A Low Complexity Approximation of the MIMO Broadcast Channel Capacity Region

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Abstract— **Points on the boundary of the MIMO broadcast channel (BC) capacity region are achieved by a combination of dirty paper coding (DPC) and linear precoding. The linear precoding determines the covariance matrices of the transmitted signals. Determining the optimum covariance matrices may lead to an undesirably high computational complexity for systems of high dimension. In this paper, an approach to approximate the MIMO BC capacity region is proposed. The proposed method combines DPC with sub-optimum linear precoding matrices that can be computed with low complexity but provide close to optimum performance. Motivated by multiobjective optimization, an efficient algorithm developed for sum-rate maximization is generalized to computing an achievable rate region. Simulation results show that this rate region, which has low complexity in terms of computing the precoding matrices, well approximates the capacity region of the MIMO BC.**

I. Introduction

The term broadcast channel denotes a setting in which a unique transmitter sends information to a number of non-cooperating receivers. In particular, we focus on broadcast channels in which the transmitter sends independent information to every user, as it occurs in the downlink of a wireless cellular system, and consider the general case of having multiple antennas or inputs at the transmitter and multiple antennas or outputs at each receiver. Assuming static channel matrices and Gaussian distributed noise vectors, this model generally corresponds to a non-degraded Gaussian BC for which the capacity region has remained unknown until very recently [1]. The boundary of this region can be reached by successively encoding information for the different users and neutralizing at each step known interference caused by already encoded users using an appropriate codebook [2].

Directly finding transmit covariance matrices that achieve the boundary of the capacity region in the BC constitutes a non-convex optimization problem difficult to solve. However, duality results with the multiple access channel (MAC) allow to convert this problem into a convex optimization problem, whose solution can be transformed back into the solution of the original problem [3]. To solve the convex optimization problem in the MAC a gradient based algorithm is presented in [4]. For the particular case of computing transmit covariance matrices that achieve the maximum sum capacity point, more efficient algorithms have been presented in [5] and [6].

The algorithms in [4], [5], and [6] are iterative, and, although they converge to the optimum solution, little can be said about their convergence rates. Indeed, the algorithm presented in [4] shows very slow convergence for broadcast channels with large numbers of inputs and outputs such as those resulting from employment of multicarrier transmission schemes [7].

The complexity involved in computing the optimum covariance matrices of the transmitted signals motivates the development of sub-optimum approaches that significantly reduce the complexity required for computing the covariance matrices while providing a close approximation of the capacity region. Such a low complexity achievable rate region is of particular interest for computing an achievable rate region for systems of high dimension, for which the convergence of the optimum algorithms may be too slow.

The paper focuses on the complexity involved in computing the transmit covariance matrices. The proposed rate regions still rely on dirty paper coding. The complexity induced by DPC is not taken into account, due to the fact that for computing rate regions, there is no need to actually carry out the coding. For practical implementation, the proposed method for computing transmit covariance matrices is of interest in conjunction with real-world "approximations" of DPC, such as Tomlinson-Harashima precoding [8], [9].

II. System Model

A MIMO broadcast channel with K receivers is considered. The transmitter has N transmit antennas, while receiver k is equipped with M_k receive antennas. The transmitter sends independent information to each of the receivers.

The received signal at receiver k is given by

$$\boldsymbol{y}_k = \boldsymbol{H}_k \sum_{i=1}^{K} \boldsymbol{x}_i + \boldsymbol{\eta}_k,$$

where $\boldsymbol{H}_k \in \mathbb{C}^{M_k \times N}$ is the channel to receiver k and $\boldsymbol{x}_k \in \mathbb{C}^N$ is the signal transmitted to receiver k. Furthermore, $\boldsymbol{\eta}_k$ is the circularly symmetric complex Gaussian noise at receiver k, with $\boldsymbol{\eta}_k \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{1}_{M_k})$.

The transmit covariance matrix of user k is given by $\boldsymbol{Q}_k = \mathrm{E}\left[\boldsymbol{x}_k \boldsymbol{x}_k^{\mathrm{H}}\right]$. Due to the fact that independent information is transmitted to the receivers, the total transmitted power is given by $\mathrm{tr}\left(\sum_{k=1}^{K} \boldsymbol{Q}_k\right)$. The total transmit power has to satisfy the power constraint $\mathrm{tr}\left(\sum_{k=1}^{K} \boldsymbol{Q}_k\right) \leq P_{\mathrm{tr}}$.

