

A General Rate Duality of the MIMO Multiple Access Channel and the MIMO Broadcast Channel

Raphael Hunger and Michael Joham

Associate Institute for Signal Processing, Technische Universität München, 80290 Munich, Germany

Telephone: +49 89 289-28508, Fax: +49 89 289-28504, Email: hunger@tum.de

Abstract—We present a general rate duality between the multiple access channel (MAC) and the broadcast channel (BC) which is applicable to systems with and without nonlinear interference cancellation. Different to the state-of-the-art rate duality with interference subtraction from Vishwanath et al., the proposed duality is filter-based instead of covariance-based and exploits the arising unitary degree of freedom to decorrelate every point-to-point link. Therefore, it allows for noncooperative stream-wise decoding which reduces complexity and latency. Moreover, the conversion from one domain to the other does not exhibit any dependencies during its computation making it accessible to a parallel implementation instead of a serial one. We additionally derive a rate duality for systems with multi-antenna terminals when linear filtering *without* interference (pre-)subtraction is applied and the different streams of a single user are *not* treated as self-interference. Both dualities are based on a framework already applied to a mean-square-error duality between the MAC and the BC. Thanks to this novel rate duality, any rate-based optimization with linear filtering in the BC can now be handled in the dual MAC where the arising expressions lead to more efficient algorithmic solutions than in the BC due to the alignment of the channel and precoder indices.

I. INTRODUCTION

In the past few years, dualities were successfully employed as the linking element between the multiple access channel (MAC) and the broadcast channel (BC). Thanks to various versions of dualities, many regions of the MAC and the BC were classified to be identical under a sum-power constraint. First, the signal-to-interference-and-noise-ratio (SINR) regions under single-stream transmission per user were shown to be identical in [1], [2]. Second, the mean-square-error (MSE) regions of the MAC and the BC coincide which has been proven by means of the SINR duality in [3] and later in [4] or directly in [5], [6]. And third, the rate regions of the MAC and the BC under Gaussian signaling and nonlinear interference cancellation have recently been shown to be the same, see [7] for the single-antenna case, [8] for the multi-antenna case, and [9] for the coincidence of the dirty-paper coding rate region and the capacity region. A stream-wise duality with power constraints on subsets of antennas which holds for the optimum filters of a quality-of-service power minimization was presented in [10] for systems with and without nonlinear interference cancellation. Due to its stream-wise nature, conversion from one domain to the dual is complicated since it is not clear how to allocate the SINRs to the users in case of multi-antenna terminals. Besides the capability of proving congruency of two regions, dualities also deliver

explicit conversion formulas how to switch from one domain to the other. In case of the rate duality in [8], (arbitrary) optimum receive filters generating sufficient statistics are assumed both in the MAC and in the BC. Given transmit covariance matrices in the MAC are converted to transmit covariance matrices in the dual BC. Dependencies during these transformations prevent a parallel processing and force a serial implementation. In addition, the received data streams have to be decoded jointly which entails a high computational complexity.

Our contribution in this paper is twofold. First, we present a novel rate duality for systems with nonlinear interference cancellation. One of the key steps involved is the change from the covariance matrices to the transmit filters by which we gain an isometry as degree of freedom. This degree of freedom is then used to decorrelate every point-to-point link thus making a fast parallel stream-wise decoding possible. As the streams of a single user now do not interfere with each other, we can employ an SINR duality in the style of our MSE duality in [5], [6]. Therein, the transmit filters in the dual domain are scaled receivers of the primal domain and the receive filters are scaled transmitters of the primal domain. We end up with a system of linear equations to determine these scaling factors.

Our second contribution is a rate duality for linear filtering applicable to multi-antenna terminals where different streams of a user are not treated as self-interference. Up to now, such a duality did not exist and hitherto existing dualities for linear filtering treat different streams of a user as virtual users contributing interference to the user under consideration, see [1], [2], [11] for example. In general, the maximum possible rate cannot be obtained when a duality based on virtual users is applied. The underlying framework for the proposed linear duality is similar to the proposed nonlinear duality presented in the following. Key observation is again the fact that decorrelation allows for a stream-wise decoding which also achieves the rate that is possible under joint decoding.

II. SYSTEM MODEL

Two systems are considered, namely the MAC where K multi-antenna users send their data to a common base station which is equipped with N antennas, and the BC where the signal flow is reversed, i.e., the base station serves the users. In the former case the transmission between the k th user and the base station is described by the channel matrix $\mathbf{H}_k \in \mathbb{C}^{N \times r_k}$ with r_k denoting the number of transmit antennas at user k . The BC link, however, is characterized by

the Hermitian channel matrix \mathbf{H}_k^H . User k multiplexes L_k data streams. If interference cancellation is applied in the MAC, we assume for the sake of readability that the decoding order is chosen such that user 1 is decoded last, whereas the reversed encoding order is chosen in the BC, i.e., user 1 is precoded first. For different sortings, the users have to be relabeled correspondingly. Under these assumptions, the rate of user k in the MAC with nonlinear interference cancellation reads as [12]

$$R_k^{\text{MAC}} = \log_2 \frac{|\sigma_\eta^2 \mathbf{I}_N + \sum_{\ell \leq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H|}{|\sigma_\eta^2 \mathbf{I}_N + \sum_{\ell < k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H|}, \quad (1)$$

