A Combinatorial Approach to Maximizing the Sum Rate in the MIMO BC with Linear Precoding

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Abstract— **We investigate the sum rate maximization in the MIMO broadcast channel when linear filtering is applied. Our particular interest lies in usually not considered system configurations where the base station is equipped with less antennas than the user terminals have in sum. For such scenarios, conventional block-diagonalization or zero-forcing techniques cannot be applied and even asymptotic results on the sum rate do not seem to exist. We show that the nonconvex problem at hand is closely related to a combinatorial stream selection and that choosing the number of active streams too high reduces the multiplexing gain and can even lead to a saturation of the sum rate, i.e., a multiplexing gain of zero. Besides the full featured reference algorithm probing all feasible stream allocation combinations, a reduced complexity successive version is presented as well. Both variants are based on the projected gradient algorithm.**

I. State of the Art & Contributions

While the sum rate maximization with nonlinear dirty-paper coding is a well-known convex problem for which several efficient algorithms exist (see for example [1], [2], [3]), maximizing the sum rate constrained to linear filtering is a highly nonconvex problem featuring several suboptimum local maxima. Indeed, asymptotic results on the rate loss of linear precoding compared to dirty paper coding are available when the base station is equipped with enough antennas [4], [5], but MIMO scenarios where the users' terminals have more antennas in sum than the base station have only been tackled by simulations so far. Suitable only for the scenarios with enough antennas, the block-diagonalization and zero-forcing approaches are usually designed in the downlink [6], [7], whereas algorithms being capable of handling also the latter, more difficult antenna configurations with less antennas at the base utilize a duality to create a dual multiple access channel offering simpler rate expressions. The algorithms employing a duality can furthermore be classified either to work in a stream-wise fashion where the individual streams of a single user are treated as interference, or in a user-wise fashion, where all streams of a single user are decoded jointly.

The authors in [8] make use of their stream-wise *mean square error* (MSE) duality in [9] and apply a complete *sequential quadratic programming* (SQP) procedure in every iteration of their algorithm to find a power allocation that locally maximizes the sum rate for given beamformers. Moreover, they switch between the virtual uplink and the downlink in every iteration yielding a computationally complex algorithm. A similar approach is presented in [10], where instead of an SQP an interior point problem solver is used to minimize the product of the streams' MSEs for fixed transmit and receive beamformers. This geometric program has to be solved with high accuracy in every single iteration in order to let the outer optimization come close to a local maximum of the sum-rate. In addition, it features a very high computational complexity. Again, frequent conversions from the uplink to the downlink via the stream-wise MSE duality in [11] increase the computational complexity. In contrast to the former two stream-wise optimizations, a projected gradient based sum-rate maximization was proposed in [12] where all streams of an individual user are decoded jointly. Since the duality of [13] employed by [12] does not preserve the individual rates, when the users decode their streams jointly, the approach in [12] delivers only valid results in the dual uplink. However, the rate duality of [14] for the joint stream decoding enables an extension to the BC. In [15], an efficient implementation of a zero-forcing based successive stream allocation is presented.

Contributions on the sum-rate maximization for single-antenna terminals intuitively limit the number of active streams to the number of available transmit antennas at the base station, when more antennas are present at the terminals than at the base station, see e.g. [16], [17]. We prove in the following that allocating more active streams than antennas available at the base is indeed suboptimum also in the MIMO case since it reduces the multiplexing gain, but can also lead to a saturation of the achieved sum rate when the transmit power is increased. An immediate consequence is that the multiplexing gain can be zero in the worst case. Noticing that at most as many streams are active as the base station has antennas and that the sum rate maximization with linear beamforming is highly nonconvex, a combinatorial search over all possible stream allocations becomes inevitable when seeking for the global optimum. We present a gradient projection based local rate maximization algorithm for fixed stream initializations and probe all possible combinations of active streams in order to get a reference algorithm for the performance evaluation of other sum rate maximizing algorithms. In order to overcome the high computational effort resulting from the complete combinatorial search, we additionally present a successive stream allocation policy probing significantly less combinations without noticeable rate loss.

