On Performance Limits of Optimum Reduced Rank Channel Estimation

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***Abstract*-** **In a wireless communication link with one transmit and multiple receive antennas we optimally reduce the spatio-temporal channel rank at the receiver based on long-term (average) channel properties. The channel is estimated and symbols are detected using this reduced rank approximation of the received signal. Examples for this receiver architecture are the temporal/spatio-temporal Rake, Space-Time Eigenrake, or beamspace processing. We discuss the fundamental trade-offs in reduced rank channel estimation and detection and propose an appropriate model for describing the spatio-temporal channel structure. Based on these insights and definitions, we derive an analytical expression for the minimum receiver SNR, which is necessary for channel estimation. Furthermore, we state under which conditions reduced rank channel estimation is asymptotically optimum.**

I. Introduction

It is known that space-time processing improves communication quality and overall system performance [1]. Due to additional degrees of freedom in space-time processing more channel parameters have to be estimated, which increases the variance of the channel estimates. Thus, channel estimation is an important issue in wireless communications, where only a few pilot symbols are available due to the changing channel, and can become a limiting factor for transmission quality. For this reason reduced rank processing techniques were proposed [2], which reduce the number of relevant channel parameters. Reducing the rank leads to a smaller variance of the channel estimates.

There are two approaches to rank reduction: The first reduces the channel based on the instantaneous channel coefficients or their estimates, respectively [2]. In the sequel we consider the second approach, which exploits long-term channel properties, i.e. the fact that channel delays, angles of arrival and average power change slowly compared to the complex fading amplitudes of the spatial or temporal taps. This average channel knowledge is given in the spatio-temporal correlation matrices [3]. Representatives are the temporal Rake [4], where the fingers are selected based on the average power of the temporal tap, the 2D Rake filter [5], and the Space-Time Eigenrake [3].

These receivers have to determine the optimum spatio-temporal rank of the channel, as their performance critically depends on this parameter [6]. An attempt of defining the spatial rank for rank reduction based on instantaneous channel properties was made in [7], using results given by Scharf [8]. This approach does not explain the dependence of the optimum rank on the number of pilot symbols available for channel estimation. A definition of the channel rank is necessary to describe performance trade-offs in reduced rank receiver processing.

After explaining the system model in Sec. II, we state the optimum rank reduction scheme based on long-term channel properties (Sec. III). Employing the MSE of the Maximum-Likelihood reduced rank channel estimate (Sec. IV), we give a novel definition of the effective spatio-temporal rank relevant for a communication link with pilot-assisted channel estimation (Sec. V). Based on this definition and a new model for the spatio-temporal channel structure (Sec. VI), we discuss the performance limits of rank reduction in different regions of operation (Sec. VII). The proofs are given in the appendix.

II. System Model

Consider a direct-sequence spread-spectrum communication link with one transmit and N_r receive antenna elements. The received signal is sampled at the chip rate. A discrete-time baseband channel model (1) with a tapped delay line of L taps and white complex Gaussian noise $n_r[t]$ is employed and models all intra- and intercell-interference. $c[t]$ is the transmitted chip sequence.

$$r[t] = \sum_{\ell=1}^{L} h[\ell] c[t-\tau_\ell] + n_r[t] \quad \in \mathbb{C}^{N_r \times 1} \tag{1}$$

Equivalent Channel Model: For our derivations we assume that a spreading sequence with perfect autocorrelation properties is used, such that the receiver can separate all temporal paths perfectly. Thus, the channel can be defined by an equivalent flat channel model with $M = L\,N_r$ signals:

$$x[t] = hs[t] + n[t] \quad \in \mathbb{C}^{M\times 1}. \tag{2}$$

$s[t]$ contains the time-multiplexed pilot and data QAM-symbol sequences[1] with average power $P_s = \mathrm{E}[|s[t]|^2]$, which is temporally uncorrelated for simplicity. $x[t]$ contains the received and despread spatial and temporal signal components. $n[t]$ is the additive stationary zero-mean complex Gaussian random process with correlation matrix $R_n[k] = \mathrm{E}[n[t+k]\,n[t]^{\mathrm{H}}] = \delta[k]\,\sigma_n^2\,\mathbf{1}$.[2] The spatio-temporal channel vector $h = [h[1]^{\mathrm{T}}, h[2]^{\mathrm{T}}, \cdots, h[L]^{\mathrm{T}}]^{\mathrm{T}}$ is modeled as a random vector independent of $n[t]$ with correlation matrix

$$R_h = \mathrm{E}[hh^{\mathrm{H}}] = U\Lambda U^{\mathrm{H}}. \tag{3}$$

[1] In the sequel t is the integer time index w.r.t. a symbol period.
[2] $\delta[k]$ is the Kronecker function, $\mathbf{1}$ the unity matrix.

