h\rangle_{x_h}}{\|x_h - z_h\|_{x_h} \cdot \|y_h - z_h\|_{x_h}}$$

if $c_\alpha \le \Delta$

$$v^{\sigma_h}(x_h) = u_h(y_h) + \|x_h - y_h\|_{x_h}$$

else if $\Delta \le -c_\beta$

$$v^{\sigma_h}(x_h) = u_h(z_h) + \|x_h - z_h\|_{x_h}$$

else

$$v^{\sigma_h}(x_h) = u_h(y_h) + \left(c_\alpha \Delta + \sqrt{(1 - c_\alpha^2)(1 - \Delta^2)}\right) \|x_h - y_h\|_{x_h}$$

With different ideas on a discretization, exactly the same update formula has been obtained for the standard eikonal equation by Kimmel and Sethian in [**KS98**] (see also [**Set99b**, section 10.3.1]), who use for acute triangulations the methodology of [**BS98**] to construct upwind schemes on unstructured meshes, and, independently, by the geophysicist Fomel [**Fom97**], who locally uses Fermat's principle of shortest traveltimes (which is closely related to the local use of the Hopf-Lax formula).

Sethian [**Set99b**, section 10.1] shows further that this update formula generalizes the upwind finite-difference scheme on Cartesian grids given by Rouy and Tourin [**RT92**].

3.2. Iterative Methods for the Fixed-Point Problem

In this section, several iterative methods for the fixed-point problem $u_h = \Lambda_h u_h$ are analyzed. We start with the Jacobi iteration, which already served us to show the existence of the discrete solution in theorem 2.6. In subsection 3.2.1, I show the convergence of this method for arbitrary initial iterates, and give an estimate on the number of iterations required to reach a pre-defined tolerance. In [**RT92**], Rouy and Tourin use a Gauss-Seidel iteration, to solve their discretization of the Eikonal equation on a Cartesian mesh. I include this method in the presentation of iterative methods, and show, that the residual $\|u_h^n - \Lambda_h u_h^n\|_\infty$ is bounded by the difference $\left\|u_h^n - u_h^{n-1}\right\|_\infty$ between two consecutive iterates.

In subsection 3.2.4, I propose an adaptive Gauss-Seidel iteration for the fixed-point problem, modeled after a similar relaxation technique, used for linear systems in the discretization of linear elliptic boundary value problems (compare [**PR93**]). Practically it turns out, that the adaptive Gauss-Seidel method is quite fast, and can compete with the Fast Marching Method, and especially with the Ordered Upwind Method for anisotropic equations. However, in the worst case it has the same asymptotic complexity as Jacobi's method, and becomes inefficient on large grids.

Finally, in subsection 3.2.5, I provide an estimate on the iteration error, that is, the error in the finite-element function due to the termination of an iterative method, when the tolerance has been reached. It is shown, how the iteration error, and the local residual, which is controllable by an iterative method, are connected.

3.2.1. Jacobi Iteration. In section 2.2, we have shown the existence of a solution to the discrete system, given as fixed-point equation (2.8), by proving the convergence of the fixed-point iteration. Practically, the fixed-point iteration is terminated, if the residual falls below a user-defined tolerance. This results in an algorithm given as follows.

ALGORITHM 3.3 (Jacobi Iteration):
Let $\tau > 0$ denote a user-defined tolerance.

(1) Choose initial iterate $u_h^0 \in \mathcal{V}_h$, such that $u_h^0|_{\partial\Omega_h} = g|_{\partial\Omega_h}$.

(2) For $n = 0, 1, 2, \dots$, let
$$u_h^{n+1} = \Lambda_h u_h^n,$$
until $\|u_h^n - \Lambda_h u_h^n\|_\infty = \left\|u_h^n - u_h^{n+1}\right\|_\infty \le \tau$.

(3) Return u_h^{n+1}.

By proposition 2.5, the sequence of the residuals $\left\|u_h^{n+1} - u_h^n\right\|_\infty$ is monotonically decreasing. Of course also u_h^{n+1} fulfills

$$\left\|u_h^{n+1} - \Lambda_h u_h^{n+1}\right\|_\infty = \left\|\Lambda_h u_h^n - \Lambda_h u_h^{n+1}\right\|_\infty \le \left\|u_h^n - u_h^{n+1}\right\|_\infty = \|u_h^n - \Lambda_h u_h^n\|_\infty \le \tau,$$

when the algorithm terminates. As u_h^{n+1} must have been computed in order to estimate the residual for u_h^n, and as it yields a smaller residual, u_h^{n+1} is returned in step (3).

3.2.2. Total Complexity. Let (u_h^n) denote the sequence generated by algorithm 3.3, if it is not terminated after the tolerance has been reached. In theorem 2.6 it is shown, that this sequence is monotonically increasing and bounded, provided that $u_h^0 \le \Lambda_h u_h^0$. Thus the limit point exists, which has to be a fixed-point of Λ_h by continuity. In the following theorem we consider arbitrary initial iterates

u_h^0 for the Jacobi iteration, and give a rough estimate on the complexity of this method. For that purpose, we have to assume the stronger compatibility condition (H4)$'$ $H(x,0) < 0$ on $\overline{\Omega}$ for the Hamilton function, such that there is a positive lower bound ρ_* for $\rho(x,q)$ on $\overline{\Omega} \times S^{d-1}$. For the asymptotic complexity, we consider a sequence of finite-element solutions (u_h) computed on a *uniform* family of triangulations.

THEOREM 3.4: *Assume (H1)-(H3),(H4)$'$, (2.2)-(2.3). Let the initial iterates u_h^0 be uniformly bounded. Then for a fixed tolerance $\tau > 0$ the Jacobi iteration for u_h terminates after at most $n = \mathcal{O}(h^{-1})$ steps.*

PROOF. We assume, that $\|u_h^0\|_\infty \leq M$ with some $M > 0$ uniformly in h. First I show, that the whole sequence $(u_h^n)_n$ generated by the algorithm is uniformly bounded. For this reason let $v_h \in \mathcal{V}_h$ denote the solution of

$$v_h = \Lambda_h v_h, \qquad v_h|_{\partial\Omega_h} \equiv M. \tag{3.7}$$

From theorem 2.6 we infer, that the Jacobi iteration for (3.7) converges, if the initial iterate $v_h^0 \equiv M$ is chosen. By theorem 2.9, v_h is bounded by

$$\|v_h\|_\infty \leq M + C_\Omega \cdot N \cdot \theta \cdot \frac{\beta}{\alpha} \cdot \operatorname{diam}(\Omega) =: M_1,$$

and by the monotonicity of (v_h^n), we have $\|v_h^n\|_\infty \leq M_1$ for all $n \in \mathbb{N}_0$. As we have $u_h^0 \leq v_h^0$, we obtain by the monotonicity of Λ_h, shown in proposition 2.5, that

$$-M \leq u_h^n \leq v_h^n \leq M_1$$

for all $n \in \mathbb{N}_0$. Thus $\|u_h^n\|_\infty \leq M_1$ for all $n \in \mathbb{N}$ uniformly in $h \to 0$.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(t) = -e^{-t}$. Then for $n \in \mathbb{N}$ and $x_h \in \Omega_h$, the following equations are equivalent:

$$\begin{aligned} u_h^n(x_h) &= \min_{y \in \partial\omega_h(x_h)} \left\{ u_h^{n-1}(y) + \rho(x_h, x_h - y) \right\} \\ \Phi(u_h^n(x_h)) &= \min_{y \in \partial\omega_h(x_h)} \left\{ -e^{-(u_h^{n-1}(y) + \rho(x_h, x_h - y))} \right\} \\ \Phi(u_h^n(x_h)) &= \min_{y \in \partial\omega_h(x_h)} \left\{ \Phi(u_h^{n-1}(y)) e^{-\rho(x_h, x_h - y)} \right\}. \end{aligned} \tag{3.8}$$

By (H4)$'$ and lemma 1.17, we have $\rho_* = \min_{\overline{\Omega} \times S^{d-1}} \rho(x,q) > 0$, and thus by (2.3) $\rho(x_h, x_h - y) \geq \rho_* h/\theta$ for all $y \in \partial\omega_h(x_h)$. From (3.8) we obtain

$$\left\|\Phi(u_h^{n+1}) - \Phi(u_h^n)\right\|_\infty \leq e^{-\rho_* h/\theta} \cdot \left\|\Phi(u_h^n) - \Phi(u_h^{n-1})\right\|_\infty. \tag{3.9}$$

By the mean value theorem, we have

$$e^{-M_1} \left\|u_h^{n+1} - u_h^n\right\|_\infty \leq \left\|\Phi(u_h^{n+1}) - \Phi(u_h^n)\right\|_\infty.$$

Thus we obtain from (3.9), that

$$e^{-M_1} \left\|u_h^{n+1} - u_h^n\right\|_\infty \leq \left(e^{-\rho_* h/\theta}\right)^n \cdot \left\|\Phi(u_h^1) - \Phi(u_h^0)\right\|_\infty.$$

Consequently, there are constants $c, C > 0$ independent of h, such that for $n \in \mathbb{N}$

$$\left\|u_h^{n+1} - u_h^n\right\|_\infty \leq C e^{-cnh}.$$

The convergence of (u_h^n) and the asserted complexity are immediate consequences. The last inequality shows, that (u_h^n) is a Cauchy sequence, and thus convergent to some finite-element function $u_h \in \mathcal{V}_h$, which has to be a fixed-point of Λ_h, by continuity. The residual fulfills $\left\|u_h^{n+1} - u_h^n\right\|_\infty \leq \tau$ after $n = \mathcal{O}(1/h)$ iterations. □

If we consider uniform triangulations with $N = \left|\overline{\Omega}_h\right|$ vertices, where $h \sim N^{-1/d}$, the total complexity of algorithm 3.3 will be $\mathcal{O}(N^{1+1/d})$. Here we assumed, that the complexity of the Hopf-Lax update $\Lambda_h u_h(x_h)$ in a single node is independent of the grid-spacing h. Then the computation of $u_h^{n+1} = \Lambda_h u_h^n$ requires $\mathcal{O}(N)$ Hopf-Lax updates, and because of theorem 3.4, the iteration terminates after $\mathcal{O}(h^{-1})$ steps.

3.2.3. Gauss-Seidel Iteration. Let $\hat{u}_h$ denote an approximate finite-element solution. In the Jacobi method 3.3, we iteratively replace $\hat{u}_h$ by $\Lambda_h \hat{u}_h$. The first improvement of this method is to overwrite $\hat{u}_h(x_h)$ by $\Lambda_h \hat{u}_h(x_h)$, whenever an update in a single node $x_h \in \Omega_h$ is computed. This possibly leads to a smaller residual, but anyway the iterate from the preceding step can be overwritten, and thus does not have to be stored separately.

Algorithm 3.5 (Gauss-Seidel Iteration):
Let $\tau > 0$ denote a user-defined tolerance and let $\Omega_h = \{x_h^1, \dots, x_h^N\}$.

(1) Choose initial iterate $\hat{u}_h \in \mathcal{V}_h$, such that $\hat{u}_h = g$ on $\partial\Omega_h$.

(2) Repeat

$$\text{For } j = 1, \dots, N: \qquad \begin{cases} r_j & = (\Lambda_h \hat{u}_h)(x_h^j) - \hat{u}_h(x_h^j) \\ \hat{u}_h(x_h^j) & = \hat{u}_h(x_h^j) + r_j, \end{cases}$$

until $\max_j |r_j| \leq \tau$.

(3) Return $\hat{u}_h$.

Let $\{x_h^1, \dots, x_h^N\} = \Omega_h$ be some enumeration of Ω_h. We denote by

$$(\Lambda_h^j u_h)(x_h^k) = \begin{cases} (\Lambda_h u_h)(x_h^k), & k = j \\ u_h(x_h^k), & \text{otherwise} \end{cases}$$

the update in x_h^j, defined as function $\Lambda_h^j : \mathcal{V}_h \to \mathcal{V}_h$. Then the iteration 3.5 with the initial value u_h^0 generates the sequence (u_h^n), where

$$u_h^{n+1} = \Lambda_h^N \cdots \Lambda_h^1 u_h^n,$$

and the update operator in the Gauss-Seidel iteration can be written as

$$\Lambda_h^{GS} : \mathcal{V}_h \to \mathcal{V}_h, \quad u_h \mapsto \Lambda_h^N \cdots \Lambda_h^1 u_h.$$

Needless to say, $\Lambda_h^1, \dots, \Lambda_h^N$, and therefore also Λ_h^{GS}, fulfill the properties (1) and (2) from proposition 2.5, that is, $v_h \leq w_h$ implies $\Lambda_h^{GS} v_h \leq \Lambda_h^{GS} w_h$, and moreover $\left\|\Lambda_h^{GS} v_h - \Lambda_h^{GS} w_h\right\|_\infty \leq \|v_h - w_h\|_\infty$. Hence we have, as for the Jacobi iteration,

$$\left\|u_h^{n+1} - u_h^n\right\|_\infty \leq \left\|u_h^n - u_h^{n-1}\right\|_\infty,$$

that is, the difference between two consecutive iterates is decreasing. But in general the sequence of the residuals $\|\Lambda_h u_h^n - u_h^n\|_\infty$ is *not* necessarily decreasing in every step, unlike to the Jacobi iteration.

The Gauss-Seidel iteration 3.5 is terminated, when $\left\|u_h^{n+1} - u_h^n\right\| \leq \tau$. From the next lemma it follows, that then also the residual

$$\left\|\Lambda_h u_h^{n+1} - u_h^{n+1}\right\| \leq \tau$$

fulfills the tolerance condition.

Lemma 3.6: *Let (u_h^n) denote the sequence generated by algorithm 3.5. Then*

$$\left\|\Lambda_h u_h^{n+1} - u_h^{n+1}\right\|_\infty \leq \left\|u_h^{n+1} - u_h^n\right\|_\infty.$$

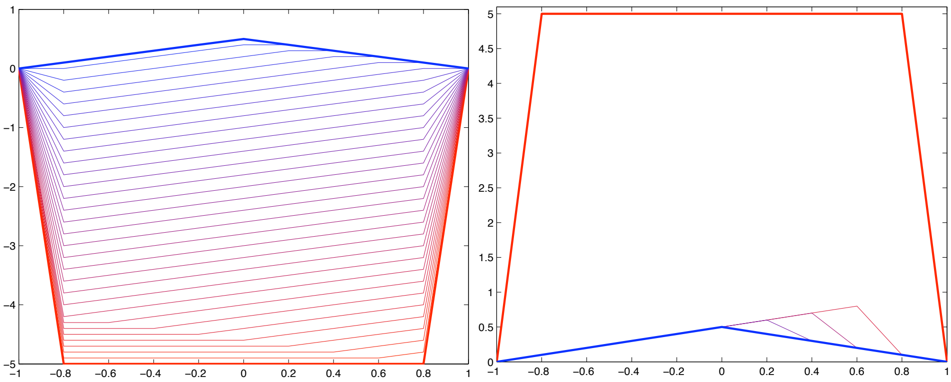

FIGURE 2. The sequence (u_h^n) generated by the Gauss-Seidel method, for two initial iterates (thick red lines) in comparison.

PROOF. For $x_h^j \in \Omega_h$, we have

$$\left|(\Lambda_h u_h^{n+1} - u_h^{n+1})(x_h^j)\right| = \left|(\Lambda_h u_h^{n+1} - \Lambda_h^{GS} u_h^n)(x_h^j)\right| \le \max_{y_h \in \partial\omega_h(x_h^j)} \left|(u_h^{n+1} - \Lambda_h^{j-1} \cdots \Lambda_h^1 u_h^n)(y_h)\right|,$$

where $\Lambda_h^{j-1} \cdots \Lambda_h^1 u_h^n(y_h) = u_h^{n+1}(y_h)$, if $y_h = x_h^k$ for some $k \in \{1, \dots, j-1\}$. Otherwise, if $y_h \notin \left\{x_h^1, \dots, x_h^{j-1}\right\}$, then $\Lambda_h^{j-1} \cdots \Lambda_h^1 u_h^n(y_h) = u_h^n(y_h)$. Thus

$$\left|(\Lambda_h u_h^{n+1} - u_h^{n+1})(x_h^j)\right| \le \max_{y_h \in \partial\omega_h(x_h^j)} \left|(u_h^{n+1} - u_h^n)(y_h)\right|,$$

and the assertion follows. □

Similar to the proof of theorem 3.4 it can be shown, that the Gauss-Seidel iteration has at most the same asymptotic complexity as the Jacobi iteration. Practically, it turns out to be faster, as in step (2), the value in some grid-point x_h^j may benefit from the values in $x_h^1, \dots, x_h^{j-1}$, which were updated in the same step. This effect depends, of course, on the enumeration of the grid-points, that is, on the order in which the updates are computed, and on the choice of the initial iterate u_h^0. Figure 2 shows the iterates (u_h^n) for the discretization of the one dimensional problem $|u'| = 1$, $u(-1) = u(1) = 0$ on $[-1, 1]$. The 11 grid-points were numbered from the left to the right. For the left frame, I chose $u_h^0 = -5$ on Ω_h. 31 iterations were necessary in order to compute the finite-element solution. With $u_h^0 = +5$ on Ω_h, the solution was obtained after only 5 iterations. If I had numbered the points by their distance from the boundary points $-1, 1$, only one iteration would have been enough. As we can see, the order of the updates and the initial iterate should be chosen in such way, that the problem's inherent flow of information from the boundary inwards the domain Ω is maintained.

3.2.4. Adaptive Gauss-Seidel Iteration. This method is an adaptive version of the Gauss-Seidel iteration, which pays attention to the flow of information and which additionally enables us to conserve a lot of unnecessary updates.

In the initial iterate u_h^0, only correct information on the boundary $\partial\Omega_h$ is available. Therefore, the update process should start in those points, which are adjacent to some boundary point. The order of the updates in the adaptive Gauss-Seidel iteration is organized in such way, that the information about the solution u_h, which initially exists only in the boundary points, may propagate inwards the domain Ω_h, along adjacent grid-points.

In step (2) of the Gauss-Seidel iteration, updates are computed in every point $x_h \in \Omega_h$, even if in some of them the tolerance condition $|(u_h - \Lambda_h u_h)(x_h)| \le \tau$ is already fulfilled. In the adaptive Gauss-Seidel iteration, a point $x_h \in \Omega_h$ is not touched anymore, if the residual in x_h has fallen below the tolerance, unless a neighbor of x_h is assigned a different value (which might have influence on the residual in x_h). The algorithm is given as follows.

ALGORITHM 3.7 (Adaptive Gauss-Seidel Iteration):
Let $\tau > 0$ denote a user-defined tolerance.

(1) Let the initial iterate be defined by

$$\hat{u}_h(x_h) = \begin{cases} \infty, & x_h \in \Omega_h \\ g(x_h), & x_h \in \partial\Omega_h \end{cases}$$

and let $\mathcal{Q}$ be the list of all points $x_h \in \Omega_h$, that are adjacent to some boundary point (in fixed, but arbitrary order).

(2) Remove the first point x_h from the list $\mathcal{Q}$ and compute

$$\hat{u} = (\Lambda_h \hat{u}_h)(x_h).$$

If $|\hat{u} - \hat{u}_h(x_h)| > \tau$, then update $\hat{u}_h(x_h) = \hat{u}$ and append all not yet enqueued neighbors y_h of x_h to the list $\mathcal{Q}$. Repeat (2) until $\mathcal{Q} = \emptyset$.

(3) Return $\hat{u}_h$.

The choice of the initial iterate $\hat{u}_h$ in step (1) ensures, that an update is computed at least once in every point $x_h \in \Omega_h$, as the first time an update is computed in x_h, the value of the residual is ∞ (practically, instead of ∞, we could use any upper bound for u_h). Alternatively, a marker list can be used, where we store for every point x_h, whether an update in x_h has been computed or not. This becomes necessary, if the algorithm shall be passed an initial iterate $\hat{u}_h$, which might already be a good approximation to u_h. The order, in which the updates are performed, follows the FIFO (first in, first out) character of $\mathcal{Q}$.

In the next theorem, I show, that the algorithm terminates after finitely many steps, with an approximate solution $\hat{u}_h$, which actually fulfills the tolerance condition. For this purpose let (u_h^n) denote the sequence, generated by algorithm 3.7, where we interpret the grid-function, obtained by performing a single update on u_h^n, as the next iterate u_h^{n+1}.

THEOREM 3.8 ([BR06, Theorem 13]): *The adaptive Gauss-Seidel method generates a monotonically decreasing sequence (u_h^n) of grid-functions. It terminates after finitely many steps with an approximate finite element solution $\hat{u}_h \in \mathcal{V}_h$, such that $\|\hat{u}_h - \Lambda_h \hat{u}_h\| \le \tau$.*

PROOF. (From [BR06].) The initialization $u_h^0|_{\Omega_h} \equiv \infty$ ensures that every point x_h is updated at least once, as the residual is ∞ when the first update value in x_h is computed. After the first update, $u_h(x_h)$ is assigned a finite value, since x_h has a neighbor in $\partial\Omega_h$ or a neighbor, for which a finitely valued update has already been computed. By induction on n we get that at each later update of a nodal point x_h, all neighbors of x_h that have been changed over the last update can only have been assigned a lower value of u_h. From the monotonicity of Λ_h we thus get the first assertion.

Since an update in step (2) only affects the residual in the neighboring points, which are are immediately enqueued, it holds that

$$\{x_h \in \Omega_h \ : \ u_h^n(x_h) < \infty \text{ and } |\Lambda_h u_h^n - u_h^n| > \tau\} \ \subseteq \ \mathcal{Q}$$

for every $n \geq 0$. So if the algorithm terminates with $\mathcal{Q} = \emptyset$, the tolerance has been reached.

Otherwise, if the iteration does not terminate, then there is at least one nodal point x_h^* that appears infinitely often as the first element of the queue $\mathcal{Q}$ and gets updated at steps $n_j \to \infty$, $j \to \infty$. Hence, there must be $|u_h^{n_j}(x_h^*) - u_h^{n_j-1}(x_h^*)| > \tau$ in contradiction to the convergence of $u_h^{n_j}(x_h^*)$ as $j \to \infty$ which is implied by the monotonicity and the trivial lower bound $u_h^n \geq \min_{x \in \partial\Omega} g(x)$. □

Though it takes advantage of the fact, that information propagates from $\partial\Omega_h$ inside Ω_h, when the discretization is solved iteratively, the total complexity of the adaptive Gauss-Seidel iteration behaves in the worst case as the complexity of the Jacobi iteration. However, numerical examples reveal, that the adaptive method is considerably faster than the Jacobi or the Gauss-Seidel iteration.

3.2.5. Choice of the Tolerance. The complexity was analyzed in theorem 3.4, by considering a fixed tolerance, while $h \to 0$. Of course the tolerance has to be diminished along with the grid-spacing h. A connection between the residual $\|\hat{u}_h - \Lambda_h \hat{u}_h\|_\infty$ of an approximate finite-element solution, and the iteration error $\|u_h - \hat{u}_h\|_\infty$ may be obtained from lemma 2.16.