where σ_η^2 is the noise variance per antenna and $\mathbf{Q}_\ell \in \mathbb{C}^{r_\ell \times r_\ell}$ denotes the transmit covariance matrix of user ℓ . Contrary, user k 's rate in the BC with nonlinear dirty paper coding is [8]

$$R_k^{\text{BC}} = \log_2 \frac{|\sigma_\eta^2 \mathbf{I}_{r_k} + \mathbf{H}_k^H \sum_{\ell \geq k} \mathbf{S}_\ell \mathbf{H}_k|}{|\sigma_\eta^2 \mathbf{I}_{r_k} + \mathbf{H}_k^H \sum_{\ell > k} \mathbf{S}_\ell \mathbf{H}_k|}, \quad (2)$$

where $\mathbf{S}_\ell \in \mathbb{C}^{N \times N}$ is the BC transmit covariance matrix of user ℓ . If only linear filtering without interference subtraction is applied, user k experiences interference from all other users.

III. RATE DUALITY FOR SYSTEMS UTILIZING INTERFERENCE SUBTRACTION

A. Benefits of the Rate Duality with Interference Cancellation

Besides the ability to show congruency between the two capacity regions, the decisive reason for utilizing the rate duality is that all rate expressions are concave functions of the transmit covariance matrices in the MAC but not in the BC. Moreover, the optimal sorting of the users can easily be obtained in the MAC. As a consequence, many rate-based maximizations can be solved with efficient algorithms converging to the global optimum in the MAC and afterwards converted to the BC by means of the duality conversion formulas.

B. State-of-the-Art Duality

By means of the MAC-to-BC conversion, we illustrate the state-of-the-art rate duality from [8]. Both in the MAC and in the BC, all rate expressions depend only on the transmit covariance matrices and not on the matrix valued receive filters since they are implicitly assumed to generate sufficient statistics. Based on these statistics, the L_k data streams of user k have to be decoded *jointly*. Given a set of transmit covariance matrices $\{\mathbf{Q}_k\}$ in the MAC which fulfills a total transmit power constraint and obtains a rate tuple $R_1^{\text{MAC}}, \dots, R_K^{\text{MAC}}$ under the assumption of optimum receive filters, the duality in [8] generates a set of transmit covariance matrices $\{\mathbf{S}_k\}$ for the BC that fulfills the same total transmit power constraint and achieves the same rate tuple $R_1^{\text{BC}}, \dots, R_K^{\text{BC}}$. In the BC, optimum receivers yielding sufficient statistics are again required and all streams of every individual user have to be decoded jointly as well.

Two key methods utilized are the *effective channel* and the *flipped channel* idea. The former one implies that the capacity of a point-to-point MIMO system with channel matrix

\mathbf{H} subject to an additive Gaussian distortion (noise plus independent interference) with covariance matrix \mathbf{X} equals the capacity of a point-to-point system with *effective channel* matrix $\mathbf{L}^{-1}\mathbf{H}$ subject to additive Gaussian distortion with identity covariance matrix if $\mathbf{X} = \mathbf{L}\mathbf{L}^H$. Given an arbitrary effective channel of a point-to-point system, a system with reversed signal flow and Hermitian effective channel (*flipped channel*) has the same capacity [13]. According to (1), the rate of user k in the MAC can be expressed as

$$R_k^{\text{MAC}} = \log_2 |\mathbf{I}_N + \mathbf{X}_k^{-1} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H|, \quad (3)$$

with the substitution $\mathbf{X}_k = \sigma_\eta^2 \mathbf{I}_N + \sum_{\ell=1}^{k-1} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H$. Introducing the Cholesky decomposition $\mathbf{X}_k = \mathbf{L}_k \mathbf{L}_k^H$, applying the determinant equality $|\mathbf{I}_a + \mathbf{A}\mathbf{B}| = |\mathbf{I}_b + \mathbf{B}\mathbf{A}|$ for arbitrary \mathbf{A} and \mathbf{B} of appropriate dimensions, and inserting two identity matrices $\mathbf{I}_{r_k} = \mathbf{F}_k^{-1} \mathbf{F}_k = \mathbf{F}_k^H \mathbf{F}_k^{-H}$, (3) can be expressed as

$$R_k^{\text{MAC}} = \log_2 |\mathbf{I}_N + \mathbf{L}_k^{-1} \mathbf{H}_k \mathbf{F}_k^{-1} \mathbf{F}_k \mathbf{Q}_k \mathbf{F}_k^H \mathbf{F}_k^{-H} \mathbf{H}_k^H \mathbf{L}_k^{-H}|.$$

Now, $\mathbf{L}_k^{-1} \mathbf{H}_k \mathbf{F}_k^{-1}$ can be regarded as the effective channel for the covariance matrix $\mathbf{F}_k \mathbf{Q}_k \mathbf{F}_k^H$. How \mathbf{F}_k must be chosen will be clarified below. Flipping the channel, outcomes in [8] ensure the existence of a covariance matrix $\mathbf{Z}_k \in \mathbb{C}^{N \times N}$ with

$$R_k^{\text{MAC}} = \log_2 |\mathbf{I}_{r_k} + \mathbf{F}_k^{-H} \mathbf{H}_k^H \mathbf{L}_k^{-H} \mathbf{Z}_k \mathbf{L}_k^{-1} \mathbf{H}_k \mathbf{F}_k^{-1}|, \quad (4)$$

$$\text{tr}(\mathbf{Z}_k) \leq \text{tr}(\mathbf{F}_k \mathbf{Q}_k \mathbf{F}_k^H).$$