Eqn. (3) gives the eigenvalue decomposition of $\boldsymbol{R}_h$ with a unitary matrix $\boldsymbol{U}$ containing the eigenvectors and a diagonal matrix $\boldsymbol{\Lambda} = \mathrm{diag}\{[\lambda_1, \lambda_2, \cdots, \lambda_M]\}$ of the eigenvalues λ_i with $\lambda_i \geq \lambda_{i+1}$. Refer to [3] for a discussion about the estimation of $\boldsymbol{R}_h$.

Now, we reduce the spatio-temporal rank of the channel and received signal, respectively, to R dimensions with the matrix $\boldsymbol{W} \in \mathbb{C}^{M\times R}$ and obtain the reduced rank version of the signal

$$\boldsymbol{y}[t] = \boldsymbol{W}^{\mathrm{H}}\boldsymbol{x}[t] \quad \in \mathbb{C}^{R\times 1} \tag{4}$$

and the channel vector

$$\boldsymbol{h}_{\mathrm{red}} = \boldsymbol{W}^{\mathrm{H}}\boldsymbol{h} \quad \in \mathbb{C}^{R\times 1}. \tag{5}$$

For processing and estimation we consider a block of N symbols $\boldsymbol{s} \in \mathbb{C}^{N\times 1}$ and write the system model as

$$\boldsymbol{X} = \boldsymbol{h}\boldsymbol{s}^{\mathrm{T}} + \boldsymbol{N} \quad \in \mathbb{C}^{M\times N} \tag{6}$$

$$\boldsymbol{Y} = \boldsymbol{W}^{\mathrm{H}}\boldsymbol{X} \quad \in \mathbb{C}^{R\times N} \tag{7}$$

with $\boldsymbol{N} = [\boldsymbol{n}[1], \boldsymbol{n}[2], \cdots, \boldsymbol{n}[N]]$ and $\boldsymbol{X}$, $\boldsymbol{Y}$ defined accordingly (Figure 1).

The optimum coherent combiner under the assumptions from above is a maximum ratio combiner, which uses the estimates of the instantaneous channel $\hat{\boldsymbol{h}}_{\mathrm{red}}$ (short-term processing, Figure 1)

$$\hat{\boldsymbol{s}}^{\mathrm{T}} = \hat{\boldsymbol{h}}_{\mathrm{red}}^{\mathrm{H}}\boldsymbol{Y}. \tag{8}$$

ML Channel Estimator: If the sequence of N transmitted symbols is a pilot sequence known to the receiver, the Maximum-Likelihood channel estimator [9] for the reduced channel coefficients is given by

$$\hat{\boldsymbol{h}}_{\mathrm{red}} = \frac{1}{N P_s}\boldsymbol{Y}\boldsymbol{s}^*. \tag{9}$$

This estimator is minimum variance and unbiased w.r.t. $\boldsymbol{h}_{\mathrm{red}}$, but biased w.r.t. estimation of $\boldsymbol{h}$.

Note, that perfect delay estimates were assumed above, when despreading the temporal paths, as they change slowly and, thus, can be estimated reliably averaging over a long period. Furthermore, the channel order L is assumed to be known.

III. Optimum Rank Reduction Based on Long-Term Channel Properties

For rank reduction we rely on the observation that the temporal and spatial channel properties, i.e. the delays and the directions of arrival, change slowly compared to the (fast fading) complex amplitudes in $\boldsymbol{h}$. These spatio-temporal long-term properties can be described by the correlation matrix $\boldsymbol{R}_h$, which contains the "average" information about the channel. Thus, $\boldsymbol{R}_h$ can be estimated very accurately, when averaging over many blocks [3].