LEMMA 3.9: *Assume (H1)′,(H2),(H3),(H4)′, (2.2)-(2.3) and let $\hat{u}_h$ denote some approximate finite-element solution computed with an iterative method, such that*

$$\|\hat{u}_h - \Lambda_h \hat{u}_h\|_\infty \leq \frac{\rho_*}{4\theta} \cdot \mathrm{tol} \cdot h,$$

where $\mathrm{tol} \leq 1$. *Then, with the solution u_h of (2.8), we have*

$$\|\hat{u}_h - u_h\|_\infty \leq \mathrm{tol}.$$

PROOF. Lemma 2.16 and its proof. □

By choosing $\mathrm{tol} \approx h$, the iteration error $\|\hat{u}_h - u_h\|_\infty$ will be of the same order as the approximation error in the best case (compare theorem 2.17). Of course, if we choose, for example, $\tau \approx \rho^* \cdot h$ in algorithm 3.3, and start with the constant initial iterate $u_h^0 \equiv 0$, then the algorithm terminates after only one step, as

$$u_h^1(x_h) = \min_{y \in \partial\omega_h(x_h)} \left\{ u_h^0(y) + \rho(x_h, x_h - y) \right\} \leq 0 + \rho^* \cdot h,$$

and thus $|u_h^1(x_h) - u_h^0(x_h)| \leq \rho^* \cdot h = \tau$. Anyway, the consideration above implies, that one should choose $\tau \approx h^2$, that is, τ should be chosen within the range of the local truncation error.

3.3. The Fast Marching Method

For isotropic Hamilton-Jacobi equations (where $H(x,p) = f(x, \|p\|)$), the order of dependence in the discretization is known, which makes it possible to construct a fast method for solving the fixed-point equation $u_h = \Lambda_h u_h$, likewise the solution of a triangular system by forward/backward substitution in linear algebra. For that purpose, only $\mathcal{O}(N)$ updates $\Lambda_h u_h(x_h)$ have to be computed, where N denotes the number of grid-points. In an implementation, additional computational effort leads to a total complexity of $\mathcal{O}(N \log N)$.

For a variational discretization of the Eikonal equation on Cartesian meshes, this method was proposed in [Tsi95] and for a finite-difference approximation of the Eikonal equation (which yields the same approximate solution) in [Set96]. Its denomination as *Fast Marching Method* is due to Sethian in [Set96]. In [KS98] a generalization of this method for triangulated domains was developed.

Beginning with the discrete causality principle in subsection 3.3.1, I introduce the Fast Marching Method in subsection 3.3.2 for our discretization of Hamilton-Jacobi equations. As the causality principle holds only on acute triangulations, it requires additional effort to extend the Fast Marching Method to general triangulations. In subsection 3.3.3, I briefly sketch the idea of virtual updates, which was suggested by Kimmel and Sethian in [KS98].

A few aspects of the implementation are subject of subsection 3.3.4. With the idea from [YBS06], I show in 3.3.5, how the total complexity of the method can be diminished from $\mathcal{O}(N \log N)$ to $\mathcal{O}(N)$, when the sequencing is done by an inexact (or untidy) priority queue. The introduced error is carefully analyzed, and I proved an estimate on the difference between the finite-element solutions computed with exact and with inexact sequencing.

3.3.1. Discrete Causality. We consider Hamilton-Jacobi equations, where $H(x,p)$ is isotropic with respect to p, that is $H(x,p) = f(x, \|p\|)$ with some continuous function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. Then the support function ρ, given by (1.9), takes the form $\rho(x,q) = n(x)\,\|q\|$, where $n \in \mathcal{C}(\Omega)$ by lemma 1.18.

The construction of the Fast Marching Method is based on the following lemma. It states, in short, that the value $u_h(x_h)$ of the finite-element solution depends only on those values in the neighboring grid-points $y_h \in \mathcal{N}(x_h)$, for which $u_h(y_h) < u_h(x_h)$, provided that the triangulation is acute. We refer to this property as the discrete causality principle.

Lemma 3.10 ([Ras02, lemma 5.2]): *Assume (H1)-(H4) and $H(x,p) = f(x, \|p\|)$ with some $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R})$. Let Σ_h denote a triangulation of $\overline{\Omega}$, such that for $x_h \in \Omega_h$*

$$\langle y_h - x_h, z_h - x_h \rangle \geq 0 \tag{3.10}$$

for all $y_h, z_h \in \overline{\Omega}_h$, such that x_h, y_h, z_h are vertices of a common simplex $\sigma_h \in \Sigma_h$. Let $u_h \in \mathcal{V}_h$ denote a finite-element function and let $y_* \in \partial\omega_h(x_h)$ be such, that

$$u_h(y_*) + \rho(x_h, x_h - y_*) = \min_{y \in \partial\omega_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}.$$

If y_* is expressed as convex combination in some simplex $\sigma_h = [x_h, x_1, \ldots, x_d]$, which contains y_* in its boundary face $\partial\omega_h(x_h) \cap \sigma_h = [x_1, \ldots, x_d]$, that is

$$y_* = \sum_{i=1}^{d} t_i^* x_i,$$

then $t_i^* > 0$ implies $u_h(x_h) > u_h(x_i)$.

Proof. If $t_i^* = 1$, then $y_* = x_i$, and as $\rho(x_h, x_h - x_i) > 0$ by lemma 1.17,

$$u_h(x_h) = u_h(x_i) + \rho(x_h, x_h - x_i) > u_h(x_i).$$

Thus we may assume, that $0 < t_i^* < 1$. Then $t = 0$ is a local minimum of

$$t \mapsto u_h(y_*) + t(u_h(x_i) - u_h(y_*)) + \rho(x_h, x_h - y_* - t(x_i - y_*)) \tag{3.11}$$

As a consequence, the derivative at $t = 0$ vanishes. As $\rho(x,q) = n(x)\,\|q\|$, differentiation with respect to t in (3.11) and setting $t = 0$ yields

$$u_h(x_i) - u_h(y_*) + n(x_h)\frac{\langle x_h - y_*, y_* - x_i \rangle}{\|x_h - y_*\|} = 0$$

or equivalently,

$$u_h(x_i) + n(x_h)\frac{\langle y_* - x_h, x_i - x_h \rangle}{\|x_h - y_*\|} = u_h(y_*) + \rho(x_h, x_h - y_*) = u_h(x_h).$$

The assertion follows, as we have, by condition (3.10),

$$\langle y_* - x_h, x_i - x_h \rangle \geq t_i^* \|x_i - x_h\|^2 > 0.$$

□

Condition (3.10) is a restriction on the angles in the triangulation at x_h, which is fulfilled, if every two edges adjacent to x_h, that belong to a common simplex, enclose an acute angle. Provided that this condition is fulfilled for all $x_h \in \Omega_h$, the Fast Marching Method, introduced in the following subsection, is applicable.

3.3.2. The Fast Marching Method for Acute Triangulations. The idea of the Fast Marching Method is to compute the finite-element solution u_h, beginning at the boundary and traversing the computational domain along increasing values of u_h. The discrete causality property, which holds true on acute triangulations, ensures, that the value of u_h for $x_h \in \Omega_h$ depends only on the lesser values $u_h(y_h) < u_h(x_h)$, that have already been computed, or that are initially known, if they are boundary values.

The grid-points $\overline{\Omega}_h$ are divided into three categories. First the set of *alive* points A_h, where the value of u_h is known to be exact. Next the set of *trial* points T_h (sometimes called the *narrow band*), where trial values have been computed, that might be changed later-on. All the other points form the set of the *far away* points (which will not be explicitly needed in the algorithm).

Initially, we put $A_h = \partial\Omega_h$, as we know that $u_h = g(x_h)$ on $\partial\Omega_h$. Next, we compute Hopf-Lax updates $u_h(x_h)$ in all points x_h, which are adjacent to some boundary point, where we only use the values of the points, that are alive. We obtain trial values on $T_h = \mathcal{N}(A_h) \setminus \partial\Omega_h$.

We infer from the last lemma, that the smallest trial value $u_h(x_h)$ for $x_h \in T_h$ has to be exact, as it depends only on the smaller values of u_h, which already belong to A_h. Thus the node x_h with the smallest value can be removed from T_h, and becomes alive. Afterwards, new trial values in all neighbors of x_h, that are not yet alive, can be assigned.

Algorithm 3.11 (Fast Marching Method):
Assume (H1)-(H4) and $H(x,p) = f(x, \|p\|)$ with some $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R})$, (3.10) should be fulfilled for all $x_h \in \Omega_h$.

(1) Let $A_h = \partial\Omega_h$, $u_h(x_h) = g(x_h)$ on $\partial\Omega_h$ and $u_h(x_h) = \infty$ for $x_h \in \Omega_h$. Let T_h be the set of all points, which are adjacent to some boundary point $y_h \in A_h$. For all $x_h \in T_h$ compute trial values

$$u_h(x_h) = (\Lambda_h u_h)(x_h).$$

(2) Let $x_h^* \in T_h$ be the point with the smallest trial value in T_h.
(3) Remove x_h^* from T_h and add it to A_h.
(4) Re-compute the values for all neighbors of x_h^*, that are not alive, and add them to T_h.
(5) If $T_h \neq \emptyset$, goto (2).

For the computation of updates in the algorithm, it suffices to use only values $u_h(x_h)$ in those neighboring points, which are alive. The algorithm terminates after $N = |\Omega_h|$ cycles, as in every cycle (2)-(5) one point is added to the list of alive points, which contains the points x_h, where $u_h(x_h)$ is already exact. Altogether, only $\mathcal{O}(N)$ updates in single points have to be computed, in order to obtain the exact finite-element solution. The actual complexity of this method is $\mathcal{O}(N \log N)$, as a priority queue has to be administered in an implementation, to allow for a fast access to the trial point with the smallest trial value in step (2).

In the next lemma, the correctness of the Fast Marching Method is proved by induction.

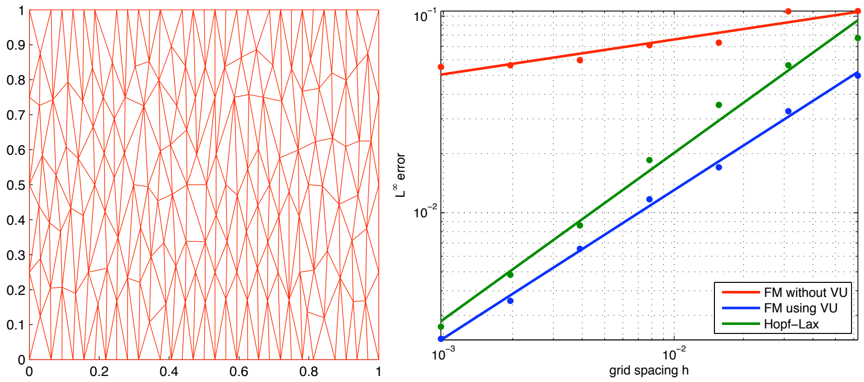

FIGURE 3. *Left.* Degenerate mesh obtained by stretching in y direction. *Right.* Accuracy of the Fast Marching Method with and without virtual updates, and the Hopf-Lax finite-element solution.

LEMMA 3.12: *With the assumptions in algorithm 3.11, this method terminates after $\mathcal{O}(N)$ steps with the exact solution u_h of (2.8).*

PROOF. Let u_h denote the solution produced by the algorithm, and let v_h denote the solution of (2.8). Let $\Omega_h = \{x_h^1, \dots, x_h^N\}$ be an enumeration of Ω_h in the same order, in which the trial points are accepted in step (2) of the algorithm. We inductively show, that $u_h = v_h$. Clearly, we always have $v_h \le u_h$ by the initialization of the algorithm and the monotonicity of the Hopf-Lax update.

Let $x_h^* \in \Omega_h$ denote some point with the smallest value of v_h among the points $x_h \in \Omega_h$. From lemma 3.10 we infer, that $v_h(x_h^*)$ only depends on the values of v_h on the boundary $\partial\Omega_h$ (furthermore, x_h^* must be adjacent to some boundary node). When x_h^1 becomes alive, the values on the boundary are already available. Thus, as $u_h(x_h^1)$ is the smallest trial value in the first cycle,

$$v_h(x_h^1) \le u_h(x_h^1) \le v_h(x_h^*) \le v_h(x_h^1).$$

If $u_h(x_h^j) = v_h(x_h^j)$ for $j = 1, \dots, n$ with some $n < N$, one can show by the same argument, that $u_h(x_h^{n+1}) = v_h(x_h^{n+1})$. If $x_h^* \in \Omega_h$ is some point with the smallest value $v_h(x_h)$ among the points $\{x_h^{n+1}, \dots, x_h^N\}$, then $v_h(x_h^*)$ only depends on the values $v_h(x_h^1), \dots, v_h(x_h^n)$ by lemma 3.10. As these values are available when x_h^{n+1} becomes alive, we have

$$v_h(x_h^{n+1}) \le u_h(x_h^{n+1}) \le v_h(x_h^*) \le v_h(x_h^{n+1}).$$

□

3.3.3. Extension on Non-Acute Triangulations. The assumption on the acuteness of the triangulation can be dropped, if a virtual update strategy is used to ensure, that a computed value $u_h(x_h)$ depends only on the smaller values of the finite-element function. This strategy is described in [KS98] for the two dimensional problem. We remark, that it may be connected with a slight loss of accuracy, and the computed solution not necessarily fulfills (2.8) in every point.

EXAMPLE 3.13: For an example, consider with $g = \|x\|$ the Eikonal equation

$$\|Du(x)\| = 1, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

where $\Omega =]0,1[^2$, which has the unique viscosity solution $u = \|x\|$. I calculated approximate solutions using the Fast Marching Method first with virtual updates, then without virtual updates, and finally I computed the Hopf-Lax finite-element

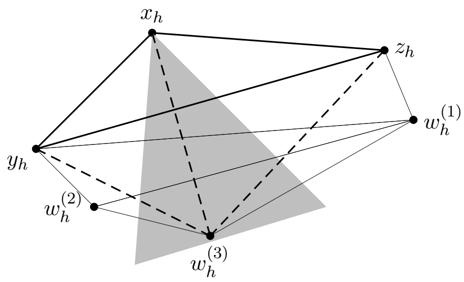

FIGURE 4. Traversal of adjacent triangles, seeking some point in the splitting section.

solution (the solution of (2.8)) using an iterative method. A quite degenerate mesh was chosen, with many obtuse angles, in which the discrete causality principle (compare lemma 3.10) is violated. The results can be found in the following table, and are visualized in figure 3.

$h_{\max}$	$h_{\min}$	$\gamma_{\max}$	$\gamma_{\min}$	$n_{\max}$	FM (VU)	FM (no VU)	Hopf-Lax
0.30806	0.03125	163.82°	6.78°	16	0.04979	0.10620	0.07730
0.15891	0.01490	165.08°	6.48°	16	0.03284	0.10591	0.05622
0.08181	0.00719	166.47°	5.38°	17	0.01700	0.07331	0.03534
0.03999	0.00347	167.28°	5.75°	18	0.01170	0.07113	0.01853
0.02094	0.00172	167.90°	5.06°	23	0.00654	0.05973	0.00861
0.01049	0.00086	167.90°	5.16°	24	0.00356	0.05626	0.00483
0.00531	0.00043	167.65°	5.01°	27	0.00228	0.05521	0.00263

Here, $h_{\max}$ and $h_{\min}$ denote the maximal and minimal lengths of an edge in the triangulation, respectively, and $\gamma_{\max}$ and $\gamma_{\min}$ the maximal and minimal angles. The next column contains the maximal number $n_{\max}$ of triangles, which had to be traversed for the virtual update strategy. The next column shows the results of the Fast Marching Method using the virtual update strategy described below, followed by the Fast Marching Method without virtual updates, that is, I used algorithm 3.11 without taking care of the fact, that the violated acute angle condition is required for the discrete causality principle. The last column shows the error of the Hopf-Lax finite-element solution. It is remarkable, that the Fast Marching Method using virtual updates yields a better approximation on the considered grids. The higher error constant of the Hopf-Lax approximation is due to the interpolatory error on the edges in the degenerate mesh.

Now let us assume, that in the Fast Marching Method, we have to compute an update in some point x_h, from the triangle $[x_h, y_h, z_h]$, that is

$$u = \min_{y \in [y_h, z_h]} \{u_h(y) + \rho(x_h, x_h - y)\}, \tag{3.12}$$

and suppose that $\langle y_h - x_h, z_h - x_h \rangle < 0$, such that the half-lines, running from x_h through y_h and z_h enclose an obtuse angle. We define the *splitting section* to be the cone

$$S = \left\{ w \in \overline{\Omega}\, ; \langle w - x_h, y_h - x_h \rangle \geq 0 \text{ and } \langle w - x_h, z_h - x_h \rangle \geq 0 \right\},$$

that is the set of points, such that the triangles $[x_h, y_h, w]$ and $[x_h, z_h, w]$ possess an acute angle at x_h. By traversing adjacent triangles, we seek for a grid-point inside this splitting section (see figure 4). If a suitable point w_h can be found, we define the update from triangle $[x_h, y_h, z_h]$ to be

$$u = \min\Big(\min_{y\in[y_h,w_h]} \{u_h(y) + \rho(x_h, x_h - y)\}, \min_{y\in[w_h,z_h]} \{u_h(y) + \rho(x_h, x_h - y)\} \Big).$$

In both *virtual* triangles $[x_h, y_h, w_h]$ and $[x_h, z_h, w_h]$, the angle condition from lemma 3.10 is fulfilled, and thus the computed update fulfills the discrete causality principle, and the Fast Marching Method remains applicable. In [**KS98**] it is shown, that the maximal number of triangles, that have to be traversed in order to find an appropriate intersection point, is bounded by a constant, which depends only on the regularity constant θ of the family of triangulations, and not on the grid-spacing h, when $h \to 0$. However, while traversing nearby triangles, one could reach the boundary $\partial\Omega$ of the computational domain. Assume, for example, that in figure 4 the edge $[w_h^{(1)}, w_h^{(2)}]$ was a boundary edge, such that the splitting point $w_h^{(3)}$ wouldn't have been found. As we know the values of u_h in the boundary points $w_h^{(1)}, w_h^{(2)}$ (they are provided by the Dirichlet boundary condition), we compute updates u_1, u_2, u_3 from the triangles $[x_h, y_h, w_h^{(2)}]$, $[x_h, w_h^{(2)}, w_h^{(3)}]$ and $[x_h, z_h, w_h^{(1)}]$, respectively, and define $u = \min(u_1, u_2, u_3)$. The possible obtuse angle in triangle $[x_h, w_h^{(2)}, w_h^{(3)}]$ is disregarded, as the boundary values are known.

Whenever an update is compute in some grid-point x_h from $[x_h, y_h, z_h]$ with an obtuse angle at x_h, the virtual update strategy yields the same subdivision into virtual triangles. Let $\tilde{\omega}_h(x_h)$ denote the virtual neighborhood patch of x_h, that is the union of all adjacent triangles, that have an acute angle at x_h, in union with all virtual triangles generated by the virtual update method at x_h. Then the Fast Marching Method with virtual updates computes a solution of the discrete system

$$u_h(x_h) = \min_{y\in\partial\tilde{\omega}_h(x_h)} \{u_h(y) + \rho(x_h, x_h - y)\}, \quad x_h \in \Omega_h,$$

with $u_h \equiv g$ on the boundary, that is, the Hopf-Lax solution, if we had a priori considered the virtual patches $\tilde{\omega}_h(x_h)$ instead of $\omega_h(x_h)$. By the quoted result in [**KS98**], the diameters of the virtual patches are still bounded by a multiple of the grid-spacing h, in a regular sequence of refined triangulations. Let (u_h) denote the sequence of finite-element functions, computed by the Fast Marching Method with virtual updates. One could follow the convergence theory in section 2.3 in order to prove analogously the uniform convergence of (u_h) to the viscosity solution u of the underlying Eikonal equation.

3.3.4. Implementation, Complexity. Subject of this subsection is the implementation and complexity of the Fast Marching Method, that is algorithm 3.11, where I restrict myself, once again, to the two dimensional case. Typically, the grid-points will be numbered in some way, that is $\overline{\Omega}_h = \{x_h^1, \dots, x_h^N\}$, such that the edges and triangles can be stored as pairs or triplets of indices, respectively. As we have to compute updates from adjacent triangles, it is useful to store those triangles for every grid-point, and likewise the adjacent grid-points for every grid-point, to allow for a fast access to them in step (4). The trial points T_h are kept in an index based priority queue, which is often realized as a binary heap, that holds the index of the trial point with the minimal trial value in its root. Insertion of an element or deletion of the element, with the highest priority (the lowest value of u_h) typically costs $\mathcal{O}(\log n)$ operations, where n denotes the total number of indices in the priority queue. As the number of trial points in the Fast Marching Method is at most $\mathcal{O}(N)$ – or more precisely, $\mathcal{O}(\sqrt{N})$, the deletion of x_h^* from the priority queue in step (2), and the insertion of the neighboring points in step (4) cost at most $\mathcal{O}(\log N)$ time.

It often occurs, that the re-computation of trial values in step (4) yields new trial values. By the monotonicity of the Hopf-Lax update function, and by the initialization ($u_h \equiv \infty$ on Ω_h), such a re-computation may only yield a *smaller* trial value. Then the priority queue has to be reordered, which can also be done in $\mathcal{O}(\log N)$ steps, provided, that we have stored the position in the heap for all grid-points.

On a regular sequence of triangulations, the number of neighboring points is bounded by a constant for each grid-point, independently of h, as the minimal angles in such a sequence of triangulations are bounded from below. Thus one cycle (2)-(5) in the Fast Marching Method costs at most $\mathcal{O}(\log N)$ operations, if the updates are computed in $\mathcal{O}(1)$ time by the Eikonal update formula. We summarize the result on the complexity in the following theorem.

THEOREM 3.14: *Let (2.1) and (2.2) be fulfilled. Then the complexity of the Fast Marching Method is at most $\mathcal{O}(N \log N)$, where N denotes the number of grid-points.*

For details on the implementation and the complexity of a heap-based priority queue see, for example, [**Sed98**].

3.3.5. Sequencing by Untidy Priority Queues. In the Fast Marching Method a priority queue is used, in order to keep track of the trial point with the minimal trial value. As an alternative, Yatziv, Bartesaghi and Sapiro suggest in [**YBS06**] to implement the narrow band T_h as an untidy priority queue. Loosely speaking, they use a bucket sort technique for the narrow band, to allow for a fast access to the trial point with the approximately smallest trial value. This method is applicable for the Eikonal equation

$$\|Du(x)\| = f(x), \quad x \in \Omega, \quad u|_{\partial\Omega} \equiv 0 \tag{3.13}$$

with constant Dirichlet boundary data (or, equivalently, for the isotropic Hamilton-Jacobi equation, compare lemma 3.10). Let $f_*, f^* > 0$ denote lower and upper bounds on f, respectively, such that $f(x) < f^*$ on Ω. In the next proposition it is shown, that the range of the trial values in the narrow band is $\mathcal{O}(h)$.