II. System Model and Dual Uplink Problem

The rate duality in [14] allows us to tackle the sum rate maximization in the dual uplink and convert the obtained solution back to the broadcast channel. In contrast to the approaches in [8] and [10], the duality therefore has to be invoked only once and not in every iteration. Let $\boldsymbol{H}_k \in \mathbb{C}^{N\times r_k}$ denote the channel matrix describing the propagation from the kth user to the base station in the dual uplink with N being the number of antennas at the base and r_k being the number of antennas of user k. A precoder matrix $\boldsymbol{T}_k \in \mathbb{C}^{r_k\times B_k}$ maps the B_k data streams of user k onto his r_k antennas. Under these assumptions, the rate of user k seeing interference from all other users reads as (see [14])

$$R_k = \log_2 \left| \mathbf{I}_{B_k} + \boldsymbol{T}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \Big(\mathbf{I}_N + \sum_{\ell\neq k} \boldsymbol{H}_\ell \boldsymbol{T}_\ell \boldsymbol{T}_\ell^{\mathrm{H}} \boldsymbol{H}_\ell^{\mathrm{H}} \Big)^{-1} \boldsymbol{H}_k \boldsymbol{T}_k \right| \quad (1)$$

with the total power consumption $\sum_{k=1}^{K} \|\boldsymbol{T}_k\|_{\mathrm{F}}^2 = P_{\mathrm{Tx}}$.

III. Multiplexing Gain Degradation When Serving Too Many Streams

While the rate expression (1) is in general only valid if the individual users jointly decode their own streams, the precoders $\boldsymbol{T}_k$ can be extended by isometries $\boldsymbol{W}_k$ decorrelating the received signal such that the same rate can also be achieved by a stream-wise decoding operation, see [14]. However, these isometries lead to the fact that user k has at most r_k active data streams with nonzero power. Thus we have to ensure that $B_k \le r_k$ holds for all users $k \in \{1,\dots,K\}$ all the time, which intuitively makes sense. For the proof that allocating more than N active streams is suboptimum and can lead to a saturation of the rate when the transmit power P_{Tx} is increased, we can therefore restrict ourselves to a stream-wise decoding which simplifies the proof. Treating each of the b transmitted streams as a virtual user, we can interpret the dual uplink system as a multi-user SIMO system with $b = \sum_{k=1}^{K} B_k$ virtual users, where the ith virtual user is characterized by an effective *vector*-valued channel $\boldsymbol{c}_i = \boldsymbol{H}_{m[i]}\boldsymbol{u}_i$, where $m[\cdot]$ maps a virtual user index to the actual user index and $\boldsymbol{u}_i$ is the corresponding unit norm beamformer, i.e., the respective normalized column of $\boldsymbol{T}_{m[i]}\boldsymbol{W}_{m[i]}$. Thus, the rate of the ith stream can be expressed as

$$\begin{aligned} r_i &= \log_2 \Big(1 + p_i \boldsymbol{c}_i^{\mathrm{H}} \big(\mathbf{I}_N + \sum_{j\neq i} \boldsymbol{c}_j \boldsymbol{c}_j^{\mathrm{H}} p_j \big)^{-1} \boldsymbol{c}_i \Big) \\ &= -\log_2 \Big(1 - p_i \boldsymbol{c}_i^{\mathrm{H}} \big(\mathbf{I}_N + \sum_{j=1}^{b} \boldsymbol{c}_j \boldsymbol{c}_j^{\mathrm{H}} p_j \big)^{-1} \boldsymbol{c}_i \Big). \end{aligned} \quad (2)$$