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A linear precoding structure

$$\boldsymbol{x}_k = \boldsymbol{V}_k \boldsymbol{P}_k^{\frac{1}{2}} \boldsymbol{s}_k$$

is used to form the transmitted signal $\boldsymbol{x}_k$ from the data signal $\boldsymbol{s}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{1}_{Q_k})$, with a unitary matrix $\boldsymbol{V}_k \in \mathbb{C}^{N \times Q_k}$ and a diagonal as well as positive definite power allocation matrix $\boldsymbol{P}_k$. Note that the number Q_k of transmitted data streams to receiver k corresponds to the rank of the covariance matrix $\boldsymbol{Q}_k$, which has to be chosen appropriately for every point in the achievable rate region.

III. Zero Forcing Successive Encoding

In [10], an algorithm is presented that decomposes the MIMO broadcast channel into a set of decoupled scalar subchannels. The decoupling is achieved by a combination of successive encoding of data streams and linear zero-forcing of the remaining interference. This algorithm, denoted as *cooperative Zero-Forcing with Successive encoding and Successive allocation* (ZFSS), represents a central building block of the results to follow. This section provides a short summary of ZFSS.

The algorithm assigns spatial dimensions to users in a successive manner. A spatial dimension is characterized by a unit norm transmit vector and a unit norm receive vector. Let $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ denote the transmit and receive vector assigned at the ith iteration, respectively. The following steps are taken to determine $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$: First, given the transmit vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}$ assigned at the previous iterations, the channel matrices are projected as follows:

$$\boldsymbol{H}_k^i = \boldsymbol{H}_k \left(\mathbf{1}_N - \sum_{\ell=1}^{i-1} \boldsymbol{v}_\ell \boldsymbol{v}_\ell^{\mathrm{H}} \right), \quad \forall k. \tag{1}$$

Next, a singular value decomposition of each projected matrix is performed:

$$\boldsymbol{H}_k^i = \boldsymbol{U}_k^i \boldsymbol{\Sigma}_k^i \boldsymbol{V}_k^{i,\mathrm{H}}, \quad \forall k.$$

Finally, $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ are chosen from one of the matrices $\boldsymbol{V}_k^i$ and $\boldsymbol{U}_k^i$ according to some spatial allocation rule $r(\bullet)$:

$$\boldsymbol{v}_i = \boldsymbol{V}_{k_i}^i \boldsymbol{e}_{c_i}, \quad \boldsymbol{u}_i = \boldsymbol{U}_{k_i}^i \boldsymbol{e}_{c_i},$$

where

$$(k_i, c_i) = r(\boldsymbol{H}_1^i, \dots, \boldsymbol{H}_K^i, \boldsymbol{\mu}).$$

The allocation rule $r(\bullet)$ is discussed in more detail in Section V.

By construction of $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$, transmission over $\boldsymbol{v}_i$ causes no interference to streams transmitted over $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}$. Interference caused to streams $i+1, \dots, Q$ is canceled by successive encoding. As a result, a set of Q decoupled subchannels results, each with gain

$$g_i^2 = |\boldsymbol{u}_i^{\mathrm{H}} \boldsymbol{H}_{k_i} \boldsymbol{v}_i|^2.$$

Define an allocation $\pi : \{1, \dots, Q\} \to \{1, \dots, K\}$ such that

$$\pi(i) = k_i.$$

Given an allocation π and a power allocation

$$\boldsymbol{p} = [p_1, \dots, p_Q]^{\mathrm{T}},$$

an achievable rate vector is given by

$$\boldsymbol{R}(\pi, \boldsymbol{p}) = [R_1(\pi, \boldsymbol{p}), \dots, R_K(\pi, \boldsymbol{p})]^{\mathrm{T}},$$

with

$$R_k(\pi, \boldsymbol{p}) = \sum_{i:\pi(i)=k} \log_2(1 + g_i^2 p_i).$$

IV. Achievable Rate Region

Let $\mathcal{S}$ denote the set of all possible subchannel allocations to users. Moreover, let $\mathcal{P}$ denote the set of all feasible power allocations $\boldsymbol{p}$:

$$\mathcal{P} = \{\boldsymbol{p} \in \mathbb{R}_+^Q : \|\boldsymbol{p}\|_1 \le P_{\mathrm{tr}}\}.$$