PROPOSITION 3.15: *Assume (2.2), let algorithm 3.11 be applicable and $f(x) < f^*$ on Ω. At any stage of the Fast Marching Method, the following estimate on the narrow band holds:*

$$\max_{x_h \in T_h} u_h(x_h) - \min_{x_h \in T_h} u_h(x_h) \quad < \quad f^* \cdot h.$$

PROOF. Initially, the set of alive points equals $\partial\Omega_h$, and for all boundary points y_h, it is $u_h(y_h) = 0$. Let x_h denote some point adjacent to a boundary point y_h. Then the computed trial value $\tilde{u}_h(x_h)$ fulfills

$$\tilde{u}_h(x_h) \le u_h(y_h) + \rho(x_h, x_h - y_h) = g(y_h) + f(x_h) \cdot \|x_h - y_h\| < f^* \cdot h.$$

In order to show the asserted estimate by induction, let us assume, that the inequality in the proposition is fulfilled at the beginning of step (2) in the Fast Marching algorithm, and let $\tilde{T}_h$ denote the narrow band after steps (2)-(4). Let $x_h^* \in T_h$ denote the grid-point with the smallest trial value in step (2). This point is removed from T_h, while the values in the grid-points x_h adjacent to x_h^* are re-computed, and, if this hasn't been done before, those points x_h are added to the narrow band. Let x_h denote some trial neighbor of x_h^*. Then we deduce for the trial value $\tilde{u}_h(x_h)$,

$$\begin{aligned}\tilde{u}_h(x_h) \le u_h(x_h^*) + \rho(x_h, x_h - x_h^*) &= \min_{x_h \in T_h} u_h(x_h) + f(x_h) \cdot \|x_h - x_h^*\| \\ &< \min_{x_h \in T_h} u_h(x_h) + f^* \cdot h.\end{aligned}$$

On the other hand, the re-computation of the trial value in some grid-point x_h adjacent to x_h^* cannot yield a smaller trial value than $u_h(x_h^*)$, by the discrete causality principle (lemma 3.10). Thus the minimal value in $\tilde{T}_h$ is greater or equal than the minimal value in T_h, and the assertion is proved. □

Let x_h^* denote the point with the minimal trial value in step (2) of the Fast Marching Method. The key, that leads to the untidy priority – and the $\mathcal{O}(N)$ complexity, is the observation, that we could have chosen some point $\tilde{x}_h^*$ with an approximately minimal trial value instead of x_h^*, accepting an error within the range of the local error $\mathcal{O}(h^2)$ of the discretization. This would lead, by the error propagation lemma, to no considerable loss in the total accuracy of the method.

For this purpose, we store the narrow band in a collection of disjoint buckets B_i, with $T_h = B_0 \cup \cdots \cup B_{d-1}$, where $d \in \mathbb{N}$ is suitably chosen, such that one bucket B_i holds trial values within the range of the local error $\mathcal{O}(h^2)$. By the last proposition, we should use $d = \mathcal{O}(1/h)$ buckets, and quantize the trial values as follows: Define $\delta = f^* \cdot h/d$, such that $\delta = \mathcal{O}(h^2)$. In the bucket B_i we store all trial grid-points x_h with trial values $\tilde{u}_h(x_h)$, such that $i = \lfloor \tilde{u}_h(x_h)/\delta \rfloor \bmod d$. This way, we obtain $|\tilde{u}_h(x_h) - \tilde{u}_h(y_h)| < \delta$ for all trial points $x_h, y_h \in B_i$, as the trial values in the narrow band differ from each other for at most $f^* \cdot h$. For my implementation, I realized the buckets as doubly-linked lists, which bears the advantages that the buckets are easily dynamically resizable, and that arbitrary elements can be removed in $\mathcal{O}(1)$ time, if their position in the list is known.

During the Fast Marching Method, we have to keep track of the bucket, that holds the grid-point with the smallest trial value. Let s denote the number of this bucket. Initially, all trial values are in the range of $]0, f^*h[$, and s is the number of the first non-empty bucket. The untidy priority queue provides the following operations:

- *Insertion of some point x_h:* Compute $i = \lfloor u_h(x_h)/\delta \rfloor \bmod d$, and set $B_i \leftarrow B_i \cup \{x_h\}$. This operation costs $\mathcal{O}(1)$.
- *Deletion of the point $\tilde{x}_h^*$ with the approximately minimal trial value:* The index s should hold the number of the bucket with the smallest trial value. If B_s is empty, we search cyclically for the next non-empty bucket $B_{s'}$, passing from bucket to bucket, substituting s by $\lfloor s+1 \rfloor \bmod d$. If all buckets are empty, the Fast Marching Method terminates. Otherwise, we simply return the first element in B_s (instead of the point with the minimal value). If the bucket B_s runs empty, we increase s by 1, ($s \leftarrow \lfloor s+1 \rfloor \bmod d$). The deletion typically costs $\mathcal{O}(1)$ operations, however, if the cost of the search for the non-empty bucket is taken into account, the total cost can rise up to $\mathcal{O}(d)$ operations.
- *Re-ordering of the queue, when trial values are updated:* Let us assume, that a re-computation of the trial value of x_h yields a smaller value, such that we have to move x_h to a different bucket. As we store for every trial point x_h its position in the queue, that is the bucket B_i containing x_h, and the point's position within the bucket B_i, it is possible to remove x_h from B_i in $\mathcal{O}(1)$ time. Re-insertion of x_h costs $\mathcal{O}(1)$, as indicated above.

Clearly, the re-ordering of the queue is necessary, if the re-computation of trial values yields smaller values. However, this necessity is not regarded at all in [YBS06]. What the complexity of the delete operation concerns, let us consider the worst case, where a lot of buckets are empty, say, $s = 1$, and B_{d-1} is the next non-empty bucket. Then B_{d-1} contains nearly the whole narrow band, and s has to be increased $d-2$ times, until the next non-empty bucket B_{d-1} has been found. Provided, that the size of the narrow band is $\mathcal{O}(d) = \mathcal{O}(1/h)$, we would not have to search for a non-empty bucket in the following delete operations. Thus in the statistical mean, also the deletion costs $\mathcal{O}(1)$. In the next lemma, the asymptotic complexity of the Fast Marching Method with an untidy priority queue is shown.

LEMMA 3.16: *Assume (2.2), (2.3). Then the asymptotic complexity of the Fast Marching Method with an untidy priority queue consisting of $d = \mathcal{O}(h^{-1})$ buckets is $\mathcal{O}(N)$, where N denotes the number of grid-points.*

PROOF. Because of the uniformity assumptions (2.2) and (2.3) on the triangulation, the area A_Δ of every triangle is bounded by

$$\frac{h^2}{2\theta^2} \leq A_\Delta \leq \frac{h^2}{2}.$$

Thus the triangulation consists of $\mathcal{O}(h^{-2})$ triangles, and accordingly, of $N = \mathcal{O}(h^{-2})$ grid-points. By theorem 2.9, the finite-element solutions (u_h) are uniformly bounded. As I will show in the following lemma, the difference between the grid-solution computed with an untidy priority queue and the exact finite-element solution is $\mathcal{O}(h)$. Hence the grid-functions computed with an untidy queue are uniformly bounded by a constant $M > 0$ for a sequence $h \to 0$. As every bucket of the untidy queue contains points with function values, which differ from each other for at most $\delta = f^* \cdot h/d$, where we chose $d = \mathcal{O}(h^{-1})$, a total number of $M/\delta = \mathcal{O}(h^{-2}) = \mathcal{O}(N)$ buckets have to be traversed during the Fast Marching Method[1]. Thus also the delete operation from the untidy queue has an average complexity of $\mathcal{O}(1)$, likewise the insertion and re-ordering. □

Finally, in the following lemma, it is shown, that the total error in the finite-element function is $\mathcal{O}(h)$, when the priority queue is substituted by an untidy priority queue. For that purpose, we have to choose at least $d \geq 1/h$ buckets, and quantize the grid-function values by $\delta = f^* \cdot h/d$. Let me remark, that an increase in the number of buckets yields a better accuracy at the price of a higher complexity of the method.

LEMMA 3.17: *Let u_h denote the finite-element solution of (3.13) computed with the Fast Marching Method with exact sequencing on an acute triangulation, and let $\tilde{u}_h$ denote the solution obtained by using an untidy priority queue with $d \geq 1/h$ buckets. Then we have*

$$\|u_h - \tilde{u}_h\|_\infty \ < \ 2\theta \cdot \frac{f^*}{f_*} \cdot h \cdot \|\tilde{u}_h\|_\infty \,.$$

PROOF. An error is introduced, when computing the Fast Marching Method with the untidy priority queue technique, described above, as in step (2) not the point x_h^* with the minimal trial value becomes alive, but some point $\tilde{x}_h^*$, such that $\tilde{u}_h(\tilde{x}_h^*) - \tilde{u}_h(x_h^*) < \delta$. By the causality principle (lemma 3.10), the value $\tilde{u}_h(\tilde{x}_h^*)$ depends only on the lower function values in the neighboring grid-points, and therefore possibly on the smaller trial values, that would have been accepted before $\tilde{u}_h(\tilde{x}_h^*)$, if an exact sequencing had been used. However, all points y_h, that become alive after $\tilde{x}_h^*$ fulfill $u_h(y_h) \geq \tilde{u}_h(x_h^*)$. Thus

$$\tilde{u}_h(\tilde{x}_h^*) - \delta < \tilde{u}_h(x_h^*) \leq (\Lambda_h \tilde{u}_h)(\tilde{x}_h^*) \leq \tilde{u}_h(\tilde{x}_h^*).$$

The assertion follows from lemma 2.16, and its proof. As $\rho(x,q) = f(x) \cdot \|q\|$, we have $\rho_* = f_*$. □

As we saw in theorem 2.17, the convergence rate is $\mathcal{O}(h)$, provided that a smooth solution of (3.13) exists. Generally, the convergence rate will be even lower. Thus the error introduced by untidy sequencing does not affect the convergence rate.

[1] This argument was communicated by Thomas Satzger and will be part of his diploma thesis.

3.4. The Ordered Upwind Method

In [SV03], Sethian and Vladimirsky describe an extension of the Fast Marching Method to Hamilton-Jacobi equations of Eikonal type, denoted as the *Ordered Upwind Method* (OUM). Therein, they consider the equation

$$F(x, Du(x)) = 1, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g,$$

where F is Lipschitz-continuous with respect to x, and fulfills (F2)-(F4) on page 18. The anisotropy coefficient is defined to be the quotient

$$\Upsilon = \max_{\overline{\Omega}\times S^{d-1}} F(x,p) \Big/ \min_{\overline{\Omega}\times S^{d-1}} F(x,p).$$

The Ordered Upwind Method for Hamilton-Jacobi equations (OUM) uses a similar discretization as the Hopf-Lax discretization, but does not actually compute a solution of (2.8). Compared to the accuracy of the finite-element solution defined herein, the accuracy of the solution computed by the OUM is worse by the factor Υ, and the complexity of the OUM is $\mathcal{O}(\Upsilon N \log N)$, where $N = |\Omega_h|$.

In order to transfer the causality property discussed in lemma 3.10 to more general Hamilton-Jacobi equations, the updates in the OUM are not computed from the neighborhood patches $\omega_h(x_h)$, but from the whole accepted front, which corresponds to the set of those alive points, which are adjacent to some trial point in the Fast Marching Method. Not the whole accepted front is relevant for computing an update in some trial point x_h, but only those points in the accepted front, which are at most $\mathcal{O}(\Upsilon h)$ away from x_h, where h denotes the grid-spacing. This may give a pointer to the asserted complexity and accuracy of the OUM, compared to the Fast Marching Method. In this section I introduce the OUM following the discussion in [SV03]. However, the method is treated therein more in the context of min-time optimal control problems. Some parts of the theory in [SV03] seemed a bit fragmentary, for example, in the proof of the uniform Lipschitz continuity of the finite-element solutions, no compatibility condition for the boundary data is regarded.

3.4.1. Anisotropy of Hamilton-Jacobi Equations.

The Ordered Upwind Method is applicable to Hamilton-Jacobi equations

$$H(x, Du(x)) = 0, \quad x \in \Omega, \qquad u|_{\partial\Omega} = g, \tag{3.14}$$

where H fulfills the convexity and coercivity conditions (H2) and (H3), and is Lipschitz-continuous with respect to the state variable, and fulfills the strict compatibility condition, that is, (H1)′ and (H4)′ on page 34 are assumed. With the support function $\rho(x,q)$ of the zero level-sets of H, we measure the anisotropy of the underlying Hamilton-Jacobi equation by

$$\Upsilon = \frac{\max_{\overline{\Omega}\times S^{d-1}} \rho(x,q)}{\min_{\overline{\Omega}\times S^{d-1}} \rho(x,q)} = \frac{\rho^*}{\rho_*} \geq 1,$$

which we refer to as the *anisotropy coefficient*. By the strict compatibility condition (H4)′, the lower bound $\rho_* = \min_{\overline{\Omega}\times S^{d-1}} \rho(x,q) > 0$. The upper bound ρ^* satisfies $\rho^* \leq \frac{\beta}{\alpha}$ with the growth constants from lemma 1.1. With the lower and upper bounds on the support function, it holds by homogeneity, that

$$\rho_* \cdot \|q\| \;\leq\; \rho(x,q) \;\leq\; \rho^* \cdot \|q\|, \quad \text{for all } x \in \overline{\Omega},\; q \in \mathbb{R}^d.$$

The distance function $\delta(x,y)$, defined by (1.11), satisfies the estimate

$$\rho_* \cdot \|x-y\| \;\leq\; \delta(x,y) \;\leq\; C_\Omega \cdot \rho^* \cdot \|x-y\|, \quad \text{for all } x, y \in \overline{\Omega},$$

where C_Ω denotes the Lipschitz bound for $\partial\Omega$ (compare lemma 1.21.) Given a compatible boundary data $g : \partial\Omega \to \mathbb{R}$, the viscosity solution of the Hamilton-Jacobi equation is given by

$$u(x) = \min_{y\in\partial\Omega} \{g(y) + \delta(x, y)\}, \tag{3.15}$$

as it was shown in theorem 1.25. Consider a simple, closed curve $\Gamma \subset \Omega$. As it can be seen easily, we have for all x within Γ:

$$u(x) = \min_{y\in\Gamma} \{u(y) + \delta(x, y)\}. \tag{3.16}$$

The following lemma will become important for the motivation of the Ordered Upwind Method. The corresponding result in [SV03] involves the value function of an optimal control problem on a convex domain Ω.

Lemma 3.18 ([SV03, Lemma 3.4]): *With the assumptions made above, let u be defined by (3.15), and with $\min_{\overline{\Omega}} u < \lambda < \max_{\overline{\Omega}} u$ let $\mathcal{L} = \{x \in \Omega\, ;\ u(x) = \lambda\}$ denote some level-set of u. Then, if for some $\overline{x}$ with $u(\overline{x}) > \lambda$,*

$$u(\overline{x}) = \min_{y\in\mathcal{L}} \{u(y) + \delta(\overline{x}, y)\} = \lambda + \delta(\overline{x}, \overline{y}),$$

with $\overline{y} \in \mathcal{L}$, we have

$$\|\overline{x} - \overline{y}\| \le C_\Omega \cdot \Upsilon \cdot \mathrm{dist}(\overline{x}, \mathcal{L}).$$

Proof. (After [SV03].) Let $\tilde{y}$ be such, that $\mathrm{dist}(\overline{x}, \mathcal{L}) = \|\overline{x} - \tilde{y}\|$. Then $u(\overline{x}) \le \lambda + \delta(\overline{x}, \tilde{y})$, and by the estimate on the optical distance δ, we have

$$\lambda + \rho_* \cdot \|\overline{x} - \overline{y}\| \ \le\ u(\overline{x}) = \lambda + \delta(\overline{x}, \overline{y}) \ \le\ \lambda + C_\Omega \cdot \rho^* \cdot \|\overline{x} - \tilde{y}\|,$$

which yields the assertion. □

Consequently, if one wants to determine the value $u(\overline{x})$ for some point $\overline{x}$ near the level-set $\mathcal{L}$, only a small part of the level-set is relevant for $\overline{x}$, that is, all points y on the level-set with $\|\overline{x} - y\|$ bounded by a multiple of the distance of $\overline{x}$ to the level-set. We can also gain an estimate on the angle between the direction of the optimal trajectory and the gradient of u. If, for some point $\overline{x} \in \Omega$, there exists an optimal trajectory $\xi \in \mathcal{C}^\infty([0, 1]; \Omega)$, such that $\overline{x} = \xi(0)$ and $\xi(1) = y \ne \overline{x}$, and

$$u(\overline{x}) = u(y) + \delta(\overline{x}, y) = u(y) + \int_0^1 \rho\big(\xi(t), -\dot{\xi}(t)\big)\, dt,$$

then for all $0 \le \tau \le 1$ Bellman's optimality principle holds, that is,

$$u\big(\overline{x} = \xi(0)\big) \ =\ u(\xi(\tau)) + \int_0^\tau \rho\big(\xi(t), -\dot{\xi}(t)\big)\, dt. \tag{3.17}$$

Passing to the limit, we get the following statement.

Lemma 3.19: *With the notation from above, let u be differentiable in $\overline{x}$. Then*

$$\cos\Big(\measuredangle(Du(\overline{x}), -\dot{\xi}(0))\Big) \ \ge\ \frac{1}{\Upsilon}.$$

Proof. Division by τ and passing to the limit in (3.17) yields

$$\Big\langle Du(\overline{x}), -\dot{\xi}(0)\Big\rangle = \rho\big(\overline{x}, -\dot{\xi}(0)\big) \ge \rho_* \cdot \Big\|\dot{\xi}(0)\Big\|.$$

By lemma 1.20, we have

$$\langle Du(\overline{x}), Du(\overline{x})\rangle \le \rho\big(\overline{x}, Du(\overline{x})\big) \le \rho^* \cdot \|Du(\overline{x})\|,$$

and the assertion is ready to obtain. □

The last result also holds true, if we substitute Υ by the *local anisotropy coefficient*,

$$\upsilon(x) = \frac{\max_{\|q\|=1} \rho(x,q)}{\min_{\|q\|=1} \rho(x,q)} = \frac{\max\{\|p\| \;;\; H(x,p) \leq 0\}}{\min\{\|p\| \;;\; H(x,p) = 0\}}, \tag{3.18}$$

which measures the anisotropic deformation of the zero level-set of $p \mapsto H(x,p)$ for every $x \in \Omega$. In the case of the Eikonal equation, where $H(x,p) = \|p\| - f(x)$, we would have $\upsilon(x) \equiv 1$. For the Eikonal equation, the directions of $-Du(\overline{x})$ and the optimal trajectory $\dot{\xi}(0)$ coincide.

3.4.2. Generalized Hopf-Lax Updates. For simplicity of presentation, we will discuss the two dimensional case in the following. The given method and results extend naturally to arbitrary space dimensions. Let Σ_h denote a triangulation of $\Omega \subset \mathbb{R}^2$ of diameter $h > 0$, and assume, that we have already computed the values of a finite-element approximation $u_h \in \mathcal{V}_h$ to the viscosity solution of (3.14) in a neighborhood of the boundary, that is in some set $A_h \subset \overline{\Omega}_h$, which we refer to as the set of *active points*. We denote by AF_h the set of all edges $[y_h, z_h]$ of the triangulation, such that y_h and z_h are active points, and are both adjacent to some not yet active point $x_h \in \overline{\Omega}_h \setminus A_h$. AF_h will be called the *accepted front*. A point x_h, which is adjacent to an active point in A_h is called a *trial point*, and the set of all trial points will be denoted by T_h. In view of (3.16), we define the generalized Hopf-Lax update for some trial point $x_h \in T_h$ by

$$u_h(x_h) = \min_{e_h \in AF_h} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\}. \tag{3.19}$$

The difference to the Hopf-Lax update defined in subsection 2.2.2 is that the whole accepted front may contribute to the value $u_h(x_h)$. In the OUM, the accepted front approximates the level-sets of u, while a sequencing similar as in the Fast Marching Method is used, in oder to keep the updates local. We obtain the following discrete analog of lemma 3.18.

LEMMA 3.20 ([SV03, Lemma 7.1]): *With the notation used above, assume that we have computed trial values for all trial points $x_h \in T_h$ by formula (3.19). Let $\overline{x}_h \in T_h$ denote the point with the smallest trial value, that is $u_h(\overline{x}_h) \leq u_h(x_h)$ for all $x_h \in T_h$. Then*

$$u_h(\overline{x}_h) = \min_{e_h \in NF_h(\overline{x}_h)} \min_{y \in e_h} \{u_h(y) + \rho(\overline{x}_h, \overline{x}_h - y)\},$$

where the near front *of x_h is the part of the accepted front, defined by*

$$NF_h(\overline{x}_h) = \{e_h \in AF_h \,;\; \operatorname{dist}(\overline{x}_h, e_h) \leq \Upsilon \cdot h\}.$$

PROOF. (After [SV03].) As $u_h(\overline{x}_h)$ is computed by formula (3.19), there is an edge $\tilde{e}_h \in AF_h$ and some point $\tilde{y} \in \tilde{e}_h$, such that

$$u_h(\overline{x}_h) = u_h(\tilde{y}) + \rho(\overline{x}_h, \overline{x}_h - \tilde{y}) \geq u_h(\tilde{y}) + \rho_* \cdot \|\overline{x}_h - \tilde{y}\|.$$

There is some trial point x_h adjacent to $\tilde{e}_h$. As the trial value in x_h is computed by (3.19), we have

$$u_h(x_h) \leq u_h(\tilde{y}) + \rho(x_h, x_h - \tilde{y}) \leq u_h(\tilde{y}) + \rho^* \cdot \|x_h - \tilde{y}\| \leq u_h(\tilde{y}) + \rho^* \cdot h$$

From the assumption $u_h(\overline{x}_h) \leq u_h(x_h)$, we deduce that

$$u_h(\tilde{y}) + \rho_* \cdot \|\overline{x}_h - \tilde{y}\| \leq u_h(\tilde{y}) + \rho^* \cdot h.$$

□

In every step of the OUM, the trial point $\overline{x}_h$ with the smallest trial value is accepted. From the last lemma we read off, that only a small part of the whole accepted front is indeed relevant for $\overline{x}_h$. This is the main observation, which leads to the construction of the OUM.

3.4.3. The Algorithm. The aim is once again to construct a finite-element approximation u_h to the viscosity solution of the anisotropic Hamilton-Jacobi equation (3.14). Initially, we know the values of u_h in the boundary points, that is the set of active points consists at the beginning of all boundary points $y_h \in \partial\Omega_h$. Typically, the accepted front AF_h consists initially of all boundary edges. Trial values for points $x_h \in \Omega_h$, adjacent to some boundary point are then computed by

$$u_h(x_h) = \min_{e_h \in NF_h(x_h)} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\}, \tag{3.20}$$

where $NF_h(x_h) = \{e_h \in AF_h\,;\ \mathrm{dist}(x_h, e_h) \le \Upsilon \cdot h\}$ denotes the near front for x_h. Then in every step, the trial point with the smallest value of u_h is accepted, similar as in the Fast Marching Method. The algorithm goes as follows:

ALGORITHM 3.21 (Ordered Upwind Method):
We assume (H1)$'$,(H2),(H3),(H4)$'$.