The rate of user k in the BC is (cf. Eq. 2)

$$R_k^{\text{BC}} = \log_2 |\mathbf{I}_{r_k} + \mathbf{Y}_k^{-1} \mathbf{H}_k^H \mathbf{S}_k \mathbf{H}_k|$$

$$= \log_2 |\mathbf{I}_{r_k} + \mathbf{F}_k^{-H} \mathbf{H}_k^H \mathbf{S}_k \mathbf{H}_k \mathbf{F}_k^{-1}|, \quad (5)$$

with the substitution $\mathbf{Y}_k = \sigma_\eta^2 \mathbf{I}_{r_k} + \sum_{\ell=k+1}^K \mathbf{H}_\ell^H \mathbf{S}_\ell \mathbf{H}_\ell = \mathbf{F}_k^H \mathbf{F}_k$. Equality between R_k^{MAC} in (4) and R_k^{BC} in (5) holds, if

$$\mathbf{S}_k = \mathbf{L}_k^{-H} \mathbf{Z}_k \mathbf{L}_k^{-1}. \quad (6)$$

Implicitly, \mathbf{Z}_k depends on \mathbf{F}_k as will be shown soon. Thus, \mathbf{S}_k depends on \mathbf{Y}_k which itself is a function of all \mathbf{S}_ℓ with $\ell > k$. These dependencies require that \mathbf{S}_k has to be computed before \mathbf{S}_{k-1} and consequently, one has to start with the computation of \mathbf{S}_K followed by $\mathbf{S}_{K-1}, \dots, \mathbf{S}_1$.

It remains to determine the matrices $\mathbf{Z}_k \forall k$. Introducing the reduced *singular-value-decomposition* (rSVD)

$$\mathbf{L}_k^{-1} \mathbf{H}_k \mathbf{F}_k^{-1} = \mathbf{U}_k \mathbf{D}_k \mathbf{V}_k^H \in \mathbb{C}^{N \times r_k} \quad (7)$$

with the two (sub-)unitary matrices $\mathbf{U}_k \in \mathbb{C}^{N \times \text{rank}(\mathbf{H}_k)}$ and $\mathbf{V}_k \in \mathbb{C}^{r_k \times \text{rank}(\mathbf{H}_k)}$, the matrix \mathbf{Z}_k reads as

$$\mathbf{Z}_k = \mathbf{U}_k \mathbf{V}_k^H \cdot \mathbf{F}_k \mathbf{Q}_k \mathbf{F}_k^H \cdot \mathbf{V}_k \mathbf{U}_k^H. \quad (8)$$

The proof for the sum-power conservation can be found in [8]. From the MAC-to-BC conversion, it can be concluded that every rate tuple in the MAC can also be achieved in the dual BC. Conversely, the transformation from the BC to the MAC which follows from the same framework, states that every rate tuple in the BC can also be achieved in the MAC. Hence, the duality of these two domains is proven and as a consequence, their capacity regions are congruent. Summing up, the state-of-the-art rate duality including interference cancellation is serial

in two senses: First, it requires a serial implementation of the covariance matrix conversion due to the dependencies of \mathbf{S}_k on \mathbf{S}_ℓ with $\ell > k$. Second, the application of the duality requires that the different streams associated to a user are decoded jointly or, at the best, in a serial fashion.

C. Proposed Filter-Based Duality

The previously described state-of-the-art rate duality is mainly deduced from information theoretic considerations, where optimum receivers generate sufficient statistics and capacity is achieved via joint decoding with inter- and intra-user successive interference cancellation. Approaching from a signal processing point of view enables us to derive a novel intuitive duality of low complexity. Switching from arbitrary sufficient statistics generating optimum receivers to MMSE receivers, we are able to express all rates in terms of error covariance matrices, which in turn only depend on the transmit covariance matrices, i.e., on the outer product of the precoding filters. The remaining degree of freedom is a unitary rotation and we utilize this isometry in order to decorrelate every single point-to-point link. Doing so, the error covariance matrix becomes diagonal and capacity is achieved with *separate* stream-wise decoding making intra-user interference cancellation superfluous. The fact that stream-wise encoding/decoding achieves capacity has already been observed in [2], [14]. There, however, intra-user successive decoding must be applied and all streams are decoded one by one.

As all rates can now be expressed as functions of the SINRs of the individual streams, we apply a low-complexity SINR duality in the style of our MSE duality in [6], [5]. In a nutshell, the scaled MMSE receivers are used as precoders in the dual domain and scaled precoding filters serve as the receive filters in the dual domain. This dual domain features the same SINR values as the original one and therefore achieves the same user rates. In the following, we give an elaborate derivation of the MAC-to-BC conversion.