Consider the rate region of ZFSS achievable by varying over all possible subchannel and power allocations:

$$\mathcal{R}_{\text{ZFSS}} = \bigcup_{\substack{\pi \in \mathcal{S} \\ \boldsymbol{p} \in \mathcal{P}}} \boldsymbol{R}(\pi, \boldsymbol{p}). \tag{2}$$

With the usual time-sharing argument, we define the achievable rate region of ZFSS as

$$\bar{\mathcal{R}}_{\text{ZFSS}} = \mathrm{Co}\left(\mathcal{R}_{\text{ZFSS}}\right), \tag{3}$$

where $\mathrm{Co}\left(\mathcal{R}\right)$ denotes the convex hull of the set $\mathcal{R}$. While (2) and (3) allow us to formally define the achievable rate region, varying over all feasible subchannel and power allocations is practically infeasible. This motivates to look for ways to characterize $\bar{\mathcal{R}}_{\text{ZFSS}}$ in a more efficient manner.

Consider the boundary of $\mathcal{R}_{\text{ZFSS}}$ where the rate of one user can only be further increased by decreasing the rate of at least one other user:

$$\mathcal{E} = \{\boldsymbol{R} \in \mathcal{R}_{\text{ZFSS}} : (\nexists \boldsymbol{R}' \in \mathcal{R}_{\text{ZFSS}} : \boldsymbol{R}' \succ \boldsymbol{R})\}, \tag{4}$$

where the strict partial order $\succ$ is defined as

$$\boldsymbol{R}' \succ \boldsymbol{R} \;\Leftrightarrow\; R_k' \ge R_k, \forall k \,\wedge\, \exists k : R_k' > R_k.$$

Notably, the set $\mathcal{E}$ contains all relevant information about $\bar{\mathcal{R}}_{\text{ZFSS}}$, in the sense that

$$\bar{\mathcal{R}}_{\text{ZFSS}} = \mathrm{Co}\left(\{\mathcal{E}, \mathbf{0}_K\}\right).$$

According to (4), the set $\mathcal{E}$ is the so-called efficient set of the *multiobjective optimization* problem

$$\max_{\substack{\pi \in \mathcal{S} \\ \boldsymbol{p} \in \mathcal{P}}} \boldsymbol{R}(\pi, \boldsymbol{p}). \tag{5}$$

In the following, a low-complexity approximation of the achievable rate region $\bar{\mathcal{R}}_{\text{ZFSS}}$ is derived which is based on an approximation of the efficient set $\mathcal{E}$ by a finite number of rate vectors. Finally, simulation results are used to demonstrate that the approximation of $\bar{\mathcal{R}}_{\text{ZFSS}}$ also well-approximates the capacity region.

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V. Low Complexity Rate Region

As stated in Section IV, the efficient set $\mathcal{E}$ captures all relevant characteristics of the achievable rate region and corresponds to the solution of the multiobjective optimization problem (5). A well-known technique to solve multiobjective optimization methods is the weighting method [11], which corresponds to scalarizing the optimization problem by multiplying the objective function with a weight vector $\boldsymbol{\mu}$. Applying the weighting method, problem (5) is transformed into

$$\max_{\substack{\pi\in\mathcal{S}\\ \boldsymbol{p}\in\mathcal{P}}} \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{R}(\pi,\boldsymbol{p}), \tag{6}$$

with $\boldsymbol{\mu}\in\mathbb{R}^K, \|\boldsymbol{\mu}\|=1$. For each choice of the weight vector $\boldsymbol{\mu}$, we obtain maximizers $\pi_{\boldsymbol{\mu}}$ and $\boldsymbol{p}_{\boldsymbol{\mu}}$ such that

$$\boldsymbol{R}(\pi_{\boldsymbol{\mu}},\boldsymbol{p}_{\boldsymbol{\mu}})\in\mathcal{E}. \tag{7}$$

Problem (6), however, can only be solved by an exhaustive search over all possible subchannel allocations $\pi\in\mathcal{S}$. Therefore, we propose to solve a simplified optimization problem. We first derive a sequential algorithm for determining a subchannel allocation. In a second step, the optimum power allocation, given a subchannel allocation, is found.