We investigate linear power allocation policies of the form $p_i = \bar{p}_i P_{\mathrm{Tx}}$, where $P_{\mathrm{Tx}} = \sum_i p_i$ denotes the total sum power and $0 \le \bar{p}_i \le 1$ is no function of P_{Tx}. Introducing the canonical unit vector $\boldsymbol{e}_i$ of appropriate dimension with the one at the ith position, the effective $N \times b$ channel matrix $\boldsymbol{C}_b = [\boldsymbol{c}_1, \dots, \boldsymbol{c}_b]$, and the diagonal nonnegative $b \times b$ power matrix $\boldsymbol{P}_b = \mathrm{diag}\{p_i\}_{i=1}^{b}$, we can reformulate (2) as

$$\begin{aligned} r_i &= -\log_2 \boldsymbol{e}_i^{\mathrm{T}} \Big(\mathbf{I}_b - \boldsymbol{P}_b \boldsymbol{C}_b^{\mathrm{H}} \big(\mathbf{I}_N + \boldsymbol{C}_b \boldsymbol{P}_b \boldsymbol{C}_b^{\mathrm{H}} \big)^{-1} \boldsymbol{C}_b \Big) \boldsymbol{e}_i \\ &= -\log_2 \boldsymbol{e}_i^{\mathrm{T}} \big(\mathbf{I}_b + \boldsymbol{P}_b^{\frac{1}{2}} \boldsymbol{C}_b^{\mathrm{H}} \boldsymbol{C}_b \boldsymbol{P}_b^{\frac{1}{2}} \big)^{-1} \boldsymbol{e}_i. \end{aligned} \quad (3)$$

For an appropriate choice of the decorrelation isometries $\boldsymbol{W}_k$, we have $\sum_{i=1}^{b} r_i = \sum_{k=1}^{K} R_k$, see [14], i.e., the system with stream-wise decoding can achieve the same rates as the jointly decoded system with user rates in (1). Now let us assume that we serve at most as many virtual users as we have antennas at the base station, i.e., $b \le N$ and $p_1, \dots, p_b > 0$. As there are $r = \sum_{k=1}^{K} r_k$ antennas at the transmitter side available, we can freely select b out of the r possible virtual streams to be the active ones with nonzero power. Clearly, the specific choice of the active streams influences the sum rate of the system, and this degree of freedom can later on be chosen to increase the performance. The sum rate obtained by the b active streams reads as

$$R_{\mathrm{tot}} = -\log_2 \prod_{i=1}^{b} \boldsymbol{e}_i^{\mathrm{T}} \big(\mathbf{I}_b + \boldsymbol{P}_b^{\frac{1}{2}} \boldsymbol{A}_b \boldsymbol{P}_b^{\frac{1}{2}} \big)^{-1} \boldsymbol{e}_i, \quad (4)$$

where $\boldsymbol{A}_b = \boldsymbol{C}_b^{\mathrm{H}} \boldsymbol{C}_b$. Letting P_{Tx} become large and assuming that the active virtual users have been chosen such that their effective channels are linearly independent, the sum rate R_{tot} asymptotically reads as

$$R_{\mathrm{tot}} \cong b \log_2 P_{\mathrm{Tx}} - \log_2 \prod_{i=1}^{b} \big[\boldsymbol{A}_b^{-1} \big]_{i,i} + \log_2 \prod_{i=1}^{b} \bar{p}_i, \quad (5)$$

where $\bar{p}_k = p_k / P_{\mathrm{Tx}} > 0$ is the normalized power of the kth virtual stream and the asymptotic equivalence $a \cong \mathrm{b}$ means that the difference $a - b$ vanishes in the high power limit. The third summand of (5) is maximized by evenly distributing the normalized powers to the virtual users, i.e., $\bar{p}_i = 1/b$, cf. [18]. It can immediately be seen that the achieved multiplexing gain is b, and the multiplexing gain can be increased with b until $b = N$. If $b > N$, $\boldsymbol{A}_b$ is rank deficient and above calculation is no longer valid. Finally, the second summand influences the rate offset and depends only on the effective channels of the virtual users, i.e., on the choice which virtual users are served and on their beamforming vectors. However, it is independent of the power allocation policy.

In the following, we show that $b > N$ decreases the multiplexing gain from N to $N-1$ or even less. In the worst case, the multiplexing gain is zero.