(1) Let $A_h = \partial\Omega_h$, $u_h(x_h) = g(x_h)$ on $\partial\Omega_h$, initialize the accepted front AF_h, and compute trial values for all $x_h \in T_h$ by (3.20).
(2) Let $\overline{x}_h \in T_h$ be the point with the smallest trial value in T_h.
(3) Remove $\overline{x}_h$ from T_h, add it to A_h, and update the accepted front.
(4) Re-compute the values for all $x_h \in T_h$, such that the near front $NF_h(x_h)$ for x_h changed in step (3). Update the set of trial points, and compute the values of the new trial points by (3.20).
(5) If $T_h \neq \emptyset$, goto (2).

Steps (2)-(5) are repeated in the algorithm, until the set of trial points is empty. As one point becomes active in every cycle, the algorithm terminates after $|\Omega_h|$ cycles. In step (3) the accepted front has to be updated. Any edge adjacent to $\overline{x}_h$ has to be examined, whether it still belongs, or has to be attached to the accepted front.

The values of all trial points y_h, within a distance of $\|\overline{x}_h - y_h\| \le (\Upsilon + 1) \cdot h$ from x_h, have to be re-computed, as they might depend on some edge, which was added or removed from the accepted front in step (3). Let me remark, that in the Ordered Upwind algorithm given in [SV03] the possible dependence of $u_h(y_h)$ on an edge, that is removed from the accepted front, is not taken into account. If a new edge is added to the accepted front, and also to the near front $NF_h(y_h)$ for all trial points y_h within some neighborhood, the re-computation of the trial values may yield only smaller values. If some edge is removed from the accepted front, and the tentative value $u_h(y_h)$ of some trial point y_h was computed from that edge, the new trial value might be larger.

In the next lemma, we show a monotonicity property of the accepted front AF_h, which enables us to prove the Lipschitz continuity of the approximate solution u_h constructed by the OUM. We will also show an estimate on the accepted front, and deduce, that the evolution of the accepted front follows the level-sets of u.

For that object, let $\{x_h^1, \ldots, x_h^n\} = \Omega_h$ denote the enumeration of Ω_h in the same order, as the grid-points become active in step (3) of the algorithm, and let AF_h^{k-1} denote the accepted front immediately before x_h^k is accepted, such that

$$u_h(x_h^k) = \min_{e_h \in AF_h^{k-1}} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\}.$$

So AF_h^{k-1} is the accepted front, when $\{x_h^1, \dots, x_h^{k-1}\}$ are already active, and $\{x_h^{k+1}, \dots, x_h^n\}$ are not.

LEMMA 3.22 ([**SV03**, Lemma 7.3]): *With the above notation, the following estimates hold:*

$$u_h(x_h^{k+1}) \le \min_{AF_h^k} u_h + \rho^* \cdot h, \tag{3.21}$$

$$\min_{AF_h^k} u_h \le \min_{AF_h^{k+1}} u_h, \tag{3.22}$$

for all $k = 0, \dots, n-1$*, and, if* $\max_{AF_h^k} u_h \le \min_{AF_h^k} u_h + \rho^* \cdot h$*, we also have*

$$\max_{AF_h^{k+1}} u_h \le \min_{AF_h^{k+1}} u_h + \rho^* \cdot h.$$

PROOF. (After [**SV03**].) In order to prove the first equation (3.21), let $\tilde{x}_h$ denote some grid-point on AF_h^k, where u_h takes its minimal value. By definition of the accepted front, $\tilde{x}_h$ is adjacent to some trial point x_h, and the trial value u in x_h, when x_h^{k+1} is about to be accepted, fulfills

$$u = \min_{e_h \in NF_h^k(x_h)} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\},$$

where NF_h^k is the near front, which is part of AF_h^k. As $\tilde{x}_h$ belongs to the near front for x_h, we have

$$u \le u_h(\tilde{x}_h) + \rho(x_h, x_h - \tilde{x}_h) \le \min_{AF_h^k} u_h + \rho^* \cdot \|x_h - \tilde{x}_h\| \le \min_{AF_h^k} u_h + \rho^* \cdot h.$$

As the smallest trial value is accepted, $u_h(x_h^{k+1}) \le u$, and the first equation is proved.

Of course, we have $\min_{AF_h^k} u_h < u_h(x_h^{k+1})$. When x_h^{k+1} becomes an active point, some grid-points may be removed from the accepted front in step (3) of the OUM, but x_h^{k+1} is the only point, that might be added to the accepted front. Thus the second equation (3.22) holds true.

To show the last property, we assume that $\max_{AF_h^k} u_h \le \min_{AF_h^k} u_h + \rho^* \cdot h$. We deduce, by (3.21), and as x_h^{k+1} is the only possible new point in AF_h^{k+1}, that

$$\max_{AF_h^{k+1}} u_h \le \max\left(\max_{AF_h^k} u_h,\ u_h(x_h^{k+1})\right) \le \min_{AF_h^k} + \rho^* \cdot h.$$

□

Initially, the accepted front AF_h consists of the boundary edges with grid-points in $\partial\Omega_h$. If homogeneous Dirichlet boundary data is provided, that is $g \equiv 0$, then $\max_{AF_h^0} = \min_{AF_h^0}$, and the last property of lemma 3.22 holds, by induction, for all $k = 0, \dots, n-1$. Thus the accepted front follows the level-sets of u_h, and as u_h is an approximation to the viscosity solution, what we will see below, the accepted front approximates in some way the level-sets of the viscosity solution u of (3.14).

3.4.4. Convergence of the OUM. Let $u_h \in \mathcal{V}_h$ denote the finite-element function computed by algorithm 3.21. Of course u_h is no finite-element solution of the Hamilton-Jacobi equation in the sense of definition 2.2, but u_h is also a suitable approximation to the viscosity solution. We follow the convergence theory in section 2.3 in order to show, that the sequence of finite-element functions (u_h) constructed by the OUM on refined meshes also converges to the viscosity solution of the Hamilton-Jacobi equation. First, we show the uniform Lipschitz continuity.

THEOREM 3.23: *Assume (H1)′,(H2),(H3),(H4)′, and let* (Σ_h) *denote a uniform family of triangulations of* $\Omega \subset \mathbb{R}^2$, *with (2.2), (2.3) being fulfilled. Let the boundary function g fulfill the following discrete compatibility requirement*

$$|g(x) - g(y)| \;\leq\; \frac{\rho_*}{\theta \cdot (\Upsilon + 1)} \cdot \|x - y\| \;\; \textit{for all } x, y \in \partial\Omega.$$

Then the sequence of finite-element functions (u_h) *generated by algorithm 3.21 is uniformly Lipschitz-continuous, with a Lipschitz constant bounded by* $2C_\Omega \cdot \theta^2 \cdot \rho^*$, *and uniformly bounded on* $\overline{\Omega}$.

PROOF. Just like in theorem 2.9, where we showed the Lipschitz continuity of the sequence of Hopf-Lax discrete solutions, we will first show, that for adjacent grid-points x_h and y_h, we have $|u_h(x_h) - u_h(y_h)| \leq \rho^* \cdot \theta \cdot \|x_h - y_h\|$. The rest of the proof is identical to the proof of theorem 2.9, and will be omitted. Let us assume, that x_h and y_h are adjacent, and that x_h becomes active, before y_h does. Then, when y_h is about to be accepted, x_h already belongs to the accepted front, and particularly to the near front for y_h. Thus we have

$$u_h(y_h) \leq u_h(x_h) + \rho(x_h, x_h - y_h) \leq u_h(x_h) + \rho^* \cdot \|x_h - y_h\|\,.$$

If $x_h \in \Omega_h$, let $AF_h(x_h)$ denote the accepted front, immediately before x_h becomes active, and let accordingly denote $AF_h(y_h)$ the accepted front for y_h. Then, by (3.21) and (3.22) in lemma 3.22, we have

$$\begin{aligned} u_h(x_h) \leq \min_{AF_h(x_h)} u_h + \rho^* \cdot h \leq \min_{AF_h(y_h)} u_h + \rho^* \cdot h \\ \leq u_h(y_h) + \rho^* \cdot h \leq u_h(y_h) + \rho^* \cdot \theta \cdot \|x_h - y_h\|\,. \end{aligned}$$

If $x_h \in \partial\Omega_h$ is a boundary point, we must argue in a different way. Let $\tilde{y}$ denote a point on some edge $\tilde{e}_h$ from the near front for y_h, such that

$$u_h(y_h) = u_h(\tilde{y}) + \rho(y_h, y_h - \tilde{y}),$$

when y_h is about to be accepted. From lemma 3.20 we read off, that $\|y_h - \tilde{y}\| \leq \Upsilon \cdot h$. At least for one endpoint $\tilde{y}_h$ of $\tilde{e}_h$, we have $u_h(\tilde{y}_h) \leq u_h(\tilde{y}) < u_h(y_h)$, and thus, as $\|y_h - \tilde{y}\| \geq h/\theta$ because of the uniformity of the triangulation, we have

$$u_h(y_h) \geq u_h(\tilde{y}_h) + \rho_* \cdot \|y_h - \tilde{y}\| \geq u_h(\tilde{y}_h) + \frac{\rho_*}{(\Upsilon + 1) \cdot \theta} \cdot \|y_h - \tilde{y}_h\|\,.$$

Following the argumentation in the proof of theorem 2.9, the construction yields a path of grid-points $y_h^1 = y_h, y_h^2 = \tilde{y}_h, \ldots$ with strictly decreasing values of u_h, which meets a boundary point after finitely many steps. Let $y_h^m \in \partial\Omega_h$ denote the point, in which the boundary is reached. Then

$$\begin{aligned} u_h(y_h) &\geq g(y_h^m) + \frac{\rho_*}{(\Upsilon + 1) \cdot \theta} \cdot \|y_h - y_h^m\| \\ &\geq g(x_h) - \frac{\rho_*}{(\Upsilon + 1) \cdot \theta} \cdot \Big(\|x_h - y_h^m\| - \|y_h - y_h^m\| \Big), \end{aligned}$$

and thus

$$g(x_h) \leq u_h(y_h) + \rho^* \cdot \big(\|x_h - y_h^m\| - \|y_h - y_h^m\| \big) \leq u_h(y_h) + \rho^* \, \|x_h - y_h\|\,.$$

□

Let me remark, that the strict compatibility condition for g could be weakened. Nevertheless the restrictive bound facilitates the argumentation. In absence of a compatibility condition, the Lipschitz property remains for the union of all triangles in Ω, that don't adjoin the boundary. This was shown in [**SV03**, Lemma 7.5], while a compatibility condition for the boundary data is not considered therein. But the

Lipschitz continuity on $\overline{\Omega}$ becomes important in the convergence theorem, where the theorem of Arzelà-Ascoli and the comparison principle (theorem 1.13) are used.

THEOREM 3.24 (Convergence of the OUM, [**SV03**, Theorem 7.7]): *Under the assumptions of theorem 3.23, the sequence of finite-element functions* (u_h) *generated by the Ordered Upwind Method converges uniformly to the viscosity solution of (3.14).*

PROOF. (After [**SV03**].) The proof is very similar to the discussion in section 2.3. By the theorem of Arzelà-Ascoli, (u_h) has a uniformly convergent subsequence, which we also denote by (u_h). Let u denote the limit function. In the first part of the proof we will see, that u is a viscosity super-solution of 3.14. It is a bit trickier to prove the sub-solution property, where the observation of lemma 3.19 is taken into account.

First Part - Super-solution: Let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and $x_0 \in \Omega$ be such, that $u - \varphi$ attains a strict local minimum in x_0. Then $u_h - \varphi_h$ converges uniformly to $u - \varphi$, with the linear interpolant φ_h of φ. Following proposition 1.7, there is a sequence of grid-points $x_h \in \Omega_h$ with $x_h \to x_0$ for $h \to 0$, such that, after passing to a subsequence if necessary,

$$(u_h - \varphi_h)(x_h) \leq (u_h - \varphi_h)(y) \quad \text{for all } y \in B \tag{3.23}$$

on some neighborhood $B = B(x_0, \delta)$ of x_0, with $\delta > 0$ (as $u_h - \varphi_h$ is piecewise linear, the local minimum is taken in some grid-point). For sufficiently small h (at least if $\Upsilon \cdot h < \delta$, according to lemma 3.20), there is some point $\tilde{y}_h \in B$ on the near front $NF_h(x_h)$ (not necessarily a grid-point), such that

$$u_h(x_h) = \min_{e_h \in NF_h(x_h)} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\} = u_h(\tilde{y}_h) + \rho(x_h, x_h - \tilde{y}_h).$$

By (3.23) we deduce, that

$$\varphi_h(x_h) \geq u_h(x_h) - (u_h - \varphi_h)(\tilde{y}_h) = \varphi_h(\tilde{y}_h) + \rho(x_h, x_h - \tilde{y}_h).$$

Let $q_h = \frac{x_h - \tilde{y}_h}{\|x_h - \tilde{y}_h\|}$. Then division of the last equation by $\|x_h - \tilde{y}_h\|$ yields, similar as in lemma 2.1

$$\langle D\varphi(x_h), q_h \rangle - \rho(x_h, q_h) \geq -c \cdot h,$$

with some constant c depending on the regularity constant of the mesh, on the second derivative of φ and on the anisotropy coefficient Υ. By compactness of S^1, the sequence (q_h) has a convergent subsequence with limit $\tilde{q}$ for $h \to 0$, and we finally obtain

$$\max_{\|q\|=1} \{\langle D\varphi(x_0), q \rangle - \rho(x_0, q)\} \geq \langle D\varphi(x_0), \tilde{q} \rangle - \rho(x_0, \tilde{q}) \geq 0$$

Thus, by lemma 1.20, u is indeed a super-solution.

Second Part - Sub-solution: Now we assume, that x_0 is a strict local maximum point of $u - \varphi$, and obtain, analogously as above, a sequence (x_h) of grid-points, where $x_h \in \Omega_h$ for $h \to 0$, such that

$$(u_h - \varphi_h)(x_h) \geq (u_h - \varphi_h)(y) \quad \text{for all } y \in B \tag{3.24}$$

on some neighborhood $B = B(x_0, \delta)$ of x_0. We have to show, that

$$\max_{\|q\|=1} \{\langle D\varphi(x_0), q \rangle - \rho(x_0, q)\} \leq 0.$$

Let the maximum be realized in some $\tilde{q}$, with $\|\tilde{q}\| = 1$. Then one obtains by the argumentation in the proof of lemma 3.19, that

$$\langle D\varphi(x_0), \tilde{q} \rangle \geq \frac{\|D\varphi(x_0)\|}{\Upsilon}. \tag{3.25}$$

In the following we show, that the half-line $\sigma(t) = x_h - t\tilde{q}$ intersects the accepted front for x_h in some point $\tilde{y}_h$, which is contained in B, for sufficiently small $h \to 0$. Then we had

$$u_h(x_h) \leq u_h(\tilde{y}_h) + \rho(x_h, x_h - \tilde{y}_h),$$

and thus, by (3.24), we could deduce

$$\varphi_h(x_h) \leq \varphi_h(\tilde{y}_h) + \rho(x_h, x_h - \tilde{y}_h),$$

which yields $\langle D\varphi(x_h), \tilde{q}\rangle - \rho(x_h, \tilde{q}) \leq ch$, and letting $h \to 0$, the assertion would be proved by lemma 1.20 and by the choice of $\tilde{q}$.

In order to show the intersection property of the half-line, let $AF(x_h)$ denote the accepted front, immediately before x_h becomes active in algorithm 3.21. By lemma 3.22, we get a lower bound on grid function values on the accepted front,

$$u_h(x_h) - h \cdot \rho^* \leq \min_{AF_h(x_h)} u_h. \tag{3.26}$$

Let m denote an upper bound for $\|D^2\varphi(x)\|$ on $\overline{\Omega}$. Then it holds, by lemma 2.1 and equation (3.24), if t is sufficiently small, such that $\sigma(t) \in B$,

$$\begin{aligned}
u_h(\sigma(t)) - u_h(x_h) &\leq \varphi_h(\sigma(t)) - \varphi_h(x_h) \\
&\leq \varphi(\sigma(t)) - \varphi(x_h) + m \cdot h^2 \\
&\leq -t \cdot \langle D\varphi(x_h), \tilde{q}\rangle + m \cdot (t^2 + h^2) \\
&\leq -t \cdot \frac{\|D\varphi(x_0)\|}{\Upsilon} + m \cdot (t^2 + h^2 + 2t\|x_h - x_0\|),
\end{aligned}$$

where we used (3.25). Let $t = r \cdot h$ with some $r > 0$ (which will be defined below). Then, for sufficiently small h,

$$u_h(\sigma(t)) - u_h(x_h) \leq -rh \cdot \frac{\|D\varphi(x_0)\|}{\Upsilon} + m \cdot (r^2h^2 + h^2 + 2rh\|x_h - x_0\|). \tag{3.27}$$

In view of equation (3.26), r and h should be chosen such, that

$$-r \cdot \frac{\|D\varphi(x_0)\|}{\Upsilon} + m \cdot (r^2h + h + 2r\|x_h - x_0\|) < -\rho^*. \tag{3.28}$$

There is some $h_0 > 0$, such that $\|x_h - x_0\| \leq \|D\varphi(x_0)\| \big/ 4\Upsilon m$ for $h < h_0$. Choosing $r \geq 2\Upsilon(\rho^* + 1) \big/ \|D\varphi(x_0)\|$, we get for $h < h_0$

$$-r \cdot \frac{\|D\varphi(x_0)\|}{\Upsilon} + m \cdot (r^2h + h + 2r\|x_h - x_0\|) \leq -(\rho^* + 1) + m(r^2 + 1) \cdot h,$$

and we can diminish h, such that $m(r^2+1) \cdot h \leq 1/2$. For such h, (3.28) is fulfilled. If h is additionally small enough, that $\sigma(r \cdot h) \in B$, then we obtain, by (3.27), that

$$u_h(\sigma(r \cdot h)) < u_h(x_h) - \rho^* \cdot h \leq \min_{AF_h(x_h)} u_h.$$

As all points y_h, which are accepted after x_h in the OUM, fulfill $u_h(y_h) \geq \min_{AF_h(x_h)}$ by the monotonicity property (3.22), the half-line $\sigma(t)$ must actually intersect the accepted front $AF_h(x_h)$ in some point $\tilde{y}_h = \sigma(\tilde{t})$ with $\tilde{t} < r \cdot h$, and the prove is complete. □

3.4.5. Implementation and Possible Extensions. In this subsection I will briefly discuss my implementation of the Ordered Upwind Method (algorithm 3.21). Just like in the Fast Marching Method, the narrow band, that is, the set of trial points T_h is stored in a priority queue. As an alternative, the idea of [YBS06] could be used, where the narrow band is stored in an untidy priority queue, reducing the total runtime to $\mathcal{O}(N)$, where N denotes the number of grid-points. We have discussed this approach in subsection 3.3.5. In the following, we incorporate the

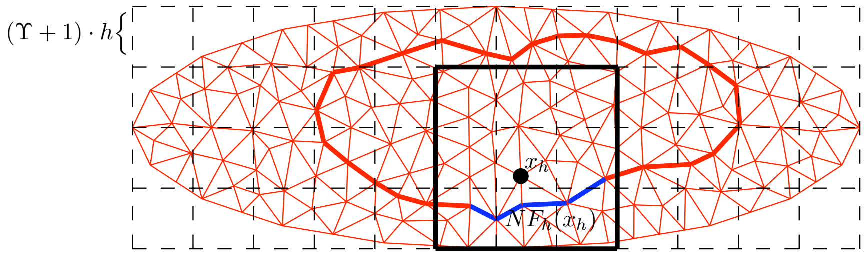

FIGURE 5. Some grid-point x_h, the accepted front (red line) and the near front for x_h (blue line).

anisotropy coefficient Υ in the investigation of the complexity, as it affects the runtime of the Ordered Upwind Method.

Computing a Hopf-Lax update in a single grid-point requires the determination of the near front for x_h, that is $NF_h(x_h) = \{e_h \in AF_h\,;\ \mathrm{dist}(x_h, e_h) \le \Upsilon \cdot h\}$, which means, we have to provide a fast access to the accepted front within a certain distance. Let $x_{\min}, x_{\max}, y_{\min}, y_{\max}$ denote the minimal/maximal abscissa/ordinate for points within $\overline{\Omega}$, respectively, such that $\overline{\Omega} \subseteq [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$. We choose $n \times n$ boxes

$$I_{i,j} = [x_{\min} + i\delta, x_{\min} + (i+1)\delta[\, \times [y_{\min} + j\delta, y_{\min} + (j+1)\delta[$$

covering $\overline{\Omega}$, such that $\delta \approx (\Upsilon + 1) \cdot h$ (see figure 5). For every index pair (i, j), where $i, j = 0, \dots, n-1$, we store the part of the accepted front lying in $I_{i,j}$, as well as the trial points within $I_{i,j}$. If for some (i, j) we have $x_h \in I_{i,j}$, then the part of the accepted front relevant for x_h is contained in the union of the boxes $I_{k,l}$, where $k = i-1, i, i+1$ and $l = j-1, j, j+1$. As the diameter of the boxes is $\mathcal{O}(\Upsilon h)$, the maximum number of grid-points within every box is bounded by a constant ($\propto \Upsilon^2$), on a uniform family of refined triangulations. However, only at most $\mathcal{O}(\Upsilon)$ points of the accepted front are contained within each box. Thus, an update in a single point costs $\mathcal{O}(\Upsilon)$ updates from edges, where the methods from section 3.1 can be used. Let me remark, that the utilization of an iterative method for the computation of updates affects the total complexity (compare subsection 3.1.2).

In step (4) of the algorithm, all trial values of points within a distance of $(\Upsilon + 1) \cdot h$ from the accepted point $\overline{x}_h$ have to be re-computed. With the above subdivision only the trial points in the neighboring boxes of the box $I_{i,j}$ containing $\overline{x}_h$ have to be considered. As every box contains approximately $\mathcal{O}(\Upsilon^2)$ grid-points, but only $\mathcal{O}(\Upsilon)$ trial points, we deduce, that $\mathcal{O}(\Upsilon)$ points have to be re-computed in step (4), and thereafter re-ordered in the priority queue, such that the total complexity will be $\mathcal{O}(N \cdot \Upsilon(\Upsilon + \log N))$, where N denotes the number of grid-points. It is possible, to reduce the complexity of the re-computation of the trial values, which leads to a total complexity of $\mathcal{O}(\Upsilon N \log N)$. When $\overline{x}_h$ is accepted, several edges may be removed from the accepted front, while other edges adjacent to $\overline{x}_h$ are added. Let R denote the set of the removed edges, and A the set of the newly added edges. If some trial point x_h has been previously updated from an edge within R, its trial value has to be completely re-computed. If, however, x_h did not depend on some edge within R, the re-computation of the trial value may be reduced to

$$\tilde{u}_h(x_h) = \min\left(\hat{u}_h(x_h), \min_{e_h \in A} \min_{y \in e_h} \{u_h(y) + \rho(x_h, x_h - y)\}\right),$$

where $\hat{u}_h(x_h)$ denotes the old trial value, and $\tilde{u}_h(x_h)$ the new one. This way, the cost of the re-computation will in most cases not be affected by Υ.