A. Subchannel Allocation

The weighted problem (6) corresponds to a weighted sum-rate maximization. In particular, for $\mu_k = 1/K, \forall k$, (6) corresponds to maximizing the sum-rate. Sum-rate maximization for ZFSS is addressed in [12]. In order to avoid the exhaustive search over all possible subchannel allocations that is required for solving

$$\max_{\substack{\pi\in\mathcal{S}\\ \boldsymbol{p}\in\mathcal{P}}} \sum_{k=1}^{K} R_k(\pi,\boldsymbol{p}), \tag{8}$$

in [12] a sequential algorithm for allocating subchannels to users is proposed. At step i of the subchannel allocation procedure, the following heuristic allocation rule is applied:

$$(k_i,c_i) = \underset{k,c}{\operatorname{argmax}}\, \sigma^i_{k,c}, \tag{9}$$

where $\sigma^i_{k,c}$ is the cth singular value of the projected channel matrix $\boldsymbol{H}^i_k$ defined in (1).

Aiming at a maximization of the weighted sum-rate, we modify the allocation rule (9), which aims at sum-rate maximization, as follows:

$$(k_i,c_i) = \underset{k,c}{\operatorname{argmax}}\, \mu_k\sigma^i_{k,c}. \tag{10}$$

Note that (10) represents a heuristic rule that does not necessarily guarantee an optimum subchannel allocation. However, simulation results show that (10) provides a good approximation at low complexity.

B. Generalized Waterfilling

Given a weight vector $\boldsymbol{\mu}$ and a subchannel allocation $\pi_{\boldsymbol{\mu}}$ obtained according to Subsection V-A, the optimum power allocation $\boldsymbol{p}_{\boldsymbol{\mu}}$ is found by solving

$$\boldsymbol{p}_{\boldsymbol{\mu}} = \underset{\boldsymbol{p}\in\mathcal{P}}{\operatorname{argmax}}\, \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{R}(\pi_{\boldsymbol{\mu}},\boldsymbol{p}). \tag{11}$$

For notational convenience, in the following we write π instead of $\pi_{\boldsymbol{\mu}}$. Problem (11) can be rewritten as

$$\begin{aligned}\boldsymbol{p}_{\boldsymbol{\mu}} = \underset{\boldsymbol{p}}{\operatorname{argmax}} &\sum_{i=1}^{Q}\mu_{\pi(i)}\log_2(1+g_i^2p_i)\\ \text{s.t.} \quad &\sum_{i=1}^{Q} p_i \le P_{\text{tr}}, \quad p_i\ge 0, 1\le i\le Q.\end{aligned}$$

Evaluating the KKT conditions yields

$$p_i = \left(\eta\mu_{\pi(i)} - g_i^{-2}\right)^+,$$

where η is chosen such that the power constraint is satisfied with equality. The structure of the classical waterfilling solution can be obtained by substituting $p_i = \mu_{\pi(i)}\tilde{p}_i$. With $\lambda_i = \mu_{\pi(i)}g_i^2$, this yields

$$\tilde{p}_i = \left(\eta - \lambda_i^{-1}\right)^+,$$

where η is now chosen such that

$$\sum_{i=1}^{Q}\mu_{\pi(i)}\tilde{p}_i = P_{\text{tr}}.$$

C. Approximate Rate Region

For each weight vector $\boldsymbol{\mu}$, we obtain a rate vector

$$\boldsymbol{R}(\boldsymbol{\mu}) \equiv \boldsymbol{R}(\pi_{\boldsymbol{\mu}},\boldsymbol{p}_{\boldsymbol{\mu}}).$$

Using S different weight vectors $\boldsymbol{\mu}^1,\ldots,\boldsymbol{\mu}^S$ yields an approximation of $\mathcal{E}$ by S rate vectors:

$$\hat{\mathcal{E}} = \left\{\boldsymbol{R}(\boldsymbol{\mu}^1),\ldots,\boldsymbol{R}(\boldsymbol{\mu}^S)\right\}.$$

The set $\hat{\mathcal{E}}$ is an approximation of $\mathcal{E}$ for two reasons: First, we only use a finite number of rate vectors to represent the efficient set. Second, due to the fact that the subchannel allocation obtained by the sequential allocation is not necessarily optimum, it is not guaranteed that (7) holds.

