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Embedding large graphs

The Bollobás-Komlós conjecture and beyond

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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Die Dissertation wurde am 28.4.2009 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 17.6.2009 angenommen.

Abstract

This thesis is concerned with embedding problems for large graphs under various types of degree conditions in the host graph.

A conjecture by Bollobás and Komlós states that every graph with sufficiently high minimum degree contains all spanning bounded-degree graphs with sublinear bandwidth.

We prove this conjecture, consider several variants as well as a bipartite analogue for sparse host graphs. In addition, we characterise graph classes embraced by these results and confirm a conjecture of Schelp on a Ramsey-type problem for trees. Our proofs are based on the regularity method.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Einbettung großer Graphen unter verschiedenen Bedingungen an die Knotengrade eines Trägergraphen.

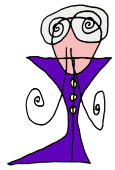
Eine Vermutung von Bollobás und Komlós besagt, dass jeder Graph mit hinreichend hohem Minimalgrad alle aufspannenden Graphen mit beschränktem Maximalgrad und sublinearer Bandweite enthält.

Die Arbeit liefert einen Beweis dieser Vermutung, betrachtet verschiedene Variationen der Fragestellung sowie ein Analogon für bipartite Graphen in dünnen Trägergraphen. Darüber hinaus wird eine Charakterisierung der von diesen Resultaten umfassten Graphenfamilien erstellt und eine Vermutung von Schelp bezüglich eines Ramsey-Problems für Bäume bestätigt. Die Beweise basieren auf der Regularitätsmethode.

Contents

Chapter 1

Introduction



How do different parameters of combinatorial structures influence each other? And what are the local or global properties they enforce? These are questions common to extremal combinatorics, Ramsey theory, and the theory of random structures. They form also the starting point of this thesis.

The problems we consider concern characteristics that ensure the existence of certain substructures. Such problems are called *embedding problems* because they can alternatively be seen as the task to embed a given structure into an object with these characteristics. Their study has led to a variety of results that are not only influential in many areas of mathematics and computer science, but often beautiful as well. On many occasions they formulate simple conditions for the embedding and connect seemingly unrelated parameters, thus exposing charming and unexpected mathematical structure.

In graph theory the classical Erdős–Stone theorem is an eminent result of this type. It considers the question how many edges are needed in a host graph G to ensure that a given graph H can be embedded into G. The surprising answer is that neither (directly) the number of vertices of H nor the number of its edges is crucial here, but its chromatic number.

The Erdős–Stone theorem applies to graphs H that are much smaller than the host graphs G into which they are embedded. A series of other classical results in extremal combinatorics (some of which we shall discuss in Section 1.1.1) concern the existence of so-called spanning graphs H, i.e., graphs that have as many vertices as G. An interesting aspect of this kind of results is that they assert a particular *global* structure (the spanning graph H) by way of certain *local* properties of the host graph G, for example its minimum degree. A conjecture of Bollobás and Komlós claims that the chromatic number is of similar importance for this type of embedding problems as for the Erdős–Stone theorem. More precisely, this conjecture states that the minimum degree needed to embed a very general class of spanning graphs H, again, solely depends on the chromatic number of H.

In this thesis we will prove the conjecture of Bollobás and Komlós, consider several variations of this result, and discuss a number of implications.

In nature the term regularity usually signifies that a structure or arrangement constitutes a particularly symmetrical or harmonious, often aesthetically pleasing pattern obeying certain rules.

The regularity lemma of Szemerédi [91] is a remarkable result on the asymptotic structure of graphs, which lies at the heart of the methodology used throughout this thesis, and describes a phenomenon of a similar character. Roughly speaking, this lemma states that *every* graph is built from relatively few pieces with a very regular structure and some negligible additional "noise". More precisely, the vertex set of a graph G can be partitioned into a constant number of equally sized classes such that the following holds for most pairs (A, B) of such classes: The edges running between the classes A and B form a bipartite graph with uniform, random-like edge distribution. Such a partition is also called a *regular partition* and the random-like bipartite graphs are the *regular pairs* of this partition. As we will see, a regular pair is an object well-suited for detecting certain structural features of the graph G that contains this regular pair.

At first sight this might seem somewhat contradictory: we want to guarantee certain structural properties but deal with "random-like" objects that, one might be tempted to think, could be anything. However, one important characteristic of random graphs is that they are rich in substructure: they usually contain many copies of all graphs H of constant size. And in some sense this property carries over to the regular pairs (A, B) provided by the regularity lemma. This motivates why partitions into regular pairs are extremely useful for verifying that some host graph G contains a given graph H as a subgraph, which means that H embeds into G.

Before stating our results in Section 1.2 we will give a short account of the relevant previous work in the area in Section 1.1. In Section 1.3 we will then sketch and explain some techniques that are used in many parts of this thesis and that pivot around the application of the regularity lemma.

1.1 History

In this section we provide some of the most important background material concerning extremal graph theory (see Section 1.1.1) and Ramsey theory[1] (see Section 1.1.2). This will enable us to put our results in context later. We distinguish between questions and results for dense graphs (Sections 1.1.1 and 1.1.2) and such for sparser graphs (Section 1.1.3) as the latter ones are of a slightly different flavour.

1.1.1 Extremal graph theory

A well-known result in computer science states that it is computationally difficult to decide whether a graph has a Hamilton cycle. This problem is NP-complete. Similar results can be formulated for the appearance of substructures other than cycles. It is therefore valuable to determine particularly simple conditions that are at least sufficient for the appearance of such substructures. Establishing such conditions is a central topic in extremal graph theory.

A number of results in this area concern the existence of subgraphs of different kinds under various vertex degree conditions. In other words, one is interested in embedding a given graph into a so-called host graph fulfilling some specified properties, e.g., having high average degree or high minimum degree.

Many early results in extremal graph theory study the appearance of a subgraph H in an n-vertex graph G_n where H is typically much smaller than G_n. We will briefly explain the most important of them in the following paragraph. Then we shift our attention to the analogous problem for growing subgraphs H_n, i.e., the case when the graph H_n that we want to embed into G_n depends on n.

[1] These areas are certainly not disjoint. In addition, it seems to the author, there is no obvious consensus on where their respective boundaries are nor whether any of the two is a (proper) subset of the other. But for the results discussed here the distinction seems rather natural.

The subgraph containment problem

The birth of extremal graph theory was marked by a result which is by now a classic: Turán's theorem [93] investigates how many edges a graph may have without containing a clique on r vertices. The contra-positive of this question asks what average degree in a graph enforces a complete graph K_r on r vertices as subgraph. This formulation shows that Turán's theorem can also be seen as a primal example for the type of embedding problems we described above. It states that K_r can be embedded into each n-vertex graph G with average degree strictly greater than $\frac{r-2}{r-1}n$.

Theorem 1.1 (Turán [93])**.** *Every graph G on n vertices with average degree $d(G) > \frac{r-2}{r-1}n$ contains a complete graph K_r on r vertices as a subgraph.* □

The question what happens if K_r is replaced by some different fixed graph H was considered by Erdős, Stone, and Simonovits [31, 33] and led to one of the many generalisations of this theorem. Their result, sometimes also called the fundamental theorem of extremal graph theory, roughly states the following. The average degree of a host graph needed to guarantee an embedding of a fixed graph H depends only on the *chromatic number* of H, i.e., the minimal number of colours needed to colour the vertices of H in such a way that no two neighbouring vertices receive the same colour (we will review all essential terminology in Chapter 2 or, alternatively, in the chapter where we need it).

Theorem 1.2 (Erdős, Stone [31])**.** *For every constant $\gamma > 0$ and every fixed graph H with chromatic number $r \geq 2$ there is a constant $n_0 \in \mathbb{N}$ such that every graph G with $n \geq n_0$ vertices and average degree $d(G) \geq \left(\frac{r-2}{r-1} + \gamma\right)n$ contains a copy of H as a subgraph.* □

Spanning subgraphs

When trying to translate Theorem 1.2 into a setting where the graphs H and G have the same number of vertices, two changes are obviously necessary.

First of all, the average degree condition must be replaced by one involving the minimum degree $\delta(G)$ of G, since we need to be able to control every single vertex of G. Also, for some graphs H it is clear that the lower bound has to be raised at least to $\delta(G) \geq \frac{r-1}{r}n$: simply consider the example where G is the complete r-partite graph with partition classes almost, but not exactly, of the same size (thus G has minimum degree almost $\frac{r-1}{r}n$) and let H be the union of $\lfloor n/r \rfloor$ vertex disjoint r-cliques (see Figure 1.1).

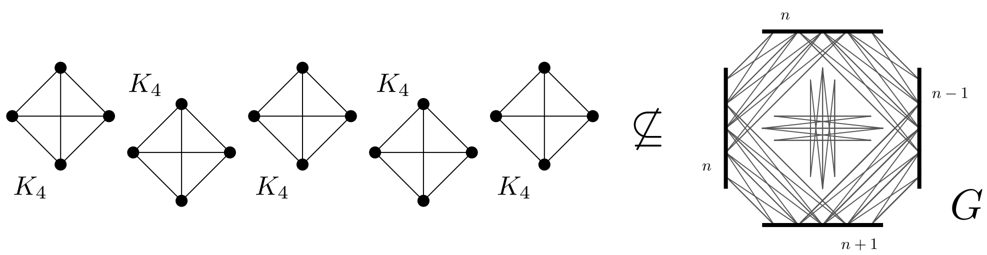


Figure 1.1: The complete 4-partite graph G on $4n$ vertices with partition classes of size $n-1$, n, n, $n+1$ cannot contain n vertex disjoint copies of K_4.

There are a number of results where a minimum degree of $\frac{r-1}{r}n$ is indeed sufficient to guarantee the existence of a certain spanning subgraph H. A well-known example is Dirac's theorem [29]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle, i.e., a cycle on n vertices.

Another early result on large r-chromatic subgraphs of graphs with minimum degree $\frac{r-1}{r}n$ follows from a theorem of Corrádi and Hajnal [26] that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles, a so-called (almost) spanning triangle factor.

Theorem 1.3 (Corrádi, Hajnal [26]). *Let G be a graph on n vertices with minimum degree $\delta(G) \geq \frac{2}{3}n$. Then G contains a triangle factor on $3\lfloor n/3 \rfloor$ vertices.* □

This was generalised by Hajnal and Szemerédi [48], who proved that every graph G with $\delta(G) \geq \frac{r-1}{r}n$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of order r.

Theorem 1.4 (Hajnal, Szemerédi [48]). *Let G be a graph on n vertices with minimum degree $\frac{r-1}{r}n$. Then G contains a K_r factor on $r\lfloor n/r \rfloor$ vertices.* □

A further extension of Theorem 1.3 was suggested by Pósa (see, e.g., [32]), who indicated how Dirac's Theorem concerning the existence of a Hamilton cycle and the spanning union of many disjoint triangles could actually fit into a common framework. He conjectured that at the same degree threshold $\delta(G) \geq \frac{2}{3}n$ where the theorem of Corrádi and Hajnal (Theorem 1.3) promises the existence of a spanning triangle factor, a graph G must indeed contain a much more rigid substructure – the square of a Hamilton cycle (where the r-th power of a graph is obtained by inserting an edge between every two vertices with distance at most r in the original graph, and the square of a graph is its second power, see Figure 1.2). Observe that the square of a cycle on $3t$ vertices contains t vertex disjoint triangles.

Conjecture 1.5 (Pósa). *An n-vertex graph G with minimum degree $\delta(G) \geq 2n/3$ contains the square of a Hamilton cycle.*

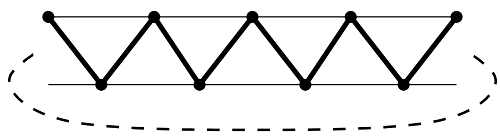


Figure 1.2: The square of a Hamilton cycle.

The following approximate version of this conjecture for the case $r = 3$ was proved by Fan and Kierstead [34]: For every constant $\gamma > 0$ there is a constant n_0 such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (2/3 + \gamma)n$ contains the square of a Hamilton cycle. Fan and Kierstead [35] later also gave a proof for the exact statement (i.e., with $\gamma = 0$ and $n_0 = 1$) for the square of a Hamilton path. In fact, they showed that for the existence of a square of a Hamilton path, the minimum degree condition $\delta(G) \geq (2n-1)/3$ is sufficient and sharp.

Theorem 1.6 (Fan, Kierstead [35]). *Every graph G on $n \geq n_0$ vertices with minimum degree $\delta(G) \geq (2n-1)/3$ contains the square of a Hamilton path.* □

Finally, Komlós, Sarközy, and Szemerédi [64] proved Pósa's conjecture (Conjecture 1.5) for large values of n.

A higher-chromatic analogue of Pósa's Conjecture was proposed by Seymour [90] who conjectured that the same statement remains true when "square" is replaced by "r-th power" and $\delta(G) \geq 2n/3$ by $\delta(G) \geq (r-1)n/r$ (this is also often called the Pósa–Seymour conjecture). Notice that this is, again, exactly the degree threshold where Theorem 1.4 asserts a spanning K_r-factor. Komlós, Sárközy, and Szemerédi [67] first proved an approximate version of this result. Later the same authors [64, 68] gave a proof of the Seymour conjecture for fixed r and sufficiently large graphs G.

Theorem 1.7 (Komlós, Sarközy, Szemerédi [64]). *For every integer $r \geq 1$ there is an integer n_0 such that every graph G on $n > n_0$ vertices with minimum degree $\delta(G) \geq (r-1)n/r$ contains the $(r-1)$-st power of a Hamilton cycle.*

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs (see the excellent survey [71] and the references therein). As an example we give the following theorem due to Kühn and Osthus [72] about spanning planar subgraphs.

Theorem 1.8 (Kühn, Osthus [72]). *There is n_0 such that every graph G with $n \geq n_0$ vertices and minimum degree at least $\frac{2}{3}n$ contains a spanning triangulation.* □

1.1.2 Ramsey theory

Most results in this thesis are *density* type results that, such as the theorems stated above, guarantee a certain substructure by way of conditions on the vertex degrees in a host graph. In Chapter 8 however we obtain a *Ramsey*-type theorem. Such results consider partitions of objects and usually guarantee that in an arbitrary partition one can find a given structure which is entirely in one of the partition classes.

As an example, the famous theorem of Ramsey [82] states that for any fixed integer r there is an integer m such that in any edge-colouring of K_m with red and green there is a monochromatic copy of K_r, that is, a copy entirely in red or in green. We write $R(r)$ for the smallest number m such that this is true.

Theorem 1.9 (Ramsey [82]). *For every integer r there is an m such that any colouring of the edges of K_m with two colours contains a monochromatic copy of K_r.* □

This concept (and result) naturally generalises to graphs F different from K_r and, more generally, to finite families of graphs $\mathcal{F}$: Let $R(\mathcal{F})$ denote the smallest integer m such that in any edge-colouring of K_m with red and green either there are copies of *all* members of $\mathcal{F}$ in red or, or copies of all of them in green. In this case we also write $K_m \to \mathcal{F}$, say that K_m is Ramsey for $\mathcal{F}$, and call $R(\mathcal{F})$ the Ramsey number of $\mathcal{F}$.

The investigation of Ramsey numbers has received much attention, and the structural theory around extensions and generalisations of Theorem 1.9 and its analogues for different structures has grown into a field on its own. For surveys on the topic we refer the reader to [44, 80].

Here we will concentrate on the case when $\mathcal{F}$ is a family of trees. Let $\mathcal{T}_t$ denote the class of trees on t vertices, and let $\mathcal{T}_t^\Delta$ be its restriction to trees of maximum degree at most Δ. Haxell, Łuczak, and Tingley [51] determined the asymptotically correct Ramsey number for trees with small maximum degree.

Theorem 1.10 (Haxell, Łuczak, Tingley [51])**.** *For all $\gamma > 0$ there is an integer t_0 and a positive η such that each tree $T \in \mathcal{T}_t^\Delta$ with bipartition classes of sizes $t_1 \le t_2$ such that $t_2 \ge t_0$ and $\eta t_2 \ge \Delta$ satisfies $R(T) \le (1+\gamma)t$, where $t := \max\{2t_2, 2t_1 + t_2\}$.* □

For general trees, Ajtai, Komlós, Simonovits, and Szemerédi [3] announced a proof of a long-standing conjecture by Erdős and Sós, which implies that $K_{2t-2} \to \mathcal{T}_t$ for large even t and $K_{2t-3} \to \mathcal{T}_t$ for large odd t. This is best possible. For the case of odd t this is also a consequence of a theorem by Zhao [94] concerning a conjecture of Loebl (see also [52]).

Theorem 1.11. *For t sufficiently large, $K_{2t-2} \to \mathcal{T}_t$.* □

Let us now move on to a modification of the original Ramsey problem. The graph $K_{R(\mathcal{F})}$ is obviously a Ramsey graph for $\mathcal{F}$ with as few vertices as possible. However, one may still ask whether there exist graphs with fewer edges which are Ramsey for $\mathcal{F}$. The minimal number of such edges is also called *size Ramsey number* and denoted by $R_s(\mathcal{F})$. Trivially $R_s(\mathcal{F}) \le \binom{R(\mathcal{F})}{2}$, but it turns out that this inequality is often far from tight. For the class of bounded-degree trees $\mathcal{T}_t^\Delta$, for example, a result of Friedman and Pippenger [36] implies a linear size Ramsey number of order $R_s(\mathcal{T}_t^\Delta) = \mathcal{O}(\Delta^4 t)$. Haxell and Kohayakawa [50] improved on this bound and replaced it with $\mathcal{O}(\Delta t)$.

More generally, Beck [14] asked whether the size Ramsey number of bounded-degree graphs is always linear. Rödl and Szemerédi [88] showed that this is not true. They constructed n-vertex graphs H with maximum degree 3 and with size Ramsey number at least $n \log^c n$ for some positive constant c. A non-trivial upper bound, on the other hand, was established by Kohayakawa, Rödl, Schacht, and Szemerédi [61]. Their result shows that the class of n-vertex graphs with maximum degree Δ has size Ramsey number at most $O(n^{2-1/\Delta} \log^{1/\Delta} n)$.

1.1.3 Sparse universal graphs

The graph embedding problems we discussed so far all fit into a common framework. Given a class of graphs $\mathcal{H}$, we are interested in identifying host graphs G which have the property that they contain all graphs from $\mathcal{H}$ as subgraphs. Such graphs G are called *universal* for $\mathcal{H}$.

In this language, most results presented in Section 1.1.1 guarantee universality via different degree conditions. However, they require the host graph to be *dense* and in fact often even *everywhere dense*, inasmuch as every vertex of G has at least a well-specified positive fraction of all other vertices as neighbours. This raises the question which sparser structures can serve as host graphs.

Sparse random graphs have proven to provide a suitable setting for constructing universal graphs G with few edges (although for several concrete families the best known constructions of universal graphs can be obtained without randomness, we will see below). Indeed, Alon, Capalbo, Kohayakawa, Rödl and Ruciński [8] showed that a sparse random balanced bipartite graph $\mathcal{G}_{n,n,p}$ (on $2n$ vertices where every edge running between the two partition classes, which are of size n each, exists independently with probability p) is universal for the class of spanning balanced bipartite bounded-degree graphs on $2n$ vertices if $p = p(n) = o(1)$ is sufficiently large. For the exact formulation, let $\mathcal{H}(n,n,\Delta)$ denote the class of all balanced bipartite graphs on $2n$ vertices with maximum degree Δ.

Theorem 1.12 (Alon, Capalbo, Kohayakawa, Rödl, Ruciński, Szemerédi [8])**.** *There is a positive c such that for all $\Delta \ge 2$ the random balanced bipartite graph $\mathcal{G}_{n,n,p}$ with $p \ge c(\log n/n)^{1/2\Delta}$ is asymptotically almost surely universal for $\mathcal{H}(n,n,\Delta)$.* □

In addition the same authors demonstrated that even more can be said when one considers copies of smaller bipartite graphs. Assume that $\mathcal{H}(n,n,\Delta)$ in the theorem above is replaced by the class $\mathcal{H}(\lceil\eta n\rceil,\lceil\eta n\rceil,\Delta)$ of all balanced bipartite graphs on $2\lceil\eta n\rceil$ vertices with maximum degree Δ for a small constant η. Then the same random graph $\mathcal{G}_{n,n,p}$ with p as in Theorem 1.12 is *fault-tolerant* in the sense that even the deletion of a substantial proportion of its edges does not destroy universality for $\mathcal{H}(\lceil\eta n\rceil,\lceil\eta n\rceil,\Delta)$. In other words, $\mathcal{G}_{n,n,p}$ contains *many* copies of all graphs from $\mathcal{H}(\lceil\eta n\rceil,\lceil\eta n\rceil,\Delta)$ *everywhere*.

Theorem 1.13 (Alon, Capalbo, Kohayakawa, Rödl, Ruciński, Szemerédi [8])**.** *For every $\gamma\in(0,1]$ and every integer $\Delta\geq 2$ there are constants $\eta>0$ and $c>0$ such that for $p\geq c(\log n/n)^{1/2\Delta}$ the random balanced bipartite graph $\Gamma=\mathcal{G}_{n,n,p}$ asymptotically almost surely has the following property. Let G be any subgraph of Γ such that $|E(G)|\geq\gamma|E(\Gamma)|$. Then G is universal for $\mathcal{H}(\lceil\eta n\rceil,\lceil\eta n\rceil,\Delta)$.* □

This result can also be interpreted in the context of Ramsey-type questions. It states that any colouring of the edges of $\mathcal{G}_{n,n,p}$ with $\lfloor 1/\gamma\rfloor$ colours has a colour class that is universal for $\mathcal{H}(\lceil\eta n\rceil,\lceil\eta n\rceil,\Delta)$.

Turning to non-bipartite graphs Alon and Capalbo [6, 7] gave explicit constructions of graphs that are universal for the class $\mathcal{H}(n,\Delta)$ of all graphs with n vertices and maximum degree Δ. They improved on several other constructions of $\mathcal{H}(n,\Delta)$-universal graphs (see [8, 9]). Universality of sparse *random* graphs for $\mathcal{H}(n,\Delta)$ was studied in [8, 27].

Theorem 1.14 (Alon, Capalbo [7])**.** *For every $\Delta\geq 3$ and every integer n there is a constant c and a graph G with at most $cn^{2-2/\Delta}$ edges that is universal for $\mathcal{H}(n,\Delta)$.* □

With regard to Ramsey-type questions Kohayakawa, Rödl, Schacht, and Szemerédi [61] recently obtained a result of similar flavour as Theorem 1.13 for $\mathcal{G}_{n,p}$ (on vertex set $\{1,\dots,n\}$ where every edge exists independently with some sufficiently high probability p). Their theorem states that in every 2-colouring of the edges of $\mathcal{G}_{n,p}$ there is a colour class that is universal for $\mathcal{H}(\lceil\eta n\rceil,\Delta)$.

Theorem 1.15 (Kohayakawa, Rödl, Schacht, Szemerédi [61])**.** *For every $\Delta\geq 2$ there exist constants $\eta>0$ and $c>0$ such that for $p\geq c(\log n/n)^{1/\Delta}$ the random graph $\mathcal{G}_{n,p}$ asymptotically almost surely is Ramsey for $\mathcal{H}(\lceil\eta n\rceil,\Delta)$.* □

This theorem implies the bound on the size Ramsey number of bounded-degree graphs mentioned in the previous section. Observe that the bound on p in this theorem is better than the one in Theorems 1.12 and 1.13.

1.2 Results

In this section we present the main results contained in this thesis and fit them into the picture drawn in the previous section. We will describe various embedding results concerning large subgraphs in dense graphs (Section 1.2.1) as well as in sparse random graphs (Section 1.2.4). In addition we will discuss a characterisation of the graph classes covered by these results (Section 1.2.2) and prove a Ramsey-type result for trees (Section 1.2.3). All results will be repeated in the respective chapters that cover their proofs.

1.2.1 Spanning subgraphs of sublinear bandwidth

In the beginning of Section 1.1.1 we considered the question what minimum degree $\delta(G)$ of an n-vertex graph G forces a particular graph H to be a spanning subgraph of G. We explained that we certainly need to have $\delta(G) \geq \frac{r-1}{r}n$ and saw different concrete examples where this is already sufficient.

In an attempt to move away from results that concern only graphs H with a special, rigid structure (such as powers of Hamilton cycles), a naïve conjecture could be that $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ suffices to guarantee that G contains a spanning copy of any r-chromatic graph H of bounded maximum degree. However, the following simple example shows that this fails in general. Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every vertex set X of size $(1/2 - \gamma)n$ has more than $2\gamma n$ neighbours outside X. In short, the obstruction here is that H has good expansion properties.

The conjecture of Bollobás and Komlós

One way to rule out such expansion properties for H is to restrict the *bandwidth* of H (as we will explain more in detail in Section 1.2.2 and Chapter 4). A graph G has bandwidth at most b, if there exists a labelling of the vertices by numbers $1, \dots, n$, such that for every edge ij of the graph we have $|i - j| \leq b$. *Bollobás and Komlós* [62, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth limited by $o(n)$, can be embedded into any graph G on n vertices with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. In Chapter 5 we provide a proof of this conjecture.

Theorem (Theorem 5.1)**.** *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, if H is an r-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$, then G contains a copy of H.*

Note that for some results mentioned in the last section, the additional term γn in the minimum degree condition is not needed (or can be replaced by a smaller term). In the general setting, however, this is not possible: Abbasi [2] showed that if $\gamma \to 0$ and $\Delta \to \infty$, then β must tend to 0 in Theorem 5.1.

Building on the techniques developed for obtaining this result, we will consider different ways to weaken the minimum degree assumption in Theorem 5.1, which we describe in the following paragraphs.

Bipartite host graphs

Theorem 5.1 implies in particular that for any $\gamma > 0$, every balanced bipartite graph H on $2n$ vertices with bounded degree and sublinear bandwidth appears as a subgraph of any $2n$-vertex graph G with minimum degree $(1 + \gamma)n$, provided that n is sufficiently large.

For Hamilton cycles H it is known that this minimum degree condition can be replaced by a much weaker condition if we have additional structural information about the host graph G: If G is a balanced bipartite graph then $\delta(G) \geq n/2$ suffices to find a Hamilton cycle in G (see [78]). In Chapter 6 we show that a slightly bigger minimum degree threshold already forces all balanced bipartite graphs H with bounded degree and sublinear bandwidth.

Theorem (Theorem 6.2)**.** *For every γ and Δ there is a positive constant β and an integer n_0 such that for all $n \geq n_0$ the following holds. Let G and H be balanced bipartite graphs on $2n$ vertices such that G has minimum degree $\delta(G) \geq (\frac{1}{2} + \gamma)n$ and H has maximum degree Δ and bandwidth at most βn. Then G contains a copy of H.*

Results of a similar nature have recently been established by Zhao [95], and by Hladký and Schacht [53] who considered the special case of coverings of G with disjoint copies of complete bipartite graphs.

Ore conditions

Recall that Dirac's theorem asserts the existence of a Hamilton cycle in n-vertex graphs of minimum degree at least $n/2$. Ore [81] realised that this conclusion remains true even under a weaker condition. He showed that it is not necessary to control the degree of every vertex independently but that it suffices to guarantee a high degree sum for pairs of non-adjacent vertices.

Theorem 1.16 (Ore [81])**.** *Every n-vertex graph $G = (V, E)$ with $\deg(u) + \deg(v) \geq n$ for all $xy \notin E$ contains a Hamilton cycle.* □

Similarly one might ask whether it is possible to replace the minimum degree condition in Theorem 5.1 by an *Ore-type condition.* In Chapter 7 we show that this is indeed the case for 3-colourable graphs G. We obtain the following theorem.

Theorem (Theorem 7.2)**.** *For all $\Delta, \gamma > 0$ there are $\beta, n_0 > 0$ such that for all $n \geq n_0$ the following holds. Let $G = (V, E)$ and H be n-vertex graphs such that H is 3-colourable, has maximum degree $\Delta(H) \leq \Delta$ and bandwidth at most βn, and G satisfies $\deg(u) + \deg(v) \geq (\frac{4}{3} + \gamma)n$ for all $xy \notin E$. Then G contains a copy of H.*

The proof of this theorem uses a recent result due to Kierstead and Kostochka [56] that establishes an Ore-type version of the Hajnal–Szemerédi theorem (Theorem 1.4).

1.2.2 Characterising sublinear bandwidth

In the preceding paragraphs we discussed a number of results concerning the embedding of graphs H with sublinear (or small linear) bandwidth. By definition such graphs H admit a vertex ordering in which the neighbourhood of every vertex v is restricted to some few vertices which immediately precede or follow v. This is beneficial in (and heavily used by) the proofs of the aforementioned embedding results, insofar as they construct the required embedding sequentially, following such an ordering. In this sense the bandwidth constraint helps to "localise" the dependencies one needs to take into account during the embedding.

Recall also that the bandwidth constraint was introduced because results such as Theorem 5.1 do not hold without this extra condition. However, one may wonder whether it cannot be replaced by other constraints. In addition, in view of our embedding results, it is natural to ask which graph classes actually have sublinear bandwidth.

We will pursue these questions in Chapter 4 and show, for example, that planar graphs of bounded maximum degree have sublinear bandwidth.

Theorem (Theorem 4.1)**.** *Suppose $\Delta \geq 4$. Let G be a planar graph on n vertices with maximum degree at most Δ. Then the bandwidth of G is bounded from above by $15n/\log_\Delta(n)$.*

Similar results can be formulated for graphs of any fixed genus (i.e., graphs that can be drawn without crossings on other surfaces than the plane) and, more generally, for any graph class defined by a fixed set of forbidden minors (see Section 4.2.1).

The following example, that also showed that Theorem 5.1 becomes false if not restricted to graphs H of sublinear bandwidth, demonstrates that not all graphs with small maximum degree have sublinear bandwidth: with high probability a random bipartite graph G on n vertices with bounded maximum degree cannot have small bandwidth since in any linear ordering of its vertices there will be an edge between the first $n/4$ and the last $n/4$ vertices in this ordering. The reason for this obstacle is that G has good expansion properties.

This implies that graphs with sublinear bandwidth cannot exhibit good expansion properties. One may ask whether the converse is also true, i.e., whether the absence of large expanding subgraphs in bounded-degree graphs must lead to small bandwidth.

In fact, Theorem 4.1 is a consequence of the proof of the following result that characterises sublinear bandwidth in various ways, among them expansion properties. This result relates different graph parameters (whose definition we defer to Chapter 2) and proves that the concepts of sublinear bandwidth $\mathrm{bw}(H)$, sublinear treewidth (denoted by $\mathrm{tw}(H)$), bad expansion properties (which are measured by the so-called non-expansion $\mathrm{b}_\varepsilon(H)$), and sublinear separators (whose size is quantified by $\mathrm{s}(H)$) are equivalent for graphs of bounded maximum degree. A class of graphs is hereditary if it is closed under taking induced subgraphs.

Theorem (Theorem 4.6)**.** *Let Δ be an arbitrary but fixed positive integer and consider a hereditary class $\mathcal{C}$ of graphs such that all graphs in $\mathcal{C}$ have maximum degree at most Δ. Let $\mathcal{C}_n$ be the set of those graphs in $\mathcal{C}$ with n vertices. Then the following four properties are equivalent:*

(*1*) *For all $\beta_1 > 0$ there is n_1 such that $\mathrm{tw}(H) \le \beta_1 n$ for all $H \in \mathcal{C}_n$ with $n \ge n_1$.*
(*2*) *For all $\beta_2 > 0$ there is n_2 such that $\mathrm{bw}(H) \le \beta_2 n$ for all $H \in \mathcal{C}_n$ with $n \ge n_2$.*
(*3*) *For all $\beta_3, \varepsilon > 0$ there is n_3 such that $\mathrm{b}_\varepsilon(H) \le \beta_3 n$ for all $H \in \mathcal{C}_n$ with $n \ge n_3$.*
(*4*) *For all $\beta_4 > 0$ there is n_4 such that $\mathrm{s}(H) \le \beta_4 n$ for all $H \in \mathcal{C}_n$ with $n \ge n_4$.*

Consequently the bandwidth constraint in our embedding results is in some sense precisely the “right” constraint: The embedding results become false if this constraint is omitted because we cannot hope to embed graphs with good expansion properties, and restricting the bandwidth rules out exactly these graphs.

1.2.3 Trees in tripartite graphs

The size Ramsey number of a class $\mathcal{F}$ of graphs asks for the minimal number of edges in a graph that is Ramsey for $\mathcal{F}$. A question of similar flavour is what happens when we do not only want to find Ramsey graphs for $\mathcal{F}$ with few *edges*, but require in addition that the number m of *vertices* is very close to $R(\mathcal{F})$. This question has two aspects: a quantitative one (i.e., how many edges can be deleted from the complete graph K_m so that the remaining graph is still Ramsey) and a structural one (i.e., what is the structure of edges which may be deleted). Questions of a similar nature were explored by Gyárfás, Sárközy, and Schelp [47] when $\mathcal{F}$ consists of an odd cycle and by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [46] when $\mathcal{F}$ consists of a path. Our focus here is on the case when $\mathcal{F}$ is a class of trees.

Schelp [89] posed the following Ramsey-type conjecture about trees in tripartite graphs: For n sufficiently large the tripartite graph $K_{n,n,n}$ is Ramsey for the class $\mathcal{T}_t^\Delta$ of trees on $t \le (3-\varepsilon)n/2$ vertices with maximum degree at most Δ for constant Δ. The conjecture thus asserts that we can delete three cliques of size $m/3$ from the graph K_m with m only slightly larger than $R(\mathcal{T}_t^\Delta)$ while maintaining the Ramsey property. In addition Schelp asked whether the same remains true when the constant maximum degree bound in the conjecture above is replaced by $\Delta \leq \frac{2}{3}t$ (which is easily seen to be best possible). In Chapter 8 we prove a result that is situated in-between these two cases, solving the problem for trees of maximum degree n^α for some small α and hence, in particular, answering the first conjecture above.

Theorem (Theorem 8.1)**.** *For all $\varepsilon > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$K_{n,n,n} \to \mathcal{T}_t^\Delta,$$

with $\Delta \leq n^\alpha$ and $t \leq (3-\varepsilon)n/2$.

As for most embedding results in this thesis, we use the regularity lemma to establish this theorem. Due to the nature of the methods related to this lemma it follows that Theorem 8.1 remains true when $K_{n,n,n}$ is replaced by a much sparser graph: For any fixed $\mu \in (0,1]$ a random subgraph of $K_{n,n,n}$ with edge probability μ allows for the same conclusion, as long as n is sufficiently large (see Section 8.7).

1.2.4 Sparse random graphs

Another way to formulate Theorem 5.1 (see page 8) is to say that the complete graph K_n is *robust* with respect to the containment of spanning bounded-degree subgraphs with sublinear bandwidth in the following sense: An adversary may arbitrarily delete edges of the graph K_n such that each vertex looses at most an $\frac{1}{r} - \gamma$ fraction of its incident edges. Then the resulting graph still contains every r-chromatic graph with bounded maximum degree and sublinear bandwidth as a subgraph. This raises the question whether the same remains true when K_n is replaced by a much sparser (but in some sense regular) graph Γ, i.e. by an n-vertex graph with e edges such that e/n^2 tends to 0 as n goes to infinity. Clearly, K_n cannot be replaced by just any sparse graph Γ and hence it seems natural to ask whether a random graph $G_{n,p}$ can play its rôle.

In Chapter 9 we will prove that this is true when we are interested in the embedding of *almost spanning* bipartite graphs H.

Theorem (Theorem 9.1)**.** *For every $\eta, \gamma > 0$, $\Delta > 1$ there exist positive constants β and c such that the following holds asymptotically almost surely for $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$. Every spanning subgraph $G = (V,E)$ of Γ with $\deg_G(v) \geq (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$ contains a copy of every graph H on $m = (1-\eta)n$ vertices with maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$.*

Observe that this result uses the same bound on p as Theorem 1.15. In fact we utilise methods developed in [61] for proving this theorem and combine them with suitably adapted methods for the proof of Theorem 1.13 in [8]. Note that in contrast to both of these theorems we embed graphs H that are almost as big as the host graph. The price we have to pay, however, is that we need to restrict the number of deleted edges at each vertex separately.

1.3 Techniques

In this section we will briefly describe some of the central ideas underlying our methods for proving the embedding results given in the previous section. We have already indicated that many of our proofs are based on the regularity lemma and that a regular partition is suitable for the embedding of graphs. In these proofs we will often also use the so-called blow-up lemma, a tool for embedding graphs into regular pairs (which we explain more in detail in Section 1.3.1). In order to successfully apply the regularity lemma and the blow-up lemma a sequence of preparatory and intermediate steps is necessary which usually require a fair amount of work. Our goal in this section is to discuss some of the necessary steps while omitting most technical details. We start with a more detailed account of regular partitions.

1.3.1 Regular partitions and the blow-up lemma

As explained earlier, the regularity lemma guarantees the existence of ε-regular partitions of graphs. Let us shed some more light on this concept and provide a few definitions that will be necessary for the following considerations.

The *density* $d(A,B)$ of a bipartite graph (A,B) with partition classes A and B is the number of its edges divided by the number of all possible edges $|A||B|$. The pair (A,B) is called *ε-regular* if all subsets A' of A and all subsets B' of B have the property that their density $d(A',B')$ differs from the density $d(A,B)$ by at most ε. An *ε-regular partition* of a graph $G=(V,E)$ is a partition of $V \setminus V_0$ with $|V_0| \le \varepsilon|V|$ into vertex sets $V_1,\dots,V_k$, so-called *clusters*, such that (V_i,V_j) forms an ε-regular pair for all but at most an ε-fraction of all possible index pairs i,j. The regularity lemma then guarantees the existence of an ε-regular partition of G into k clusters where k only depends on ε but not on the size of G.

While an empty bipartite graph is ε-regular for all ε, it will of course not be useful in applications concerning the embedding of a graph. For these applications we need regular pairs that have many edges (*dense* pairs). The information which pairs in a regular partition are dense is captured in the so-called *reduced graph* R of the partition, which contains a vertex for each cluster and an edge for every ε-regular pair with density at least d for some small constant d (much bigger than ε).

At the very beginning of this chapter we indicated that regular pairs are "random-like" in the sense that they contain any (bipartite) graph of fixed size. What is more surprising is that we can even guarantee much bigger structures in dense regular pairs (A,B). The so-called blow-up lemma [65]—the other crucial ingredient to many proofs in this thesis besides the regularity lemma—implies that (A,B) is almost as rich in spanning bounded-degree subgraphs as the corresponding complete bipartite graph if (A,B) satisfies the additional property that every vertex has a d-fraction of all possible neighbours. Such pairs are called *(ε,d)-super-regular*. This additional condition is no severe restriction as any regular pair can easily be transformed into a super-regular pair by omitting a few vertices (see Proposition 3.6).

More generally, the blow-up lemma allows for the embedding of spanning graphs into *systems* of super-regular pairs. To make this more precise, assume that we find, say, a triangle in the reduced graph. This corresponds to a triple of clusters in G such that each cluster pair is a dense regular pair. As explained we can remove a small number of vertices to obtain super-regular pairs. Assume further that we are given a 3-colourable graph H whose colour classes smaller than the clusters. If, in addition, the maximum degree of H is bounded by a constant Δ, then the blow-up lemma guarantees a copy of H in the subgraph of G induced by the three clusters in the triangle (if ε is sufficiently small). This method naturally generalises to graphs H with higher chromatic number: If we want to embed an r-colourable graph H into G, then we can use a complete graph K_r on r vertices in the reduced graph.

1.3.2 Connected factors and bandwidth

With the strategy described in the previous section we can embed small but linear-sized graphs H into a host graph $G = (V, E)$ by finding a single dense super-regular "spot" in a regular partition of G. If we find many such spots, then we can use the blow-up lemma for each of them separately. Assume, for example, that the reduced graph has a perfect matching M and that the graph H consists of small vertex-disjoint bipartite graphs, say, copies of $K_{23,23}$. Then we could map the first copies of $K_{23,23}$ to the first edge of M, the next few copies to the next edge of M, and so on, and in this way embed an almost spanning graph H on as many as $(1 - \varepsilon)|V|$ vertices (we might not be able to use the vertices in the exceptional set V_0).

If we want to embed (almost) spanning *connected* graphs H, then using *disconnected* matching edges in the reduced graph obviously no longer suffices. But if we manage to decompose H into small segments by removing some vertices (this is for example possible if H has small bandwidth), then we can still use a similar strategy. Consider again the example of a bipartite graph H and a matching M with m edges in the reduced graph. We cut H into m segments and assign each segment to an edge of M. Then we need to show that the segments can be connected to form a copy of H. This, however, is possible if M is a *connected matching*, that is, if each pair of M-edges is connected by a path P in R. Roughly speaking, for two segments H_1 and H_2 assigned to edges e_1 and e_2 of M that need to be connected we can simply do the following. We use a few vertices of the segments H_1 and H_2, which we call *connecting vertices* and assign them to the clusters of the path P (but do not yet embed them). In doing so, we make sure that neighbouring vertices of H are assigned to neighbouring clusters on P (this is again possible if H has small bandwidth).

In order to obtain the embedding of H into G we will then, in a first step, embed the few connecting vertices, using a greedy strategy. Then, in a second step, we embed the remaining vertices of each segment with the help of the blow-up lemma.

The possibility of using connected matchings in this way was pointed out by Łuczak [76]. If H is not bipartite but 3-colourable, then the concept of a connected matching has to be replaced by a *connected triangle factor*. A connected triangle factor is a covering of the reduced graph R with triangles that are connected by sequences of triangles where each pair of consecutive triangles in such a sequence shares an edge. This was used by Komlós, Sarközy, and Szemerédi [64] to prove Pósa's conjecture for large graphs and can easily be generalised to higher-chromatic H.

In our proofs we usually apply the blow-up lemma in the way just explained (this method is captured in Lemma 3.12). The blow-up lemma only treats graphs H with constant maximum degree. This entails the constant maximum degree bounds in most of our results. An exception is Theorem 8.1 where we embed trees, which are graphs of a structure simple enough so that we can develop an alternative embedding lemma (Lemma 8.12) dealing with growing maximum degrees.

1.3.3 Regularity and inheritance of graph properties

The question now is how we guarantee the existence of the substructures exploited in the previous section, such as connected matchings or connected triangle factors, in the reduced graph. One of the important observations is that the reduced graph $R = (V_R, E_R)$ "inherits" certain properties from the host graph G. For example, it is well known that minimum degree conditions translate from G to R in this way: if G has minimum degree $\alpha|V|$ then R

has minimum degree at least $\alpha'|V_R|$ with α' only slightly smaller than α (see Lemma 3.4). Accordingly, if we were interested in showing that a particular, say, 2-colourable graph H is contained in every graph G with minimum degree $\alpha|V|$, then we could alternatively prove that every graph $R = (V_R, E_R)$ with minimum degree $\alpha'|V_R|$ contains *some* large connected matching. Applying this result to the reduced graph R of G would then help to show the former result.

In this way, we reduced the original embedding problem to a simpler one. This can indeed be considered as one of the main strengths of the regularity method for embedding problems: Instead of looking for a particular substructure in a host graph G we search for a member of a more general class of substructures in the reduced graph corresponding to G.

1.3.4 Balancing

There is a problem we have been ignoring so far. When assigning the vertices of one segment H_i of H to an edge (or a triangle) in the reduced graph R, it might happen that the clusters in this edge receive very different amounts of vertices because H_i has colour classes of very different sizes (i.e., it is a very unbalanced graph). If we want to embed spanning or almost spanning graphs, however, this is fatal: As all clusters have the same size, we will over- or under-fill them. To solve this problem, we will refine the partition of H into segments that we used so far and cut H into much smaller pieces. We will then distribute these small pieces among the super-regular spots in the reduced graph, guaranteeing a more balanced assignment of vertices to clusters. For approaching this last step we use one of the following methods, which we explain for the example of a bipartite graph H where we have a connected matching M in the reduced graph.

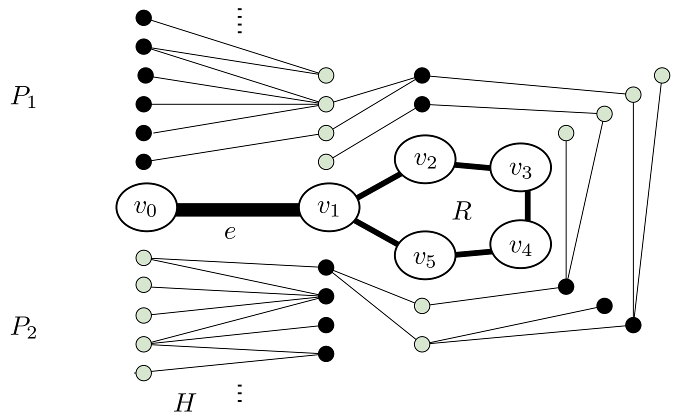


Figure 1.3: Balancing by mapping some vertices between two neighbouring pieces P_1 and P_2 of H to the vertices of a 5-cycle v_1, v_2, v_3, v_4, v_5 in the reduced graph R while still mapping most vertices of P_1 and P_2 to the edge $e = v_0v_1$.

One idea is to still assign all pieces of one segment to the same edge e of M but to change the "orientation" of the colour classes in the pieces from time to time. It turns out that this can be done with the help of an odd cycle in the reduced graph by assigning some vertices of H to the clusters of this odd cycle, as is illustrated in Figure 1.3 for the case when H is a tree. (If H is not bipartite but r-chromatic, then we can use a copy of K_{r+1} instead of an odd cycle, as is shown in Chapter 5.)

The application of this procedure, however, requires additional structure (an odd cycle in the reduced graph), which we cannot always guarantee (as in Chapter 6, where we consider bipartite graphs and thus get bipartite reduced graphs). But if the graph H is itself balanced (which does not prevent the segments from being unbalanced), then we can still use the following observation. A random assignment of the small pieces to the edges e of M will ensure that for each e the pieces assigned to e form roughly a balanced subgraph of H. So, using the connecting strategy described above, we are done.

Yet another possibility is to replace the (balanced bipartite) matching M by a structure that represents the ratio of the colour classes of H in a better way. This is performed in the proof of Theorem 8.1 in Chapter 8, where we substitute the edges of M by unbalanced bipartite graphs $K_{1,r}$.

1.3.5 Adjusting regular partitions

The balancing method described above provided us with a homomorphism from H to the reduced graph that maps roughly the same number of vertices to each cluster—but not quite. While this is no problem if the graph H is slightly smaller than G (recall that all clusters have the same size), we might get into trouble if H and G are of the same size because, obviously, the number of vertices we can embed into a cluster is limited by its size.

For this reason we might have to modify the partition of G slightly and shift some vertices to different clusters until the cluster sizes match the number of assigned H-vertices exactly. (In addition we will have to integrate the vertices in the extra set V_0 into the clusters, but this we ignore here.) This shifting, however, needs to be done rather carefully: While dense regularity is robust towards such small alterations, super-regularity is not. Suppose that we want to find a vertex u that we can shift from a cluster $V_{i'}$ to a cluster V_i, and let (V_i, V_j) form a *super-regular* pair. To preserve the super-regularity of (V_i, V_j) we need to ensure that u has sufficiently many neighbours in V_j. For this it would help if $(V_{i'}, V_j)$ forms a dense *regular* pair, since then most vertices in $V_{i'}$ have many neighbours in V_j and could thus serve as candidates for u.

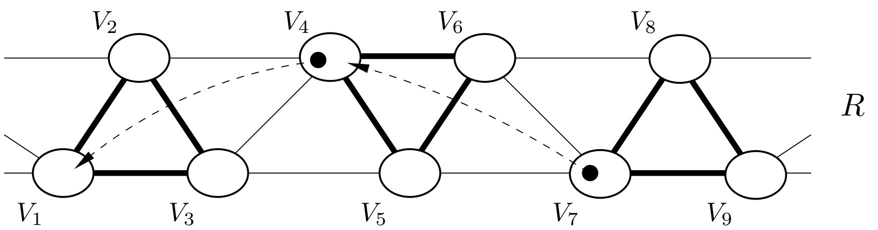


Figure 1.4: Moving a vertex from V_7 to V_4 and then one from V_4 to V_1 thus decreasing the size of V_7 and increasing the size of V_1. All continuous lines represent regular pairs in the reduced graph, super-regular pairs are denoted by thicker lines.

Hence, besides the super-regular spots needed for the application of the blow-up lemma, we would like our reduced graph to have some rich and well-connected structure of regular pairs. As an example, consider the reduced graph R depicted in Figure 1.4 where the thicker lines represent super-regular pairs. If we wanted to reduce the size of cluster V_7 and increase the size of cluster V_1 in R then we could use the strategy outlined above: We find a vertex in V_7 with many neighbours in V_5 and V_6 (which exists because (V_7, V_5) and (V_7, V_6) are regular pairs) and move it to V_4, thus maintaining super-regularity of (V_4, V_5) and (V_4, V_6). Then, we find a (presumably different) vertex in V_4 with many neighbours in V_2 and V_3 and move it to V_1.

1.3.6 A lemma for H and a lemma for G

In our proofs we will often use a pair of lemmas that 'prepare' the graphs H and G for the embedding, which we will call *lemma for H* and *lemma for G*, respectively. The lemma for G will typically construct a regular partition of the graph G (using the regularity lemma as described in Section 1.3.1), guarantee the necessary structural properties of this partition (as explained in Sections 1.3.2 and 1.3.3), and adjust the partition to the needs of H (see Section 1.3.5). The lemma for H, on the other hand, provides a partition of H that is compatible to this partition of G in view of the embedding task (using the bandwidth constraint and the balancing method outlined in Section 1.3.4).

After these preparations we can then apply regularity-based embedding lemmas (such as the blow-up lemma) to embed H into G.

1.3.7 Regularity in sparse graphs

For sparse graphs (such as the graph G in Theorem 9.1) the conclusion of the regularity lemma becomes trivial—any pair with density at most ε is ε-regular. This suggests a modification of this concept for sparse graphs. The idea is to consider a "scaled" version of regularity and measure all densities (and hence regularity) *relatively* to the overall density of the graph under study (see Section 3.4). A regularity lemma for this sparse version of regularity was observed independently by Kohayakawa and Rödl (see [57, 59]).

Moreover, the development of a corresponding regularity method suitable for dealing with embedding problems in sparse graphs recently gained much attention (see, e.g., the survey by Gerke and Steger [41]). Even so, much less is known compared to dense graphs. In particular embedding lemmas (such as the blow-up lemma) are not available in full generality. When the host graph G is a subgraph of a sparse random graph, however, a variety of embedding-type results were obtained. In Chapter 9 we illustrate that in this case the approach described above can partly be transferred to the sparse setting.

1.4 Organisation

This thesis is structured as follows.

- We split the explanation of the notation and concepts that we use into two parts. Basic definitions are provided in Chapter 2. Most notions concerning regularity are treated together with the regularity method in Chapter 3.
- Next we discuss structural properties of graphs with sublinear bandwidth and give the proof of Theorems 4.1 and 4.6 in Chapter 4. This is based on joint work with Andreas Würfl, Klaas Pruessmann, and Anusch Taraz [19].
- Then we turn to embedding results. In Chapter 5 we prove the conjecture of Bollobás and Komlós and obtain Theorem 5.1. This is based on joint work with Mathias Schacht and Anusch Taraz [20, 21].
- In Chapter 6 we consider a bipartite version of this theorem in the form of Theorem 6.2 which is based on joint work with Peter Heinig and Anusch Taraz.
- Ore conditions and the proof of Theorem 7.2 are covered in Chapter 7. This result is based on joint work with Sybille Müller [18].
- The Ramsey-type result on trees, Theorem 8.1, is established in Chapter 8. It is based on joint work with Jan Hladký and Diana Piguet [17].
- In Chapter 9 we move to sparse graphs and present the proof of Theorem 9.1 which was obtained in joint work with Yoshiharu Kohayakawa and Anusch Taraz.
- In Chapter 10 we close with some remarks on recent related work.

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Chapter 2

Definitions

In this chapter we provide the definitions of most concepts that we shall use throughout the thesis. For all elementary graph theoretic concepts not defined in this chapter we refer the reader to, e.g., [28, 92]. In addition, we defer most definitions that are not used in many chapters to later and only introduce them once we need them. In particular, all terminology concerning the regularity method is introduced in Chapter 3.

2.1 Basic notions

Graphs. All graphs in this thesis are finite, simple and undirected, unless noted otherwise. Let $G = (V, E)$ be a graph. We also denote the vertices of G by $V(G) := V$ and their number, which is also called the *order* of G, by $v(G) := |V|$. Similarly, $E(G) := E$ are the edges of G, and their number is $e(G) := |E|$. Edges $\{u, v\}$ of G are frequently simply denoted by uv. Let $A, B, C \subseteq V$ be pairwise disjoint vertex sets in G. We define $E(A) := E \cap \binom{A}{2}$ and write $E(A, B)$ for the set of edges with one end in A and one end in B and $e(A, B)$ for the number of such edges. Moreover, $G[A]$ is the graph with vertex set A and edge set $E(A)$. Analogously, $G[A, B]$ is the bipartite graph with vertex set $A \dot\cup B$ and edge set $E(A, B)$ and $G[A, B, C]$ is the tripartite graph with vertex set $A \dot\cup B \dot\cup C$ and edge set $E(A, B) \dot\cup E(B, C) \dot\cup E(C, A)$. For convenience we frequently identify graphs G with their edge set $E(G)$ and vice versa. For an edge set F we denote by $A \cap F$ the set of vertices from A that appear in some edge of F. For a vertex $v \in V$ we write $N_G(v)$ or simply $N(v)$ for the *neighbourhood* of v in G and $N_B(v)$ denotes the set of neighbours of v in B. The union of all neighbourhoods of vertices in A is also denoted by $N_G(A)$ or simply $N(A)$, and $N_B(A) := N_G(A) \cap B$. The *common* or *joint neighbourhood* of two vertices u and v, i.e., the set of all vertices that are neighbours of both u and v, is denoted by $N^\cap_G(\{u, v\})$ or simply $N^\cap_G(u, v)$ or $N^\cap(u, v)$. More generally, $N^\cap_G(A)$ is the joint neighbourhood of all vertices in A and $N^\cap_B(A) := N^\cap_G(A) \cap B$. For $v \in V$ we let $d_G(v) := |N_G(v)|$ be the *degree* of v in G and, similarly, $d_B(v) := |N_B(v)|$.

For basic graph parameters we use standard notation. In particular, $\delta(G)$ is the *minimum degree* and $\Delta(G)$ the *maximum degree* of a graph. The maximum size of a clique in G is the *clique number* $\omega(G)$. By $\alpha(G)$ we denote the *independence number* of a graph, i.e., the size of a largest stable (or independent) set in G. The *diameter* $\mathrm{diam}(G)$ is the maximal distance between two vertices in G.

A *partition* of a graph G is a partition of its vertex set $V(G)$ into disjoint sets. A (proper) *vertex colouring* or simply *colouring* σ of a graph is a colouring of V such that all pairs of vertices u and v with $uv \in E$ receive different colours. The *colour classes* of σ, i.e., the sets formed by vertices of the same colour, are also called *partition classes* of this colouring. A colouring of G can equivalently be regarded as a partition of G into independent sets. The *chromatic number* $\chi(G)$ is the minimal number of colours needed for a proper vertex colouring of G. And an *equitable colouring* of G is a colouring, such that for any two colour classes X, $Y \subseteq V$ we have $|X| - |Y| \leq 1$. Notice that in the Chapter 8, where we consider problems from Ramsey theory, we typically consider colourings of the edges of G and do not impose any (general) restriction on these colourings.

A graph H is a *subgraph* of G if there is an *edge-preserving* injective mapping f from $V(H)$ to $V(G)$, i.e., an injective function $f\colon V(H) \to V(G)$ such that for every edge $\{u,v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(G)$. In this case we also say that G contains a *copy* of H and that f is an *embedding* of H into G, and write $H \subseteq G$. The subgraph or copy is *induced* if non-edges of H are mapped to non-edges of G and *spanning* if H and G have the same number of vertices. We say that a subgraph H of G *covers* a vertex v of G if v is contained in some edge of H. A graph G is called *universal* for a class of graphs $\mathcal{H}$ if G contains all graphs from $\mathcal{H}$ as subgraphs. A *packing* of two graphs H and G is an embedding of H into the complement of G. A *graph homomorphism* or simply *homomorphism* from H to G is an edge-preserving (not necessarily injective) mapping $h\colon V(H) \to V(G)$. All these notions naturally generalise to hypergraphs.

Among others, we shall use the following special graphs. A path on n vertices is denoted by P_n, a cycle on n vertices by C_n. The graph K_n is the complete graph on n vertices. The graph K_3 is also called *triangle*. $K_{n,m}$ is the complete bipartite graph with partition classes of size n and m, and similarly $K_{n,m,k}$ is the complete tripartite graph on $n+m+k$ vertices. A *Hamilton cycle* is a cycle on n vertices and a *Hamilton path* a path on n vertices.