Theorem III.1: *Allocating more than N active streams in a MAC or BC communication scenario with linear transceivers where the terminals of the users have more antennas in sum than the N-antenna base station, and where the number of streams per user is upper bounded by its number of antennas, yields a multiplexing gain between zero and $N-1$.*

For the proof of above theorem, the $N+1$-th virtual user with $p_{N+1} > 0$ is added to the system. The sum rate changes to

$$R_{\mathrm{tot}} = -\log_2 \prod_{i=1}^{N+1} \boldsymbol{e}_i^{\mathrm{T}} \big(\mathbf{I}_{N+1} + \boldsymbol{P}_{N+1}^{\frac{1}{2}} \boldsymbol{A}_{N+1} \boldsymbol{P}_{N+1}^{\frac{1}{2}} \big)^{-1} \boldsymbol{e}_i, \quad (6)$$

where $\boldsymbol{A}_{N+1} = \boldsymbol{C}_{N+1}^{\mathrm{H}} \boldsymbol{C}_{N+1}$ which has only rank N making the asymptotic analysis slightly more complicated. Introducing the matrix partitioning

$$\boldsymbol{A}'_{N+1} = \boldsymbol{P}_{N+1}^{-1} + \boldsymbol{A}_{N+1} = \begin{bmatrix} \boldsymbol{A}'_N & \boldsymbol{b} \\ \boldsymbol{b}^{\mathrm{H}} & d' \end{bmatrix} \tag{7}$$

with $\boldsymbol{b} = \boldsymbol{C}_N^{\mathrm{H}} \boldsymbol{c}_{N+1}$ and the substitutions

$$\boldsymbol{A}'_N = \boldsymbol{C}_N^{\mathrm{H}} \boldsymbol{C}_N + P_{\mathrm{Tx}}^{-1} \bar{\boldsymbol{P}}_N^{-1}, \tag{8}$$

$$d' = \|\boldsymbol{c}_{N+1}\|_2^2 + \frac{1}{P_{\mathrm{Tx}} \cdot \bar{p}_{N+1}}, \tag{9}$$

where $\boldsymbol{P}_N = P_{\mathrm{Tx}} \cdot \bar{\boldsymbol{P}}_N$, the sum rate can be expressed as

$$R_{\mathrm{tot}} = -\log_2 \Big(\frac{1}{P_{\mathrm{Tx}}^{N+1}} \prod_{i=1}^{N+1} \bar{p}_i^{-1} \big[\boldsymbol{A}'^{-1}_{N+1} \big]_{i,i} \Big). \tag{10}$$

In the following, we compute the diagonal entries of the matrix $\boldsymbol{A}'^{-1}_{N+1}$ when P_{Tx} becomes large. Applying the inversion rules for block partitioned matrices, we find that the first N diagonal entries of $\boldsymbol{A}'^{-1}_{N+1}$ read with $i \in \{1, \ldots, N\}$ as

$$\big[\boldsymbol{A}'^{-1}_{N+1}\big]_{i,i} = \big[\boldsymbol{A}'^{-1}_N\big]_{i,i} + \frac{|\boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{A}'^{-1}_N \boldsymbol{b}|^2}{d' - \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}'^{-1}_N \boldsymbol{b}}, \tag{11}$$

whereas the $N+1$-th diagonal entry of the inverse is

$$\big[\boldsymbol{A}'^{-1}_{N+1}\big]_{N+1,N+1} = \frac{1}{d' - \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}'^{-1}_N \boldsymbol{b}}. \tag{12}$$

Due to

$$\boldsymbol{A}'^{-1}_N = \boldsymbol{A}_N^{-1} - \boldsymbol{A}_N^{-1} \boldsymbol{P}_N^{-\frac{1}{2}} \big(\mathbf{I} + \boldsymbol{P}_N^{-\frac{1}{2}} \boldsymbol{A}_N^{-1} \boldsymbol{P}_N^{-\frac{1}{2}} \big)^{-1} \boldsymbol{P}_N^{-\frac{1}{2}} \boldsymbol{A}_N^{-1},$$