A low complexity approximation of the achievable rate region $\mathcal{R}_{\text{ZFSS}}$ is given by

$$\hat{\mathcal{R}}_{\text{ZFSS}} = \text{Co}\left(\{\hat{\mathcal{E}},\boldsymbol{0}_K\}\right).$$

The main complexity of the algorithm lies in the computation of the SVDs of the matrices $\boldsymbol{H}^i_k$. At each of the Q steps, at most K SVDs have to be computed, resulting in QK SVDs in total. Importantly, the algorithm finishes after Q steps, and, except for computing the SVDs, no numerical iterations are required. This clearly distinguishes our approach from the capacity-optimum algorithms. In [4], a gradient-search based algorithm is proposed for obtaining arbitrary points on the boundary of the capacity region. While optimum, an a priori

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unknown number of numerical iterations is required for each point. For systems of large dimension (e.g., MIMO OFDM), we have observed extremely slow convergence [7], resulting in a large number of iterations and a high computational complexity. For the computation of the point in the capacity region that maximizes the sum-rate, more efficient algorithms than [4] have been proposed [5],[6]. These algorithms, however, also require an a priori unknown number of numerical iterations. Moreover, they are specifically tailored to sum-rate maximization and cannot be generalized to computing the entire capacity region [6].

The quality of the approximation $\hat{\mathcal{R}}_{\text{ZFSS}}$ strongly depends on the choice of the weight vectors $\boldsymbol{\mu}^1, \ldots \boldsymbol{\mu}^S$. Let $\mathcal{U}$ denote the set of feasible weight vectors:

$$\mathcal{U} = \left\{ \boldsymbol{\mu} \in \mathbb{R}^K_{0,+} : \|\boldsymbol{\mu}\|_1 = 1 \right\}.$$

Distributing the S weight vectors uniformly on $\mathcal{U}$ generally does not result in a uniform distribution of the corresponding rate vectors, a behavior that is well known for the weighting method [13]. Instead, a uniform distribution of the weight vectors usually yields a few regions in which rate vectors are densely clustered. Based on this observation, it is desirable to find a method for choosing the weight vectors $\boldsymbol{\mu}$ such that a better distribution of the rate vectors $\boldsymbol{R}(\boldsymbol{\mu})$ results.

D. Weight Adaption

In order to obtain a good distribution of the sample rate vectors $\boldsymbol{R}(\boldsymbol{\mu}^1), \ldots, \boldsymbol{R}(\boldsymbol{\mu}^S)$, we propose to adapt the generation of the weight vectors $\boldsymbol{\mu}^1, \ldots, \boldsymbol{\mu}^S$ to changes in the subchannel allocation. We first limit our considerations to the case of $K = 2$ users. The first weight vector is chosen as

$$\boldsymbol{\mu}^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}.$$

With this weight vector, the algorithm defined in Subsection V-A is carried out, resulting in a set of projected matrices $\boldsymbol{H}_k^i$ and the corresponding singular value decompositions. Denote the singular values corresponding to the weight vector $\boldsymbol{\mu}^w$ as $\sigma_{k,c}^{i,w}$. Define

$$\sigma_k^{i,w} = \max_c \sigma_{k,c}^{i,w}.$$

According to (10), with $\mu_1^1 = 1$, the first r_1 subchannels are allocated to user 1, as

$$\mu_1^1 \sigma_1^{i,1} \geq (1 - \mu_1^1) \sigma_2^{i,1}, \forall i. \tag{12}$$

In the next step, μ_1^1 is decreased to obtain a second weight vector $\boldsymbol{\mu}^2$. Let

$$\bar{\mu}_1^1 = \max_i \frac{\sigma_2^{i,1}}{\sigma_1^{i,1} + \sigma_2^{i,1}}.$$

While $\bar{\mu}_1^1$ yields the same subchannel allocation as μ_1^1, any $\mu_1 < \bar{\mu}_1^1$ yields a different allocation. In general, let $\bar{\mu}_1^w$ be the smallest μ_1 that yields the same subchannel allocation as μ_1^w. A change in the subchannel allocation indicates a significant change in the corresponding rate vector. Therefore, given weight vector $\boldsymbol{\mu}^w$, the weight vector $\boldsymbol{\mu}^{w+1}$ is chosen as

$$\boldsymbol{\mu}^{w+1} = \begin{bmatrix} \bar{\mu}_1^w & 1 - \bar{\mu}_1^w \end{bmatrix}^{\mathrm{T}}.$$

We proceed until the first M_2 subchannels are allocated to user 2. The final weight vector is set to

$$\boldsymbol{\mu}^W = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

By this procedure, we obtain W weight vectors. If $W < S$, the remaining $S - W$ weight vectors are obtained by a uniform distribution of μ_1 on each of the $W - 1$ intervals $[\bar{\mu}_1^w, \bar{\mu}_1^{w+1}]$.