The r-th *power* G^r of a graph $G = (V, E)$ is the graph on vertex set V and with edges uv for all vertices u and v of distance at most r in G. The second power G^2 is also called the *square* of G. A *square-path* on n vertices is the square of P_n, and a *square-cycle* on n vertices the square of C_n. A vertex set $S \subseteq V(G)$ is called *s-independent* in G if S is independent in the s-th power G^s of G.

Let H be a fixed graph. An *H-factor* in a graph G is a subgraph of G consisting of vertex disjoint H-copies. A perfect H-factor is an spanning H-factor. A K_r-factor is also called *r-factor*. A matching is a 2-factor. We shall frequently identify a matching M with its edges (but say so in this case). The *size* $|M|$ of the matching M is the number of its edges.

Numbers. For positive integers a and b with $a \leq b$ we let $[a]$ denote the set $\{1, \dots, a\}$ and $[a, b] := \{a, \dots, b\}$. Further, $(a, b] = \{a+1, \dots, b\}$ and the sets $[a, b)$ and (a, b) are defined accordingly. For numbers x, y, and z we write $x = y \pm z$ if $|x - y| \leq z$.

Let $n, k, r \in \mathbb{N}$. An *integer partition* of n is a sequence of natural numbers that add up to n. We call an integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n (with $n_{i,j} \in \mathbb{N}$ for all $i \in [k]$ and $j \in [r]$) *r-equitable*, if $|n_{i,j} - n_{i,j'}| \leq 1$ for all $i \in [k]$ and $j, j' \in [r]$.

To simplify the presentation, addition and subtraction in the index of an integer partition $(n_i)_{i\in[k]}$ is modulo k, unless stated otherwise. This means, for example, that n_{k+1} denotes n_1, n_{k+2} denotes n_2, n_0 denotes n_k, and n_{-1} denotes n_{k-1}. The same rule applies to other partitions and sequences indexed by $[k]$ or by $[k] \times [r]$.

In addition, we shall frequently omit floor and ceiling signs $\lfloor\rfloor$ and $\lceil\rceil$ when they are not relevant in a particular calculation or argument.

Sets. An *ℓ-set* is a set with exactly ℓ elements. Let I be an index set. A *family of indexed sets* or an *indexed set system* is a sequence $\mathcal{X} = \big(X_i \colon i \in I\big)$ of (not necessarily distinct) sets.

For finite I we define $|\mathcal{X}| = |I|$ and write $\mathcal{X}' \subseteq \mathcal{X}$ if $\mathcal{X}'$ is a *subfamily* of $\mathcal{X}$. A *system of distinct representatives* for the family $\mathcal{X}$ is a sequence of distinct elements $(x_i : i \in I)$ such that $x_i \in X_i$ for all $i \in I$.

Hall's theorem states that a bipartite graph $G = (A \dot\cup B, E)$ contains a perfect matching covering A if and only if for all subsets $A' \subseteq A$ we have that $|N_G(A')| \geq |A'|$. This inequality is also called *Hall's condition* and can alternatively be formulated for finite set systems: the indexed set system $\mathcal{X} = \big(X_i : i \in I\big)$ has a system of distinct representatives if and only if for every subfamily $\mathcal{X}'$ of $\mathcal{X}$ we have $|\bigcup \mathcal{X}'| \geq |\mathcal{X}'|$.

Asymptotics. For asymptotic notation we use the Landau symbols $\mathcal{O}(n)$, $o(n)$, $\Omega(n)$. The symbol $\ll$, however, is reserved for relations between small constants. For example, when we write $\varepsilon \ll \varepsilon'$ for two positive real numbers ε and ε' then we mean that $\varepsilon \leq \varepsilon'$ and that we can make ε' arbitrarily small by choosing ε sufficiently small.

2.2 Graph parameters

In this section we define a series of graph parameters that describe different aspects of the global structure of a graph. We start with the *bandwidth* of a graph, a concept central to this thesis. Roughly speaking, this parameter measures how well a graph resembles a path.

Definition 2.1 (bandwidth)**.** *Let $G = (V, E)$ be a graph on n vertices. The bandwidth of G is denoted by* $\mathrm{bw}(G)$ *and defined to be the minimum positive integer b, such that there exists a labelling of the vertices in V by numbers $1, \dots, n$ so that the labels of every pair of adjacent vertices differ by at most b.*

Another way to say this is that G is a subgraph of the $\mathrm{bw}(G)$-th power of a path (and not a subgraph for any lesser power). To give an example, a cycle $C_n = v_1, v_2, \dots, v_n$ has bandwidth 2 as the following labelling shows: simply assign the numbers $[n]$ to the vertices of C_n in the order v_1, v_n, v_2, v_{n-1}, v_3, v_{n-2}, v_4,. . . , $v_{\lceil (n+1)/2 \rceil}$.

Next, we will introduce the notions of *tree decomposition* and *treewidth*. A tree decomposition tries to arrange the vertices of a graph in a tree-like manner and the treewidth measures how well this can be done.

Definition 2.2 (treewidth)**.** *Let $G = (V, E)$ be a graph. A* tree decomposition *of G is a pair $(\{X_i : i \in I\},\ T = (I, F))$ where $\{X_i : i \in I\}$ is a family of subsets $X_i \subseteq V$ and $T = (I, F)$ is a tree such that the following holds:*

(a) $\bigcup_{i \in I} X_i = V$,

(b) for every edge $\{v, w\} \in E$ there exists $i \in I$ with $\{v, w\} \subseteq X_i$,

(c) for every $i, j, k \in I$: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.

The width of $(\{X_i : i \in I\}, T = (I, F))$ is defined as $\max_{i \in I} |X_i| - 1$. *The* treewidth $\mathrm{tw}(G)$ *of G is the minimum width of a tree decomposition of G.*

It follows directly from the definition that $\mathrm{tw}(G) \leq \mathrm{bw}(G)$ for any graph G: if the vertices of G are labelled by numbers $1, \dots, n$ such that the labels of adjacent vertices differ by at most b, then $I := [n - b]$, $X_i := \{i, \dots, i + b\}$ for $i \in I$ and $T := (I, F)$ with $F := \{\{i - 1, i\} : 2 \leq i \leq n - b\}$ define a tree decomposition of G with width b.

A *separator* in a graph is a small cut set that splits the graph into components of limited size.

Definition 2.3 (separator, separation number)**.** *Let $0 < \alpha < 1$ be a real number, $s \in \mathbb{N}$ and $G = (V, E)$ a graph. A subset $S \subseteq V$ is said to be an* (s, α)-separator *of G, if there exist subsets $A, B \subseteq V$ such that*

(*a*) $V = A \dot\cup B \dot\cup S$,
(*b*) $|S| \leq s$, $|A|, |B| \leq \alpha|V|$, *and*
(*c*) $E(A, B) = \emptyset$.

We also say that S separates G into A and B. The separation number $\mathrm{s}(G)$ *of G is the smallest s such that all subgraphs G' of G have an $(s, 2/3)$-separator.*

A vertex set is said to be expanding if it has many external neighbours. We call a graph *non-expanding*, if every sufficiently large subgraph contains a subset which is not expanding.

Definition 2.4 (expander, non-expanding)**.** *Let $\varepsilon > 0$ be a real number, $b \in \mathbb{N}$ and consider graphs $G = (V, E)$ and $G' = (V', E')$. We say that G' is an* ε-expander *if all subsets $U \subseteq V'$ with $|U| \leq |V'|/2$ fulfil $|N(U)| \geq \varepsilon|U|$. (Here $N(U)$ is the set of neighbours of vertices in U that lie outside of U.) The graph G is called* (b, ε)-non-expanding*, if no subgraph $G' \subseteq G$ with $|V'| \geq b$ vertices is an ε-expander. Finally, we define the* ε-non-expansion $\mathrm{b}_\varepsilon(G)$ *of G to be the minimum b for which G is $(b+1, \varepsilon)$-non-expanding.*

There is a wealth of literature on this class of graphs (see e.g. [54]). In particular, it is known that for example (bipartite) random graphs with bounded maximum degree form a family of ε-expanders. We also loosely say that such graphs have *good expansion properties*.

Finally, a graph is *planar* if it can be drawn in the plane without edge crossings and the genus of a graph G counts the minimum number of handles that must be added to the plane to embed the graph without edge crossings.

2.3 Random variables and random graphs

The notation and terminology we use for probabilities, random variables, and related concepts is standard (in the theory of random graphs) and follows [55].

By $\mathrm{Bi}(n, p)$ we denote the *binomial distribution* with parameters n and p. The term *Chernoff's inequality* (or *Chernoff bound*) collects different exponentially decreasing bounds on tail distributions of sums of independent random variables. In this thesis we shall use the following versions for a binomially distributed random variable Y and real numbers $t \geq 0$ and $s \geq 6\,\mathbb{E}\,Y$ (see [55, Chapter 2]):

$$\mathbb{P}[Y \geq 7\,\mathbb{E}\,Y] \leq \exp(-7\,\mathbb{E}\,Y)\,, \tag{2.1}$$

$$\mathbb{P}[|Y| \geq \mathbb{E}\,Y + t] \leq 2\exp(-2t^2/n)\,, \tag{2.2}$$

$$\mathbb{P}[|Y| \geq \mathbb{E}\,Y + s] \leq \exp(-s)\,. \tag{2.3}$$

We further use the following formulation of a concentration bound sometimes attributed to Hoeffding.

Theorem 2.5 (Hoeffding bound [13, Theorem A.1.16])**.** *Let $X_1, \dots, X_s$ be independent random variables with $\mathbb{E}\,X_i = 0$ and $|X_i| \leq 1$ for all $i \in [s]$ and let X be their sum. Then $\mathbb{P}[|X| \geq a] \leq 2\exp(-a^2/(2s))$.* □

The study of random graphs is a lively area with many surprising results, far-reaching applications and elegant techniques. We will just introduce (and need) some of the more basic definitions here. For more information see [55, 13].

Let $p\colon \mathbb{N} \to [0,1]$ be a function. The *random graph* $\mathcal{G}_{n,p}$ is generated by including each of the $\binom{n}{2}$ possible edges on n vertices with probability $p = p(n)$ independently at random. By linearity of expectation, the expected number of edges incident to a vertex of $\mathcal{G}_{n,p}$ equals $(n-1)p$. Therefore, the parameter p governs the average degree, or what we call the *density* of the graph.

A class of graphs that is closed under isomorphism is called a *graph property*. A *hereditary* graph property is closed under taking induced subgraphs.

For a fixed function p we say that the random graph $\mathcal{G}_{n,p}$ has a graph property $\mathcal{P}$ *asymptotically almost surely* (abbreviated a.a.s.) if the probability that $\mathcal{G}_{n,p} \in \mathcal{P}$ tends to 1 as n goes to infinity.

We close with the following Lemma which collects some well known facts about the edge distribution in random graphs $\mathcal{G}_{n,p}$. This lemma follows directly from the Chernoff bound for binomially distributed random variables.

Lemma 2.6. *If $\log^4 n/(pn) = o(1)$ then a.a.s. the random graph $\Gamma = \mathcal{G}_{n,p}$ has the following properties. For all vertex sets X, Y, $Z \subseteq V(\Gamma)$ with $X \cap Y = \emptyset$ and $|X|, |Y|, |Z| \geq \frac{n}{\log n}$, $|Z| \leq n - \frac{n}{\log n}$ we have*

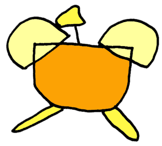
(i) $e_\Gamma(X) = (1 \pm \frac{1}{\log n})p\binom{|X|}{2}$,

(ii) $e_\Gamma(X, Y) = (1 \pm \frac{1}{\log n})p|X||Y|$,

(iii) $\sum_{z \in Z} \deg_\Gamma(z) = (1 \pm \frac{1}{\log n})p|Z|n$. □

Chapter 3

The regularity method



As explained in the introduction many proofs in the following chapters rely on the celebrated regularity lemma of Szemerédi [91]. This lemma was originally (in the late 1970s) motivated by a famous question concerning the existence of arithmetic progressions in sets with positive density, but has since then turned out to be useful for a variety of important questions. In particular it has been the key instrument for the solution of a number of long-standing open problems (such as, e.g, Theorem 1.7) in extremal graph theory (see the surveys [70, 71] for a more detailed account on this topic).

In this chapter we will state the regularity lemma (Section 3.1), formulate some relevant consequences (in Sections 3.1 and 3.2 where we treat regularity and super-regularity, respectively), and introduce some related tools, such as the blow-up lemma (Section 3.3). We conclude the chapter by discussing a sparse analogue of regularity and a sparse version of the regularity lemma (Section 3.4).

3.1 Regular partitions of graphs

The regularity lemma relies on the concept of an ε-regular pair. To define this, let $G = (V, E)$ be a graph. For disjoint nonempty vertex sets $A, B \subseteq V$ the *density* $d(A, B) := e(A, B)/(|A||B|)$ of the pair (A, B) is the number of edges that run between A and B divided by the number of possible edges between A and B. In the following let $\varepsilon, d \in [0, 1]$.

Definition 3.1 (ε-regular)**.** *The pair (A, B) is* ε-regular*, if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ it is true that $|d(A, B) - d(A', B')| \leq \varepsilon$. An ε-regular pair (A, B) is called (ε, d)-regular, if it has density at least d.*

Next, we will state a version of Szemeredi's regularity lemma that is useful for our purposes, the so-called degree form (see, e.g., [70, Theorem 1.10]). As explained earlier, the regularity lemma asserts that graphs have regular partitions. The following definition makes this concept (which we shall only use in a later version of the regularity lemma, Lemma 3.4, however) precise.

Definition 3.2 (regular partition, reduced graph)**.** *An* (ε, d)-regular partition *of G with* reduced graph $R = (V_R, E_R)$ *is a partition $V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_k$ of V with $|V_0| \leq \varepsilon|V|$, $V_R = [k]$, such that (V_i, V_j) is an (ε, d)-regular pair in G whenever $ij \in E_R$. If such a partition exists, we also say that R is an* (ε, d)-reduced graph *of G or that $V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_k$ is* regular on R. *Moreover, R is the* maximal *(ε, d)-reduced graph of the partition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_k$ if there is no $ij \notin E_R$ with $i, j \in [k]$ such that (V_i, V_j) is (ε, d)-regular. The partition classes V_i with $i \in [k]$ are called* clusters *of G and V_0 is the* exceptional set*.*

We sometimes also call a vertex i of the reduced graph a cluster and occasionally identify it with its corresponding set V_i. When the exceptional set V_0 is empty (or we want to ignore it as well as its size) then we frequently omit it and say that $V_1\dot\cup\dots\dot\cup V_k$ forms an (ε,d)-regular partition or $V_1\dot\cup\dots\dot\cup V_k$ is regular on R. Finally, a partition $V_0\dot\cup V_1\dot\cup\dots\dot\cup V_k$ of V is an *equipartition* if $|V_i|=|V_j|$ for all $i,j\in[k]$.

The degree form of the regularity lemma now takes any sufficiently large graph G and generates a "blueprint" G' of G that differs only slightly from G. The graph G' consists of constantly many regular pairs *all* of which are either dense or do not contain any edges at all. In our formulation of the lemma the graph G' does not appear explicitly; it is the graph obtained by taking out all edges of G that are not part of some dense regular pair.

Lemma 3.3 (regularity lemma, degree form). *For all $\varepsilon>0$ and k_0 there is k_1 such that for every $d\in[0,1]$ every graph $G=(V,E)$ on $n\ge k_1$ vertices has an equipartition $V=V_0\dot\cup V_1\dot\cup\dots\dot\cup V_k$ with $k_0\le k\le k_1$ and $|V_0|\le\varepsilon n$ such that for each $v\in V_i$ with $i\in[k]$ there are at most $(d+\varepsilon)n$ edges $e\in E$ with $v\in e$ that are not in some (ε,d)-regular pair (V_i,V_j) with $j\in[k]$.* □

In other words, every graph G has an (ε,d)-regular partition with reduced graph R on $k\le k_1$ vertices such that most edges of G lie on edges of R. The crucial point here is that the upper bound k_1 on the number of clusters does not depend on the order n of G but only on ε. However, this dependence is rather unfortunate: Proofs of the regularity lemma bound k_1 by a tower of 2s with height proportional to ε^{-5} (where ε^{-5} cannot be replaced by anything better than $\log(1/\varepsilon)$ as was shown by Gowers [43]). As a consequence, results obtained with the help of this lemma typically talk about huge graphs only. This is also true for the embedding results in this thesis. A good description for the order of magnitude is "larger than the number of atoms in the universe". The more the merrier.

In the introduction we explained that the reduced graph inherits certain properties from the graph G and that this is useful in applications of the regularity lemma. The next lemma illustrates this inheritance for minimum degree conditions. This is a simple corollary of the degree form of the regularity lemma (see, e.g., [74, Proposition 9]).

Lemma 3.4 (regularity lemma, minimum degree version). *For every $\gamma>0$ there exist $d>0$ and $\varepsilon_0>0$ such that for every $0<\varepsilon\le\varepsilon_0$ and every integer k_0 there exists k_1 so that the following holds. For every $\nu\ge0$ every graph G on $n\ge k_1$ vertices with $\delta(G)\ge(\nu+\gamma)n$ has an (ε,d)-regular equipartition $V=V_0\dot\cup V_1\dot\cup\dots\dot\cup V_k$ with reduced graph R such that $k_0\le k\le k_1$ and $\delta(R)\ge(\nu+\gamma/2)k$.* □

In Chapter 7 we will see another example of this inheritance principle (see Lemma 7.4). In general, we shall use different versions and corollaries of the regularity lemma in different chapters, that we only introduce as we need them (see, e.g., Lemma 8.6).

3.2 Super-regularity

For the blow-up lemma (Lemma 3.9), which we will describe below, we need the concept of a *super-regular pair*. Roughly speaking, a regular pair is super-regular if every vertex has a sufficiently large degree.

Definition 3.5 (super-regular pair). *An (ε,d)-regular pair (A,B) in a graph $G=(V,E)$ is* (ε,d)-super-regular *if every vertex $v\in A$ has degree* $\deg_B(b)\ge d|B|$ *in B and every $v\in B$*

has $\deg_A(v) \geq d|A|$ *in* A. *A partition* $V = V_0\dot\cup V_1\dot\cup \dots \dot\cup V_k$ *is* (ε, d)-super-regular on a graph $R = ([k], E_R)$, *if all pairs* (V_i, V_j) *with* $ij \in E_R$ *are* (ε, d)*-super-regular.*

Similarly as for (ε, d)-regular partitions we may also omit the set V_0 here and say that the partition $V_1\dot\cup \dots \dot\cup V_k$ is super-regular on R.

Lemma 3.3 asserts that graphs have *regular* partitions. However, for embedding spanning graphs we need to construct a partition that is also *super-regular* for certain cluster pairs (cf. Section 1.3.2). The following two propositions indicate how to connect these two worlds and find such super-regular pairs in a regular partition. The first proposition implies that every (ε, d)-regular pair (A, B) contains a "large" super-regular sub-pair (A', B').

Proposition 3.6. *Let* (A, B) *be an* (ε, d)*-regular pair and* B' *be a subset of* B *of size at least* $\varepsilon|B|$. *Then there are at most* $\varepsilon|A|$ *vertices* v *in* A *with* $|N(v) \cap B'| < (d - \varepsilon)|B'|$.

Proof. Let $A' = \{v \in A\colon\ |N(v) \cap B'| < (d-\varepsilon)|B'|\}$ and assume to the contrary that $|A'| > \varepsilon|A|$. But then $d(A', B') < ((d - \varepsilon)|A'||B'|)/(|A'||B'|) = d - \varepsilon$ which is a contradiction since (A, B) is (ε, d)-regular. □

We will say that all other vertices in A are (ε, d)-*typical* with respect to B' (or simply *typical*, when ε and d are clear from the context). Repeating Proposition 3.6 a fixed number of times, we obtain the following proposition (see, e.g., [74, Proposition 8]).

Proposition 3.7. *Given* $\varepsilon, d > 0$ *and* $\Delta \in \mathbb{N}$ *set* $\varepsilon' := 2\varepsilon\Delta/(1 - \varepsilon\Delta)$ *and* $d' := d - 2\varepsilon\Delta$. *Let* G *be a graph with an* (ε, d)*-regular equipartition with reduced graph* R *and let* R' *be a subgraph of* R *with* $\Delta(R') \leq \Delta$. *Then* G *has an* (ε', d')*-regular equipartition with reduced graph* R *which is super-regular on* R'. □

We close this section with the following useful observation. It states that the notion of regularity is "robust" in view of small alterations of the respective vertex sets.

Proposition 3.8. *Let* (A, B) *be an* (ε, d)*-regular pair and let* $(\hat{A}, \hat{B})$ *be a pair such that* $|\hat{A} \triangle A| \leq \alpha|\hat{A}|$ *and* $|\hat{B} \triangle B| \leq \beta|\hat{B}|$ *for some* $0 \leq \alpha, \beta \leq 1$. *Then,* $(\hat{A}, \hat{B})$ *is an* $(\hat{\varepsilon}, \hat{d})$*-regular pair with*

$$\hat{\varepsilon} := \varepsilon + 3\big(\sqrt{\alpha} + \sqrt{\beta}\big) \quad \text{and} \quad \hat{d} := d - 2(\alpha + \beta)\,.$$

If, moreover, (A, B) *is* (ε, d)*-super-regular and each vertex* v *in* $\hat{A}$ *has at least* $d|\hat{B}|$ *neighbours in* $\hat{B}$ *and each vertex* v *in* $\hat{B}$ *has at least* $d|\hat{A}|$ *neighbours in* $\hat{A}$, *then* $(\hat{A}, \hat{B})$ *is* $(\hat{\varepsilon}, \hat{d})$*-super-regular with* $\hat{\varepsilon}$ *and* $\hat{d}$ *as above.*

Proof. Let A, B, $\hat{A}$ and $\hat{B}$ be as above. First we estimate the density of $(\hat{A}, \hat{B})$. Let $d' := d(A, B) \geq d$ be the density of (A, B). If $(\hat{A}, \hat{B})$ had the same density as (A, B), we would have $e(\hat{A}, \hat{B}) = d'|\hat{A}||\hat{B}|$. The actual value of $e(\hat{A}, \hat{B})$ can deviate by at most

$$\begin{aligned} |\hat{A} \triangle A| \cdot |\hat{B} \cup B| + |\hat{B} \triangle B| \cdot |\hat{A} \cup A| &\leq \alpha|\hat{A}| \cdot (1 + \beta)|\hat{B}| + \beta|\hat{B}| \cdot (1 + \alpha)|\hat{A}| \\ &\leq 2(\alpha + \beta)|\hat{A}||\hat{B}| \end{aligned}$$

from this value. So, clearly

$$\hat{d} = d - 2(\alpha + \beta) \leq d' - 2(\alpha + \beta) \leq d(\hat{A}, \hat{B}) \leq d' + 2(\alpha + \beta)\,.$$

Now let $\hat{A}' \subseteq \hat{A}$ and $\hat{B}' \subseteq \hat{B}$ be sets of sizes $|\hat{A}'| \geq \hat{\varepsilon}|\hat{A}|$ and $|\hat{B}'| \geq \hat{\varepsilon}|\hat{B}|$. Denote $\hat{A}' \cap A$ by A' and $\hat{B}' \cap B$ by B' and observe that

$$|A'| \geq |\hat{A}'| - \alpha|\hat{A}| \geq (\hat{\varepsilon} - \alpha)|\hat{A}| \geq (\varepsilon + \sqrt{\hat{\alpha}})|\hat{A}| \geq \varepsilon(1+\alpha)|\hat{A}| \geq \varepsilon|A|.$$

Similarly, $|B'| \geq \varepsilon|B|$. It follows that $d' - \varepsilon \leq d(A', B') \leq d' + \varepsilon$. Moreover, $|A'| \leq |\hat{A}'|$ and

$$|A'| \geq |\hat{A}'| - \alpha|\hat{A}| \geq |\hat{A}'| - \alpha\frac{|\hat{A}'|}{\hat{\varepsilon}} \geq (1 - \sqrt{\alpha})|\hat{A}'|\,,$$

where the last inequality follows from the definition of $\hat{\varepsilon}$. The same calculations yield

$$(1 - \sqrt{\beta})|\hat{B}'| \leq |B'| \leq |\hat{B}'|\,.$$

For the number of edges between A' and B' we therefore get

$$\begin{aligned} e(\hat{A}', \hat{B}') &\geq e(A', B') \geq (d' - \varepsilon)|A'||B'| \geq (d' - \varepsilon)(1 - \sqrt{\alpha})(1 - \sqrt{\beta})|\hat{A}'||\hat{B}'| \\ &\geq (d' - \varepsilon - \sqrt{\alpha} - \sqrt{\beta})|\hat{A}'||\hat{B}'| \end{aligned}$$

since $\alpha, \beta \leq 1$. Similarly,

$$\begin{aligned} e(\hat{A}', \hat{B}') &\leq e(A', B') + (|\hat{A}'| - |A'|)|\hat{B}'| + (|\hat{B}'| - |B'|)|\hat{A}'| \\ &\leq (d' + \varepsilon)|A'||B'| + \sqrt{\alpha}|\hat{A}'||\hat{B}'| + \sqrt{\beta}|\hat{A}'||\hat{B}'| \\ &\leq (d' + \varepsilon + \sqrt{\alpha} + \sqrt{\beta})|\hat{A}'||\hat{B}'|\,. \end{aligned}$$

With this we can now compare the densities of $(\hat{A}', \hat{B}')$ and $(\hat{A}, \hat{B})$:

$$\begin{aligned} d(\hat{A}, \hat{B}) - d(\hat{A}', \hat{B}') &\leq (d' + 2(\alpha + \beta)) - (d' - \varepsilon - \sqrt{\alpha} - \sqrt{\beta}) \leq \hat{\varepsilon}, \\ d(\hat{A}', \hat{B}') - d(\hat{A}, \hat{B}) &\leq (d' + \varepsilon + \sqrt{\alpha} + \sqrt{\beta}) - (d' - 2(\alpha + \beta)) \leq \hat{\varepsilon}, \end{aligned}$$

This implies that $(\hat{A}, \hat{B})$ is $(\hat{\varepsilon}, \hat{d})$-regular. The second part of the proposition follows immediately from Definition 3.5, since $\hat{d}|\hat{A}| \leq d|\hat{A}|$ and $\hat{d}|\hat{B}| \leq d|\hat{B}|$. □

3.3 Embedding lemmas

Now we turn to the regularity-related embedding results that we will apply. We first introduce the blow-up lemma (Section 3.3.1). Then we complement this lemma with another embedding result, which we call partial embedding lemma (Section 3.3.2). In the last part of this section we combine these two embedding lemmas to obtain a "general" embedding lemma which captures how we will use the blow-up lemma and the partial embedding lemma later in our proofs (Section 3.3.3).

3.3.1 The blow-up lemma

One important feature of super-regular pairs is that a powerful lemma, the *blow-up lemma* proven by Komlós, Sárközy and Szemerédi [65] (see also [85] for an alternative proof), guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs. In fact, the statement is more general and allows the embedding of graphs

H into partitions that are super-regular on some graph R if there is a homomorphism from H to R that does not send too many vertices of H to each cluster of R.

For specifying the homomorphism from H to R in the language of partitions we use the following definition. Let $R = ([k], E_R)$ and $H = (W, F)$ be graphs. A partition $W = W_1\dot\cup \dots \dot\cup W_k$ of H is an *R-partition* if $xy \in F$ for $x \in W_i$ and $y \in W_j$ implies $ij \in E(R)$.

Lemma 3.9 (Blow-up lemma [65]). *For every d, Δ, $c > 0$ and $r \in \mathbb{N}$ there exist constants $\varepsilon = \varepsilon(d, \Delta, c, r)$ and $\alpha = \alpha(d, \Delta, c, r)$ such that the following holds for all $r' \leq r$. Let $R = ([r'], E_R)$, $G = (V, E)$, and $H = (W, F)$ be graphs with $\Delta(H) \leq \Delta$ such that G has a partition $V = V_1\dot\cup \dots \dot\cup V_{r'}$ which is (ε, d)-super-regular on R and H has an R-partition $W = W_1\dot\cup \dots \dot\cup W_{r'}$ where $|W_i| \leq |V_i| = n_i$ for every $i \in [r']$. Moreover, suppose that in each class W_i there is a set of at most $\alpha \cdot \min_{j\in[r']} n_j$* special vertices *$y$, each of which is equipped with a* candidate set *$C_y \subseteq V_i$ with $|C_y| \geq c|V_i|$. Then there is an embedding of H into G such that each special vertex is mapped to a vertex in its candidate set.* □

We also say that the special vertices y in Theorem 3.9 are *image restricted to C_y*.

3.3.2 The partial embedding lemma

In the introduction (see page 13) we described that our embedding strategy will usually be to apply the blow-up lemma on local super-regular spots in the reduced graph for some large segments of the graph H separately. In order to connect these segments to form a copy of the whole graph H we will use the following weaker embedding lemma (concerning only linear sized, but not spanning embeddings) which is formulated in the less restrictive environment of (ε, d)-regular pairs. This lemma takes a small subgraph B of H on vertex set $X\dot\cup Y$ and produces a *partial* embedding of this graph: it embeds the vertices in X and creates (sufficiently large) sets C_y for the vertices $y \in Y$ suitable for the future embedding of these y. In the next section we will explain how this is used in conjunction with the blow-up lemma.

A lemma similar to Lemma 3.10, in a slightly different context, was first obtained by Chvátal, Rödl, Szemerédi, and Trotter [25] (see also [28, Lemma 7.5.2]). The only difference between Lemma 3.10 and their embedding lemma is that we only embed *some* of the vertices of the graph B into G.

Lemma 3.10 (Partial embedding lemma). *For every integer $\Delta > 0$ and every $d \in (0, 1]$ there exist positive constants $c = c(\Delta, d)$ and $\varepsilon = \varepsilon(\Delta, d)$ such that the following is true. Let $G = (V, E)$ be an n-vertex graph that has an (ε, d)-regular partition $V = V_1\dot\cup \dots \dot\cup V_k$ with reduced graph R on k vertices. Let, furthermore, $B = (V_B, E_B)$ be a graph with $\Delta(B) \leq \Delta$ that admits a homomorphism $h\colon V_B \to V_R$ into R satisfying $|h^{-1}(i)| \leq 2\varepsilon|V_i|$ for all $i \in [k]$ and let $V_B = X\dot\cup Y$ be an arbitrary partition of its vertex set. Then there exist an embedding $f\colon X \to V$ of $B[X]$ into G and sets $C_y \subseteq V_{h(y)} \setminus f(X)$ for all $y \in Y$ such that for all $y \in Y$ we have*

(*i*) *$f(x) \in V_{h(x)}$ for all $x \in X$,*
(*ii*) *$C_y \subseteq N_G(f(x))$ for all $x \in N_B(y) \cap X$, and*
(*iii*) *$|C_y| \geq c|V_{h(y)}|$.*

In other words, Lemma 3.10 provides a mapping f for those vertices $x \in X$ of B into the cluster $V_{h(x)}$ required by h, respecting the edges between such vertices. Moreover, for the other vertices $y \in Y$ of B, it prepares sufficiently large sets $C_y \subseteq V_{h(y)} \setminus f(X)$ such that, no matter where y will later be embedded in C_y, it will be adjacent to any of its already embedded neighbours $x \in N_B(y) \cap X$. We will also call these sets C_y *candidate sets* for the vertices in Y (they will be used as candidate sets for the blow-up lemma in the following section).

The proof of Lemma 3.10 follows very much along the lines of the embedding lemma from [25]. We also proceed iteratively, embedding the vertices in X into G one by one.

Proof. Given Δ and d, choose $c := (d/2)^\Delta/2$ and $\varepsilon := c/\Delta$. Note, that this implies $\varepsilon \leq (d/2)^\Delta/4 \leq d/2$. Let G, R and B with $V(B) = X\dot{\cup}Y$ be graphs as required. For the size of X we have $|X| \leq 2\varepsilon|V_i|$ for all $i \in [k]$ by assumption. We now construct the embedding $f : X \to V(G)$. For this, we will create sets C_b not only for the vertices in Y, but for all vertices $b \in V(B)$. First, set $C_b := V_{h(b)}$ for all $b \in V(B)$. Then, repeat the following steps for each $x \in X$:

(a) For all $b \in N_B(x)$, delete all vertices $v \in C_x$ with $|N_G(v) \cap C_b| < (d-\varepsilon)|C_b|$.
(b) Then, choose one of the vertices remaining in C_x as $f(x)$.
(c) For all $b \in N_B(x)$, delete all vertices $v \in C_b$ with $v \notin N_G(f(x))$.
(d) For all $b \in V(B)$, delete $f(x)$ from C_b.

We claim, that at the end of this procedure, f and the C_y with $y \in Y$ are as desired. Indeed, f is an embedding of $B[X]$ into G and conditions (i) and (ii) are satisfied by construction. It remains to prove that condition (iii) is satisfied and that $f(x)$ can be chosen in step (a) throughout the entire procedure.

We start by showing, that we always have $|C_b| \geq c|V_{h(b)}|$ for all $b \in V(B)$. This implies condition (iii). In total, step (d) removes at most $|X \cap V_{h(b)}|$ vertices from each C_b. By the choice of $f(x)$ in step (a) and (b), an application of step (c) to a vertex $b \in N_B(x)$, reduces the size of C_b at most by a factor of $d-\varepsilon$. Since each vertex in $b \in B$ has at most Δ neighbours, we always have

$$|C_b| \geq (d-\varepsilon)^\Delta|V_{h(b)}| - |X \cap V_{h(b)}| \geq ((d/2)^\Delta - 2\varepsilon)|V_{h(b)}| \geq \tfrac{1}{2}(d/2)^\Delta|V_{h(b)}| = c|V_{h(b)}|.$$

Finally we consider step (a). The last inequality shows that we always have $|C_b| \geq c|V_{h(b)}| \geq \varepsilon|V_{h(b)}|$ for every vertex $b \in V(B)$. Consequently, by Proposition 3.6, at most $\Delta\varepsilon|V_{h(x)}|$ vertices are deleted from C_x in step (a). Since $\Delta\varepsilon|V_{h(x)}| \leq (c/2)|V_{h(x)}| < |C_x|$, the set C_x does not become empty and thus $f(x)$ can be chosen in step (b). □

3.3.3 A general embedding lemma

When embedding a spanning graph H into a host graph G a possible strategy is as follows: Assume we are given a regular partition of G with cluster graph R and a subgraph $R' \subseteq R$ which consists of many small components such that the partition of G is super-regular on each of these components. Assume further that we have a partition of H that is "compatible" with the partition of G (exact definitions follow below). Then we will first use the fact that G is regular on R and apply the partial embedding lemma (Lemma 3.10) to embed some few special vertices of H and define candidate sets for their neighbours. In a second step we will use the blow-up lemma (Lemma 3.9) separately on each component of R' to embed all remaining vertices of H into clusters of G and the edges between them on super-regular pairs corresponding to edges of R'.

This joint application of partial embedding lemma and blow-up lemma is encapsulated in the next lemma, the general embedding lemma. Before stating it we need to make the notion of "compatible" partitions precise. This is addressed in the following definition where we require the partition of H to have smaller partition classes than the partition of G (condition (*i*)) and to be an R-partition. This means that edges of H run only between partition classes that correspond to a dense regular pair in G (condition (*ii*)). Further, in each partition class W_i of H we identify two subsets X_i and Y_i that are both supposed to be small (condition (*iii*)). The set X_i contains those vertices that send edges over pairs that do not belong to the super-regular pairs specified by R' and Y_i contains neighbours of such vertices.

Definition 3.11 (ε-compatible). *Let $H = (W, E_H)$ and $R = ([k], E_R)$ be graphs and let $R' = ([k], E_{R'})$ be a subgraph of R. We say that a vertex partition $W = (W_i)_{i\in[k]}$ of H is* ε-compatible *with an integer partition $(n_i)_{i\in[k]}$ of n and with $R' \subseteq R$ if the following holds. For $i \in [k]$ let X_i be the set of vertices in W_i with neighbours in some W_j with $ij \notin E_{R'}$, set $X := \bigcup X_i$ and $Y_i := N_H(X) \cap W_i \setminus X$. Then for all $i, j \in [k]$ we have that*

(*i*) $|W_i| \le n_i$,

(*ii*) *$xy \in E_H$ for $x \in W_i$ and $y \in W_j$ implies $ij \in E_R$,*

(*iii*) *$|X_i| \le \varepsilon n_i$ and $|Y_i| \le \varepsilon \cdot \min\{n_j : i$ and j are in the same component of $R'\}$.*

The partition $W = (W_i)_{i\in[k]}$ of H is ε-compatible with a partition $V = (V_i)_{i\in[k]}$ of a graph G and with $R' \subseteq R$ if $W = (W_i)_{i\in[k]}$ is ε-compatible with $(|V_i|)_{i\in[k]}$ and with $R' \subseteq R$.

The general embedding lemma asserts that a bounded-degree graph H can be embedded into a graph G if H and G have compatible partitions in the sense explained above.

Lemma 3.12 (general embedding lemma). *For all $d, \Delta, r > 0$ there is $\varepsilon = \varepsilon(d, \Delta, r) > 0$ such that the following holds. Let $G = (V, E)$ be an n-vertex graph that has an (ε, d)-regular partition $V = (V_i)_{i\in[k]}$ with reduced graph R on $[k]$ which is (ε, d)-super regular on a graph $R' \subseteq R$ connected components having at most r vertices each. Further, let $H = (W, E_H)$ be an n-vertex graph with maximum degree $\Delta(H) \le \Delta$ that has a vertex partition $W = (W_i)_{i\in[k]}$ which is ε-compatible with $V = (V_i)_{i\in[k]}$ and $R' \subseteq R$. Then $H \subseteq G$.*

Proof. Given d, Δ, and r we request constants ε_{PEL} and c_{PEL} from Lemma 3.10 with input d and Δ. Further, Lemma 3.9 with input d, Δ, c_{PEL}, and r provides constants ε_{BL} and α_{BL}. We set $\varepsilon := \min\{\varepsilon_{\text{PEL}}, \varepsilon_{\text{BL}}, \frac{1}{2}\alpha_{\text{BL}}\}$. Now, let the graphs $G = (V, E)$ and $H = (W, E_H)$, the partitions $V = V_1 \dot\cup \cdots \dot\cup V_k$ and $W = W_1 \dot\cup \cdots \dot\cup W_k$, the reduced graph $R = (V_R, E_R)$ and its subgraph R' be such that the conditions of the lemma are satisfied.

Our first goal is to apply the partial embedding lemma (Lemma 3.10). To this end, for all $i \in [k]$, let X_i and Y_i be the sets in Definition 3.11, set $X := \bigcup X_i$, $Y := \bigcup Y_i$, and $B := H[X \dot\cup Y]$. Let $h \colon V(B) \to V_R$ be the mapping that maps each vertex $v \in V(B) = X \dot\cup Y$ to the unique cluster i in R with $v \in W_i$. This is a homomorphism because $W_1 \dot\cup \cdots \dot\cup W_k$ is ε-compatible with $R' \subseteq R$. Moreover, for all $i \in [k]$ we have $|h^{-1}(i)| \le 2\varepsilon|V_i|$ since $h^{-1}(i) \subseteq X_i \dot\cup Y_i$ by definition of h and, again by ε-compatibility, $|X_i \cup Y_i| \le 2\varepsilon|V_i|$. Since additionally $\Delta(B) \le \Delta(H) \le \Delta$, the graph B, and the homomorphism h satisfy the conditions of Lemma 3.10. As $\varepsilon \le \varepsilon_{\text{PEL}}$, we can apply this lemma to the graph G, its partition $V_1 \dot\cup \cdots \dot\cup V_k$ with reduced graph R, and to B and h, and obtain an embedding $f \colon X \to V$ of $B[X] = H[X]$ into G with $f(x) \in V_{h(x)}$ for all $x \in X$ and candidate sets $C_y \subseteq V_{h(y)} \setminus f(X)$ for all $y \in Y$. We have now already embedded all the vertices in X but not yet those in Y.

We will embed the remaining vertices of H by using the blow-up lemma separately on each component of R'. For this purpose let $R'_1, \dots, R'_s$ be the components of R'. Now, fix $j \in [s]$, let $r'_j := |R'_j|$, let $i(1), \dots, i(r'_j)$ be the vertices of R'_j, and observe that for each vertex i of R'_j the partial embedding lemma generated candidate sets C_y for at most $|Y_i| \le \varepsilon \cdot \min_{i' \in V(R'_j)} |V_{i'}| \le \alpha_{\text{BL}} \cdot \min_{i' \in V(R'_j)} |V_{i'} \setminus X_{i'}|$ vertices $y \in W_i$ with $|C_y| \ge c_{\text{PEL}}|V_i| \ge c_{\text{PEL}}|V_i \setminus X_i|$. Since $\varepsilon \le \alpha_{\text{BL}}, \varepsilon_{\text{BL}}$ and R'_j has at most r vertices we can provide the blow-up lemma with these candidate sets together with the graphs R'_j, $G_j := G[V_{i(1)} \cup \dots \cup V_{i(r')} \setminus X]$ and $H_j := H[W_{i(1)} \cup \dots \cup W_{i(r')} \setminus f(X)]$ and obtain an embedding f_j of H_j into G_j.

We claim that the united embedding $g \colon W \to V$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \\ f_j(x) & \text{if } x \in V(H_j) \end{cases}$$

is an embedding of H into G. Indeed, let xy be an edge of H. If both $x, y \in X$ then $g(x)g(y) = f(x)f(y)$, which is an edge of G because f is an embedding. Similarly, if x and y are both in the same graph H_j, then $g(x)g(y) = f_j(x)f_j(y)$ is an edge of G as f_j is an embedding. By the definition of X it remains to consider the case $x \in X, y \notin X$ which implies that y lies in some graph H_j and $y \in Y$ by the definition of Y. Accordingly, the partial embedding lemma (Lemma 3.10) defined a candidate set C_y for y with $C_y \subseteq N_G(g(x))$ and the blow-up lemma (Lemma 3.9) guaranteed $f_j(y) \in C_y$. It follows that $g(x)g(y) = f(x)f_j(y)$ is also an edge of G in this case. □

3.4 Sparse graphs

In this section we will consider a sparse version of the regularity lemma, developed by Kohayakawa and Rödl (see [57, 59]). Before stating this lemma we introduce the necessary definitions.

In Section 1.3.7 we explained that the key to a meaningful regularity concept for sparse graphs G is to scale all densities (that we determine or compare) by dividing them by a scaling factor p. This scaling factor p can be thought of as the density of the graph G (or some super-graph Γ of G) or, more abstractly, as a parameter measuring how sparse the graphs are that we are interested in, which will usually depend on the number n of vertices in G. The following definitions provide scaled versions (and hence sparse analogues) of density and regularity.

Let $G = (V, E)$ be a graph, $p \in (0, 1]$, and $\varepsilon, d > 0$ be reals. For disjoint nonempty $U, W \subseteq V$ the *p-density* of the pair (U, W) is defined by $d_{G,p}(U, W) := e_G(U, W)/(p|U||W|)$.

Definition 3.13 ((ε, d, p)-dense). *The pair (U, W) is* (ε, d, p)-dense *if $d_{G,p}(U', W') \ge d - \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \ge \varepsilon|U|$ and $|U'| \ge \varepsilon|U|$. Omitting the parameters d, or ε and d, we may also speak of* (ε, p)-dense *pairs, or* p-dense *pairs.*

Let us remark that the two-sided error bound we saw in the definition of (dense) regularity is replaced by a one-sided error bound in this definition. More precisely, instead of requiring $|d_{G,p}(U, W) - d_{G,p}(U', W')| \le \varepsilon$ (which were a "true" sparse analogue of regularity, cf. Definition 3.1) we demand only $d_{G,p}(U', W') \ge d - \varepsilon$. But for our embedding applications it is only this lower bound that is of interest (we do not mind if there are "too many" edges).

We now turn to the definition of a p-dense partition.

Definition 3.14 (p-dense partition). *An* (ε, p)-dense partition *of $G = (V, E)$ is a partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_r$ of V with $|V_0| \le \varepsilon|V|$ such that (V_i, V_j) is an (ε, p)-dense pair in G for all but at most $\varepsilon\binom{r}{2}$ pairs $ij \in \binom{[r]}{2}$.*

As before, the partition classes V_i with $i \in [r]$ are called the *clusters* of the partition and V_0 is the *exceptional set*. Moreover, an (ε, p)-dense partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_r$ is an (ε, d, p)-*dense partition* of G with *reduced graph* $R = ([r], E(R))$ if the pair (V_i, V_j) is (ε, d, p)-dense in G whenever $ij \in E(R)$.

The sparse regularity lemma asserts p-dense partitions for sparse graphs G without dense spots. To quantify this latter property we need the following notion. Let $\eta > 0$ be a real number and $K > 1$ an integer. We say that $G = (V, E)$ is (η, K)-*bounded with respect to* p if for all disjoint sets $X, Y \subseteq V$ with $|X|, |Y| \geq \eta|V|$ we have $e_G(X, Y) \leq Kp|X||Y|$.

Lemma 3.15 (sparse regularity lemma)**.** *For each $\varepsilon > 0$, each $K > 1$, and each $r_0 \geq 1$ there are constants r_1, ν, and n_0 such that for any $p \in (0, 1]$ the following holds. Any graph $G = (V, E)$ with $|V| \geq n_0$, and which is (ν, K)-bounded with respect to p admits an (ε, p)-dense equipartition with r clusters for some $r_0 \leq r \leq r_1$.* □

In analogy to the (dense) regularity lemma this lemma produces a partition of a sparse graph into a constant number of p-dense pairs (with the restriction that we consider only bounded graphs). Moreover, as is true for dense regular pairs, it follows directly from the definition that sub-pairs of p-dense pairs again form p-dense pairs (Proposition 3.8 discusses a similar property of dense regular pairs but is much more general).

Proposition 3.16. *Let (X, Y) be (ε, d, p)-dense, $X' \subseteq X$ with $|X'| = \mu|X|$. Then (X', Y) is $(\frac{\varepsilon}{\mu}, d, p)$-dense.* □

Also Proposition 3.6 has a sparse counterpart: Neighbourhoods of most vertices in a p-dense pair are not much smaller than expected. Again, this is a direct consequence of the definition of p-dense pairs.

Proposition 3.17. *Let (X, Y) be (ε, d, p)-dense. Then less than $\varepsilon|X|$ vertices $x \in X$ have $|N_Y(x)| < (d - \varepsilon)p|Y|$.* □

But in contrast to the non-sparse case the size of such neighbourhoods may be tiny (if $p = o(1)$). Recall, for example, that the iterative embedding procedure in the proof of the partial embedding lemma presented above (Lemma 3.10) was based on the fact that typical vertices in a regular pair have neighbourhoods of linear size. Therefore it is not surprising that establishing embedding results for p-dense pairs (or partitions) is often more difficult. As we shall discuss in Chapter 9, however, such results do exist if we consider p-dense pairs inside a random graph. This is also the setting of the following and last lemma that we shall formulate in this chapter.

Similarly as for dense graphs some properties of the graph G translate to certain properties of the reduced graph R of the partition constructed by the sparse regularity lemma. An example of this phenomenon is given in the following lemma, Lemma 3.18, which is a minimum degree version of the sparse regularity lemma. The minimum degree version of the dense regularity lemma (Lemma 3.4) can be formulated in the following way. If an n-vertex graph G is such that that each of its vertices has an α-proportion of all the $n - 1$ neighbours it has in the complete graph K_n as neighbours in G then the reduced graph R of a regular partition of this graph has minimum degree only slightly smaller than $\alpha|V(R)|$. The sparse analogue, Lemma 3.18, now replaces the complete graph K_n in this formulation by a random graph $\Gamma = \mathcal{G}_{n,p}$. Consequently, the graph G is a subgraph of Γ such that each vertex v of G satisfies the following condition. The neighbourhood of v in the graph G is at least an α-proportion of the neighbourhood of v in the random graph Γ. Then, as the lemma asserts, G has an (ε, d, p)-dense partition with reduced graph R on r vertices and minimum degree almost αr.

Lemma 3.18 (sparse regularity lemma, minimum degree version for $\mathcal{G}_{n,p}$)**.** *For all $\alpha \in [0,1]$, $\varepsilon > 0$, and every integer r_0, there is an integer $r_1 \geq 1$ such that for all $d \in [0,1]$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ if $\log^4 n/(pn) = o(1)$. Let $G = (V, E)$ be a spanning subgraph of Γ with $\deg_G(v) \geq \alpha \deg_\Gamma(v)$ for all $v \in V$. Then there is an (ε, d, p)-dense partition of G with reduced graph R of minimum degree $\delta(R) \geq (\alpha - d - \varepsilon)|V(R)|$ with $r_0 \leq |V(R)| \leq r_1$.*

Notice that, in contrast to the dense minimum degree version of the regularity lemma (Lemma 3.4), we do observe "more" than a mere inheritance of properties here: the graph G we started with is *sparse*, but the reduced graph R we obtain is *dense*. This will enables us, as we shall illustrate in Chapter 9, to apply results obtained for dense graphs to the reduced graph R, and hence use such dense results to draw conclusions about sparse graphs.

Proof. For the proof we shall use the sparse regularity lemma (Lemma 3.15) and the facts about the edge distribution in random graphs provided by Lemma 2.6.

Given α, ε, and r_0 let r_1, ν, and n_0 be as provided by Lemma 3.15 for input

$$\varepsilon' := \varepsilon^2/100\,, \quad K := 1 + \varepsilon'\,, \quad \text{and} \quad r_0' := \min\{2r_0, \lceil 1/\varepsilon' \rceil\}\,.$$

Let further d be given and assume that n is such that $n \geq n_0$, $\log n \geq 1/\varepsilon'$, and $\log n \geq 1/\nu$. Let Γ be a typical graph from $\mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$, i.e., a graph satisfying properties (*i*)–(*iii*) of Lemma 2.6. We will show that then Γ also satisfies the conclusion of Lemma 3.18.

To this end we consider an arbitrary subgraph $G = (V, E)$ of Γ that satisfies the assumptions of this lemma. By property (*ii*) of Lemma 2.6 the graph $G \subseteq \Gamma$ is $(1/\log n, 1 + 1/\log n)$-bounded with respect to p. Because $1 + 1/\log n \leq 1 + \varepsilon' = K$ the sparse regularity lemma (Lemma 3.15) with input ε', K, and r_0' asserts that G has an (ε', p)-dense equipartition $V = V_0' \dot\cup V_1' \dot\cup \dots \dot\cup V_{r'}'$ for some $r_0' \leq r' \leq r_1$. Observe that there are at most $r'\sqrt{\varepsilon'}$ clusters in this partition which are contained in more than $r'\sqrt{\varepsilon'}$ pairs that are not (ε', p)-dense. We add all these clusters to V_0', denote the resulting set by V_0 and the remaining clusters by $V_1, \dots, V_r$. Then $r_0 \leq r'/2 \leq r \leq r_1$ and we claim that the partition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_r$ has the desired properties.

Indeed, $|V_0| \leq \varepsilon' n + r'\sqrt{\varepsilon'}(n/r') \leq \varepsilon n$ and the number of pairs in $V_1 \dot\cup \dots \dot\cup V_r$ which are not (ε, p)-dense is at most $r \cdot r'\sqrt{\varepsilon'} \leq 2r^2\sqrt{\varepsilon'} \leq \varepsilon\binom{r}{2}$. It follows that $V_1 \dot\cup \dots \dot\cup V_r$ is an (ε, p)-dense partition and hence an (ε, d, p)-dense partition. Let R be the (maximal) corresponding reduced graph, i.e., R has vertex set $[r]$ and edges ij for exactly all (ε, d, p)-dense pairs (V_i, V_j) with i, $j \in [r]$. It remains to show that we have $\delta(R) \geq (\alpha - d - \varepsilon)|R|$.

To see this, define $L := |V_i|$ for all $i \in [r]$ and consider arbitrary disjoint sets $X, Y \subseteq V(G)$. Then $\sum_{x \in X} \deg_G(x) = 2e_G(X) + e_G(X, Y) + e_G(X, V \setminus (X \cup Y))$ and therefore

$$e_G(X, Y) \geq \Big(\alpha \sum_{x \in X} \deg_\Gamma(x)\Big) - 2e_\Gamma(X) - e_\Gamma\big(X, V \setminus (X \cup Y)\big).$$

By properties (*i*)–(*iii*) of Lemma 2.6 this implies

$$\begin{aligned} e_G(X, Y) &\geq \alpha\Big(1 - \frac{1}{\log n}\Big)p|X|n - 2\Big(1 + \frac{1}{\log n}\Big)p\binom{|X|}{2} \\ &\qquad - \big(1 + \frac{1}{\log n}\Big)p|X|\Big(n - |X| - |Y|\Big) \\ &\geq \big(\alpha(1 - \varepsilon')n - (1 + \varepsilon')(n - |Y|)\big)p|X|\,, \end{aligned} \tag{3.1}$$

as long as $|X| \geq n/\log n$ and $|X \cup Y| \leq n - n/\log n$. Now fix $i \in [r]$ and let $\bar{V_i} := V \setminus (V_0 \cup V_i)$. Then

$$e_G(V_i, \bar{V_i}) \leq \left(\deg_R(i) + 2r\sqrt{\varepsilon'}\right)\left(1+\varepsilon'\right)pL^2 + \left(r - \deg_R(i)\right)dpL^2$$

since each cluster is in at most $r'\sqrt{\varepsilon'} \leq 2r\sqrt{\varepsilon'}$ irregular pairs, since R is an (ε', d, p)-reduced graph and $G \subseteq \Gamma$ is $(1/\log n, 1+\varepsilon')$-bounded with respect to p. On the other hand (3.1) implies

$$\begin{aligned} e_G(V_i, \bar{V_i}) &\geq \Big(\alpha(1-\varepsilon')n - (1+\varepsilon')\big(|V_0| + |V_i|\big)\Big)p|V_i| \\ &\geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'}\Big)pLn \end{aligned}$$

where we use $|V_0| \leq (\varepsilon' + \sqrt{\varepsilon'})n$ and $|V_i| \leq n/r'_0 \leq \varepsilon' n$. We conclude that

$$\Big(\deg_R(i)(1+\varepsilon'-d) + 2r\sqrt{\varepsilon'}(1+\varepsilon') + rd\Big)pL^2 \geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'}\Big)prL^2$$

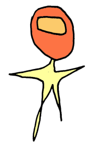
since $n/L \geq r$. This gives

$$\begin{aligned} \deg_R(i) \geq \deg_R(i)(1+\varepsilon'-d) &\geq \Big(\alpha(1-\varepsilon') - (1+\varepsilon')3\sqrt{\varepsilon'} - 2\sqrt{\varepsilon'}(1+\varepsilon') - d\Big)r \\ &\geq \Big(\alpha - \alpha\varepsilon' - 9\sqrt{\varepsilon'} - d\Big)r \geq (\alpha - d - \varepsilon)|R|\,. \end{aligned}$$

Hence the (ε, d, p)-dense partition $V = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_r$ has a reduced graph R with $\delta(R) \geq (\alpha - d - \varepsilon)|R|$. □

Chapter 4

Bandwidth, expansion, treewidth, and universality



The purpose of this chapter is to establish relations between the bandwidth and the treewidth of bounded-degree graphs H, and to connect these parameters to the size of a separator of H as well as the size of an expanding subgraph of H. Our results imply that if one of these values is sublinear in the number of vertices of H then so are all the others (see Theorem 4.6 in Section 4.1). This implies for example that graphs of fixed genus have sublinear bandwidth or, more generally, a corresponding result for graphs with any fixed forbidden minor (see Corollary 4.9 in Section 4.2).

4.1 Relations

There are a number of different parameters in graph theory which measure how well a graph can be organized in a particular way, where the type of desired arrangement is often motivated by geometrical properties, algorithmic considerations or specific applications. Well-known examples of such parameters are the genus, the bandwidth, or the treewidth of a graph. While the the genus characterizes the surfaces on which a particular graph can be drawn without crossings, the other two parameters describe how well a graph can be laid out in a path-like, respectively tree-like, manner. The central topic of this chapter is to discuss the relations between such parameters. We would like to determine how they influence each other and what causes them to be large. To this end we will mostly be interested in distinguishing between the case where these parameters are linear in n, where n is the number of vertices in the graph under investigation, and the case where they are sublinear in n.

Clearly one reason for a graph to have high bandwidth are vertices of high degree. Thus the star $K_{1,n-1}$ illustrates that in general even trees may have a bandwidth of order $\Omega(n)$. In [24] Chung proved however that any n-vertex tree T with maximum degree Δ has bandwidth at most $5n/\log_\Delta(n)$. The following theorem extends Chung's result to planar graphs.