Optimality Criterion: As the interference $\boldsymbol{n}[t]$ is white, we aim to maximize the signal power in $\boldsymbol{y}[t]$ for a given Rank R,

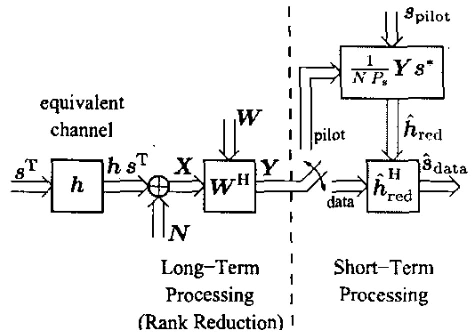

Fig. 1. Receiver with reduced rank channel estimation based on long-term properties of the wireless communication channel.

i.e. $\max_{\boldsymbol{W}} \mathrm{E}[\|\boldsymbol{W}^{\mathrm{H}}\boldsymbol{h}\, s[t]\|_2^2]$ [6]. Thus, the optimum rank reducing transformation $\boldsymbol{W}$ is chosen according to

$$\max_{\boldsymbol{W}} \mathrm{trace}\{\boldsymbol{W}^{\mathrm{H}}\boldsymbol{R}_h\boldsymbol{W}\} \quad \text{s.t.} \quad \mathrm{rank}\,\boldsymbol{W} = R \quad \text{and} \quad \boldsymbol{W}^{\mathrm{H}}\boldsymbol{W} = \mathbf{1}. \tag{10}$$

The solution is not unique. The columns of $\boldsymbol{W}$ are a unitary basis of the subspace spanned by the eigenvectors of $\boldsymbol{R}_h$ corresponding to the R largest eigenvalues:

$$\boldsymbol{W} = \boldsymbol{U}_{\mathrm{red}}\boldsymbol{Q} \quad \text{with} \quad \boldsymbol{U}_{\mathrm{red}} = \boldsymbol{U}[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_R] \quad \text{and} \quad \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{H}} = \mathbf{1},\ \boldsymbol{Q} \in \mathbb{C}^{R\times R}. \tag{11}$$

$\boldsymbol{e}_i$ is the i-th column of the unity matrix. The proof follows from [8, p.337], for example. The considered signal power in the reduced space of $\boldsymbol{y}[t]$ is $\mathrm{tr}\,\boldsymbol{W}^{\mathrm{H}}\boldsymbol{R}_h\boldsymbol{W} = \sum_{i=1}^{R}\lambda_i$. [3]

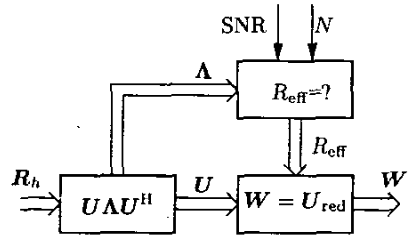

Fig. 2. Calculation of the matrix $\boldsymbol{W}$ for rank reduction based on the long-term (average) channel properties $\boldsymbol{R}_h$ and on the effective spatio-temporal rank R_{eff}.

IV. Mean Square Error of Channel Estimation

To describe the trade-off for a good choice of R analytically we introduce the mean square error (MSE) of the channel estimator (Eqn. 9) w.r.t. the true channel $\boldsymbol{h}$

$$\mathrm{MSE} = \frac{1}{P_{\mathrm{h}}}\mathrm{E}[\|\boldsymbol{h} - \boldsymbol{W}\hat{\boldsymbol{h}}_{\mathrm{red}}\|_2^2]. \tag{12}$$

[3] Refer to [6] for applications of criterion (10) to the UMTS-uplink.

The MSE is normalized by $P_{\mathrm{h}} = \mathrm{tr}\, \boldsymbol{R}_h = \sum_{i=1}^{M} \lambda_i$ to normalize the channel power. For unitary columns in $\boldsymbol{W}$ Eqn. (12) yields

$$\mathrm{MSE} = \underbrace{\frac{1}{P_{\mathrm{h}}} \sum_{i=R+1}^{M} \lambda_i}_{\text{Bias}} + \underbrace{\frac{R}{\gamma}}_{\text{Variance}} \tag{13}$$

where γ is the effective SNR defined as

$$\gamma \;=\; N \cdot \mathrm{SNR} \quad \text{with} \tag{14}$$

$$\mathrm{SNR} \;=\; \frac{\mathrm{E}[\|\boldsymbol{h}\, s[t]\|_2^2]}{\mathrm{E}[\|\boldsymbol{n}[t]\|_2^2]/M} = \frac{P_{\mathrm{h}}\, P_{\mathrm{s}}}{\sigma_n^2}. \tag{15}$$