we can use the asymptotic relation

$$\boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}'^{-1}_N \boldsymbol{b} \cong \boldsymbol{b}^{\mathrm{H}} \Big(\boldsymbol{A}_N^{-1} - \boldsymbol{A}_N^{-1} \boldsymbol{P}_N^{-1} \boldsymbol{A}_N^{-1} \Big) \boldsymbol{b}$$

and find with $\boldsymbol{A}_N^{-1} = \boldsymbol{C}_N^{-1} \boldsymbol{C}_N^{-\mathrm{H}}$

$$d' - \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}'^{-1}_N \boldsymbol{b} \cong \frac{1}{P_{\mathrm{Tx}}} \left[\frac{1}{\bar{p}_{N+1}} + \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}_N^{-1} \bar{\boldsymbol{P}}_N^{-1} \boldsymbol{A}_N^{-1} \boldsymbol{b} \right], \tag{13}$$

since $\|\boldsymbol{c}_{N+1}\|_2^2 = \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}_N^{-1} \boldsymbol{b}$. Combining (13) and (12), it can be observed that the $N+1$-th diagonal element of $\boldsymbol{A}'^{-1}_{N+1}$ increases linearly in P_{Tx}. For an asymptotic analysis of (11), we exploit the asymptotic equivalences $\boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{A}'^{-1}_N \boldsymbol{b} \cong \boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{A}_N^{-1} \boldsymbol{b}$ and $[\boldsymbol{A}'^{-1}_N]_{i,i} \cong [\boldsymbol{A}_N^{-1}]_{i,i}$. The ith diagonal element of $\boldsymbol{A}'^{-1}_{N+1}$ with $i \in \{1, \ldots, N\}$ then asymptotically reads as

$$\big[\boldsymbol{A}'^{-1}_{N+1}\big]_{i,i} \cong \big[\boldsymbol{A}_N^{-1}\big]_{i,i} + \frac{|\boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{A}_N^{-1} \boldsymbol{b}|^2}{d' - \boldsymbol{b}^{\mathrm{H}} \boldsymbol{A}'^{-1}_N \boldsymbol{b}}. \tag{14}$$

While the first summand is constant, the second one increases linearly with P_{Tx} in the limit as the numerator is no function of P_{Tx} and the denominator is inversely proportional to P_{Tx} in the limit, see (13). The only way to prevent from the linear increase is when $|\boldsymbol{e}_i^{\mathrm{T}} \boldsymbol{A}_N^{-1} \boldsymbol{b}|^2$ is zero. This is obtained when the $N+1$-th effective channel $\boldsymbol{c}_{N+1}$ is orthogonal to the projection of $\boldsymbol{c}_i$ onto $\mathcal{N}_i$, where $\mathcal{N}_i$ is the orthogonal complement of the span of $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N\} \setminus \{\boldsymbol{c}_i\}$. In other words, $\boldsymbol{c}_{N+1}$ has to be orthogonal to that part of $\boldsymbol{c}_i$ which does not lie in the span of the other effective channels $\boldsymbol{c}_j$ for all $j \in \{1, \ldots, N\} \setminus \{i\}$. Then, (14) is constant and asymptotically equivalent to $[\boldsymbol{A}_N^{-1}]_{i,i}$. Summing up, each of the first N diagonal elements of the inverse in (10) asymptotically increases linearly with P_{Tx} irrespective of the normalized power distribution $\bar{p}_1, \ldots, \bar{p}_{N+1} > 0$ except when the effective channel $\boldsymbol{c}_{N+1}$ of the $N+1$-th stream is orthogonal to one or more effective channels $\boldsymbol{c}_i$ of the first N users *after projection* onto $\mathcal{N}_i$. In such a case, the respective diagonal elements do not increase with P_{Tx}. In conjunction with the prefactor $1/P_{\mathrm{Tx}}^{N+1}$, we can conclude that the multiplexing gain is reduced by one for every stream to whose *projected* effective channel the added stream's channel is not orthogonal. In the worst case, the multiplexing gain is zero. Having proven that for large enough transmit power one has to allocate exactly N active streams (if the users have less than N antennas in sum, then every user has to activate as many streams as he has transmit antennas, i.e., $B_k = r_k$), we conjecture that even for moderate transmit power, serving more than N active streams can never be optimum in the sum rate sense.