1) *Derivation:* Assuming that every MAC covariance matrix $\mathbf{Q}_k = \mathbf{T}_k \mathbf{T}_k^H$ is generated by the precoder $\mathbf{T}_k \in \mathbb{C}^{r_k \times L_k}$, the symbol estimate of user k in the MAC is

$$\hat{\mathbf{s}}_k = \mathbf{G}_k \left[\mathbf{H}_k \mathbf{T}_k \mathbf{s}_k + \sum_{\ell > k} \mathbf{H}_\ell \mathbf{T}_\ell \mathbf{s}_\ell + \sum_{\ell < k} \mathbf{H}_\ell \mathbf{T}_\ell \mathbf{s}_\ell + \boldsymbol{\eta} \right],$$

where \mathbf{G}_k denotes the receive filter of user k , \mathbf{s}_k its data vector with identity covariance matrix, and $\boldsymbol{\eta}$ the additive noise. Since interference caused by users $l > k$ is removed by successive interference cancellation, the MMSE receiver for user k is

$$\mathbf{G}_k = \mathbf{T}_k^H \mathbf{H}_k^H \left(\sum_{\ell \leq k} \mathbf{H}_\ell \mathbf{T}_\ell \mathbf{T}_\ell^H \mathbf{H}_\ell^H + \sigma_\eta^2 \mathbf{I}_N \right)^{-1}. \quad (9)$$

Using (9) and the matrix-inversion lemma, the MMSE error covariance matrix $\mathbf{C}_k = \mathbb{E}[(\mathbf{s}_k - \hat{\mathbf{s}}_k)(\mathbf{s}_k - \hat{\mathbf{s}}_k)^H]$ reads as

$$\mathbf{C}_k = \mathbf{I}_{L_k} - \mathbf{G}_k \mathbf{H}_k \mathbf{T}_k = \left[\mathbf{I}_{L_k} + \mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}_k^{-1} \mathbf{H}_k \mathbf{T}_k \right]^{-1}, \quad (10)$$

with $\mathbf{X}_k = \sigma_\eta^2 \mathbf{I}_N + \sum_{\ell=1}^{k-1} \mathbf{H}_\ell \mathbf{T}_\ell \mathbf{T}_\ell^H \mathbf{H}_\ell^H$. The rate of user k can be expressed in terms of its error covariance matrix

$$R_k^{\text{MAC}} = \log_2 |\mathbf{C}_k^{-1}| = -\log_2 |\mathbf{C}_k|, \quad (11)$$

cf. (3). Note that the rate of user k is invariant to a unitary matrix \mathbf{W}_k multiplied from the right hand side to \mathbf{T}_k yielding $\mathbf{T}'_k = \mathbf{T}_k \mathbf{W}_k$. Moreover, the rate expressions of other users only depend on the transmit covariance matrices and not on the filters themselves therefore being also invariant to this isometry. Last but not least, the transmit power $\text{tr}(\mathbf{Q}_k) = \text{tr}(\mathbf{T}_k \mathbf{T}_k^H) = \text{tr}(\mathbf{T}'_k \mathbf{T}'_k{}^H)$ is invariant under this isometry \mathbf{W}_k . Although \mathbf{W}_k does not influence the interference covariance matrix experienced by any other user, it can be used as a spatial decorrelation filter for every point-to-point link which in conjunction with the MMSE receiver $\mathbf{G}'_k = \mathbf{W}_k^H \mathbf{G}_k$ diagonalizes the error-covariance matrix \mathbf{C}_k . To this end, \mathbf{W}_k must be chosen as the eigenbasis of $\mathbf{G}_k \mathbf{H}_k \mathbf{T}_k$ which is also the eigenbasis of $\mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}_k^{-1} \mathbf{H}_k \mathbf{T}_k$. Due to the decorrelation, all point-to-point links from the users to the base station achieve capacity without intra-user successive interference cancellation thus making separate stream decoding possible. This way, the rate of user k can be expressed as the sum of the individual streams' rates, i.e., $R_k^{\text{MAC}} = \sum_{i=1}^{L_k} R_{k,i}^{\text{MAC}}$, where

$$R_{k,i}^{\text{MAC}} = \log_2(1 + \text{SINR}_{k,i}^{\text{MAC}}).$$

Let $\mathbf{t}'_{k,i}$ be the i th column of \mathbf{T}'_k and $\mathbf{g}'_{k,i}$ be the i th row of \mathbf{G}'_k , then the general SINR definition in the MAC

$$\text{SINR}_{k,i}^{\text{MAC}} = \frac{|\mathbf{g}'_{k,i}{}^T \mathbf{H}_k \mathbf{t}'_{k,i}|^2}{\mathbf{g}'_{k,i}{}^T \left(\mathbf{X}_k + \sum_{m \neq i} \mathbf{H}_k \mathbf{t}'_{k,m} \mathbf{t}'_{k,m}{}^H \mathbf{H}_k^H \right) \mathbf{g}'_{k,i}^*} \quad (12)$$

reduces for the special choice of the decorrelation filter \mathbf{W}_k to

$$\text{SINR}_{k,i}^{\text{MAC}} = \frac{|\mathbf{g}'_{k,i}{}^T \mathbf{H}_k \mathbf{t}'_{k,i}|^2}{\sigma_\eta^2 \|\mathbf{g}'_{k,i}\|_2^2 + \sum_{\ell < k} \sum_{m=1}^{L_\ell} |\mathbf{g}'_{k,i}{}^T \mathbf{H}_\ell \mathbf{t}'_{\ell,m}|^2} \quad (13)$$