Trade-off: The rank R determines the numerical complexity of the short-term processing stage (maximum ratio combining, Figure 1) as well as the performance of the communication link [6], which is determined by the SNR at the output of the combiner Eqn. (8) and the variance of the channel estimator in flat fading channels. On the one hand the bias in Eqn. (12) gives the fraction of the signal power $\sum_{R+1}^{M} \lambda_i$ in the subspace orthogonal to span$\{\boldsymbol{W}\}$, which is neglected for a given rank. This is the *bias* of the estimator, i.e. a systematic error, and depends on the eigenvalue spectrum of $\boldsymbol{R}_h$ and the rank R. It increases with smaller R. On the other hand the variance of the estimated channel coefficients is reduced, as only $R \leq M$ parameters have to be estimated. The estimation *variance* in the *second term* of Eqn. (12) increases linearly with the number of parameters R to estimate. The slope is given by the SNR and the length of the pilot sequence N. We conclude: (i) There is a rank $R = R_{\mathrm{eff}}$, which results in best performance, i.e. the optimum trade-off between neglected signal power and estimation variance. (ii) If complexity is a major issue, we want to choose the smallest rank R, which still gives satisfactory performance.

V. The Effective Spatio-Temporal Rank

The correlation matrix $\boldsymbol{R}_h$ of typical communication channels has full algebraic rank $R = M$, although its eigenvalues decrease with rather a steep slope [6]. As discussed above the optimum rank for receiver processing based on long-term channel properties is an integer in the interval $0 \leq R \leq M$. We define the *effective spatio-temporal rank* of the channel as the channel rank or the number of channel dimensions, which are relevant to the receiver in order to achieve optimum performance.

Definition: Assume a single-input/multiple-output system (Figure 1) as defined in Eqn. (2) with rank reduction according to Eqn. (4) and (10), and a ML-estimator to estimate the channel coefficients (Eqn. 9). *The effective Rank R_{eff}* of this system is defined as the Rank R, which achieves optimum system performance.

We choose the MSE (Eqn. 12) as a measure for receiver performance. Thus the effective rank is given by

$$R_{\mathrm{eff}} = \arg\min_{R} \mathrm{MSE}\,(\gamma, R, M, \Lambda)\,. \tag{16}$$

Further investigations showed good correspondence between the location of the minimum MSE and the minimum uncoded bit error rate.

VI. Modeling the Eigenvalue Profile

The eigenvalue profile of $\boldsymbol{R}_h$ is modeled as

$$\lambda_i = \begin{cases} \lambda_{\max} & , \quad 1 \leq i \leq i_0 \\ \lambda_{\max} \exp(-\frac{i-i_0}{\tau}) & , \quad i_0 \leq i \leq M \end{cases}, \tag{17}$$

which includes the uniformly distributed (i.e. uncorrelated channel) and exponentially decaying power profiles as special cases. Figure 3 shows the good least-squares fit of model (17) to the spatial scenario with a uniform linear array (half wavelength spacing *d*), M antenna elements and angle spread $\Delta\phi$.[4]

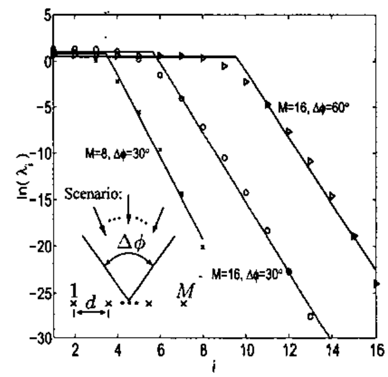

Fig. 3. Modeling the eigenvalue profile for $\{M = 8, \Delta\phi = 30°\}$, $\{M = 16, \Delta\phi = 30°\}$ and $\{M = 16, \Delta\phi = 60°\}$ (markers: $\ln(\lambda_i)$ as calculated for the scenario depicted in the left corner, solid line: least squares fit of model (17)).