Theorem 4.1. *Suppose $\Delta \geq 4$. Let H be a planar graph on n vertices with maximum degree at most Δ. Then the bandwidth of H satisfies*

$$\mathrm{bw}(H) \leq \frac{15n}{\log_\Delta(n)}.$$

It is easy to see that the bound in Theorem 4.1 is sharp up to the multiplicative constant—since the bandwidth of any graph H is bounded from below by $(n-1)/\operatorname{diam}(H)$, it suffices to consider for example the complete binary tree on n vertices. (We remark in passing that Theorem 4.1 implies that graphs with maximum degree $1 \leq \Delta \leq 3$ must satisfy an upper bound of $\mathrm{bw}(H) \leq 20n/\log_\Delta(n)$.) Theorem 4.1 is used in [23] to infer a result about the geometric realizability of planar graphs $H = (V, E)$ with $|V| = n$ and $\Delta(H) \leq \Delta$.

As explained in the introduction we will show a more general theorem (Theorem 4.6) which proves that the concepts of sublinear bandwidth, sublinear treewidth, bad expansion properties, and sublinear separators are equivalent for graphs of bounded maximum degree. In order to establish this theorem, we will now discuss quantitative relations between the parameters involved. Since planar graphs are known to have small separators [75], we will get Theorem 4.1 as a byproduct of these results in Section 4.2.1.

Let us start with the following well known theorem due to Robertson and Seymour [83] that relates the treewidth and the separation number of a graph.[1]

Theorem 4.2 (treewidth→separator, [83]). *All graphs H have separation number*

$$\mathrm{s}(H) \leq \mathrm{tw}(H) + 1\,. \qquad \square$$

This theorem states that graphs with small treewidth have small separators. By repeatedly extracting separators, one can show that (a qualitatively different version of) the converse also holds: $\mathrm{tw}(H) \leq \mathcal{O}(\mathrm{s}(H) \log n)$ for a graph H on n vertices (see e.g. [16], Theorem 20). We shall use a similar but more involved argument to show that one can establish the following relation linking the separation number with the bandwidth of graphs with bounded maximum degree.

Theorem 4.3 (separator→bandwidth). *For each $\Delta \geq 4$ every graph H on n vertices with maximum degree $\Delta(H) \leq \Delta$ has bandwidth*

$$\mathrm{bw}(H) \leq \frac{6n}{\log_\Delta(n/\,\mathrm{s}(H))}.$$

The proof of this theorem is provided in Section 4.3.1. Observe that Theorems 4.2 and 4.3 together with the obvious inequality $\mathrm{tw}(H) \leq \mathrm{bw}(H)$ tie the concepts of treewidth, bandwidth, and separation number well together. Apart from the somewhat negative statement of *not having* a small separator, what can we say about a graph with large tree- or bandwidth? The next theorem states that such a graph must contain a big expander.

Theorem 4.4 (non-expansion→treewidth). *Let $\varepsilon > 0$ be constant. All graphs H on n vertices have treewidth $\mathrm{tw}(H) \leq 2\,\mathrm{b}_\varepsilon(H) + 2\varepsilon n$.*

A result with similar implications was recently proved by Grohe and Marx in [45]. It shows that $b_\varepsilon(H) < \varepsilon n$ implies $\mathrm{tw}(H) \leq 2\varepsilon n$. For the sake of being self contained we present our (short) proof of Theorem 4.4 in Section 4.3.2. In addition, it is not difficult to see that conversely the non-expansion of a graph can be estimated via its bandwidth—which we prove in Section 4.3.2, too.

Proposition 4.5 (bandwidth→non-expansion). *Let $\varepsilon > 0$ be constant. All graphs H on n vertices have $\mathrm{b}_\varepsilon(H) \leq 2\,\mathrm{bw}(H)/\varepsilon$.*

A qualitative consequence summarizing the four results above is given in the following theorem. It states that if one of the parameters bandwidth, treewidth, separation number, or non-expansion is sublinear for a family of graphs, then so are the others.

[1] In fact, their result states that any graph H has a $(\mathrm{tw}(H)+1, 1/2)$-separator, and doesn't talk about subgraphs of H. But since every subgraph of H has treewidth at most $\mathrm{tw}(H)$, it thus also has a $(\mathrm{tw}(H) + 1, 1/2)$-separator and the result, as stated here, follows.

Theorem 4.6 (sublinear equivalence theorem). *Let Δ be an arbitrary but fixed positive integer and consider a hereditary class of graphs $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have maximum degree at most Δ. Denote by $\mathcal{C}_n$ the set of those graphs in $\mathcal{C}$ with n vertices. Then the following four properties are equivalent:*

(*1*) *For all $\beta_1 > 0$ there is n_1 such that $\mathrm{tw}(H) \leq \beta_1 n$ for all $H \in \mathcal{C}_n$ with $n \geq n_1$.*

(*2*) *For all $\beta_2 > 0$ there is n_2 such that $\mathrm{bw}(H) \leq \beta_2 n$ for all $H \in \mathcal{C}_n$ with $n \geq n_2$.*

(*3*) *For all $\beta_3, \varepsilon > 0$ there is n_3 such that $\mathrm{b}_\varepsilon(H) \leq \beta_3 n$ for all $H \in \mathcal{C}_n$ with $n \geq n_3$.*

(*4*) *For all $\beta_4 > 0$ there is n_4 such that $\mathrm{s}(H) \leq \beta_4 n$ for all $H \in \mathcal{C}_n$ with $n \geq n_4$.*

Proof. (*1*)⇒(*4*): Given $\beta_4 > 0$ set $\beta_1 := \beta_4/2$, let n_1 be the constant from (*1*) for this β_1, and set $n_4 := \max\{n_1, 2/\beta_4\}$. Now consider $H \in \mathcal{C}_n$ with $n \geq n_4$. By assumption we have $\mathrm{tw}(H) \leq \beta_1 n$ and thus we can apply Theorem 4.2 to conclude that $\mathrm{s}(H) \leq \mathrm{tw}(H) + 1 \leq \beta_1 n + 1 \leq (\beta_4/2 + 1/n)n \leq \beta_4 n$.

(*4*)⇒(*2*): Given $\beta_2 > 0$ let $d := \min\{4, \Delta\}$, set $\beta_4 := d^{-6/\beta_2}$, get n_4 from (*4*) for this β_4, and set $n_2 := n_4$. Let $H \in \mathcal{C}_n$ with $n \geq n_2$. We conclude from (*4*) and Theorem 4.2 that

$$\mathrm{bw}(H) \leq \frac{6n}{\log_d n - \log_d \mathrm{s}(H)} \leq \frac{6n}{\log_d n - \log_d(d^{-6/\beta_2} n)} = \beta_2 n.$$

(*2*)⇒(*3*): Given $\beta_3, \varepsilon > 0$ set $\beta_2 := \varepsilon\beta_3$, get n_2 from (*2*) for this β_2 and set $n_3 := n_2$. By (*2*) and Proposition 4.5 we get for $H \in \mathcal{C}_n$ with $n \geq n_3$ that $\mathrm{b}_\varepsilon(H) \leq 2\,\mathrm{bw}(H)/\varepsilon \leq 2\beta_2 n/\varepsilon \leq \beta_3 n$.

(*3*)⇒(*1*): Given $\beta_1 > 0$, set $\beta_3 := \beta_1/4$, $\varepsilon := \beta_1/4$ and get n_3 from (*3*) for this β_3 and ε, and set $n_1 := n_3$. Let $H \in \mathcal{C}_n$ with $n \geq n_4$. Then (*3*) and Theorem 4.4 imply $\mathrm{tw}(H) \leq 2\,\mathrm{b}_\varepsilon(H) + 2\varepsilon n \leq 2\beta_3 n + 2(\beta_1/4)n = \beta_1 n$. □

4.2 Applications

For many interesting bounded-degree graph classes (non-trivial) upper bounds on the bandwidth are not at hand. A wealth of results however has been obtained about the existence of sublinear separators. This illustrates the importance of Theorem 4.6. In this section we will give examples of such separator theorems and provide applications of them in conjunction with Theorem 4.6.

4.2.1 Separator theorems

A classical result in the theory of planar graphs concerns the existence of separators of size $2\sqrt{2n}$ in any planar graph on n vertices proven by Lipton and Tarjan [75] in 1977. Clearly, together with Theorem 4.3 this result implies Theorem 4.1 from the introduction. This motivates why we want to consider some generalizations of the planar separator theorem in this section. The first such result is due to Gilbert, Hutchinson, and Tarjan [42] and deals with graphs of arbitrary genus. [2]

Theorem 4.7 (Gilbert, Hutchinson, Tarjan [42]). *An n-vertex graph H with genus $g \geq 0$ has separation number $\mathrm{s}(H) \leq 6\sqrt{gn} + 2\sqrt{2n}$.* □

[2] Again, the separator theorems we refer to bound the size of a separator in H. Since the class of graphs with genus less than g (or, respectively, of F-minor free graphs) is closed under taking subgraphs however, this theorem can also be applied to such subgraphs and thus the bound on $\mathrm{s}(H)$ follows.

For fixed g the class of all graphs with genus at most g is closed under taking minors. Here F is a *minor* of H if it can be obtained from H by a sequence of edge deletions and contractions. A graph H is called F-minor free if F is no minor of H. The famous graph minor theorem by Robertson and Seymour [84] states that any minor closed class of graphs can be characterized by a finite set of forbidden minors (such as $K_{3,3}$ and K_5 in the case of planar graphs). The next separator theorem by Alon, Seymour, and Thomas [12] shows that already forbidding one minor enforces a small separator.

Theorem 4.8 (Alon, Seymour, Thomas [12])**.** *Let F be an arbitrary graph. Then any n-vertex graph H that is F-minor free has separation number* $\mathrm{s}(H) \leq |F|^{3/2}\sqrt{n}$. □

We can apply these theorems to draw the following conclusion concerning the bandwidth of bounded-degree graphs with fixed genus or some fixed forbidden minor from Theorem 4.3.

Corollary 4.9. *Let g be a positive integer, $\Delta \geq 4$ and F be an h-vertex graph and H an n-vertex graph with maximum degree $\Delta(H) \leq \Delta$.*

(*a*) *If H has genus g then* $\mathrm{bw}(H) \leq 15n/\log_{\Delta}(n/g)$.

(*b*) *If H is F-minor free then* $\mathrm{bw}(H) \leq 12n/\log_{\Delta}(n/h^3)$. □

4.2.2 Universality

In the previous section we argued that certain interesting graph classes have sublinear bandwidth. Clearly, this knowledge has different consequences in conjunction with the embedding results for graphs of sublinear bandwidth established in the later chapters of this thesis. In this section we will briefly outline the nature of such implications. We concentrate on corollaries of Theorem 5.1 here. Similar corollaries can of course be formulated concerning the other embedding results.

Recall that a graph G that contains copies of all graphs $H \in \mathcal{H}$ for some class of graphs $\mathcal{H}$ is also called *universal for $\mathcal{H}$*. Theorem 4.6 states that in bounded-degree graphs, the existence of a big expanding subgraph is in fact the only obstacle which can prevent sublinear bandwidth and thus the only possible obstruction for a universality result as in Theorem 5.1. More precisely we immediately get the following corollary from Theorem 4.6.

Corollary 4.10. *If the class $\mathcal{C}$ meets one (and thus all) of the conditions in Theorem 4.6, then the following is also true. For every $\gamma > 0$ and $r \in \mathbb{N}$ there exists n_0 such that for all $n \geq n_0$ and for every graph $H \in \mathcal{C}_n$ with chromatic number r and for every graph G on n vertices with minimum degree at least $(\frac{r-1}{r} + \gamma)n$, the graph G contains a copy of H.* □

By Theorem 4.1 we infer as a special case that all sufficiently large graphs with minimum degree $(\frac{3}{4} + \gamma)n$ are universal for the class of bounded-degree planar graphs. Universal graphs for bounded-degree planar graphs have also been studied in [15, 22].

Corollary 4.11. *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds:*

(*a*) *Every 3-chromatic planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{2}{3} + \gamma)n$.*

(*b*) *Every planar graph on n vertices with maximum degree at most Δ can be embedded into every graph on n vertices with minimum degree at least $(\frac{3}{4} + \gamma)n$.* □

This extends Theorem 1.8 due to Kühn, Osthus & Taraz, which states that for every graph G with minimum degree at least $(\frac{2}{3}+\gamma)n$ *there exists a particular* spanning triangulation H that can be embedded into G. Using Corollary 4.9 it is moreover possible to formulate corresponding generalizations for graphs of fixed genus and for F-minor free graphs for any fixed F.

4.3 Proofs

4.3.1 Separation and bandwidth

For the proof of Theorem 4.3 we will use a decomposition result which roughly states the following. If the removal of a small separator S decomposes the vertex set of a graph H into relatively small components $R_i \dot\cup P_i$ such that the vertices in P_i form a "buffer" between the vertices in the separator S and the set of remaining vertices R_i in the sense that $\mathrm{dist}_H(S, R_i)$ is sufficiently big, then the bandwidth of H is small.

Lemma 4.12 (decomposition lemma)**.** *Let $H = (V, E)$ be a graph and S, P, and R be vertex sets such that $V = S \dot\cup P \dot\cup R$. For $b, r \in \mathbb{N}$ with $b \geq 3$ assume further that there are decompositions $P = P_1 \dot\cup \dots \dot\cup P_b$ and $R = R_1 \dot\cup \dots \dot\cup R_b$ of P and R, respectively, such that the following properties are satisfied:*

- (*i*) $|R_i| \leq r$,
- (*ii*) $e(R_i \dot\cup P_i, R_j \dot\cup P_j) = 0$ *for all* $1 \leq i < j \leq b$,
- (*iii*) $\mathrm{dist}_H(u, v) \geq b/2$ *for all* $u \in S$ *and* $v \in R_i$ *with* $i \in [b]$.

Then $\mathrm{bw}(H) \leq 2(|S| + |P| + r)$.

Proof. Assume we have $H = (V, E)$, $V = S \dot\cup P \dot\cup R$ and $b, r \in \mathbb{N}$ with the properties stated above. Our first goal is to partition V into pairwise disjoint sets $B_1, \dots, B_b$, which we call *buckets*, and that satisfy the following property:

$$\text{If } \{u, v\} \in E \text{ for } u \in B_i \text{ and } v \in B_j \text{ then } |i - j| \leq 1. \tag{4.1}$$

To this end all vertices of R_i are placed into bucket B_i for each $i \in [b]$ and the vertices of S are placed into bucket $B_{\lceil b/2 \rceil}$. The remaining vertices from the sets P_i are distributed over the buckets according to their distance from S: vertex $v \in P_i$ is assigned to bucket $B_{j(v)}$ where $j(v) \in [b]$ is defined by

$$j(v) := \begin{cases} i & \text{if } \mathrm{dist}(S, v) \geq |b/2 - i|, \\ \lceil b/2 \rceil - \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < b/2 - i \\ \lceil b/2 \rceil + \mathrm{dist}(S, v) & \text{if } \mathrm{dist}(S, v) < i - b/2. \end{cases} \tag{4.2}$$

This placement obviously satisfies

$$|B_j| \leq |S| + |P| + |R_i| \leq |S| + |P| + r \tag{4.3}$$

by construction and condition (*i*). Moreover, we claim that it guarantees condition (4.1). Indeed, let $\{u, v\} \in E$ be an edge. If u and v are both in S then clearly (4.1) is satisfied. Thus it remains to consider the case where, without loss of generality, $u \in R_i \dot\cup P_i$ for some $i \in [b]$. By condition (*ii*) this implies $v \in S \dot\cup R_i \dot\cup P_i$. First assume that $v \in S$. Thus

$\mathrm{dist}(u,S) = 1$ and from condition (iii) we infer that $u \in P_i$. Accordingly u is placed into bucket $B_{j(u)} \in \{B_{\lceil b/2\rceil-1}, B_{\lceil b/2\rceil}, B_{\lceil b/2\rceil+1}\}$ by (4.2) and v is placed into bucket $B_{\lceil b/2\rceil}$ and so we also get (4.1) in this case. If both $u, v \in R_i \dot\cup P_i$, on the other hand, we are clearly done if $u, v \in R_i$. So assume without loss of generality, that $u \in P_i$. If $v \in P_i$ then we conclude from $|\,\mathrm{dist}(S,u) - \mathrm{dist}(S,v)| \le 1$ and (4.2) that u is placed into bucket $B_{j(u)}$ and v into $B_{j(v)}$ with $|j(u) - j(v)| \le 1$. If $v \in R_i$, finally, observe that $|\,\mathrm{dist}(S,u) - \mathrm{dist}(S,v)| \le 1$ together with condition (iii) implies that $\mathrm{dist}(S,u) \ge b/2 - 1$ and so u is placed into bucket $B_{j(u)}$ with $j(u) \in \{i(v), i(v) - 1\}$ by (4.2) and v is assigned to B_i with $i = i(v)$. Thus we also get (4.1) in this last case.

Now we are ready to construct an ordering of V respecting the desired bandwidth bound. We start with the vertices in bucket B_1, order them arbitrarily, proceed to the vertices in bucket B_2, order them arbitrarily, and so on, up to bucket B_b. By condition (4.1) this gives an ordering with bandwidth at most twice as large as the largest bucket and thus we conclude from (4.3) that $\mathrm{bw}(H) \le 2(|S| + |P| + r)$. □

A decomposition of the vertices of H into buckets as in the proof of Lemma 4.12 is also called a path partition of H and appears e.g. in [30].

Before we get to the proof of Theorem 4.3, we will establish the following technical observation about labelled trees.

Proposition 4.13. *Let b be a positive real, $T = (V,E)$ be a tree with $|V| \ge 3$, and $\ell : V \to [0,1]$ be a real valued labelling of its vertices such that $\sum_{v\in V} \ell(v) \le 1$. Denote further for all $v \in V$ by $L(v)$ the set of leaves that are adjacent to v and suppose that $\ell(v) + \sum_{u\in L(v)} \ell(u) \ge |L(v)|/b$. Then T has at most b leaves in total.*

Proof. Let $L \subseteq V$ be the set of leaves of T and $I := V \setminus L$ be the set of internal vertices. Clearly

$$1 \ge \sum_{v\in V} \ell(v) = \sum_{v\in I} \left(\ell(v) + \sum_{u\in L(v)} \ell(u) \right) \ge \sum_{v\in I} \frac{|L(v)|}{b} = \frac{|L|}{b}$$

which implies the assertion. □

The idea of the proof of Theorem 4.3 is to repeatedly extract separators from H and the pieces that result from the removal of such separators. We denote the union of these separators by S, put all remaining vertices with small distance from S into sets P_i, and all other vertices into sets R_i. Then we can apply the decomposition lemma (Lemma 4.12) to these sets S, P_i, and R_i. This, together with some technical calculations, will give the desired bandwidth bound for H.

Proof of Theorem 4.3. Let $H = (V,E)$ be a graph on n vertices with maximum degree at most $\Delta \ge 4$. Observe that the desired bandwidth bound is trivial if $\log_\Delta n - \log_\Delta \mathrm{s}(H) \le 6$, so assume in the following that $\log_\Delta n - \log_\Delta \mathrm{s}(H) > 6$. Define

$$\beta := \log_\Delta n - \log_\Delta \mathrm{s}(H) \qquad \text{and} \qquad b := \lceil \beta \rceil \ge 7 \tag{4.4}$$

and observe that with this choice of β our aim is to show that $\mathrm{bw}(H) \le 6n/\beta$.

The goal is to construct $V = S \dot\cup P \dot\cup R$ with the properties required by Lemma 4.12. For this purpose we will recursively use the fact that H and its subgraphs have separators of size at most $\mathrm{s}(H)$. In the i-th round we will identify separators $S_{i,k}$ in H whose removal splits H into parts $V_{i,1},\dots,V_{i,b_i}$. The details are as follows.

In the first round let $S_{1,1}$ be an arbitrary $(\mathrm{s}(H),2/3)$-separator in H that separates H into $V_{1,1}$ and $V_{1,2}$ and set $b_1:=2$. In the i-th round, $i>1$, consider each of the sets $V_{i-1,j}$ with $j\in[b_{i-1}]$. If $|V_{i-1,j}|\le 2n/b$ then let $V_{i,j'}:=V_{i-1,j}$, otherwise choose an $(\mathrm{s}(H),2/3)$-separator $S_{i,k}$ that separates $H[V_{i-1,j}]$ into sets $V_{i,j'}$ and $V_{i,j'+1}$ (where k and j' are appropriate indices, for simplicity we do not specify them further). Let S_i denote the union of all separators constructed in this way (and in this round). This finishes the i-th round. We stop this procedure as soon as all sets $V_{i,j'}$ have size at most $2n/b$ and denote the corresponding i by i^*. Then b_{i^*} is the number of sets $V_{i^*,j'}$ we end up with in the last iteration. Let further x_S be the number of separators $S_{i,k}$ extracted from H during this process in total.

Claim 4.14. *We have $b_{i^*}\le b$ and $x_S\le b-1$.*

We will postpone the proof of this fact and first show how it implies the theorem. Set $S:=\bigcup_{i\in[i^*]}S_i$, for $j\in[b_{i^*}]$ define

$$P_j:=\{v\in V_{i^*,j}:\mathrm{dist}(v,S)<\beta/2\}\qquad\text{and}\qquad R_j=V_{i^*,j}\setminus P_j,$$

set $P_j=R_j=\emptyset$ for $b_{i^*}<j\le b$ and finally define $P:=\bigcup_{j\in[b]}P_j$ and $R:=\bigcup_{j\in[b]}R_j$.

We claim that $V=S\dot\cup P\dot\cup R$ is a partition that satisfies the requirements of the decomposition lemma (Lemma 4.12) with parameter b and $r=2n/b$. To check this, observe first that for all $i\in[i^*]$ and $j,j'\in[b_i]$ we have $e(V_{i,j},V_{i,j'})=0$ since $V_{i,j}$ and $V_{i,j'}$ were separated by some $S_{i',k}$. It follows that $e(R_j\dot\cup P_j,R_{j'}\dot\cup P_{j'})=e(V_{i^*,j},V_{i^*,j'})=0$ for all $j,j'\in[b_{i^*}]$. Trivially $e(R_j\dot\cup P_j,R_{j'}\dot\cup P_{j'})=0$ for all $j\in[b]$ and $b_{i^*}<j'\le b$ and therefore we get condition (ii) of Lemma 4.12. Moreover, condition (iii) is satisfied by the definition of the sets P_j and R_j above. To verify condition (i) note that $|R_j|\le|V_{i^*,j}|\le 2n/b=r$ for all $j\in[b_{i^*}]$ by the choice of i^* and $|R_j|=0$ for all $b_{i^*}<j\le b$. Accordingly we can apply Lemma 4.12 and infer that

$$\mathrm{bw}(H)\le 2\left(|S|+|P|+\frac{2n}{b}\right). \tag{4.5}$$

In order to establish the desired bound on the bandwidth, we thus need to show that $|S|+|P|\le n/\beta$. We first bound the size of S. By Claim 4.14 at most $x_S\le b-1$ separators have been extracted in total, which implies

$$|S|\le x_S\cdot\mathrm{s}(H)\le(b-1)\,\mathrm{s}(H). \tag{4.6}$$

Furthermore all vertices $v\in P$ satisfy $\mathrm{dist}_H(v,S)<\beta/2$ by definition. As H has maximum degree Δ there are at most $|S|(\Delta^{\beta/2}-1)/(\Delta-1)$ vertices $v\in V\setminus S$ with this property and hence

$$\begin{aligned}|S|+|P|&\le|S|(1+\frac{\Delta^{\beta/2}-1}{\Delta-1})\le|S|\frac{\Delta^{\beta/2}}{\Delta-2}\le\frac{(b-1)\,\mathrm{s}(H)}{(\Delta-2)}\sqrt{\frac{n}{\mathrm{s}(H)}}\\&=\frac{(b-1)n}{(\Delta-2)}\sqrt{\frac{\mathrm{s}(H)}{n}}\end{aligned}$$

where the third inequality follows from (4.4) and (4.6). It is easy to verify that for any $x\ge1$ and $\Delta\ge4$ we have $(\Delta-2)\sqrt{x}\ge\log^2_\Delta x$. This together with (4.4) gives $(\Delta-2)\sqrt{n/\mathrm{s}(H)}\ge\beta^2$

and hence we get

$$|S| + |P| \leq \frac{(b-1)n}{\beta^2} \leq \frac{n}{\beta}.$$

Together with (4.5) this yields the assertion of the theorem.

It remains to prove Claim 4.14. Notice that the process of repeatedly separating H and its subgraphs can be seen as a binary tree T on vertex set W whose internal nodes represent the extraction of a separator $S_{i,k}$ for some i (and thus the separation of a subgraph of H into two sets $V_{i,j}$ and $V_{i,j'}$) and whose leaves represent the sets $V_{i,j}$ that are of size at most $2n/b$. Clearly the number of leaves of T is b_{i^*} and the number of internal nodes x_S. As T is a binary tree we conclude $x_S = b_{i^*} - 1$ and thus it suffices to show that T has at most b leaves in order to establish the claim. To this end we would like to apply Proposition 4.13. Label an internal node of T that represents a separator $S_{i,k}$ with $|S_{i,k}|/n$, a leaf representing $V_{i,j}$ with $|V_{i,j}|/n$ and denote the resulting labelling by ℓ. Clearly we have $\sum_{w \in W} \ell(w) = 1$. Moreover we claim that

$$\ell(w) + \sum_{u \in L(w)} \ell(w) \geq |L(w)|/b \qquad \text{for all } w \in W \tag{4.7}$$

where $L(w)$ denotes the set of leaves that are children of w. Indeed, let $w \in W$, notice that $|L(w)| \leq 2$ as T is a binary tree, and let u and u' be the two children of w. If $|L(w)| = 0$ we are done. If $|L(w)| > 0$ then w represents a $(2/3, s(H))$-separator $S(w) := S_{i-1,k}$ that separated a graph $H[V(w)]$ with $V(w) := V_{i-1,j} \geq 2n/b$ into two sets $U(w) := V_{i,j'}$ and $U'(w) := V_{i,j'+1}$ such that $|U(w)| + |U'(w)| + |S(w)| = |V(w)|$. In the case that $|L(w)| = 2$ this implies

$$\ell(w) + \ell(u) + \ell(u') = \frac{|S(w)| + |U(w)| + |U'(w)|}{n} = \frac{|V(w)|}{n} \geq 2/b$$

and thus we get (4.7). If $|L(w)| = 1$ on the other hand then, without loss of generality, u is a leaf of T and $|U'(w)| > 2n/b$. Since $S(w)$ is a $(2/3, s(H))$-separator however we know that $|V(w)| \geq \frac{3}{2}|U'(w)|$ and hence

$$\begin{aligned}\ell(w) + \ell(u) &= \frac{|S(w)| + |U(w)|}{n} = \frac{|S(w)| + |V(w)| - |U'(w)| - |S(w)|}{n} \\ &\geq \frac{\frac{3}{2}|U'(w)| - |U'(w)|}{n} \geq \frac{\frac{1}{2}(2n/b)}{n}\end{aligned}$$

which also gives (4.7) in this case. Therefore we can apply Proposition 4.13 and infer that T has at most b leaves as claimed. □

4.3.2 Non-expansion

In this section we study the relation between non-expansion, bandwidth and treewidth. We first give a proof of Proposition 4.5.

Proof of Proposition 4.5. We have to show that for every graph H and every $\varepsilon > 0$ the inequality $b_\varepsilon(H) \leq 2\,\mathrm{bw}(H)/\varepsilon$ holds. Suppose that H has n vertices and let $\sigma : V \to [n]$ be an arbitrary labelling of H. Furthermore assume that $V' \subseteq V$ with $|V'| = b_\varepsilon(H)$ induces an ε-expander in H. Define $V^* \subseteq V'$ to be the first $b_\varepsilon(H)/2 = |V'|/2$ vertices of V' with respect to the ordering σ. Since V' induces an ε-expander in H there must be at least $\varepsilon b_\varepsilon(H)/2$ vertices in $N^* := N(V^*) \cap V'$. Let u be the vertex in N^* with maximal $\sigma(u)$ and $v \in V^* \cap N(u)$. As $u \notin V^*$ and $\sigma(u') > \sigma(v')$ for all $u' \in N^*$ and $v' \in V^*$ by the choice of V^* we have $|\sigma(u) - \sigma(v)| \geq |N^*| \geq \varepsilon b_\varepsilon(H)/2$. Since this is true for every labelling σ we can deduce that $b_\varepsilon(H) \leq 2\,\mathrm{bw}(H)/\varepsilon$. □

The remainder of this section is devoted to the proof of Theorem 4.4. We will use the following lemma which establishes a relation between non-expansion and certain separators.

Lemma 4.15 (non-expansion→separator)**.** *Let H be a graph on n vertices and let $\varepsilon > 0$. If H is $(n/2, \varepsilon)$-non-expanding then H has a $(2\varepsilon n/3, 2/3)$-separator.*

Proof of Lemma 4.15. Let $H = (V, E)$ with $|V| = n$ be $(n/2, \varepsilon)$-non-expanding for $\varepsilon > 0$. It follows that every subset $V' \subseteq V$ with $|V'| \geq n/2$ induces a subgraph $H' \subseteq H$ with the following property: there is $W \subseteq V'$ such that $|W| \leq |V'|/2$ and $|N_{H'}(W)| \leq \varepsilon|W|$. We use this fact to construct a $(2\varepsilon n/3, 2/3)$-separator in the following way:

1. Define $V_1 := V$ and $i := 1$.
2. Let $H_i := H[V_i]$.
3. Find a subset $W_i \subseteq V_i$ with $|W_i| \leq |V_i|/2$ and $|N_{H_i}(W_i)| \leq \varepsilon|W_i|$.
4. Set $S_i := N_{H_i}(W_i)$, $V_{i+1} := V_i \setminus (W_i \cup S_i)$.
5. If $|V_{i+1}| \geq \frac{2}{3}n$ then set $i := i + 1$ and go to step (2).
6. Set $i^* := i$ and return

$$A := \bigcup_{i=1}^{i^*} W_i, \quad B := V_{i^*+1}, \quad S := \bigcup_{i=1}^{i^*} S_i.$$

This construction obviously returns a partition $V = A \dot\cup B \dot\cup S$ with $|B| < \frac{2}{3}n$. Moreover, $|V_{i^*}| \geq \frac{2}{3}n$ and $|W_{i^*}| \leq |V_{i^*}|/2$ and hence

$$|A| = n - |B| - |S| = n - |V_{i^*+1}| - |S| = n - (|V_{i^*}| - |W_{i^*}| - |S_{i^*}|) - |S| \leq n - \frac{|V_{i^*}|}{2} \leq \frac{2}{3}n.$$

The upper bound on $|S|$ follows easily since

$$|S| = \sum_{i=1}^{i^*} |N_{H_i}(W_i)| \leq \sum_{i=1}^{i^*} \varepsilon|W_i| = \varepsilon|A| \leq \frac{2}{3}\varepsilon n.$$

It remains to show that S separates H. This is indeed the case as $N_H(A) \subseteq S$ by construction and thus $E(A, B) = \emptyset$. □

Now we can prove Theorem 4.4. As remarked earlier, Grohe and Marx [45] independently gave a proof of an equivalent result which employs similar ideas but doesn't use separators explicitly.

Proof of Theorem 4.4. Let $H = (V, E)$ be a graph on n vertices, $\varepsilon > 0$, and let $b \geq b_\varepsilon(H)$. It follows immediately from the definition of non-expansion that every subgraph $H' \subseteq H$ with $H' = (V', E')$ and $|V'| \geq 2b$ also has $b_\varepsilon(H') \leq b$.

We now prove Theorem 4.4 by induction on the size of H. $\mathrm{tw}(H) \leq 2\varepsilon n + 2b$ trivially holds if $n \leq 2b$. So let H have $n > 2b$ vertices and assume that the theorem holds for all graphs with less than n vertices. Then H is (b,ε)-non-expanding and thus has a $(2\varepsilon n/3, 2/3)$-separator S by Lemma 4.15. Assume that S separates H into the two subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$. Let $(\mathcal{X}_1, T_1)$ and $(\mathcal{X}_2, T_2)$ be tree decompositions of H_1 and H_2, respectively, and assume that $\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$. We use them to construct a tree decomposition $(\mathcal{X}, T)$ of H as follows. Let $\mathcal{X} = \{X_i \cup S : X_i \in \mathcal{X}_1\} \cup \{X_i \cup S : X_i \in \mathcal{X}_2\}$ and $T = (I_1 \cup I_2, F = F_1 \cup F_2 \cup \{e\})$ where e is an arbitrary edge between the two trees. This is indeed a tree decomposition of H: Every vertex $v \in V$ belongs to at least one $X_i \in \mathcal{X}$ and for every edge $\{v, w\} \in E$ there exists $i \in I$ (where I is the index set of $\mathcal{X}$) with $\{v, w\} \subseteq X_i$. This is trivial for $\{v, w\} \subseteq V_i$ and follows from the definition of $\mathcal{X}$ for $v \in S$ and $w \in V_i$. Since S separates H there are no edges $\{v, w\}$ with $v \in V_1$ and $w \in V_2$. For the same reason the third property of a tree decomposition holds: if j lies on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$ as the intersection is S if X_i, X_k are subsets of V_1 and V_2 respectively.

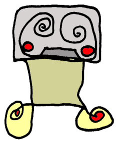
We have seen that $(\mathcal{X}, T)$ is a tree decomposition of H and can estimate its width. This gives $\mathrm{tw}(H) \le \max\{\mathrm{tw}(H_1), \mathrm{tw}(H_2)\} + |S|$. With the induction hypothesis we have

$$\begin{aligned}\mathrm{tw}(H) &\le \max\{2\varepsilon \cdot |V_1| + 2b,\ 2\varepsilon \cdot |V_2| + 2b\} + |S| \\ &\le 2\varepsilon n + 2b.\end{aligned}$$

where the second inequality is due to the fact that $|V_i| \le (2/3)n$ and $|S| \le (2\varepsilon n)/3$. □

Chapter 5

The bandwidth conjecture of Bollobás and Komlós



Minimum degree conditions that enforce the appearance of certain large subgraphs H in a host graph G are the theme of several prominent results in extremal combinatorics. As we saw in the introduction (Section 1.1.1) several theorems of this type assert the existence of specific graphs H (such as Hamilton cycles in Dirac's theorem or spanning K_r-factors in the Hajnal-Szemerédi theorem) if G has minimum degree at least $\delta(G) = n \cdot (\chi(H) - 1)/\chi(H)$ where n is the number of vertices in G. The Bollobás-Komlós conjecture (already discussed in Section 1.2.1) formulates a similar criterion for the existence of a more general class of subgraphs. In the present chapter we will prove this conjecture. We establish the following theorem.

Theorem 5.1. *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. If H is an r-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.*

Obviously, Hamilton cycles and their powers have constant bandwidth and are thus embraced by this theorem. In addition, we saw in the last chapter that, for example, bounded-degree F-minor-free graphs for any F have sublinear bandwidth (Corollary 4.9) and, more generally, that a hereditary class of bounded-degree graphs has sublinear bandwidth if and only if it does not contain expanders of linear order (Theorem 4.6). In the introduction we furthermore explained that expanders H in fact show that the claim of Theorem 5.1 no longer holds when the bandwidth restriction on H is omitted.

This indicates that Theorem 5.1 can be regarded as a common generalisation of some of the results concerning Hamilton cycles, their powers, or spanning K_r-factors mentioned above (and others, see also Section 1.1.1). Observe, however, that for $(r-1)$-st powers of Hamilton cycles H this is true only if r divides n, since otherwise $\chi(H) = r + 1$. We will return to this topic in Section 5.1 where we discuss a more general result than Theorem 5.1 (which we shall subsequently prove, see Theorem 5.2) that actually also includes those cases. In Section 5.2 we shall then describe the ideas and main lemmas used in the proof, which is presented in Section 5.3. The remaining sections of this chapter cover the proofs of these lemmas. Some technical results (as well as several ideas) obtained along the way will also be reused in later chapters and hence prepare us for the proofs of other embedding results such as Theorem 7.2 in Chapter 7.

Before starting let us note that the analogue of Theorem 5.1 for bipartite H was announced by Abbasi [1] in 1998, and a short proof based on our methods can be found in [49]. Moreover, as mentioned already in the introduction, Abbasi [2] showed that the additional term γn in the minimum degree condition in Theorem 5.1 cannot be omitted.

5.1 The rôle of the chromatic number

The Pósa-Seymour conjecture (see Section 1.1.1) implies that the $(r-1)$-st power of a Hamilton cycle on an odd number of vertices is forced as a spanning subgraph in any graph of minimum degree $\frac{r-1}{r}n$ (although it is $(r+1)$- and not r-chromatic). In the same way other $(r+1)$-chromatic graphs are also forced already for this minimum degree. It seems that one important question here is whether all $r+1$ colours are needed by *many* vertices. For instance, the *critical chromatic number* $\chi_{cr}(H)$ of a graph H is defined as

$$\chi_{cr}(H) := \frac{(\chi(H)-1)|V(H)|}{|V(H)|-\sigma},$$

where σ is the minimum size of the smallest colour class in a colouring of H with $\chi(H)$ colours. Obviously, $\chi(H)-1 < \chi_{cr}(H) \le \chi(H)$, with (approximate) equality for σ tending to 0 or $|V(H)|/\chi(H)$, respectively. This concept was introduced by Komlós [63], who proved that a minimum degree condition of $\delta(G) \ge (\chi_{cr}(H)-1)n/\chi_{cr}(H)$ suffices to find a family of disjoint copies of H covering all but εn vertices of G. Kühn and Osthus [73] further investigated this question and managed to determine the corresponding minimum degree condition (up to an additive constant) for the containment of a spanning H-factor for every H.

The following example shows however, that we cannot replace $\chi(H)$ in Theorem 5.1 by $\chi_{cr}(H)$. Let b, r, m, and n be positive integers such that r divides n and $(b(r-1)+1)m = n$. Consider the graph H that is obtained from m vertex disjoint copies of the complete r-partite graph K with one colour class of size one and $r-1$ colour classes of size b by adding all edges between different colour classes of the i-th and $(i+1)$-st such copy for all $i \in [m-1]$ (this graph is similar to the graph C^r_m defined below, see page 49). Furthermore, let K_r^- be the graph K_r minus an edge and let G be the graph obtained from K_r^- by replacing the two non-adjacent vertices by cliques Z_1 and Z_2 of size n/r each, all other vertices by independent sets of size n/r, and all edges of K_r^- by complete bipartite graphs. Then H and G are graphs on n vertices with $\delta(G) = \frac{r-1}{r}n - 1$, $\Delta(H) \le 3br$, $\mathrm{bw}(H) \le 3br$, $\chi(H) = r$, and $\chi_{cr} = r - 1 + \frac{1}{b}$. It is not difficult to check that H is not a subgraph of G because H is so "well connected" that any potential copy of H in G could use only vertices in one of the two cliques Z_1 and Z_2.

Nevertheless our methods allow an extension of Theorem 5.1 that goes into a somewhat similar direction. Assume that the vertices of H are labelled $1, \dots, n$. For two positive integers x, y, an $(r+1)$-colouring $\sigma : V(H) \to \{0, \dots, r\}$ of H is said to be *(x, y)-zero free* with respect to such a labelling, if for each $t \in [n]$ there exists a t' with $t \le t' \le t + x$ such that $\sigma(u) \ne 0$ for all $u \in [t', t'+y]$. We also say that the interval $[t', t'+y]$ is *zero free*.

The following theorem states that the assertion of Theorem 5.1 remains true for $(r+1)$-colourable graphs H if the $(r+1)$-st colour is used by few vertices and does only appear rarely, at well separated places in the bandwidth ordering of H.

Theorem 5.2. *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ the following holds. Let H be a graph with $\Delta(H) \le \Delta$ whose vertices are labelled $1, \dots, n$ such that, with respect to this labelling, H has bandwidth at most βn, an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free, and uses colour 0 for at most βn vertices in total. If G is a graph on n vertices with minimum degree $\delta(G) \ge (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.*

Obviously Theorem 5.2 implies Theorem 5.1, and the remaining part of this chapter is devoted to the proof of Theorem 5.2.

5.2 The main lemmas and an outline of the proof

In this section we introduce the central lemmas that are needed for the proof of our main theorem. Our emphasis is to explain how they work together to give the proof of Theorem 5.2, which itself is then presented in full detail in the subsequent section, Section 5.3. We will use the strategy outlined in Sections 1.3 and 1.4, and apply the general embedding lemma together with two structural lemmas that provide partitions, suitable for the general embedding lemma, of the graph H and G, respectively. For these lemmas we need a few more definitions.

Suppose that m and r are integers. Let C_m^r be the mr-vertex graph obtained from a path on m vertices by replacing every vertex by a clique of size r and replacing every edge by a complete bipartite graph minus a perfect matching (see Figure 5.1). More precisely,

$$V(C_m^r) = [m] \times [r] \tag{5.1}$$

and

$$\{(i,j),(i',j')\} \in E(C_m^r) \qquad \text{iff} \qquad i = i' \quad \text{or} \quad |i - i'| = 1 \ \wedge \ j \neq j'. \tag{5.2}$$

Let K_m^r be the graph on vertex set $[m] \times [r]$ that is formed by the disjoint union of m complete graphs on r vertices. Then $K_m^r \subseteq C_m^r$ and we call the complete graph on vertices $(i,1),\ldots,(i,r)$ the *i-th component of K_m^r* for $i \in [m]$. Note moreover, that $\sigma\colon\ [m] \times [r] \to [r]$ with $\sigma(i,j) := j$ for $i \in [m]$ and $j \in [r]$ is a valid r-colouring of C_m^r. We will later consider vertex partitions $(V_{i,j})_{i\in[m],j\in[r]}$ that are (ε,d)-regular on C_m^r for some ε and d. Then we will also say, that *$V_{i,j}$ has colour j*.

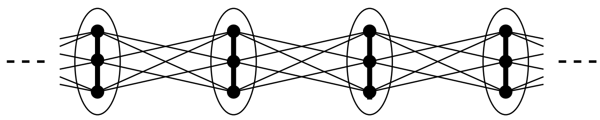


Figure 5.1: The graph C_m^3. The encircled triples of vertices form cliques of size 3 in C_m^3.

Remark. *In what follows we shall frequently consider graphs R_m^r on vertex set $[m] \times [r]$ and say that $K_m^r \subseteq C_m^r \subseteq R_m^r$. Then we implicitly assume that the vertices of R_m^r are labelled such that this is consistent with these copies of K_m^r and C_m^r, i.e., the vertex set $\{i\} \times [r]$ forms the i-th component of K_m^r for each $i \in [m]$ and $[m] \times \{j\}$ forms colour class j of C_m^r for each $j \in [r]$.*

We can now state (and then explain) our first main lemma which asserts a regular partition of the graph G with structural properties that will be suitable for embedding H into G.

Lemma 5.3 (Lemma for G)**.** *For all $r \in \mathbb{N}$ and $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for every positive $\varepsilon \le \varepsilon_0$ there exist k_1 and $\xi_0 > 0$ such that for all $n \ge k_1$ and for every graph G on vertex set $[n]$ with $\delta(G) \ge ((r-1)/r + \gamma)n$ there exist $k \in \mathbb{N} \setminus \{0\}$ and a graph R_k^r on vertex set $[k] \times [r]$ with*

- (R1) *$k \le k_1$,*
- (R2) *$\delta(R_k^r) \ge ((r-1)/r + \gamma/4)kr$,*
- (R3) *$K_k^r \subseteq C_k^r \subseteq R_k^r$, and*
- (R4) *there is an r-equitable integer partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} \ge (1-\varepsilon)n/(kr)$ such that the following holds.*

For every partition $(n_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} - \xi_0 n \le n_{i,j} \le m_{i,j} + \xi_0 n$ there exists a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V with

(V1) $|V_{i,j}| = n_{i,j}$,

(V2) $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-regular on R^r_k, and*

(V3) $(V_{i,j})_{i\in[k],j\in[r]}$ *is (ε, d)-super-regular on K^r_k.*

We give the proof of Lemma 5.3, which borrows ideas from [69], in Section 5.5. To illustrate what this lemma says let us first assume for simplicity that $n_{i,j} = m_{i,j}$. In this case, Lemma 5.3 would guarantee a partition of the vertex set of G in such a way that the partition classes form many (super-)regular pairs, and that these pairs are organised in a sort of backbone, namely in the form of a C^r_k for the regular pairs, and, contained therein, a spanning family K^r_k of disjoint complete graphs for the super-regular pairs.

However, the lemma says more. When we come to the point (R4), the lemma 'has in mind' the partition we just described, but does not exhibit it. Instead, it only discloses the sizes $m_{i,j}$ and allows us to wish for small amendments: for every $i \in [k]$ and $j \in [r]$, we can now look at the value $m_{i,j}$ and ask for the size of the corresponding partition class to be adjusted to a new value $n_{i,j}$, differing from $m_{i,j}$ by at most $\xi_0 n$.

When proving Lemma 5.3, one thus needs to alter the partition by shifting a few vertices. This is where the main difficulty lies. The strategy to solve this problem was outlined already in Section 1.3.5. We will see that the existence of an almost spanning copy of C^r_m in the reduced graph allows us to perform the alteration just described. In order to guarantee such a structure in the reduced graph we will first solve the following special case of Theorem 5.1, which asserts the copy of C^r_m in a graph of high minimum degree.

Lemma 5.4 (backbone lemma)**.** *For all integers $r \ge 1$ and positive constants γ and ε there exists $n_0 = n_0(r, \gamma, \varepsilon)$ such that for every $n \ge n_0$ the following holds. If G is an n-vertex graph with minimum degree $\delta(G) \ge ((r-1)/r + \gamma)n$, then G contains a copy of C^r_m with $rm \ge (1-\varepsilon)n$.*

Now we come to the second main lemma. It prepares the graph H so that it can be embedded into G. This is exactly the place where, given the values $m_{i,j}$, the new values $n_{i,j}$ in the setting described above are specified.

Lemma 5.5 (Lemma for H)**.** *Let $r, k \ge 1$ be integers and let β, $\xi > 0$ satisfy $\beta \le \xi^2/(3026r^3)$.*

Let H be a graph on n vertices with $\Delta(H) \le \Delta$, and assume that H has a labelling of bandwidth at most βn and an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free with respect to this labelling, and uses colour 0 for at most βn vertices in total. Let R^r_k be a graph with $V(R^r_k) = [k] \times [r]$ such that $\delta(R^r_k) > (r-1)k$ and $K^r_k \subseteq C^r_k \subseteq R^r_k$. Furthermore, suppose $(m_{i,j})_{i\in[k],j\in[r]}$ is an r-equitable integer partition of n with $m_{i,j} \ge 35\beta n$ for every $i \in [k]$ and $j \in [r]$.

Then there exists a mapping $f\colon V(H) \to [k] \times [r]$ and a set of special vertices $X \subseteq V(H)$ with the following properties

(*a*) $|X| \le kr\xi n$,

(*b*) $m_{i,j} - \xi n \le |W_{i,j}| := |f^{-1}(i,j)| \le m_{i,j} + \xi n$ *for every $i \in [k]$ and $j \in [r]$,*

(*c*) *for every edge $\{u, v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(R^r_k)$, and*

(*d*) *if $\{u, v\} \in E(H)$ and, moreover, u and v are both in $V(H) \setminus X$, then $\{f(u), f(v)\} \in E(K^r_k)$.*

In other words, Lemma 5.5 receives a graph H as input and, from Lemma 5.3, a reduced graph R^r_k (with $K^r_k \subseteq C^r_k \subseteq R^r_k$), an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n, and a parameter ξ. Again we emphasise that this is all what Lemma 5.5 needs to know about G. It then provides us with a function f which maps the vertices of H onto the vertex set $[k] \times [r]$ of R^r_k in such a way that (i, j) with $i \in [k]$, $j \in [r]$ receives $n_{i,j} := |W_{i,j}|$ vertices from H, with $|n_{i,j} - m_{i,j}| \leq \xi n$. Although the vertex partition of G is not known exactly at this point, we already have its reduced graph R^r_k. Lemma 5.5 guarantees that the endpoints of an edge $\{u, v\}$ of H get mapped into vertices $f(u)$ and $f(v)$ of R^r_k, representing future partition classes $V_{f(u)}$ and $V_{f(v)}$ in G which will form a *super-regular* pair (see (d)) – except for those few edges with one or both endpoints in some small special set X. But even these edges will be mapped into pairs of classes in G that will form at least *regular* pairs (see (c)). Lemma 5.5 will then return the values $n_{i,j}$ to Lemma 5.3, which will finally produce a corresponding partition of the vertices of G.

In the next section we give the precise details how Theorem 5.2 can be deduced from Lemma 5.3 and Lemma 5.5 following the outline discussed above.

5.3 Proof of Theorem 5.2

In this section we give the proof of Theorem 5.2 based on the general embedding lemma (Lemma 3.12), the lemma for G (Lemma 5.3), and the lemma for H (Lemma 5.5). In particular, we will use Lemma 5.3 for partitioning G, and Lemma 5.5 for assigning the vertices of H to the parts of G. For this, it will be necessary to split the application of Lemma 5.3 into two phases. The first phase is used to set up the parameters for Lemma 5.5. With this input, Lemma 5.5 then defines the sizes of the parts of G that are constructed during the execution of the second phase of Lemma 5.3. We shall see that the two partitions we obtain are compatible partitions (see Definition 3.11). This will enable us to apply the general embedding lemma, Lemma 3.12 to embed H into G.

Proof of Theorem 5.2. Given r, Δ, and γ, let d and ε_0 be as asserted by Lemma 5.3 for input r and γ. Let $\varepsilon_{\text{GEL}} = \varepsilon_{\text{GEL}}(d, \Delta, r)$ be as given by Lemma 3.12 and set

$$\varepsilon := \min\{\varepsilon_0, \varepsilon_{\text{GEL}}, 1/4\}\,. \tag{5.3}$$

Then, Lemma 5.3 provides constants k_1 and ξ_0 for this ε. We define

$$\xi := \min\{\xi_0, \varepsilon_{\text{GEL}}/(4k_1^2 r^2(\Delta + 1)^2)\} \tag{5.4}$$

as well as $\beta := \min\{\xi^2/(3026r^3), (1 - \varepsilon)/(35k_1 r)\}$ and $n_0 := k_1$, and consider arbitrary graphs H and G on $n \geq n_0$ vertices that meet the conditions of Theorem 5.2.

Applying Lemma 5.3 to G we get an integer k with $0 < k \leq k_1$, graphs $K^r_k \subseteq C^r_k \subseteq R^r_k$ on vertex set $[k] \times [r]$, and an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n such that (R1)–(R4) are satisfied. Now all constants that appear in the proof are fixed. To summarise, this is how they are related:

$$\frac{1}{\Delta}, \gamma \gg d \gg \varepsilon \gg \frac{1}{k_1} \gg \xi \gg \beta\,.$$

Before continuing with Lemma 5.3, we would like to apply the Lemma 5.5. Note that owing to (R4) and the choice of β above, we have $m_{i,j} \geq (1-\varepsilon)n/(kr) \geq 35\beta n$ for every $i \in [k]$, $j \in [r]$. Consequently, for constants k, β, and ξ, graphs H and $K^r_k \subseteq C^r_k \subseteq R^r_k$, and the partition

$(m_{i,j})_{i\in[k],j\in[r]}$ of n we can indeed apply Lemma 5.5. This yields a mapping $f\colon V(H)\to[k]\times[r]$ and a set of special vertices $X_H\subseteq V(H)$. These will be needed later. For the moment we are only interested in the sizes $n_{i,j}:=|W_{i,j}|=|f^{-1}(i,j)|$ for $i\in[k]$ and $j\in[r]$. Condition (b) of Lemma 5.5 and the choice of $\xi\le\xi_0$ in (5.4) imply that

$$m_{i,j}-\xi_0 n\le m_{i,j}-\xi n\le n_{i,j}\le m_{i,j}+\xi n\le m_{i,j}+\xi_0 n$$

for every $i\in[k]$, $j\in[r]$. Accordingly, we can continue with Lemma 5.3 to obtain a partition $V=(V_{i,j})_{i\in[k],j\in[r]}$ with $|V_{i,j}|=n_{i,j}$ that satisfies conditions (V1)–(V3) of Lemma 5.3. Note that

$$|V_{i,j}|=n_{i,j}\ge m_{i,j}-\xi n\overset{\text{(R4)}}{\ge}(1-\varepsilon)\frac{n}{kr}-\xi n=(1-(\varepsilon+\xi kr))\frac{n}{kr}\overset{(5.3),(5.4)}{\ge}\frac{1}{2}\frac{n}{kr}. \tag{5.5}$$

Now, we have partitions $V(H)=(W_{i,j})_{i\in[k],j\in[r]}$ of H and $V(G)=(V_{i,j})_{i\in[k],j\in[r]}$ of G with $|W_{i,j}|=|V_{i,j}|=n_{i,j}$ for all $i\in[k]$, $j\in[r]$. Furthermore, by (V2) and (V3) the partition $V(G)=(V_{i,j})_{i\in[k],j\in[r]}$ is an (ε,d)-regular partition with reduced graph R^r_k that is (ε,d)-super-regular on K^r_k with $\varepsilon\le\varepsilon_{\text{GEL}}$ by (5.3).

To finish the proof we will use the general embedding lemma, Lemma 3.12, on the graphs H and G with their respective partitions and the reduced graph $R=R^r_k$ and its subgraph $R'=K^r_k$ (with components of size r). For applying this lemma it remains to verify that $(W_{i,j})_{i\in[k],j\in[r]}$ is ε_{GEL}-compatible with $(V_{i,j})_{i\in[k],j\in[r]}$ and $K^r_k\subseteq R^r_k$, i.e., we need to check conditions (i)–(iii) of Definition 3.11. We just saw that (i) is satisfied, and (ii) holds by (c) of Lemma 5.5. To see (iii) we define $X':=X_H\cup N_H(X_H)$ where X_H is the set of special vertices generated by Lemma 5.5. Observe that this implies $|X'|\le(\Delta+1)kr\xi n$ by (a) of Lemma 5.5. Now let the set X be as in Definition 3.11 and set $X_{i,j}:=X\cap W_{i,j}$ and $Y_{i,j}:=N_H(X)\cap W_{i,j}\setminus X$ for all $i\in[k]$ and $j\in[r]$. Owing to (d) of Lemma 5.5, we have $X_{i,j}\subseteq X\subseteq X'$ and hence

$$|X_{i,j}|\le|X'|\le(\Delta+1)kr\xi n\overset{(5.4)}{\le}\frac{1}{\Delta}\varepsilon_{\text{GEL}}\frac{1}{2}\frac{n}{kr}\overset{(5.5)}{\le}\frac{1}{\Delta}\varepsilon_{\text{GEL}}|V_{i,j}|$$

which implies the first part of (iii) in Definition 3.11. For the second part, we use that $|Y_{i,j}|\le\Delta|X|\le\Delta|X'|$ and so the same calculation implies $|Y_{i,j}|\le\varepsilon_{\text{GEL}}|V_{i',j'}|$ for all $i,i'\in[k]$ and $j,j'\in[r]$. Accordingly all requirements of the general embedding lemma, Lemma 3.12, are satisfied and we get that H is a subgraph of G. $\square$

Algorithmic embeddings

We note that the proof of Theorem 5.1 presented above yields an algorithm, which finds an embedding of H into G, if H is given along with a valid r-colouring and a labelling of the vertices respecting the bandwidth bound βn. This follows from the observation that the proof above is constructive, and all the lemmas used in the proof (Lemma 5.3, Lemma 5.5, and the general embedding lemma, Lemma 3.12, which in turn combines the blow-up lemma, Lemma 3.9, and the partial embedding lemma, Lemma 3.10) have algorithmic proofs. Algorithmic versions of the blow-up lemma were obtained in [66, 87]. In [66] a running time of order $\mathcal{O}(n^{3.376})$ was proved. The key ingredient of Lemma 5.3 is Szemerédi's regularity lemma for which an $\mathcal{O}(n^2)$ algorithm exists [60] (for an $\mathcal{O}(n^{2.376})$ algorithm see [10]). All other arguments in the proof of Lemma 5.3 can be done algorithmically in $\mathcal{O}(n^2)$ (see Section 5.5). Similarly, the proof of Lemma 5.5 is constructive if an r-colouring of H and an ordering respecting the bandwidth bound is given (see Section 5.6). Finally, we note that the proof of the partial embedding lemma (following along the lines of [25]) gives rise to an $\mathcal{O}(n^3)$ algorithm. Thus there is an $\mathcal{O}(k \times ((1/k + \xi_0)n)^{3.376} + n^2 + n^3) = \mathcal{O}(n^{3.376})$ embedding algorithm, where the implicit constant depends on γ and Δ only.