The weight vector adaption based on changes in the subchannel allocation has the following advantageous properties: First, a dense clustering of the generated rate vectors is avoided. Moreover, we are able to capture every change in the subchannel allocation, which helps to avoid undesirably large "jumps" between subsequent rate vectors.

The idea of adapting the weight vector to changes in the subchannel allocation can be extended to more than two users. The basic algorithm corresponds to searching for changes in the subchannel allocation while varying $\boldsymbol{\mu}$ over the set $\mathcal{U}$. For two users, this search is relatively simple, as $\mathcal{U}$ is a line. For more than two users, the increased dimensionality of $\mathcal{U}$ renders this search a more difficult task. A possible approach is to choose the K boundary points of $\mathcal{U}$ where $\mu_k = 1$ as starting points and then to search over the vertices connecting these points. A detailed investigation of this higher-dimensional search is left as an open problem.

The proposed method provides a low-complexity approximation of the capacity region. Clearly, the complexity-performance trade-off achieved by the proposed method is the most significant figure of merit. So far, we have not been able to provide an analytical assessment of the performance of the proposed method. Instead, we have to resort to a simulative assessment, which is provided in the next section.

VI. Simulation Results

In Figs. 1 and 2 the ZFSS region $\bar{\mathcal{R}}_{\text{ZFSS}}$, the suboptimum ZFSS region $\hat{\mathcal{R}}_{\text{ZFSS}}$ and the actual capacity region are compared for a scenario with $N = 4$ transmit antennas, $K = 2$ receivers and $M_k = 2$ receive antennas at each of the two receivers. For Fig. 1 the entries of the channel matrices are

$$\boldsymbol{H}_1 = \begin{bmatrix} 0.0112 & 0.8057 & -0.9898 & 0.2895 \\ -0.6451 & 0.2316 & 1.3396 & 1.4789 \end{bmatrix},$$
$$\boldsymbol{H}_2 = \begin{bmatrix} 1.1380 & -1.2919 & -0.3306 & 0.4978 \\ -0.6841 & -0.0729 & -0.8436 & 1.4885 \end{bmatrix}.$$

For Fig. 2 channel coefficients are obtained by multiplying matrix $\boldsymbol{H}_1$ by 2 and matrix $\boldsymbol{H}_2$ by 0.5. This corresponds to an unbalanced situation that might arise when users are at different distances from the transmitter. The transmit power is chosen such that $10 \log_{10} \frac{P_{\text{tr}}}{\sigma^2} = 10$ dB.

To plot the boundary of the capacity region the algorithm presented in [4] is used. Points on the boundary are computed by incrementing the weight μ_1 from 0 to 1 in increments of

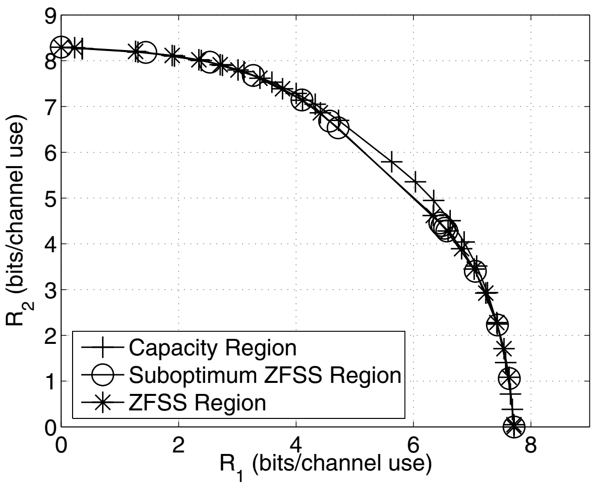

Fig. 1. Capacity and ZFSS regions for a balanced BC.