i.e., the summation over m in the denominator of (12) vanishes as $\mathbf{G}'_k \mathbf{H}_k \mathbf{T}'_k$ is diagonal. Inserting \mathbf{G}'_k into (13) yields

$$\text{SINR}_{k,i}^{\text{MAC}} = \mathbf{t}'_{k,i}{}^H \mathbf{H}_k^H \mathbf{X}_k^{-1} \mathbf{H}_k \mathbf{t}'_{k,i}, \quad (14)$$

according to the diagonal entries of $\mathbf{W}_k^H \mathbf{C}_k^{-1} \mathbf{W}_k$, see (10).

In the dual BC with Hermitian channels, dirty paper coding for inter-user interference presubtraction is applied with reversed order. The receivers perform a stream-wise decoding based on the outputs of the receive filters $\mathbf{B}_k \forall k$. Given precoders $\mathbf{P}_1, \dots, \mathbf{P}_K$, the SINR of user k 's stream i is

$$\text{SINR}_{k,i}^{\text{BC}} = \frac{|\mathbf{b}_{k,i}{}^T \mathbf{H}_k^H \mathbf{p}_{k,i}|^2}{\mathbf{b}_{k,i}{}^T \left(\mathbf{Y}_k + \sum_{m \neq i} \mathbf{H}_k^H \mathbf{p}_{k,m} \mathbf{p}_{k,m}^H \mathbf{H}_k \right) \mathbf{b}_{k,i}^*}, \quad (15)$$

and the rate of user k in the BC with stream-wise decoding reads as $R_k^{\text{BC}} = \sum_{i=1}^{L_k} \log_2(1 + \text{SINR}_{k,i}^{\text{BC}})$. Besides the decorrelation, the flipping of transmit and receive filters is the core of our duality: Scaled transmit matrices including the decorrelation in the MAC act as receive filters in the BC and scaled receivers in the MAC act as transmit filters in the BC:

$$\mathbf{p}_{k,i} = \alpha_{k,i} \mathbf{g}'_{k,i}^* \quad \text{and} \quad \mathbf{b}_{k,i} = \alpha_{k,i}^{-1} \mathbf{t}'_{k,i}^*. \quad (16)$$

Plugging (16) into the general BC SINR expression (15) we obtain by means of the diagonal structure of $\mathbf{G}'_k \mathbf{H}_k \mathbf{T}'_k$

$$\text{SINR}_{k,i}^{\text{BC}} = \frac{\alpha_{k,i}^2 |\mathbf{g}'_{k,i}{}^T \mathbf{H}_k \mathbf{t}'_{k,i}|^2}{\sigma_\eta^2 \|\mathbf{t}'_{k,i}\|_2^2 + \sum_{\ell > k} \sum_{m=1}^{L_\ell} |\mathbf{g}'_{\ell,m}{}^T \mathbf{H}_\ell \mathbf{t}'_{\ell,m}|^2 \alpha_{\ell,m}^2}.$$

Equating $\text{SINR}_{k,i}^{\text{BC}}$ with the MAC SINR from (13), we get

$$\begin{aligned} & \alpha_{k,i}^2 \left[\sigma_\eta^2 \|\mathbf{g}'_{k,i}\|_2^2 + \sum_{\ell < k} \sum_{m=1}^{L_\ell} |\mathbf{g}'_{k,i}{}^T \mathbf{H}_\ell \mathbf{t}'_{\ell,m}|^2 \right] \\ & - \sum_{\ell > k} \sum_{m=1}^{L_\ell} \alpha_{\ell,m}^2 |\mathbf{g}'_{\ell,m}{}^T \mathbf{H}_k \mathbf{t}'_{k,i}|^2 = \sigma_\eta^2 \|\mathbf{t}'_{k,i}\|_2^2, \end{aligned} \quad (17)$$

which needs to hold for all users k and all streams $i \in \{1, \dots, L_k\}$ thus generating the system of linear equations

$$\mathbf{M} [\alpha_{1,1}^2, \dots, \alpha_{K,L_K}^2]^T = \sigma_\eta^2 [\|\mathbf{t}'_{1,1}\|_2^2, \dots, \|\mathbf{t}'_{K,L_K}\|_2^2]^T \quad (18)$$

with the $\sum_{k=1}^K L_k \times \sum_{k=1}^K L_k$ block upper triangular matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{1,1} & \cdots & \mathbf{M}_{1,K} \\ \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_{K,K} \end{bmatrix}. \quad (19)$$

The off-diagonal blocks with $a < b$ read as (cf. Eq. 17)

$$\mathbf{M}_{a,b} = -(\mathbf{G}'_b \mathbf{H}_a \mathbf{T}'_a)^H \odot (\mathbf{G}'_b \mathbf{H}_a \mathbf{T}'_a)^T \in \mathbb{R}^{L_a \times L_b} \quad (20)$$

with the *Hadamard* product \odot , and $\mathbf{M}_{a,a}$ is diagonal with

$$[\mathbf{M}_{a,a}]_{i,i} = \sigma_\eta^2 \|\mathbf{g}'_{a,i}\|_2^2 - \sum_{\ell < a} \sum_{m=1}^{L_\ell} [\mathbf{M}_{\ell,a}]_{m,i}. \quad (21)$$