IV. Combinatorial Search for the Optimum Stream Selection

In the low power regime, it is well known that the sum rate maximizing transmission strategy is to allocate only a single active stream belonging to the strongest eigenmode of all users' channels. Raising the transmit power, more and more streams have to be allocated until in the asymptotic limit, $b = N$ out of r streams are active. Note that the nonconvexity of the problem leads to the fact that the optimum stream allocation cannot be obtained by an iterative sum rate maximizing algorithm from all stream initialization points in general. This is a drawback which one does not have to face when dirty paper coding or successive interference cancellation is applied instead of linear filtering.

In order to find out which sum rates can at least be achieved by means of a linear precoding scheme, our reference algorithm probes all reasonable stream allocations. To this end, we distribute b active streams over the K users, where b ranges from 1 to N. Keeping in mind that the rate expression (1) depends only on the covariance matrices $\boldsymbol{T}_\ell \boldsymbol{T}_\ell^{\mathrm{H}}$ of all users ℓ, it does not make sense to allocate more streams to a single user than he has transmit antennas as any full rank $r_k \times r_k$ covariance matrix can be achieved with a rank r_k precoder. Thus, the constraint $B_k \leq r_k$ must be satisfied for all users k. To illustrate this, Table I shows the possible stream allocations to the users in a system where $K = 3$ users each having $r_1 = r_2 = r_3 = 2$ antennas communicate with an $N = 4$ antenna base station.

Depending on the number of active streams per user, the precoders are differently initialized. Here, we apply truncated identity matrices for the active precoding matrices. Initialization examples for the precoder $\boldsymbol{T}_k$ of user k with either 0, 1, or 2 active streams are therefore

$$\boldsymbol{T}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{T}_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ or } \boldsymbol{T}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{15}$$

respectively. Given a fixed stream allocation as initialization, we apply a projected gradient ascent algorithm to increase the

Number of streams b	Stream allocation for the users
1	**100**, **010**, **001**
2	**200**, 020, 002, **110**, **101**, 011
3	**111**, **120**, 102, **210**, 201, 012, 021
4	**220**, 202, 022, **211**, 121, 112

TABLE I

POSSIBLE STREAM ALLOCATIONS FOR $K = 3$ USERS, $N = 4$ ANTENNAS AT THE BASE STATION, AND $r_k = 2\,\forall k$ ANTENNAS AT THE TERMINALS.

sum rate, where we update the precoders of all users according to their gradients and afterwards project the resulting precoders on the transmit power constraint set defined by the equality $\sum_{k=1}^{K} \|\boldsymbol{T}_k\|_{\mathrm{F}}^2 = P_{\mathrm{Tx}}$. Hence, the projection simply scales the magnitude of all precoders by an iteration-dependent common factor $\alpha^{(n)}$. Doing so, the update rule for the precoder of user m in iteration $n+1$ reads as

$$\boldsymbol{T}_m^{(n+1)} = \alpha^{(n)} \Big(\boldsymbol{T}_m^{(n)} + p^{(n)} s^{(n)} \cdot \frac{\partial \sum_{k=1}^{K} R_k}{\partial \boldsymbol{T}_m^*} \Big|_{\boldsymbol{T}_m = \boldsymbol{T}_m^{(n)}} \Big), \tag{16}$$

where the iteration-dependent preconditioning scalar is

$$p^{(n)} = \frac{\sqrt{P_{\mathrm{Tx}}}}{\sqrt{\sum_{m=1}^{K} \left\| \frac{\partial \sum_{k=1}^{K} R_k}{\partial \boldsymbol{T}_m^*} \Big|_{\boldsymbol{T}_m = \boldsymbol{T}_m^{(n)}} \right\|_{\mathrm{F}}^2}}, \tag{17}$$