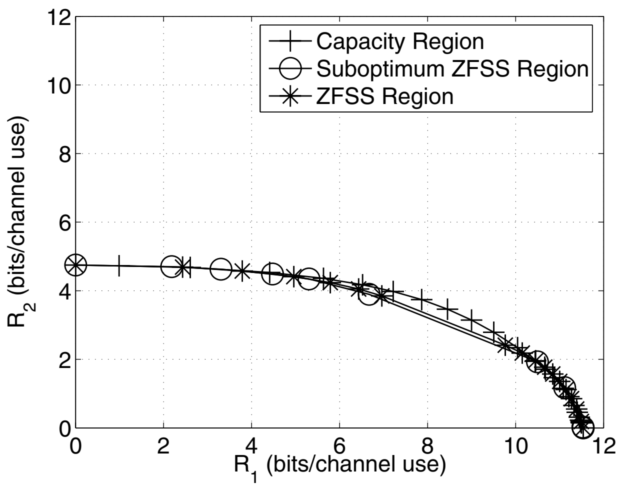

Fig. 2. Capacity and ZFSS regions for an unbalanced BC.

0.05 and then carrying out the algorithm from [4] for each μ_1, yielding a total of 22 points. Note that for $\mu_1 = \mu_2 = 1/2$ two different points result corresponding to two different precoding orders. The boundary of the ZFSS is computed by building the convex hull over the union of regions that are obtained by considering all possible allocations. The boundary of the suboptimum ZFSS region is plotted using a total of $S = 21$ samples determined as described in Section V-D.

In both examples, it can be observed that the difference between $\bar{\mathcal{R}}_{\rm ZFSS}$ and $\hat{\mathcal{R}}_{\rm ZFSS}$ regions is very slight and the loss of any of these with respect to the capacity region is almost negligible.

A. Average Results

In order to avoid being misled by particular examples, average results over a number of different channel realizations are required. Averaging of rate regions can be done by considering straight lines going through the origin and defined by particular

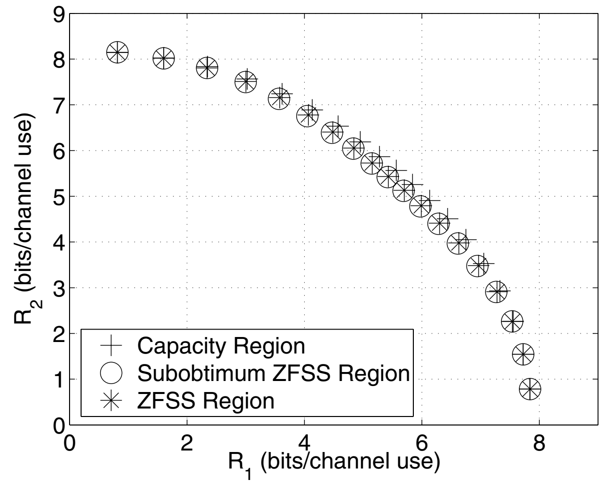

Fig. 3. Average capacity and ZFSS regions for uncorrelated balanced BC.

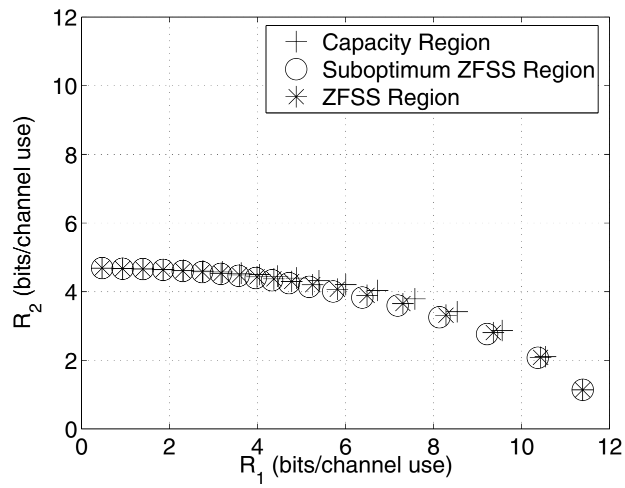

Fig. 4. Average capacity and ZFSS regions for uncorrelated unbalanced BC.

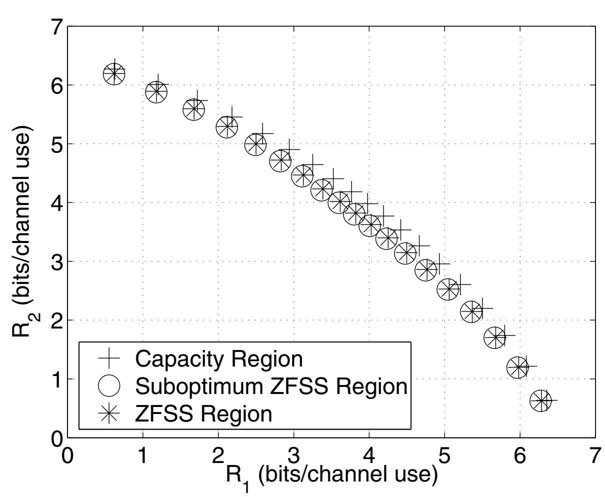

Fig. 5. Average capacity and ZFSS regions for correlated balanced BC.