5.4 The backbone lemma

In this section we prove Lemma 5.4. The proof is a simple consequence of the following result concerning the Pósa–Seymour conjecture [68] that we already mentioned in the introduction and state (and need) here in a weaker form only (cf. Section 1.1.1). In what follows we denote by P_k^{r-1} the $(r-1)$-st power of a path on kr vertices, where

$$V(P_k^{r-1}) = [k] \times [r] \tag{5.6}$$

and

$$\{(s,t),(s',t')\} \in E(P_k^{r-1}) \qquad \text{iff} \qquad s = s' \quad \text{or} \quad s' = s+1 \ \wedge \ t' < t. \tag{5.7}$$

Observe that this notation is slightly non-standard: P_k^{r-1} is $(r-1)$-st power of a path P_{kr} (and *not* of a path on k vertices). Similarly, recall that C_m^r is the graph on mr vertices defined in (5.1) and (5.2) (and *not* the r-th power of a cycle C_m).

Theorem 5.6 (Komlós, Sárközy, and Szemerédi)**.** *For every $r \geq 2$ there exists k_0 such that every graph R on $kr \geq k_0$ vertices with minimum degree $\delta(R) \geq \frac{(r-1)}{r}kr = (r-1)k$ contains a copy of P_k^{r-1}.*

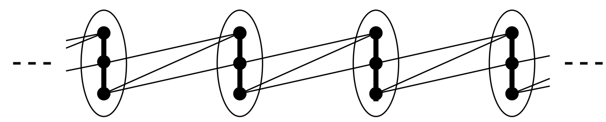


Figure 5.2: The graph P_m^2. The encircled triples of vertices form cliques of size 3 in P_m^2. Vertices $(s,1)$ are at the botom, vertices $(s,2)$ in the middle, and vertices $(s,3)$ at the top, $s \in [m]$.

Notice that P_m^{r-1} is a subgraph of the graph C_m^r (see Figure 5.2). On the other hand, there is an "equipartite" homomorphism from C_m^r to the $(r-1)$-st power of a Hamilton path.

Proposition 5.7. *Let $k \geq 1$ and $\ell \geq r \geq 1$. Let $C_{k\ell}^r$ be the graph defined in* (5.1) *and* (5.2) *and let P_k^{r-1} be the graph defined in* (5.6) *and* (5.7)*. Then there exists a graph homomorphism $\varphi\colon V(C_{k\ell}^r) \to V(P_k^{r-1})$ such that*

$$\ell - r < \left|\varphi^{-1}\big((s,t)\big)\right| < \ell + r$$

for all $(s,t) \in [k] \times [r] = V(P_k^{r-1})$.

Proof. It is straight-forward to check that the following map

$$\varphi\big((i,j)\big) = \left(\left\lceil \frac{\max\{i-j,0\}+1}{\ell} \right\rceil, j\right)$$

is a graph homomorphism from $C_{k\ell}^r$ to P_k^{r-1} with the desired property. □

Now Lemma 5.4 follows from a joint application of the regularity lemma (in form of Lemma 3.4), Theorem 5.6, Proposition 3.7, Proposition 5.7, and the general embedding lemma (Lemma 3.12). More precisely, we first apply Lemma 3.4 to the graph G with $\delta(G) \geq ((r-1)/r + \gamma)n$ and infer that the corresponding reduced graph R satisfies $\delta(R) \geq ((r-1)/r)|V(R)|$. Consequently, Theorem 5.6 implies that $R \supseteq P_k^{r-1}$. Since $\Delta(P_k^{r-1}) \leq 3(r-1)$ we can, owing to Proposition 3.7 applied with $R' = P_k^{r-1}$, remove about $\varepsilon|V(G)|$ vertices from G such that edges of P_k^{r-1} correspond to super-regular pairs in the adjusted partition. Finally, thanks to Proposition 5.7 we find a partition of P_k^{r-1} that is compatible with the partition of G. Thus we can apply Lemma 3.12 and conclude that G contains an almost spanning copy of C_m^r. Below we give the technical details of this proof.

Proof of Lemma 5.4. For $r = 1$ the lemma is trivial, as C_m^1 is simply an independent set. Hence, let $r \geq 2$ and γ, $\varepsilon > 0$ be given. We apply Lemma 3.4 with γ and obtain constants d, $\varepsilon_0 > 0$. We set $d_{\text{GEL}} = d/2$, $\Delta_{\text{GEL}} = 3(r-1)$, and $r_{\text{GEL}} = r$ and get $\varepsilon_{\text{GEL}} = \varepsilon_{\text{GEL}}(d, \Delta, r)$ from Lemma 3.12. We then set

$$\varepsilon_{\text{RL}} := \frac{\min\{\varepsilon, \varepsilon_0, \varepsilon_{\text{GEL}}, d\}}{100r}.$$

Moreover, let k_0 be given by Theorem 5.6 for r and set

$$k_0' := \max\{rk_0 + r, 4r/\gamma, 10r/\varepsilon_{\text{RL}}\}.$$

Next, we continue the application of Lemma 3.4 with $\frac{1}{2}\varepsilon_{\text{RL}}$ and k_0' and obtain k_1. Finally, we let $n_0 = \lceil 10k_1^2 r^3/\varepsilon_{\text{RL}}\rceil$. After we fixed all constants, we consider the input graph G on $n \geq n_0$ vertices from Lemma 5.4. We have $\delta(G) \geq ((r-1)/r + \gamma)n$. Consequently, Lemma 3.4, applied with γ, $\frac{1}{2}\varepsilon_{\text{RL}}$ and k_0' fixed above, yields an integer k', $k_0' \leq k' \leq k_1$, an $(\frac{1}{2}\varepsilon_{\text{RL}}, d)$-regular equipartition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_{k'} = V(G)$ with reduced graph R_G on vertex set $[k']$. Without loss of generality, we may assume that $k' = kr$ for some integer $k \geq k_0$, since otherwise, we simply unite V_0 with up to at most $r-1$ vertex classes V_i $(i > 0)$ and obtain an exceptional set V_0', which obeys

$$|V_0'| \leq |V_0| + (r-1)\tfrac{n}{k'} \leq \left(\tfrac{1}{2}\varepsilon_{\text{RL}} + \tfrac{r-1}{k_0'}\right)n \leq \varepsilon_{\text{RL}} n.$$

Note that, since $k' \geq k_0' \geq 4r/\gamma$, the resulting reduced graph R_G still satisfies

$$\delta(R_G) \geq \big((r-1)/r + \gamma/2\big)k' - (r-1) \geq \big((r-1)/r + \gamma/4\big)kr.$$

Moreover, as $k = \lfloor k'/r \rfloor \geq \lfloor k_0'/r \rfloor \geq k_0$ (where k_0 came from Theorem 5.6), we infer by Theorem 5.6 that $P_k^{r-1} \subseteq R_G$. Observe further that $K_k^r \subseteq P_k^{r-1}$.

We now apply Proposition 3.7 with ε_{RL}, with d, and $R' = K_k^r \subseteq R_G$. This way we get an (ε', d')-regular partition $V_0' \dot\cup V_1' \dot\cup \dots \dot\cup V_{kr}'$ with clusters of size at least $L := (1-\varepsilon')n/kr$ and reduced graph R_G that is (ε', d')-super-regular on K_k^r with

$$\varepsilon' = 2\varepsilon_{\text{RL}} r/(1 - \varepsilon_{\text{RL}} r) \leq \varepsilon_{\text{GEL}} \quad \text{and} \quad d' = d - 2\varepsilon_{\text{RL}} r > d/2.$$

Hence, in view of Proposition 5.7 we can apply the general embedding lemma, Lemma 3.12, to $G[V \setminus V_0']$ with $(\varepsilon_{\text{GEL}}, d/2)$-regular partition $V_1' \dot\cup \dots \dot\cup V_{kr}'$ with reduced graph $R = P_k^{r-1} \subseteq R_G$, and to $H = C_{k\ell}^r$ for $\ell = L - r$ where H is partitioned as dictated by the homomorphism φ from Proposition 5.7. It is easy to check that these partitions and reduced graphs are ε_{GEL}-compatible (the set X contains only the $2(r+1)r(k-1)$ vertices with neighbours in two

K_r's of the P_k^{r-1} under the homomorphism φ). Consequently, G contains a copy of C_m^r (with $m = k\ell$) on $rm = rk\ell$ vertices. Moreover, we have

$$rm = rk\ell = rk(L - r) = rk\left((1 - \varepsilon')\frac{n}{kr} - r\right) \geq (1 - \varepsilon_{\mathrm{GEL}})n - r^2 k \geq (1 - \varepsilon)n,$$

because $r^2 k \leq \frac{1}{2}\varepsilon n$ by our choice of n_0. □

In this proof we used the fact that the reduced graph R of our host graph G contains the $(r-1)$-st power of a spanning path. It is not difficult to verify (by repeating almost the same proof) that this spanning path power in R could easily be replaced by an almost spanning path power in R, and that this is indeed all we need for proving the existence of an almost spanning copy of C_m^r in G. This yields the following lemma which we will use only in chapter 7.

Lemma 5.8. *For all integers $r \geq 1$ and positive reals μ and d there exists a positive ε_0 such that for all $\varepsilon \leq \varepsilon_0$ and integers k_1 there is an integer $n_0 > k_1$ such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices that has an (ε, d)-regular equipartition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ with reduced graph R and $r \leq k \leq k_1$. If R contains the $(r-1)$-st power P_ℓ^{r-1} of a path on $r\ell \geq (1-\varepsilon)k$ vertices then G contains a copy of C_m^r for some m with $rm \geq (1-\mu)n$.* □

5.5 The lemma for G

The main ingredients for the proof of Lemma 5.3 are Szemerédi's regularity lemma, which provides a reduced graph R for G, and the backbone lemma which guarantees the copy of a C_k^r in R. This subgraph is sparse enough, so that we can transform the corresponding regular pairs into super-regular pairs. On the other hand, its structure is rich enough so that we can use it to develop a strategy for moving vertices between the clusters of R in order to adjust the sizes of these clusters (as outlined in Section 1.3.5). We will first consider the special case of Lemma 5.3 that $n_{i,j} = m_{i,j}$ for all $i \in [k]$, $j \in [r]$. This is captured by the following proposition.

Proposition 5.9. *For all $r \in \mathbb{N}$ and $\gamma > 0$ there exist $d > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exists k_1 such that for all $n \geq k_1 r$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq ((r-1)/r + \gamma)n$ there exists $k \in \mathbb{N} \setminus \{0\}$, and a graph R_k^r on vertex set $[k] \times [r]$ with*

- (R1) *$k \leq k_1$,*
- (R2) *$\delta(R_k^r) \geq ((r-1)/r + \gamma/4)kr$,*
- (R3) *$K_k^r \subseteq C_k^r \subseteq R_k^r$, and*
- (R4) *there is an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n with $m_{i,j} \geq (1-\varepsilon)n/(kr)$ such that the following holds.*

There is a partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V with

- (U1) *$|U_{i,j}| = m_{i,j}$,*
- (U2) *$(U_{i,j})_{i\in[k],j\in[r]}$ is (ε, d)-regular on R_k^r, and*
- (U3) *$(U_{i,j})_{i\in[k],j\in[r]}$ is (ε, d)-super-regular on K_k^r.*

Notice that once we have Proposition 5.9, the only thing that is left to be done when proving Lemma 5.3 is to show that the sizes of the classes $U_{i,j}$ can be slightly changed from $m_{i,j}$ to $n_{i,j}$ without "destroying" properties (U2) and (U3).

For the proof of Proposition 5.9 we proceed in three steps. From the regularity lemma we first obtain a partition $U'_0\dot\cup U'_1\dot\cup\cdots\dot\cup U'_{k'}$ of $V(G)$ with reduced graph R such that $K^r_k \subseteq C^r_k \subseteq R$. According to this occurrence, we will then rename the vertices of R from $[k']$ to $[k]\times[r]$ and thus obtain R^r_k. In a similar manner we rename the clusters in the partition. We then use Proposition 3.7 to get a new partition $U''_0\dot\cup(U''_{i,j})_{i\in[r],j\in[k]}$ that is super-regular on K^r_k (and still regular on R^r_k). In a last step we distribute the vertices in U''_0 to the clusters $U''_{i,j}$ with $i\in[k]$ and $j\in[r]$, while maintaining the super-regularity. The partition obtained in this way will be the desired r-equitable partition $(U_{i,j})_{i\in[r],j\in[k]}$.

Proof of Proposition 5.9. We first fix all constants necessary for the proof. For $r=1$ the Proposition holds trivially. Let $r\ge 2$ and $\gamma>0$ be given. The regularity lemma in form of Lemma 3.4 applied with $\gamma'=\gamma$ yields positive constants d' and ε'_0. We fix the promised constants d and ε_0 for Proposition 5.9 by setting

$$d := \min\{d'/3, \gamma/4\} \quad \text{and} \quad \varepsilon_0 := \varepsilon'_0\,. \tag{5.8}$$

Now let some positive $\varepsilon\le\varepsilon_0$ be given, for which Proposition 5.9 asks us to define k_1. For that first define

$$\varepsilon' := \min\{\varepsilon^4, (d')^2, \gamma^2\}\cdot 10^{-3} r^{-4} \tag{5.9}$$

and let k_0 be sufficiently large so that we can apply the backbone lemma, Lemma 5.4, with r, $\gamma/2$ and ε' to graphs R' on $k'\ge rk_0/(1-\varepsilon')$ vertices with minimum degree $\delta(R')\ge((r-1)/r+\gamma/2)k'$. We then define an auxiliary constant k'_0 by

$$k'_0 := \max\{rk_0/(1-\varepsilon'), 10r/\gamma\} + r/(1-\varepsilon') \tag{5.10}$$

and let k'_1 be given by Lemma 3.4 applied with γ', ε', and k'_0. We finally set $k_1 := \lceil k'_1/r\rceil$ for Proposition 5.9. After we have defined k_1, let $G=(V,E)$ be a graph satisfying the assumptions of Proposition 5.9.

Since $\varepsilon'\le\varepsilon\le\varepsilon_0=\varepsilon'_0$, and by the choice of ε'_0 and d', Lemma 3.4 applied with input $\gamma'=\gamma$, ε', k'_0 and $\nu' := (r-1)/r$ yields an (ε',d')-regular equipartition $U'_0\dot\cup U'_1\dot\cup\cdots\dot\cup U'_{k'}=V$ of G with reduced graph R' on vertex set $[k']$ such that $\delta(R')\ge((r-1)/r+\gamma/2)k'$ and $k'_0\le k'\le k'_1$. By the choice of k_0, Lemma 5.4 implies that $C^r_k\subseteq R'$ with $k\ge\lfloor(1-\varepsilon')k'/r\rfloor$. Let R^r_k be the graph induced by the kr vertices corresponding to this occurrence of C^r_k in R' and rename the vertex set of R^r_k to $[k]\times[r]$ accordingly. We clearly have $K^r_k\subseteq C^r_k\subseteq R^r_k$ and thus we get (R3). In addition, we will also rename the clusters of G' accordingly in order to obtain a vertex partition $(U'_{i,j})_{i\in[k],j\in[r]}$. Observe, that $kr\le k'\le k'_1\le k_1 r$. Therefore R^r_k satisfies property (R1) of Proposition 5.9. Moreover, R^r_k is an (ε',d')-reduced graph for $G[U'_1\dot\cup\cdots\dot\cup U'_{kr}]$ with

$$|V(R^r_k)| = kr \ge (1-\varepsilon')k' - r \ge (1-\varepsilon')k'_0 - r \overset{(5.10)}{\ge} rk_0 \tag{5.11}$$

and

$$\delta(R^r_k) \ge \delta(R') - \varepsilon' k' - r \ge ((r-1)/r + \gamma/2 - \varepsilon')k' - r \overset{(5.9),(5.10)}{\ge} ((r-1)/r+\gamma/4)kr.$$

Thus, we also have property (R2). Proposition 3.7 applied with $R_{3.7} := R'$, with $R'_{3.7} := K^r_k$ and accordingly $\Delta(R'_{3.7})\le r$ implies that there is an (ε'',d'')-regular partition $U''_0\cup(U''_{i,j})_{i\in[k],j\in[r]}$ of

G with reduced graph R' that is (ε'', d'')-super-regular on K_k^r where $\varepsilon'' = 2\varepsilon' r/(1-\varepsilon' r) \le 10\varepsilon' r$ and $d'' = d' - 2\varepsilon' r \ge d'/2$ by (5.9). Let L'' denote the size of the clusters in this partition. Then

$$\frac{n}{kr} \ge L'' \ge (1-10\varepsilon' r)\frac{n}{kr} \qquad \text{and} \qquad |U_0''| \le 10\varepsilon' rn\,. \tag{5.12}$$

In order to obtain the required partition of V with clusters $U_{i,j}$ for $i \in [k]$, $j \in [r]$ we will distribute the vertices in U_0'' to the clusters $U_{i,j}''$ so that the resulting partition is r-equitable and still (ε, d)-regular on R_k^r and (ε, d)-super-regular on K_k^r.

For this purpose, let u be a vertex in U_0''. The i-th component of K_k^r is called *u-friendly*, if u has at least $dn/(kr)$ neighbours in each of the clusters $U_{i,j}''$ with $j \in [r]$. We claim that each $u \in U_0''$ has at least γk components that are u-friendly. Indeed, assume for a contradiction that there were only $x < \gamma k$ such u-friendly components for some u. Then, since u has less than $(r-1)L'' + dn/(kr)$ neighbours in clusters of components that are not u-friendly, we can argue that

$$\begin{aligned}
|N_G(u)| &< xrL'' + (k-x)\left((r-1)L'' + \frac{dn}{kr}\right) + |U_0''| \\
&= k(r-1)L'' + xL'' + (k-x)\frac{d}{kr}n + |U_0''| \overset{(5.12)}{<} k(r-1)\frac{n}{kr} + \gamma\frac{n}{r} + \frac{d}{r}n + 10\varepsilon' rn \\
&\qquad \overset{(5.8),(5.9)}{\le} \frac{r-1}{r}n + \frac{\gamma}{r}n + \frac{\gamma}{4r}n + \frac{\gamma}{4}n \overset{r\ge 2}{\le} \left(\frac{r-1}{r} + \gamma\right)n,
\end{aligned}$$

which is a contradiction.

In a first step we now assign the vertices $u \in U_0''$ as evenly as possible to u-friendly components of K_k^r. Since each vertex $u \in U_0''$ has at least γk such u-friendly components, each component of K_k^r gets assigned at most $|U_0''|/(\gamma k)$ vertices.

Then, in a second step, in each component we distribute the vertices that have been assigned to this component as evenly as possible among the r clusters of this component. It follows immediately that the resulting partition is r-equitable. Moreover, every cluster $U_{i,j}''$ with $i \in [k]$, $j \in [r]$ gains at most

$$\frac{|U_0''|}{\gamma k} \overset{(5.12)}{\le} \frac{10\varepsilon' rn}{\gamma k} \overset{(5.12)}{\le} \frac{10\varepsilon' r^2}{\gamma(1-10\varepsilon' r)}L'' \overset{(5.9)}{\le} \frac{20\varepsilon' r^2}{\gamma}|U_{i,j}''| \overset{(5.9)}{\le} \sqrt{\varepsilon'}|U_{i,j}''| \tag{5.13}$$

vertices from U_0'' during this process. The resulting partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V satisfies properties (U1)–(U3). Indeed, define

$$m_{i,j} := |U_{i,j}| \ge |U_{i,j}''| = L'' \overset{(5.12)}{\ge} (1-10\varepsilon' r)n/(kr) \overset{(5.9)}{\ge} (1-\varepsilon)n/(kr),$$

and note that for this choice (R4) and (U1) of Proposition 5.9 hold. Moreover, recall that $(U_{i,j}'')_{i\in[k],j\in[r]}$ is $(10\varepsilon' r, d'/2)$-regular on R_k^r and $(10\varepsilon' r, d'/2)$-super-regular on K_k^r. By (5.13), Proposition 3.8 with $\alpha = \beta = \sqrt{\varepsilon'}$ assures that $(U_{i,j})_{i\in[k],j\in[r]}$ is $(\hat{\varepsilon}, \hat{d})$-regular on R_k^r and $(\hat{\varepsilon}, \hat{d})$-super-regular on K_k^r, where

$$\hat{\varepsilon} := 10\varepsilon' r + 6\sqrt[4]{\varepsilon'} \quad \text{and} \quad \hat{d} := \frac{d'}{2} - 4\sqrt{\varepsilon'}\,.$$

Since $10\varepsilon' r + 6\sqrt[4]{\varepsilon'} \le \varepsilon$ and $d'/2 - 4\sqrt{\varepsilon'} \ge d'/3 \ge d$ by (5.8) and (5.9), this implies (U2) and (U3) and concludes the proof of Proposition 5.9. □

It remains to show how to deduce the lemma for G (Lemma 5.3) from Proposition 5.9. As mentioned earlier, we need to show that the sizes of the clusters can be slightly changed. In order to achieve this, we will develop a technique for adapting the cluster sizes step by step by moving one vertex at a time from one cluster to another cluster until each cluster has exactly the right number of vertices.

As explained already in Section 1.3.5 the problem is that *every* vertex that is moved to a new cluster which is part of a super-regular component of K_k^r must make sure that it has sufficiently many neighbours inside the neighbouring clusters within the component. For this, we will exploit the high minimum degree of R_k^r as well as the structure of C_k^r. The following two facts will allow us to move vertices between different clusters. The first observation will be useful to address imbalances within clusters of one colour class of C_k^r.

Fact 5.10. *Suppose that $(V_{i,j})_{i\in[k],j\in[r]}$ is a vertex partition that is $(\varepsilon,\frac{1}{2}d)$-regular on C_k^r and satisfies $|V_{i,j}|\geq n/(2kr)$ for all $i\in[k]$ and $j\in[r]$. Now, fix $i\in[k]$ and $j\in[r]$. Then, there are at least $(1-2r\varepsilon)n/(2kr)$ "good" vertices $v\in V_{i,j}$ that have at least $(\frac{1}{2}d-\varepsilon)n/(2kr)$ neighbours in each set $V_{i',j'}$ with $i'\in\{i-1,i+1\}$ and $j'\in[r]\setminus\{j\}$.*

Proof. Note that (i,j) is connected to each of the (i',j') in C_k^r. Since $(V_{i,j})_{i\in[k],j\in[r]}$ is $(\varepsilon,\frac{1}{2}d)$-regular on C_k^r we can apply Proposition 3.6 with input ε, $\frac{1}{2}d$, $A=V_{i,j}$, and $B=B'=V_{i',j'}$ for each $i'\in\{i-1,i+1\}$ and $j'\in[r]\setminus\{j\}$. It follows that at least $|V_{i,j}|-2(r-1)\varepsilon|V_{i,j}|$ vertices of $V_{i,j}$ have more than $(\frac{1}{2}d-\varepsilon)|V_{i',j'}|$ neighbours in each $V_{i',j'}$. This implies the assertion of Fact 5.10, because

$$|V_{i,j}|-2(r-1)\varepsilon|V_{i,j}|\geq(1-2(r-1)\varepsilon)\frac{n}{2kr}\geq(1-2r\varepsilon)\frac{n}{2kr} \tag{5.14}$$

and

$$(\tfrac{1}{2}d-\varepsilon)|V_{i',j'}|\geq(\tfrac{1}{2}d-\varepsilon)\frac{n}{2kr}\,.$$

□

Before we move on, let us quickly illustrate how Fact 5.10 is used in the proof of Lemma 5.3. For this purpose assume further that $(V_{i,j})_{i\in[k],j\in[r]}$ is also super-regular on K_k^r. Now suppose that for some $i<i'\in[k]$ and $j\in[r]$ we would like to decrease the size of $V_{i,j}$ and increase the size of $V_{i',j}$. Then by Fact 5.10 there is some vertex v (in fact many vertices) in $V_{i,j}$ which has "many" neighbours in each $V_{i+1,j'}$ with $j'\in[r]\setminus\{j\}$. Hence, we can move v from $V_{i,j}$ to $V_{i+1,j}$ without loosing the super-regularity on K_k^r. Repeating this process by moving a vertex from $V_{i+1,j}$ to $V_{i+2,j}$ and so on, we will eventually reach $V_{i',j}$ (see Figure 5.3). Observe that it is of course not necessarily the vertex $v\in V_{i,j}$ we started with, which is really moved all the way to $V_{i',j}$ during this process, but rather a sequence of vertices each moving one cluster further. After such a sequence of applications of Fact 5.10, we end up with a new partition with the following properties. The cardinality of $V_{i,j}$ decreased by one and $|V_{i',j}|$ increased by one. All other clusters do not change their size. Therefore such a sequence of moves, decreases the imbalances within clusters of colour j in C_k^r and we say that we *moved a vertex along colour class j of C_k^r from $V_{i,j}$ to $V_{i',j}$.*

The next simple fact allows to address imbalances across different colours. More precisely, it will be used for moving a vertex v from cluster $V_{i,j}$ to a cluster $V_{i^*,j'}$ with $j\neq j'$.

Fact 5.11. *Let R_k^r be a graph on vertex set $[k]\times[r]$ and suppose that $(V_{i,j})_{i\in[k],j\in[r]}$ is a vertex partition that is $(\varepsilon,\frac{1}{2}d)$-regular on R_k^r and satisfies $|V_{i,j}|\geq n/(2kr)$ for all $i\in[k]$ and $j\in[r]$.*

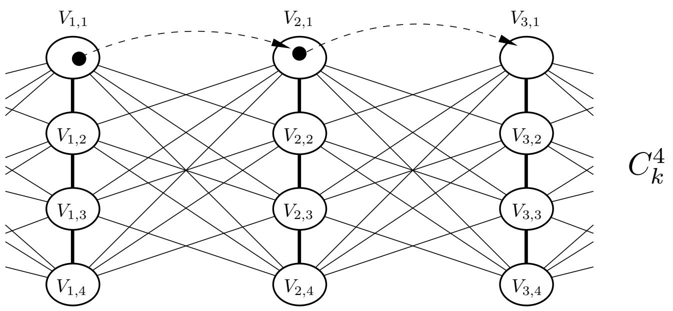


Figure 5.3: Moving a vertex from $V_{1,1}$ to $V_{3,1}$ along colour class 1 of C_k^4 and thus decreasing the size of $V_{1,1}$ and increasing the size of $V_{3,1}$.

Now, fix $i, i^* \in [k]$ and $j^+, j^- \in [r]$. If for each $j' \in [r]$ with $j' \neq j^-$ the vertex (i^*, j') is a neighbour of (i, j^+) in R_k^r then there are at least $(1 - 2r\varepsilon)n/(2kr)$ "good" vertices $v \in V_{i,j^+}$ that have at least $(\frac{1}{2}d - \varepsilon)n/(2kr)$ neighbours in each $V_{i^*,j'}$ with $j' \in [r]$, $j' \neq j^-$.

Proof of Fact 5.11. The existence of the vertices v follows similarly as in the proof of Fact 5.10. Indeed, by Proposition 3.6, there are at least

$$|V_{i,j^+}| - (r-1)\varepsilon|V_{i,j^+}| \geq (1 - r\varepsilon)n/(2kr) \geq (1 - 2r\varepsilon)n/(2kr)$$

such vertices (cf. (5.14)). □

The idea of the technique for adapting the cluster sizes now is as follows. We pick one cluster C^+ that has too many vertices compared to the desired partition and one cluster C^- that has too few vertices at a time. If C^+ and C^- have the same colour then we can move a vertex along C_k^r from C^+ to C^- by repeatedly applying Fact 5.10. If C^+ and C^- are of different colours j^+ and j^- on the other hand we first find a cluster D^+ that has the same colour as C^+ and a component i^* of K_k^r such that the vertex (i, j^+) corresponding to D^+ in R_k^r has edges to all vertices (i^*, j') in R_k^r with $j' \in [r]$, $j' \neq j^-$. Next, we use Fact 5.11 in order to move a vertex from D^+ to the cluster D^- corresponding to the vertex (i^*, j^-) of R_k^r. Then we can proceed as before and move one vertex along C_k^r from C^+ to D^+ and one from D^- to C^- (see Figure 5.4). We repeat this process until every cluster has exactly the right size.

This is only possible, however, if we can guarantee the existence of the cluster D^+ in R_k^r in each step. If this is the case then we say that R_k^r is colour adjustable.

Definition 5.12 (colour adjustable)**.** *Let R_k^r be a graph on vertex set $[k] \times [r]$ with $C_k^r \subseteq R_k^r$. We say that C_k^r is* colour adjustable *in R_k^r if for all $j^+, j^- \in [r]$ with $j^+ \neq j^-$ there are $i, i^* \in [k]$ such that (i, j^+) has edges to all (i^*, j') in R_k^r with $j' \neq j^-$ and $j' \in [r]$.*

In our setting the copy of C_k^r in the reduced graph R_k^r is colour adjustable thanks to its high minimum degree as we will show in the proof of Lemma 5.3 below. Never the less, we shall formulate the adjustment technique just described for general colour adjustable graphs in the following lemma. This lemma will be reused in Chapter 7 where we do not deal with reduced graphs R_k^r of sufficiently high minimum degree anymore.

Lemma 5.13 (adjusting lemma)**.** *For all integers r and constants d, $\varepsilon > 0$ there is $\varepsilon' > 0$ such that for all integers k there is a $\xi > 0$ such that for all integers n the following holds. Let*

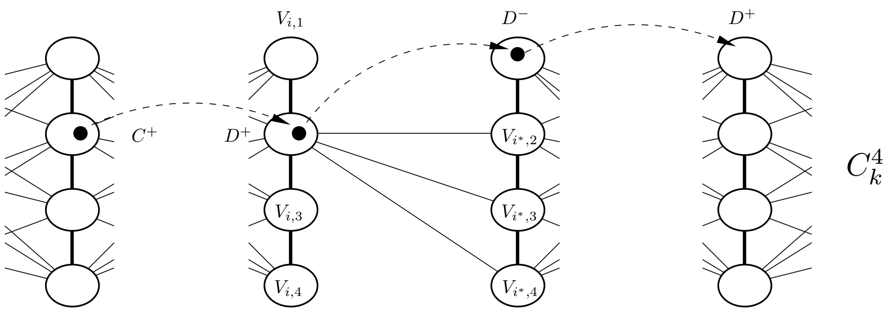


Figure 5.4: Moving a vertex from C^+ to D^+, then from D^+ to D^-, then from D^- to C^- and thus decreasing the size of C^+ and increasing the size of C^-.

$G = (V, E)$ be an n-vertex graph that has an (ε', d)-regular partition $V = (U_{i,j})_{i\in[k],j\in[r]}$ with reduced graph R_k^r on vertex set $[k]\times[r]$ such that $|U_{i,j}| \geq n/(2rk)$ for all $i \in [k], j \in [r]$. Assume further that $K_k^r \subseteq C_k^r \subseteq R_k^r$, the copy of C_k^r is colour adjustable in R_k^r, and $(U_{i,j})_{i\in[k],j\in[r]}$ is (ε', d)-super regular on K_k^r. Let $(n_{i,j})_{i\in[k],j\in[r]}$ be an integer partition of n with $n_{i,j} = |U_{i,j}| \pm \xi n$. Then there is an $(\varepsilon, \frac{1}{2}d)$-regular partition $V = (V_{i,j})_{i\in[k],j\in[r]}$ with reduced graph R_k^r that is $(\varepsilon, \frac{1}{2}d)$-super-regular on K_k^r and satisfies $|V_{i,j}| = n_{i,j}$ for all $i \in [k], j \in [r]$.

Proof. Given r, d, and ε we assume without loss of generality that $\varepsilon \leq 1/(3r)$ and first choose an auxilliary constant ξ' and then ε' such that the following inequalities are satisfied

$$12\xi' \leq \tfrac{1}{2}d\,, \quad 6\sqrt{3\xi'} \leq \tfrac{1}{2}\varepsilon\,, \quad \xi' \leq \tfrac{1}{2}(1-2r\varepsilon)\,, \quad \text{and} \quad \varepsilon' \leq \tfrac{1}{2}\varepsilon\,. \tag{5.15}$$

Then we receive the constant k as input and fix ξ with

$$2kr\xi \leq \xi'/(kr)\,. \tag{5.16}$$

Let G, its partition $V = (U_{i,j})_{i\in[k],j\in[r]}$ and the integer partition $(n_{i,j})_{i\in[k],j\in[r]}$ be given, assume that all assumptions of the lemma are satisfied, and let $(m_{i,j})_{i\in[k],j\in[r]}$ be the integer partition defined by $m_{i,j} := |U_{i,j}|$ for all $i \in [k]$, $j \in [r]$. Our goal is to modify the partition $V = (U_{i,j})_{i\in[k],j\in[r]}$ gradually until we get a partition $V = (V_{i,j})_{i\in[k],j\in[r]}$ with the desired properties.

We initially set $V_{i,j} := U_{i,j}$ for all $i \in [k]$, $j \in [r]$. In the following, we shall perform several steps to move vertices out of some clusters and into some other clusters. For this purpose we will use Facts 5.10 and 5.11. During this so-called balancing process we will call a cluster $V_{i,j}$ *deficient*, if $|V_{i,j}| < n_{i,j}$, and *excessive*, if $|V_{i,j}| > n_{i,j}$. In the end we will neither have deficient clusters nor excessive clusters and thus obtain the desired partition.

As indicated earlier, one iteration of the balancing process is as follows. Choose an arbitrary excessive cluster V_{i^+,j^+} and a deficient cluster V_{i^-,j^-}. Note that there are deficient clusters as long as there are excessive clusters by definition, and vice versa. We distinguish two cases. If $j^+ = j^-$ we use Fact 5.10 for moving a vertex along colour class j^+ of C_k^r from cluster V_{i^+,j^+} to cluster V_{i^-,j^-}. (We will argue below why the hypothesis of Fact 5.10 is satisfied.)

Otherwise, we first use that R_k^r is colour adjustable and conclude that there are indices $i, i^* \in [k]$ such that (i, j^+) has edges to all (i^*, j') with $j' \neq j^-$ and $j' \in [r]$. Hence we can apply Fact 5.11 to cluster V_{i,j^+} and move one of the vertices $v \in V_{i,j^+}$ with many neighbours in all $V_{i^*,j'}$ with $j' \neq j^-$ and $j' \in [r]$ from cluster V_{i,j^+} to V_{i^*,j^-}. Then, we can proceed similarly as in the previous case and move one vertex along colour class j^+ of C_k^r from cluster V_{i^+,j^+} to cluster V_{i,j^+} and one vertex along colour class j^- of C_k^r from V_{i^*,j^-} to V_{i^-,j^-} with Fact 5.10.

In total at most

$$\sum_{i=1}^{k}\sum_{j=1}^{r} |n_{i,j} - m_{i,j}| \leq kr\xi n$$

iterations have to be performed in order to guarantee that $|V_{i,j}| = n_{i,j}$ for all $i \in [k]$ and $j \in [r]$. Moreover, in each iteration not more than one vertex gets moved out of each $V_{i,j}$ with $i \in [k]$, $j \in [r]$, and at most one vertex gets moved into each $V_{i,j}$. So, throughout the process we have

$$|U_{i,j} \triangle V_{i,j}| \leq 2 \cdot kr\xi n \overset{(5.16)}{\leq} \xi' \frac{n}{kr}, \tag{5.17}$$

for all $i \in [k]$, $j \in [r]$.

Note that, since by (5.15) we have $(1 - 2r\varepsilon)n/(2kr) \geq \xi' n/(kr)$, in every step the "moving" vertex v can be chosen from the set of $(1-2r\varepsilon)n/(2kr)$ "good" vertices guaranteed by Facts 5.10 and 5.11. In addition it follows that

$$|V_{i,j}| \geq |U_{i,j}| - |U_{i,j} \triangle V_{i,j}| \overset{(5.17)}{\geq} \left(\tfrac{1}{2} - \xi'\right) \frac{n}{kr} \overset{(5.15)}{\geq} \frac{n}{3kr} \tag{5.18}$$

after (and throughout) the balancing process for all $i \in [k]$, $j \in [r]$. Recall that $(U_{i,j})_{i\in[k],j\in[r]}$ is (ε', d)-regular on R_k^r and (ε', d)-super-regular on K_k^r. Therefore, we can apply Proposition 3.8 with input ε', d, $A := U_{i,j}$, $\hat{A} := V_{i,j}$, and $B := U_{i',j'}$, $\hat{B} := V_{i',j'}$ for any neighbouring vertices (i, j) and (i', j') in R_k^r. For this, we set

$$\alpha := \beta := 3\xi' \geq \frac{|U_{p,q} \triangle V_{p,q}|}{|V_{p,q}|} \quad \text{for all } p \in [k] \text{ and } q \in [r]\,, \tag{5.19}$$

where the inequality follows from (5.17) and (5.18). With

$$\hat{\varepsilon} := \varepsilon' + 3\big(\sqrt{\alpha} + \sqrt{\beta}\big) \overset{(5.19)}{=} \varepsilon' + 6\sqrt{3\xi'} \overset{(5.15)}{\leq} \varepsilon$$

and

$$\hat{d} := d - 2(\alpha + \beta) \overset{(5.19)}{=} d - 12\xi' \overset{(5.15)}{\geq} \tfrac{1}{2}d.$$

we deduce from Proposition 3.8 that $(V_{i,j})_{i\in[k],j\in[r]}$ remains $(\varepsilon, \frac{1}{2}d)$-regular on R_k^r and, since we only moved "good" vertices, $(V_{i,j})_{i\in[k],j\in[r]}$ remains $(\varepsilon, \frac{1}{2}d)$-super-regular on K_k^r throughout the entire process. This also justifies that the hypotheses of Facts 5.10 and 5.11 are satisfied and we could therefore indeed apply these facts throughout the entire balancing process. Observe that $(V_{i,j})_{i\in[k],j\in[r]}$ has all required properties. □

Now Lemma G is a straightforward consequence of Proposition 5.9 and Lemma 5.13.

Proof of Lemma 5.3. We first fix the constants involved in the proof. Let r and $\gamma > 0$ be given by Lemma 5.3. For r and γ, Proposition 5.9 yields constants $d' > 0$ and $\varepsilon'_0 > 0$. For Lemma 5.3 we set $\varepsilon_0 := \min\{\varepsilon'_0, 1/2\}$ and $d := d'/2$. For given $\varepsilon \leq \varepsilon_0$, we let $\varepsilon'_{5.13}$ be the constant given by Lemma 5.13 for input r, d replaced by $d' = 2d$, and ε. Then we fix

$$\varepsilon' := \min\left\{\varepsilon'_{5.13}, \varepsilon'_0, \varepsilon\right\}\,. \tag{5.20}$$

As $\varepsilon' \leq \varepsilon_0 \leq \varepsilon'_0$ we can apply Proposition 5.9 with r, γ, and ε' to obtain k_1. Next, for all $k' \in [k_1]$ we let $\xi_{k'}$ be the constant provided by Lemma 5.13 for input r and d', ε, and k'. Finally, we define the constant ξ_0 promised by Lemma 5.3 to be the minimum of these $\xi_{k'}$, $k' \in [k_1]$.

Having fixed all the constants, let $G = (V, E)$ be a graph on $n \geq k_1$ vertices with $\delta(G) \geq ((r-1)/r + \gamma)n$. We now apply Proposition 5.9 with r, γ, and ε' to the input graph G and get a positive integer $k \leq k_1$, a graph R^r_k, and a partition $(U_{i,j})_{i\in[k],j\in[r]}$ of V so that (R1)–(R4) and (U1)–(U3) of Proposition 5.9 hold with ε replaced by ε' and d replaced by $d' = 2d$. Since $\varepsilon \geq \varepsilon'$, this shows that k, R^r_k, and $m_{i,j} = |U_{i,j}|$ for all $i \in [k]$, $j \in [r]$ also satisfy properties (R1)–(R4) of Lemma 5.3.

It remains to prove the 'second part' of Lemma 5.3. For that let $(n_{i,j})_{i\in[k],j\in[r]}$ be an integer partition of $n = |V|$ satisfying $n_{i,j} = m_{i,j} \pm \xi_0 n$ for every $i \in [k]$, $j \in [r]$. We would like to apply Lemma 5.13 with parameters $d' = 2d$, ε, and k in order to modify the partition $(U_{i,j})_{i\in[k],j\in[r]}$ of G with reduced graph R^r_k and obtain a new partition $(V_{i,j})_{i\in[k],j\in[r]}$ with cluster sizes as prescribed by $(n_{i,j})_{i\in[k],j\in[r]}$. We first need to check that all preconditions of this lemma are satisfied.

By (R3) we have $K^r_k \subseteq C^r_k \subseteq R^r_k$, by (U2) and (U3) the partition $(U_{i,j})_{i\in[k],j\in[r]}$ is $(\varepsilon', 2d)$-regular on R and $(\varepsilon', 2d)$-regular on K^r_k, and from (R4) we get that $|U_{i,j}| \geq n/(2kr)$. Furthermore, by (R2), the minimum degree $\delta(R^r_k)$ of the reduced graph is at least $((r-1)/r + \gamma/4)kr$ which implies that the copy of C^r_k in R^r_k is colour adjustable. Indeed, let $j^+ \in [r]$ be arbitrary. Then, for *any* $i \in [k]$ there is a component i^* of K^r_k such that (i, j^+) has edges to *all* vertices (i^*, j') in this component ($j' \in [r]$) because $\delta(R^r_k) > (r-1)k$. (This condition is stronger than the one we need to be colour adjustable.)

Accordingly we can use Lemma 5.13 with parameters $d' = 2d$, ε, and k as planned. By our choice of ε' in (5.20) we obtain a partition $(V_{i,j})_{i\in[k],j\in[r]}$ of V with $|V_{i,j}| = n_{i,j}$ for all $i \in [k]$, $j \in [r]$ that is (ε, d)-regular on R^r_k and (ε, d)-super-regular on K^r_k. Therefore $(V_{i,j})_{i\in[k],j\in[r]}$ satisfies (V1)–(V3) and this concludes the proof of Lemma 5.3. □

5.6 The lemma for H

In order to prove the lemma for H (Lemma 5.5), we need to exhibit a mapping $f\colon V(H) \to [k] \times [r]$ with properties (a)–(d). Basically, we would like to use the fact that H is almost r-colourable, visit the vertices of H in the order of the bandwidth labelling and arrange that f maps the first vertices of colour 1 to $(1, 1)$, the first vertices of colour 2 to $(1, 2)$, the first vertices of colour 3 to $(1, 3)$, and so on. Ignoring the vertices of colour 0, it would be ideal if in this way, at more or less the same moment, we would have dealt with $m_{1,1}$ vertices of colour 1, $m_{1,2}$ vertices of colour 2 and so on, since we could then move on and let f assign vertices to the next component of $K^r_k \subseteq C^r_k$.

However, the problem is that although the $m_{i,j}$ are r-equitable, i.e., almost identical, the colour classes of H may vary a lot in size. Therefore, the basic idea of our proof of Lemma 5.5 will be to find a recolouring of H with more or less balanced colour classes (besides colour 0).

We emphasise that everything in this section is completely elementary (i.e. it does not use any advanced machinery from the regularity method) but at times a bit technically cumbersome. Therefore we split it into a series of simple propositions.

Proposition 5.14. *Let $c_1, \dots, c_r \in \mathbb{R}$ be such that $c_1 \leq c_2 \leq \dots \leq c_r \leq c_1 + x$ and $c'_1, \dots, c'_r \in \mathbb{R}$ such that $c'_r \leq c'_{r-1} \leq \dots \leq c'_1 \leq c'_r + x$. If we set $c''_i := c_i + c'_i$ for all $i \in [r]$*

then

$$\max_i\{c_i''\} \le \min_i\{c_i''\} + x.$$

Proof. It clearly suffices to show that $c_i + c_i' \le c_j + c_j' + x$ for all $i, j \in [r]$. For $i \le j$ this follows from $c_i \le c_j$ and $c_i' \le c_r' + x \le c_j' + x$. Similarly, for $i > j$ we have that $c_i \le c_1 + x \le c_j + x$ and $c_i' \le c_j'$. □

Now assume that the vertices of H are labelled $1, \dots, n$. Recall that for an $(r+1)$-colouring $\sigma : V(H) \to \{0, \dots, r\}$ of H an interval $[s,t] \subseteq [n]$ is called zero free, if $\sigma(u) \ne 0$ for all $u \in [s,t]$. Moreover, the colouring σ is called (x,y)-zero free on the interval $[a,b] \subseteq [n]$, if for each $t \in [n]$ there exists an interval $[t', t'+y] \subseteq [t, t+x+y]$ such that $[t', t'+y] \cap [a,b]$ is zero free.

The following proposition investigates under what conditions a colouring remains (x,y)-zero free when a few more vertices receive colour 0.

Proposition 5.15. *Assume that the vertices of H are labelled $1, \dots, n$. Let y be a positive integer, $a \in [n]$ and suppose that $\sigma : V(H) \to \{0, \dots, r\}$ is an $(r+1)$-colouring that is $(8y,y)$-zero free on $[n]$ as well as $(2y,y)$-zero free on $[a,n]$ with respect to this labelling.*

Let $a + 3y \le b \le a + 5y$ and suppose that σ' is another $(r+1)$-colouring that differs from σ in that some of the vertices in the interval $(b, b+y)$ now have colour 0, i.e., $(\sigma')^{-1}(0) \subseteq \sigma^{-1}(0) \cup (b, b+y)$.

Then σ' must still be $(8y,y)$-zero free on $[n]$ and $(2y,y)$-zero free on $[a+6y, n]$.

Proof. By definition $[b, b+y] \subseteq [a+3y, a+6y]$ and thus

$$\text{(i)} \quad \sigma'\big|_{[1,a+3y]} \equiv \sigma\big|_{[1,a+3y]} \quad \text{and} \quad \text{(ii)} \quad \sigma'\big|_{[b+y,n]} \equiv \sigma\big|_{[b+y,n]}\,. \tag{5.21}$$

First note that the second claim of the proposition is trivial, because $b + y \le a + 6y$ and part (ii) of (5.21) show that the fact that σ is $(2y,y)$-zero free on $[a,n]$ implies that σ' is $(2y,y)$-zero free on $[a+6y, n]$.

As for the first claim, we need to show that for every $t \in [n]$ there exists an interval $[t', t'+y] \subseteq [t, t+9y]$ which is zero-free under σ'. Here we need to distinguish several cases.

$t < a - 6y$: By part (i) of (5.21) the assertion follows from the fact that σ is $(8y,y)$-zero free on $[n]$.

$a - 6y \le t < a$: The fact that σ is $(2y,y)$-zero free on $[a,n]$ implies (when applied to the vertex a) that there is a zero free interval $[t', t'+y] \subseteq [a, a+3y] \subseteq [t, t+9y]$ under σ. By part (i) of (5.21), $[t', t'+y]$ is also zero free under σ'.

$a \le t < b + y$: The fact that σ is $(2y,y)$-zero free on $[a,n]$ implies (when applied to the vertex $b+y$) that there is a zero free interval $[t', t'+y] \subseteq [b+y, b+4y] \subseteq [t, a+9y] \subseteq [t, t+9y]$ under σ. By part (ii) of (5.21), $[t', t'+y]$ is also zero free under σ'.

$b + y \le t$: Here the assertion follows because part (ii) of (5.21) shows that the fact that σ is $(8y,y)$-zero free on $[n]$ implies that σ' is $(8y,y)$-zero free on $[b+y, n]$.

□

Now we introduce the notion of *switching* two colours $l, l' \in [r]$ at some given vertex s, which will be essential to transform the given colouring of H into one that uses the colours $1, \dots, r$ in a more or less balanced manner. Basically, all vertices of colour l after s are coloured by l'

and vice versa. In order to avoid adjacent vertices of the same colour, we use the bandwidth condition and colour vertices in the interval $s-\beta n, s+\beta n$ that previously had colour l with colour 0.

Proposition 5.16. *Assume that the vertices of H are labelled $1,\dots,n$ with bandwidth at most βn with respect to this labelling. Let $s \in [n]$ and suppose further that $\sigma\colon [n] \to \{0,\dots,r\}$ is a proper $(r+1)$-colouring of $V(H)$ such that $[s-2\beta n, s+2\beta n]$ is zero free.*

Then for any two colours $l, l' \in [r]$ the mapping $\sigma'\colon [n] \to \{0,\dots,r\}$ defined by

$$\sigma'(v) := \begin{cases} l & \text{if } \sigma(v) = l', s < v \\ l' & \text{if } \sigma(v) = l, s+\beta n < v \\ 0 & \text{if } \sigma(v) = l, s-\beta n \le v \le s+\beta n \\ \sigma(v) & \text{otherwise} \end{cases}$$

is a proper *$(r+1)$-colouring of H. (We will say that σ' is obtained from σ by an $(l,l',\beta n)$*-switch at vertex *s.)*

Note that we only introduced new vertices of colour 0 in the interval $[s-\beta n, s+\beta n]$ and that all these vertices are non-adjacent since they have colour l in σ.

Proof. Indeed, as σ' is derived from the proper colouring σ by interchanging the colours l and l' after the vertex s and introducing some new vertices of colour 0 in $[s-\beta n, s+\beta n]$, the only monochromatic edges that σ' could possibly yield are edges $\{u,v\}$ with either $u \le s$ and $s < v$ and $\{\sigma(u),\sigma(v)\} = \{l,l'\}$ or with $\sigma'(u) = \sigma'(v) = 0$. The second case is clearly ruled out by the facts that H has bandwidth at most βn, that $[s-2\beta n, s+2\beta n]$ is zero free under σ and that there are no edges between new vertices of colour 0. For the first case, since H has bandwidth at most βn, we must have that $u \in [s-\beta n, s]$ and $v \in [s+1, s+\beta n]$. But if $\sigma(u) = l$ and $\sigma(v) = l'$, then $\sigma'(u) = 0$ and $\sigma'(v) = l$. If $\sigma(u) = l'$ and $\sigma(v) = l$ on the other hand, then $\sigma'(u) = l'$ and $\sigma'(v) = 0$. Hence, σ' is a proper $(r+1)$-colouring. □

The next and final proposition is based on repeated applications of the three preceding ones and sums up what we have achieved so far. For that we need one more definition: For $x \in \mathbb{N}$, a colouring $\sigma\colon [n] \to \{0,\dots,r\}$ is called *x-balanced*, if for each interval $[a,b] \subseteq [n]$ and each $l \in [r]$, we have

$$\frac{b-a}{r} - x \le \left|\sigma^{-1}(l) \cap [a,b]\right| \le \frac{b-a}{r} + x.$$

Proposition 5.17. *Assume that the vertices of H are labelled $1,\dots,n$ with bandwidth at most βn and that H has an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free with respect to this labelling, which uses at most βn vertices of colour 0 in total. Let ξ be a constant with $\beta < \xi^2/(48r)$ and assume that $1/\xi$ is an integer. Then there exists a proper $(r+1)$-colouring $\sigma\colon V(H) \to \{0,\dots,r\}$ that is $(32r\beta n, 4r\beta n)$-zero free and $5\xi n$-balanced.*

The idea of the proof is as follows. We cut H into pieces of length ξn and proceed by induction. Suppose that we have found a colouring that is zero free and balanced on the first p pieces. Then permute the colours on the remaining pieces such that the *largest* colour class of the union of pieces 1 to p has the same colour as the *smallest* colour class of the $(p+1)$-st piece, and vice versa (again, ignoring colour 0). Now glueing the colourings together (as in Proposition 5.16), the new colouring will be roughly as balanced on the first $p+1$ pieces (see Proposition 5.14) and as zero free (see Proposition 5.15) as the old one.

Proof. Suppose that H, β and ξ are given with the required properties. In the first part of the proof, we will prove the following statement by induction (on p): for all integers $p \in [1/\xi]$ there exists a proper $(r+1)$-colouring $\sigma_p \colon [n] \to \{0, \dots, r\}$ of the vertices of H with the following properties:

$$\sigma_p \text{ is } (32r\beta n, 4r\beta n)\text{-zero free on } [n], \tag{5.22}$$

$$\sigma_p \text{ is } (8r\beta n, 4r\beta n)\text{-zero free on } [p\xi n, n], \tag{5.23}$$

and for all $j \in [p]$

$$\max_{l \in [r]} \left\{ |\sigma_p^{-1}(l) \cap [j\xi n]| \right\} \leq \min_{l \in [r]} \left\{ |\sigma_p^{-1}(l) \cap [j\xi n]| \right\} + \xi n + 24rj\beta n. \tag{5.24}$$

For $p = 1$, we let σ_1 be the original $(8r\beta n, 4r\beta n)$-zero free $(r+1)$-colouring of H. Hence (5.22), (5.23), and (5.24) hold trivially.

Next suppose that σ_p is given. For $i \in [r-1]$, we will fix colours l_i, l_i' and positions s_i and then obtain σ_{p+1} from σ_p by a series of $r-1$ appropriate $(l_i, l_i', \beta n)$-switches at positions s_i. For this purpose recall that by induction (5.23) guarantees that σ_p is $(8r\beta n, 4r\beta n)$-zero free on the interval $[p\xi n, n]$. When applied to the vertex $t := p\xi n + 12r\beta n$, there exists a vertex $t' \in [p\xi n + 12r\beta n, p\xi n + 20r\beta n]$ such that $[t', t' + 4r\beta n]$ is zero free. Now choose positions $s_1, \dots, s_{r-1}$ by letting $s_1 := t' + 4\beta n$ and $s_i := s_{i-1} + 4\beta n$ for all $2 \leq i \leq r-1$. Thus

$$\begin{aligned} p\xi n + 12r\beta n &\leq t' < s_1 - 2\beta n \leq s_1 \leq \dots \leq s_{r-1} \leq s_{r-1} + 2\beta n \\ &= t' + 4(r-1)\beta n + 2\beta n < t' + 4r\beta n. \end{aligned} \tag{5.25}$$

Now let c_i be the number of vertices in $[p\xi n]$ with colour i under σ_p for $i \in [r]$ and suppose w.l.o.g. that $c_1 \leq \dots \leq c_r$. For some (not yet specified) colours l_i, l_i' we will obtain σ_{p+1} from σ_p by consecutive $(l_i, l_i', \beta n)$-switches at s_i for all $i \in [r-1]$ and denote by c_i' the number of vertices in the interval

$$I := [t' + 4r\beta n, (p+1)\xi n].$$

which have colour i under σ_{p+1} for $i \in [r]$. Observe that by (5.25), all switches occur before the interval I, so since every permutation of the set $[r]$ can be written as the composition of at most $r-1$ transpositions, it is clear that we can choose the colours $l_1, l_1', \dots, l_{r-1}, l_{r-1}' \in [r]$ such that $c_r' \leq \dots \leq c_1'$.

Again by (5.25) we have $[s_1 - 2\beta n, s_{r-1} + 2\beta n] \subseteq [t', t' + 4r\beta n]$, which, by the choice of t', is zero free under σ_p at the beginning of the switches. Moreover, the switch at s_{i-1} introduces new vertices of colour 0 only in the interval $[s_{i-1} - \beta n, s_{i-1} + \beta n]$ which (by definition of the s_i) is disjoint from $[s_i - 2\beta n, s_{r-1} + 2\beta n]$.

Thus we can be sure that before we apply the switch at s_i, the interval $[s_i - 2\beta n, s_i + 2\beta n]$ is zero free. Hence we can apply Proposition 5.16 for each of the $r-1$ switches and obtain that σ_{p+1} is again a proper $(r+1)$-colouring of H.