$s^{(n)}$ is the adaptive step-size which is initialized with one and multiplied by a factor smaller than one (we chose $2/3$) every time the utility tends to decrease. The individual rates R_k are defined in (1). Making use of the two substitutions $\boldsymbol{X} = \mathbf{I}_N + \sum_{\ell=1}^{K} \boldsymbol{H}_\ell \boldsymbol{T}_\ell \boldsymbol{T}_\ell^{\mathrm{H}} \boldsymbol{H}_\ell^{\mathrm{H}}$ and $\boldsymbol{X}_k = \boldsymbol{X} - \boldsymbol{H}_k \boldsymbol{T}_k \boldsymbol{T}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}}$, the partial derivative of the sum rate can be expressed very efficiently as

$$\frac{\partial \sum_{k=1}^{K} R_k}{\partial \boldsymbol{T}_m^*} = \frac{1}{\ln 2} \boldsymbol{H}_m^{\mathrm{H}} \Big(K \boldsymbol{X}^{-1} - \sum_{k \neq m} \boldsymbol{X}_k^{-1} \Big) \boldsymbol{H}_m \boldsymbol{T}_m. \tag{18}$$

According to (16) and (18), the updated precoder $\boldsymbol{T}_m^{(n+1)}$ follows from $\boldsymbol{T}_m^{(n)}$ via left-hand multiplication with another matrix. Thus, the rank of $\boldsymbol{T}_m^{(n+1)}$ cannot become larger than the rank of $\boldsymbol{T}_m^{(n)}$, so the initially chosen number of active streams cannot increase during the gradient ascent steps. Although we probe all possible initialization combinations, it is not clear whether the global optimum can be obtained by the gradient ascent or not. Because of the nonconvexity of the problem, only a local maximum in the vicinity of the initialization can be reached. However, simulations revealed that brute-forcing all possible allocations aided by the gradient projection algorithm outperforms the hitherto existing rate maximizing algorithms [10], [12]. Note that above algorithm can easily be extended to maximize the *weighted* sum rate.

V. SUCCESSIVE STREAM SELECTION WITH REDUCED COMPLEXITY

Even for the simple system configuration with only $K = 3$ two antenna users communicating with an $N = 4$ antenna base station, 22 different stream allocations have to be probed (see Table I). In order to reduce this high computational complexity, we propose a successive variant which adds one more stream to one of the K users in every iteration subject to the constraints that at most N streams are active, no user features more streams B_k than he has antennas r_k, and the updated stream allocation differs from that in the previous iteration for only one user. For example, assuming that the gradient ascent algorithm achieves a larger rate for the particular stream initialization 100 than for 010 and 001, we do not have to test the allocations 020, 002, and 011 when a second stream is added. Assuming furthermore, that 110 is the best initialization in the second iteration with $b = 2$, and 210 is optimum for $b = 3$, only the bold stream distributions in Table I have to be checked. Only 11 out of the 22 possible stream allocations have to be probed during the successive stream allocation halvening the computational complexity. If, for example, the sum rate for $b = 3$ active streams is not larger than for $b = 2$ streams, $b = 4$ active streams do not have to be probed at all.

VI. SIMULATION RESULTS

For a system configuration where $K = 3$ two-antenna users are served by a base station with $N = 4$ antennas, we simulated the sum rate achieved by the presented combinatorial approach in combination with the projected gradient algorithm (dashed curve), its drastically less complex successive version (solid curve), the geometric programming based approach in [10] (square marker), and the algorithm in [12] (circle marker). As a reference, we added the sum rate curve achieved via dirty paper coding (triangle-up marker). Due to the huge computational complexity of the algorithm in [10], we averaged over only 50 channel realizations. In the low power regime in Fig. 1, all linear schemes show almost the same performance, since only few of the four possible streams are activated. This behavior changes when going to a higher transmit power, see Fig. 2. First of all, we observe that the successive stream allocation algorithm presented in the previous section features almost no performance degradation with respect to the full featured combinatorial search, i.e., the dashed and the solid curve coincide. Despite the complexity reduction, there is almost no loss of rate. The geometric programming based algorithm in [10] shows a performance degradation for transmit powers larger than 20 dB. However, its computational load to compute the optimum precoders is very high. Finally, the approach in [12] also has to face problems in the high SNR regime. Since the users have all their streams activated (6 in sum) during the initialization in this algorithm, the optimum stream allocation apparently is not achieved by their gradient approach. Due to the similarity of the algorithm in [8] with the one in [10], we expect them to have equivalent performance.