In [**SV03**], Sethian and Vladimirsky discuss the possibility of using the local anisotropy coefficient $\upsilon(x_h)$, defined in (3.18), instead of Υ, for the restriction of the accepted front to the part $NF(x_h)$ relevant for x_h. This may greatly reduce the complexity for the computation of updates. However, the arguments given are heuristic, and there is no theoretical result on the convergence of this modification.

CHAPTER 4

Applications and Extensions

In this chapter, I present a view applications of the methods, described in chapter 3. In the first section, three examples for Hamilton-Jacobi equations are given, and the methods for solving the discretization, introduced in chapter 3 are compared. In the final example, the Ordered Upwind Method turns out to be less efficient than the adaptive Gauss-Seidel method, up to a grid-size of 15000×15000 nodes. This is owed to a large anisotropy coefficient in the treated problem, which affects the complexity of the OUM.

In section 4.2, we discuss an approach to the computation of distance maps on manifolds, which has been proposed in [KS98]. The idea is to use the Fast Marching Method directly on the manifold (in detail, on a polyhedral approximation of the manifold), in order to compute the distance function. Additionally, I characterize the distance function as the unique viscosity solution of the Eikonal equation on the manifold, and show the convergence of the obtained discretization. Of course, a different approach would be to calculate the distance function in local coordinates, which leads to a generalized Eikonal equation $\langle Du(x), M(x)Du(x)\rangle = 1$, but in this case, the Fast Marching Method would not be applicable.

In section 4.3, I will follow an idea of Sethian and Vladimirsky in [SV00] to develop a second order discretization for the Eikonal equation. They use second order finite differences to approximate the directional derivatives. Additionally, they store an approximation to the gradient Du_h for every grid-point. The method works well, as long as the considered equation admits a continuously differentiable solution. However, if a shock line runs between two adjacent nodes y_h, z_h, then the computed second order update from the triangle $[x_h, y_h, z_h]$ can be far away from the true solution. In this case, we would like to use the first order formula to calculate the update, which is automatized in subsection 4.3.4, where we discuss an adaptive discretization, that chooses the second order update only in smooth subregions.

4.1. Some Applications

Three examples for static Hamilton-Jacobi equations are given, an obstacle problem for the Eikonal equation, then the calculation of first arrival times in an anisotropic medium, which leads to a generalized Eikonal equation. The last example is an exit-time optimal control problem. The different algorithms from chapter 3 are compared, and the last example reveals, that the adaptive Gauss-Seidel method, proposed in subsection 3.2.4 is by far more efficient than the elaborate Ordered Upwind Method, up to a grid with 15000×15000 mesh-points.

4.1.1. Problems with Obstacles. We consider the problem of a wave propagating around obstacles. For our computational domain Ω, let the obstacle(s) be given as a compact subset $C \subset \Omega$. One approach, proposed in [Set99b] or in [GK06] for a Cartesian mesh, would be the consideration of the Eikonal equation

$$\|Du(x)\| = f(x), \quad x \in \Omega, \quad u(x_0) = 0, \tag{4.1}$$

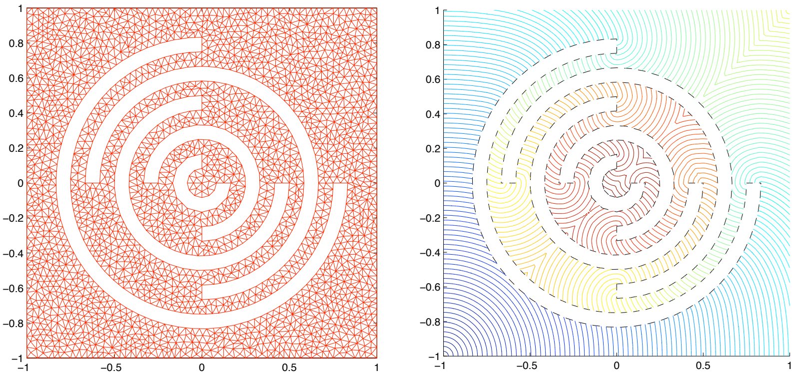

FIGURE 1. The propagation of a wave, starting at $(-1,-1)$ around some obstacle, given as segments of circles. The left figure shows the triangulation, while the right figure shows 200 contour lines of the solution, on a triangulation with diameter $h = 0.0217$.

where x_0 is the starting point of the wave front, and where we set

$$f(x) = \begin{cases} \infty, & \text{if } x \in C \\ 1, & \text{if } x \in \Omega \setminus C, \end{cases}$$

such that the wave propagates with "infinite slow speed" (that is with speed 0) inside the obstacle. However, as triangulations can be adapted for rather general geometries, I used a triangulation of $\Omega \setminus C$ to compute the numerical solution (see figure 1), that is, the distance function

$$\delta(x, x_0) = \inf \left\{ \int_0^1 \left\| \dot{\xi} \right\| \, dt \;\; \xi(0) = x_0, \; \xi(1) = x, \; \xi([0,1]) \subset \Omega \setminus C \right\},$$

where the infimum is taken over all smooth curves. This example was borrowed from [GK06, Example 3]. Therein, Gremaud and Kuster compute the solution of (4.1) with the Fast Marching Method on a Cartesian grid, which is locally adapted near the obstacle: They insert additional nodes on the boundary of C, corresponding to the intersection of the mesh lines and the obstacle‘s boundary.

The numerical results are summarized in the following table. I used the second order Fast Marching Method from subsection 4.3.4, in order to provide an error estimation.

h	time $[s]$	L^∞ error	L^1 error
0.04313651	0.08	0.04751628	0.03462862
0.02172584	0.28	0.02319735	0.01809229
0.01106919	1.15	0.01452095	0.01102806
0.00566726	5.30	0.00638592	0.00527629
0.00285967	31.94	0.00276091	0.00261857
p		1.03	0.94

The table contains in the first column the grid diameter h, then the computational time for the first oder Fast Marching Method, measured on a laptop with a 2GHz processor, and the estimated L^∞ and L^1 error in the third and fourth column. Exact initial values were provided, within a distance of 0.2 from the starting point $(-1,-1)$ (where $\delta(x, x_0) = \|x - x_0\|$, as the wave front hasn‘t reached the obstacle for $\delta(x, x_0) \leq 0.2$). Thus we could expect the observed first oder convergence. Let me remark, that the Fast Marching Method is the fastest known algorithm, to compute the distance function on triangulations. For the five grids in the above

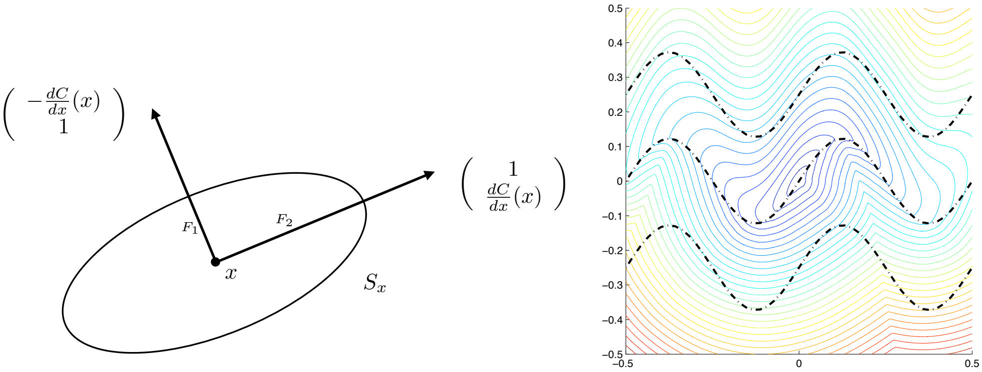

FIGURE 2. Traveltimes in an inhomogeneous anisotropic medium. The left sketch shows the speed profile for some point x (see text). The right figure shows the contours of the distance function from the origin.

table, I obtained the following computational times with the adaptive Gauss-Seidel method on the same computer: (0.18,0.76,3.84,21.78,232.66), where a tolerance of $0.1 \cdot h^2$ was chosen. Of course, the errors were of the same order.

4.1.2. Seismic Traveltimes. Next, we discuss an example from [**SV03**], involving discontinuities in the Hamilton function. We compute first arrival times in an anisotropic medium, with applications to seismic imaging. On the computational domain $\Omega = [-a, a]^2$, with some $a > 0$, we consider different material layers, separated by a sinusoids:

$$\xi_i(x_1) = (x_1, C(x_1) + b_i), \quad C(x_1) = A \sin\left(\frac{m\pi x_1}{a} + \beta\right),$$

where $i = 1, \ldots, k$. We should obtain $k + 1$ layers (for suitable constants a, b_i)

$$L_i = \{x \in \Omega\,;\; x_2 \in]C(x_1) + b_{i-1}, C(x_1) + b_i]\}\,, \quad i = 1, \ldots, k + 1$$

where $b_0 = -\infty$, and $b_{k+1} = \infty$.

In each layer, the anisotropic speed profile S_x at $x \in \Omega$ is given by an ellipse with the bigger axis of length F_2 tangential to ξ_i, and the smaller axis F_1, normal to ξ_i (see figure 2). By lemma 1.31, the support function is given by the gauge, generated from the speed profile S_x, that is,

$$\rho(x, q) = \inf\{\lambda \geq 0\,;\; q \in \lambda S_x\}\,.$$

After transforming q in the coordinate system, given by the bigger and the smaller axis of the ellipse, that is,

$$\tilde{q} = \frac{1}{\sqrt{1 + C'(x_1)^2}} \left(\begin{array}{cc} 1 & C'(x_1) \\ -C'(x_1) & 1 \end{array}\right) \cdot \left(\begin{array}{c} q_1 \\ q_2 \end{array}\right), \tag{4.2}$$

we have $q \in \lambda S_x$, if and only if

$$\frac{\tilde{q}_1^2}{F_2^2} + \frac{\tilde{q}_2^2}{F_1^2} \leq \lambda^2,$$

and thus, with $T(x)$ denoting the transformation matrix from (4.2),

$$\rho(x, q) = \sqrt{\frac{\tilde{q}_1^2}{F_2^2} + \frac{\tilde{q}_2^2}{F_1^2}} = \sqrt{q^T T(x)^T \left(\begin{array}{cc} F_2^{-2} & 0 \\ 0 & F_1^{-2} \end{array}\right) T(x) q}.$$

As ρ is of the form (3.3), the update formula from subsection 3.1.3 may be applied, within the adaptive Gauss-Seidel, or the Ordered Upwind Method for anisotropic Hamilton-Jacobi equations. I tested both methods on refined triangulations, and

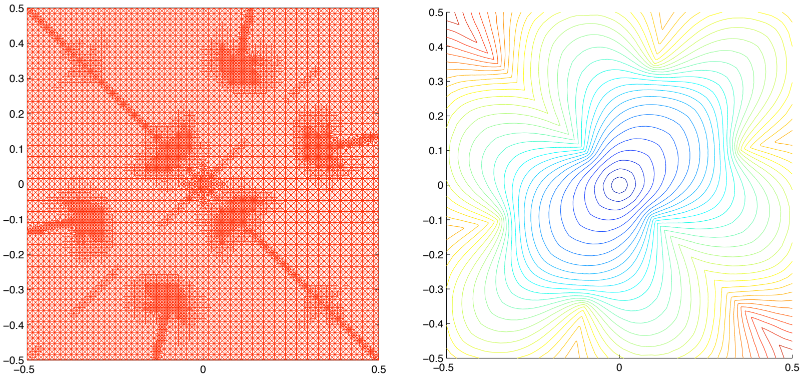

FIGURE 3. Left the adaptive mesh, obtained by a local refinement, where the estimate of $\left\|D^2u\right\|_\infty$ is large, and right the value function of an exit-time optimal control problem (see text).

summarized the results in the following table. The parameters were $a = 0.5$, $A = 0.1225$, $m = 2$, $\beta = 0$, $b_i = -0.25, 0.00, 0.25$, and the pair (F_2, F_1) was chosen to be $(1,1)$ in the lower two layers, and $(3,1)$ in the upper two layers.

	Adaptive Gauss-Seidel			Ordered Upwind		
h	time $[s]$	L^∞ error	L^1 error	time $[s]$	L^∞ error	L^1 error
2^{-5}	0.06	0.05718105	0.01668719	0.24	0.04640310	0.01515617
2^{-6}	0.40	0.03685709	0.01021499	0.95	0.02318624	0.00719938
2^{-7}	2.13	0.02115077	0.00583814	4.07	0.01309713	0.00392346
2^{-8}	9.83	0.01131355	0.00307966	16.97	0.00632722	0.00185433
2^{-9}	68.22	0.00553607	0.00139647	75.59	0.00230410	0.00065820
p		0.78	0.81		0.94	1.00

The errors were computed by a comparison with the finite-element solution on a mesh with about 1450×1450 grid-points. For the computation of the order, the results on the mesh with diameter $h = 2^{-9}$ were not included. One observes a better convergence rate, but a higher computational time of the Ordered Upwind Method. The adaptive Gauss-Seidel method was used with a fixed tolerance of 10^{-10}, and shows an asymptotic time complexity of $\mathcal{O}(N^{5/4})$, while the Ordered Upwind Method needs $\mathcal{O}(N \log N)$ time, where N denotes the number of grid-points, but with a much larger constant in the $\mathcal{O}$ term. Anyway, the OUM will become more efficient than the adaptive Gauss-Seidel method on large grids. A contour plot of the finite-element solution can be found in figure 2.

Though we did not consider discontinuous Hamilton functions in the convergence theory in section 2.3, this example suggests, that the discussed methods work equally well under less restrictive continuity assumptions. Anyway, for the discontinuous support function ρ from the example, the distance function

$$\delta(x, x_0) = \inf \left\{ \int_0^1 \rho(\xi(t), -\dot{\xi}(t))\, dt\,;\ \xi(0) = x_0,\ \xi(1) = x \right\}$$

is well-defined, and Lipschitz-continuous. Within each layer, the support function is Lipschitz-continuous, and the estimate from lemma 2.14 holds. Across the layer boundaries, errors are introduced of order $\mathcal{O}(h)$, which are transported, but not enlarged by the Hopf-Lax discretization.

4.1.3. Exit-Time Optimal Control. The last example, also obtained from [**SV03**], covers the most general case, an exit-time optimal control problem, where $\rho(x, q)$ won't be of the form (3.3), such that an iterative method will have to be used for the local minimization in the Hopf-Lax discretization. Additionally, the problem shows the effect of a rather large anisotropy coefficient of $\Upsilon = 19$, which affects the complexity of the Ordered Upwind Method. With the state equation

$$\dot{y}(t) = a(t) + b(y(t)), \quad y(0) = x, \quad b(y) = -0.9\sin(4\pi y_1)\sin(4\pi y_2)\cdot\frac{y}{\|y\|},$$

where the controls $a(t)$ are taken from the set $\mathcal{A} = \{a : [0,\infty[\to S^1\,;\ a \text{ measurable}\}$, we denote by $t_x(a)$ the first time, the trajectory $y(t)$ under the control $a(\cdot)$ reaches the origin (where we set $t_x(a) = \infty$, if $y(t)$ never reaches 0). Then the value function $u(x) = \inf_{a\in\mathcal{A}} t_x(a)$ is a viscosity solution of the Hamilton-Jacobi-Bellman equation

$$H(x, Du(x)) = \max_{\|a\|=1}\{\langle a + b(x), -Du(x)\rangle - 1\} = 0, \quad u(0) = 0.$$

Here, the speed profile at $x \in \mathbb{R}^2$ is given by $S_x = \{b(x) + a\,;\ \|a\| = 1\}$, and we obtain the support function with the help of lemma 1.31. Obviously, we have $q \in \lambda S_x$ (with some $\lambda \geq 0$), if and only if

$$\|q - \lambda b\| = \lambda \quad \Leftrightarrow \quad \|q\|^2 - 2\lambda\langle q, b\rangle + \lambda^2\|b\|^2 = \lambda^2,$$

where $b = b(x)$. Thus we obtain λ as the (unique) non-negative solution of the quadratic equation above, and

$$\rho(x, q) = \lambda = \sqrt{\langle q, b\rangle^2 + \|q\|^2\left(1 - \|b(x)\|^2\right)} - \langle q, b(x)\rangle\,.$$

The value function, and the underlying adaptive mesh, used for the computation, are depicted in figure 3. Starting with a regular 91×91 mesh, the adaptive mesh was obtained through a local refinement based on estimates of the second derivative $\|D^2 u_h\|_\infty$.

The obtained error on regular grids, estimated by a comparison with the finite-element solution on a fine 1450×1450 grid, and the computational times for the adaptive Gauss-Seidel, the standard Gauss-Seidel, and the Ordered Upwind method, are given in the following table:

	Adaptive Gauss-Seidel		Ordered Upwind		Gauss-Seidel	
h	time $[s]$	L^∞ error	time $[s]$	L^∞ error	time $[s]$	L^∞ error
2^{-5}	0.27	0.01864	2.06	0.01872	1.20	0.01863
2^{-6}	1.25	0.00983	7.61	0.00998	9.31	0.00981
2^{-7}	5.74	0.00508	29.95	0.00506	73.69	0.00502
2^{-8}	29.59	0.00242	120.17	0.00232	584.97	0.00227
2^{-9}	166.01	0.00126	489.28	0.00081	4670.50	0.00079
p		0.98		1.00		1.01

The Gauss-Seidel Iteration shows an asymptotic time complexity of $\mathcal{O}(N^{3/2})$, as predicted in theorem 3.4 (that is, the same time complexity as the Jacobi method discussed in section 3.2). The adaptive Gauss-Seidel method is by far the fastest algorithm in this example, although its asymptotic complexity is worse than the $\mathcal{O}(N\log N)$ of the OUM. By extrapolating the results I found, that the OUM would be faster on mesh with at least 15000×15000 grid-points.

4.2. Distance Maps on Manifolds

As the discussion of this application is quite lengthy, I decided to devote an own section to the computation of distance maps on manifolds. While an algorithm for this problem has already been given in [**KS98**], there has been no theoretical result on the convergence of this method. Contrary to this approach, I will first recall some concepts known from differential geometry, which can be found in, for example, [**BG05**]. The aim is to compute the distance map M on a smooth and compact surface in $\mathbb{R}^3$. This distance map $u : M \to \mathbb{R}$ can be characterized as the unique viscosity solution of the Eikonal equation $\|\nabla_p u\| = 1$ on the manifold, with $u(q) = 0$ for some point $q \in M$, what will be shown in subsection 4.2.2.

In the next subsection, I will introduce the Hopf-Lax discretization on the manifold (where M will be approximated by a polyhedron M_h). Theorem 4.5 shows the convergence of the grid-functions $u_h : M_h \to \mathbb{R}$ to the distance function u on the manifold. For any sequence (p_h) of grid-points on refined polyhedral approximations of M, such that $p_h \to p \in M$, we have $u(p) = \lim_{h\to 0} u_h(p_h)$. I will further point out the difference between the discretization of the pullback equation on the parameter plane under some chart, and the Hopf-Lax discretization on the manifold.

The Fast Marching Method with virtual update strategy, introduced in [KS98], can be used to compute the grid-solution $u_h : M_h \to \mathbb{R}$ efficiently in $\mathcal{O}(N \log N)$ time, where N denotes the number of grid-points. The application of this method to the Hopf-Lax discretization on manifolds will be discussed in subsection 4.2.5, and I added a proposal for the calculation of the involved unfolding of triangles on the tangent plane. The final subsection contains a few examples.

4.2.1. Differentiable Submanifolds. We consider two-dimensional submanifolds of $\mathbb{R}^3$, that is surfaces in $\mathbb{R}^3$. Let $d \le n$ denote natural numbers. Generally, a *d-dimensional smooth submanifold of* $\mathbb{R}^n$ is a connected subset $M \subseteq \mathbb{R}^n$, such that every point $p \in M$ has a relatively open neighborhood $V \subseteq M$, which is homeomorphic to an open subset $U \subseteq \mathbb{R}^d$, such that the homeomorphism $\phi : U \to V$ is $\mathcal{C}^\infty(U)$, and its derivative (the Jacobi matrix) $D\phi(x)$ has rank d for every $x \in U$. In this context, ϕ is called a *(coordinate) chart*.

An *atlas* is a family $\{U_i, \phi_i\}_{i\in I}$ of charts $\phi_i : U_i \to M$, for which the $(U_i)_{i\in I}$ constitute an open covering of M. Provided, that the images V_1 and V_2 of two charts $\phi_i : U_i \to M$ have a non-empty intersection, then the *chart transition*

$$\tau = \phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(V_1 \cap V_2) \to \phi_2^{-1}(V_1 \cap V_2)$$

is differentiable of class $\mathcal{C}^\infty$. Since $\phi_1^{-1} \circ \phi_2$ is the inverse of $\phi_2^{-1} \circ \phi_1$, chart transitions are even diffeomorphisms of class $\mathcal{C}^\infty$.

A map $f : M \to \mathbb{R}$ is *differentiable in* $p \in M$, if there exists a coordinate chart $\phi : U \to M$, such that $\phi(x) = p$ for some $x \in U$, and $f \circ \phi : U \to \mathbb{R}$ is differentiable in x. As usual, we can define $\mathcal{C}^k$-differentiable maps on M.

A vector $v \in \mathbb{R}^n$ will be called a *tangent vector of* M *in* $p \in M$, if there exists a smooth curve $c :]-\epsilon, \epsilon[\to M$, such that $c(0) = p$ and $\dot{c}(0) = v$. The set of tangent vectors T_pM in some point $p \in M$ is a d-dimensional subspace of $\mathbb{R}^n$, and is called the *tangent space of* M *in* p. To see this, let $\phi : U \to V$ denote a chart with $p = \phi(x)$. Then we define a curve c_i by $t \mapsto \phi(x + te_i)$, where $e_1, \ldots, e_d$ denote the standard basis vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\mathbb{R}^d$. Thus $\dot{c}_i(0) = D\phi(x) \cdot e_i$, that is, $\dot{c}_i(0)$ is the ith column of the Jacobi matrix of ϕ. The column vectors of the Jacobi matrix form a basis of the tangent space. In the context of abstract manifolds, those basis vectors are often denoted as $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$. Some tangent vector $v \in T_pM$ can be expressed in *local coordinates*, that is $(v_1, \ldots, v_d)^T = \frac{d}{dt}(\phi^{-1} \circ c)(0) \in \mathbb{R}^d$, where $c(0) = p$ and $\dot{c}(0) = v$. Then, by chain rule,

$$v = \frac{d}{dt}(\phi \circ \phi^{-1} \circ c)(0) = D\phi(\phi^{-1}(p)) \cdot (v_1, \ldots, v_d)^T = \sum_{i=1}^{d} v_i \cdot \frac{\partial}{\partial x_i}.$$

We used above the differentiability of $\phi^{-1} \circ c$, which can be shown by extending ϕ to a diffeomorphism $\tilde{\phi} : \tilde{U} \to \tilde{V}$ of open subsets $\tilde{U}, \tilde{V}$ of $\mathbb{R}^n$. If $\overline{\phi} : \overline{U} \to \overline{V}$ is another chart on a neighborhood of p, then

$$\frac{d}{dt}(\overline{\phi}^{-1} \circ c)(0) = \frac{d}{dt}(\overline{\phi}^{-1} \circ \phi \circ \phi^{-1} \circ c)(0) = D(\overline{\phi}^{-1} \circ \phi)(x) \cdot (v_1, \ldots, v_d)^T,$$

thus the local coordinates are transformed by $D(\overline{\phi}^{-1} \circ \phi)(x)$.