Since all off-diagonal elements of \mathbf{M} are nonpositive and all diagonal elements are nonnegative, \mathbf{M} is a *Z-matrix* [15]. For $\sigma_\eta^2 > 0$, \mathbf{M} is column diagonally dominant. So, \mathbf{M} is an *M-matrix* such that its inverse exists with nonnegative entries [15] yielding valid solutions $\alpha_{k,i}^2 \geq 0$. Because of the block upper triangular structure of \mathbf{M} we can quickly solve for $\alpha_{1,1}^2, \dots, \alpha_{K,L_K}^2$ via back-substitution, in particular since the diagonal blocks $\mathbf{M}_{k,k}$ are diagonal matrices. Note that a rank-deficient precoder \mathbf{T}_m manifests in zero columns and zero rows in \mathbf{M} which have to be removed before inversion. The respective $\alpha_{m,\cdot}^2$ and $\|\mathbf{t}'_{m,\cdot}\|_2^2$ in (18) also have to be removed, and finally, $\mathbf{p}_{m,\cdot} = \mathbf{0}$ and $\mathbf{b}_{m,\cdot} = \mathbf{0}$ must be chosen.

Summing up the rows of (18), we obtain

$$\sum_{k=1}^K \sum_{i=1}^{L_k} \underbrace{\alpha_{k,i}^2 \|\mathbf{g}'_{k,i}\|_2^2}_{\|\mathbf{p}_{k,i}\|_2^2} \sigma_\eta^2 = \sigma_\eta^2 \sum_{k=1}^K \sum_{i=1}^{L_k} \|\mathbf{t}'_{k,i}\|_2^2, \quad (22)$$

stating that the dual BC consumes the same power as the MAC. Thus, the same or larger (if MMSE receivers are chosen for $\mathbf{B}_1, \dots, \mathbf{B}_K$) rates can be achieved in the dual BC as in the primal MAC under the same transmit power constraint. The reverse direction of the duality transforming BC filters to the MAC can be handled with the same framework. Due to its similarity, we skip its derivation. From this direction of the duality, it follows that the BC rate region is a subset of the MAC capacity region. In combination with the former result of the MAC-to-BC conversion stating that the MAC capacity region is a subset of the BC rate region, the following theorem becomes evident with the aid of [9] (cf. [8]):

Theorem III.1: *The capacity regions of the MAC and the BC are congruent under a sum-power constraint.*

As a consequence, any optimization in the BC can be solved in the MAC, which offers concave rate expressions suitable for efficient globally convergent algorithms. Since both capacity regions are congruent, we optimize over the same region and therefore, do not introduce any suboptimality at this point. Having found the solution in the MAC we can convert it back to the BC by means of the duality. Optimality in one domain translates itself to optimality in the other domain. The main advantage of the proposed filter-based duality compared to the state-of-the-art duality in [8] is that both the conversion and the decoding in the dual domain can be parallelized and need not be applied serially as in [8]. The computation of the transmit and receive filters features no dependencies and the decoding process does not require intra-user interference cancellation or intra-user joint decoding of the streams, all streams of a user can be decoded independently in parallel.

2) *Algorithmic Implementation:* Given arbitrary precoding filters $\mathbf{T}_k \forall k$ in the MAC, MMSE receivers \mathbf{G}_k are first computed via (9) for all k , see Line 2 in Alg. 1. The decorrelation filter \mathbf{W}_k is chosen as the eigenbasis of $\mathbf{G}_k \mathbf{H}_k \mathbf{T}_k$ and afterwards, the transmit and receive filters are adapted, see Lines 3 and 4. Thereby, a parallel stream-wise decoding is possible without intra-user interference cancellation. Having set up the linear system of equations in (18) which ensures the conservation of the SINRs in the BC, the precoders \mathbf{P}_k and receivers \mathbf{B}_k are computed with (16), cf. Line 8.

IV. RATE DUALITY FOR SYSTEMS WITHOUT INTERFERENCE SUBTRACTION

In case of linear filtering, i.e., when nonlinear inter-user interference cancellation is not applied, user k experiences interference from all other users $\ell \neq k$. Up to now, a *rate* duality for the linear case without interference subtraction does not exist in the literature when multi-antenna terminals are involved and different streams shall *not* be treated as self-interference. By jointly decoding the streams in the MAC, user k can achieve the rate

$$\begin{aligned} R_k^{\text{MAC}} &= \log_2 \left| \mathbf{I}_N + \left(\sum_{\ell \neq k} \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H + \sigma_\eta^2 \mathbf{I}_N \right)^{-1} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right| \\ &= -\log_2 \left| \mathbf{I}_N - \mathbf{X}^{-1} \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^H \right|, \end{aligned} \quad (23)$$

with the substitution $\mathbf{X} = \sigma_\eta^2 \mathbf{I}_N + \sum_{\ell=1}^K \mathbf{H}_\ell \mathbf{Q}_\ell \mathbf{H}_\ell^H$. In contrast to systems with interference cancellation described in the previous section, this matrix is common to MMSE receivers