VII. ALGORITHMIC IMPLEMENTATION

A pseudo-code implementation of the successive stream selection algorithm from Section V can be found in Algorithm 1. Note that $\mathcal{B}^{(m)}$ denotes the ordered set of active streams $B_1^{(m)}, \ldots, B_K^{(m)}$ in iteration m.

Algorithm 1 Successive Stream Selection Pseudo Code

$\mathcal{B}^{(0)} \leftarrow (0, \ldots, 0)$
for $m = 1$ to $\min(N, r)$ **do**
  $\mathcal{B}^{(m)}$ is feasible if
  $\cdot\ \exists i : B_i^{(m)} = B_i^{(m-1)} + 1$
  $\cdot\ B_k^{(m)} = B_k^{(m-1)}\ \forall k \neq i$
  $\cdot\ B_k^{(m)} \leq r_k\ \forall k$
  for all $\mathcal{B}^{(m)}$ that are feasible **do**
    Initialize precoders according to (15)
    Compute maximum sum rate via projected gradient algorithm using (16) - (18)
  end for
  Save best $\mathcal{B}^{(m)}$ with maximum sum rate $R_{\max}^{(m)}$
  Stop, if $R_{\max}^{(m)} \leq R_{\max}^{(m-1)}$
end for

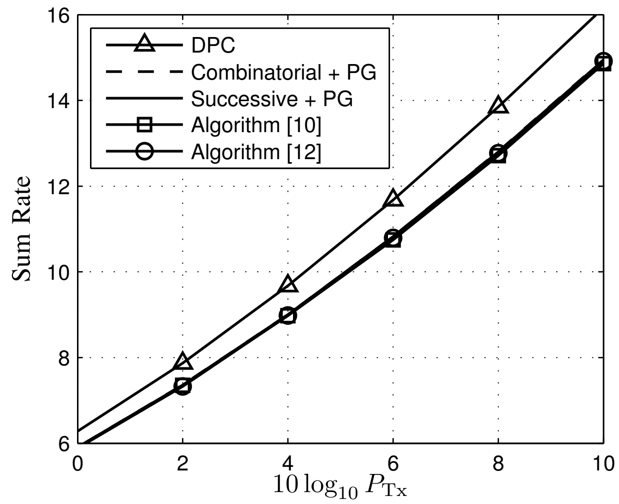

Fig. 1. Sum rate vs. transmit power for a system with $K = 3$ two-antenna users and a $N = 4$ antenna base station - low power regime.

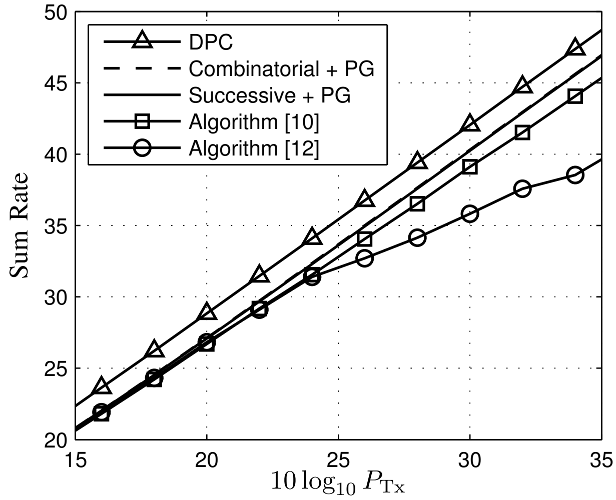

Fig. 2. Sum rate vs. transmit power for a system with $K = 3$ two-antenna users and a $N = 4$ antenna base station - high power regime.

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