VII. Performance Limits of Rank Reduction

In the following, we will use effective rank analysis to derive some interesting limits of performance of optimum reduced rank channel estimation. First we state the following

Theorem 1: For the eigenvalue profile specified in (17), the effective Rank R_{eff} is given as

$$R_{\mathrm{eff}} = \begin{cases} M & \text{for} \quad \gamma \geq \gamma_{\mathrm{free}} \\ i_0 + \tau \cdot \ln(\gamma/\gamma_0) & \text{for} \quad \gamma_0 < \gamma < \gamma_{\mathrm{free}} \\ i_0 & \text{for} \quad \gamma_{\mathrm{cutoff}} \leq \gamma \leq \gamma_0 \\ 0 & \text{for} \quad \gamma < \gamma_{\mathrm{cutoff}} \end{cases},$$

with (18)

$$\gamma_{\mathrm{free}} \;=\; \gamma_0 \cdot \exp\left((M - i_0)/\tau\right), \tag{19}$$

$$\gamma_0 \;=\; \gamma_{\mathrm{cutoff}} \cdot \tau \cdot (\exp(1/\tau) - 1) \tag{20}$$

$$\gamma_{\mathrm{cutoff}} \;=\; \mathrm{tr}\, \boldsymbol{R}_h \,/\, \lambda_{\max}. \tag{21}$$

The proof is given in the appendix. □

[4] For reasons of brevity we consider a spatial scenario only.

A. Estimation Cutoff

If the effective SNR γ drops below a certain limit, which we call the *cutoff SNR* γ_{cutoff}, the effective rank drops to *zero*. This means that, for $\gamma < \gamma_{\text{cutoff}}$ it is best in MSE sense, to *cut off* channel estimation altogether and report the communication link as *broken*. Note, that γ_{cutoff} does *not* depend on the parameters i_0 and τ of the eigenvalue profile used. Furthermore, there is $1 \leq \gamma_{\text{cutoff}} \leq M$, where the lower limit is taken on for a fully correlated (rank one) channel, and the upper one in the uncorrelated case.

B. Minimum Operational Rank Region

For an effective SNR γ in the *minimum operational rank* region: $\gamma_{\text{cutoff}} \leq \gamma \leq \gamma_0$, the effective rank is equal to i_0, independent of γ. Note, that for a given channel power $\text{tr}\boldsymbol{R}_h$, the upper limit γ_0 is *independent* of i_0. For $i_0 = 1$, in a spatio-temporal scenario, we conclude that, inside the minimum operational rank region, *beamforming* is the optimum linear preprocessing procedure to channel estimation. [5]

C. Full Rank Processing Region

For an effective SNR growing above the value γ_{free}, the effective rank is equal to M, which means, that *no* reduction of rank takes place anymore. Note, that γ_{free} depends explicitly on M, i_0 and τ and grows at least exponentially in M (Eqn. 19).

D. Asymptotic Optimality of Rank Reduction

The parameters i_0 and τ usually are not constants but actually depend on the value M in some way or another. In the asymptotic case, when M approaches infinity, we can make a general assertion on the effective rank, in the following

Theorem 2: If there exist real numbers $0 < a \leq 1$, $b > 0$ and $\beta > 1$, such that

$$i_0(M) \; < \; a \cdot M, \quad \text{and} \tag{22}$$

$$\tau(M) \; < \; b \cdot M^{1/(2 \cdot \beta)}, \tag{23}$$

for all $M \geq M_0$ with some $M_0 \geq 1$, then

$$\lim_{M \to \infty} \frac{R_{\text{eff}}}{M} < a \tag{24}$$

for any finite value of the ratio $\gamma/\gamma_{\text{cutoff}}$.
The proof is given in the appendix. □

As long as i_0 is not climbing stronger than linearly with M, and τ is not climbing stronger than the square root of M, the effective rank will never grow above i_0 as M approaches infinity. In the case that, $i_0 < M$ it follows, that for large enough M, an actual reduction of rank *will* occur, for any ratio $\gamma/\gamma_{\text{cutoff}}$.

[5] assuming the temporal channel taps are uncorrelated, which leads to a block diagonal matrix $\boldsymbol{R}_h$. The eigenvectors then correspond to the spatial eigenvectors for each temporal tap. Selecting the dominant eigenvector is therefore equivalent to beamforming for the one temporal tap with highest SNR.