For $v \in T_pM$, the derivative of a smooth function $f : M \to \mathbb{R}$ in direction v at p is defined to be $v(f)(p) = \frac{d}{dt} f(c(t))|_{t=0}$, for a curve c with $c(0) = p$ and $\dot{c}(0) = v$. The derivatives of f in direction $\frac{\partial}{\partial x_i}$ are simply the partial derivatives of $f \circ \phi$.

A *Riemannian metric* on a smooth (sub)manifold M is given by a scalar product $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pM, which depends smoothly on the base point p. A smooth submanifold M endowed with a Riemannian metric is called a *Riemannian manifold*. Let $\phi : U \to V$ denote a chart with $p \in V$. Then the metric can be expressed by local coordinates. If $(v_1, \ldots, v_d)$ are the coordinates of v, defined as above, and similarly $(w_1, \ldots, w_d)$ are the coordinates of $w \in T_pM$, then

$$\langle v, w \rangle_p = \sum_{i,j=1}^{d} v_i w_j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p = \sum_{i,j=1}^{d} g_{ij}(x) v_i w_j,$$

where $\phi(x) = p$ and $G(x) = (g_{ij}(x))$ is a positive definite, symmetric matrix, which depends smoothly on x.

On a Riemannian manifold, we can define the gradient of a differentiable function $f : M \to \mathbb{R}$ at p as an element of the tangent space T_pM. The gradient of f at p, denoted by $\nabla_p f$, is defined to be the tangent vector, such that $\langle \nabla_p f, v \rangle_p = v(f)(p)$ for every $v \in T_pM$. In local coordinates, and in view of the above paragraph, $\nabla_p f$ can be expressed as

$$\nabla_p f = \sum_{i=1}^{d} \Big(\sum_{j=1}^{d} g^{ij}(x) \frac{\partial}{\partial x_j}(f)(p) \Big) \frac{\partial}{\partial x_i},$$

where $G(x)^{-1} = (g^{ij}(x))$ denotes the inverse of the matrix $G(x) = (g_{ij}(x))$ associated to the Riemann metric. As one readily verifies, this definition is independent of the choice of the coordinate chart.

For a submanifold of $\mathbb{R}^n$, the tangent space can be expressed as a subspace $T_pM \subseteq \mathbb{R}^n$, and we can consider the Riemannian metric, induced by the Euclidean scalar product on $\mathbb{R}^n$. Then the associated matrix, which represents the metric in local coordinates, is given by $G(x) = D\phi(x)^T D\phi(x)$, and the gradient of $f : M \to \mathbb{R}$ is given by

$$\nabla_p f = D\phi(x) \cdot G(x)^{-1} \cdot D(f \circ \phi)(x) \in \mathbb{R}^n.$$

By means of the gradient, defined above, the Eikonal equation on the manifold can be written as

$$\|\nabla_p u\| = 1, \quad p \in N,$$

where N denotes some subset of M.

4.2.2. The Distance Function on Submanifolds. On a smooth manifold $M \subseteq \mathbb{R}^n$, we consider the distance function, given by

$$\text{(4.3)} \quad \delta_M(p, q) = \inf \Big\{ \int_0^1 \|\dot{\gamma}(t)\| \; dt\,; \quad \gamma : [0,1] \to M \text{ a piecewise smooth curve with } \gamma(0) = p, \gamma(1) = q \Big\},$$

where $p, q \in M$. As M is connected, any two points on M can be joined by a piecewise smooth curve (compare the argument given in [Jos95] on page 14). If $p, q \in V \subseteq M$, such that $\phi : U \to V$ is a chart on M, then the length of a curve γ joining p and q in V can be expressed in local coordinates. Let $\xi = \phi^{-1} \circ \gamma$ denote the pullback curve. Then we obtain, with $G(x) = D\phi(x)^T D\phi(x)$,

$$\text{(4.4)} \qquad L(\gamma) = \int_0^1 \sqrt{\left\langle \dot{\xi}(t), G(\xi(t))\dot{\xi}(t) \right\rangle}\, dt.$$

We collect some properties of the distance function in the following lemma.

LEMMA 4.1: *Let $M \subseteq \mathbb{R}^n$ denote a smooth manifold. Then δ_M defined by (4.3) fulfills the following properties for all $p, q, r \in M$:*

(1) *$\delta_M(p,q) \geq 0$, and $\delta_M(p,q) = 0$ implies $p = q$ (definiteness)*
(2) *$\delta_M(p,q) = \delta_M(q,p)$ (symmetry)*
(3) *$\delta_M(p,q) \leq \delta_M(p,r) + \delta_M(r,q)$ (triangle inequality)*

Moreover, for every compact subset of M, there is a constant $C > 0$, such that

$$\|p - q\| \leq \delta_M(p,q) \leq C \cdot \|p - q\|,$$

for all p, q within the compact subset.

PROOF. The proof of (1)-(3) can be found in [**Jos95**, Lemma 1.4.1]. For some point $p_0 \in M$ there exists a chart $\phi : U \to V$ on M with $p_0 \in V$. There is a subset $\tilde{U}$ of U containing $x_0 = \phi^{-1}(p_0)$, and an open subset W of $\mathbb{R}^{n-d}$ containing the origin, such that ϕ can be extended to a diffeomorphism $\Phi : \tilde{U} \times W \to \tilde{V}$, where $\tilde{V} \subset \mathbb{R}^n$, and $M \cap \tilde{V} = \Phi(\tilde{U} \times \{0\})$. Next, there is a closed ball $\overline{B} \subset \tilde{U}$ containing x_0 as an interior point, and we obtain $\|x - y\| \leq C_1 \cdot \|p - q\|$ for all points $x, y \in \overline{B}$ and $p = \phi(x)$, $q = \phi(y)$, where C_1 depends on the partial derivatives of Φ^{-1}. For the matrix $G(x) = D\phi(x)^T D\phi(x)$, there is a bound $C_2 > 0$, such that $\langle \xi, G(x)\xi \rangle \leq C_2^2 \cdot \|\xi\|^2$ for all $\xi \in \mathbb{R}^d$, and $x \in \overline{B}$. For the straight line $\xi(t)$ passing through x and y in $\overline{B}$ in the coordinate space, we deduce by (4.4), that, with $p = \phi(x)$, $q = \phi(y)$,

$$d(p,q) \leq C_2 \int_0^1 \left\| \dot{\xi}(t) \right\| \, dt = C_2 \cdot \|x - y\| \leq C_1 \cdot C_2 \cdot \|p - q\|.$$

The other estimate $d(p,q) \geq \|p - q\|$ holds because of Jensen's inequality. □

Let $q_0 \in M$. I would like to show, that $u : M \to \mathbb{R}$, defined by $u(p) = \delta_M(p, q_0)$ is a viscosity solution of the Eikonal equation

$$\|\nabla_p u\| = 1, \quad p \in M_{q_0} = M \setminus \{q_0\}, \quad u(q_0) = 0. \tag{4.5}$$

Given some chart $\phi : U \to V$, this means, that $u \circ \phi$ is a viscosity solution in $U \subseteq \mathbb{R}^d$. The details are given in the next lemma, and the subsequent proof.

LEMMA 4.2: *For every $q_0 \in M$, the distance function $u(p) = \delta_M(p, q_0)$ on M is a local Lipschitz-continuous viscosity solution of the Eikonal equation (4.5). Given some chart $\phi : U \to V$ with $q_0 \notin V$, then $(u \circ \phi) : U \to \mathbb{R}$ is a viscosity solution of the generalized Eikonal equation*

$$\left\langle D(u \circ \phi)(x), G(x)^{-1} D(u \circ \phi)(x) \right\rangle = 1, \quad x \in U, \tag{4.6}$$

where $G(x) = D\phi(x)^T D\phi(x)$.

PROOF. Let $\psi \in \mathcal{C}^\infty(M)$, such that $u - \psi$ attains a local maximum at $p_0 \in M_{q_0}$. Let $\phi : U \to V$ denote a chart with $p_0 \in V$, and let $B \subset U$ denote some open ball containing $x_0 = \phi^{-1}(p_0)$, such that $q_0 \notin \phi(\overline{B})$. Then

$$(u \circ \phi)(x) = \min_{y \in \partial B} \{(u \circ \phi)(y) + \delta(x,y)\},$$

for all $x \in B$, where

$$\delta(x,y) = \inf \Big\{ \int_0^1 \sqrt{\left\langle \dot{\xi}(t), G(\xi(t)) \dot{\xi}(t) \right\rangle} \, dt \, ; \ \xi : [0,1] \to \overline{B} \text{ piecewise smooth and } \xi(0) = x, \xi(1) = y \Big\}.$$

By theorem 1.25 and example 1.28, $(u \circ \phi) : U \to \mathbb{R}$ is a viscosity solution of the Eikonal equation

$$\langle D\tilde{u}(x), G(x)^{-1} D\tilde{u}(x)\rangle = 1, \quad x \in B.$$

As $(u \circ \phi - \psi \circ \phi)$ has a local maximum at x_0, we obtain

$$\langle D(\psi \circ \phi)(x_0), G(x_0)^{-1} D(\psi \circ \phi)(x_0)\rangle \leq 1$$

or equivalently, $\|\nabla_{p_0}\psi\| \leq 1$. The super-solution property may be shown analogously, and the local Lipschitz continuity of u is a consequence of the triangle inequality and the last inequality in lemma 4.1. □

For a compact manifold, the viscosity solution of (4.5) is actually unique, as the following lemma shows.

LEMMA 4.3: *Let $M \subset \mathbb{R}^n$ denote a compact Riemannian manifold. Then the distance function $u(p) = \delta_M(p, q_0)$ is the* unique *viscosity solution of (4.5).*

PROOF. The result is not surprising, as M is compact, and q_0 is the relative boundary of M_{q_0}, one can apply the uniqueness proof for static Hamilton-Jacobi equations on bounded domains, for example [BCD97, theorem 2.5.9]. I will briefly sketch the main ideas. Let u, v denote two viscosity solutions of (4.5), and assume by contradiction, that $u(p) > v(p)$ for some $p \in M_{q_0}$. Then there exists some number $0 < t < 1$, such that $tu(p) > v(p)$. We consider the function $u^t : M \to \mathbb{R}$, defined by $u^t(p) = t \cdot u(p)$, with some (fixed, but arbitrary) $0 < t < 1$. Then $\|\nabla_p u^t\| \leq t$ in the viscosity sense. Let $\delta = \sup_{p\in M}(u^t - v)(p) > 0$. One considers the test function

$$\Psi_\epsilon(p, q) = u^t(p) - v(q) - \frac{\|p - q\|^2}{2\epsilon}.$$

Let $O = \{p \in M\,;\ (u^t - v)(p) > \delta/2\}$. Then O is a relatively open set, with $q_0 \in M \setminus O$. We denote by (p_ϵ, q_ϵ) the maximum point of Ψ on $\overline{O} \times \overline{O}$. We obtain the following:

- u and v are Lipschitz-continuous: One can show by passing to local coordinates, that u, v are locally Lipschitz-continuous, hence also globally, as M is compact.
- $\|p_\epsilon - q_\epsilon\| = \mathcal{O}(\epsilon)$, where the constant in the $\mathcal{O}$ term depends on the Lipschitz bounds of u and v.
- For small enough $\epsilon \to 0$, we have $(p_\epsilon, q_\epsilon) \in O \times O$.

Let me remark, that for a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, the restriction $f|_M$ of f on the manifold has the gradient

$$\nabla_p f|_M = \Pi(p) \cdot Df(p) = D\phi(x)(D\phi(x)^T D\phi(x))^{-1} D\phi(x)^T Df(p),$$

where ϕ denotes some chart with $\phi(x) = p$, $Df(p)$ is the gradient of f (expressed as a column vector) and $\Pi(p) \in \mathbb{R}^{n\times n}$ is the projection map on the tangent space T_pM.

Since p_ϵ is a local maximum of $p \mapsto u^t(p) - \left(v(q_\epsilon) + \|p - q_\epsilon\|^2 / 2\epsilon\right) = u^t(p) - \psi(p)$, and as u^t fulfills $\|\nabla_p u^t\| \leq t$ in the viscosity sense, we obtain

$$\|\nabla_{p_\epsilon}\psi|_M\| = \frac{1}{\epsilon}\|\Pi(p_\epsilon) \cdot (p_\epsilon - q_\epsilon)\| \leq t. \tag{4.7}$$

On the other hand, q_ϵ is a local maximum of $q \mapsto v(q) - \left(u^t(p_\epsilon) - \|p_\epsilon - q\|^2 / 2\epsilon\right) = v(q) - \vartheta(q)$, thus

$$\|\nabla_{q_\epsilon}\vartheta|_M\| = \frac{1}{\epsilon}\|\Pi(q_\epsilon) \cdot (p_\epsilon - q_\epsilon)\| \geq 1. \tag{4.8}$$

From equations (4.7), (4.8), we obtain

$$1 - t \leq \frac{1}{\epsilon} \|\Pi(q_\epsilon) \cdot (p_\epsilon - q_\epsilon)\| - \frac{1}{\epsilon} \|\Pi(p_\epsilon) \cdot (p_\epsilon - q_\epsilon)\| \leq \frac{\|p_\epsilon - q_\epsilon\|}{\epsilon} \cdot \|\Pi(q_\epsilon) - \Pi(p_\epsilon)\|,$$

and therefore the desired contradiction, when $\epsilon \to 0$, by the continuity of Π and as $\|p_\epsilon - q_\epsilon\| = \mathcal{O}(\epsilon)$. □

Let $N \subset M$ denote a bounded, (relatively) open and connected subset of M. Of course, we could treat in a similar way the boundary value problem

$$\|\nabla_p u\| = f(p), \quad p \in N, \quad u(q) = g(q), \quad q \in \partial N,$$

or even consider more general Hamilton-Jacobi equations. See also [MM02] for Hamilton-Jacobi equations on manifolds, and the theorem 3.1 therein, which is a generalization of my lemma 4.3. Mantegazza and Mennucci provide a uniqueness proof, which follows the same conception, but with a different (and slightly more complicated) test function Ψ, as they treat general smooth manifolds.

4.2.3. The Hopf-Lax Discretization on Submanifolds. An application of the Fast Marching Method is the computation of the distance function, the viscosity solution of (4.5), on a smooth surface $M \subset \mathbb{R}^3$. We consider therefore a compact, smooth surface M (or a compact subset of a smooth surface M), and approximate M by polyhedra M_h with triangular faces.

We suppose, that M can be covered by finitely many charts $\phi_i : U_i \to M$, with $\phi_i \in \mathcal{C}^\infty(\overline{U}_i)$, such that there exist regular triangulations Σ_h^i of $\overline{U}_i$, and

$$M_h = \bigcup_i \bigcup_{[x_h, y_h, z_h] \in \Sigma_h^i} [\phi_i(x_h), \phi_i(y_h), \phi_i(z_h)],$$

with (2.1) and (2.2) being fulfilled for all Σ_h^i. Of course the triangulations Σ_h^i have to be compatible, such that every vertex, edge, or triangular face of M_h in $\phi_i(\overline{U}_i)$ has exactly one corresponding vertex, edge or face in Σ_h^i.

Let the vertices of M_h be denoted by P_h. Similar as in subsection 2.2.2, we consider the Hopf-Lax discretization of (4.5), where we assume, that $q_0 \in P_h$:

$$(4.9) \quad u_h(p_h) = (\Lambda_h u_h)(p_h) \text{ for all } p_h \in P_h \setminus \{q_0\}, \quad u_h(q_0) = 0,$$
$$(\Lambda_h u_h)(p_h) = \min_{\substack{[p_h,\, q_h^1,\, q_h^2] \\ \text{a face in } M_h}} \min_{q \in [q_h^1, q_h^2]} \{u_h(q) + \|p_h - q\|\}.$$

Of course, p_h and q are points in $\mathbb{R}^3$, where generally $q \notin M$, and

$$u_h(q) = \frac{\|q - q_h^2\|}{\|q_h^1 - q_h^2\|} \cdot u_h(q_h^1) + \frac{\|q - q_h^1\|}{\|q_h^1 - q_h^2\|} \cdot u_h(q_h^2)$$

is determined by linear interpolation, if $q \in [q_h^1, q_h^2]$. By similar arguments as in theorems 2.6 and 2.7 one can show, that (4.9) admits a unique solution $u_h : M_h \to \mathbb{R}$, where the intermediate values are determined by linear interpolation. Based on the regularity assumptions about the triangulation in the coordinate plane, we can show the consistency of the discretization, in analogy to subsection 2.3.3.

Lemma 4.4: *With the above assumptions, let $\psi \in C^\infty(M)$, $p \in M_{q_0}$, and $p_h \in P_h$ be a sequence of grid-points converging to p as $h \to 0$. Then*

$$\psi(p_h) \leq (\Lambda_h \psi)(p_h) \;\; \text{for all } h \quad \Rightarrow \quad \|\nabla_p \psi\| \leq 1,$$
$$\psi(p_h) \geq (\Lambda_h \psi)(p_h) \;\; \text{for all } h \quad \Rightarrow \quad \|\nabla_p \psi\| \geq 1,$$

where ψ is linearly interpolated on the faces of M_h, in order to obtain $\Lambda_h \psi$.

PROOF. Let $\phi : U \to V = \phi(U)$ be some chart with $p \in V$, and $(p_h) \subset V$ (after passing to a subsequence). Let Σ_h denote the regular triangulations of $\overline{U}$, which are mapped onto M_h, such that the regularity assumptions (2.1) and (2.2) are fulfilled. Given a function $f : \overline{U} \to M$, we consider the piecewise linear interpolant

$$(I_h f)(x) = a_1 \cdot f(x_h) + a_2 \cdot f(y_h) + a_3 \cdot f(z_h), \text{ if } x = a_1 x_h + a_2 y_h + a_3 z_h, \quad a_1 + a_2 + a_3 = 1, \quad a_i \geq 0,$$

if $x \in [x_h, y_h, z_h] \in \Sigma_h$. With $x_h = \phi^{-1}(p_h)$ and $x = \phi^{-1}(p)$, we obtain

$$(\psi \circ \phi)(x_h) \begin{matrix} \leq \\ \geq \end{matrix} \min_{y \in \partial\omega_h(x_h)} \{ I_h(\psi \circ \phi)(y) + \|\phi(x_h) - I_h\phi(y)\| \}$$

in the two cases of the assertion. For simplicity, I_h denotes also the interpolation operator for a scalar valued function in the last equation. For the ith coordinate of ϕ, one obtains by Taylor expansion (just like in the proof of lemma 2.1), that

$$\phi^i(x_h) - I_h\phi^i(y) = \langle D\phi^i(x_h), x_h - y \rangle + R_i,$$

where the remainder is bounded by $\left\|D^2\phi^i\right\|_\infty \cdot h^2/2$, if $y \in \partial\omega_h(x_h)$. Thus

$$\Big| \|\phi(x_h) - I_h\phi(y)\| - \|D\phi(x_h)(x_h - y)\| \Big| \leq \sqrt{R_1^2 + R_2^2 + R_3^2} = C \cdot h^2,$$

with some constant C depending on the second derivatives of ϕ. Thus we obtain in both cases, as

$$\|D\phi(x_h)(x_h - y)\| = \sqrt{\langle (x_h - y), G(x_h)(x_h - y) \rangle} = \rho(x_h, x_h - y),$$

which is the support function of the generalized Eikonal equation (4.6), that

$$(\psi \circ \phi)(x_h) \begin{cases} \leq \min_{y \in \partial\omega_h(x_h)} \{ I_h(\psi \circ \phi)(y) + \rho(x_h, x_h - y) \} + Ch^2 \\ \geq \min_{y \in \partial\omega_h(x_h)} \{ I_h(\psi \circ \phi)(y) + \rho(x_h, x_h - y) \} - Ch^2 \end{cases}$$

Likewise the theorem 2.11 and its proof, we deduce, by passing to the limit $h \to 0$,

$$\|\nabla_p \psi\|^2 = \left\langle D(\psi \circ \phi)(x), G(x)^{-1} D(\psi \circ \phi)(x) \right\rangle \begin{matrix} \leq \\ \geq \end{matrix} 1,$$

respectively. □

We can follow the theory in section 2.3, to show the convergence of the sequence (u_h) obtained from (4.9).

THEOREM 4.5: *Let $M \subset \mathbb{R}^n$ denote a compact Riemannian manifold. For some sequence $h \to 0$ let $u_h : P_h \to \mathbb{R}$ denote the grid-functions defined by (4.9). Then we obtain a continuous function $u : M \to \mathbb{R}$ by*

$$u(p) = \lim u_h(p_h), \text{ where } p_h \in P_h, \quad p_h \to p.$$

The value $u(p)$ does not depend on the choice of the sequence $p_h \to p$. Moreover, $u(p) = \delta_M(p, q_0)$ is the unique viscosity solution of (4.5).