It is now easy to check that σ_{p+1} satisfies the requirements (5.22), (5.23), and (5.24), with p replaced by $p+1$. Indeed, properties (5.22) and (5.23) follow by evoking Proposition 5.15 with $y := 4r\beta n$, $a := p\xi n$, and $b := t' \in [p\xi n + 12r\beta n, p\xi n + 20r\beta n]$. To prove (5.24), observe that as $\sigma_{p+1}(v) = \sigma_p(v)$ for all $v \leq p\xi n$, we know by induction that (5.24) with σ_{p+1} in the place of σ_p still holds for all $j \leq p$. Moreover, we have $|\sigma_{p+1}^{-1}(i) \cap [p\xi n]| = c_i$. Using that $c_1 \leq \dots \leq c_r$ together with, again, (5.24) from the induction for $j = p$, we now have

$$\begin{aligned} |\sigma_{p+1}^{-1}(1) \cap [p\xi n]| &\leq |\sigma_{p+1}^{-1}(2) \cap [p\xi n]| \leq \dots \leq |\sigma_{p+1}^{-1}(r) \cap [p\xi n]| \\ &\leq |\sigma_{p+1}^{-1}(1) \cap [p\xi n]| + \xi n + 24rp\beta n. \end{aligned}$$

On the other hand, we have $|\sigma_{p+1}^{-1}(i) \cap I| = c'_i$. Using that $c'_r \leq \cdots \leq c'_1$ and $|I| \leq \xi n \leq \xi n + 24rp\beta n$, we obtain

$$\begin{aligned} |\sigma_{p+1}^{-1}(r) \cap I| \leq |\sigma_{p+1}^{-1}(r-1) \cap I| \leq \cdots \leq |\sigma_{p+1}^{-1}(1) \cap I| \\ \leq |\sigma_{p+1}^{-1}(r) \cap I| + \xi n + 24rp\beta n. \end{aligned}$$

we can now apply Proposition 5.14 with $x := \xi n + 24rp\beta n$ to see that

$$\begin{aligned} &\max_{l\in[r]} \left\{|\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]|\right\} \\ &\qquad \leq \min_{l\in[r]} \left\{|\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]|\right\} + \xi n + 24rp\beta n + \underbrace{\left|[p\xi n, (p+1)\xi n] \setminus I\right|}_{\leq t'+4r\beta n - p\xi n \leq 24r\beta n}, \\ &\qquad\qquad \leq \min_{l\in[r]} \left\{|\sigma_{p+1}^{-1}(l) \cap [(p+1)\xi n]|\right\} + \xi n + 24r(p+1)\beta n \end{aligned}$$

which implies equation (5.24) for $j = p+1$ as well. This completes the inductive proof of statements (5.22), (5.23), and (5.24). Recall moreover that the switch at s_i introduces new vertices of colour 0 only in the interval $[s_i - \beta n, s_i + \beta n]$ for all $i \in [r-1]$. Therefore each of these switches introduces at most $2\beta n$ new vertices of colour 0. Since σ_1 has at most βn vertices of colour 0 it follows that σ_j colours at most $j(r-1)2\beta n + \beta n \leq 2rj\beta n$ vertices with 0.

For the second part of the proof, set $p := 1/\xi$ and consider the $(r+1)$-colouring $\sigma := \sigma_p$ whose existence we have proven in the first part. Recall that by (5.22) and (5.24) we know that

$$\sigma \text{ is } (32r\beta n, 4r\beta n)\text{-zero free on } [n] \tag{5.26}$$

and for all integers $1 \leq j \leq 1/\xi$

$$\max_{l\in[r]} \left\{|\sigma^{-1}(l) \cap [j\xi n]|\right\} \leq \min_{l\in[r]} \left\{|\sigma^{-1}(l) \cap [j\xi n]|\right\} + \xi n + 24rj\beta n. \tag{5.27}$$

It remains to prove that σ is $5\xi n$-balanced. Let i_+ and i_- be the colours in $[r]$ that are used most and least often in the interval $[j\xi n]$ by σ, respectively; and denote by c_{i_+} and c_{i_-} the number of vertices of colour i_+ and i_- in $[j\xi n]$, respectively. Set $\Lambda := \xi n + 24rj\beta n$ and rewrite property (5.27) as $c_{i_+} \leq c_{i_-} + \Lambda$. Thus, since σ uses at most $2rp\beta n$ vertices of colour 0 on $[j\xi n]$, we obtain that for all $l \in [r]$

$$\frac{j\xi n - 2rp\beta n}{r} - \Lambda \leq c_{i_+} - \Lambda \leq c_{i_-} \leq |\sigma^{-1}(l) \cap [j\xi n]| \leq c_{i_+} \leq c_{i_-} + \Lambda \leq \frac{j\xi n}{r} + \Lambda.$$

Since $\beta < \xi^2/(48r)$, we infer that for every $j \in [1/\xi]$

$$\frac{j\xi n}{r} - 2\xi n < |\sigma^{-1}(l) \cap [j\xi n]| < \frac{j\xi n}{r} + 2\xi n\,. \tag{5.28}$$

Now for an arbitrary interval $[a,b] \subseteq [n]$, we choose $j,\ j' \in [p]$ such that

$$a - \xi n \leq j\xi n \leq a \leq b \leq j'\xi n \leq b + \xi n\,.$$

This yields that

$$|\sigma^{-1}(l) \cap [(j+1)\xi n, (j'-1)\xi n]| \leq \left|\sigma^{-1}(l) \cap [a,b]\right| \leq |\sigma^{-1}(l) \cap [j\xi n, j'\xi n]|.$$

The lower bound is equal to

$$
\begin{aligned}
&|\sigma^{-1}(l) \cap [(j'-1)\xi n]| - |\sigma^{-1}(l) \cap [(j+1)\xi n]| \\
&\qquad\qquad \overset{(5.28)}{\geq} \left(\frac{(j'-1)\xi n}{r} - 2\xi n\right) - \left(\frac{(j+1)\xi n}{r} + 2\xi n + 1\right) \\
&\qquad\qquad \geq \left(\frac{b-\xi n}{r} - 2\xi n\right) - \left(\frac{a+\xi n}{r} + 2\xi n\right) - 1 \geq \frac{b-a}{r} - 5\xi n.
\end{aligned}
$$

Similarly, the upper bound equals

$$
\begin{aligned}
&|\sigma^{-1}(l) \cap [j'\xi n]| - |\sigma^{-1}(l) \cap [j\xi n)| \\
&\qquad\qquad \overset{(5.28)}{\leq} \left(\frac{j'\xi n}{r} + 2\xi n\right) - \left(\frac{j\xi n}{r} - 2\xi n - 1\right) \\
&\qquad\qquad \leq \left(\frac{b+\xi n}{r} + 2\xi n\right) - \left(\frac{a-\xi n}{r} - 2\xi n\right) + 1 \leq \frac{b-a}{r} + 5\xi n.
\end{aligned}
$$

Thus, σ is $5\xi n$-balanced, which completes the proof of Proposition 5.17. □

After these preparations, the proof of the lemma for H (Lemma 5.5) will be straightforward and the basic idea can be described as follows. We will take the $(r+1)$-colouring σ of H which is guaranteed by Proposition 5.17. Next we partition $V(H) = [n]$ into k intervals, where the i-th interval will have length roughly $m_{i,1} + \cdots + m_{i,r}$. In order to prove the lemma, define $f : V(H) \to V(R^r_k) = [k] \times [r]$ in such a way that it maps all vertices in the i-th interval with colour $j \neq 0$ to (i, j), i.e., the j-th vertex of the i-th component of K^r_k. Obviously the bandwidth condition implies that two adjacent vertices u, v will either lie in the same or in neighbouring intervals. If, for example, two adjacent vertices u, v both lie in the i-th interval, then $f(u)$ and $f(v)$ are connected by an edge in $E(K^r_k)$, as required by (d) in the lemma. If, on the other hand, u and v lie in neighbouring intervals, then $f(u)$ and $f(v)$ are vertices of different colours in neighbouring components of K^r_k, and as such connected by an edge of $E(C^r_k) \subseteq E(R^r_k)$ as needed by (c); and for this case we will need to define the set X to make sure that (d) will not be required here. Finally, a little more care is needed for the vertices that receive colour 0 by σ.

Proof of Lemma 5.5. Given r, k and β, let ξ, R^r_k and H be as required. Assume w.l.o.g. that the vertices of H are labelled $1, \ldots, n$ with bandwidth at most βn and that H has an $(8r\beta n, 4r\beta n)$-zero free $(r+1)$-colouring with respect to this labelling. Set $\xi' = \xi/(11r)$, and note that $\beta \leq \xi^2/(3026r^3) < (\xi')^2/(48r)$. Therefore, by Proposition 5.17 with input β, ξ', and H, there is a $(32r\beta n, 4r\beta n)$-zero free and $5\xi' n$-balanced colouring $\sigma\colon V(H) \to \{0, \ldots, r\}$ of H.

Observe that for each set of r vertices in R^r_k, the common neighbourhood of these vertices is nonempty, because $\delta(R^r_k) > (r-1)k$. It follows that for each $i \in [k]$ there exists a vertex $r_i \in V(R^r_k) = [k] \times [r]$ that is adjacent in R^r_k to each vertex of the i-th component of K^r_k:

$$\{r_i, (i, j)\} \in E(R^r_k) \qquad \forall j \in [r]. \tag{5.29}$$

The vertices r_i will be needed to construct the mapping f.

Given an r-equitable partition $(m_{i,j})_{i\in[k],j\in[r]}$ of n, set

$$M_i := \sum_{j\in[r]} m_{i,j}$$

for $i \in [k]$. Now let $t_0 := 0$ and $t_k := n$, and for every $i = 1, \dots, k-1$ choose a vertex

$$t_i \in \left[\sum_{i'=1}^{i} M_{i'}, \sum_{i'=1}^{i} M_{i'} + 33r\beta n\right]$$

such that σ is zero free on $[t_i-\beta n, t_i+\beta n]$. Indeed, such a t_i exists since σ is $(32r\beta n, 4r\beta n)$-zero free. We say that $(t_{i-1}, t_i]$ is the i-th interval of H. Vertices $v \in V(H)$ with $v \in [t_i-\beta n, t_i+\beta n]$ for some $i \in [k]$ are called *boundary vertices* of H. Observe that the choice of the t_i implies that boundary vertices are never assigned colour 0 by σ.

Using σ, we will now construct $f : V(H) \to [k] \times [r]$ and $X \subseteq V(H)$. For each $i \in [k]$, and each $v \in (t_{i-1}, t_i]$ in the i-th interval of H we set

$$f(v) := \begin{cases} r_i & \text{if } \sigma(v) = 0, \\ (i, \sigma(v)) & \text{otherwise,} \end{cases}$$

and

$$X := \big\{v \in V(H) : \sigma(v) = 0\big\} \cup \big\{v \in V(H) : v \text{ is a boundary vertex}\big\}.$$

It remains to show that f and X satisfy properties (a)–(d) of Lemma 5.5.

Since σ is $5\xi'n$-balanced, $(n/r) - 5\xi'n \le |\sigma^{-1}(l)|$ for all $l \in [r]$. Consequently

$$|\{v \in [n] : \sigma(v) = 0\}| \le r \cdot 5\xi' n. \tag{5.30}$$

Moreover, there are exactly $k \cdot 2\beta n$ boundary vertices and so we can bound

$$|X| \le 5r\xi'n + 2k\beta n \le 6kr\xi'n \le kr\xi n,$$

which yields (a).

For (b), we need to estimate $|W_{i,j}|$, the number of vertices in H that are mapped by f to (i, j) for each $i \in [k]$ and $j \in [r]$. First, the number of vertices of colour 0 that are mapped to (i, j) can obviously be bounded from above by the bound in (5.30). Furthermore, the mapping f sends all vertices v in the i-th interval of H with $\sigma(v) = j \neq 0$ to (i, j), which are at most $(t_i - t_{i-1})/r + 5\xi'n$ vertices, because σ is $5\xi'n$-balanced. Thus, by the choice of t_{i-1} and t_i, and making use of the fact that $|m_{i,j} - M_i/r| \le 1$ (because the $m_{i,j}$ are known to be r-equitable), we can bound

$$|W_{i,j}| \le \frac{t_i - t_{i-1}}{r} + 5\xi'n + 5r\xi'n \le \frac{M_i + 33r\beta n}{r} + 10r\xi'n \le m_{i,j} + 11r\xi'n = m_{i,j} + \xi n.$$

Similarly, $|W_{i,j}| \ge m_{i,j} - \xi n$ and this shows (b).

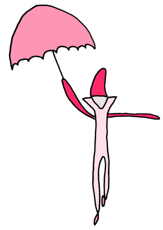
Now, we turn to (c) and (d). For a vertex $u \in V(H)$, let $i(u)$ be the index in $[k]$ for which $u \in (t_{i(u)-1}, t_{i(u)}]$. Let $\{u, v\}$ be an edge of H. Since σ is a proper colouring, this implies that $\sigma(u) \neq \sigma(v)$.

We will first consider the case that u and v are in the same interval of H and not of colour 0, i.e., $i := i(u) = i(v)$ and $\sigma(u) \neq 0 \neq \sigma(v)$. By the definition of f, we have $f(u) = (i, \sigma(u))$ and $f(v) = (i, \sigma(v))$ and hence $\{f(u), f(v)\} \in E(K^r_k)$, which proves ($c$) and ($d$) for this case.

Next we consider the case where u and v are in the same interval $i = i(u) = i(v)$ of H and one of them, say u, has colour 0. Here, by definition of X, we do not need to worry about (d) and only need to verify (c). Indeed, $f(u) = r_i$ and $f(v) = (i, \sigma(v))$. Hence, by (5.29), $\{f(u), f(v)\} \in E(R_k^r)$.

It remains to consider the case where u and v are in different intervals of H. Then both of them are boundary vertices, because the bandwidth of H is at most βn, so again by definition of X, we only need to verify (c). Moreover, $\sigma(u) \neq 0 \neq \sigma(v)$ because by the choice of the t_i boundary vertices are never coloured with 0. Assume w.l.o.g. that $u < v$. It follows that $i(v) = i(u) + 1$ and so $f(u) = (i(u), \sigma(u))$ and $f(v) = (i(u) + 1, \sigma(v))$. This implies that $\{f(u), f(v)\} \in E(C_k^r) \subseteq E(R_k^r)$, which yields (c). □

Chapter 6



Variation 1: Bipartite graphs

With the proof of the Bollobás–Komlós conjecture in the previous chapter we obtained a minimum degree bound for the containment of r-chromatic spanning subgraphs H (cf. Theorem 5.1). This bound is essentially best possible for an almost trivial reason: There are graphs G with minimum degree just slightly smaller that are $(r-1)$-colourable (see also Section 1.2.1). Such G clearly do not contain H as a subgraph. These graphs are simply too different in structure from H (the graph G can be partitioned into $r-1$ independent sets, H cannot).

One may ask, however, whether it is possible to lower the minimum degree threshold in Theorem 5.1 for graphs G and H that are structurally more similar and, in particular, have the same chromatic number. In this chapter we will pursue this question for the case of balanced bipartite graphs, i.e., bipartite graphs on $2n$ vertices with n vertices in each colour class.

Recall that Dirac's theorem implies that a $2n$-vertex graph G with minimum degree at least n contains a Hamilton cycle. It follows from a result of Moon and Mooser [78] that this threshold can be cut (almost) in half if G is balanced bipartite.

Theorem 6.1 (Moon & Moser [78])**.** *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq \frac{n}{2}+1$, then G contains a Hamilton cycle.* □

We prove that slightly increasing this minimum degree bound suffices already to obtain all balanced bipartite graphs with bounded maximum degree and sublinear bandwidth as subgraphs. We establish the following bipartite analogue of Theorem 5.1, halving the minimum degree threshold in that result.

Theorem 6.2. *For all γ and Δ there is a positive constant β and an integer n_0 such that for all $n \geq n_0$ the following holds. Let G and H be balanced bipartite graphs on $2n$ vertices such that G has minimum degree $\delta(G) \geq (\frac{1}{2}+\gamma)n$ and H has maximum degree Δ and bandwidth at most βn. Then $H \subseteq G$.*

The proof of this theorem is given in Section 6.2. It follows the proof scheme that we saw in the last section, but the arguments need to be adapted to the bipartite setting. In particular we have to use a different strategy for "balancing" the graph H here (see Section 6.4; the concept of "balancing" is explained in Section 1.3.4). Before providing the proof of Theorem 6.2 we introduce in the next section a version of the regularity lemma designed for use with bipartite graphs.

6.1 Bipartite regular partitions

In this chapter we deal with bipartite graphs and with regular partitions of such graphs. In our proofs it will be essential that these regular partitions respect the bipartition of the graph that we start with, i.e., they should *refine* this bipartition. Since proofs of the regularity lemma (see, e.g., [28]) iteratively construct refining partitions until a regular partition is obtained, however, this is no problem. We can simply pass our bipartition to the regularity lemma as initial partition. Accordingly we obtain "bipartite analogues" of the minimum degree version of the regularity lemma (Lemma 3.4) or of Proposition 3.7 which, we recall, provides a regular partition that is moreover super-regular on some subgraph of the reduced graph. This is summarized in Lemma 6.3 below (whose proof is standard and will just be sketched here). Before stating this lemma we will adapt the notation concerning regular partitions to this bipartite setting.

The concepts of regular partitions and reduced graphs essentially remain as introduced in Chapter 3, but all regular partitions that we consider in this chapter refine some bipartition and all reduced graphs are bipartite. More precisely, for a bipartite graph $G = (A\dot\cup B, E)$ we will obtain an *ε-regular partition* $(A_0\dot\cup B_0)\dot\cup A_1\dot\cup B_1\dot\cup \dots \dot\cup A_k\dot\cup B_k$ such that $A = A_0\dot\cup \dots \dot\cup A_k$ and $B = B_0\dot\cup \dots \dot\cup A_k$. In particular we have two different *exceptional sets* now, one in A called A_0 and one in B called B_0, each of size εn at most. Hence, such a partition is an *equipartition* if $|A_1| = |B_1| = |A_2| = \dots = |A_k| = |B_k|$.

In addition, we consider only regular pairs running between the bipartition classes, i.e., pairs of the form (A_i, B_j). Consequently, all *reduced graphs* (also the maximal reduced graph of a partition) in this chapter are bipartite.

We now state the version of the regularity lemma that we will use in this chapter.

Lemma 6.3 (regular partitions of bipartite graphs)**.** *For every $\varepsilon' > 0$ and for every $\Delta, k_0 \in \mathbb{N}$ there exists $k_1 = k_1(\varepsilon', k_0) \in \mathbb{N}$ such that for every $0 \le d' \le 1$, for*

$$\varepsilon'' := \frac{2\Delta\varepsilon'}{1-\varepsilon'\Delta} \qquad \text{and} \qquad d'' := d' - 2\varepsilon'\Delta\,,$$

and for every bipartite graph $G = (A\dot\cup B, E)$ with $|A| = |B| \ge k_1$ and $\delta(G) \ge \nu|G|$ for some $0 < \nu < 1$ there exists a graph R and an integer k with $k_0 \le k \le k_1$ with the following properties:

(*a*) *R is a reduced graph of an (ε', d')-regular equipartition of G with $|V(R)| = 2k$.*

(*b*) *$\delta(R) \ge (\nu - d' - \varepsilon'')|R|$.*

(*c*) *For every subgraph $R^* \subseteq R$ with $\Delta(R^*) \le \Delta$ there is an (ε'', d'')-regular equipartition*

$$A\dot\cup B = A''_0\dot\cup B''_0\dot\cup A''_1\dot\cup B''_1\dot\cup \dots \dot\cup A''_k\dot\cup B''_k$$

with $A''_i \subseteq A$ and $B''_i \subseteq B$ for all $0 \le i \le k$ and (ε'', d'')-reduced graph R, which in addition is (ε'', d'')-super-regular on R^*.

Proof (sketch). As a first step we simulate the proof of the degree-form of the regularity lemma (Lemma 3.4) starting with $A\dot\cup B$ as the initial partition (see also [28, Chapter 7.4], or the survey [70]). This yields a partition into clusters $A_0, \dots, B_k$ such that for all vertices $v \notin A_0 \cup B_0$ there are at most $(d + \varepsilon)n$ edges $e \in E$ with $v \in e$ such that e is not in some (ε', d')-regular pair (A_i, B_j). Hence we get (*a*). Let R be the maximal (bipartite) (ε', d')-reduced graph of this partition. Then, analogously as in Lemma 3.4 the graph R inherits the minimum degree condition of G (except for a small loss). This yields (*b*). Finally, we use Proposition 3.7 with R' replaced by R^* to obtain (*c*). □

We remark that in the following $A\dot\cup B$ will denote the vertex set of the host graph G while $\tilde A\dot\cup\tilde B$ is the vertex set of the bipartite graph H we would like to embed. The letters A_i and B_i with $i \in [k]$ for some integer k will be used for the clusters of a regular partition of G as well as for the vertices of a corresponding reduced graph.

6.2 Proof of Theorem 6.2

The proof of Theorem 6.2 is structured similarly as the proof of Theorem 5.2. We will use the general embedding lemma (Lemma 3.12). For applying this lemma we need compatible partitions of the graphs G and H which are provided by the next two lemmas. These lemmas are bipartite version of Lemmas 5.3 and 5.5 (see also the explanations for these lemmas on page 49 in Section 5.2). We start with the lemma for G which constructs a regular partition of G whose reduced graph R contains a perfect matching within a Hamilton cycle of R. The lemma guarantees, moreover, that the regular partition is super-regular on this perfect matching (see Figure 6.1) and that the cluster sizes in the partition can be slightly changed.

Lemma 6.4 (Lemma for G)**.** *For every $\gamma > 0$ there exists $d_{\mathrm{LG}} > 0$ such that for every $\varepsilon > 0$ and every $k_0 \in \mathbb{N}$ there exist $k_1 \in \mathbb{N}$ and $\xi_{\mathrm{LG}} > 0$ with the following properties: For every $n \geq k_1$ and for every balanced bipartite graph $G = (A\dot\cup B, E)$ on $2n$ vertices with $\delta(G) \geq \big(1/2 + \gamma\big)n$ there exists $k_0 \leq k \leq k_1$ and a partition $(n_i)_{i\in[k]}$ of n with $n_i \geq n/(2k)$ such that for every partition $(a_i)_{i\in[k]}$ of n and $(b_i)_{i\in[k]}$ of n satisfying $a_i \leq n_i + \xi_{\mathrm{LG}}n$ and $b_i \leq n_i + \xi_{\mathrm{LG}}n$, for all $i \in [k]$, there exist partitions*

$$A = A_1\dot\cup\cdots\dot\cup A_k \quad \textit{and} \quad B = B_1\dot\cup\cdots\dot\cup B_k$$

such that

(G1) *$|A_i| = a_i$ and $|B_i| = b_i$ for all $i \in [k]$,*
(G2) *(A_i, B_i) is $(\varepsilon, d_{\mathrm{LG}})$-super-regular for every $i \in [k]$.*
(G3) *(A_i, B_{i+1}) is $(\varepsilon, d_{\mathrm{LG}})$-regular for every $i \in [k]$.*

The proof of this Lemma is presented in Section 6.3. The following lemma, which we will prove in Section 6.4, constructs the corresponding partition of H.

Lemma 6.5 (Lemma for H)**.** *For every $k \in \mathbb{N}$ and every $\xi > 0$ there exists $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and for every balanced bipartite graph $H = (\tilde A\dot\cup\tilde B, F)$ on $2n$ vertices having $\mathrm{bw}(H) \leq \beta n$ and for every integer partition $n = n_1 + \cdots + n_k$ with $n_i \leq n/8$ there exists a set $X \subseteq V(H)$ and a graph homomorphism $f\colon V(H) \to V(C)$, where C the cycle on vertices $A_1, B_2, A_2, \ldots, B_k, A_k, B_1, A_1$, such that*

(H1) *$|X| \leq \xi \cdot 2k \cdot n$,*
(H2) *for every $\{x, y\} \in F$ with $x \in \tilde A\backslash X$ and $y \in \tilde B\backslash X$ there is $i \in [k]$ such that $f(x) \in A_i$ and $f(y) \in B_i$,*
(H3) *$|f^{-1}(A_i)| < n_i + \xi n$ and $|f^{-1}(B_i)| < n_i + \xi n$ for every $i \in [k]$,*

With these lemmas at our disposal, we are ready to give the proof of the main theorem.

Proof of Theorem 6.2. Given γ and Δ let $d = d_{\mathrm{LG}}$ be the constant provided by Lemma 6.4 for input γ. Let ε be the constant Lemma 3.12 returns for input d, Δ, and $r = 2$. We continue the application of Lemma 6.4 with input ε and $k_0 := 2$ and get constants k_1 and ξ_{LG} and

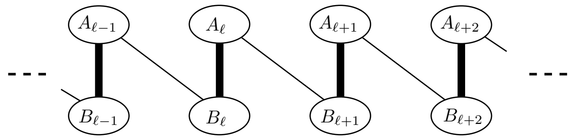


Figure 6.1: The regular partition constructed by Lemma 6.4 with super-regular pairs (A_i, B_i) and regular pairs (A_i, B_{i+1}).

set $\xi_{\text{LH}} := \xi_{\text{LG}}\varepsilon/(100\Delta k_1^2)$. Let further β be the minimum of all the values β_k and n_0' be the maximum of all the values n_k that Lemma 6.5 returns for input k and ξ, where k runs from k_0 to k_1. Finally we set $n_0 := \max\{n_0', k_1\}$.

Let $G = (A\dot\cup B, E)$ and $H = (\tilde{A}\dot\cup\tilde{B}, F)$ be balanced bipartite graphs on $2n$ vertices each with $n \geq n_0$, $\delta(G) \geq (\frac{1}{2} + \gamma)n$, $\Delta(H) \leq \Delta$, and $\text{bw}(H) \leq \beta n$. We apply Lemma 6.4 to the graph G in order to obtain an integer k and an integer partition $(n_i)_{i\in[k]}$ with $n_i \geq \frac{1}{2}n/k$ for all $i \in [k]$. Next, we apply Lemma 6.5 to the graph H and the integer partition $(n_i)_{i\in[k]}$ and get a vertex set $X \subseteq \tilde{A} \cup \tilde{B}$ and a homomorphism f from H to the cycle C on vertices $A_1, B_1, A_2, \ldots B_k, A_k, B_1, A_1$ such that (H1)–(H3) are satisfied. With this we can define the integer partitions $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ required for the continuation of Lemma 6.4: Set $a_i := |f^{-1}(A_i)|$ and $b_i := |f^{-1}(B_i)|$ for all $i \in [k]$. By (H3) we have $a_i \leq n_i + \xi_{\text{LH}}n \leq n_i + \xi_{\text{LG}}n$ and $b_i \leq n_i + \xi_{\text{LG}}n$ for all $i \in [k]$. It follows that Lemma 6.4 now gives us vertex partitions $A = (A_i)_{i\in[k]}$ and $B = (B_i)_{i\in[k]}$ for G such that (G1)–(G3) hold. We complement this with vertex partitions $\tilde{A} = (\tilde{A}_i)_{i\in[k]}$ and $\tilde{B} = (\tilde{B}_i)_{i\in[k]}$ for H defined by $\tilde{A}_i := f^{-1}(A_i)$ and $\tilde{B}_i := f^{-1}(B_i)$ and claim that we can use the general embedding lemma (Lemma 3.12) for these vertex partitions of G and H.

Indeed, observe first that (G2) and (G3) imply that the partition $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ is (ε, d)-regular with reduced graph C because $d = d_{\text{LG}}$. Further, by (G3) this partition is (ε, d)-super regular on the graph R' on the same vertices as C and with edges A_iB_i for all $i \in [k]$ (i.e., R' is a perfect matching in C). The components of R' have size $r = 2$. It follows that we can apply Lemma 3.12 if the vertex partition $V(H) = (\tilde{A}_i)_{i\in[k]}\dot\cup(\tilde{B}_i)_{i\in[k]}$ is ε-compatible with the partition $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ and with $R' \subseteq C$. To check this note first that by (G1) we have $|A_i| = a_i = |\tilde{A}_i|$ and $|B_i| = b_i = |\tilde{B}_i|$ for all $i \in [k]$ and thus Property (i) of an ε-compatible partition is satisfied. Since f is a homomorphism from H to C we also immediately get Property (ii) for $(\tilde{A}_i)_{i\in[k]}\dot\cup(\tilde{B}_i)_{i\in[k]}$. In addition, because $|A_i| = a_i \leq n_i + \xi_{\text{LH}}n$ for all $i \in [k]$, we also have $|A_i| \geq n_i - k\xi_{\text{LH}}n \geq \frac{1}{2}n/k - k\xi_{\text{LH}}n \geq \Delta\xi_{\text{LH}}2kn/\varepsilon$ by the choice of ξ_{LH}. This together with (H1) implies that $|X \cap A_i| \leq \xi_{\text{LH}}2kn \leq \varepsilon|A_i|$ and $|N_H(X) \cap A_i| \leq \Delta|X| \leq \Delta\xi_{\text{LH}}2kn \leq \varepsilon|A_j|$ for all $i, j \in [k]$. Similarly we get $|X \cap B_i| \leq \varepsilon|B_i|$ and $|N_H(X) \cap B_i| \leq \varepsilon|B_j|$ for all $i, j \in [k]$. This clearly implies Property (iii) of an ε-compatible partition.

Accordingly we can apply Lemma 3.12 to the graphs G and H with their partitions $V(G) = (A_i)_{i\in[k]}\dot\cup(B_i)_{i\in[k]}$ and $V(H) = (\tilde{A}_i)_{i\in[k]}\dot\cup(\tilde{B}_i)_{i\in[k]}$, respectively, which implies that H is a subgraph of G. □

6.3 A regular partition with spanning cycle

In this section we will prove the Lemma for G. This lemma is a consequence of the regularity lemma (Lemma 6.3), Theorem 6.1, and the following lemma which states that, under certain circumstances, we can adjust a (super)-regular partition in order to meet a request for slightly differing cluster sizes. This is a bipartite version of Lemma 5.13.

Lemma 6.6. *Let $k \geq 1$ be an integer, $0 < \xi \leq 1/(20k^2)$ and let $G = (A\dot{\cup}B, E)$ be a balanced bipartite graph on $2n$ vertices with partitions $A = A'_1\dot{\cup}\cdots\dot{\cup}A'_k$ and $B = B'_1\dot{\cup}\cdots\dot{\cup}B'_k$ such that $|A'_i|, |B'_i| \geq n/(2k)$ and (A'_i, B'_i) is (ε', d')-super-regular and (A'_i, B'_{i+1}) is (ε', d')-regular for all $i \in [k]$. Let $(a'_i)_{i\in[k]}$ and $(b'_i)_{i\in[k]}$ be integers such that $a'_i, b'_i \leq \xi n$ for all $i \in [k]$ and $\sum_{i\in[k]} a'_i = \sum_{i\in[k]} b'_i = 0$. Then there are partitions $A = A_1\dot{\cup}\cdots\dot{\cup}A_k$ and $B = B_1\dot{\cup}\cdots\dot{\cup}B_k$ with $|A_i| = |A'_i| + a'_i$ and $|B_i| = |B'_i| + b'_i$ and such that (A_i, B_i) is (ε, d)-super-regular and (A_i, B_{i+1}) is (ε, d)-regular for all $i \in [k]$ where $\varepsilon := \varepsilon' + 100k\sqrt{\xi}$ and $d := d' - 100k^2\sqrt{\xi} - \varepsilon'$.*

The proof of this lemma is deferred to the end of the section. First we will use it to prove Lemma 6.4. To this end we will apply Lemma 6.3 to the input graph G. By (a) and (b) of this lemma we obtain a regular partition with a bipartite reduced graph R of high minimum degree. Theorem 6.1 then guarantees the existence of a Hamilton cycle in R which will imply property (G3). This Hamilton cycle serves as R^* in Lemma 6.3(c), which promises a regular partition of G that is super-regular on R^*. For finishing the proof we will use a greedy strategy for distributing the vertices in the exceptional sets over the clusters of this partition (without destroying the super-regularity required for (G2)) and then apply Lemma 6.6 to adjust the cluster sizes as needed for (G1).

Proof of Lemma 6.4. Let $0 < \gamma < 1/2$ be given and set $d_{\mathrm{LG}} := \gamma^2/100$. Now let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ be given. We assume that $\varepsilon \leq \gamma^2/1000$, since otherwise we can set $\varepsilon := \gamma^2/1000$, prove the lemma, and all statements will still hold for a larger ε.

Our next task is to choose ε' and d'. For this, consider the following functions in ε' and d':

$$\begin{aligned} \varepsilon'' &:= \frac{\varepsilon'}{1-2\varepsilon'}\,, & \hat{\varepsilon} &:= \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')}\,, \\ d'' &:= d' - 4\varepsilon'\,, & \hat{d} &:= d'' - 4\varepsilon''/\gamma(1-\varepsilon'')\,. \end{aligned} \tag{6.1}$$

Observe that

$$\varepsilon' \ll \varepsilon'' \ll \hat{\varepsilon} \qquad \text{and} \qquad \hat{d} \ll d'' \ll d'\,.$$

It is not difficult to see that $\varepsilon' > 0$ and $d' > 0$ can be chosen so that the following inequalities are all satisfied:

$$\hat{\varepsilon} \leq \tfrac{1}{10}\varepsilon\,, \qquad \hat{d} - \varepsilon \geq 2d_{\mathrm{LG}}\,, \qquad \gamma - d' - \varepsilon'' > 0 \tag{6.2}$$

$$(\tfrac{1}{2} + \gamma - \varepsilon'')(1-d'')^{-1} \geq \tfrac{1}{2} + \tfrac{2}{3}\gamma\,, \qquad d''(1-d'')^{-1} \leq \tfrac{1}{6}\gamma\,. \tag{6.3}$$

Next, using (6.2), we can choose an integer k'_0 with $k_0 \leq k'_0$ such that for all integers k with $k'_0 \leq k$ we have

$$(\gamma - d' - \varepsilon'')k \geq 1\,. \tag{6.4}$$

Apply Lemma 6.3 with ε', $\Delta := 2$, and with k_0 replaced by k'_0, to obtain k_1. Choose $\xi_{\mathrm{LG}} > 0$ such that

$$100k_1\sqrt{\xi_{\mathrm{LG}}} \leq \tfrac{1}{10}\varepsilon, \quad 100(k_1)^2\sqrt{\xi_{\mathrm{LG}}} \leq d_{\mathrm{LG}}. \tag{6.5}$$

Now let G be given. Feed d' and G into Lemma 6.3 and obtain $k \in \mathbb{N}$ with $k_0 \le k_0' \le k \le k_1$ together with an (ε', d')-regular equipartition of G into $2k+2$ classes and an (ε', d')-reduced graph R on $2k$ vertices by (a) of Lemma 6.3. By assumption $\delta(G) \ge (\frac{1}{2} + \gamma)n$, so setting $\nu := 1/2 + \gamma$ and making use of part (b) of Lemma 6.3, we get

$$\delta(R) \ge (\tfrac{1}{2} + \gamma - d' - \varepsilon'')|V(R)| = \tfrac{1}{2}|V(R)| + (\gamma - d' - \varepsilon'')k \overset{(6.4)}{\ge} \tfrac{1}{2}|V(R)| + 1.$$

We infer from Lemma 6.1 that R contains a Hamilton cycle R^*. Now apply part (c) of Lemma 6.3 and obtain an (ε'', d'')-regular equipartition of G with classes

$$A = A_0''\dot\cup \dots \dot\cup A_k'' \quad \text{and} \quad B = B_0''\dot\cup \dots \dot\cup B_k''.$$

Obviously, R and thus R^* are bipartite and so, without loss of generality (renumbering the clusters if necessary), we can assume that R^* consists of the vertices representing the classes

$$A_1'', B_2'', A_2'', B_3'', \dots, B_k'', A_k'', B_1'', A_1''$$

with edges in this order. Therefore, we know that the pairs (A_i'', B_i'') and (A_i'', B_{i+1}'') are (ε'', d'')-super-regular for all $i \in [k]$. Let $L := |A_i''| = |B_i''|$ and observe that

$$(1-\varepsilon'')\frac{n}{k} \le L \le \frac{n}{k}\,.$$

Our next aim is to get rid of the classes A_0'' and B_0'' by moving their vertices to other classes. We will do this, roughly speaking, as follows. When moving a vertex $x \in A_0''$ to some class A_i'', say, we will move an arbitrary vertex $y \in B_0''$ to the corresponding class B_i'' at the same time. We will also make sure that x has at least $d''|B_i''|$ neighbours in B_i'' and y has at least $d''|A_i''|$ neighbours in A_i''. Here are the details for this operation. For an arbitrary pair $(x, y) \in A_0'' \times B_0''$ we define

$$I(x,y) := \Big\{ i \in [k]\colon \quad |N_G(x) \cap B_i''| \ge d''\ |B_i''| \quad \text{and} \quad |N_G(y) \cap A_i''| \ge d''\ |A_i''| \Big\}\,.$$

We claim that for every $(a, b) \in A_0'' \times B_0''$ we have $|I(x, y)| \ge \gamma k$. To prove this claim, first recall that $L = |A_i''| = |B_i''|$ for all $i \in [k]$. Define

$$I(x) := \big\{ i \in [k]\colon |N_G(x) \cap B_i''| \ge d''|B_i''| \big\}\,, \quad I(y) := \big\{ i \in [k]\colon |N_G(y) \cap A_i''| \ge d''|A_i''| \big\}\,.$$

As $|A_0''| = |B_0''| \le \varepsilon'' n$ we have

$$\begin{aligned}(\tfrac{1}{2} + \gamma)n \le \deg_G(x) &\le |I(x)|L + (k - |I(x)|)\, d''L + \varepsilon'' n \\ &= |I(x)|(1-d'')L + kd''L + \varepsilon'' n\,.\end{aligned}$$

and hence

$$\begin{aligned}|I(x)| &\ge \frac{(\frac{1}{2}+\gamma)n - kd''L - \varepsilon'' n}{(1-d'')L} = \frac{(\frac{1}{2} + \gamma - \varepsilon'')}{1-d''}\frac{n}{L} - \frac{d''}{1-d''}k \\ &\overset{(6.3)}{\ge} (\tfrac{1}{2} + \tfrac{2}{3}\gamma)k - \tfrac{1}{6}\gamma k = (\tfrac{1}{2} + \tfrac{1}{2}\gamma)k\,.\end{aligned}$$

Similarly, $|I(y)| \ge (\frac{1}{2} + \frac{1}{2}\gamma)k$. Since $I(x)$ and $I(y)$ are both subsets of $[k]$, this implies that $|I(x, y)| = |I(x) \cap I(y)| \ge \gamma k$, which proves the claim.

We group the vertices in $A''_0 \cup B''_0$ into (at most $\varepsilon'' n$) pairs $(x,y) \in A''_0 \times B''_0$ and choose an index $i \in I(x,y)$ which has the property that (A''_i, B''_i) has so far received a minimal number of additional vertices. Then we move x into A''_i and y into B''_i. Hence, at the end, every cluster A''_i, or B''_i gains at most $\varepsilon'' n/(\gamma k)$ additional vertices. Denote the final partition obtained in this way by

$$A \dot\cup B = \hat{A}_1 \dot\cup \hat{B}_1 \dot\cup \ldots \dot\cup \hat{A}_k \dot\cup \hat{B}_k .$$

Set $\alpha := \beta := \varepsilon''/\gamma(1-\varepsilon'')$ and observe that

$$\frac{\varepsilon'' n}{\gamma k} = \alpha(1-\varepsilon'')\frac{n}{k} \le \alpha L .$$

So Proposition 3.8 tells us that for all $i \in [k]$ the pairs $(\hat{A}_i, \hat{B}_i)$ are still $(\hat\varepsilon, \hat d)$-super-regular and the pairs $(\hat{A}_i, \hat{B}_{i+1})$ are still $(\hat\varepsilon, \hat d)$-regular, because

$$\hat\varepsilon \overset{(6.1)}{=} \varepsilon'' + 6\sqrt{\varepsilon''/\gamma(1-\varepsilon'')} = \varepsilon'' + 3(\sqrt{\alpha} + \sqrt{\beta}) \qquad \text{and}$$
$$\hat d \overset{(6.1)}{=} d'' - 4\varepsilon''/\gamma(1-\varepsilon'') = d'' - 4\alpha = d'' - 2(\alpha+\beta) .$$

Now return to the statement of Lemma 6.4. We set $n_i := |\hat{A}_i| = |\hat{B}_i|$ for all $i \in [k]$. Let $(a_i)_{i\in[k]}$ and $(b_i)_{i\in[k]}$ be given and set $a''_i := a_i - n_i$ and $b''_i := b_i - n_i$. Then

$$a''_i \le \xi_{\text{LG}} n, \quad b''_i \le \xi_{\text{LG}} n, \quad \sum_i a''_i = \sum_i a_i - \sum_i n_i = n - n = 0 = \sum_i b''_i .$$

Therefore we can apply Lemma 6.6 with parameter ξ_{LG} to the graph G with partitions $\hat{A}_1 \dot\cup \ldots \dot\cup \hat{A}_k$ and $\hat{B}_1 \dot\cup \ldots \dot\cup \hat{B}_k$. Since

$$\hat\varepsilon + 100k\sqrt{\xi_{\text{LG}}} \overset{(6.2),(6.5)}{\le} \tfrac{1}{10}\varepsilon + \tfrac{1}{10}\varepsilon \le \varepsilon \qquad \text{and}$$
$$\hat d - 100k^2\sqrt{\xi_{\text{LG}}} - \varepsilon \overset{(6.2),(6.5)}{\ge} 2d_{\text{LG}} - d_{\text{LG}} = d_{\text{LG}} ,$$

we obtain sets A_i and B_i for each $i \in [k]$ such that $|A_i| = |\hat{A}_i| + a''_i = n_i + a''_i = a_i$ and $|B_i| = b_i$, and with the property that (A_i, B_i) is (ε, d)-super-regular and (A_i, B_{i+1}) is (ε, d)-regular. This completes the proof of Lemma 6.4. □

It remains to prove Lemma 6.6.

Proof of Lemma 6.6. The lemma will be proved by performing a simple redistribution algorithm that will iteratively adjust the cluster sizes. Throughout the process, we denote by A_i and B_i the changing clusters, beginning with $A_i := A'_i$ and $B_i := B'_i$, and we call A_i a *sink* when $|A_i| < |A'_i| + a'_i$ and a *source* when $|A_i| > |A'_i| + a'_i$ and analogously for B'_i. Each iteration of the algorithm will have the effect that the number of vertices in a single source decreases by one, the number of vertices in a single sink increases by one, and all other cluster cardinalities stay the same.

We start by describing one iteration of the algorithm. Obviously, as long as not every cluster in A has exactly the desired size, there is at least one source. We choose an arbitrary source A_i, and, as will be further explained below, the regularity of the pair (A_i, B_{i+1}) implies that within A_i there is a large set of vertices each of which can be added to the neighbouring cluster A_{i+1} while preserving the super-regularity of the pair (A_{i+1}, B_{i+1}). We do this with one arbitrary vertex from this set. Thereafter, within A_{i+1} there is again a large set of vertices (the newly arrived vertex may or may not be one of them) suitable for being moved into A_{i+2} while preserving the super-regularity of the pair (A_{i+2}, B_{i+2}), and we again do this with one arbitrary vertex from this set. We then continue in this way until for the first time we move a vertex into a sink. (Typically it is not the vertex we initially took out of A_i that arrives in the sink.) This is the end of the iteration.

We repeat such iterations as long as there are sources, i.e. we choose an arbitrary source (which may or may not be the one we have just removed a vertex from) and repeat what we have just described. Since each iteration ends with adding a vertex to a sink while not changing the cardinality of the clusters visited along the way, we do not increase the number of vertices in any source, let alone create a new source, and hence after a finite number of iterations (which we will estimate below) the algorithm ends with no sources remaining and therefore all clusters within A having exactly the desired size.

We then repeat what we have just described for the clusters within B, the only difference being that vertices get moved from B_i into B_{i-1}, not B_{i+1}, since only in this direction a regular pair can be used ((A_{i-1}, B_i) is regular, (A_{i+1}, B_i) need not be regular).

We now analyse the algorithm quantitatively. Clearly, the total number of iterations (we call it t) is at most the sum of all positive a'_i and all positive b'_i. Obviously, both the sum of all positive a'_i and the sum of all positive b'_i can be each at most $\frac{1}{2}k\xi n$ hence

$$t \leq \frac{1}{2}k\xi n + \frac{1}{2}k\xi n = k\xi n. \tag{6.6}$$

We will now use this bound together with Proposition 3.8 to estimate the effect of the redistribution on the regularity and density parameters. Since in each iteration each cluster receives at most one vertex and loses at most one vertex, for every $i \in [k]$ and after any step of the algorithm, we have

$$|A_i \Delta A'_i| \leq 2t \leq 2k\xi n,$$

and analogously $|B_i \Delta B'_i| \leq 2k\xi n$. We now invoke Proposition 3.8 on the pairs (A_i, B_i) and (A_i, B_{i+1}), once with $\hat{A} := A_i$, $\hat{B} := B_i$ then with $\hat{A} := A_i$, $\hat{B} := B_{i+1}$ and we claim that we may use $\alpha := \beta := 16k^2\xi$. This is so because $|A_i| \geq |A'_i| - t \geq n/(2k) - 2k\xi n$ and because $\xi \leq 1/(20k^2)$ implies $2k\xi n \leq 5k\xi n - 20k^3\xi^2 n$, hence $|A_i \Delta A'_i| \leq 2k\xi n \leq (5k\xi - 20k^3\xi^2)n = 10k^2\xi(n/(2k) - 2k\xi n) \leq \alpha|A'_i|$, and analogously $|B_i \Delta B'_i| \leq \beta|B'_i|$. By Proposition 3.8, every pair (A_i, B_i) and (A_i, B_{i+1}) is $(\hat{\varepsilon}, \hat{d})$-regular with $\hat{\varepsilon} := \varepsilon + 24k\sqrt{\xi}$ and $\hat{d} := d' - 64k^2\xi$, hence $\hat{\varepsilon} \leq \varepsilon$ and $\hat{d} \geq d$, proving the parameters claimed in the lemma, as far as mere regularity goes.

As for the claimed super-regularity of the vertical pairs, let A_i, B_i and B_{i+1} be clusters at an arbitrary step of the algorithm. Using Proposition 3.6 and (6.6) we know that the pairs (A_i, B_i) and (A_i, B_{i+1}) being $(\hat{\varepsilon}, \hat{d})$-regular implies that there are at least $(1-\hat{\varepsilon})|A_i|$ vertices in A_i having at least $(\hat{d}-\hat{\varepsilon})|B_{i+1}| - t \geq (\hat{d}-\hat{\varepsilon})|B_{i+1}| - 2k\xi n$ neighbours in B_{i+1}, and it remains to prove that $(\hat{d} - \hat{\varepsilon})|B_{i+1}| - 2k\xi n \geq d|B_{i+1}|$ which is equivalent to $2k\xi n/|B_{i+1}| \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\xi$. Because of $2k\xi n/|B_{i+1}| \leq 2k\xi n/(|B'_{i+1}| - t) \leq 2k\xi n/(n/2k - 2k\xi n) = 4k^2\xi/(1 - 4k^2\xi)$ it is therefore sufficient that $4k^2\xi/(1 - 4k^2\xi) \leq 100k^2\sqrt{\xi} - 64k^2\xi - 24k\sqrt{\xi}$ and it is easy to check that this is true by the hypothesis on ξ. □

6.4 Distributing H among the edges of a cycle

In this section we will provide the proof of the Lemma for H (Lemma 6.5). The idea is to cut H into small pieces along its bandwidth order. These pieces are then distributed to the edges A_iB_i of the cycle C in such a way that the following holds. Let $\tilde{A}_i$ be all the vertices from $\tilde{A}$ and $\tilde{B}_i$ all the vertices from $\tilde{B}$ that were assigned to the edge A_iB_i. Then we require that $\tilde{A}_i$ and $\tilde{B}_i$ are roughly of size n_i. Observe that this goal would be easy to achieve if H was *locally balanced*, i.e., if each of the small pieces had colour classes of equal size. While this need not be the case, we know that H itself is a *balanced* bipartite graph. Therefore we use a probabilistic argument to show that the pieces of H can be grouped in such a way that the resulting packages form balanced bipartite subgraphs of H (which we can then distribute to the edges A_iB_i, meeting the above requirement). The details of this argument are given below (see Lemma 6.7).

After the distribution of the pieces to the edges A_iB_i we will construct the desired homomorphism f in the following way. We will map most vertices of $\tilde{A}_i$ to A_i and most vertices of $\tilde{B}_i$ to B_i. The remaining vertices will be mapped to other cycle vertices in C. This is necessary because there might be edges between a piece assigned to A_iB_i and a piece assigned to $A_{i'}B_{i'}$ with $i \neq i'$ that we need to take care of. In the proof of Lemma 6.5 on page 80 we will explain how this is done.

Balancing H locally

We explained above that our goal is to group small pieces $W_1, \ldots, W_\ell$ of the balanced bipartite graph H on $2n$ vertices into packages $P_1, \ldots, P_k$ that form balanced bipartite subgraphs of H. This is equivalent to the following problem. Given the sizes a_j and b_j of the colour classes of each piece W_j (i.e., a_j counts the vertices of W_j that are in $\tilde{A}$ and b_j those that are in $\tilde{B}$) we know that the a_j sum up to n and the b_j sum up to n. Then we would like to have a mapping $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ the a_j with $j \in \varphi^{-1}(i)$ sum up approximately to the same value as the b_j with $j \in \varphi^{-1}(i)$. The following lemma asserts that such a mapping φ exists. The package P_i will then (in the proof of Lemma 6.5) consist of all pieces W_j with $j \in \varphi^{-1}(i)$.

Lemma 6.7. *For all $0 < \xi \le 1/4$ and all positive integers k there exists $\ell \in \mathbb{N}$ such that for all integers $n \ge \ell$ the following holds. Let $(n_i)_{i\in[k]}$, $(a_j)_{j\in[\ell]}$, and $(b_j)_{j\in[\ell]}$ be integer partitions of n such that $n_i \le \frac{1}{8}n$ and $a_j + b_j \le (1+\xi)\frac{2n}{\ell}$ for all $i \in [k]$, $j \in [\ell]$. Then there is a map $\varphi : [\ell] \to [k]$ such that for all $i \in [k]$ and $\bar{a}_i := \sum_{j\in\varphi^{-1}(i)} a_j$ and $\bar{b}_i := \sum_{j\in\varphi^{-1}(i)} b_j$ we have*

$$\bar{a}_i < n_i + \xi n \qquad \text{and} \qquad \bar{b}_i < n_i + \xi n\,. \tag{6.7}$$

In the proof of lemma 6.7 we will use the Hoeffding-bound for sums of independent random variables given in Theorem 2.5.

Proof. For the proof of this lemma we use a probabilistic argument and show that under a suitable probability distribution a random map satisfies the desired properties with positive probability. For this purpose set $\ell := \lceil k^5/\xi^2 \rceil$ and construct a random map $\varphi\colon [\ell] \to [k]$ by choosing $\varphi(j) = i$ with probability n_i/n for $i \in [k]$, independently for each $j \in [\ell]$. For showing that this map satisfies (6.7) with positive probability we first estimate the sum of all a_j's and b_j's assigned to a fixed $i \in [k]$. To this end, let $\mathbb{1}_j$ be the indicator variable for the event $\varphi(j) = i$ and define a random variable $S_i := \sum_{j\in[\ell]} \mathbb{1}_j$. Clearly S_i is binomially distributed, we have $\mathbb{E}\, S_i = \ell\frac{n_i}{n}$, and by the Chernoff bound $\mathbb{P}[|S_i| \ge \mathbb{E}\, S_i + t] \le 2\exp(-2t^2/\ell)$ (see (2.2)) we get

$$\mathbb{P}\left[\left|S_i - \ell\frac{n_i}{n}\right| \ge \tfrac{1}{2}\xi\ell\right] \le 2\exp(-\tfrac{1}{2}\xi^2\ell).$$

Next, we examine the difference between the sum of the a_j's assigned to i and the sum of the b_j's assigned to i. We define random variables $D_{i,j} := \frac{\ell}{3n}(a_j - b_j)(\mathbb{1}_j - \frac{n_i}{n})$ and set $D_i := \sum_{j\in[\ell]} D_{i,j}$. Then $\mathbb{E}\, D_{i,j} = 0$ and as $a_j + b_j \le \frac{3n}{\ell}$ we have $|D_{i,j}| \le 1$. Thus Theorem 2.5 implies

$$\mathbb{P}\left[|D_i| \ge \tfrac{1}{6}\xi\ell\right] \le 2\exp(-\tfrac{1}{72}\xi^2\ell).$$

By the union bound, the probability that we have

$$|S_i - \ell\tfrac{n_i}{n}| < \tfrac{1}{2}\xi\ell \qquad \text{and} \qquad |D_i| < \tfrac{1}{6}\xi\ell \qquad \text{for all } i \in [k] \tag{6.8}$$

is therefore at least $1 - k\cdot 2\exp(-\frac{1}{2}\xi^2\ell) - k\cdot 2\exp(-\frac{1}{72}\xi^2\ell)$ which is strictly greater than 0 by our choice of ℓ. Therefore there exists a map φ with property (6.8). We claim that this map satisfies (6.7). To see this, observe first that $\frac{3n}{\ell}D_i = \sum_{j\in\varphi^{-1}(i)}(a_j - b_j) = \bar{a}_i - \bar{b}_i$ by definition of D_i. Together with (6.8) this implies $\bar{a}_i - \bar{b}_i < \xi n$. Moreover, we have $S_i = |\varphi^{-1}(i)|$ and

$$\begin{aligned}\bar{a}_i &= \tfrac{1}{2}(\bar{a}_i + \bar{b}_i) + \tfrac{1}{2}(\bar{a}_i - \bar{b}_i) \le \tfrac{1}{2}(1+\xi)\tfrac{2n}{\ell}|\varphi^{-1}(i)| + \tfrac{1}{2}\cdot\tfrac{1}{2}\xi n \\ &\overset{(6.8)}{<} \frac{1}{2}(1+\xi)\frac{2n}{\ell}\Big(\ell\frac{n_i}{n} + \frac{1}{2}\xi\ell\Big) + \frac{1}{4}\xi n \le n_i + \xi n\end{aligned}$$

where the last inequality follows from $\xi \le \frac{1}{4}$ and $n_i \le \frac{1}{8}n$. Since an entirely analogous calculation shows that $\bar{b}_i < n_i + \xi n$, this completes the proof of (6.7). □

The Lemma for H

For the proof of Lemma H we will now use Lemma 6.7 as outlined above its statement. In this way we obtain an assignment of pieces $W_1, \dots W_\ell$ of H to edges A_iB_i of C. This assignment, however, does not readily give a homomorphism from H to C as there might be edges between pieces W_j and W_{j+1} that end up on edges A_iB_i and $A_{i'}B_{i'}$ which are not neighbouring on C. Never the less (owing to the small bandwidth of H) we will be able to transform it into a homomorphism by assigning some few vertices of W_{j+1} to other vertices of C along the path between A_iB_i and $A_{i'}B_{i'}$ on C.

Proof of Lemma 6.5. Let k and ξ be given. Give $\xi' := \xi/4$ and k to Lemma 6.7, get ℓ, set $\beta := \xi'/(4\ell k)$ and $n_0 := \lceil \ell/(2\xi)\rceil$, and let H and $(n_i)_{i\in[k]}$ be given as in the statement of the present lemma.

We assume that the vertices of H are given a bandwidth labelling, partition $V(H)$ along this labelling into ℓ sets $W_1, \dots, W_\ell$ of as equal sizes as possible and define $x_i := |W_i \cap \tilde{A}|$ and $y_i := |W_i \cap \tilde{B}|$. Then $x_i + y_i = |W_i| \le \lceil 2n/\ell\rceil \le 2n/\ell + 1 \le (1+\xi)2n/\ell$ and since $n_i \le n/8$ by hypothesis we can give $(n_i)_{i\in[k]}$, $(x_i)_{i\in[\ell]}$ and $(y_i)_{i\in[\ell]}$ to Lemma 6.7 and get a $\varphi\colon [\ell] \to [k]$ with (6.7).

Let us discuss the main difficulty in our proof. Since the map φ is obtained via the probabilistic method, there is no control over how far apart in the Hamilton cycle C two sets $W_{\varphi(i)}$ and $W_{\varphi(i+1)}$ will be assigned by φ. Hence these sets might end up in non-contiguous vertices of the cycle C. If there are edges between $W_{\varphi(i)}$ and $W_{\varphi(i+1)}$ we need to guarantee, however, that these edges are mapped to edges of C in order to obtain the desired homomorphism f. Therefore, we resort to a greedy linking process which constructs f and (thanks to the bandwidth condition) needs to alter the assignment proposed by φ only slightly.

Let w_i be the first vertex in W_i and define sets of *linking vertices* by

$$L^i_j := [w_i + (j-1)\beta n, w_i + j\beta n)$$

for every $j \in [2k]$, and set $L^i := \bigcup_{j \in [2k]} L^i_j$. Then all L^i_j have the common cardinality βn and $|L^i| = 2k\beta n$. Since $\beta \leq 1/(4k\ell)$ implies that $2k\beta n + \beta n \leq \lfloor 2n/\ell \rfloor \leq |W_i|$ for every $i \in [\ell]$, we have $L^i_j \subsetneq W_i$ for every $i \in [\ell]$ where $|W_i \backslash L^i| \geq \beta n$, i.e. to the right of every set W_i there are at least βn non-linking vertices.

We now construct a map f by defining, for every $i \in [\ell]$,

$$f(x) := \begin{cases} A_{\varphi(i-1)+\lfloor j/2 \rfloor} & \text{if } x \in L^i_j \text{ with } j \in \big[2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)\big], \\ A_{\varphi(i)} & \text{else,} \end{cases} \tag{6.9}$$

for every $x \in W_i \cap \tilde{A}$, and

$$f(y) := \begin{cases} B_{\varphi(i-1)+\lceil j/2 \rceil} & \text{if } y \in L^i_j \text{ with } j \in \big[2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)\big], \\ B_{\varphi(i)} & \text{else,} \end{cases} \tag{6.10}$$

for every $y \in W_i \cap \tilde{B}$, and show that this is indeed a homomorphism. To do this it is convenient to note that a set $\{A_i, B_{i'}\}$ is an edge of C if and only if $0 \leq i' - i \leq 1$.