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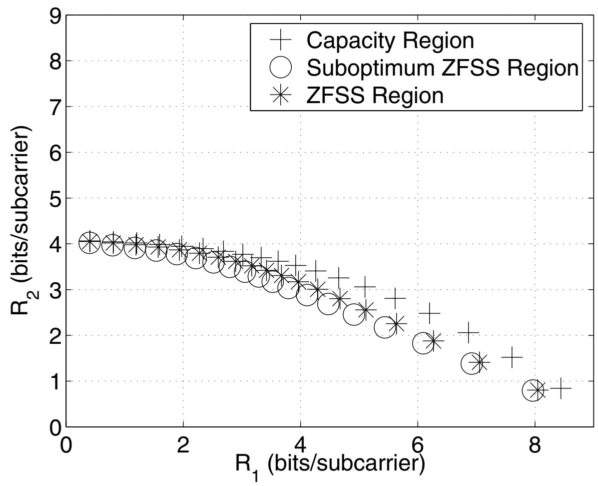

Fig. 6. Average capacity and ZFSS regions for correlated unbalanced BC.

ratios between rates of different users. The intersection points of these lines with the boundary of any region are computed and averaged over different channel realizations.

Four scenarios have been simulated with the settings listed above, i.e. $N = 4$, $K = 2$, $M_k = 2$ and a transmit SNR of 10 dB. Fig. 3 corresponds to a balanced situation in which channel coefficients are independently drawn according to a zero-mean circularly symmetric complex Gaussian distribution with unit variance. Fig. 4 corresponds to an unbalanced situation in which the variance of the distributions is 4 for user 1 and 1/4 for user 2. Coefficients are also drawn independent of each other. Figs. 5 and 6 respectively represent the balanced and unbalanced situations mentioned above but now some correlation is introduced between transmit elements. For the balanced situation the eigenvalue profile of the transmit correlation matrix, defined as $\boldsymbol{R}_{\mathrm{Tx}} = \mathrm{E}\{\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{H}_k\}$, has been chosen to be

$$\boldsymbol{\lambda} = 2 \cdot [\ 3.65 \quad 0.34 \quad 0.01 \quad 0\ ].$$

The same profile has been chosen for the unbalanced situation scaled by 4 for user 1 and by 1/4 for user 2.

For averaging purposes straight lines have been considered that go through the origin and have slopes $R_1/R_2 = 0.1 \times n$, and $R_2/R_1 = 0.1 \times n$, where $1 \leq n \leq 10$, $n \in \mathbb{N}$. The intersection points of these lines with the boundaries of the regions under study were averaged over 100 randomly drawn channel realizations.

It can be observed that for uncorrelated channels (Figs. 3, 4) the suboptimum ZFSS region $\hat{\mathcal{R}}_{\mathrm{ZFSS}}$ is practically as large as the actual ZFSS region $\bar{\mathcal{R}}_{\mathrm{ZFSS}}$. Also the difference between any of these regions and the capacity region is very small. The same occurs for correlated channels in the balanced situation (Fig. 5). The largest difference is found in Fig. 6 and corresponds to the case of having correlated channels in an unbalanced situation. Due to the strong correlation and the different channel powers, this scenario almost corresponds to a degraded BC. In this kind of channels our zero-forcing approach becomes equivalent to time sharing between the users, i.e. the boundary of the ZFSS is the straight line passing through the single user capacity points on the axes. The optimum approach however admits interuser interference and the capacity region shows a tangential slope of 1 at the sum capacity point of the region, which is equal to the capacity of the user with the strongest channel.

VII. Conclusions

A method for computing a low-complexity approximation of capacity region of the MIMO broadcast channel was presented. In contrast to optimum algorithms, an achievable rate region is computed in a non-iterative fashion. Special emphasis is put on obtaining a good approximation with only a small number of computed points. The quality of the approximation is validated by means of simulations, which show a small loss with respect to the capacity region for the settings under consideration. An analytical investigation of the performance of the proposed method is subject of future research.

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