$$\mathbf{G}_k = \mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}^{-1} \quad (24)$$

for all users k and therefore has to be computed only once. Applying \mathbf{G}_k , user k experiences the error covariance matrix

$$\mathbf{C}_k = \mathbf{I}_{L_k} - \mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}^{-1} \mathbf{H}_k \mathbf{T}_k, \quad (25)$$

which is again decorrelated by the isometry \mathbf{W}_k since the rate $R_k^{\text{MAC}} = -\log_2 |\mathbf{C}_k|$ is again invariant under this unitary degree of freedom. Choosing \mathbf{W}_k as the eigenbasis of $\mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}^{-1} \mathbf{H}_k \mathbf{T}_k$, we adapt the receive filter $\mathbf{G}'_k = \mathbf{W}_k^H \mathbf{G}_k$ and the transmit filter $\mathbf{T}'_k = \mathbf{T}_k \mathbf{W}_k$. Due to the decorrelation, the error covariance matrix $\mathbf{W}_k^H \mathbf{C}_k \mathbf{W}_k$ is diagonalized and

all L_k streams of user k can be decoded separately yielding the rate $R_k^{\text{MAC,lin}} = \sum_{i=1}^{L_k} R_{k,i}^{\text{MAC,lin}}$, with the rate

$$R_{k,i}^{\text{MAC,lin}} = \log_2(1 + \text{SINR}_{k,i}^{\text{MAC,lin}}) \quad (26)$$

of user k 's stream i . Its SINR now reads as

$$\text{SINR}_{k,i}^{\text{MAC,lin}} = \frac{|g_{k,i}'^T \mathbf{H}_k \mathbf{t}_{k,i}'|^2}{\sigma_\eta^2 \|g_{k,i}'\|_2^2 + \sum_{\ell \neq k} \sum_{m=1}^{L_\ell} |g_{k,i}'^T \mathbf{H}_\ell \mathbf{t}_{\ell,m}'|^2}.$$

We apply the same rule for finding the precoding and receive filters \mathbf{P}_k and \mathbf{B}_k of user k in the BC as we do in case of interference cancellation, i.e., $\mathbf{p}_{k,i} = \alpha_{k,i} g_{k,i}'^*$ and $\mathbf{b}_{k,i} = \alpha_{k,i}^{-1} \mathbf{t}_{k,i}'^*$, see (16). With these transformations, the BC SINR reads as

$$\text{SINR}_{k,i}^{\text{BC,lin}} = \frac{\alpha_{k,i}^2 |g_{k,i}'^T \mathbf{H}_k \mathbf{t}_{k,i}'|^2}{\sigma_\eta^2 \|t_{k,i}'\|_2^2 + \sum_{\ell \neq k} \sum_{m=1}^{L_\ell} |g_{\ell,m}'^T \mathbf{H}_k \mathbf{t}_{k,i}'|^2 \alpha_{\ell,m}^2}.$$

Equating the BC and MAC SINRs yields the system of linear equations (18), where the matrix \mathbf{M} is not block upper triangular as in (19), since inter-user interference cancellation is not applied:

$$\mathbf{M} = \begin{bmatrix} M_{1,1} & \cdots & M_{1,K} \\ \vdots & \ddots & \vdots \\ M_{K,1} & \cdots & M_{K,K} \end{bmatrix}. \quad (27)$$

For this reason, (18) is solved via *LU-factorization* [16, Section 3.2.5] and forward-backward substitution. The diagonal blocks of \mathbf{M} are diagonal matrices with diagonal entries

$$[M_{a,a}]_{i,i} = \sigma_\eta^2 \|g_{a,i}'\|_2^2 - \sum_{\ell \neq a} \sum_{m=1}^{L_\ell} [M_{\ell,a}]_{m,i}, \quad (28)$$

such that \mathbf{M} is again an *M-matrix* satisfying the power conservation equation (22). With slight modifications, Alg. 1 can be used to perform the MAC-to-BC conversion without nonlinear inter-user interference cancellation. In Line 2, \mathbf{G}_k must be computed according to (24), and in Line 7, the matrix \mathbf{M} follows from (27), (20), and (28). Again, the converse direction of the duality underlies the same framework and completes the proof of the duality in case of linear filtering without inter-user interference cancellation:

Theorem IV.1: *The MIMO MAC and the MIMO BC share the same rate region under linear filtering and a sum-power constraint both for separate and joint de-encoding of each user's data streams.*

This novel rate duality for systems without interference cancellation allows us to convert any rate-based optimization from the BC to the MAC without loss of optimality. An immediate benefit is that we can switch from the rate expression

$$R_k^{\text{MAC,interference}} = -\log_2 \prod_i [\mathbf{I}_{L_k} - \mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}^{-1} \mathbf{H}_k \mathbf{T}_k]_{i,i}$$

with separate stream decoding and hence self-interference to the one in (23) with joint stream decoding

$$R_k^{\text{MAC,lin}} = -\log_2 |\mathbf{I}_{L_k} - \mathbf{T}_k^H \mathbf{H}_k^H \mathbf{X}^{-1} \mathbf{H}_k \mathbf{T}_k|,$$

which is always larger than or equal to $R_k^{\text{MAC,interference}}$. Moreover, the channel and precoder indices are aligned in

Algorithm 1 Novel stream-wise MAC-to-BC conversion.