Let arbitrary vertices $x \in \tilde{A}$ and $y \in \tilde{B}$ with $\{x, y\} \in F$ be given. Since the sets W_i are defined along the bandwidth labelling, either x and y are both within the same W_i or x and y lie in consecutive sets W_i and W_{i+1}. We will now distinguish several cases. For brevity let $I_i := \big[2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)\big]$.

Case 1. Both x and y lie within the same set W_i.
Case 1.1. There is $j \in I_i$ with $x \in L^i_j$, hence $f(x) = A_{\varphi(i-1)+\lfloor j/2 \rfloor}$. Due to the bandwidth condition together with $|L^i_j| = \beta n$, if $y \notin L^i_j$ and $j+1 \in I_i$, then necessarily $y \in L^i_{j+1}$, which explains the following three sub-cases.
Case 1.1.1. We have $y \in L^i_j$, hence $f(y) = B_{\varphi(i-1)+\lceil j/2 \rceil}$, hence the difference of the indices of $f(x)$ and $f(y)$ is $\lceil j/2 \rceil - \lfloor j/2 \rfloor$, which is either 0 or 1 according to whether j is even or odd, hence $\{f(x), f(y)\} \in E(C)$.
Case 1.1.2. We have $y \notin L^i_j$ and $j+1 \in I_i$, hence $y \in L^i_{j+1}$, hence $f(y) = \varphi(i-1) + \lceil (j+1)/2 \rceil$, hence the difference of indices of $f(y)$ and $f(x)$ is $\lceil (j+1)/2 \rceil - \lfloor j/2 \rfloor$, and this is always 1, whether j is even or odd, so $\{f(x), f(y)\} \in E(C)$.
Case 1.1.3. We have $y \notin L^i_j$ and $j+1 \notin I_i$, hence $f(y) = B_{\varphi(i)}$. Here, $j+1 \notin L^i_j$ implies that $j \geq 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$ while being within Case 1.1 implies $j \in I_i$, hence $j \leq 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$, so we have $j = 2 \cdot \big((\varphi(i) - \varphi(i-1)) \mod k\big)$, thus $f(x) = A_{\varphi(i-1)+\lfloor j/2 \rfloor} = A_{\varphi(i)}$, the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.
Case 1.2. There is no $j \in I_i$ with $x \in L^i_j$, hence $f(x) = A_{\varphi(i)}$. Being within Case 1, i.e. $y \in W_i$, it follows that there are exactly two cases.
Case 1.2.1. If y precedes x in the bandwidth labelling, then $y \in L^i_{2 \cdot ((\varphi(i) - \varphi(i-1)) \mod k)}$, hence $f(y) = B_{\varphi(i)}$, so the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.
Case 1.2.2. If y succeeds x in the bandwidth labelling, then, since $y \in W_i$ by being within Case 1, there is no $j \in I_i$ with $y \in I_i$, hence $f(y) = B_{\varphi(i)}$, so again the index difference between $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$.

Case 2. We have $x \in W_i$ and $y \in W_{i+1}$. Then, by the bandwidth condition and size of the sets of linking vertices, we must have $y \in L^{i+1}_1$, hence $f(y) = B_{\varphi((i+1)-1)+\lceil 1/2 \rceil} = B_{\varphi(i)+1}$, and since there are at least βn non-linking vertices to the right of W_i, the vertex x cannot lie in a L^i_j, hence $f(x) = A_{\varphi(i)}$, so the index difference of $f(y)$ and $f(x)$ is 1 and $\{f(x), f(y)\} \in E(C)$.

Case 3. We have $y \in W_i$ and $x \in W_{i+1}$. Then, by the bandwidth condition and size of the sets of linking vertices, we must have $x \in L_1^{i+1}$, hence $f(x) = A_{\varphi((i+1)-1)+\lfloor 1/2 \rfloor} = A_{\varphi(i)}$, and since there are at least βn non-linking vertices to the right of W_i, the vertex y cannot lie in a L_j^i, hence $f(y) = B_{\varphi(i)}$, so the index difference of $f(y)$ and $f(x)$ is 0 and $\{f(x), f(y)\} \in E(C)$. This completes the proof that f is a homomorphism.

We now prove (H1) and (H2). Define $X := \bigcup_{i \in [\ell]} L^i$. Then $|X| \leq \ell \cdot 2k \cdot \beta n \leq \ell \cdot 2k \cdot (\xi'/(2\ell k)) \cdot n = \xi' n \leq \xi n$, which shows (H1), and (H2) is obvious from the definitions of X and the map f above.

We now prove (H3). For this it suffices to note, rather crudely, that for every $j \in [k]$, no preimage $f^{-1}(A_j)$ can become larger than the sum of the sizes of all sets W_i assigned to A_j by φ (which by the definition of f equals the sum of all $x_i = |\tilde{A} \cap W_i|$ with $\varphi(i) = j$) plus the total number of linking vertices, i.e. for every $j \in [k]$, using the choice of β and using that φ has the property promised by Lemma 6.7, we have $|f^{-1}(A_j)| \leq \left(\sum_{i \in \varphi^{-1}(j)} x_i\right) + |\bigcup_{i \in [\ell]} L^i| \leq n_j + \xi' n + \ell \cdot |L^i| = n_j + \xi' n + 2k\ell\beta n \leq n_j + 2\xi' n = n_j + \xi n$, completing the proof of (H3). □

Chapter 7

Variation 2: Ore conditions

Ore's theorem (Theorem 1.16) formulates a criterion for the existence of a Hamilton cycle that is different from the minimum degree conditions we saw in the last two chapters. In this theorem only degree sums of non-adjacent vertices in a host graph G are considered and required to be large. In the following we will call degree conditions of this type *Ore conditions*. We define the *Ore degree* $\delta_{\emptyset}(G)$ as the biggest number q such that all pairs of non-adjacent vertices u, v of G satisfy $\deg(u) + \deg(v) \geq q$.

In this chapter we determine an Ore condition that allows for the embedding of 3-colourable bounded-degree graphs of sublinear bandwidth. We will prove an analogue of Theorem 5.1 under this weaker Ore condition for the case of 3-chromatic graphs (Theorem 7.2). The proof applies a recent result of Kierstead and Kostochka [56] (Theorem 7.1) about the existence of spanning triangle factors under Ore conditions as well as a version of the regularity lemma adapted to this setting (Lemma 7.4).

After introducing these two results in Section 7.1 we will investigate some structural properties of graphs with high Ore degree in Section 7.2. Then, in Section 7.3, we turn to a lemma concerning the existence of square-paths in graphs with high Ore degree (Lemma 7.3). This lemma will be used in our proof of Theorem 7.2, which is provided in the remaining sections.

7.1 Embedding under Ore conditions

The question whether the minimum degree condition in the theorem of Hajnal and Szemerédi (Theorem 1.4) can be replaced by an Ore condition, was considered by Kierstead and Kostochka [56]. Recall that this theorem states that any n-vertex graph G with minimum degree $\delta(G) \geq \frac{r-1}{r}n$ contains a so-called spanning K_r*-factor*, that is, a family of $\lfloor n/r \rfloor$ vertex disjoint r-cliques.

Theorem 7.1 (Kierstead, Kostochka [56])**.** *For all r, every n-vertex graph G with $\delta_{\emptyset}(G) \geq 2\frac{r-1}{r}n - 1$ contains a spanning K_r-factor.* □

Here we are interested in replacing the minimum degree condition in the 3-chromatic version of the Bollobás-Komlós conjecture (see Theorem 5.1) by an Ore condition. We establish the following theorem.

Theorem 7.2. *For all $\Delta, \gamma > 0$ there are $\beta, n_0 > 0$ such that for all $n \geq n_0$ the following holds. Let G and H be n-vertex graphs such that H is 3-colourable, has maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$, and G satisfies $\delta_{\emptyset}(G) \geq (\frac{4}{3} + \gamma)n$. Then G contains a copy of H.*

In contrast to the Ore-type results mentioned above we use the regularity lemma for proving this theorem. The proof method is similar to the one presented in Chapter 5 but we need to cope with the weaker Ore condition now and hence new ideas are necessary.

On the way to the proof of Theorem 7.2 we first show the following lemma, which asserts the existence of the square of an almost spanning path in a graph with sufficiently high Ore degree. This may be regarded as a special case of Theorem 7.2. As we will, see this special case helps us to deduce the general case.

Lemma 7.3. *For all $\gamma, \mu > 0$ there is an n_0 such that for all $n \geq n_0$ every n-vertex graph $G = (V, E)$ satisfying $\delta_{\varnothing}(G) \geq (\frac{4}{3} + \gamma)n$ contains a square-path on at least $(1 - \mu)n$ vertices.*

As discussed in the introduction (Section 1.1.1), the question which minimum degree condition enforces a *spanning* square of a cycle (and hence also the square of a path) is the subject of Pósa's conjecture that was resolved for large n by Komlós, Sárközy, and Szemerédi [64] with the help of the regularity lemma. In order to prove Lemma 7.3 we use a strategy similar to the one in [64]. Again, we need to refine this method to deal with the weaker Ore condition (see Section 7.3).

One crucial observation for both the proof of Theorem 7.2 and the proof of Lemma 7.3 is an inheritance principle for Ore degrees. Recall that the minimum degree version of the regularity lemma (Lemma 3.4) states that graphs with high minimum degree have reduced graphs with high minimum degree. The following corollary of the regularity lemma states that also Ore conditions are passed on in this way: Graphs with high Ore degree have reduced graph with high Ore degree.

Lemma 7.4 (regularity lemma, Ore version)**.** *For every $\varepsilon > 0$ and every integer k_0 there is k_1 such that every graph $G = (V, E)$ on $n > k_1$ vertices with $\delta_{\varnothing}(G) \geq \eta n$ has an (ε, d)-regular equipartition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ with reduced graph R and $k_0 \leq k \leq k_1$ with the following property. For each $v \in V_i$ with $i \in [k]$ there are at most $(\varepsilon + d)n$ edges in E incident to v that are not in some (ε, d)-regular pair (V_i, V_j) with $j \in [k]$ and $\delta_{\varnothing}(R) \geq (\eta - 2(\varepsilon + d))k$.*

Proof. For ε and k_0 let k_1 be given by Lemma 3.3. Consider an arbitrary graph G on $n > k_1$ vertices with $\delta_{\varnothing}(G) \geq \eta n$. By Lemma 3.3 the graph G has an (ε, d)-regular equipartition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ with $k_0 \leq k \leq k_1$ such that for each $v \in V_i$ with $i \in [k]$ there are at most $(\varepsilon + d)n$ edges in E incident to v that are not in some (ε, d)-regular pair (V_i, V_j) with $j \in [k]$. Let R be a *maximal* reduced graph of this (ε, d)-regular partition, i.e., each pair $i, j \in [k]$ with $ij \notin E(R)$ corresponds to a pair (V_i, V_j) that is not (ε, d)-regular.

Assume for a contradiction that $\delta_{\varnothing}(R) < (\eta - 2(\varepsilon + d))k$. Then there are clusters $i, j \in [k]$ with $ij \notin E(R)$ and $\deg_R(i) + \deg_R(j) < (\eta - 2(\varepsilon + d))k$. By the maximality of R the pair (V_i, V_j) is not (ε, d)-regular and hence there are vertices $v_i \in V_i$ and $v_j \in V_j$ with $v_i v_j \notin E$. It follows from $L := |V_1| = \ldots = |V_k|$ that $\deg_G(v) \leq \deg_R(\ell)L + (\varepsilon + d)n$ for all $v \in V_\ell$, $\ell \in [k]$. Therefore

$$\begin{aligned}\deg_G(v_i) + \deg_G(v_j) &\leq (\deg_R(i) + \deg_R(j))\, L + 2(\varepsilon + d)n < \\ &< (\eta - 2(\varepsilon + d))kL + 2(\varepsilon + d)n \leq \eta n,\end{aligned}$$

which is a contradiction. □

7.2 Graphs with high Ore degree

For warming up (and for later use) we collect some structural observations about graphs with certain Ore conditions in this section. The first simple proposition shows that an Ore condition induces a (much weaker) minimum degree condition.

Proposition 7.5. *Any n-vertex graph G has minimum degree $\delta(G) \geq \delta_{\varnothing}(G) - n$.*

Proof. Let v be a vertex of $G = (V, E)$. Either $\deg(v) = n - 1 \geq \delta_{\varnothing}(G) - n$ (since trivially $\delta_{\varnothing}(G) \leq 2(n-1) \leq 2n-1$) or there is a vertex u such that $uv \notin E$. But then $\deg(v) \geq \delta_{\varnothing}(G) - \deg(u) \geq \delta_{\varnothing}(G) - n$. □

The next proposition states that we can say even more: If we delete all vertices with degree somewhat higher than promised by Proposition 7.5 from a graph the remaining vertices form a clique. For making this precise we will use the following definitions. The *big vertices* of a graph G are those that have degree at least $\delta_{\varnothing}(G)/2$, all other vertices are called *small vertices*.

Proposition 7.6. *The small vertices of a graph form a clique.*

Proof. Let $G = (V, E)$ be a graph and $u, v \in V$ with $uv \notin E$. By definition $\deg(u) + \deg(v) \geq \delta_{\varnothing}(G)$ and so at least one of these vertices is big. □

The following proposition is of technical nature (but will prove useful later). Recall that an *equipartition* of a set V is a partition of V into sets of equal size. The proposition asserts that for every equipartition of a graph G with high Ore degree into three parts the following holds. Whenever we pick two of the parts then there is a vertex in G with many neighbours in the union of these two parts.

Proposition 7.7. *Let $G = (V, E)$ be a graph on $3n$ vertices with $\delta_{\varnothing}(G) > 4n$ and let $V = V_1 \dot\cup V_2 \dot\cup V_3$ be an equipartition of V. Then, for all $\ell, \ell' \in [3]$ there is a vertex in V_ℓ with more than n neighbours in $V_\ell \cup V_{\ell'}$.*

Proof. If there is a big vertex in $v \in V_\ell$ then $\deg_G(v) > 2n$ trivially implies the claim. Hence assume, all vertices in V_ℓ are small and thus form a clique by Proposition 7.6. We claim that then each vertex $v' \in V_{\ell'}$ has at least two neighbours in V_ℓ: if v' is big this follows from $\deg_G(v') > 2n$ and if v' is small from Proposition 7.6. But this implies that there is a vertex $v \in V_\ell$ that has at least two neighbours in $V_{\ell'}$ and hence satisfies the claim. □

For stating the next (and last) two structural observations, which address the distribution of triangles inside a graph with high Ore degree, we need some more definitions.

Definition 7.8 (triangle path, triangle connected)**.** *Let $G = (V, E)$ be a graph. A* triangle walk *in G is a sequence of edges $e_1, \dots, e_p$ in G such that e_i and e_{i+1} share a triangle of G for all $i \in [p-1]$. We say that e_1 and e_p are* triangle connected *in G and the* length *of the triangle walk $e_1, \dots, e_p$ is $p - 1$.*

For each triangle walk $e_1, \dots, e_p$ there is a sequence of vertices $w_1, \dots, w_{p+1}$ that naturally *corresponds* to this triangle walk, that is, this sequence results from "walking" along $e_1, \dots, e_p$: Set $w_1 := v_1$, $w_2 := v_2$, and let $w_3 := e_2 \setminus e_1$, $w_4 := e_3 \setminus e_2$, and so on. Similarly we can *associate* a sequence of triangles $t_1, \dots, t_{p-1}$ with the triangle walk, where t_i is the triangle that contains the edges e_i and e_{i+1} for each $i \in [p-1]$.

A *connected triangle component* or simply *triangle component* in G is a set of edges $C \subseteq E$ such that each pair of edges in C is triangle connected. Finally, we say that G has *triangle diameter* at most d if there is a pair of vertex disjoint triangles in G and each pair of vertex disjoint triangles t, t' in G is connected by a triangle walk of length at most d, i.e., there are edges $e \in t$ and $e' \in t'$ with a triangle walk of this length between them. If there is no d such that G has triangle diameter at most d then G has infinite triangle diameter.

The first of our two observations concerning triangle walks and triangle components in graphs G with high Ore degree guarantees that in every (non-trivial) triangle component of G we find a copy of K_4.

Proposition 7.9. *Let G be a graph on $3n$ vertices with $\delta_\varnothing(G) \geq 4n+3$ and t be a triangle in G. Then G contains a K_4 which is in the same triangle component as t.*

Proof. We consider three cases. *Case 1*: If all three vertices of t are big and hence have degree more than $2n$, then the common neighbourhood of these three vertices is non-empty and so we are done.

Case 2: If there is one small vertex u and two big vertices v and w in t then clearly the common neighbourhood of v and w is non-empty and thus contains some vertex u'. If u' is small then Proposition 7.6 implies that u and u' are joined by an edge and we are done. If u' is big, on the other hand, the triangle t' on u', v, w contains only big vertices and hence Case 1 implies that u', v, w lies in a K_4.

Case 3: If t contains at least two small vertices u and v then, if there are at least two other small vertices in G then we are done by Proposition 7.6. Accordingly we can assume that at most 1 other vertex is small. Let w be the third vertex in t. By Proposition 7.5 the vertices u, v, and w have degree at least $n+3$ each and thus at least n neighbours in $V(G) \setminus \{u, v, w\}$. We conclude that there are $x, x' \in V(G) \setminus \{u, v, w\}$ such that x is a neighbour of at least two vertices in $\{u, v, w\}$ and the same holds for x'. By our assumption one of x and x' is big. This means that there is a triangle t' on vertices u', v', and w' that shares two vertices with t and has at least one big vertex, say w'. If either u' or v' are big we are done by Cases 1 and 2, hence we assume that u' and v' are small. Then however, u' and w' have degree at least $2n-2$ and $n-1$, respectively, in $V(G) \setminus (t \cup t')$. Since $|V(G) \setminus (t \cup t')| = 3n-4$ it follows that u' and w' have a common neighbour v'' in $V(G) \setminus (t \cup t')$ which is big by our assumption. Thus, we are again done by Case 1 and 2 as u', v'', w' form a triangle with at least two big vertices in the same triangle component as t. □

The last result we present in this section states that a sufficiently high Ore degree forces the triangle diameter to be finite. For the proof of this lemma we will use two auxiliary propositions that we provide and explain below.

Lemma 7.10. *An n-vertex graph G with $\delta_\varnothing(G) > \frac{4}{3}n$ has triangle diameter at most* 7.

Observe that the following example shows that already for $\delta_\varnothing(G) = \frac{4}{3}(n-1)$ the triangle diameter may be infinite: Let G be a graph on $3m+1$ vertices whose vertex set is partitioned into two cliques of size m and an independent set $m+1$, and that contains in addition all edges running between the independent set and the cliques.

Lemma 7.10 claims that each pair of vertex disjoint triangles t and t' in an n-vertex graph G with $\delta_\varnothing(G) > \frac{4}{3}n$ is connected by a triangle walk of length at most 7. For proving this lemma we will construct two growing triangle walks with associated triangle sequences $T = t_0, t_1, ..., t_m$ and $T' = t'_0, t'_1, ..., t'_{m'}$ where $t_0 = t$ and $t'_0 = t'$. When constructing these triangle walks we will guarantee that the number of edges between the last triangles t_m and $t'_{m'}$ increases with m and m' until we force a *direct* triangle connection between t_m and t'_m (i.e., a triangle walk only using vertices from t_m and $t'_{m'}$). In this process, a triangle t_{m+1} such that there are more edges between t_{m+1} and $t'_{m'}$ than between t_m and $t'_{m'}$ is called *better* than t_m for $t'_{m'}$.

In the next two propositions we will develop a strategy for finding suitable better triangles t_{m+1} or $t'_{m'+1}$. We start with the following simple observation.

Proposition 7.11. *For a bipartite graph $F = (A\dot{\cup}B, E)$ with $|A| = |B| = 3$ at least one of the following holds.*

- (a) *There is a path of length 3 in F.*
- (b) *There is a perfect matching in $\bar{F}[A, B]$ where $\bar{F}$ is the complement of F.*
- (c) *F has exactly three edges, all sharing one vertex $v \in A\dot{\cup}B$.*

Proof. If there is a vertex $v \in A\dot{\cup}B$ with $\deg(v) = 3$ then either (c) holds, or there is an edge $e \in E$ with $v \notin e$, which implies that (a) is true. If $\deg(v) \leq 2$ for all $v \in A\dot{\cup}B$ and (a) is false, on the other hand, we immediately get a perfect matching in $\bar{F}[A, B]$ and hence (b). □

This motivates the following case distinction in the construction of our two triangle sequences T and T'. If the bipartite graph F formed by the edges between the last two triangles t_m and $t'_{m'}$ in these sequences satisfies property (a) of Lemma 7.11, then t_m and $t'_{m'}$ are (directly) triangle connected and we are done. Otherwise F satisfies (b) or (c) of Lemma 7.11. We will show that in both cases we can find a triangle t^b that is either better than t_m for $t'_{m'}$ and shares an edge with t_m or vice versa. This argument (which is presented below in the proof of Lemma 7.10) is prepared by the following proposition.

Proposition 7.12. *Let $G = (V, E)$ be an n-vertex graph with $\delta_{\varnothing}(G) > \frac{4}{3}n$ and A, B be vertex disjoint subsets of V with $|A| = |B| = 3$.*

- (i) *If (b) of Lemma 7.11 is true for $G[A, B]$ then there is a vertex $c \in V\backslash(A\dot{\cup}B)$ with at least five neighbours in $A\dot{\cup}B$.*
- (ii) *If (c) of Lemma 7.11 is true for $G[A, B]$ and a vertex $v \in A$ then for any $v' \in B$ there is $c \in V\backslash(A\dot{\cup}B)$ with at least four neighbours in $A\dot{\cup}B \setminus \{v'\}$.*

Proof. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. We start with (i). Assume without loss of generality that $a_1b_1, a_2b_2, a_3b_3 \notin E$. Let V' be the set of vertices in $V\backslash(A\dot{\cup}B)$ with at least five neighbours in $A\dot{\cup}B$, let V'' the set of vertices in $V\backslash(A\dot{\cup}B)$ without this property, and set $n' := |V'|$. With $e(A, B) \leq 6$ it follows that

$$\begin{aligned} 4n &< \big(\deg(a_1) + \deg(b_1)\big) + \big(\deg(a_2) + \deg(b_2)\big) + \big(\deg(a_3) + \deg(b_3)\big) \\ &= 2e(A\dot{\cup}B) + \sum_{v \in V'} |N_{A\dot{\cup}B}(v)| + \sum_{v \in V''} |N_{A\dot{\cup}B}(v)| \\ &\leq 24 + 6n' + 4(n - 6 - n') = 2n' + 4n, \end{aligned}$$

which implies $n' > 0$.

For (ii) we argue similarly. Assume without loss of generality that $v = a_1$ and $v' = b_1$ and set $B' := \{b_2, b_3\}$. Let V' be the set of vertices in $V\backslash(A\dot{\cup}B')$ with at least four neighbours in $A\dot{\cup}B'$ and V'' the set of vertices in $V\backslash(A\dot{\cup}B')$ without this property and set $n' := |V'|$.

Observe that $b_1 \notin V'$ because b_1 neither forms an edge with a_2 nor with a_3. By Proposition 7.5 we have $\deg(a_1) > \frac{1}{3}n$. As $e(A, B') = 2$ it follows that

$$\begin{aligned} 3n &< \deg(a_1) + \big(\deg(a_2) + \deg(b_2)\big) + \big(\deg(a_3) + \deg(b_3)\big) \\ &\leq 12 + 5n' + 3(n - 5 - n') \leq 2n' + 3n. \end{aligned}$$

which, again, implies that $n' > 0$. □

For constructing the triangle t^b that is better (as promised above) than t_m for $t'_{m'}$, or vice versa, we will use the vertex c asserted by Proposition 7.12. With this we are ready to give the proof of Lemma 7.10.

Proof of Lemma 7.10. Let t and t' be two arbitrary vertex disjoint triangles in G. (Clearly, the Ore-condition forces the existence of two such triangles.) We need to show that there is a triangle walk of length at most 7 between t and t'. For this purpose we will construct two sequences of triangles $T = t_0, t_1, ..., t_m$ and $T' = t'_0, t'_1, ..., t'_{m'}$ associated with two triangle walks as follows. First set $m = m' := 0$, $t_0 := t$, and $t'_0 := t'$. Then, in each step, if Proposition 7.11(a) holds for the bipartite graph between t_m and $t'_{m'}$, then we stop. Otherwise we claim that we find a vertex $x \in V \setminus (t_m \dot\cup t'_{m'})$ such that either x and an edge of t_m form a triangle t_{m+1} that is better than t_m for $t'_{m'}$, or x and an edge of $t'_{m'}$ form a triangle $t'_{m'+1}$ that is better than $t'_{m'}$ for t_m. This claim is verified below. If the first (respectively second) case occurs then we add the triangle t_{m+1} (or $t'_{m'+1}$) to T (or T') and increase m (or m') by 1.

Clearly, this procedure terminates after at most 5 steps since Proposition 7.11(a) holds as soon as we have 5 edges between t_m and $t'_{m'}$. This however implies that there is a sequence $T_* = t_0, \ldots, t_m, t_*, t'_*, t'_{m'}, \ldots, t'_0$ of at most 9 triangles such that two consecutive triangles in this sequence share an edge, where t_* and t'_* are obtained from the path of length 3 asserted by Proposition 7.11(a). By omitting $t_0 = t$ and $t'_0 = t'$ from T_* we obtain a sequence of at most 7 triangles that corresponds to a triangle-walk of length at most 7 between t and t'.

It remains to show that the vertex x in the procedure above can always be chosen. Let $A = \{a_1, a_2, a_3\}$ be the vertices of t_m and $B = \{b_1, b_2, b_3\}$ be those of $t'_{m'}$ in some step of the procedure and assume that Proposition 7.11(a) does not hold for $G[A, B]$. This implies that ($*$) there is at most one vertex in each of the triangles t_m and $t'_{m'}$ that has more than one neighbour in the (respectively) other triangle.

Now, assume first, that Proposition 7.11(b) holds for $G[A, B]$. Then Proposition 7.12(i) asserts the existence of a vertex c outside t_m and $t'_{m'}$ with at least 5 neighbours in $A \dot\cup B$. Without loss of generality, only b_1 is possibly not connected to c. By ($*$) either a_1 or a_2, say a_1, has at most one neighbour in $t'_{m'}$. But then the triangle t_{m+1} on vertices a_2, a_3, and c is better for $t'_{m'}$ than t_m (a_1 has at most one neighbour in $t'_{m'}$ while c has at least two) and thus we can choose $x = c$.

If, on the other hand, Proposition 7.11(c) holds for $G[A, B]$ and, say, $v = a_1$ then we can argue similarly. Indeed, for $v' := b_1$ Proposition 7.12(ii) asserts a vertex c outside t_m and $t'_{m'}$ with at least 4 neighbours in $A \dot\cup B \setminus \{v'\}$. If c has an edge to a_1 and at least one other vertex of A, say a_2, then the triangle t_{m+1} on vertices a_1, a_2, and c is better for $t'_{m'}$ than t_m (a_3 has no neighbour in $t'_{m'}$ while c has at least one). Otherwise c has both, b_2 and b_3, as neighbours and the triangle $t'_{m'+1}$ on vertices b_2, b_3, and c is better for t_m than $t'_{m'}$. Hence, again, we can choose $x = c$. □

7.3 Square-paths

In this section we will explain the proof of Lemma 7.3. As indicated earlier the idea is to follow the strategy of [64]. Accordingly our plan is not to repeat all the (technical) details (they can be found in [79]) but rather to outline the method, indicate the differences to our setting, capture those parts we will reuse in the form of a lemma (see Lemma 7.13 below), and fill the remaining gaps.

In the proof of Theorem 1.7 that is provided in [64] the existence of the square of a Hamilton cycle is verified via a joint application of the regularity lemma and the blow-up lemma. More precisely, for an n-vertex graph G with minimum degree $(\frac{2}{3}+\gamma)n$ for $\gamma > 0$ and when one is only interested in finding a square-path on $(1-\mu)n$ vertices (which is clearly contained in the square of a Hamilton cycle) this method can be roughly described as follows. Firstly, an application of the degree version of the regularity lemma (Lemma 3.3) to the graph G (with $\varepsilon \ll d \ll \gamma, \mu$) yields an (ε, d)-regular partition of G with reduced graph $R = (V_R, E_R)$ on k vertices such that $\delta(R) \geq (\frac{2}{3}+\frac{\gamma}{2})n$. Secondly, with the help of a classical theorem, the Theorem of Corrádi and Hajnal [26], it is then possible to infer that R can be covered by $t := \lfloor k/3 \rfloor$ triangles which we number from 1 to t and whose union we denote by T. Thirdly, each pair of these triangles can be connected by a short triangle walk within R. Fourthly, this implies that for all $i \in [t-1]$ there is a constant length square-path, a so-called connecting path, starting in G-vertices of the triangle number i and ending in G-vertices in the triangle number $i+1$ of T; all vertices of such a connecting path are subsequently moved to the exceptional set of the regular partition. Fifthly, one can obtain an (ε', d')-regular partition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ of G (with $\varepsilon' \ll d' \ll \mu$) with reduced graph R that is super-regular on all triangles in T by using Proposition 3.7. And finally sixthly, for each triangle $ij\ell$ of R the blow-up lemma asserts that one can find a square-path $P(i,j,\ell)$ in G covering $V_i \dot\cup V_j \dot\cup V_\ell$ such that these square-paths $P(i,j,\ell)$ together with the connecting paths form a square-path in G covering at least $|V_1 \dot\cup \dots \dot\cup V_k| \geq (1-\varepsilon')n \geq (1-\mu)n$ vertices.

To sum up, the following lemma crystallises steps 4 to 6 out of this method. In the statement of this lemma we use the following definition which is based on concept of triangle components introduced in Section 7.2. A *spanning connected triangle factor* in a graph G is a spanning triangle factor with all edges in the same triangle component of G.

Lemma 7.13 (Komlós, Sárközy, Szemerédi [64])**.** *For all positive reals μ and d there exists a positive ε_0 such that for all $\varepsilon \leq \varepsilon_0$ and integers k_1 there is an integer $n_0 > k_1$ such that the following holds. Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices that has an (ε, d)-regular equipartition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ with reduced graph R and $3 \leq k \leq k_1$. If R contains a spanning connected triangle factor. Then G contains a square-path on $(1-\mu)n$ vertices.* □

In [64] the correctness of steps 1 to 3 of the agenda described above relies on the minimum degree condition in the problem treated there. Here we need to adjust these steps to our setting with Ore condition. For steps 1 and 2 we will use the Ore version of the regularity lemma (Lemma 7.4) and Theorem 7.1, respectively (the details are presented directly in the proof of Lemma 7.3 below). For adapting step 3 we use the fact that also in a graph with sufficiently high Ore degree each pair of triangles can be connected by a short sequence of triangles which follows from Lemma 7.10.

Proof of Lemma 7.3. Given γ and μ we set $d := \gamma/10$ and let ε_0 be the constant provided by Lemma 7.13 for input μ and d. Then we choose $\varepsilon := \min\{\varepsilon_0, \gamma/10\}$ and ask Lemma 7.4 about its k_1 of choice for ε and $k_0 := 3$. With ε and k_1 at hand we can continue the application of Lemma 7.13 and obtain an n_0.

Now, let $G = (V, E)$ be a graph on $n \geq n_0$ vertices with Ore degree $\delta_\varnothing(G) \geq (\frac{4}{3} + \gamma)n$. From the regularity lemma, Lemma 7.4, we obtain an (ε, d)-regular equipartition of V with reduced graph R on k vertices where $3 \leq k \leq k_1$ and

$$\delta_\varnothing(R) \geq \big(\tfrac{4}{3} + \gamma - 2(d+\varepsilon)\big)k \geq (\tfrac{4}{3} + \tfrac{1}{2}\gamma)k > \tfrac{4}{3}k\,.$$

It follows that we can apply Theorem 7.1 to R and conclude that R contains a spanning triangle factor which is connected by Lemma 7.10. Hence G contains a square-path of length $(1-\mu)n$ by Lemma 7.13 as claimed. □

For the proof of Theorem 7.2 we will not apply Lemma 7.3 directly but we will use the following corollary. The reason is that the square of a path is not "connected well enough" for our application and we instead need to be able to guarantee (under the same conditions as those in Lemma 7.3) a graph with a more robust structure – the graph C^3_m defined (and used) in Chapter 5 (see page 49). The following corollary of Lemma 7.3 asserts the existence of such a C^3_m-copy in a graph with high Ore degree.

Corollary 7.14. *For all $\gamma, \mu > 0$ there is an n_0 such that for all $n \geq n_0$ every n-vertex graph $G = (V, E)$ satisfying $\delta_\varnothing(G) \geq (\frac{4}{3}+\gamma)n$ contains a copy of C^3_m for some m with $3m \geq (1-\mu)n$.*

For reducing this corollary from Lemma 7.3 we will use Lemma 5.8 from Chapter 5. With this lemma we obtain Corollary 7.14 by another application of the regularity lemma and as an easy consequence of Lemma 7.3.

Proof of Corollary 7.14. For arbitrary constants $\gamma > 0$, $\mu > 0$ and $0 < d \leq \frac{\gamma}{8}$ let ε_0 be as given by Lemma 5.8 applied with $r := 3$, μ, and d. Choose $\varepsilon > 0$ small enough such that

$$2(\varepsilon + d) \leq \frac{\gamma}{2}\,. \tag{7.1}$$

From Lemma 7.3 applied with $\frac{\gamma}{2}$ and μ replaced by ε we get an integer k_0'. For input ε and $k_0 := \max\{k_0, 3\}$ Lemma 7.4 supplies us with a k_1 for which Lemma 5.8 provides an integer $n_0 > k_1$.

Let $G = (V, E)$ be a graph on $n \geq n_0$ vertices satisfying $\delta_\varnothing(G) \geq (\frac{4}{3} + \gamma)n$. It follows from Lemma 7.4 applied to the graph G and $\eta = \frac{4}{3} + \gamma$ that G has an (ε, d)-regular equipartition with reduced graph R on k vertices such that $k_0 \leq k \leq k_1$ and

$$\delta_\varnothing(R) \geq \left(\frac{4}{3} + \gamma - 2(\varepsilon + d)\right)k \overset{(7.1)}{\geq} \left(\frac{4}{3} + \frac{\gamma}{2}\right)k\,.$$

Hence an application of Lemma 7.3 to the graph R finds a square-path on at least $(1-\varepsilon)k$ vertices in R, i.e., a copy of P^2_ℓ on $3\ell \geq (1-\varepsilon)k$ vertices. By Lemma 5.8 this implies that G contains a copy of C^3_m for some m with $3m \geq (1-\mu)n$ □

7.4 Proof of Theorem 7.2

The proof of Theorem 7.2 relies on the general embedding lemma (Lemma 3.12) and is similar in structure (and philosophy) to the proof of Theorem 5.1 in Chapter 5. For applying the general embedding lemma, we again use two complementary lemmas, the lemma for G (Lemma 7.15) which, given a graph G with the necessary Ore condition, provides us with an adequate system of regular pairs, and the lemma for H (Lemma 7.16) which produces the corresponding compatible partition of bounded-degree graphs H with small bandwidth.

For the formulation of these lemmas, recall the definition of a colour adjustable copy of the graph C_k^3 (see Definition 5.12 on page 59). Recall, furthermore, that the graph K_k^3 on vertex set $[k] \times [3]$ is the union of k vertex disjoint triangles on vertices $\{i\} \times [3]$, $i \in [k]$ (see page 49), and the *i-th triangle* in K_k^3 is $K_k^3[(i,1),(i,2),(i,3)]$.

The lemma for G now states that a graph G with sufficiently high Ore degree has an (ε, d)-regular partition with reduced graph R that contains a colour adjustable copy of $C_k^3 \supseteq K_k^3$ as well as a K_4 that is triangle connected to this C_k^3. The colour adjustability of C_k^3 will be used by Lemma 5.13 which adjusts this partition to the partition of H later. The copy of K_4 is needed by the lemma for H as we will explain below.

We say that a vertex partition $(V_{i,j})_{i\in[k],j\in[3]}$ is *equitable* if $|V_{i,j}|$ and $|V_{i,j'}|$ differ by at most 1 for all $i \in [k]$ and $j, j' \in [3]$.

Lemma 7.15 (Lemma for G)**.** *For all $\gamma > 0$ there is $d > 0$ such that for all $\varepsilon > 0$ there is $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$ the following holds. Let $G = (V, E)$ be an n-vertex with $\delta_{\varnothing}(G) \geq (\frac{4}{3} + \gamma)n$. Then G has an equitable (ε, d)-regular partition $V = (V_{i,j})_{i\in[k],j\in[3]}$ with reduced graph R on $3k \leq k_1$ vertices such that*

- *(G1) $K_4 \subseteq R$ and $K_k^3 \subseteq C_k^3 \subseteq R$ such that K_4 and K_k^3 are in the same triangle component of G and C_k^3 is colour-adjustable in R,*
- *(G2) $(V_{i,j})_{i\in[k],j\in[3]}$ is (ε, d)-super regular on K_k^3,*
- *(G3) $|V_{i,j}| \geq (1-\varepsilon)n/(3k)$ for all $i \in [k]$, $j \in [3]$.*

This Lemma is almost a standard consequence of the Ore version of the regularity lemma (Lemma 7.4) and Lemma 7.3. For distributing the vertices in the exceptional set however we need some more work. The proof is given in Section 7.6.

We next state the lemma for H. This lemma receives an integer partition $(n_{i,j})_{i\in[k],j\in[3]}$ as input, which will encode the sizes of the partition classes of G later, and tries to set up the sizes of the partition classes it constructs for H according to $(n_{i,j})_{i\in[k],j\in[3]}$. However, it does not succeed completely but can only get close to this goal (by an error of ξn). In the proof of Theorem 7.2 we will make use of the adjusting lemma, Lemma 5.13 from Chapter 5, to compensate this difference.

Lemma 7.16 (Lemma for H)**.** *For all $\xi > 0$ and $k \in \mathbb{N}$ there is $\beta_0 > 0$ such that for all $\beta \leq \beta_0$ the following holds. Let $H = (\tilde{V}, \tilde{E})$ be a 3-colourable n-vertex graph which has maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$. Further, let R be a graph on $3k$ vertices that contains a spanning connected triangle factor K_k^3 as well as a copy of K_4 in the same triangle component. Let $(n_{i,j})_{i\in[k],j\in[3]}$ be an equitable integer partition of n with $n_{i,j} \geq n/(6k)$ and set $n'_{i,j} := n_{i,j} + \xi n$. Then H has a vertex partition $\tilde{V} = (\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ that is ξ-compatible with $(n'_{i,j})_{i\in[k],j\in[3]}$ and $K_k^3 \subseteq R$.*

The proof of this lemma is given in section 7.5. The idea is to follow the strategy outlined in Section 1.3.4 of the introduction, cut H into small pieces along its bandwidth order, assign these pieces to the triangles of $K_k^3 \subseteq R$ in the reduced graph, and balance this assignment, if necessary, with the help of the K_4-copy in R.

Using the lemma for G, the lemma for H, the general embedding lemma, and the adjusting lemma we can now give the proof of Theorem 7.2.

Proof of Theorem 7.2. Given Δ and γ let d be the constant provided by Lemma 7.15 for this γ and assume without loss of generality that $d \leq \frac{1}{4}$. With d and Δ at hand we request a constant ε_{GEL} from Lemma 3.12 with input $d_{\text{GEL}} := d/2$, Δ, and $r := 3$. We feed $= 3$, d and ε_{GEL} into Lemma 5.13 to get an $\varepsilon'_{5.13}$ and fix

$$\varepsilon_G := \min\{\varepsilon'_{5.13}, \varepsilon_{\text{GEL}}, 1/10\}\,. \tag{7.2}$$

Then we pass ε_G to Lemma 7.15 to get k_1. Next, we continue the application of Lemma 5.13. For each $0 < k' \leq k_1$ this lemma provides a constant $\xi_{k'}$ and we choose the smallest of these $\xi_{k'}$ as $\xi_{5.13}$. Similarly, for

$$\xi_H := \min\{\varepsilon_{\text{GEL}}, \xi_{5.13}/k_1\}$$

and each $0 < k' \leq k_1$ Lemma 7.16 replies with a $\beta_{k'}$ and we choose the smallest of these $\beta_{k'}$ as β. We set $n_0 := k_1$ and have now fixed all constants.

Next, we receive input graphs $G = (V, E)$ and $H = (\tilde{V}, \tilde{E})$ on $n \geq n_0$ vertices with $\Delta(H) \leq \Delta$, $\text{bw}(H) \leq \beta n$, and $\delta_{\varnothing}(G) \geq (\frac{4}{3} + \gamma)n$. We hand G on to Lemma 7.15 to obtain an (ε_G, d)-regular partition $V = (U_{i,j})_{i\in[k],j\in[3]}$ with reduced graph R on $3k \leq k_1$ vertices fulfilling (*G1*)–(*G3*). Let $(n_{i,j})_{i\in[k],j\in[3]}$ be the integer partition of n given by $n_{i,j} := |U_{i,j}|$ and observe that, by (*G2*) this partition is equitable and by (*G1*) the graph R contains K_4 as well as a spanning connected triangle factor K^3_k. Moreover, by (*G3*) and (7.2) we have $n_{i,j} \geq (1 - \varepsilon_G)n/(3k) \geq n/(6k)$ for all $i \in [k]$, $j \in [3]$. Hence we can appeal to Lemma 7.16 with input ξ_H and k, the graphs H and R and the partition $(n_{i,j})_{i\in[k],j\in[3]}$. We obtain a vertex partition $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ of H which is ξ_H-compatible (and thus ε_{GEL}-compatible) with the integer partition $(n'_{i,j})_{i\in[k],j\in[3]}$ defined by $n'_{i,j} := n_{i,j} + \xi_H n$ and with $K^3_k \subseteq R$.

By (*G1*) the reduced graph R fulfils $K^3_k \subseteq C^3_k \subseteq R$ and C^3_k is colour-adjustable in R. By (*G2*) the graph G is (ε_G, d)-super-regular on K^3_k and by Lemma 7.16 the partition of H is ξ_H-compatible with $(n'_{i,j})_{i\in[k],j\in[3]}$ and $K^3_k \subseteq R$. This implies $|\tilde{V}_{i,j}| \leq n'_{i,j} = n_{i,j} + \xi_H$ and hence $|\tilde{V}_{i,j}| = n_{i,j} \pm k\xi_H n = |U_{i,j}| \pm \xi_{5.13}n$ for all $i \in [k]$, $j \in [3]$. It follows that the conditions of Lemma 5.13 are satisfied for the graph G, its (ε_G, d)-regular partition $(U_{i,j})_{i\in[k],j\in[3]}$ with reduced graph R and $\varepsilon_G \leq \varepsilon'_{5.13}$, and for the integer partition $(\tilde{n}_{i,j})_{i\in[k],j\in[3]}$ defined by $\tilde{n}_{i,j} := |\tilde{V}_{i,j}|$. Since $d_{\text{GEL}} = \frac{1}{2}d$ this lemma gives us an $(\varepsilon_{\text{GEL}}, d_{\text{GEL}})$-regular partition $V = (V_{i,j})_{i\in[k],j\in[3]}$ with reduced graph R that is $(\varepsilon_{\text{GEL}}, d_{\text{GEL}})$-super-regular on K^3_k and satisfies $|V_{i,j}| = \tilde{n}_{i,j} = |\tilde{V}_{i,j}|$ for all $i \in [k], j \in [3]$.

This in turn prepares us for the finish. Observe that the partitions $(V_{i,j})_{i\in[k],j\in[3]}$ and $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ of G and H, respectively, satisfy the properties required by the general embedding lemma (Lemma 3.12) due to the choice of ε_{GEL}. Hence we can apply this lemma to the graphs G and H with their partitions $(V_{i,j})_{i\in[k],j\in[3]}$ and $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$, respectively, and to R and $R' := K^3_k$. Therefore H is a subgraph of G. □

7.5 Partitions of 3-colourable graphs with small bandwidth

In this section we prove of the Lemma for H (Lemma 7.16). In this proof we will perform the following steps. Let H be given together with a graph R that contains a connected spanning triangle factor K^3_k as well as a K_4 in the same triangle component and an integer partition

$(n_{i,j})_{i\in[k],j\in[3]}$. We first cut H into k segments with sizes as dictated by the integer partition $(n_{i,j})_{i\in[k],j\in[3]}$ and assign each segment to a triangle of K_k^3. Afterwards we further cut each segment into much smaller pieces and use the 3-colouring of H to construct homomorphisms from each of these pieces to the clusters of the triangle a piece was assigned to. The most difficult step of the proof will then be to "connect" the homomorphisms of the pieces and obtain a homomorphism from H to R. For convenience this step is taken care of by the following proposition, which we will prove first. Its proof relies on the fact that K_4 and K_k^3 are in the same triangle component of R.

Proposition 7.17. *Let $R = (V_R, E_R)$ be a graph on $3k$ vertices, let t and t' be two triangles that are in the same triangle component of R which also contains a K_4. Let H be a 3-colourable graph on vertex set $[n]$ with bandwidth at most βn whose vertices are given in bandwidth order and let $\varphi : [n] \to t \cup t'$ be a mapping from the vertices of H to the vertices of the triangles t and t'. Assume further that there is a vertex $x \in [n]$ such that for $\tilde{V} := [x]$ and $\tilde{V}' := (x, n]$ the restriction of φ to $\tilde{V}$ is a homomorphism from $H[\tilde{V}]$ to t and the restriction of φ to $\tilde{V}'$ is a homomorphism from $H[\tilde{V}']$ to t'.*

Then there is a homomorphism $h : [n] \to V_R$ from H to R such that if $h(\tilde{u}) \neq \varphi(\tilde{u})$ then $\tilde{u} \in [x, x + 100k^2\beta n]$.

Proof. Assume the graphs H and R satisfy the requirements of the proposition. Let v_1^*, v_2^*, v_3^*, $v_4^* \in V_R$ be the vertices of the K_4 that is in the same triangle component as t and t'. Let f be the restriction of φ to $\tilde{V}$ and f' be the restriction of φ to $\tilde{V}'$.

Let σ be a 3-colouring of H that is "compatible" with the homomorphisms f and f' in the following sense: If $f(\tilde{u}) = f(\tilde{v})$ for $\tilde{u}, \tilde{v} \in \tilde{V}$ or $f'(\tilde{u}) = f'(\tilde{v})$ for $\tilde{u}$, $\tilde{v} \in \tilde{V}'$ then $\sigma(\tilde{u}) = \sigma(\tilde{v})$. This colouring σ exists because f is a homomorphism from $H[\tilde{V}]$ to the triangle t and f' is a homomorphism from $H[\tilde{V}]$ to the triangle t'. Accordingly we can denote the vertices of t by v_1, v_2, and v_3 in such a way that all vertices in $x \in f^{-1}(u_j)$ have colour $\sigma(x) = j$ for $j \in [3]$. Similarly we can denote the vertices of t' by v_1', v_2', and v_3' such that $f'(x) = v_j'$ iff $\sigma(x) = j$. We also say that σ *induces colour* j on the vertices v_j and v_j' of R for $j \in [3]$.

Next we will define how, for an arbitrary triangle walk $w = e_1, \dots e_p$ starting in $e_1 = v_1v_2$ these induced colours "propagate" along this triangle walk. We remark that the propagated colours will later describe how a vertex $\tilde{u}$ of H is assigned to a vertex in such a triangle walk depending on the colour $\sigma(\tilde{u})$. Let $w_1, \dots, w_{p+1}$ with $w_1 = v_1$ and $w_2 = v_2$ be the vertex sequence corresponding to the triangle walk w (see Definition 7.8). Consistently with the definition of induced colours, we say that colour $c_1 := 1$ is *propagated by* w to $v_1 = w_1$ and colour $c_2 := 2$ to $v_2 = w_2$. Inductively, for a vertex w_i with $w \in [p+1]$ we say that colour $c_i \in [3]$ is propagated to w_i if the following holds: Recall that $w_i := e_{i-1} \setminus e_{i-2}$ and let t_i be the triangle containing e_{i-1} and e_{i-2} (which exists by the definition of a triangle walk). We will set up c_i such that on the triangle t_i all three colours appear. For this purpose let $w_i, w_{i'}, w_{i''}$ be the vertices of t_i. Then $i', i'' \leq i$ and so the colours $c_{i'}$ and $c_{i''}$ propagated to $w_{i'}$ and $w_{i''}$, respectively, were previously defined (and are distinct). Then let c_i be the colour such that $\{c_i, c_{i'}, c_{i''}\} = [3]$. Observe that by this definition, different colours may be propagated to a vertex v of R; but in the following we will usually talk about colours propagated to the final vertices w_p and w_{p+1} of a triangle walk w, by which we mean the colours c_p and c_{p+1}, respectively. Moreover, we analogously define colours propagated on triangle paths starting in $v_1'v_2'$.

Claim 7.18. *There is a triangle walk $w^* = e_1^*, \dots, e_{p^*}^*$ starting in $e_1^* = v_1v_2$ and ending in*

$e^*_{p^*} = v'_1v'_2$ *such that the colours propagated by* w *to* v'_1 *and* v'_2 *coincide with the colours induced on* v'_1 *and* v'_2 *by* σ *and such that* $p^* \le 2k^2 + 20$.

Proof. For proving this claim we will choose triangle walks w and w' starting in v_1v_2 and $v'_1v'_2$, respectively, and ending in the edge $v^*_1v^*_2$ of K_4. Then, potentially "walking" around the K_4 a couple of times, we will connect these two triangle walks such that the resulting triangle walk propagates colours that are consistent with the induced colours.

Let $e = v_1v_2$, $e' = v'_1v'_2$, $e^* = v^*_1v^*_2$ be edges of t, t', and K_4, respectively. Since t, t', and K_4 are in the same triangle component of R there are triangle walks $w = e_1, \dots, e_p$ and $w' = e'_1, \dots, e'_{p'}$ in R with $e_1 = e$, $e'_1 = e'$, and $e_p = e'_{p'} = e^*$. Since R has less than k^2 edges we can further assume that $p, p' \le k^2$. Let $w_1, \dots, w_{p+1}$ and $w'_1, \dots, w'_{p'+1}$ be the vertex sequences corresponding to the triangle walks w and w', respectively, and note that we have $w_pw_{p+1} = e_p = w'_pw'_{p+1} = e'_{p'} = e^*$. Now w propagates colours $c(v^*_1)$ and $c(v^*_2)$ to v^*_1 and v^*_2, respectively, and w' propagates colours $c'(v^*_1)$ and $c'(v^*_2)$ to these vertices. It is not difficult to check that, using the edges of K_4 the triangle walk w can be extended to a triangle walk $\hat{w}$ starting in e and ending in e^* such that $\hat{w}$ propagates colours $c'(v^*_1)$ and $c'(v^*_2)$ to its final vertices v^*_1 and v^*_2. Indeed, if we denote by w^+ the triangle walk in K_4 with corresponding vertex sequence $v^*_3, v^*_4, v^*_1, v^*_2$ and by w^- the triangle walk in K_4 with corresponding vertex sequence $v^*_4, v^*_3, v^*_2, v^*_1$ then $\hat{w}$ can be obtained by appending at most 3 times a copy of either w^+ or w^- to w (mixtures of w^+ and w^- are allowed). It follows that the triangle path w^* resulting from the concatenation of $\hat{w}$ and w' satisfies the claim because the length of w' is at most $k^2 - 1$ and that of $\hat{w}$ is at most $k^2 - 1 + 16$. □

We will use this triangle walk $w^* = e^*_1, \dots, e^*_{p^*}$ for constructing the desired homomorphism $h : [n] \to V_R$. Let $c^*_1, \dots, c^*_{p^*+1}$ be the sequence of colours propagated along w^* and $t^*_1, \dots, t^*_{p^*-1}$ be the sequence of triangles defined by w^*, i.e., t^*_i is the triangle of R that contains e^*_i and e^*_{i+1}. Further, for all $i \in [p^* - 1], j \in [3]$ let $u_{i,j}$ be the vertex $w^*_{i'}$ in triangle t^*_i such that colour $c^*_{i'} = j$ was propagated to $w^*_{i'}$ by w^*. Observe that by the definition of the colour propagation and the choice of w^* we have

$$u_{1,1} = v_1, \quad u_{1,2} = v_2 \qquad \text{and} \qquad u_{p^*-1,1} = v'_1, \quad u_{p^*-1,2} = v'_2. \tag{7.3}$$

Now we have to change the images of some vertices of H in order to transform the mapping φ into a homomorphism h. For this reason we define *linking sets* $L_1, \dots, L_{p^*-1} \subseteq [n]$ in H with $L_i := (x + (i-1)2\beta n, x + i \cdot 2\beta n]$. The idea then is to let h map all vertices of L_i to triangle t^*_i in w^* in such a way that vertices with colour j are mapped to the vertex $u_{i,j}$ in triangle t^*_i to which colour j was propagated. Accordingly we define

$$h(\tilde{u}) := \begin{cases} u_{i,j} & \text{if } \tilde{u} \in L_i \text{ and } \sigma(\tilde{u}) = j \\ \varphi(\tilde{u}) & \text{if } \tilde{u} \notin \bigcup L_i \end{cases}$$

and claim that this is a homomorphism satisfying the assertions of the Proposition. It is easy to see from the definition of h that $h(\tilde{u}) \ne \varphi(\tilde{u})$ implies $\tilde{u} \in (x, x + 100k^2\beta n]$ because $p^* \cdot 2\beta n \le (2k^2 + 20)2\beta n \le 100k^2\beta n$. Hence it remains to show that h is a homomorphism. To see this let $\tilde{u}\tilde{v}$ be an arbitrary edge of H. We consider several cases. If neither $\tilde{u}$ nor $\tilde{v}$ is in a linking set then $h(\tilde{u}) = \varphi(\tilde{u})$ and $h(\tilde{v}) = \varphi(\tilde{v})$. Moreover $\mathrm{bw}(H) \le \beta n$ implies that either both $\tilde{u}, \tilde{v} \in \tilde{V}$ or both $\tilde{u}, \tilde{v} \in \tilde{V}'$ and so $h(\tilde{u})h(\tilde{v})$ is an edge of one of the triangles t and t'. As second case assume that $\tilde{u}$ is in a linking set L_i with $i \in [p^* - 1]$. Now, if $\tilde{v}$ is in the same linking set L_i then $\tilde{u}$ and $\tilde{v}$ are both mapped to vertices of the triangle t_i^* by h. As $\tilde{u}$ and $\tilde{v}$ have different colours in σ they are mapped to distinct vertices of t_i. Hence assume that $\tilde{v} \notin L_i$. Then either $\tilde{v} \le x$ and $i = 1$, or $\tilde{v} \in L_{i-1} \dot{\cup} L_{i+1}$, or $\tilde{v} > x + (p^* - 1)2\beta n$ and $i = p^* - 1$. We will only consider the last of these three cases here, which is the most difficult one. The other two cases follow similarly. In this last case we have

$$h(\tilde{u}) = u_{p^*-1,j} \text{ with } j = \sigma(\tilde{u}) \qquad \text{and} \qquad h(\tilde{v}) = \varphi(\tilde{v}) = v'_{j'} \text{ with } j' = \sigma(\tilde{v}),$$

where we recall that $v'_{j'}$ is the vertex of the triangle t' on which σ induces colour j'. Since $\tilde{u}$ and $\tilde{v}$ have different colours in σ we have $j \neq j'$. Moreover, w^* is a triangle walk ending in $v'_1 v'_2$ and so its last triangle $t^*_{p^*-1}$ and the triangle t' share the edge $v'_1 v'_2$. By the definition of the colour propagation and the resulting definition of h we thus have the following situation: If $h(\tilde{u}) = u_{p^*-1,j}$ is a vertex of t' we have $j \in \{1, 2\}$ and $h(\tilde{u}) = u_{p^*-1,j} = v'_j$ by (7.3). Therefore $h(\tilde{u})h(\tilde{v})$ is an edge of t'. Otherwise, by (7.3), we have $j = 3$ and so $j' \in \{1, 2\}$. It follows that $h(\tilde{v}) = v'_{j'}$ also is a vertex of the triangle $t^*_{p^*-1}$, more precisely by (7.3) it is the vertex $u_{p^*-1,j'}$. Consequently $h(\tilde{u})h(\tilde{v})$ is an edge of $t^*_{p^*-1}$ and we are done. □

Now we can give the proof of Lemma 7.16. In this proof we make use of the simple Proposition 5.14 (see page 62) from Chapter 5.