PROOF. We can pass to local coordinates, in order to show, that the sequence of finite-element functions (v_h^i), obtained by interpolating $u_h \circ \phi_i$ on $\overline{U}_i$, has a convergent subsequence, which converges uniformly on $\overline{U}_i$ to a viscosity solution v^i of (4.6) on $U_i \setminus \left\{\phi_i^{-1}(q_0)\right\}$. Indeed the functions (v_h^i) are uniformly bounded, and Lipschitz-continuous, as $h \to 0$. The remaining part is a consequence of theorem 2.12 and of lemma 4.4. Let us assume, that there are N charts $\phi_i : U_i \to V_i$, such that the triangulations Σ_h^i of $\overline{U}_i$ provide the approximation of M by polyhedra M_h. Passing N times to a subsequence, we obtain, first a subsequence $(v_{h_1}^1)$ of (v_h^1) converging uniformly on $\overline{U}_1$, second a subsequence $(v_{h_2}^2)$ of $(v_{h_1}^2)$ converging uniformly on $\overline{U}_2$, and so on. For the subsequence $h_N \to 0$, each sequence $(v_{h_N}^i)$

converges uniformly on $\overline{U}_i$. Then $u(p) = v^i(\phi_i^{-1}(p))$, if $p \in V_i$, is a well-defined continuous function on M. To show this, let $p \in V_i \cap V_j$ and consider some sequence $(p_{h_N}) \subset V_i \cap V_j$, with $p_{h_N} \in P_{h_N}$, converging to p. Then

$$v^i_{h_N}(\phi_i^{-1}(p_{h_N})) = u_{h_N}(p_{h_N}) = v^j_{h_N}(\phi_j^{-1}(p_{h_N})) \quad \stackrel{h_N \to 0}{\Longrightarrow} \quad v^i(\phi_i^{-1}(p)) = v^j(\phi_j^{-1}(p)).$$

As u is a viscosity solution of (4.5), we deduce $u(p) = \delta_M(p, q_0)$ for all $p \in M$, by lemma 4.3. If some sequence $(v^{i+1}_{h_i})$ had a limit point $\tilde{v}^{i+1}$ different from v^{i+1}, then there were subsequences $h'_{i+1} \to 0, \ldots, h'_N \to 0$, such that $v^j_{h'_N} \to \tilde{v}^j$ for $j = i+1, \ldots, N$, and we would obtain by the same construction as above a different viscosity solution $\tilde{u}(p)$ on M, contrary to the uniqueness lemma 4.3. □

4.2.4. Computation in Local Coordinates. Given some chart $\phi : U \to V$, the Eikonal equation $\|\nabla_p u\| = 1$ can be expressed in local coordinates. We obtain, as in lemma 4.1, the generalized Eikonal equation

$$\langle Dv(x), G(x)^{-1} Dv(x) \rangle = 1, \quad x \in U$$

With $H(x,p) = \langle p, G(x)^{-1} p \rangle - 1$, the support function is given by $\rho(x,q) = \langle q, G(x) q \rangle^{1/2}$ (compare example 1.28). Here $G(x) = D\phi(x)^T D\phi(x)$ denotes the first fundamental form. If U admits a triangulation Σ_h, then the Hopf-Lax discretization of the local equation becomes

$$(\Lambda_h v_h)(x_h) = \min_{y \in \partial\omega_h(x_h)} \left\{ v_h(y) + \langle (x_h - y), G(x_h)(x_h - y) \rangle^{1/2} \right\}. \tag{4.10}$$

Unfortunately, the assigned value at x_h depends on $G(x_h)$, and therefore on the choice of the chart $\phi : U \to V$. Nevertheless, we could compute an approximation to the distance function from some point $x_0 \in U$, at least in a smaller neighborhood $\tilde{U} \subset U$ of x_0, by solving (4.10) with boundary condition $v_h(x_0) = 0$. But if M is covered by more than one chart, difficulties arise, where the charts $V_i = \phi_i(U_i)$ overlap. Moreover, if the same grid-point $\phi_i(x_h) = \phi_j(\tilde{x}_h)$ is expressed in different local coordinates, there is an ambiguity in (4.10), as the value depends on the choice of the chart. The discretization (4.9) considered in the last subsection overcomes these difficulties. As we have already seen in the proof of lemma 4.4, the expression of (4.9) in local coordinates leads to a discretization *different* from (4.10):

$$\begin{aligned}(\Lambda_h v_h)(x_h) &= \min_{y \in \partial\omega_h(x_h)} \{ v_h(y) + \|\Phi(x_h) - (I_h \Phi)(y)\| \} \\ &= \min_{[x_h, y_h, z_h] \in \Sigma_h} \min_{0 \le t \le 1} \{ t v_h(y_h) + (1-t) v_h(z_h) + \|\Phi(x_h) - t\Phi(y_h) - (1-t)\Phi(z_h)\| \}\end{aligned}$$

In the last equation δ_M is approximated by the exact distance on M_h, whereas in (4.10) the neighborhood patch $\partial\omega_h(x_h)$ is mapped on the tangent space $T_{p_h} M$ at $p_h = \Phi(x_h)$.

4.2.5. The Fast Marching Method on Manifolds. The Fast Marching Method can be used, in order to compute a solution of (4.9) on the manifold. For the computation of updates from triangles, that is,

$$(\Lambda_h u_h)(p_h) = \min_{\substack{[p_h, q_h^1, q_h^2] \\ \text{a face in } M_h}} \min_{q \in [q_h^1, q_h^2]} \{ u_h(q) + \|p_h - q\| \}, \tag{4.11}$$

we can use the update formula derived in subsection 3.1.3. However, the Fast Marching Method relies on the causality property, which states, that the value $u_h(p_h)$ at p_h depends only on the smaller grid-function values in the neighboring grid-points. As we saw in lemma 3.10, the causality property holds, if all angles in the triangulation are acute. This assertion immediately carries over to the Hopf-Lax discretization of the Eikonal equation on manifolds. Let u_h denote the solution

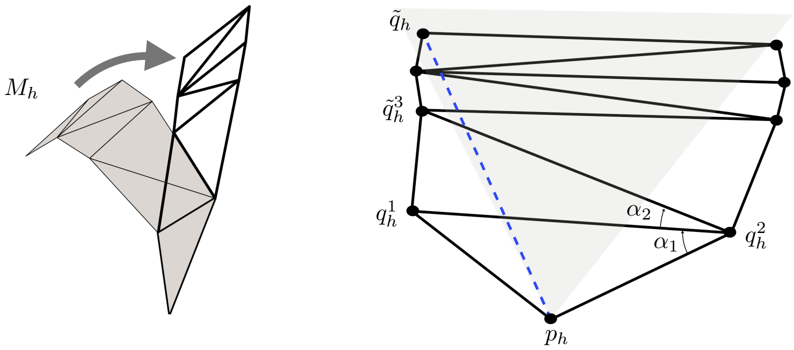

FIGURE 4. The virtual update procedure on manifolds. Adjacent triangles are traversed and unfolded, until a suitable point $\tilde{q}_h$ in the splitting section has been found.

of (4.9). If the value at p_h is computed from the triangle $[p_h, q_h^1, q_h^2]$, where the angle at p_h is acute, such that

$$u_h(p_h) = t_1 u_h(q_h^1) + t_2 u_h(q_h^2) + \left\|p_h - t_1 q_h^1 - t_2 q_h^2\right\|, \quad t_i \geq 0, \quad t_1 + t_2 = 1,$$

then $t_i > 0$ implies $u_h(q_h^i) < u_h(p_h)$. Thus, if all triangular faces of M_h are acute, then the causality property holds, and the Fast Marching Method (algorithm 3.11) is applicable.

For general polyhedra with some obtuse triangular faces, a virtual update strategy can be used to save the causality property, which has been proposed by Kimmel and Sethian in [KS98]. This technique has already been considered in subsection 3.3.3, and will now be extended to the discretization on M_h. Let us assume, that we encounter some triangle $[p_h, q_h^1, q_h^2]$ with an obtuse angle at p_h during the Fast Marching Method, from which the update function (4.11) has to be evaluated. Just like in subsection 3.3.3, we could search for a grid-point q_h in the splitting section

$$S = \left\{q \in \mathbb{R}^3 \,;\ \left\langle q - p_h, q_h^1 - p_h\right\rangle \geq 0 \text{ and } \left\langle q - p_h, q_h^2 - p_h\right\rangle \geq 0\right\},$$

which is the intersection of two closed half-spaces, and instead of (4.11), we would update the value at p_h from the acute virtual triangles $[p_h, q_h^1, q_h]$ and $[p_h, q_h, q_h^2]$. However, this approach does not yield an appropriate method for two reasons: first, there is no guarantee, that a grid-point in the splitting section exists, and second, if there was a suitable point q_h in S, then $\|p_h - q_h\|$ could widely differ from the distance $\delta_M(p_h, q_h)$ on a curved manifold.

Let E denote the plane through p_h, q_h^1, q_h^2. Instead of searching for a grid-point q_h in S, the neighboring triangles are unfolded on E, until a vertex $\tilde{q}_h$ in $S \cap E$ has been found (see figure 4). Then p_h and $\tilde{q}_h$ are connected by a virtual edge, and the update at p_h is computed from the acute triangles $[p_h, q_h^1, \tilde{q}_h]$ and $[p_h, \tilde{q}_h, q_h^2]$ in E. Just like in the two dimensional case in subsection 3.3.3, there exists an upper bound on the number of triangles, that have to be traversed, until a suitable point in the splitting section has been found, as shown by Kimmel and Sethian in [KS98]. In detail, if h denotes the maximal length of an edge, $h_{\min}$ the minimal altitude, and $\gamma_{\max}, \gamma_{\min}$ the maximal/minimal angles in the triangular faces of M_h, then the number of triangles that are needed to be unfolded is approximately bounded by

$$m = \frac{h^2}{\gamma_{\min} \cdot (\pi - \gamma_{\max})^3 \cdot h_{\min}^2}.$$

(Practically it turns out, that this bound is quite pessimistic.)

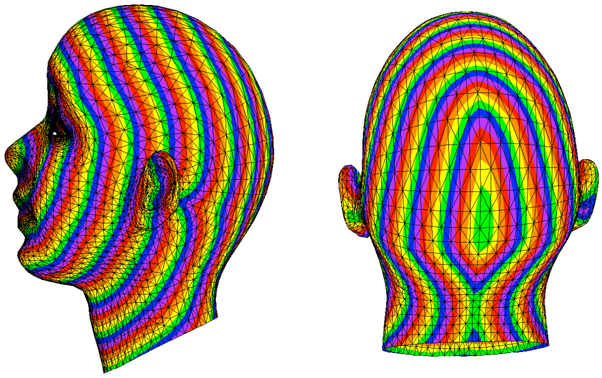

FIGURE 5. The contours of the distance function $\delta_M(p, \text{nose})$ on a triangulated human head.

When unfolding the triangles on E, we have to compute the new coordinates $\tilde{q}_h^k$ of the vertices. As there is no suggestion in [**KS98**] on how to perform the unfolding numerically, I will briefly sketch my approach. I have avoided the computation of cross products, and surface normals, because of potential numerical problems. We denote by q_h^3 the third vertex in the triangle adjacent to $[p_h, q_h^1, q_h^2]$ on the opposite edge of p_h (see figure 4). Then the coordinates $\tilde{q}_h^3$ of the vertex in the unfolded triangle are given by

$$\tilde{q}_h^3 = q + \left\| q_h^3 - q_h^2 \right\| \cdot \sin \alpha_2 \cdot v,$$

where q denotes the intersection point of the perpendicular to the edge $[q_h^1, q_h^2]$ through the point q_h^3, and v denotes the unit vector, also orthogonal to $[q_h^1, q_h^2]$, but lying in the plane E, and pointing outward from the triangle $[p_h, q_h^1, q_h^2]$. The point q is given by

$$q = q_h^2 + \left\| q_h^3 - q_h^2 \right\| \cdot \cos \alpha_2 \cdot \frac{q_h^1 - q_h^2}{\| q_h^1 - q_h^2 \|}$$

and v can be computed as follows:

$$\tilde{v} = q_h^2 - p_h + \left\| p_h - q_h^2 \right\| \cdot \cos \alpha_1 \cdot \frac{q_h^1 - q_h^2}{\| q_h^1 - q_h^2 \|}, \qquad v = \frac{\tilde{v}}{\| \tilde{v} \|}.$$

The transformation in the remaining steps, until a suitable point has been found, can be done analogously.

4.2.6. Examples. Figures 5 and 6 show two examples of distance functions on manifolds. First, I computed the distance function from the nose on a polyhedral model of a human head. The second example shows the distance function on the torus, both on the three dimensional model, and on the parameter plane. The torus can be covered by four charts $\phi_i :]0,1[\times]0,1[\to \mathbb{R}^3$, where

$$\phi_i(\xi, \eta) = \phi(\xi + \alpha_i, \eta + \beta_i), \quad \alpha_i, \beta_i \text{ suitable},$$
$$\phi(\xi, \eta) = \Big(\cos(2\pi\xi)(R + r\cos(2\pi\eta)),\ \sin(2\pi\xi)(R + r\cos(2\pi\eta)),\ \sin(2\pi\eta) \Big),$$

and $0 < r < R$ are the radii of the torus. As ϕ is 1-periodic in both arguments ξ and η, two points (ξ_1, η_1) and (ξ_2, η_2) represent the same point on the manifold under ϕ, if $\xi_1 - \xi_2 \in \mathbb{Z}$ and $\eta_1 - \eta_2 \in \mathbb{Z}$. In figure 6 (right), the points $(0,0), (0,1), (1,0)$ and $(1,1)$ all represent the same point on the torus, namely the point, from where the distance function was computed.

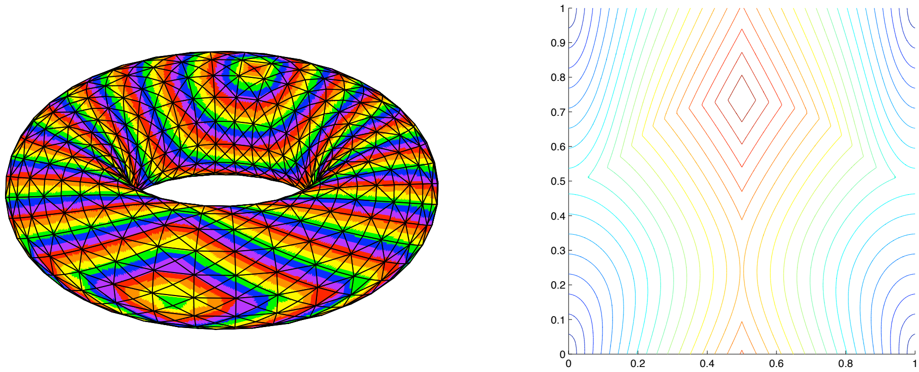

FIGURE 6. The contours of the distance function from a point on the torus (the initial point resides on the back side, from where the concentric circles spread out). *Right:* The contours of the distance function in the parameter plane (see text).

4.3. A Higher Order Scheme for the Fast Marching Method

Sethian and Vladimirsky propose in [SV00] a second order variant of the Fast Marching Method, allowing for the computation of a second order approximation to a smooth solution of the Eikonal equation $\|Du(x)\| = f(x)$ in $\mathcal{O}(N \log N)$ time, where N denotes the total number of grid-points. I will discuss this approach, and provide a Hopf-Lax type formulation of the second order scheme. The idea is to use second order finite differences for the directional derivatives, incorporating the gradient information, which is additionally stored for every grid-point.

An estimate on the obtained local error is given in subsection 4.3.2, and I briefly discuss the applicability of this method, in the case, where the viscosity solution forms shocks. After exploring the connection to the discretization proposed in [SV03] in subsection 4.3.3, I introduce an adaptive variant of the discretization, which takes care of possible shock lines in the solution, and switches back to the first order discretization, if necessary. As numerical experiments reveal in the last subsection, we obtain a method, that shows second order convergence in the L^1 norm, even for solutions with discontinuities in the first derivative.

4.3.1. Idea: Second Order Finite Differences. We follow the approach by Sethian and Vladimirsky in [SV00], in order to construct a second order approximation to the viscosity solution of the static Hamilton-Jacobi equation. For the moment, I will not restrict myself to the Eikonal equation, as this approach may work equally well for more general Hamilton-Jacobi equations, together with the Ordered Upwind Method.

In view of subsection 3.1.1, let us consider the Hopf-Lax approximation from a single simplex $\sigma_h = [x_h, x_1, \ldots, x_d]$ adjacent to x_h:

$$v^{\sigma_h}(x_h) = \min \left\{ \sum_{i=1}^{d} t_i u_h(x_i) + \rho\Big(x_h, x_h - \sum_{i=1}^{d} t_i x_i\Big) ; \quad \sum_{i=1}^{d} t_i = 1,\ t_i \geq 0 \right\}.$$

If the minimization on the boundary of σ_h yields the value $u_h(x_h)$ of the finite-element solution, we say that σ_h is *defining* for $u_h(x_h)$. Provided, that σ_h is defining for u_h at x_h, we can rewrite this formula, denoting by $\nu_i = (u_h(x_h) -$

$u_h(x_i))\big/\ \|x_h - x_i\|$ the directional derivative of u_h in the direction of an edge adjacent to x_h:

$$\max\left\{\sum_{i=1}^{d} t_i\nu_i\,\|x_h - x_i\| - \rho\Big(x_h, x_h - \sum_{i=1}^{d} t_i x_i\Big)\right\} = 0, \tag{4.12}$$

where the maximization is done over all convex combinations as above. The gradient of $u_h|_{\sigma_h}$ will be denoted by $Du_h(\sigma_h)$. Then the last equation is equivalent to the following formulation:

$$\max\left\{\langle Du_h(\sigma_h), x_h - y\rangle - \rho(x_h, x_h - y)\right\} = 0,$$

where the maximum is taken over all $y \in \partial\sigma_h$ on the face opposite of x_h (compare also the remark 2.3). The idea to obtain a higher order approximation, is to substitute the gradient Du_h of the finite-element function by a higher order approximation of the gradient, or equivalently, to provide better approximations of the directional derivatives ν_i. Sethian and Vladimirsky propose to store an approximation of the gradient for every grid-point x_h, and to use second order finite differences

$$\nu_i \approx \frac{2\varphi(x_h) - 2\varphi(x_i) - \langle D\varphi(x_i), x_h - x_i\rangle}{\|x_h - x_i\|} + \mathcal{O}(h^2) \tag{4.13}$$

for the directional derivative. Let us assume, that the grid-function values and gradients in the neighboring points in σ_h have already been calculated. Then, in view of equation (4.12), $v^{\sigma_h}(x_h)$ is assigned the value

$$\begin{aligned} v^{\sigma_h}(x_h) = \min\Big\{ &\sum_{i=1}^{d} t_i u_h(x_i) + \frac{1}{2}\sum_{i=1}^{d} t_i \left\langle \hat{D}u_h(x_i), x_h - x_i\right\rangle \\ &+ \frac{1}{2}\rho\Big(x_h, x_h - \sum_{i=1}^{d} t_i x_i\Big)\,;\quad \sum_{i=1}^{d} t_i = 1,\ t_i \geq 0\Big\}. \end{aligned} \tag{4.14}$$

As we have the following relation for a differentiable function φ

$$\begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} = \begin{pmatrix} \frac{(x_h - x_1)^T}{\|x_h - x_1\|} \\ \vdots \\ \frac{(x_h - x_d)^T}{\|x_h - x_d\|} \end{pmatrix} \cdot D\varphi(x_h) = P \cdot D\varphi(x_h), \tag{4.15}$$

we obtain an approximation to the gradient by $\hat{D}u_h(x_h) = P^{-1}\hat{\nu}$, where $\hat{\nu}$ denotes the vector containing the second order finite difference approximations of the directional derivatives (4.13). For a grid-function u_h, we define the second order Hopf-Lax update in some grid-point x_h:

$$(\Lambda_h^{(G)} u_h)(x_h) = \min_{\sigma_h \text{ adjacent to } x_h} v^{\sigma_h}(x_h), \quad v^{\sigma_h}(x_h) \text{ obtained from (4.14)}. \tag{4.16}$$

Then, an approximation of the gradient in x_h from some neighboring simplex $\sigma_h = [x_h, x_1, \ldots, x_d]$ can be obtained by

$$\hat{D}^{\sigma_h} u_h(x_h) = P^{-1}\cdot\hat{\nu}, \quad \hat{\nu}_i = \frac{2u_h(x_h) - 2u_h(x_i) - \left\langle \hat{D}u_h(x_i), x_h - x_i\right\rangle}{\|x_h - x_i\|}. \tag{4.17}$$

Remark 4.6: The update formula (4.14) has the following form

$$v^{\sigma_h}(x_h) = \min\Big\{\sum_{i=1}^{d} t_i w(x_i) + \frac{1}{2}\rho\Big(x_h, x_h - \sum_{i=1}^{d} t_i x_i\Big)\,;\quad \sum_{i=1}^{d} t_i = 1,\ t_i \geq 0\Big\},$$

In the case of the generalized Eikonal equation $\langle Du, M(x)Du\rangle = 1$, the update can be calculated using the formula from subsection 3.1.3.

4.3.2. The Local Error for Smooth Solutions. As the following lemma shows, the local error of this discretization is of third order for classical solutions of the Hamilton-Jacobi equation.

LEMMA 4.7: *Assume (H1)-(H4), (2.1),(2.2). If $u \in \mathcal{C}^2(\overline{\Omega})$ is a classical solution of $H(x, Du(x)) = 0$, then*

$$\left| u(x_h) - (\Lambda_h^{(G)} u)(x_h) \right| = \mathcal{O}(h^3)$$

for all $x_h \in \Omega_h$. Additionally, we have for all neighboring simplices σ_h,

$$\left\| Du(x_h) - \hat{D}^{\sigma_h} u(x_h) \right\| = \mathcal{O}(h^2)$$

for the approximation (4.17) of the gradient.

PROOF. If $(\Lambda_h^{(G)} u)(x_h) = v^{\sigma_h}(x_h)$ with some simplex $\sigma_h = [x_h, x_1, \dots, x_d]$ adjacent to x_h, then

$$u(x_h) - (\Lambda_h^{(G)} u)(x_h) = \frac{1}{2} \max \Big\{ \sum t_i (2u(x_h) - 2u(x_i) - \langle Du(x_i), x_h - x_i \rangle) \\ - \rho(x_h, x_h - \sum t_i x_i)\, ; \quad \sum_{i=1}^{d} t_i = 1,\ t_i \geq 0 \Big\},$$

and denoting by ν_i the directional derivatives of u as above, we obtain by (4.13),

$$\begin{aligned} u(x_h) - (\Lambda_h^{(G)} u)(x_h) &= \frac{1}{2} \max \Big\{ \sum t_i \nu_i - \rho(x_h, x_h - \sum t_i x_i) \Big\} + \mathcal{O}(h^3) \\ &= \frac{1}{2} \max \Big\{ \sum t_i \langle Du(x_h), x_h - x_i \rangle - \rho(x_h, x_h - \sum t_i x_i) \Big\} + \mathcal{O}(h^3) \\ &\leq \mathcal{O}(h^3), \end{aligned}$$

because of lemma 1.20, and as $H(x_h, Du(x_h)) = 0$. For the other estimate, we choose, in view of lemma 1.20, some $q \in S^{d-1}$, such that $\langle Du(x_h), q\rangle = \rho(x_h, q)$. Then $-q$ points from x_h in some adjacent simplex σ_h, and we can find $t_i \geq 0$, $\sum t_i = 1$, such that

$$q = \frac{x_h - \sum t_i x_i}{\|x_h - \sum t_i x_i\|} = \frac{x_h - y}{\|x_h - y\|}, \quad y = \sum t_i x_i.$$

Then we deduce the other estimate

$$\begin{aligned} 2((\Lambda_h^{(G)} u)(x_h) - u(x_h)) &\leq - \langle Du(x_h), x_h - y\rangle + \rho(x_h, x_h - y) + \mathcal{O}(h^3) \\ &\leq \mathcal{O}(h^3). \end{aligned}$$

By the regularity assumption (2.1), there is some $c > 0$ depending on the regularity constant, such that $\|Pq\| \geq c\,\|q\|$, thus $\left\|P^{-1}\right\|$ is uniformly bounded from above. This shows, together with (4.13), the second assertion. □

There are several problems concerning the discretization (4.16), (4.17). First, the consistency analysis in the last lemma requires a smooth solution of the Hamilton-Jacobi equation. Of course, if two adjacent grid-points x_j and x_k are separated by a shock front, that is, a sharp discontinuity in Du, then we cannot expect to obtain an useful approximation to the gradient by (4.17). However, we can use the values of u_h and $\hat{D}u_h$ on $[x_j, x_k]$ to check for a jump in the first derivative, and switch back to the first oder scheme, if necessary (compare the update strategy (4.24), which will be introduced in subsection 4.3.4). Even if shock lines appear, the information propagates into the shocks, and we will observe second order convergence away from the shock lines.