In Figure 4 the dependency of τ and i_0 on M for a fixed spatial scenario shows that conditions (22) and (23) are satisfied, whereas τ grows much more slowly than required by Eqn. (23). Figure 5 shows R_{eff}/M as a function of M and $\gamma/\gamma_{\text{cutoff}}$. The dependency of R_{eff}/M on $\gamma/\gamma_{\text{cutoff}}$ decreases with increasing M and finally vanishes for large M. For small M we can identify the minimum operational rank region with $\gamma_0/\gamma_{\text{cutoff}}$. It also shows that the bound (24) is tight.

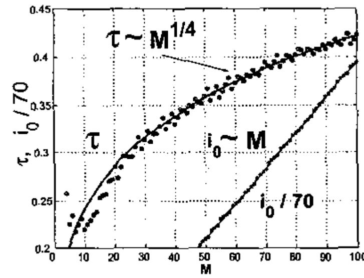

Fig. 4. We observe a dependency of the model parameters on M, such that $\tau(M) \propto M^{1/4}$ and $i_0 \propto M$, for the scenario in Fig. 3 with angle spread $\Delta\phi = 30°$ (i_0 is scaled by 70 to fit in the same range).

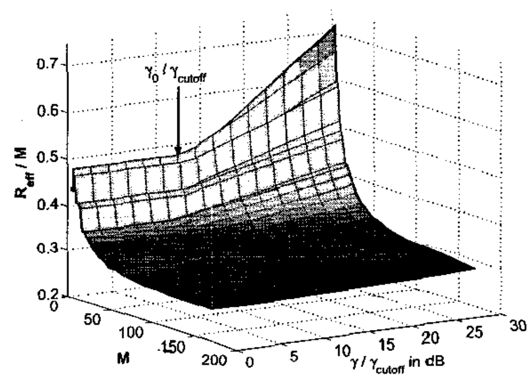

Fig. 5. R_{eff}/M converges to a for large M, i.e. a plateau independent of M and $\gamma/\gamma_{\text{cutoff}}$, for the scenario in Fig. 3 with $\Delta\phi = 30°$. From Figure 4 it turns out that $a = 0.28$, which corresponds to the height of the plateau.

VIII. Conclusions

When reducing the spatio-temporal channel rank based on long-term correlation properties of the channel, we neglect signal power (bias) and reduce the estimation variance of the instantaneous channel coefficients. The trade-off between bias and variance of the channel estimates motivates the definition of the effective spatio-temporal rank using the MSE of estimation to describe the optimum trade-off. The effective rank depends on the channel structure, the number of pilot symbols, and the SNR at the receiver. Based on a good analytical model with only three parameters for typical eigenvalue spectra, we identify relevant effective SNR (Eqn. 14) regions for receiver operation: A minimum SNR and pilot sequence length are required for channel estimation (cutoff SNR). They only depend on the maximum eigenvalue w.r.t. the total received power. Moreover, there is a region above the cutoff SNR, in which beamforming, i.e. rank one processing, can be optimum. For high effective SNR above a boundary given by the number of channel parameters M and the eigenvalue spectrum full rank processing, i.e. no rank reduction, is the best choice for receiver processing. Finally, we give conditions for the channel structure, under which rank reduction is optimum for large M independent of the SNR and the number of pilot symbols N. Using an example we illustrate that these conditions are satisfied in typical channels.

Appendix A
Proof of Theorem 1

First note, that

$$\mathrm{tr}\, \boldsymbol{R}_h \;/\; \lambda_{\max} \;=\; i_0 + \frac{1-\exp\left(-\left(M-i_0\right)/\tau\right)}{\exp\left(1/\tau\right)-1}, \qquad (25)$$

which follows immediately from (17) and the relationship

$$q + q^2 + \cdots + q^M = \frac{1-q^M}{q^{-1}-1}, \text{ for } 0 < |q|, \qquad (26)$$

which is used with $q = \exp\left(-1/\tau\right)$ for $\tau > 0$. Now, we have to deal with two cases. Let us first take $0 \le R \le i_0$. From (13), (17), and (25) we get

$$\mathrm{MSE} = 1 + R \cdot \left(\frac{1}{\gamma} - \frac{\lambda_{\max}}{\mathrm{tr}\, \boldsymbol{R}_h}\right). \qquad (27)$$