Proof of Lemma 7.16. Given ξ and k choose $\beta_0 := 10^{-4}\xi^2/k^3$, let $\beta \le \beta_0$ be given, and set $\xi' := 10^3 k^2 \beta$. Let $H = ([n], \tilde{E})$ be a 3-colourable graph with $\Delta(H) \le \Delta$ and $\mathrm{bw}(H) \le \beta n$ such that $\beta \le \beta_0$, and let σ be a 3-colouring of H. Let $R = ([k] \times [3], E_R)$ be a graph on $3k$ vertices that contains a K_4 and a spanning connected triangle factor K_k^3 (assume the vertices of R are named according to this K_k^3). Let further $(n_{i,j})_{i\in[k],j\in[3]}$ be an equitable integer partition satisfying $n_{i,j} \ge n/(6k)$ for all $i \in [k], j \in [3]$ and set $n'_{i,j} := n_{i,j} + \xi n$.

As explained in the introduction of this section we will first cut H into segments (which are assigned to different triangles of K_k^3) and then these segments into much smaller pieces of size roughly $\xi' n$. Then we will construct homomorphisms for each of these pieces to one of the triangles in K_k^3. These homomorphisms and the segments above are set up in such a way that, altogether each cluster $V_{i,j}$ receives roughly $n_{i,j}$ vertices. Finally we will use Proposition 7.17 to modify these homomorphisms slightly in order to get a homomorphism from H to R which will in turn give the desired vertex partition of H.

Assume that H is given in bandwidth order. Cut the vertices $[n]$ of H along this bandwidth order into segments $S_i \subseteq [n]$ with $i \in [k]$ where S_i has size $s_i := n_{i,1} + n_{i,2} + n_{i,3}$ and let s_i^* be the first vertex of S_i. Then assign segment S_i to the i-th triangle of K_k^3.

Next we do the following for each triangle t of K_k^3. Let S_i be the segment assigned to t and $(i, 1)$, $(i, 2)$, and $(i, 3)$ be the clusters of t. We assume for simplicity that $\xi' n$ is integer, set $p(i) := \lfloor s_i/(\xi' n) \rfloor$ define pieces $P_{i,\ell} := [\, s_i^* + (\ell - 1)\xi' n \,,\, s_i^* + \ell \xi' n \,)$ for all $\ell \in [p(i)]$, and add all (at most $\xi' n$) vertices $u \ge p_i \xi' n$ of S_i to $P_{i,p(i)}$. We will use these pieces to assign the vertices of S_i to the clusters of t. We start with the first piece $P_{i,1}$ and assign the vertices $u \in P_{i,1}$ with colour $\sigma(u) = j$ to cluster (i, j) for $j \in [3]$. We continue with the piece $P_{i,2}$ and then $P_{i,3}$ and so on as follows. Assume we are about to assign the vertices of piece $P_{i,\ell}$ in this process and so far $s_{i,j}$ vertices u were assigned to cluster (i, j) for $j \in [3]$. The idea now is to choose the smallest colour class $j \in [3]$ of piece $P_{i,\ell}$, i.e., the vertex set $\sigma^{-1}(j) \cap P_{i,\ell}$, and assign all these vertices to the cluster (i, j') of t that received most vertices so far, i.e., the cluster with biggest $s_{i,j'}$. We further assign the next smallest colour class of $P_{i,\ell}$ to the cluster of t that received the next biggest number of vertices so far and so on. To make this precise assume for simplicity of the presentation that $s_{i,1} \leq s_{i,2} \leq s_{i,3}$. Now let $s'_j := |\sigma^{-1}(j) \cap P_{i,\ell}|$ be the vertices of colour j in $P_{i,\ell}$ and let $\pi : [3] \to [3]$ be a permutation such that $s'_{\pi(1)} \geq s'_{\pi(2)} \geq s'_{\pi(3)}$. Then we assign all vertices $u \in P_{i,\ell}$ of colour $\sigma(u) = \pi(j)$ to cluster (i, j) for $j \in [3]$. Observe that Proposition 5.14 applied with $r := 3$ as well as $c_j = s_{i,j}$ and $c'_j = s'_{\pi(j)}$ for $j \in [3]$ and for $x = \xi' n$ asserts that throughout this procedure we have that the number of vertices assigned to two different clusters of t differs by $2\xi' n$ at most.

After assigning the vertices of all pieces $P_{i,\ell}$ with $i \in [k]$ and $\ell \in [p(i)]$ to clusters of K^3_k in this way denote the resulting (united) assignment by $\varphi : [n] \to [k] \times [3]$. Note that

$$|\varphi^{-1}((i,j))| \leq n_{i,j} \pm 2\xi' n \tag{7.4}$$

for all vertices (i, j) of R because $(n_{i,j})_{i\in[k],j\in[3]}$ is an equipartition, we assigned a segment of size $n_{i,1} + n_{i,2} + n_{i,3}$ to the triangle t in R consisting of vertices $(i, 1), (i, 2), (i, 3)$ and then asserted that the number of vertices assigned to two different clusters of t differ by $2\xi' n$ at most. Moreover φ is a homomorphism to R (in fact to a single triangle of K^3_k) when restricted to $P_{i,\ell}$ for all $i \in [k]$ and $\ell \in [p(i)]$.

In the remaining part of the proof we will transform the assignment φ into a homomorphism from the whole graph H to R. To this end we will use Proposition 7.17. Let t_i be the i-th triangle of K^3_k for $i \in [k]$. For each consecutive pair of pieces $P_{i,\ell}$ and $P_{i,\ell+1}$ with $\ell \in [p(i) - 1]$ we apply Proposition 7.17 on R, the triangles $t := t' := t_i$, the graph $H_{i,\ell+1} := H[P_{i,\ell} \cup P_{i,\ell+1}]$, the mapping φ restricted to the vertices of $H_{i,\ell+1}$ and to the last vertex (in the bandwidth ordering) x of $P_{i,\ell}$. Then Proposition 7.17 produces a homomorphism $h_{i,\ell+1}$ from $H_{i,\ell+1}$ to R that coincides with φ on $P_{i,\ell}$ and all but the first $100k^2\beta n$ vertices of $P_{i,\ell+1}$. Similarly, for the last and the first piece, respectively, of each two consecutive segments S_i and S_{i+1}, $i \in [k - 1]$ we apply Proposition 7.17 on R, the triangles $t = t_i$ and $t' = t_{i+1}$, the graph $H_{i+1,1} := H[P_{i,p(i)} \cup P_{i+1,1}]$, the mapping φ restricted to the vertices of $H_{i+1,1}$ and to the last vertex (in the bandwidth ordering) of $P_{i,p(i)}$. Then Proposition 7.17 produces a homomorphism $h_{i+1,1}$ from $H_{i+1,1}$ to R that coincides with φ on $P_{i,p(i)}$ and all but the first $100k^2\beta n$ vertices of $P_{i+1,1}$. Then we define the mapping $h : \tilde{V} \to V_R$ by setting $h(\tilde{u}) := h_{i,\ell}(\tilde{u})$ for all vertices $\tilde{u}$ in piece $P_{i,\ell}$ for all $i \in [k]$, $\ell \in [p(i)]$. Observe that h coincides with φ on all but the first $100k^2\beta n$ vertices of each piece $P_{i,\ell}$ which we denote by $P^*_{i,\ell}$. Moreover, as $100k^2\beta n + 100\beta n \leq \xi' n$ these non-coinciding intervals $P^*_{i,\ell}$ do not overlap but have distance more than βn, hence there are no edges between them. Thus, because each $h_{i,\ell}$ is a homomorphism from $H_{i,\ell}$ to R, the united mapping h also is a homomorphism from H to R which coincides with φ on all but at most $k \cdot 100k^2\beta n$ vertices.

For defining the vertex partition $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ we now set $\tilde{V}_{i,j} := h^{-1}((i,j))$. Since h is a homomorphism to R the partition $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ clearly is an R-partition of H. Additionally (7.4) implies $\big||\tilde{V}_{i,j}| - n_{i,j}\big| \leq 2\xi' n + 100k^3\beta n \leq \xi n$ and hence $|\tilde{V}_{i,j}| \leq n'_{i,j}$ because h coincides with φ on all but at most $100k^3\beta n$ many vertices. For $i \in [k]$, $j \in [3]$ let $Z_{i,j}$ be the set of vertices in $\tilde{V}_{i,j}$ that have neighbours in clusters that do not belong to the i-th triangle of K^3_k or that have common neighbours with such neighbours . To show that $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ is ξ-compatible it remains to show that for all $i \in [k]$, $j \in [3]$ the set $Z_{i,j}$ satisfies $|Z_{i,j}| \leq \frac{1}{2}\xi n_{i,j} \leq \frac{1}{2}\xi n'_{i,j}$. This is true because $Z_{i,j}$ can only contain vertices of the intervals $P^*_{i',\ell}$ with $i' \in [k], \ell \in [p(i)]$, their neighbourhood and their second neighbourhood. There are at most $1/\xi'$ sets $P^*_{i',\ell}$ and they are intervals in the bandwidth order of H of size $100k^2\beta n$ each. It follows that $|Z_{i,j}| \leq (100k^2\beta n + 4\beta n)/\xi' \leq \frac{1}{2}\xi n/(10k) \leq \frac{1}{2}\xi n_{i,j}$ where we use that $\beta \leq \beta_0 = 10^{-4}\xi^2/k^3$. Thus $(\tilde{V}_{i,j})_{i\in[k],j\in[3]}$ satisfies all required properties. □

7.6 Colour-adjustable partitions

In this section we prove the lemma for G. We will first apply the Ore version of the regularity lemma, Lemma 7.4 to obtain a regular partition of G and a reduced graph R with high Ore degree. This enables us then to use Corollary 7.14 and infer that the reduced graph R contains a copy of C^3_k, which will be shown to be colour adjustable by using Proposition 7.7. Moreover, the existence of a K_4 in the same triangle component follows from Proposition 7.9. We will then apply Proposition 3.7 from Chapter 3 to modify the regular partition of G into a regular equipartition (with the same reduced graph R) that is additionally super-regular on $K^3_k \subseteq C^3_k$.

It remains, and this is the most laborious step, to distribute the vertices in the exceptional set of the regular partition to the other partition classes such that neither the super-regularity nor the equipartition gets destroyed. For this we will distinguish between small and big vertices (cf. the definition on page 85) in the exceptional set. The reason for this distinction is that, as it turns out, the big vertices can easily be distributed to triangles T of K^3_k where they have many neighbours in *every* cluster of T (this was in fact already demonstrated in the distribution of the exceptional set in Section 5.5 of Chapter 5). Hence we can distribute these big vertices "equally" over the clusters of such triangles while maintaining super-regularity as well as an equipartition.

For the small vertices applying the same procedure is unfortunately not possible. As we will show, however, for each small vertex v we can find triangles T in K^3_k such that v has many neighbours in two of the three clusters of T. Thus moving v to the third cluster C of T will not destroy super-regularity. For maintaining an equipartition we will, in exchange, pick a big vertex in C and move it to the exceptional set.

Proof of Lemma 7.15. We first set up the necessary constants. Given γ let $d := \gamma/10^4$. Given ε we next fix auxiliary constants ε' and d' by setting $d' := 2d$ and choosing ε' small enough such that

$$\varepsilon' + 10^3\sqrt{\varepsilon'/\gamma} \le \varepsilon \qquad \text{and} \qquad 10^3\sqrt{\varepsilon'/\gamma} \le d\,. \tag{7.5}$$

Next we define constants $\varepsilon_{\mathrm{RL}}$ and d_{RL} for the application of the regularity lemma such that

$$\varepsilon' \ge \frac{8\varepsilon_{\mathrm{RL}}}{1-4\varepsilon_{\mathrm{RL}}}\,, \qquad d' \le d_{\mathrm{RL}} - 8\varepsilon_{\mathrm{RL}}\,, \qquad \text{and} \qquad 2(\varepsilon_{\mathrm{RL}} + d_{\mathrm{RL}}) \le \tfrac{1}{10}\gamma\,. \tag{7.6}$$

This is possible because $d' = 2d = 2\gamma/10^4$. Now we apply Corollary 7.14 with parameters $\frac{1}{2}\gamma$ and $\mu := \varepsilon_{\mathrm{RL}}$ to obtain k_0 and Lemma 7.4 with input $\varepsilon_{\mathrm{RL}}$ and this k_0 and get k_1. This finishes the definitions of the constants.

For constructing the desired partition of an input graph G satisfying the requirements of the lemma we commence by applying the regularity lemma in the form of Lemma 7.4 with parameters $\varepsilon_{\mathrm{RL}}$, k_0, $\eta := \frac{4}{3} + \gamma$, and d_{RL}. We obtain an $(\varepsilon_{\mathrm{RL}}, d_{\mathrm{RL}})$-regular equipartition with reduced graph R_{RL} on $k_0 \le k_{\mathrm{RL}} \le k_1$ vertices with $\delta_\varnothing(R_{\mathrm{RL}}) \ge (\eta - 2(\varepsilon_{\mathrm{RL}} + d_{\mathrm{RL}}))k_{\mathrm{RL}} \ge (\frac{4}{3} + \frac{1}{2}\gamma)k_{\mathrm{RL}}$. Next we apply Corollary 7.14 with $\frac{1}{2}\gamma$ and $\mu := \varepsilon_{\mathrm{RL}}$ to the graph R_{RL} and conclude that C^3_k is a subgraph of R_{RL} with $3k \ge (1-\varepsilon_{\mathrm{RL}})k_{\mathrm{RL}}$. We claim that the graph $R = (V_R, E_R)$ induced in R_{RL} on the $3k$ vertices spanned by C^3_k is still a $(2\varepsilon_{\mathrm{RL}}, d_{\mathrm{RL}})$-reduced graph for an equipartition of G and satisfies $\delta_\varnothing(R) \ge (4+\gamma)k$. Indeed, $\delta_\varnothing(R) \ge \delta_\varnothing(R_{\mathrm{RL}}) - 2\varepsilon_{\mathrm{RL}}k_{\mathrm{RL}} \ge (\frac{4}{3} + \frac{1}{3}\gamma)3k$. Moreover, assume we move those clusters (of the $(\varepsilon_{\mathrm{RL}}, d_{\mathrm{RL}})$-regular equipartition corresponding to R_{RL}) that are not covered by C^3_k to the exceptional set. Then we obtain an $(2\varepsilon_{\mathrm{RL}}, d_{\mathrm{RL}})$-regular partition of G since the exceptional set of this new partition is of size at most $\varepsilon_{\mathrm{RL}}n + \varepsilon_{\mathrm{RL}}k_{\mathrm{RL}}(n/k_{\mathrm{RL}}) = 2\varepsilon_{\mathrm{RL}}n$.

We now rename the clusters of R according to the (spanning) copy of C_k^3 to $[k] \times [3]$ and claim that C_k^3 is colour-adjustable in R. To see this let $\{j^+, j^-, j\} = [3]$ be arbitrary and apply Proposition 7.7 to R and the equipartition $V_R = ([k] \times \{1\}) \dot\cup ([k] \times \{2\}) \dot\cup ([k] \times \{3\})$. This implies that there is an $i \in [k]$ such that (i, j^+) has more than k neighbours in $[k] \times \{j^+, j\}$ and consequently there is $i^* \in [k]$ such that (i, j^+) has an edge to the two vertices (i^*, j') with $j' \neq j^-$ in R. This means that C_k^3 is colour-adjustable in R. Observe further that $K_k^3 \subseteq C_k^3$ and so all triangles of this K_k^3 lie in one triangle component of R and we get a K_4 in the same triangle component by Proposition 7.9. Consequently we have (*G1*).

We continue by applying Proposition 3.7 with parameters $2\varepsilon_{\rm RL}$, $d_{\rm RL}$, and $\Delta = 2$ to the graph G and its $(2\varepsilon_{\rm RL}, d_{\rm RL})$-reduced graph R and to $R' := K_k^3$. We obtain an (ε', d')-partition $V_0' \dot\cup (V_{i,j}')_{i\in[k], j\in[3]}$ with reduced graph R that is (ε', d')-super-regular on K_k^3 because

$$\varepsilon' \overset{(7.6)}{\geq} 8\varepsilon_{\rm RL}/(1-4\varepsilon_{\rm RL}) \qquad \text{and} \qquad d' \overset{(7.6)}{\leq} d_{\rm RL} - 8\varepsilon_{\rm RL}\,.$$

It remains to distribute the vertices in the exceptional set of this partition to the other partition classes without destroying regularity or super-regularity. Let us first describe how we perform this distribution and then determine its effects on the regularity. Roughly speaking, the strategy is as follows: We will first exchange some small vertices in V_0' with big vertices from other clusters (hence maintaining an equipartition). Then, in a second step, we will redistribute all remaining vertices in V_0' as well as the new big vertices (that we just exchanged against small vertices) to the clusters of $(V_{i,j}')_{i\in[k],j\in[3]}$. We will see that the first step is necessary in order to guarantee that the second step can be carried out in such a way that the resulting partition is still an equipartition.

For giving the details it is convenient to first introduce some definitions. Let L' be the sizes of the clusters in $(V_{i,j}')_{i\in[k],j\in[3]}$. We say that a triangle t in K_k^3 is *u-friendly* if u has at least $2d'L'$ neighbours in each cluster of t and *u-half-friendly* if t is not u-friendly but there are two clusters of t such that u has at least $2d'L'$ neighbours in these clusters.

Claim 7.19. *If the number of u-friendly triangles is less than $\frac{1}{10}\gamma k$ then u is small and there are at least $\frac{1}{10}\gamma k$ triangles that are u-half-friendly.*

Proof. If there are less than $\frac{1}{10}\gamma k$ triangles that are u-friendly, then u has degree

$$\deg(u) < (2kL' + 2d'L') + \tfrac{1}{10}\gamma kL' + \varepsilon' n \leq (\tfrac{2}{3} + d')n + \tfrac{1}{10}\gamma\tfrac{1}{3}n + \varepsilon' n \leq (\tfrac{2}{3} + \tfrac{1}{2}\gamma)n$$

and thus u is small. Therefore Proposition 7.5 implies $\deg(u) \geq (\frac{1}{3} + \gamma)n$ and hence, if X is the number of u-half-friendly triangles, we get

$$\begin{aligned}(\tfrac{1}{3} + \gamma)n &\leq \deg(u) \\ &< \tfrac{1}{10}\gamma k \cdot 3L' + X \cdot (2L' + 2d'L') + (k - \tfrac{1}{10}\gamma k - X)(L' + 2\cdot 2d'L') + \varepsilon' n \\ &= X(L' - 2d'L') + kL'\big(\tfrac{3}{10}\gamma + (1 - \tfrac{1}{10}\gamma)(1 + 4d')\big) + \varepsilon' n \\ &\leq X(n/3k) + (\tfrac{1}{10}\gamma + \tfrac{1}{3} + 2d')n + \varepsilon' n,\end{aligned}$$

We conclude that $X \geq (\frac{1}{3} + \gamma - \frac{1}{10}\gamma - \frac{1}{3} - 2d' - \varepsilon')3k \geq \frac{1}{10}\gamma k$. □

Our first aim now is to exchange all vertices u of V_0' that do not have many u-friendly triangles with big vertices from clusters in triangles that are u-half-friendly. While the preceding claim guarantees that there are many u-half-friendly triangles for these vertices, the following claim implies that we can find many big vertices in such triangles.

Claim 7.20. *If u is small and C is a cluster such that u has less than $2d'L'$ neighbours in C then there are at least $100\varepsilon'n/(\gamma k)$ big vertices in C.*

Proof. By Proposition 7.6 the small vertices of G form a clique. Hence all non-neighbours of u in C are big. The number of these non-neighbours is at least $(1-2d')L' \geq \frac{1}{2}(n/3k) \geq 100\varepsilon'n/(\gamma k)$. □

Accordingly we can now perform the following exchange procedure: Let $U \subseteq V'_0$ be the set of vertices u in V'_0 with less than $\frac{1}{10}\gamma k$ triangles that are u-friendly. By Claim 7.19 all these vertices are small and have at least $\frac{1}{10}\gamma k$ half-friendly triangles. It follows that we can assign the vertices of U to the triangles of K^3_k in such a way that each $u \in U$ is assigned to a u-half-friendly triangle and no triangle receives more than $|U|/(\frac{1}{10}\gamma k) \leq 10\varepsilon'n/(\gamma k)$ vertices. For each vertex $u \in U$ we then do the following: Let t be the triangle such that u is assigned to t. Since t is u-half-friendly, there is a cluster C in t such that u has less than $2d'L'$ neighbours in C. In the two other clusters of t the vertex u has at least $2d'L'$ neighbours. We remove u from V'_0, add it to C and instead move a big vertex of C to V'_0. This is possible since at most $10\varepsilon'n/(\gamma k)$ vertices of U were assigned to C, and by Claim 7.20 the cluster C contains at least $100\varepsilon'n/(\gamma k)$ big vertices.

Let $V''_0 \cup (V''_{i,j})_{i\in[k],j\in[3]}$ be the partition we obtain after this exchange procedure. Observe that $|V''_0| \leq \varepsilon'n$ and V''_0 only contains vertices u with at least $\frac{1}{10}\gamma k$ triangles that are u-friendly. Hence we can assign each of these vertices $u \in V''_0$ to a u-friendly triangle of K^3_k in such a way that at most $\varepsilon'n/(\frac{1}{10}\gamma k)$ vertices are assigned to each triangle. This implies that each $u \in V''_0$ is assigned to a triangle t such that u has at least $2d'L'$ neighbours in each cluster of t. Finally, for each triangle t of K^3_k we distribute all vertices of V''_0 that were assigned to t among the three clusters in t such that the sizes of the resulting clusters are as equal as possible.

We call the resulting partition $(V_{i,j})_{i\in[k],j\in[3]}$ and claim that this partition has the required properties. Verifying this claim will finish the proof of the lemma. Observe first that $|V_{i,j}| \geq |V'_{i,j}| \geq (1-\varepsilon')n/(3k) \geq (1-\varepsilon)n/(3k)$ for all $i \in [k], j \in [3]$ and so we get (*G3*). To see (*G2*) notice further that this partition is equitable: The regularity lemma produced the equipartition $(V'_{i,j})_{i\in[k],j\in[3]}$, and the exchange procedure modified this partition to get $(V''_{i,j})_{i\in[k],j\in[3]}$ and guaranteed that $|V'_{i,j}| = |V''_{i,j}|$ for all $i \in [k], j \in [3]$. Then, in the final distribution of V''_0 we distributed vertices assigned to a triangle of K^3_k as equally as possible and so $(V_{i,j})_{i\in[k],j\in[3]}$ is indeed equitable. It remains to show that $(V_{i,j})_{i\in[k],j\in[3]}$ is (ε,d)-regular on R and (ε,d)-super-regular on K^3_k. To see this, recall that at most $10\varepsilon'n/(\gamma k)$ vertices were moved out of any cluster and into any cluster throughout the exchange and distribution procedure. For $\alpha := 100\varepsilon'/\gamma$ we thus have

$$|V'_{i,j} \triangle V_{i,j}| \leq 2\cdot 10\frac{\varepsilon'n}{\gamma k} \leq \alpha(1-\varepsilon)\frac{n}{3k} \leq \alpha|V_{i,j}|$$

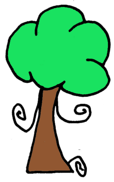
for all $i \in [k], j \in [3]$. Hence Proposition 3.8 applied with ε', d' and $\alpha =: \beta$ to all (ε',d')-regular pairs of the partition $(V'_{i,j})_{i\in[k],j\in[3]}$ implies that the partition $(V_{i,j})_{i\in[k],j\in[3]}$ is (ε,d)-regular on R by (7.5). In addition all vertices $v \in V_{i,j} \cap V''_{i,j}$ have at least

$$d'|V''_{i,j'}| - \alpha|V_{i,j'}| \geq (d'(1-2\alpha)-\alpha)|V_{i,j'}| \overset{(7.5)}{\geq} d|V_{i,j'}|$$

neighbours in all clusters $V_{i,j'}$ with $j \neq j'$. The exchange and distribution procedure asserted the same for all vertices $v \in V_{i,j} \setminus V''_{i,j}$. Accordingly, Proposition 3.8 implies additionally that $(V_{i,j})_{i\in[k],j\in[3]}$ is (ε,d)-super-regular and thus we get (*G2*). □

Chapter 8

The tripartite Ramsey number for trees



In this chapter we consider a Ramsey-type embedding problem for trees. Recall that, for a family $\mathcal{F}$ of graphs and a graph K, we say that K is *Ramsey* for $\mathcal{F}$ and write $K \to \mathcal{F}$ if in any edge-colouring of K with green and red one of the colour classes contains a copy of each member of $\mathcal{F}$.

Many classical Ramsey problems consider the question for which n the complete graph K_n is Ramsey for a particular family of graphs $\mathcal{F}$ (see Section 1.1.2). Here we are instead interested in the case that K is a complete tripartite graph $K_{n,n,n}$. We prove the following theorem about trees, confirming a conjecture of Schelp [89] (see also Section 1.2.3). Let $\mathcal{T}_t$ denote the family of trees of order t and $\mathcal{T}_t^\Delta$ be its restriction to trees of maximum degree at most Δ.

Theorem 8.1. *For every $\mu > 0$ there are $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$K_{n,n,n} \to \mathcal{T}_t^\Delta,$$

if $\Delta \leq n^\alpha$ and $t \leq (3-\mu)n/2$.

By Theorem 1.11 the graph K_{3n} is Ramsey for $\mathcal{T}_s$ with $s = \frac{3}{2}n + 1$. Theorem 8.1 asserts that if we replace $\mathcal{T}_s$ by the class $\mathcal{T}_t^\Delta$ of bounded-degree trees of site t with t only slightly smaller than s, then we can remove three n-vertex cliques from this K_{3n} and the resulting graph $K_{n,n,n}$ is still Ramsey for $\mathcal{T}_t^\Delta$.

Notice that, in contrast to all other embedding results in this thesis, the bound on the maximum degree of T in Theorem 8.1 is not constant, but grows with n. Consequently we do not apply the blow-up lemma (Lemma 3.9) when proving this result, but replace it by a greedy embedding strategy for trees that can cope with such growing maximum degrees.

The proof of Theorem 8.1 splits into a combinatorial part and a regularity-based embedding part. The lemmas we need for the combinatorial part are stated in Section 8.2 and proved in Section 8.6. As explained in the introduction (see Section 1.3.2), Łuczak [76] noted that a large connected matching in a cluster graph is a suitable structure for embedding paths. Here we extend Łuczak's idea and use what we call "odd connected matchings" and "connected fork systems" in the cluster graph to embed trees (as explained in Section 1.3.4).

For the embedding part we formulate an embedding lemma (Lemma 8.12, see Section 8.3) that provides rather general conditions for the embedding of trees with growing maximum degree. The proof of this lemma is presented in Section 8.5. First, however, we shall introduce all definitions necessary for this chapter.

8.1 Coloured graphs: definitions and tools

For convenience let us first recall some definitions. A *matching* M in a graph $G = (V, E)$ is a set of vertex-disjoint edges in E and its *size* is the number of edges in M. For a vertex set D and an edge set M, let $D \cap M$ be the set of vertices from D that appear in some edge of M. For a vertex v and a vertex set U covered by M we also write, abusing notation, $v \in M$ and $U \subseteq M$. In this chapter we sometimes also consider a matching as a bijection $M\colon V_M \to V_M$ where $V_M \subseteq V$ is the set of vertices covered by M. For $U \subseteq V_M$ we then denote by $M(U)$ the set of vertices $v \in V_M$ such that $uv \in M$ for some $u \in U$.

In addition we shall use the following convention in this chapter. To make our notation compact we sometimes use subscripts in a non-standard way as illustrated by the following example. Let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ be sets and suppose that $D \in \{A, B\}$ and $i \in [2]$. The symbol D_i then denotes the set A_i if $D = A$ and the set B_i if $D = B$.

Coloured graphs

A *coloured graph* G is a graph (V, E) together with a 2-colouring of its edges by red and green. We denote by $G(c)$ the subgraph of G formed by the edges with colour c. Two vertices are *connected* in G if they lie in the same connected component of G and are *c-connected* in G if they are connected in $G(c)$. Let G be a coloured graph and v be a vertex of G and $c \in \{\text{red}, \text{green}\}$. Then a vertex u is a *c-neighbour* of v if uv is an edge of colour c in G. The *c-neighbourhood* of v is the set of all c-neighbours of v.

Definition 8.2 (connected, odd, even)**.** *Let G' be either a subgraph of an uncoloured graph G, or a c-monochromatic subgraph of a coloured graph G. Then we say that G' is* connected *if any two vertices covered by G' are connected, respectively c-connected, in G. Further, the component of G, respectively of $G(c)$, containing G' is called the* component of G' *and is denoted by $G[G']$. Further, G' is* odd *if there is an odd cycle in $G[G']$, otherwise G' is* even.

Notice that this notion of connected subgraphs differs from the standard one. A red-connected matching is a good example to illustrate this concept: it is a matching with all edges coloured in red and with a path (in the original graph) of red colour between any two vertices covered by the matching. For subgraphs containing edges of different colours the notion of connectedness is not defined.

Definition 8.3 (fork, fork system)**.** *An r-*fork *(or simply* fork*) is the complete bipartite graph $K_{1,r}$. We also say that an r-fork has r* prongs *and one* centre *by which we refer to the vertices in the two partition classes of $K_{1,r}$. A* fork system *F in a graph G is a set of pairwise vertex disjoint forks in G (not necessarily having the same number of prongs). We say that F has* ratio *r if all its forks have at most r prongs. Then we also call F an r-fork system.*

Suppose that F is a connected fork system in G. If F is even, then the *size* f of F is the order of the bigger bipartition class of $G[F]$. If F is odd, then F has size at least f if there is a connected bipartite subgraph G' of G such that F has size f in G'. For a vertex set D in G we say that F is *centered* in D if the centres of the forks in F all lie in D.

Next, we define two properties of coloured graphs that characterise structures (in a reduced graph) suitable for the embedding of trees as we shall see later (Section 8.3). Roughly speaking, these properties guarantee the existence of large monochromatic connected matchings and fork-systems.

Definition 8.4 (m-odd, (m, f, r)-good). *Let G be a coloured graph on n vertices. Then G is called* m-odd *if G contains a monochromatic odd connected matching of size at least m. We say that G is* (m, f, r)-good *(in colour c) if G contains a c-coloured connected matching M of size at least m as well as a c-coloured connected fork system F of size at least f, and ratio at most r.*

Now we define a set of special, so-called extremal, configurations of coloured graphs that will need special treatment in our proofs. To prepare their definition, let K be a graph on n vertices and D, D' be disjoint vertex sets in K. We say that the bipartite graph $K[D, D']$ is *η-complete* if each vertex of $K[D, D']$ is incident to all but at most ηn vertices of the other bipartition class. If K is a coloured graph then $K[D, D']$ is *(η, c)-complete* for some colour c if it is η-complete and all edges in $K[D, D']$ are of colour c. We call a set A *negligible* if $|A| < 2\eta n$. Otherwise, A is *non-negligible*.

Definition 8.5 (extremal). *Let $K = (V, E)$ be a coloured graph of order $3n$. Suppose that $\eta > 0$ is given. We say that K is a pyramid configuration with parameter η if it satisfies* (E1) *below and a spider configuration if it satisfies* (E2)*. In both cases we call K* extremal with parameter η *or* η-extremal*. Otherwise we say that K is* not η-extremal.

(E1) pyramid configurations: *There are (not necessarily distinct) colours c, c' and pairwise disjoint subsets D_1, D_2, D'_1, $D'_2 \subseteq V$ of size at most n, with $|D_1|, |D_2| \geq (1-\eta)n$ and $|D'_1| + |D'_2| \geq (1-\eta)n$ where D'_1 and D'_2 are either empty or non-negligible. Further, $K[D_1, D'_1]$ and $K[D_2, D'_2]$ are (η, c)-complete and $K[D_1, D'_2]$, $K[D_2, D'_1]$, and $K[D_1, D_2]$ are η-complete.*
In addition, either $K[D_1, D_2]$ is (η, c')-complete or both $K[D_1, D'_2]$ and $K[D'_1, D_2]$ are (η, c')-complete. In the first case we say the pyramid configuration has a c'-tunnel*, and in the second case that it has a* crossing*. The pairs (D_1, D'_1) and (D_2, D'_2) are also called the* pyramids *of this configuration.*

(E2) spider configuration: *There is a colour c and pairwise disjoint subsets A_1, A_2, B_1, B_2, C_1, $C_2 \subseteq V$ such that $|D_1 \cup D_2| \geq (1-\eta)n$ and $K[D_1, D'_2]$ is (η, c)-complete for all $D, D' \in \{A, B, C\}$ with $D \neq D'$, the edges in all these bipartite graphs together form a connected bipartite subgraph K_c of K with (bi)partition classes $A_1 \dot\cup B_1 \dot\cup C_1$ and $A_2 \dot\cup B_2 \dot\cup C_2$. Further there are sets $A_B \dot\cup A_C = A_2$, $B_A \dot\cup B_C = B_2$, and $C_A \dot\cup C_B \dot\cup C_C = C_2$, each of which is either empty or non-negligible, such that the following conditions are satisfied for all $\{D, D', D''\} = \{A, B, C\}$:*

1. $|A_1| \geq |B_1| \geq |C_1 \cup C_C|$ *and* $|D_{D'}| = |D'_D| \leq n - |D''_2|$,
2. *either* $C_C = \emptyset$ *or* $A_B = \emptyset$,
3. *either* $A_2 = \emptyset$ *or* $|A_2 \cup B_2 \cup C_A \cup C_B| \leq (1-\eta)\frac{3}{2}n$,
4. *either* $C_1 = \emptyset$ *or* $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$ *or* $|B_1 \cup C_1| \leq (1-\eta)\frac{3}{4}n$.

By $\mathcal{K}_n^\eta$, finally, we denote the class of all spanning subgraphs K of $K_{n,n,n}$ with minimum degree $\delta(K) > (2-\eta)n$. We also call the graphs in this class *η-complete tripartite graphs.*

Regularity

The version of the regularity lemma that we will use in this chapter takes, similarly as the version that we saw in Chapter 6 (see Lemma 6.3), a preliminary partition as input and produces a regular partition which refines this partition. As explained in Section 6.1 standard proofs of the regularity lemma allow for this extension. We will use the following definition. Suppose that P is a partition of the vertex set V of a graph. Then we say that a partition $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_s$ of V *refines* P if for every $i \in [s]$ there exists a member $A \in P$ such that $V_i \subseteq A$. Observe that, in contrast to standard notation, we do not require that the "extra" set V_0 (which will be the exceptional set of a regular partition for our purposes) is contained in one partition class of P only.

Lemma 8.6 (Regularity lemma)**.** *For all $\varepsilon > 0$ and integers k_0 and k_* there is an integer k_1 such that for all graphs $G = (V, E)$ on $n \geq k_1$ vertices the following holds. Let G be given together with a partition $V = V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$ of its vertices. Then there is $k_0 \leq k \leq k_1$ such that G has an ε-regular equipartition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_k$ refining $V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$* □

We also say that $V = V_1^* \dot\cup \dots \dot\cup V_{k_*}^*$ is a *prepartition* of G.

Remark. Throughout this chapter use blackboard symbols such as $\mathbb{G}$ or $\mathbb{M}$ for reduced graphs and their subgraphs.

8.2 Connected matchings and fork systems

In order to prove Theorem 8.1 we will use the following structural result about coloured graphs from $\mathcal{K}_n^\eta$. It asserts that such graphs either contain large monochromatic odd connected matchings or appropriate connected fork systems. With the help of the regularity method we will then, in Section 8.3, use this result (on the reduced graph of a regular partition) to find monochromatic trees. The reason why odd connected matchings and connected fork systems are useful for this task is explained in Section 8.3.1.

Lemma 8.7. *For all $\eta' > 0$ there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Every coloured graph $K \in \mathcal{K}_n^\eta$ is either $(1-\eta')\frac{3}{4}n$-odd or $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 3\big)$-good.*

We remark that the dependence of the constant n_0 and η' is only linear, and in fact we can choose $n_0 = \eta'/200$. As we will see below, Lemma 8.7 is a consequence of the following two lemmas. The first lemma analyses non-extremal members of $\mathcal{K}_n^\eta$.

Lemma 8.8 (non-extremal configurations)**.** *For all $\eta' > 0$ there are $\eta \in (0, \eta')$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Let K be a coloured graph from $\mathcal{K}_n^\eta$ that is not η'-extremal. Then K is $(1-\eta')\frac{3}{4}n$-odd.*

The second lemma handles the extremal configurations.

Lemma 8.9 (extremal configurations)**.** *For all $\eta' > 0$ there is $\eta \in (0, \eta')$ such that the following holds. Let K be a coloured graph from $\mathcal{K}_n^\eta$ that is η-extremal. Then K is $((1-\eta')n, (1-\eta')\frac{3}{2}n, 3)$-good.*

Proofs of Lemma 8.8 and 8.9 are provided in Sections 8.6.1 and 8.6.2, respectively. We get Lemma 8.7 as an easy corollary.

Proof of Lemma 8.7. Given η' let $\eta_{8.9} < \eta'$ be the constant provided by Lemma 8.9 for input η' and let $\eta_{8.8}$ be the constant produced by Lemma 8.8 for input $\eta'_{8.8} := \eta_{8.9}$. Set $\eta := \min\{\eta_{8.9}, \eta_{8.8}\}$ and let $K \in \mathcal{K}_n^\eta$ be a given coloured graph. Then $K \in \mathcal{K}_n^{\eta_{8.8}}$ and by Lemma 8.8 the graph K is either $(1-\eta'_{8.8})3n/4$-odd (and thus $(1-\eta')3n/4$-odd as oddness is monotone) or $\eta_{8.9}$-extremal. In the first case we are done and in the second case Lemma 8.9 implies that K is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 3\big)$-good (goodness is also monotone) and we are also done. □

8.3 Proof of Theorem 8.1

In this section we will first briefly outline the main ideas for the proof of Theorem 8.1. Then we will state the remaining necessary lemmas, most notably our main embedding result (Lemma 8.12). These lemmas will be proved in the subsequent sections. At the end of this section we finally provide a proof of Theorem 8.1.

8.3.1 The idea of the proof

We apply the regularity lemma on the coloured graph $K_{n,n,n}$ with prepartition as given by the partition classes of $K_{n,n,n}$. As a result we obtain a coloured reduced graph $\mathbb{K} \in \mathcal{K}_k^\eta$ where the colour of an edge in $\mathbb{K}$ corresponds to the majority colour in the underlying regular pair. Such a regular pair is well-known to possess almost as good embedding properties as a complete bipartite graph. We apply our structural result (Lemma 8.7) and infer that $\mathbb{K}$ is either $(1-\eta')\frac{3}{4}k$-odd or $((1-\eta')k, (1-\eta')\frac{3}{2}k, 3)$-good. Accordingly, there is a colour, say green, such that $\mathbb{K}$ contains either an odd connected green matching $\mathbb{M}_o$ of size at least $(1-\eta')\frac{3}{4}k$, or it contains a connected greed matching $\mathbb{M}$ of size at least $(1-\eta')k$ and a 3-fork system $\mathbb{F}$ of size at least $(1-\eta')\frac{3}{2}k$. We shall show that using either of these structures we can embed any tree $T \in \mathcal{T}_t^\Delta$ into the green subgraph of $K_{n,n,n}$. As a preparatory step, we cut T into small subtrees (see Lemma 8.14), called shrubs.

Now let us first consider the case when we have an odd matching $\mathbb{M}_o$. Our aim is to embed each shrub S into a regular pair (A, B) corresponding to an edge $e \in \mathbb{M}_o$. Shrubs are bipartite graphs. Therefore there are two ways of assigning the colour classes of S to the clusters of e. This corresponds to two different *orientations* of S for the embedding in (A, B). Our strategy is to choose orientations for all shrubs (and hence assignments of their colour classes to clusters of edges in $\mathbb{M}_o$) in such a way that every cluster of $V(\mathbb{M}_o)$ receives roughly the same number of vertices of T. We will show that this is possible without "over-filling" any cluster. It follows that we can embed all shrubs into regular pairs corresponding to edges of $\mathbb{M}_o$. The fact that $\mathbb{M}_o$ is connected and *odd* then implies that between any pair of edges in $\mathbb{M}_o$ there are walks of both even and odd length in the reduced graph. We will show that this allows us to connect the shrubs and to obtain a copy of T in the green subgraph of $K_{n,n,n}$.

If, on the other hand, we have a matching $\mathbb{M}$ as well as a 3-fork system $\mathbb{F}$, the basic strategy remains the same. We assign shrubs to edges of $\mathbb{M}$ or $\mathbb{F}$. As opposed to the previous case, however, these substructures of the reduced graph are not odd. This means that we cannot choose the orientations of the shrubs as before. Rather, these orientations are determined by the connections between the shrubs. Therefore, we distinguish the following two situations when embedding the tree T. If the partition classes of T are reasonably balanced, then we use the matching $\mathbb{M}$ for the embedding. If T is unbalanced, on the other hand, we employ the fork system $\mathbb{F}$ and use the prongs of the forks in $\mathbb{F}$ to accommodate the bigger partition class of T and the centres for the smaller.

8.3.2 The main embedding lemma

As indicated, in the proof of the main theorem we will use the regularity lemma in conjunction with an embedding lemma (Lemma 8.12). This lemma states that a tree T can be embedded into a graph given together with a regular partition if there is a homomorphism from T to the reduced graph of the partition with suitable properties. In the following definition of a valid assignment we specify these properties. Roughly speaking, a valid assignment is a homomorphism h from a tree T to a (reduced) graph $\mathbb{G}$ such that no vertex of $\mathbb{G}$ receives too many vertices of T and that does not "spread" in the tree too quickly in the following sense: for each vertex $x \in V(T)$ we require that the neighbours of x occupy at most two vertices of $\mathbb{G}$.

Definition 8.10 (valid assignment). *Let $\mathbb{G}$ be a graph on vertex set $[k]$, let T be a tree, $\varrho \in [0,1]$ and $L \in \mathbb{N}$. A mapping $h\colon V(T) \to [k]$ is a (ϱ, L)-*valid assignment* of T to $\mathbb{G}$ if*

1. *h is a homomorphism from T to $\mathbb{G}$,*
2. *$|h(N_T(x))| \le 2$, for every $x \in V(T)$,*
3. *$|h^{-1}(i)| < (1-\varrho)L$, for every $i \in [k]$.*

In addition we need the concept of a cut of a tree, which is a set of vertices that cuts the tree into small components which we call shrubs.

Definition 8.11 (cut, shrubs). *Let $S \in \mathbb{N}$ and T be a tree with vertex set $V(T)$. A set $C \subseteq V(T)$ is an S-*cut* (or simply *cut*) of T if all components of $T - C$ are of size at most S. The components of $T - C$ are called the *shrubs* of T corresponding to C.*

Now we can state the main embedding lemma.

Lemma 8.12 (main embedding lemma). *Let G be an n-vertex graph with an (ε, d)-reduced graph $\mathbb{G} = ([k], E(\mathbb{G}))$ and let T be a tree with $\Delta(T) \le \Delta$ and an S-cut C. If T has a $\big(\varrho, (1-\varepsilon)\frac{n}{k}\big)$-valid assignment to $\mathbb{G}$ and $(\frac{1}{10}d\varrho - 10\varepsilon)\frac{n}{k} \ge |C| + S + \Delta$ then $T \subseteq G$.*

The proof of this lemma is deferred to Section 8.5. Before we can apply it for embedding a tree T in the proof of Theorem 8.1 we need to construct a valid assignment for T. This is taken care of by the following lemma which states that this is possible if the reduced graph of some regular partition contains an odd connected matching or a suitable fork system. The proof of this lemma is given in Section 8.4.

Lemma 8.13 (assignment lemma). *For all $\varepsilon, \mu > 0$ with $\varepsilon < \mu/10$ and for all $k \in \mathbb{N}$ there is $\alpha = \alpha(k) > 0$ and $n_0 = n_0(\mu, \varepsilon, k) \in \mathbb{N}$ such that for all $n \ge n_0$, all $r \in \mathbb{N}$, all graphs $\mathbb{G}$ of order k, and all trees T with $\Delta(T) \le n^{\alpha}$ the following holds. Assume that either*

- (M) *$\mathbb{G}$ contains an odd connected matching of size at least m and that $t := |V(T)| \le (1-\mu)2m\frac{n}{k}$, or*
- (F) *$\mathbb{G}$ contains a connected fork system with ratio r and size at least f, and T has colour class sizes t_1 and t_2 with $t_2 \le t_1 \le t'$ and $t_2 \le t'/r$, where $t' = (1-\mu)f\frac{n}{k}$.*

Then there is an $(\varepsilon\frac{n}{k})$-cut C of T with $|C| \le \varepsilon\frac{n}{k}$ and a $\big(\frac{1}{2}\mu, (1-\varepsilon)\frac{n}{k}\big)$-valid assignment of T to $\mathbb{G}$.

8.3.3 The proof

Now we have all tools we need to prove the main theorem.

Proof of Theorem 8.1. We start by defining the necessary constants. Given $\mu > 0$, set $\mu' := \eta'$ in such a way that

$$1 - \tfrac{\mu}{3} \le (1-\eta')^2(1-\mu'). \tag{8.1}$$

Lemma 8.7 with input $\eta' > 0$ provides us with $\eta > 0$ and $k_0 \in \mathbb{N}$. The regularity lemma, Lemma 8.6, with input

$$\varepsilon := \min\{\tfrac{1}{100}\eta^2, \tfrac{1}{10}\eta'^2, 10^{-3}\mu'\} \tag{8.2}$$

and k_0 and $k_* := 3$ returns a constant k_1. Next we apply Lemma 8.13 with input $\frac{\varepsilon}{10}$ and μ' separately for each value $3k$ with $k_0 \leq 3k \leq k_1$ and get constants $\alpha(3k)$ and $n_0'(3k)$ for each of these applications. Set $\alpha := \min\{\alpha(3k)\colon\ k_0 \leq 3k \leq k_1\}$ and $n_0' := \max\{n_0'(3k)\colon\ k_0 \leq 3k \leq k_1\}$. Finally, choose

$$n_0 := \max\{n_0', k_1, (\tfrac{k_1}{\varepsilon})^{1/(1-\alpha)}\}\,. \tag{8.3}$$

We are given a complete tripartite graph $K_{n,n,n}$ with $n \geq n_0$ as input whose edges are coloured with green and red. Our goal is to select a colour and show that in this colour we can embed every member of $\mathcal{T}_t^\Delta$ with $\Delta \leq n^\alpha$ and $t \leq (3-\mu)n/2$.

We first select the colour. To this end let G and R be the subgraphs of $K_{n,n,n}$ formed by the green and red edges, respectively. We apply the regularity lemma, Lemma 8.6, with input $\frac{\varepsilon}{10}$ on the graph G with prepartition $V_1^*\dot\cup V_2^*\dot\cup V_3^*$ as given by the three partition classes of $K_{n,n,n}$. We obtain an $\frac{\varepsilon}{10}$-regular equipartition $V = V_0\dot\cup V_1\dot\cup \dots \dot\cup V_{3k}$ refining this prepartition such that $k_0 \leq 3k \leq k_1$. Each cluster of this partition lies entirely in one of the partition classes of $K_{n,n,n}$. Let $\mathbb{K} = ([3k], E_{\mathbb{K}})$ be the graph that contains edges for all ε-regular cluster pairs that do not lie in the same partition class of $K_{n,n,n}$. Clearly, $\mathbb{K}$ is a tripartite graph. Furthermore, there are less than εk^2 pairs (V_i, V_j) in our regular partition that are not $\frac{\varepsilon}{10}$-regular in G. It follows that at most $2\sqrt{\varepsilon}k$ clusters V_i are contained in more than $\sqrt{\varepsilon}k$ irregular pairs. We move all these clusters and possibly up to $6\sqrt{\varepsilon}k$ additional clusters to the exceptional set V_0. The additional clusters are chosen in such a way that we obtain in each partition class of $\mathbb{K}$ the same number of clusters. We call the resulting exceptional set V_0' and denote the remaining clusters by $V_1'\dot\cup \dots \dot\cup V_{3k'}'$ and the corresponding subgraph of $\mathbb{K}$ by $\mathbb{K}'$. Observe that $k' \geq (1-3\sqrt{\varepsilon})k$. Because each remaining cluster forms an irregular pair with at most $\sqrt{\varepsilon}k \leq 2\sqrt{\varepsilon}k' \leq \eta' k'$ of the remaining clusters we conclude that $\mathbb{K}'$ is a graph from $\mathcal{K}_{k'}^{\eta}$. In addition, it easily follows from the definition of ε-regularity that each pair (V_i, V_j) with i, $j \in [k']$ which is ε-regular in G is also ε-regular in R. This motivates the following "majority" colouring of $\mathbb{K}'$: We colour the edges ij of $\mathbb{K}'$ by green if the ε-regular pair (V_i, V_j) has density at least $\frac{1}{2}$ and by red otherwise. In this way we obtain a coloured graph $\mathbb{K}_c' \in \mathcal{K}_{k'}^{\eta}$.

Now we are in a position to apply Lemma 8.7 to $\mathbb{K}_c'$. This lemma asserts that $\mathbb{K}_c'$ is either $(1-\eta')\frac{3}{4}k'$-odd or $\big((1-\eta')k', (1-\eta')\frac{3}{2}k', 3\big)$-good. By definition this means that in one of the colours of $\mathbb{K}_c'$, say in green, we either have

(O) an odd connected matching $\mathbb{M}_o$ of size $m_1 \geq (1-\eta')\frac{3}{4}k' \geq (1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{4}k$,

(G) or a connected matching $\mathbb{M}$ of size $m_2 \geq (1-\eta')k' \geq (1-\eta')(1-3\sqrt{\varepsilon})k$ together with a connected fork system $\mathbb{F}$ of size $f \geq (1-\eta')\frac{3}{2}k' \geq (1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{2}k$ and ratio 3.

In the following we use the matchings and fork systems we just obtained to show that we can embed all trees of $\mathcal{T}_t^\Delta$ in the corresponding system of regular pairs. For this purpose let $\mathbb{G}$ be the graph on vertex set $[3k]$ that contains precisely all green edges of $\mathbb{K}_c'$. Observe that $\mathbb{G}$ is an $(\varepsilon, 1/2)$-reduced graph for G.

Let $T \in \mathcal{T}_t^\Delta$ be a tree of order $t \leq (3-\mu)n/2$ and with maximal degree $\Delta(T) \leq n^\alpha$. Now we distinguish two cases, depending on whether we obtained configuration (O) or configuration (G) from Lemma 8.7. In both cases we plan to appeal to Lemma 8.13 to show that T has

$$\text{an } (\varepsilon\tfrac{n}{k})\text{-cut } C \text{ with } |C| \leq \varepsilon\tfrac{n}{k} \text{ and a } (\tfrac{1}{2}\mu', (1-\varepsilon)\tfrac{3n}{3k})\text{-valid assignment to } \mathbb{G}. \tag{8.4}$$

Recall that we fed constants ε, $\mu' > 0$ and $3k$ into this lemma. Assume first that we are in configuration (O). Because $m_1 \geq (1-\eta')(1-2\sqrt{\varepsilon})\frac{3}{4}k$ we have

$$t \leq (3-\mu)\frac{n}{2} \leq 3(1-\tfrac{\mu}{3})\frac{n}{2}\frac{m_1}{(1-\eta')(1-3\sqrt{\varepsilon})\frac{3}{4}k} \overset{(8.1),(8.2)}{\leq} (1-\mu')2m_1 \cdot \frac{3n}{3k}\,.$$

Hence by (M) of Lemma 8.13 applied with the matching $\mathbb{M}_o$ (with n replaced by $\tilde{n} := 3n$ and k replaced by $\tilde{k} := 3k$) we get (8.4) for T in this case, as $\Delta(T) \leq n^\alpha \leq \tilde{n}^\alpha$.

If we are in configuration (G), on the other hand, then let $t_1 \geq t_2$ be the sizes of the two colour classes of T. We distinguish two cases, using the two different structures provided in (G). Assume first that $t_2 \leq \frac{t}{3}$. Then, we calculate similarly as above that

$$t_2 \leq \tfrac{1}{3}t \leq (1-\tfrac{1}{3}\mu)\tfrac{n}{2} \leq \tfrac{1}{3}(1-\mu')f\tfrac{3n}{3k}\,, \qquad \text{and} \qquad t_1 \leq t \leq (1-\mu')f\tfrac{3n}{3k}\,.$$

Otherwise, if $t_2 \geq \frac{t}{3}$ then, similarly,

$$t_2 \leq t_1 \leq \tfrac{2}{3}t \leq (1-\tfrac{\mu}{3})n \leq (1-\mu')m_2 \cdot \tfrac{3n}{3k}\,.$$

Consequently, in both cases we can appeal to (F) of Lemma 8.13, in the first case applied to $\mathbb{F}$ and in the second to $\mathbb{M}$. We obtain (8.4) for T as desired.

We finish our proof with an application of the main embedding lemma, Lemma 8.12. As remarked earlier $\mathbb{G}$ is an $(\varepsilon, 1/2)$-reduced graph for G. We further have (8.4). For applying Lemma 8.12 it thus remains to check that $(\frac{1}{2}\cdot\frac{1}{10}\varrho - 10\varepsilon)\frac{n}{k} \geq S + |C| + \Delta$ with $\varrho = \frac{1}{2}\mu'$, $S = \varepsilon\frac{n}{k}$, $|C| \leq \varepsilon\frac{n}{k}$, and $\Delta \leq n^\alpha$. Indeed,

$$\left(\tfrac{1}{20}\varrho - 10\varepsilon\right)\tfrac{n}{k} = \left(\tfrac{1}{20}\cdot\tfrac{1}{2}\mu' - 10\varepsilon\right)\tfrac{n}{k} \overset{(8.2)}{\geq} 3\varepsilon\tfrac{n}{k} \overset{(8.3)}{\geq} \varepsilon\tfrac{n}{k} + \varepsilon\tfrac{n}{k} + n^\alpha\,.$$

So Lemma 8.12 ensures that $T \subseteq G$, i. e., there is an embedding of T in the subgraph induced by the green edges in $K_{n,n,n}$. □

8.4 Valid Assignments

In this section we will provide a proof for Lemma 8.13. The idea is as follows. Given a tree T and a graph G with reduced graph $\mathbb{G}$ we first construct a cut of T that provides us with a collection of small shrubs (see Lemma 8.14). Then we distribute these shrubs to edges of the given matching or fork-system in $\mathbb{G}$ (see Lemmas 8.15 and 8.16). Finally, we slightly modify this assignment in order to obtain a homomorphism from T to $\mathbb{G}$ that satisfies the conditions required for a valid assignment (see Lemma 8.18).

Lemma 8.14. *For every $S \in \mathbb{N}$ and for any tree T there is an S-cut of T that has size at most $\frac{|V(T)|}{S}$.*

Proof. To prove Lemma 8.14 we need the following fact.

Fact 1. *For any $S \in \mathbb{N}$ and any tree T with $|V(T)| > S$, there is a vertex $x \in V(T)$ such that the following holds. If F_x is the forest consisting of all components of $T - x$ with size at most S, then $|V(F_x)| + 1 > S$.*

To see this, root the tree T at an arbitrary vertex x_0. If x_0 does not have the required property, it follows from $|V(T)| > S$ that there exists a component T_1 in $T - x_0$ with $|V(T_1)| > S$. Set $x_1 := N(x_0) \cap V(T_1)$. Let $F(T_1 - x_1)$ be the forest consisting of the components of $T_1 - x_1$ that have size at most S. Observe that $F(T_1 - x_1)$ is a subgraph of F_{x_1}. So if $|F(T_1 - x_1)| + 1 > S$, then x_1 has the property required by Fact 1. Otherwise there exists a component T_2 in $T_1 - x_1$ of size larger than S. Observe that T_2 is also a component of $T - x_1$. Now repeat the procedure just described by setting $x_2 = N(x_1) \cap V(T_2)$ and so on, i.e., more generally we obtain trees T_i and vertices $x_i = N(x_{i-1}) \cap V(T_i)$. As the size of T_i decreases as i increases, there must be an x_i with the property required by Fact 1.