Moreover there are cases, where smooth solutions are available, at least locally near a smooth piece of the boundary $\partial\Omega$ (compare the local existence theorem, obtained by the method of characteristics [**Eva98**, Theorem 3.2.2]). Consider also the following regularity result for convex Hamiltonians, borrowed from [**BCD97**]. A function $f : \Omega \to \mathbb{R}$ is called *semiconcave*, if there is a constant $C > 0$, such that $f(p) - C\,\|p\|^2$ is concave.

LEMMA 4.8 ([**BCD97**, Proposition 2.5.7]): *Let $u \in \mathcal{C}(\Omega)$ be a viscosity solution of $H(x, Du(x)) = 0$ on the open, convex set Ω, where $p \mapsto H(x,p)$ is strictly convex for all $x \in \Omega$. Assume further, that $-u$ is semiconcave. Then $u \in \mathcal{C}^1(\Omega)$.*

A second problem is, that initial values have to be provided for the gradient in the boundary points. However, those values could be calculated, using the first order scheme within a locally refined mesh near the boundary. In the case where the boundary and the boundary value function are smooth, we could also compute initial values for the gradient, based on the boundary value function, the Hamilton-Jacobi equation and the requirement, that the boundary conditions should be non-characteristic (that is, $\langle D_p H(x_0, Du(x_0)), n(x_0)\rangle \neq 0$, where $n(x_0)$ denotes the outward unit normal to $\partial\Omega$ in $x_0 \in \partial\Omega$).

A further problem is, that it seems to be, that (4.16),(4.17) cannot be solved iteratively. Numerical experiments show, that the Jacobi iteration, or the Gauss-Seidel iteration for the scheme do not converge, even if good initial iterates are provided, and if the problem admits a smooth solution. But, as the following subsections show, we can use the Fast Marching Method to compute a second order approximation to the viscosity solution of the Eikonal equation in $\mathcal{O}(N \log N)$ steps, where N denotes the number of grid-points.

4.3.3. Connection with the Second Order Scheme in [SV03]. Let me remark, that (4.16) is only an alternate formulation of the upwind finite-difference discretization proposed by Vladimirsky and Sethian in [**SV00**] and [**SV03**]. They consider two-dimensional Hamilton-Jacobi equations $F(x, Du) = 1$ of Eikonal type, likewise the equation (1.15) in subsection 1.4.1. In view of equation (4.15), we obtain by $\hat{D}u_h(x_h) = P^{-1}\hat{\nu}$ an approximation to the gradient at x_h, where $\hat{\nu}$ is the vector, containing the finite-difference approximations to the directional derivatives:

$$\begin{aligned} \hat{\nu}_i &= \frac{u - u_h(x_i)}{\|x_h - x_i\|} \text{ (first order)}, \\ \hat{\nu}_i &= \frac{2u - 2u_h(x_i) - 2\langle Du_h(x_i), x_h - x_i\rangle}{\|x_h - x_i\|} \text{ (second order)}, \end{aligned}$$

where u is the sought-after approximation of $u(x_h)$, computed from the triangle $\sigma_h = [x_h, x_1, x_2]$. Plugging the approximate gradient into the equation yields:

$$F(x_h, \hat{D}u_h(x_h)) = 1 \quad \Leftrightarrow \quad \left\|P^{-1}\hat{\nu}\right\| F\left(x_h, \frac{P^{-1}\hat{\nu}}{\|P^{-1}\hat{\nu}\|}\right) = 1. \tag{4.18}$$

This way we obtained a (non-linear) equation for u. In case of the Eikonal equation,where $F(x,p) = f(x) \cdot \|p\|$, this equation reduces to

$$F(x_h, \hat{D}u_h(x_h)) = 1 \quad \Leftrightarrow \quad \left\|P^{-1}\hat{\nu}\right\|^2 f(x_h)^2 = 1,$$

a quadratic equation for u. Moreover Sethian and Vladimirsky consider the following upwinding criterion. Let u denote the maximal solution of (4.18), and $\hat{D}u_h(x_h)$

the obtained approximate gradient. Let q^* denote the characteristic direction, such that

$$\left\langle \hat{D}u_h(x_h), -q^* \right\rangle = \rho(x_h, -q^*)$$

(which is unique, if $q \mapsto \rho(x,q)$ is strictly convex). Then q^* should point into the simplex σ_h, where the update u was computed from, that is

$$P^{-T} q^* > 0 \quad \text{(component-wise).} \tag{4.19}$$

For the Eikonal equation, the characteristic direction coincides with $-\hat{D}u_h(x_h)$, and the upwinding criterion becomes

$$(PP^T)^{-1}\hat{\nu} > 0 \quad \text{(component-wise).} \tag{4.20}$$

Then $v^{\sigma_h}(x_h)$ is assigned the value

$$v^{\sigma_h}(x_h) = \begin{cases} \text{the solution } u \text{ of (4.18)}, & \text{if (4.19) holds,} \\ \min\left\{u_h(x_i) + \rho(x_h, x_h - x_i)\,;\; i = 1, 2\right\}, & \text{otherwise} \end{cases} \tag{4.21}$$

(subsection 8.2.2 in [SV03]). Assume, that $v^{\sigma_h}(x_h) = u$, with the upwinding criterion being fulfilled. Of course, as $\rho(x,q)$ is the polar of $F(x,p)$ (compare subsection 1.4.1), the following equations are equivalent:

$$\begin{aligned} F(x_h, P^{-1}\hat{\nu}) &= 1 \\ \max_{q \neq 0} \frac{\left\langle P^{-1}\hat{\nu}, q \right\rangle}{\rho(x_h, q)} &= 1 \\ \max_{y \in \partial\omega_h(x_h)} \left\{ \left\langle P^{-1}\hat{\nu}, x_h - y \right\rangle - \rho(x_h, x_h - y) \right\} &= 0. \end{aligned}$$

Thus in this case, we obtain the same update, as from formula (4.14) (or (3.1) for the first order scheme). For the first order scheme, both methods are equivalent. However, for the second order scheme, (4.21) uses only a second order approximation of $u(x_h)$, if the upwinding criterion fails. Furthermore, update formula (4.14) necessarily yields a value $v^{\sigma_h}(x_h)$ satisfying the upwinding criterion, if $t_1^*, \ldots, t_d^* > 0$.

4.3.4. Causality and the Fast Marching Method. For the Eikonal equation $\|Du(x)\| = f(x)$, consider the following update strategy:

$$v^{\sigma_h}(x_h) = \begin{cases} \text{the 2nd order update (4.14)}, \\ \qquad \text{if } \left\langle \hat{D}u_h(x_i), x_h - x_i \right\rangle \geq 0 \quad \forall i = 1, \ldots, d, \\ \text{the 1st order update (3.1)}, \\ \qquad \text{otherwise.} \end{cases} \tag{4.22}$$

Then the causality property holds (compare subsection 3.3.1) on an acute triangulation. That is, if the minimum is attained for $t_1^*, \ldots, t_d^*$, then $t_i^* > 0$ implies $u_h(x_h) > u_h(x_i)$, provided that every pair of edges $x_h - x_j$, $x_h - x_i$ of σ_h encloses at x_h an acute angle. This follows directly from lemma 3.10 and remark 4.6.

Note that the requirement $\langle \hat{D}u_h(x_i), x_h - x_i \rangle \geq 0$ in (4.22), which ensures the causality, is no considerable restriction, as we would like to have the upwinding criterion (4.20) being fulfilled, and as the gradient $\hat{D}u_h(x_h)$ should not differ much from the gradients $\hat{D}u_h(x_i)$ in the neighboring points, for $h \to 0$, at least in the case of a smooth solution.

Based on the update formula (4.22), we can compute a second-order approximation to the solution of the Eikonal equation using the Fast Marching Method. In algorithm 3.11, we must additionally make sure, that updates are calculated *only* from triangles $[x_h, x_1, x_2]$, where the values in the neighboring points are already alive. Moreover, the approximation to the gradient (4.17) in some $x_h \in \Omega_h$ is not computed, until x_h has been accepted. When x_h becomes alive, $\hat{D}u_h(x_h)$ is computed from the triangle, from which x_h was assigned its value $u_h(x_h)$. Otherwise, if we would compute updates based on trial values in the neighboring grid-points, the obtained gradient $\hat{D}u_h(x_h)$ could differ widely from the true gradient, as numerical experiments have shown, leading to escalating errors in the numerical solution.

The method works well, provided that the Eikonal equation admits a continuously differentiable solution. Nonetheless, discontinuities in the derivative give rise to errors in the discrete solution, and the observed convergence is even slower than for the first order scheme. But there is also a remedy. The errors occur, when an update is computed in some point x_h from a triangle $[x_h, x_1, x_2]$, such that x_1 and x_2 are separated by a shock front in the solution. Consider the cubic hermite interpolant on the edge $[x_1, x_2]$ of $f(t) = u(x_1 + t(x_2 - x_1))$ with respect to the data points $(0, f(0)), (0, f'(0)), (1, f(1)), (1, f'(1))$. For the fourth order divided difference, we have, if f is three times differentiable,

$$\begin{aligned}\{0,0,1,1\}f = f'(0) + f'(1) - 2(f(1) - f(0)) &= \frac{f^{(3)}(\tau)}{3!} \\ &= \frac{1}{6} D^3 u(x_1 + \tau(x_2 - x_1))(x_2 - x_1, x_2 - x_1, x_2 - x_1) = \mathcal{O}(h^3).\end{aligned}$$

If f is only twice differentiable, we obtain by Taylor expansion, that $\{0,0,1,1\}f = f''(\tau_1) - f''(\tau_2) = \mathcal{O}(h^2)$. Now assume, that Γ is a shock front in the solution u, and $x \in \Gamma$, and U a disk-shaped neighborhood of x, which is cut in two halves U_1, U_2 by Γ, such that u is smooth on either side of Γ. If some edge $[x_1, x_2]$ intersects Γ in some point $\xi \in]x_1, x_2[$, then we observe a jump

$$\left\| \lim_{x \to \xi, x \in [x_1, \xi[} Du(x) - \lim_{x \to \xi, x \in]\xi, x_2]} Du(x) \right\| = \delta > 0$$

in the first derivative, and the divided difference will be of order $\mathcal{O}(\delta h)$ for $h \to 0$. As we want to avoid the computation of second order updates from edges crossing a shock line, we can compute the fourth order divided difference on the edge, based on the values of u_h and $\hat{D}u_h$, and then we use the second order scheme, if

$$Du_h(x_1)(x_2 - x_1) + Du_h(x_2)(x_2 - x_1) - 2(u_h(x_2) - u_h(x_1)) \leq C \cdot \|x_2 - x_1\|^2, \tag{4.23}$$

with some suitable constant $C > 0$. This leads to the following update strategy:

$$v^{\sigma_h}(x_h) = \begin{cases} \text{the 2nd order update (4.14),} \\ \quad \text{if } \left\langle \hat{D}u_h(x_i), x_h - x_i \right\rangle \geq 0 \quad \forall i = 1, \ldots, d \\ \quad \text{and if additionally (4.23) holds,} \\ \text{the 1st order update (3.1),} \\ \quad \text{otherwise.} \end{cases} \tag{4.24}$$

With this little completion of the second order approach in [SV00], I obtained valuable results, even in the case of non-smooth solutions of the Eikonal equation, where shock fronts appear.

Let me remark, that the acceleration of the Fast Marching Method by using an untidy priority queue, as described in 3.3.5, *is not applicable here*, as locally second order errors would be introduced because of the inexact minimization. It seems to be possible to transfer the approach to the second order method, using $\mathcal{O}(h^{-2})$ buckets in the untidy queue, instead of $\mathcal{O}(h^{-1})$ buckets for the first order scheme. Then we would obtain the desired accuracy, for a high price of memory, necessary to store $\mathcal{O}(N)$ buckets, where N denotes the number of grid-points. But this would also lead to a total complexity of $\mathcal{O}(N^2)$ of the method, which makes this approach completely worthless (compare the argumentation in the proof of lemma 3.16).

4.3.5. Numerical Experiments. The first example is the boundary value problem for the Eikonal equation on $\Omega =]0,1[^2$

$$\|Du(x)\| = \pi\sqrt{\cos(\pi x_1)^2 + \cos(\pi x_2)^2 - 2\cos(\pi x_1)^2\cos(\pi x_2)^2}, \quad u|_{\partial\Omega} = 0,$$

which has the smooth solution $u(x) = \sin(\pi x_1)\sin(\pi x_2)$. (As $\|Du(x)\| = 0$ at the point $x = (1/2, 1/2)$, this equation does not have a unique solution, of course $-u(x)$ is another viscosity solution. However, by the structure of the Hopf-Lax approximation, we expect convergence to the maximal solution $u(x)$.) The following table shows the L^∞ and the L^1 errors of both the first order and the second order scheme, on refined triangular meshes with diameter h.

h	L^∞ (1st order)	L^∞ (2nd order)	L^1 (1st order)	L^1 (2nd order)
2^{-4}	0.10138960	0.00321483	0.01398418	0.00129874
2^{-5}	0.04989070	0.00080332	0.00677243	0.00032531
2^{-6}	0.02474450	0.00020081	0.00333853	0.00008137
2^{-7}	0.01232205	0.00005020	0.00165839	0.00002034
2^{-8}	0.00614847	0.00001255	0.00082671	0.00000509
p	1.01	2.00	1.02	2.00

The last row contains the order p estimated by linear regression. One observes perfect second order convergence in both the L^1 and the L^∞ norm for the approximation (4.24).

The next example is the distance function from two points, computed on $\overline{\Omega}$:

$$u(x) = \min(\|x - x_0\|, \|x - x_1\|), \quad x_0 = (0,0),\ x_1 = (1,1).$$

This function is not differentiable in x_0,x_1, and along the shock line $\Gamma : \xi + \eta = 1$ ($x = (\xi, \eta)$). For this example, I used an irregular mesh, such that the shock line is not aligned with some diagonal edges in the triangulation. For both the first and the second order method, I provided exact initial data within a distance of 0.1 from the source points x_0 and x_1. The following results were obtained:

h	L^∞ (1st order)	L^∞ (2nd order)	L^1 (1st order)	L^1 (2nd order)
0.08085013	0.01386558	0.02105527	0.00662746	0.00359176
0.04226291	0.00584294	0.00557624	0.00335249	0.00033519
0.02124246	0.00261613	0.00125870	0.00158179	0.00005528
0.01104151	0.00128329	0.00095241	0.00083484	0.00001224
0.00568431	0.00064939	0.00062027	0.00043171	0.00000483
p	1.15	1.33	1.03	2.49

While the L^∞ error of the second order method is not notably better, one obtains a higher rate of convergence in the L^1 norm. Taking a closer look on the error $|u(x) - u_h(x)|$ reveals, that first order errors in the second order scheme occur near the shock line, causing the L^∞ convergence to be of first order, and the L^1 convergence to be of second order.

Bibliography

[Alt99] Hans Wilhelm Alt, *Lineare Funktionalanalysis - eine anwendungsorientierte Einführung. (Linear functional analysis - an application oriented introduction)* , Springer, 1999.

[BCD97] Martino Bardi and Italo Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, 1997.

[Ben77] Stanley H.jun. Benton, *The Hamilton-Jacobi equation. A global approach.*, Mathematics in Science and Engineering, Vol. 131, Academic Press. XI, 1977.

[BG05] Keith Burns and Marian Gidea, *Differential geometry and topology. With a view to dynamical systems.*, Chapman & Hall/CRC, 2005.

[BJ02] Guy Barles and Espen Robstad Jakobsen, *On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations.*, M2AN, Math. Model. Numer. Anal. **36** (2002), no. 1, 33–54.

[BR06] Folkmar Bornemann and Christian Rasch, *Finite-element discretization of static Hamilton-Jacobi equations based on a local variational principle.*, Comput. Visual Sci. **9** (2006), 57–69.

[BS98] Timothy J. Barth and James A. Sethian, *Numerical schemes for the Hamilton-Jacobi and level set equations on triangulated domains.*, J. Comput. Phys. **145** (1998), no. 1, 1–40.

[CEL84] Michael G. Crandall, Lawrence C. Evans, and Pierre-Louis Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations.*, Trans. Am. Math. Soc. **282** (1984), 487–502.

[CH64] E.D. Conway and E. Hopf, *Hamilton's theory and generalized solutions of the Hamilton-Jacobi equation*, J. Math. Mech. **13** (1964), 939–986.

[CIL87] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions, *Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited.*, J. Math. Soc. Japan **39** (1987), 581–596.

[CL81] Michael G. Crandall and Pierre-Louis Lions, *Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre.*, C. R. Acad. Sci., Paris **292** (1981), 183–186 (French).

[CL83] ———, *Viscosity solutions of Hamilton-Jacobi equations.*, Trans. Am. Math. Soc. **277** (1983), 1–42.

[CL84] ———, *Two approximations of solutions of Hamilton-Jacobi equations.*, Math. Comput. **43** (1984), 1–19.

[CL86] ———, *On existence and uniqueness of solutions of Hamilton-Jacobi equations.*, Nonlinear Anal., Theory Methods Appl. **10** (1986), 353–370.

[DE04] Klaus Deckelnick and Charles M. Elliott, *Uniqueness and error analysis for Hamilton-Jacobi equations with discontinuities*, Interfaces Free Bound. **6** (2004), no. 3, 329–349.

[EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions.*, Studies in Advanced Mathematics, 1992.

[Eva98] Lawrence C. Evans, *Partial differential equations.*, Graduate Studies in Mathematics, 1998.

[FF94] Marizio Falcone and Roberto Ferretti, *Discrete time high-order schemes for viscosity solutions of Hamilton- Jacobi-Bellman equations.*, Numer. Math. **67** (1994), no. 3, 315–344.

[FF98] Maurizio Falcone and Roberto Ferretti, *Convergence analysis for a class of high-order semi-Lagrangian advection schemes.*, SIAM J. Numer. Anal. **35** (1998), no. 3, 909–940.

[Fom97] Sergey Fomel, *A variational formulation of the fast eikonal solver*, Tech. Report 95, pp. 127–149, Stanford Exploration Project, 1997.

[GK06] Pierre A. Gremaud and Christopher M. Kuster, *Computational study of fast methods for the Eikonal equation.*, SIAM J. Sci. Comput. **27** (2006), no. 6, 1803–1816.

[GT92] Roberto L.V. González and Mabel M. Tidball, *Sur l'ordre de convergence des solutions discrétisées en temps et en espace de l'équation de Hamilton-Jacobi. (On the rates of* *convergence of fully discrete solutions of Hamilton-Jacobi equations).*, C. R. Acad. Sci., Paris **314** (1992), no. 6, 479–482 (French).

[Hop50] Eberhard Hopf, *The partial differential equation $u_t + uu_x = \mu u_{xx}$.*, Commun. Pure Appl. Math. **3** (1950), 201–230.

[Hop65] E. Hopf, *Generalized solutions of non-linear equations of first order*, J. Math. Mech. **14** (1965), 951–973.

[Ish87] Hitoshi Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type*, Proc. Am. Math. Soc. **100** (1987), 247–251.

[Jos95] Jürgen Jost, *Riemannian geometry and geometric analysis.*, Springer, Berlin, 1995.

[KS98] Ron Kimmel and James A. Sethian, *Computing geodesic paths on manifolds.*, Proc. Natl. Acad. Sci. USA **95** (1998), no. 15, 8431–8435.

[Lax63] P.D. Lax, *Nonlinear hyperbolic systems of conservation laws*, Nonlinear Probl., Proc. Sympos. Madison 1962, 3-12 , 1963.

[Lio82] Pierre-Louis Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics, 1982.

[MM02] Carlo Mantegazza and Andrea Carlo Mennucci, *Hamilton-Jacobi equations and distance functions on Riemannian manifolds.*, Appl. Math. Optimization **47** (2002), no. 1, 1–25.

[PBT98] L.C. Polymenakos, D.P. Bertsekas, and J.N. Tsitsiklis, *Implementation of efficient algorithms for globally optimal trajectories.*, IEEE Trans. Autom. Control **43** (1998), no. 2, 278–283.

[PR93] C. Pflaum and U. Rüde, *Gauß' adaptive relaxation for the multilevel solution of partial differential equations on sparse grids*, Tech. report, Institut für Informatik, Technische Universität München, 1993.

[PTVF02] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical recipes in C++. The art of scientific computing. 2nd ed.*, Cambridge University Press, 2002.

[Ras02] Christian Rasch, *Fast Solution of the Eikonal Equation on Triangulated Domains*, Diploma Thesis, 2002.

[Roc97] R.Tyrrell Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, 1997.

[RT92] Elisabeth Rouy and Agnés Tourin, *A viscosity solutions approach to shape-from-shading.*, SIAM J. Numer. Anal. **29** (1992), no. 3, 867–884.

[Sed98] Robert Sedgewick, *Algorithms in C. 3rd ed.*, Addison Wesley Longman, 1998.

[Set96] James A. Sethian, *A fast marching level set method for monotonically advancing fronts.*, Proc. Natl. Acad. Sci. USA **93** (1996), no. 4, 1591–1595.

[Set99a] ———, *Fast marching methods.*, SIAM Rev. **41** (1999), no. 2, 199–235.

[Set99b] ———, *Level set methods and fast marching methods. Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science.*, Cambridge Monographs on Applied and Computational Mathematics, 1999.

[Set01] ———, *Evolution, implementation, and application of level set and fast marching methods for advancing fronts.*, J. Comput. Phys. **169** (2001), no. 2, 503–555.

[SK04] Alon Spira and Ron Kimmel, *An efficient solution to the eikonal equation on parametric manifolds.*, Interfaces Free Bound. **6** (2004), no. 3, 315–327.

[SV00] James A. Sethian and Alexander Vladimirsky, *Fast methods for the eikonal and related Hamilton-Jacobi equations on unstructured meshes.*, Proc. Natl. Acad. Sci. USA **97** (2000), no. 11, 5699–5703.

[SV01] ———, *Ordered upwind methods for static Hamilton-Jacobi equations.*, Proc. Natl. Acad. Sci. USA **98** (2001), no. 20, 11069–11074.

[SV03] ———, *Ordered Upwind Methods for static Hamilton-Jacobi equations: Theory and algorithms.*, SIAM J. Numer. Anal. **41** (2003), no. 1, 325–363.

[TCOZ03] Yen-Hsi Richard Tsai, Li-Tien Cheng, Stanley Osher, and Hong-Kai Zhao, *Fast sweeping algorithms for a class of Hamilton–Jacobi equations.*, SIAM J. Numer. Anal. **41** (2003), no. 2, 673–694.

[Tsi95] John N. Tsitsiklis, *Efficient algorithms for globally optimal trajectories.*, IEEE Trans. Autom. Control **40** (1995), no. 9, 1528–1538.

[Ver96] Rüdiger Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley-Teubner Series Advances in Numerical Mathematics, 1996.

[YBS06] Liron Yatziv, Alberto Bartesaghi, and Guillermo Sapiro, *O(N) implementation of the fast marching algorithm.*, J. Comput. Phys. **212** (2006), no. 2, 393–399.

[Zha05] Hongkai Zhao, *A fast sweeping method for eikonal equations.*, Math. Comput. **74** (2005), no. 250, 603–627.