By defining

$$\gamma_{\mathrm{cutoff}} := \mathrm{tr}\, \boldsymbol{R}_h \;/\; \lambda_{\max}, \qquad (28)$$

it is clear, that for $\gamma < \gamma_{\mathrm{cutoff}}$ the smallest MSE is achieved for $R = 0$, while for $\gamma > \gamma_{\mathrm{cutoff}}$ the minimum is reached for $R = i_0$. Now let $R > i_0$. This time we find

$$\mathrm{MSE} = \frac{\lambda_{\max}}{\mathrm{tr}\, \boldsymbol{R}_h} \cdot \frac{\exp\left(-\dfrac{R-i_0}{\tau}\right) - \exp\left(-\dfrac{M-i_0}{\tau}\right)}{\exp\left(1/\tau\right)-1} + \frac{R}{\gamma}. \qquad (29)$$

Applying (28) and locating the unique root of

$$\frac{\partial \mathrm{MSE}}{\partial R} = \frac{1}{\gamma} - \frac{\exp\left(-\left(R-i_0\right)/\tau\right)}{\gamma_{\mathrm{cutoff}} \cdot \tau \cdot \left(\exp\left(1/\tau\right)-1\right)}, \qquad (30)$$

yields the effective rank

$$R_{\mathrm{eff}} = i_0 + \tau \cdot \ln\left(\gamma/\gamma_0\right), \qquad (31)$$

since $\partial^2 \mathrm{MSE}/\partial R^2$ is positive for $R > i_0$. In (31) we have used the abbreviation

$$\gamma_0 := \gamma_{\mathrm{cutoff}} \cdot \tau \cdot \left(\exp\left(1/\tau\right)-1\right). \qquad (32)$$

The condition $R > i_0$ translates into $\gamma > \gamma_0$. Finally, the optimum rank can be at most equal to M. As (31) is strictly increasing in γ, it represents a valid solution only for $\gamma \le \gamma_{\mathrm{free}}$, where

$$\gamma_{\mathrm{free}} := \gamma_0 \cdot \exp\left(\left(M-i_0\right)/\tau\right) \qquad (33)$$

Appendix B
Proof of Theorem 2

First note, that the inequality

$$-\tau \cdot \ln\left(\exp\left(1/\tau\right) - 1\right) \;<\; \tau^2/\pi \qquad (34)$$

holds for all $\tau > 0$. This follows from the fact, that

$$\max_{\tau>0} \frac{1}{-\tau} \cdot \ln\left(\exp\left(1/\tau\right)-1\right) \approx 0.316\ldots < \frac{1}{\pi}. \qquad (35)$$

From (18) and (20) we know, that

$$R_{\mathrm{eff}} \le i_0 + \tau \cdot \ln\left(\frac{\gamma/\gamma_{\mathrm{cutoff}}}{\tau}\right) - \tau \cdot \ln\left(\exp\left(1/\tau\right)-1\right). \qquad (36)$$

Applying (34) takes us to

$$R_{\mathrm{eff}} < i_0 + \frac{\tau^2}{\pi} + \tau \cdot \ln\left(\frac{\gamma/\gamma_{\mathrm{cutoff}}}{\tau}\right), \qquad (37)$$

which can be further upper bounded:

$$\begin{aligned} R_{\mathrm{eff}} &< i_0 + \frac{\tau^2}{\pi} + \tau \cdot \ln\left(1 + \frac{\gamma/\gamma_{\mathrm{cutoff}}}{\tau}\right) \\ &< i_0 + \tau^2/\pi + \gamma/\gamma_{\mathrm{cutoff}}. \end{aligned} \qquad (38)$$

The last step makes use of the fact, that $g(\tau) := \tau \cdot \ln\left(1 + \frac{c}{\tau}\right)$, with $c > 0$, is strictly increasing in τ, with $\lim_{\tau\to\infty} g(\tau) = c$. Using (22) and (23) it follows, that

$$\frac{R_{\mathrm{eff}}}{M} < a + \frac{b^2}{\pi} \cdot \frac{1}{M^{1-1/\beta}} + \frac{\gamma/\gamma_{\mathrm{cutoff}}}{M}, \qquad (39)$$

for $M \ge M_0$. As $\beta > 1$ we finally conclude, that

$$\lim_{M\to\infty} \frac{R_{\mathrm{eff}}}{M} \;<\; a \qquad (40)$$

for any finite value of the ratio $\gamma/\gamma_{\mathrm{cutoff}}$.

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