- 1: **for** $k = 1 : K$ **do**
 - 2: $\mathbf{G}_k \leftarrow \mathbf{T}_k^H \mathbf{H}_k^H (\sum_{\ell < k} \mathbf{H}_\ell \mathbf{T}_\ell \mathbf{T}_\ell^H \mathbf{H}_\ell^H + \sigma_\eta^2 \mathbf{I}_N)^{-1}$
 - 3: $\mathbf{W}_k \leftarrow$ eigenbasis($\mathbf{G}_k \mathbf{H}_k \mathbf{T}_k$) *decorrelation matrix*
 - 4: $\mathbf{G}'_k \leftarrow \mathbf{W}_k^H \mathbf{G}_k$ and $\mathbf{T}'_k \leftarrow \mathbf{T}_k \mathbf{W}_k$ *decorrelate*
 - 5: **end for**
 - 6: set up \mathbf{M} with (19) – (21), remove zero columns/rows
 - 7: solve for $\alpha_{1,1}^2, \dots, \alpha_{K,L_K}^2$ via (18)
 - 8: $\mathbf{p}_{k,i} = \alpha_{k,i} g_{k,i}'^*$ and $\mathbf{b}_{k,i} = \frac{1}{\alpha_{k,i}} \mathbf{t}_{k,i}'^* \quad \forall k, \quad \forall i$
-

the MAC, see (23), whereas they aren't in the BC. Although (weighted) sum-rate maximization remains a nonconcave maximization in the MAC, the aforementioned indices alignment allows for simpler expressions and reduced-complexity algorithms. Last but not least, MAC precoders are characterized by only $\sum_{k=1}^K r_k^2$ variables instead of $N \sum_{k=1}^K r_k$ in the BC. Summing up, solving rate based optimizations with linear filtering in the MAC and applying the proposed duality is more efficient than solving the problem in the BC.

REFERENCES

- [1] H. Boche and M. Schubert, "A General Duality Theory for Uplink and Downlink Beamforming," in *Proc. IEEE Vehicular Techn. Conf. (VTC) fall, Vancouver, Canada*, September 2002, pp. 87 – 91 vol.1.
- [2] P. Viswanath and D. Tse, "Sum Capacity of the Vector Gaussian Broadcast Channel and Uplink-Downlink Duality," *IEEE Trans. Inf. Theory*, vol. 49, no. 8, pp. 1912–1921, August 2003.
- [3] S. Shi and M. Schubert, "MMSE Transmit Optimization for Multiuser Multiantenna Systems," in *Proc. IEEE Internat. Conf. on Acoustics, Speech, and Signal Proc. (ICASSP), Philadelphia, USA*, Mar. 2005.
- [4] A. Khachan, A. Tenenbaum, and R. S. Adve, "Linear Processing for the Downlink in Multiuser MIMO Systems with Multiple Data Streams," in *IEEE International Conf. on Comm.*, June 2006, vol. 9, pp. 4113–4118.
- [5] A. Mezghani, M. Joham, R. Hunger, and W. Utschick, "Transceiver Design for Multi-User MIMO Systems," in *Proc. ITG/IEEE WSA 2006*, March 2006.
- [6] R. Hunger, M. Joham, and W. Utschick, "On the MSE-Duality of the Broadcast Channel and the Multiple Access Channel," *Submitted to IEEE Transactions on Signal Processing*.
- [7] N. Jindal, S. Vishwanath, and A. Goldsmith, "On the Duality of Gaussian Multiple-Access and Broadcast Channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 768–783, May 2004.
- [8] S. Vishwanath, N. Jindal, and A. Goldsmith, "Duality, Achievable Rates, and Sum-Rate Capacity of MIMO Broadcast Channels," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2658–2668, October 2003.
- [9] H. Weingarten, Y. Steinberg, and S. S. Shamai, "The Capacity Region of the Gaussian Multiple-Input Multiple-Output Broadcast Channel," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3936–3964, 2006.
- [10] W. Yu and T. Lan, "Transmitter Optimization for the Multi-Antenna Downlink With Per-Antenna Power Constraints," *IEEE Trans. Signal Process.*, vol. 55, no. 6-1, pp. 2646–2660, June 2007.
- [11] B. Song, R. L. Cruz, and B. D. Rao, "Network Duality for Multiuser MIMO Beamforming Networks and Applications," *IEEE Trans. Commun.*, vol. 55, no. 3, pp. 618–630, 2007.
- [12] W. Yu, W. Rhee, S. Boyd, and J.M. Cioffi, "Iterative Water-Filling for Gaussian Vector Multiple-Access Channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 1, pp. 145–152, January 2004.
- [13] E. Telatar, "Capacity of Multi-Antenna Gaussian Channels," *European Transactions on Telecommunications*, vol. 10, pp. 585–595, 1999.
- [14] D. Tse and P. Viswanath, "On the Capacity of the Multiple Antenna Broadcast Channel," *DIMACS Workshop. American Math. Soc. 2003*, vol. 62, pp. 87–105, October 2002.
- [15] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, 1979.
- [16] G. H. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 2nd edition, 1991.