Now we prove Lemma 8.14. Set $C = \emptyset$. Repeat the following process until it stops. Choose a component T' of $T - C$ with size larger than S. Apply Fact 1 to T' and obtain a cut vertex x of T' together with a forest F_x consisting of components of $T' - x$ that have size at most S and is such that $|V(F_x) \cup \{x\}| > S$. Add x to C and repeat unless there is no component of size larger than S in $T - C$. As $|V(T - C)|$ decreases this process stops. Observe that then C is an S-cut. By the choice of C we obtain

$$|V(T)| = \sum_{x \in C} |V(F_x) \cup \{x\}| > |C| \cdot S,$$

which implies the required bound on the size of C. □

After Lemma 8.14 provided us with a cut and some corresponding shrubs we will distribute each of these shrubs T_i to an edge e of the odd matching or the fork system in the reduced graph by assigning one colour class of T_i to one end of e and the other colour class to the other end. Here our goal is to distribute the shrubs and their vertices in such a way that no cluster receives too many vertices. The next two lemmas guarantee that this can be done. Lemma 8.15 takes care of the distribution of the shrubs to the clusters of a matching M and Lemma 8.16 to those of a fork system F. We provide Lemmas 8.15 and 8.16 with numbers $a_{i,1}$ and $a_{i,2}$ as input. These numbers represent the sizes of the colour classes $A_{i,1}$ and $A_{i,2}$ of the shrub T_i. Since we do not need any other information about the shrubs in these lemmas the shrubs T_i do not explicitly appear in their statement. Both lemmas then produces a mapping φ representing the assignment of the colour classes $A_{i,1}$ and $A_{i,2}$ to the clusters of M or F.

Lemma 8.15. *Let $\{a_{i,j}\}_{i \in [s],\, j \in [2]}$ be natural numbers with sum at most t and $a_{i,1} + a_{i,2} \leq S$ for all $i \in [s]$, and let M be a matching on vertices $V(M)$. Then there is a mapping $\varphi\colon [s] \times [2] \to V(M)$ such that $\varphi(i,1)\varphi(i,2) \in M$ for all $i \in [s]$ and*

$$\sum_{(i,j) \in \varphi^{-1}(v)} a_{i,j} \leq \frac{t}{2|M|} + 2S \qquad \text{for all } v \in V(M). \tag{8.5}$$

Proof. A simple greedy construction gives the mapping φ: We consider the numbers $a_{i,j}$ as weights that are distributed, first among the edges, and then among the vertices of M. For this purpose greedily assign pairs $(a_{i,1}, a_{i,2})$ to the edges of M, in each step choosing an edge with minimum total weight. Then, clearly, no edge receives weight more than $S + t/|M|$. In a second round, do the following for each edge vw of M. For the pairs $(a_{i,1}, a_{i,2})$ that were assigned to e, greedily assign one of the weights of this pair to v and the other one to w, such that the total weight on v and on w are as equal as possible. Hence each of these vertices receives weight at most $\frac{1}{2}(S + t/|M|) + S$ and so the mapping φ corresponding to this weight distribution satisfies the desired properties. □

Lemma 8.16. *Let $\{a_{i,1}\}_{i\in[s]}$ and $\{a_{i,2}\}_{i\in[s]}$ be natural numbers with sum at most t_1 and t_2, respectively. Let $S \le t_1 + t_2 =: t$ and assume that $a_{i,1} + a_{i,2} \le S$ for all $i \in [s]$. Let F be a fork system with ratio at most r and partition classes $V_1(F)$ and $V_2(F)$ where $|V_1(F)| \ge |V_2(F)|$. Then there is a mapping $\varphi\colon [s] \times [2] \to V_1(F) \cup V_2(F)$ such that $\varphi(i,1)\varphi(i,2) \in F$ and $\varphi(i,j) \in V_j(F)$ for all $i \in [s], j \in [2]$ satisfying that for all $v_1 \in V_1(F)$, $v_2 \in V_2(F)$ we have*

$$\sum_{(i,1)\in\varphi^{-1}(v_1)} a_{i,1} \le \frac{t_1}{|F|} + \sqrt{12tS|F|} \qquad \text{and} \qquad \sum_{(i,2)\in\varphi^{-1}(v_2)} a_{i,2} \le \frac{rt_2}{|F|} + \sqrt{12tS|F|}\,. \tag{8.6}$$

In the proof of this lemma we will make use the so-called Hoeffding bound for sums of independent random variables provided by Theorem 2.5.

Proof of Lemma 8.16. For showing this lemma we use a probabilistic argument and again consider the $a_{i,j}$ as weights which are distributed among the vertices of F.

Observe first that we can assume without loss of generality that for all but at most one $i \in [s]$ we have $\frac{1}{2}S \le a_{i,1} + a_{i,2}$ since otherwise we can group weights $a_{i,1}$ together, and also group the corresponding $a_{i,2}$ together, such that this condition is satisfied and continue with these grouped weights. This in turn implies, that $s \le (2t/S) + 1 \le 3t/S$.

We start by assigning weights $a_{i,1}$ to vertices of $V_1(F)$ by (randomly) constructing a mapping $\varphi_1\colon [s] \times [1] \to V_1(F)$. To this end, independently and uniformly at random choose for each $i \in [s]$ an image $\varphi_1(i,1)$ in $V_1(F)$. Clearly, there is a unique way of extending such a mapping φ_1 to a mapping $\varphi\colon [s] \times [2] \to V_1(F) \cup V_2(F)$ satisfying $\varphi(i,1)\varphi(i,2) \in F$. We claim that the probability that φ_1 gives rise to a mapping φ which satisfies the assertions of the lemma is positive.

Indeed, for any fixed vertex $v = v_1 \in V_1(F)$ or $v = v_2 \in V_2(F)$ let $\sigma(v)$ be the event that the mapping φ does not satisfy (8.6) for v. We will show that $\sigma(v)$ occurs with probability strictly less than $1/(2|F|)$, which clearly implies the claim above. We first consider the case $v = v_1 \in V_1(F)$. For each $i \in [s]$ let $\mathbb{1}_i$ be the indicator variable for the event $\varphi(i,1) = v_1$ and define a random variable X_i by setting

$$X_i = \left(\mathbb{1}_i - \tfrac{1}{|F|}\right)\tfrac{a_{i,1}}{S}.$$

Observe that these variables are independent, and satisfy $\mathbb{E}\,X_i = 0$ and $|X_i| \le 1$ and so Theorem 2.5 applied with $a = \sqrt{12t|F|/S}$ asserts that

$$\mathbb{P}\Big[\sum_{i\in[s]} X_i > \sqrt{12t|F|/S}\Big] \le \exp\left(-\frac{12t|F|}{S\cdot 2s}\right) \le \exp\left(-2|F|\right) < \frac{1}{2|F|} \tag{8.7}$$

where we used $s \le 3t/S$. Now, by definition we have

$$X := \sum_{i\in[s]} X_i = \frac{1}{S} \sum_{(i,1)\in\varphi^{-1}(v_1)} a_{i,1} - \frac{t_1}{|F|S},$$

and so, if (8.6) did not hold for v_1, then we had $X > \sqrt{12t|F|/S}$, which by (8.7) occurs with probability less than $1/(2|F|)$.

For the case $v = v_2 \in V_2(F)$ we proceed similarly. Let $r' \le r$ be the number of prongs of the fork that contains v_2. We define indicator variables $\mathbb{1}'_i$ for the events $\varphi(i,2) = v_2$ for $i \in [s]$ and random variables

$$Y_i = \left(\mathbb{1}'_i - \tfrac{r'}{|F|}\right)\tfrac{a_{i,2}}{S}.$$

with $\mathbb{E}\, Y_i = 0$ and $|Y_i| \leq 1$. The rest of the argument showing that $\sigma(v_2)$ occurs with probability strictly less than $1/(2|F|)$ is completely analogous to the case $v = v_1$ above. With this we are done. □

As explained earlier these two previous lemmas will allow us to assign the shrubs of a tree T to edges of a reduced graph $\mathbb{G}$. By applying them we will obtain a mapping ψ from the vertices of T to those of $\mathbb{G}$ that is a homomorphism when restricted to the shrubs of T. The following lemma transforms such a ψ to a homomorphism h from the whole tree T to $\mathbb{G}$ that "almost" coincides with ψ provided the structures of T and $\mathbb{G}$ are "compatible" with respect to ψ in the sense of the following definition.

Definition 8.17 (walk condition)**.** *Let T be a tree and $C \subseteq V(T)$. A mapping $\psi\colon V(T)\setminus C \to \mathbb{G}$ satisfies the* walk condition *if for any $x, y \in V(T) \setminus C$ such that there is a path $P_{x,y}$ from x to y whose internal vertices are all in C there is a walk $\mathbb{P}_{x,y}$ between $\psi(x)$ and $\psi(y)$ in $\mathbb{G}$ such that the length of $P_{x,y}$ and the length of $\mathbb{P}_{x,y}$ have the same parity.*

Lemma 8.18. *Let T be a tree with maximal degree Δ, let C be a cut of T, and let $\mathbb{G}$ be a graph on k vertices. Let $\psi\colon V(T) \setminus C \to V(\mathbb{G})$ be a homomorphism that maps each shrub of T corresponding to C to an edge of $\mathbb{G}$ and that satisfies the walk condition. Then there is a homomorphism $h\colon V(T) \to V(\mathbb{G})$ satisfying*

(h1) *$|h(N_T(x))| \leq 2$ for all vertices $x \in V(T)$ and*
(h2) *$|\{x \in V(T) :\ h(x) \neq \psi(x)\}| \leq 3|C|\Delta^{2k+1}$.*

Observe that Property (h1) in this lemma asserts that images of neighbours of any vertex in T occupy at most two vertices in $\mathbb{G}$. By assumption, this is clearly true for ψ but we need to make sure that h inherits this feature. Property (h2) on the other hand states that h and ψ do not differ much. The assumption that ψ satisfies the walk condition is essential for the construction of the homomorphism h.

Proof of Lemma 8.18. We start with some definitions. Choose a non-empty shrub corresponding to C in T and call it *shrub* 1. Then choose a cut-vertex $x_0^* \in C$ adjacent to this shrub. We consider x_0^* as the *root* of the tree T. This naturally induces the following partial order $\prec$ on the vertices $V(T)$ of T: For vertices $x, y \in V(T)$ we have $x \prec y$ iff y is a descendant of x in the tree T with root x_0^*. Note that x_0^* is the unique minimal element of $\prec$ and the leaves of T are its maximal elements. Further, for $x \in C$ set $W_x := \{z \in V(T) :\ \mathrm{dist}_T(x, z) \leq 2k+1 \ \&\ x \prec z\}$ and let $W = C \cup \bigcup_{x\in C} W_x$. Observe that the bound on the maximal degree of T implies that $|W| \leq 2\Delta^{2k+1}|C| + |C| \leq 3\Delta^{2k+1}|C|$. For $x \in V(T) \setminus W$, we set $h(x) := \varphi(x)$. This ensures that Condition (h2) is fulfilled. In addition the following fact holds because ψ maps each shrub to an edge of $\mathbb{G}$.

Fact 1. *The mapping h restricted to $V(T) \setminus W$ is a homomorphism. For all vertices $x \in V(T) \setminus W$ all children y of x that are not cut-vertices have the same $h(y)$.*

We shall extend h to the set W. Our strategy is roughly as follows: We start by defining $h(x_0^*)$ for the root cut-vertex x_0^* in a suitable way. Recall that all children of x_0^* are contained in W. Then, we let h map all non-cut-vertex children $y \in N_T(x_0^*) \setminus C$ of x_0^* to a suitable neighbour of $h(x_0^*)$ in $\mathbb{G}$ and do the following for each of these y. Observe that y is the root of some shrub, which we will call the *shrub of y*. Now, $h(y)$ and $\psi(y)$ might be different. However, we will argue that there is a walk of even length $m \leq 2k$ between $h(y)$ and $\psi(y)$. Then we will define h for all vertices $y' \in W_{x_0^*}$ contained in the shrub of y and with distance at most m from y. More precisely we will use the walk of length m between $h(y)$ and $\psi(y)$ and let h map all y' with distance i to y to the i-th vertex of this walk. All vertices z in the shrub of y for which h is still undefined after these steps are then mapped to $h(z) := \psi(z)$. Once this has been done for all $y \in N_T(x_0^*) \setminus C$ we proceed in the same way with the next cut-vertex: We choose a cut-vertex x^* with parent x for which $h(x)$ is already defined and proceed similarly for x^* as we did for x_0^*.

We now make the procedure for the extension of h on W precise. Throughout this procedure we will assert the following property for all non-cut vertices y of T such that $h(y)$ is defined.

$$\text{There is a path of even length in } \mathbb{G} \text{ between } h(y) \text{ and } \psi(y). \tag{8.8}$$

Observe that (8.8) trivially holds for all $y \in V(T) \setminus W$.

As explained, we start our procedure with the root x_0^* of the tree T. Let x_1 be the root of shrub 1. By definition x_1 is adjacent to x_0^*. Note that, while ψ is not defined on x_0^* it is defined on x_1. Hence we can legitimately set $h(y) = \psi(x_1)$ for all neighbours $y \notin C$ of x_0^* in T and choose $h(x_0^*)$ arbitrarily in $N_{\mathbb{G}}(\psi(x_1))$. Observe that this is consistent with (8.8) because for any neighbour $y \notin C$ of x_0^* we have $h(y) = \psi(x_1)$ and $\mathrm{dist}_T(y, x_1) \in \{0, 2\}$. By assumption ψ satisfies the walk condition. Hence there is a walk in $\mathbb{G}$ with even length between $h(y) = \psi(x_1)$ and $\psi(y)$. Let $\mathbb{P}_y = v_0, v_1, \ldots, v_m$ be a walk in $\mathbb{G}$ of minimal but even length with $v_0 = h(y)$ and $v_m = \psi(y)$. As $\mathbb{G}$ has k vertices we have that $m \leq 2k$. For all vertices $z \in W$ that are in the shrub of y and satisfy $\mathrm{dist}_T(y, z) = j$ for some $j \leq m$, we then define $h(z) := v_j$. For the remaining vertices $z \in W$ in the shrub of y we set $h(z) := \psi(z)$. Observe that this is again consistent with (8.8) and in conjunction with Fact 1 implies the following condition (which we will also guarantee throughout the whole process of defining h).

Fact 2. *Let $x^* \in C$ and $y \notin C$ such that $h(x^*)$ and $h(y)$ are defined. Then the following holds:*

(i) *All children $y' \notin C$ of x^* have the same $h(y')$ and $h(x^*)h(y') \in E(\mathbb{G})$.*

(ii) *All children $y' \notin C$ of y have the same $h(y')$ and $h(y)h(y') \in E(\mathbb{G})$.*

In this way we have defined h for all shrubs adjacent to the root x_0^*.

Next we consider any vertex $x^* \in C \cap N_T(x_0^*)$ and set $h(x^*) := h(x_1)$, where x_1 is as defined above. We let z^* be the parent of x^*, i.e., $z^* = x_0^*$. Then set $h(y) := h(z^*)$ for all children $y \notin C$ of x^*. This is consistent with Fact 2. Afterwards we have the following situation: x^* and $z^* = x_0^*$ are neighbouring cut-vertices and the vertex x_1 is a non-cut-vertex neighbour of x_0^*. Let $y \in N_T(x^*) \setminus C$. Then we have $\mathrm{dist}_T(x_1, y) = 3$. Because y and x_1 are both non-cut vertices the properties of ψ imply as before that there is a walk in $\mathbb{G}$ of odd length between $\psi(x_1)$ and $\psi(y)$. By the walk condition and the facts that $h(x_1) = \psi(x_1)$ and $h(x_0^*) = h(y)$, we know that in $\mathbb{G}$ there is a walk $\mathbb{P}_y$ of even length $m \leq 2k$ between $h(y)$ and $\psi(y)$. This verifies (8.8) for y. We thus can define h for the vertices z contained in the shrub of y as above: if $\mathrm{dist}_T(y, z) \leq m$ then we use this path and set $h(z)$ according to $\mathrm{dist}_T(y, z)$ and otherwise we set $h(z) := \psi(z)$. With this we stay consistent with (8.8) and Fact 2. We then repeat the above procedure for all $x^* \in C \cap N_T(x_0^*)$ which implies that the next fact holds true.

Fact 3. *All vertices $x \in N_T(x_0^*)$ have the same $h(x)$.*

Now we are in the following situation.

Fact 4. *The mapping h is defined on all shrubs adjacent to cut vertices x^* with $h(x^*)$ defined. Moreover, for each cut vertex x^* with $h(x^*)$ undefined that has a parent z for which $h(z)$ is defined, then z has a parent z' with $h(z')$ defined and $h(z)h(z')$ is an edge of $\mathbb{G}$.*

As long as h is not defined for all $z \in V(T)$ we then repeat the following. We choose a cut vertex x^* with $h(x^*)$ undefined that is minimal with respect to this property in $\prec$. Denote the parent of x^* by z and let z' be the parent of z. Then, by Fact 4, the mapping h has already been defined for z' and z. Set $h(x^*) := h(z')$ and for all children $y \notin C$ of x^* set $h(y) := h(z)$. Because $h(z')h(z)$ is an edge of $\mathbb{G}$ by Fact 4 this gives the following property for x^* (which we, again, guarantee throughout the definition of h).

Fact 5. *For all cut vertices $x^* \in C$ with $h(x^*)$ defined we have that $h(x^*)h(z)$ is an edge of $\mathbb{G}$, where z is the parent of x^*. Moreover if $x^* \notin \{x_0\} \cup (C \cap N_T(x_0^*))$, we have that $h(x^*) = h(z')$, where z' is the parent of z.*

Moreover, the definition of $h(y)$ is consistent with (8.8), i.e. there is a path of even length in $\mathbb{G}$ between $h(y)$ and $\psi(y)$ for all children $y \notin C$ of x^*. Accordingly we can again define h for the vertices in the shrub of y as before, using this path.

This finishes the description of the definition of h. It remains to verify that h is a homomorphism and satisfies Condition (h1). For the first part it suffices to check that for any $y \in V(T) \setminus \{x_0^*\}$ with parent x we have $h(y) \in N_{\mathbb{G}}(h(x))$. If y is a vertex in some shrub then Facts 2(i) and 2(ii) imply that $h(x)h(y)$ is an edge of $\mathbb{G}$. If y is a cut-vertex, on the other hand, Fact 5 implies that $h(x)h(y)$ is an edge of $\mathbb{G}$. So h is a homomorphism.

Further, by Fact 2(i) and (ii) we get for all vertices x of T that all children $x' \notin C$ of x have the same $h(x')$. By Fact 5, if $x \neq x_0^*$ then all children $x' \in C$ of x and the parent z of x have the same $h(x') = h(z')$. Together wit Fact 3, this implies Property (h1). □

Now we are ready to prove Lemma 8.13.

Proof of Lemma 8.13. Given $\varepsilon, \mu > 0$ with $\varepsilon \leq \mu/10$ and $k \in \mathbb{N}$ we set α, n_0 and an auxiliary constant $\beta > 0$ such that

$$\alpha \cdot (2k+1) = \tfrac{1}{2}, \qquad \beta = \varepsilon\mu/(500k^3), \quad \text{and} \quad n_0 = (1500k/(\varepsilon\mu))^4. \tag{8.9}$$

Let $\mathbb{G}$ be a graph of order k that has an odd connected matching $\mathbb{M}$ of size at least m or a fork system $\mathbb{F}$ of size at least f and ratio r. Let T be a tree satisfying the respective conditions of Case (M) or (F) and let V_1 and V_2 denote the two partition classes of T with $t_1 = |V_1| \geq |V_2| = t_2$. We first construct an S-cut C for T with $S := \beta n \leq \varepsilon \frac{n}{k}$. Lemma 8.14 asserts that there is such a cut C with

$$|C| \leq \frac{|V(T)|}{S} \leq \frac{(1-\mu)2k\frac{n}{k}}{\beta n} \overset{(8.9)}{\leq} \frac{1000k^3}{\varepsilon\mu} \overset{(8.9)}{\leq} \varepsilon\frac{n}{k}. \tag{8.10}$$

Let $T_1, \ldots, T_s$ be the shrubs of T corresponding to the cut C. We now distinguish whether we are in Case (M) or (F) of the lemma. In both cases we will construct a mapping ψ that is a homomorphism from $T - C$ to either $\mathbb{M}$ or $\mathbb{F}$ and satisfies the walk condition. After this case distinction the mapping ψ will serve as input for Lemma 8.18 which we then use to finish this proof.

Case (M) : In this case we apply Lemma 8.15 in order to obtain an assignment of the shrubs to matching edges of $\mathbb{M}$ as follows. Set $a_{i,j} := |V(T_i) \cap V_j|$ for all $i \in [s]$, $j \in [2]$. This implies that $\sum_{i,j} a_{i,j} \leq |V(T)| \leq t = (1-\mu)2m\frac{n}{k}$ and, because C is an S-cut, that $a_{i,1} + a_{i,2} \leq S$ for all $i \in [s]$. Accordingly Lemma 8.15 produces a mapping $\varphi : [s] \times [2] \to V(\mathbb{M})$ satisfying $\varphi(i,1)\varphi(i,2) \in \mathbb{M}$ and (8.5).

We now use φ to construct a mapping $\psi\colon T \setminus C \to V(\mathbb{M})$. Set $\psi(v) := \varphi(a_{i,j})$ for all $v \in V(T_i) \cap V_j$. Note that this definition together with (8.5) gives

$$|\psi^{-1}(\ell)| \leq \frac{t}{2m} + 2S \leq (1-\mu)\frac{n}{k} + 2\beta n \tag{8.11}$$

for all vertices ℓ of $\mathbb{M}$. Each edge of $T - C$ lies in some shrub T_i, $i \in [s]$ and as the mapping φ sends each shrub T_i to an edge of $\mathbb{M}$, the mapping ψ is a homomorphism from $T - C$ to $\mathbb{M}$. Moreover, as $\mathbb{M}$ is an odd connected matching, for any pairs of vertices $\ell, \ell' \in V(\mathbb{M})$ there is as well an even as also an odd walk in $\mathbb{G}$ between ℓ and ℓ'. Thus ψ satisfies the walk condition.

Case (F): In this case we apply Lemma 8.16 in order to obtain an assignment of the shrubs corresponding to C to edges of $\mathbb{F}$. For this application we use parameters $t_1 = |V_1|$, $t_2 = |V_2|$ and $a_{i,j} := |V(T_i) \cap V_j|$ for all $i \in [s]$, $j \in [2]$. It follows that $\sum_i a_{i,1} = t_1$ and $\sum_i a_{i,2} = t_2$. Because C is an S-cut, we further have $a_{i,1} + a_{i,2} \leq S$ for all $i \in [s]$. Accordingly Lemma 8.16 produces a mapping $\varphi : [s] \times [2] \to V(\mathbb{F})$ satisfying $\varphi(i,1)\varphi(i,2) \in \mathbb{F}$ and (8.6).

Again, we use φ to construct the mapping $\psi\colon T \setminus C \to V(\mathbb{F})$ by setting $\psi(v) := \varphi(a_{i,j})$ for all $v \in V(T_i) \cap V_j$. By assumption we have $t_1 \leq t' = (1-\mu)f\frac{n}{k}$ and $t_2 \leq \frac{t'}{r} = (1-\mu)f\frac{n}{rk}$ and hence $t_1 + t_2 \leq (1-\mu)f\frac{n}{k}(1+\frac{1}{r})$. Together with (8.6) this implies for all vertices $\ell_1 \in V_1(\mathbb{F})$ and $\ell_2 \in V_2(\mathbb{F})$ that

$$\begin{aligned} |\psi^{-1}(\ell_1)| &\leq \frac{(1-\mu)f\frac{n}{k}}{f} + \sqrt{12(1-\mu)f\tfrac{n}{k}(1+\tfrac{1}{r})Sf} \\ &\leq (1-\mu)\tfrac{n}{k} + 2fn\sqrt{6\beta/k}\,, \end{aligned} \tag{8.12}$$

and similarly

$$|\psi^{-1}(\ell_2)| \leq \frac{r(1-\mu)f\frac{n}{rk}}{f} + 2fn\sqrt{6\beta/k} \leq (1-\mu)\frac{n}{k} + 2fn\sqrt{6\beta/k}\,. \tag{8.13}$$

Putting (8.12) and (8.13) together, we conclude for any $\ell \in V(\mathbb{F})$ that

$$|\psi^{-1}(\ell)| \leq (1-\mu)\tfrac{n}{k} + 2fn\sqrt{6\beta/k} \leq (1-\mu)\tfrac{n}{k} + 2n\sqrt{6\beta k}\,. \tag{8.14}$$

As before it is easy to see that the mapping ψ is a homomorphism from $T - C$ to $\mathbb{F}$. Moreover, as $\mathbb{F}$ is a fork system, there is an even walk between any two vertices $\ell, \ell' \in V_1(\mathbb{F})$ and between any two vertices $\ell, \ell' \in V_2(\mathbb{F})$. Because ψ maps vertices of $V_1(T)$ to $V_1(\mathbb{F})$ and vertices of $V_2(T)$ to vertices of $V_2(\mathbb{F})$, the mapping ψ also satisfies the walk condition in this case.

Applying Lemma 8.18: In both Cases (M) and (F) we now apply Lemma 8.18 in order to transform ψ into a homomorphism from the whole tree T to $\mathbb{G}$. With input T, $\Delta := n^\alpha$, C, $\mathbb{G}$, and ψ this lemma produces a homomorphism $h : V(T) \to V(\mathbb{G})$ satisfying (h1) and (h2). We claim that h is the desired $(\mu/2, (1-\varepsilon)\frac{n}{k})$-valid assignment.

Indeed, h is a homomorphism and so we have Condition 1 of Definition 8.10. Condition 2 follows from (h1). To check Condition 3 let ℓ be any vertex of $\mathbb{G}$. We need to verify that $|h^{-1}(\ell)| \leq (1-\frac{1}{2}\mu)(1-\varepsilon)\frac{n}{k}$. By (h2) we have $|h^{-1}(\ell)| \leq |\psi^{-1}(\ell)| + 3|C|\Delta^{2k+1}$. Because $|C| \leq 1000k/(\varepsilon\mu)$ by (8.10) and $\Delta^{2k+1} = n^{\alpha\cdot(2k+1)} = \sqrt{n}$ by (8.9) we infer that

$$\begin{aligned} |h^{-1}(\ell)| &\leq |\psi^{-1}(\ell)| + \frac{3000k}{\varepsilon\mu}\sqrt{n} \overset{(8.9)}{\leq} |\psi^{-1}(\ell)| + \beta n \\ &\overset{(8.11),(8.14)}{\leq} (1-\mu)\tfrac{n}{k} + \max\left\{2\beta n, 2n\sqrt{6\beta k}\right\} + \beta n \overset{(8.9)}{\leq} (1-\tfrac{1}{2}\mu)(1-\varepsilon)\tfrac{n}{k}, \end{aligned}$$

where in the last inequality we use that $\varepsilon \leq \mu/10$. □

8.5 Proof of the main embedding lemma

Our proof of Lemma 8.12 uses a greedy strategy for embedding the vertices of a tree with valid assignment into the given host graph.

Proof of Lemma 8.12. Let $V_0\dot\cup V_1\dot\cup\dots\dot\cup V_k$ be an (ε,d)-regular partition of G with reduced graph $\mathbb{G}$ and let T be a tree with $\Delta(T)\le\Delta$ and with a $(\varrho,(1-\varepsilon)\frac{n}{k})$-valid assignment h to $\mathbb{G}$. Further, let C be an S-cut of T, let $T_1,\dots,T_s$ be the shrubs of T corresponding to C, and assume that

$$(\tfrac{1}{10}d\varrho-10\varepsilon)\tfrac{n}{k}\ge |C|+S+\Delta\,. \tag{8.15}$$

As last preparation we arbitrarily divide each cluster $V_i=V_i'\dot\cup V_i^*$ into a set V_i' of size $(1-\frac{1}{2}\varrho)|V_i|$, which we will call *embedding space*, and the set of remaining vertices V_i^*, the so-called *connecting space*. Next we will first specify the order in which we embed the vertices of T into G, then describe the actual embedding procedure, and finally justify the correctness of this procedure.

Pick an arbitrary vertex $x_1^*\in C$ as root of T and order the cut vertices $C=\{x_1^*,\dots,x_c^*\}$, $c=|C|$ in such a way that on each $x_1^*-x_i^*$-path in T there are no x_j^* with $j>i$. Similarly, for each $i\in[s]$ let $t(i)$ denote the number of vertices in the shrub T_i and order the vertices $y_1,\dots,y_{t(i)}$ of T_i such that all paths in T_i starting at the root of T_i have solely ascending labels. For embedding T into G we process the cut vertices and shrubs according to these orderings, more precisely we first embed x_1^*, then all shrubs T_i that have x_1^* as parent, one after the other. For embedding T_i we embed its vertices in the order $y_1,\dots,y_{t(i)}$ defined above. Then we embed the next cut vertex x_2^* (which is a child of one of the shrubs embedded already or of x_1^*), then all child shrubs of x_2^*, and so on. Let $x_1,\dots,x_{|V(T)|}$ be the corresponding ordering of $V(T)$.

Before turning to the embedding procedure itself, observe that Property 2 of Definition 8.10 asserts the following fact. For a vertex x_j of T and for $i\in[k]$ let $N_i(x_j)$ be the set of neighbours $x_{j'}$ of x_j in T with $j'>j$ and $h(x_{j'})=i$.

Fact 1. *For all vertices x_j of T at most two sets $N_i(x_j)$ are non-empty.*

The idea for embedding T into G is as follows. We equip each vertex $x\in V(T)$ with a *candidate set* $V(x)\subseteq V_{h(x)}$ and from which x will choose its image in G. To start with, we set $V(x^*):=V^*_{h(x^*)}$ for all vertices $x^*\in C$ and $V(x):=V'_{h(x)}$ for all other vertices x. Cut vertices will be embedded to vertices in a connecting space and non-cut vertices to vertices in an embedding space. Then we will process the vertices of T in the order $x_1,\dots,x_{|V(T)|}$ defined above and embed them one by one. Whenever we embed a cut vertex x^* to a vertex v in this procedure we will set up so-called *reservoir sets* $R_i\subseteq V_i\cap N_G(v)$ for all (at most two) clusters V_i such that some child x of x^* is assigned to V_i, i.e., $h(x)=i$. Reservoir sets will be used for embedding the children of cut vertices. We (temporarily) remove the vertices in these reservoir sets from all other candidate sets but put them back after processing all child shrubs of x^*. This will ensure that we have enough space for embedding children of x^*, even after possibly embedding $\Delta-1$ child shrubs of x^*.

Now let us provide the details of the embedding procedure. Throughout, x^* will denote the cut vertex whose child-shrubs are currently processed. The set U will denote the vertices in G used so far; thus initialize this set to $U:=\emptyset$. As indicated above, initialize $V(x^*):=V^*_{h(x^*)}$ for all vertices $x^*\in C$ and $V(x):=V'_{h(x)}$ for $x\in V(T)\setminus C$, and set $R_i:=\emptyset$ for all $i\in[k]$. For constructing an embedding $f\colon V(T)\to V(G)$ of T into G, repeat the following steps:

1. Pick the next vertex x from $x_1, \dots, x_{|V(T)|}$.
2. Choose a vertex $v \in V(x) \setminus U$ that is typical with respect to $V(y) \setminus U$ for all unembedded $y \in N_T(x)$, set $f(x) = v$, and $U := U \cup \{v\}$.
3. For all unembedded $y \in N_T(x)$ set $V(y) := (V(y) \setminus U) \cap N_G(v)$.
4. If $x \in C$ then set $x^* := x$. Further, for all i with $N_i(x) \setminus C \neq \emptyset$ arbitrarily choose a reservoir set $R_i \subseteq (V_i' \setminus U) \cap N_G(v)$ of size $5\varepsilon\frac{n}{k} + \Delta$, set $V(y) := R_i$ for all $y \in N_i(x) \setminus C$, and (temporarily) remove R_i from all other candidate sets in V_i', i.e., set $V(y') := V(y') \setminus R_i$ for all $y' \in V(T) \setminus N_i(x)$.
5. After the vertices of all child shrubs of x^* are embedded put the vertices in R_i back to all candidate sets in V_i' for all $i \in [k]$, i.e., $V(y) := V(y) \cup R_i$ for all $y \in V(T) \setminus C$ with $h(y) = i$, and set $R_i := \emptyset$.

Steps 3 and 4 of this procedure guarantee for each vertex y with embedded parent x that the candidate set $V(y)$ is contained in $N_G(f(x))$. Accordingly, if we can argue that in Step 2 we can always choose an image v of x in $V(x)$ (and that we can choose the reservoir sets in Step 4) we indeed obtain an embedding f of T into G. To show this we first collect some observations that will be useful in the following analysis. The order of $V(T)$ guarantees that all child shrubs of a cut vertex are embedded before the next cut vertex. Notice that this implies the following fact (cf. Step 4 and Step 5).

Fact 2. For all $i \in [k]$, at any point in the procedure, the reservoir set R_i satisfies $|R_i| = 5\varepsilon\frac{n}{k} + \Delta$ if there is a neighbour x of the current cut-vertex x^* such that $h(x) = i$ and $|R_i| = 0$ otherwise. In addition no reservoir set gets changed before all child shrubs of x^* are embedded.

Further, since h is a $(\varrho, (1-\varepsilon)\frac{n}{k})$-valid assignment and only cut-vertices are embedded into connecting spaces V_i^*, we always have

$$|V_i' \cap U| \leq (1 - \tfrac{1}{2}\varrho)\tfrac{n}{k} \quad \text{and} \quad |V_i^* \cap U| \leq |C| \quad \text{for all } i \in [k]\,. \tag{8.16}$$

Now we check that Steps 2 and 4 can always be performed. To this end consider any iteration of the embedding procedure and suppose we are processing vertex x. We distinguish three cases.

Case 1: Assume that x is a cut-vertex. Then we had $V(x) = V^*_{h(x)}$ until the parent x' of x got embedded. In the iteration when x' got embedded then the set $V(x)$ shrunk to a set of size at least $(d-\varepsilon)|V^*_{h(x)} \setminus U|$ in Step 3 because $f(x')$ is typical with respect to $V^*_{h(x)} \setminus U$. No vertices embedded between x' and x (except for possible vertices in C) alter $V(x)$, and so by (8.16) we have

$$|V(x) \setminus U| \geq (d-\varepsilon)|V^*_{h(x)}| - |C| \geq (d-\varepsilon)\tfrac{1}{2}\varrho\tfrac{n}{k} - |C| \overset{(8.15)}{>} 4\varepsilon\tfrac{n}{k}$$

when we are about to choose $f(x)$. By Fact 1 at most two of the sets $N_i(x)$ are non-empty and each of these two sets can contain cut vertices y^* and non-cut vertices y'. We clearly have $V(y^*) = V_i^*$ and $V(y') = V_i'$ and so there are at most 4 different sets $V(y) \setminus U$, each of size at least $\frac{1}{2}\varrho\frac{n}{k} - |C| > \varepsilon\frac{n}{k}$ by (8.16) and (8.15), with respect to which we need to choose a typical $f(x)$. By Proposition 3.6 there are less than $4\varepsilon\frac{n}{k}$ vertices in $V(x) \setminus U$ (which is a subset of V_i) that do not fulfil this requirement. Hence we can choose $f(x)$ whenever $x \in C$. In addition, we can choose the reservoir sets in Step 4 of this iteration: Indeed, let i be such that $N_i(x) \setminus C \neq \emptyset$ and let $y \in N_i(x) \setminus C$ be a neighbour of x we wish to embed to V_i. In

Step 2, when we choose $f(x)$, then $V(y) = V'_i$ and so $f(x)$ is typical with respect to $V'_i \setminus U$. By Proposition 3.6 and (8.16) we thus have in Step 3 of this iteration that

$$|(V'_i \setminus U) \cap N_G(v)| \geq (d-\varepsilon)|V'_i \setminus U| \geq (d-\varepsilon)\tfrac{1}{2}\varrho\tfrac{n}{k} \overset{(8.15)}{\geq} 5\varepsilon\tfrac{n}{k} + \Delta.$$

Therefore we can choose R_i in Step 4.

Case 2: Assume that x is not in C but the child of a cut vertex x^*. Then $V(x) = R_{h(x)}$ before x gets embedded. Moreover, due to Step 4, R_i has been removed from all candidate sets besides those of the at most Δ neighbours of x^*. By Fact 2 we have $|R_{h(x)}| = 5\varepsilon\frac{n}{k} + \Delta$ and so we conclude that $|V(x) \setminus U| \geq 5\varepsilon\frac{n}{k} > 4\varepsilon\frac{n}{k}$. As in the previous case, there are at most four different sets $V(y) \setminus U$ for unembedded neighbours y of x, each of size at least $\frac{1}{2}\varrho\frac{n}{k} - |R_{h(y)}| = \frac{1}{2}\varrho\frac{n}{k} - 5\varepsilon\frac{n}{k} - \Delta \geq \varepsilon\frac{n}{k}$ by (8.15) and (8.16). Thus Proposition 3.6 guarantees that there is $v \in V(x) \setminus U$ which is typical with respect to all these sets $V(y) \setminus U$ and hence we can choose $f(x)$ in this case.

Case 3: As third and last case, let x be a vertex of some shrub T_j which is the child of a (non-cut) vertex x' of T_j. Until x' got embedded we had $V(x) = V'_{h(x)} \setminus R_{h(x)}$ and so, $v' = f(x')$ was chosen typical with respect to $V'_{h(x)} \setminus (R_{h(x)} \cup U)$ where U is the set of used vertices in G at the time when x' got embedded. In the corresponding iteration $V(x)$ shrunk to $(V'_{h(x)} \setminus (R_{h(x)} \cup U)) \cap N_G(v')$. This together with (8.16) implies that immediately after this shrinking we had

$$|V(x) \setminus U| \geq (d-\varepsilon)(\tfrac{1}{2}\varrho\tfrac{n}{k} - |R_{h(x)}|) \geq (d-\varepsilon)(\tfrac{1}{2}\varrho\tfrac{n}{k} - 5\varepsilon\tfrac{n}{k} - \Delta) \overset{(8.15)}{>} 4\varepsilon\tfrac{n}{k} + |T_j|.$$

By construction only vertices from T_j come between x' and x in the order of $V(T)$ and so when we want to embed x in the procedure above we still have $|V(x) \setminus U| > 4\varepsilon\frac{n}{k}$ where U now is the set of vertices used until the embedding of x. Similarly as in the other two cases there are at most four different types of candidate sets for non-embedded neighbours of x, all of these have more than $\varepsilon\frac{n}{k}$ vertices and so Proposition 3.6 allows us to choose an $f(x) \in V(x) \setminus U$ typical with respect to these sets. This concludes the case distinction and hence the proof of correctness of our embedding procedure. □

8.6 Coloured tripartite graphs are either good or odd

In this section we provide the proofs of Lemma 8.8 and Lemma 8.9. We start with a collection of simple propositions that will turn out useful in these proofs.

Our first two observations are about matchings in η-complete graphs. The following proposition states that a bipartite η-complete coloured graph contains a reasonably big matching in one of the two colours.

Proposition 8.19. *Let K be a coloured graph on n vertices and let D and D' be vertex sets of size at least m in K. If $K[D, D']$ is η-complete then $K[D, D']$ contains a matching M either in red or in green of size at least $\frac{m}{2} - \eta n$.*

Proof. Assume without loss of generality that $|D| \leq |D'|$. Colour a vertex $v \in D$ with red if it has more red-neighbours than green-neighbours in $K[D, D']$ and with green otherwise. By the pigeon-hole principle there is a set $X \subseteq D$ of size $\frac{1}{2}|D|$ such that all vertices in X have the same colour, say red. But then each vertex in X has at least $\frac{1}{2}|D'| - \eta n \geq |X| - \eta n$

red-neighbours in D'. Accordingly we can greedily construct a red matching of size at least $|X| - \eta n \geq \frac{m}{2} - \eta n$ between X and D'. □

The next proposition gives a sufficient condition for the existence of an almost perfect matching in a subgraph of $K \in \mathcal{K}_n^\eta$.

Proposition 8.20. *Let $K \in \mathcal{K}_n^\eta$ have partition classes A, B, and C and let $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$ with $|A'| \geq |B'| \geq |C'|$. If $|A'| \leq |B' \cup C'|$ then there is a matching in $K[A', B', C']$ covering at least $|A' \cup B' \cup C'| - 4\eta n - 1$ vertices.*

Proof. Let $x := |B'| - |C'|$ and $y := \lfloor \frac{1}{2}(|A'| - x) \rfloor$. Observe that $x \leq |B'| \leq |A'|$. Hence $y \geq 0$, $x + y \leq \frac{1}{2}(|A'| + x) \leq \frac{1}{2}(|B' \cup C'| + x) = |B'|$, and $y \leq \frac{1}{2}(|A'| - x) \leq \frac{1}{2}(|B' \cup C'| - x) = |C'|$. Choose arbitrary subsets $U_B \subseteq B'$ of size $x + y$, $U_C \subseteq C'$ of size y, set $U := U_B \cup U_C$, $W := B' \setminus U_B$ and $W' := C' \setminus U_C$. Clearly $|W'| = |C'| - y = |B'| - (x + y) = |W|$ and $|A'| - 1 \leq x + 2y = |U| \leq |A'|$. Thus we can choose a subset U' of A' of size $|U|$ that covers all but at most 1 vertex of A' and so that $K[U, U']$ and $K[W, W']$ are η-complete balanced bipartite subgraphs. A simple greedy algorithm allows us then to find matchings of size at least $|U| - \eta n$ and $|W| - \eta n$ in $K[U, U']$ and $K[W, W']$, respectively. These matchings together form a matching in $K[A', B', C']$ covering at least $|U \cup U' \cup W \cup W'| - 4\eta n \geq |A' \cup B' \cup C'| - 4\eta n - 1$ vertices. □

The following proposition shows that induced subgraphs of η-complete tripartite graphs are connected provided that they are not too small. Moreover, subgraphs that substantially intersect all three partition classes contain a triangle.

Proposition 8.21. *Let $K \in \mathcal{K}_n^\eta$ be a graph with partition classes A, B, C, and let $A' \subseteq A$, $B' \subseteq B$, $C' \subseteq C$.*

(a) *If $|A'| > 2\eta n$ then every pair of vertices in $B' \cup C'$ has a common neighbour in A'.*
(b) *If $|A'|, |B'| > 2\eta n$ then $K[A', B']$ is connected.*
(c) *If $|A'|, |B'|, |C'| > 2\eta n$ then $K[A', B', C']$ contains a triangle.*

Proof. As $K \in \mathcal{K}_n^\eta$, each vertex in $B' \cup C'$ is adjacent to at least $|A'| - \eta n > |A'|/2$ vertices in A'. Thus every pair of vertices in B' has a common neighbour in A' which gives (a). For the proof of (b) observe that by (a) every pair of vertices in B' has a common neighbour in A'. Since the same holds for pairs of vertices in A' the graph $K[A', B']$ is connected. To see (c) we use (a) again and infer that every pair of vertices in $A' \times B'$ has a common neighbour in C'. As $|A'|, |B'| > 2\eta n$ there is some edge in $A' \times B'$ and thus there is a triangle in $K[A', B', C']$. □

Similar in spirit to (c) of Proposition 8.21 we can enforce a copy of a cycle of length 5 in a system of η-complete graphs as we show in the next proposition.

Proposition 8.22. *Let K be a coloured graph on n vertices, let c be a colour, vw be a c-coloured edge of K, and let $D_1, D_2, D_3 \subseteq V(K)$ such that all graphs $K[v, D_1]$, $K[D_1, D_2]$, $K[D_2, D_3]$, and $K[D_3, w]$ are (η, c)-complete bipartite graphs. Set $D := \bigcup_{i \in [3]} D_i \cup \{v, w\}$. If $|D_i| > 2\eta n + 2$ for all $i \in [3]$ then $K[D]$ contains a c-coloured copy of C_5.*

Proof. By Proposition 8.21(a) every pair of vertices in $D_1 \cup D_3$ is connected by a path of colour c and length 2 with centre in $D_2 \setminus \{v, w\}$. Moreover, v has at least $|D_1| - \eta n \geq 1$ neighbours in D_1 and similarly w has a neighbour in D_3. Hence there is a c-coloured C_5 in $K[D]$. □

8.6.1 Non-extremal configurations

In the proof of Lemma 8.8 we will use that coloured graphs K from $\mathcal{K}_n^\eta$ have the following property P. Either one colour of K has a big odd connected matching or both colours have big connected matchings whose components are bipartite. Analysing these bipartite configurations will then lead to a proof of Lemma 8.8. Property P is a consequence of the next lemma, Lemma 8.23, which states that if all connected matchings in a colour of K are small then the other colour has bigger odd connected matchings.

Lemma 8.23 (improving lemma). *For every $\eta' > 0$ there are $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that a coloured graph $K \in \mathcal{K}_n^\eta$ is neither η'-extremal nor $\frac{3}{4}(1-\eta')n$-odd. Let M be a maximum connected matching in K of colour c. If $\eta' n < |M| < (1-\eta')n$ then K has an odd connected matching M' in the other colour satisfying $|M'| > |M|$.*

Proof. Given η' define $\tilde{\eta} := \eta'/3$ and let η be small enough and n_0 large enough such that $(\frac{1}{100}\eta' - 5\eta)n_0 > 1$ (and hence $\eta < \frac{1}{500}\eta'$). For $n \geq n_0$ let $K = (A\dot\cup B\dot\cup C, E)$ be a coloured graph from $\mathcal{K}_n^\eta$ with partition classes A, B, and C that is neither η'-extremal nor $(1-\eta')3n/4$-odd. Suppose $c =$ green and hence that K has a maximum green connected matching M with $\eta' n < |M| < (1-\eta')n$. For $D, D' \in \{A, B, C\}$ with $D \neq D'$ let $M_{DD'} := M \cap (D \times D')$. We call the $M_{DD'}$ the *blocks* of M and say that a block $M_{DD'}$ is *substantial* if $|M_{DD'}| \geq \tilde{\eta} n$. Let R be the set of vertices in K not covered by M. For $D \in \{A, B, C\}$ let $R_D := R \cap D$.

Fact 1. *We have $|R_A| - |M_{BC}| = |R_B| - |M_{CA}| = |R_C| - |M_{AB}| > \eta' n$.*

Indeed, $|R_A| + |M_{AB}| + |M_{AC}| = |R_B| + |M_{AB}| + |M_{BC}|$ and hence $|R_A| - |M_{BC}| = |R_B| - |M_{AC}| = |R_B| - |M_{CA}|$ which proves the first part of this fact. For the second part observe that $|R_A| + 2|M_{AB}| + |R_B| + 2|M_{BC}| + |R_C| + 2|M_{CA}| = 3n$. Hence we conclude from $|M| = |M_{AB}| + |M_{BC}| + |M_{CA}| < (1-\eta')n$ that

$$\begin{aligned}3(|R_A| - |M_{BC}|) &= (|R_A| - |M_{BC}|) + (|R_B| - |M_{CA}|) + (|R_C| - |M_{AB}|)\\ &= 3(n - |M_{AB}| - |M_{BC}| - |M_{CA}|) > 3\eta' n.\end{aligned}$$

This finished the proof of Fact 1.

In the remainder we assume without loss of generality that $|R_A| \geq |R_B| \geq |R_C|$. By Fact 1 this implies that $|M_{BC}| \geq \frac{1}{3}\eta' n$ since $|M| > \eta' n$ and hence M_{BC} is substantial. Our next main goal is to find a connected matching in red that is bigger than M. For achieving this goal the following fact about red connections between vertices of R will turn out useful.

Fact 2. *There is a vertex $u^* \in R_A$ such that $R - u^*$ is red connected.*

To see this, assume first that there is a vertex $u^* \in R_A$ that has more than $4\eta n$ green-neighbours in M_{BC}. Then more than $2\eta n$ of these neighbours are in, say, $M_{BC} \cap B$. Call this set of vertices B^*. Now let $u \neq u^*$ be any vertex in $R \setminus C$. By the maximality of M the vertex u has no green-neighbours in $M(B^*)$. This implies that u has at least $|M(B^*)| - \eta n > |M(B^*)|/2$ red-neighbours in $M(B^*)$. Thus any two vertices in $R \setminus C$ have a common red-neighbour in $M(B^*)$. A vertex $u \in R_C$ on the other hand has at least $|R_A| - \eta n \geq |M_{BC}| + \eta' n - \eta n > 2\eta n + 1$ neighbours in R_A where the first inequality follows from Fact 1. If at least 2 of these neighbours are red then u is red connected to $R_A - u^*$. Otherwise u has a set U of more than $2\eta n$ green-neighbours in $R_A - u^*$. But then, by the maximality of M, the graph $K[U, M_{BC} \cap B]$ is red. Since $|M_{BC}| \geq \eta' n > \eta n$ the vertex u has a neighbour v in $M_{BC} \cap B$. Since u has a green-neighbour in R_A it follows from the maximality of M that uv is red. Thus u is red connected to U and therefore to all vertices of $(R \setminus C) - u^*$.

If there is no vertex in R_A with more than $4\eta n$ green-neighbours in M_{BC} on the other hand, then any two vertices in R_A obviously have at least $|M_{BC}| - 4\eta n - 2\eta n \geq \frac{1}{3}\eta' n - 6\eta n > 0$ common red-neighbours in $B \cap M_{BC}$. Moreover, by the maximality of M, each vertex $v \in R_C \cup R_B$ is either red connected to R_A or it has only red-neighbours in M_{BC}. Thus v has a common red-neighbour with any vertex in R_A which proves Fact 2 also in this case.

Fact 3. *K has a red connected matching M' with $|M'| \geq |M| + \frac{1}{4}\eta' n$.*

Let uv be an arbitrary edge in M_{BC}. Then, by the maximality of M, one vertex of this edge, say u, has at most one green-neighbour in R_A. By Fact 1 we have $|R_A| \geq |M_{BC}| + \eta' n$ and since u has at most $\eta n < \eta' n$ non-neighbours in R_A it follows that u has at least $|M_{BC}| + 1$ red-neighbours in R_A. Thus, a simple greedy method allows us to construct a red matching M'_{BC} of size $|M_{BC}|$ between $R_A - u^*$ and such vertices u of matching edges in M_{BC}. Let R'_A be the set of vertices in R_A not covered by M'_{BC}. We repeat this process with M_{AC} and M_{AB}, respectively, to obtain red matchings M'_{AC} and M'_{AB} and sets R'_B and R'_C.

By maximality of M, for each vertex $w \in R'_A$ the following is true: either w has no green-neighbour in M_{BC}, or w has no green-neighbour in R'_B. Moreover w has at most ηn non-neighbours. Observe that $|R'_B|, |R'_A| > \eta' n$ by Fact 1 and the set X of vertices in M_{BC} that are not covered by M'_{BC} has size at least $\frac{1}{3}\eta' n$ since $|M_{BC}| = |M'_{BC}| \geq \frac{1}{3}\eta' n$ and each edge of M'_{BC} uses exactly one vertex from M_{BC}. This implies that we can again use a greedy method to construct a red matching M'_R with edges from $(R'_A - u^*) \times (R'_B \cup X)$ of size at least $\frac{1}{3}\eta' n - \eta n - 1 \geq \frac{1}{4}\eta' n$. Hence we obtain a red matching $M' := M'_{BC} \dot\cup M'_{CA} \dot\cup M'_{AB} \dot\cup M'_R$ of size at least $|M| + \frac{1}{4}\eta' n$. For establishing Fact 3 it remains to show that M' is red connected. This follows from Fact 2 since each edge of M' intersects $R - u^*$.

If the matching M' is odd then the proof of Lemma 8.23 is complete. Hence assume in the remainder that M' is even. Since M' intersects $R - u^*$ this together with Fact 2 immediately implies the next fact. For simplifying the statement as well as the following arguments we will first delete the vertex u^* from K (and let K denote the resulting graph from now on).

Fact 4. *No odd red cycle in K contains a vertex of R.*

Fact 8 below uses this observation to conclude that K is extremal, contradicting the hypothesis of Lemma 8.23. To prepare the proof of this fact we first need some auxiliary observations.

Fact 5. *For $\{D, D', D''\} = \{A, B, C\}$, if $M_{DD'}$ is a substantial block then there is a vertex $v^* \in R_{D''}$ such that $K[M_{DD'}, R_{D''} - v^*]$ is red and $K[M_{DD'}]$ is green.*

We first establish the first part of the statement. We may assume that there are vertices $v^* \in R_{D''}$ and $v \in M_{DD'}$ such that v^*v is green (otherwise we are done). Without loss of generality $v \in D$. Let $X = N(v^*)$. Then, by the maximality of M, all edges between v^* and $X \cap R$ are red. By Fact 4 this implies that all edges between $X \cap R_D$ and $X \cap R_{D'}$ are green. Since $\min\{|X \cap R_D|, |X \cap R_{D'}|\} > \eta n$, this set of edges is not empty. We use the maximality of M to infer that all edges between $M_{DD'}$ and $X \cap (R_D \cup R_{D'})$ are red. Using Fact 4 this in turn implies that edges between $Y := M_{DD'} \cap X$ and v^* are green. By the maximality of M all edges between $M(Y)$ and $R_{D''} - v^*$ are consequently red. We claim that therefore $K[R_{D'} \cap X, R_{D''} - v^*]$ is green. Indeed, assume there was a red edge $ww' \in R_{D'} \cap X \times (R_{D''} - v^*)$. Then w and w' have at least $|M(Y) \cap D| - 2\eta n \geq |M_{DD'}| - 3\eta n \geq \tilde{\eta} n - 3\eta n > 0$ common neighbours w'' in $M(Y) \cap D$. Since edges between $M(Y)$ and $R_{D''} - v^*$ and edges between

$M_{DD'}$ and $X \cap R_{D'}$ are red, so are the edges ww'' and $w'w''$ and thus we have a red triangle $ww'w''$ contradicting Fact 4. By Fact 1 we have $|R_{D'} \cap X| \geq \eta' n - \eta n > \eta n$ and so each vertex in $R_{D''} - v^*$ is connected by a green edge to some vertex in $R_{D'} \cap X$. The maximality of M implies that $K[M_{DD'}, R_{D''} - v^*]$ is red as required. For the second part of Fact 5 observe that the fact that $K[M_{DD'}, R_{D''} - v^*]$ is red and $|R_{D''}| \geq \eta' n > 2\eta n + 1$ imply that each pair of vertices in $M_{DD'}$ has a common red neighbour in $R_{D''} - v^*$ and so by Fact 4 the graph $K[M_{DD'}]$ is green. This establishes Fact 5.

Now we also delete all (at most 3) vertices from R that play the rôle of v^* in Fact 5 (and again keep the names for the resulting sets).

Fact 6. *Suppose that* $\{D, D', D''\} = \{A, B, C\}$ *and that* $M_{DD'}$ *is a substantial block. Then for one of the sets* D *and* D'*, say for* D*, the graph* $K[M_{DD'}, R_D]$ *is red and* $K[R_{D''}, R_D]$ *is green. For the other set* D' *the following is true. If* $v \in R_{D'}$ *then* $K[v, M_{DD'}]$ *and* $K[v, R]$ *are monochromatic, with distinct colours.*

We start with the first part of this fact and distinguish two cases. First, assume that there is a red edge ww' with $w \in R_{D''}$ and $w' \in R_{D'}$. We will show that in this case $K[M_{DD'}, R_D]$ is red and $K[R_{D''}, R_D]$ is green. Since $M_{DD'}$ is substantial, edges between w and $M_{DD'}$ are red by Fact 5 and hence, owing to Fact 4, edges between $M_{DD'} \cap N(w)$ and w' are green. Since $K[M_{DD'}]$ is green by Fact 5, since M is maximal, and since each vertex in $M_{DD'} \cap D'$ has some neighbour in $M_{DD'} \cap N(w')$ this implies that all edges between $M_{DD'}$ and R_D are red. Moreover, edges between $M_{DD'} \cap D'$ and $R_{D''}$ are red by Fact 5 and hence we conclude from Fact 4 that $K[R_{D''}, R_D]$ is green If, on the other hand, there is no red edge between $R_{D''}$ and $R_{D'}$ then the first part of the fact is true with D and D' interchanged: Clearly $K[R_{D''}, R_{D'}]$ is green and by maximality of M we infer that $K[M_{DD'}, R_{D'}]$ is red.

For the second part of the fact suppose that $K[M_{DD'}, R_D]$ is red and $K[R_{D''}, R_D]$ is green. Let $v \in R_{D'}$ and assume first that v has a green neighbour in $M_{DD'}$. The maximality of M then implies that $K[v, R]$ is red and since $K[R_{D''}, M_{DD'}]$ is also red (by Fact 5) we get that $K[v, M_{DD'}]$ is green. Hence it remains to consider the case that $K[v, M_{DD'}]$ is red. By Fact 5 the graph $K[R_{D''}, M_{DD'}]$ is red and so Fact 4 forces the graph $K[v, R_{D''}]$ to be green. To show that also $K[v, R_D]$ is green assume to the contrary that there is a red edge vw with $w \in R_D$. Recall that $K[v, M_{DD'} \cap D]$, $K[M_{DD'} \cap D, R_{D''}]$, $K[R_{D''}, M_{DD'} \cap D']$, and $K[M_{DD'} \cap D', w]$ are red (and clearly η-complete). Since $|M_{DD'} \cap D|, |R_{D''}|, |M_{DD'} \cap D'| \geq \tilde{\eta} n - 1 \geq 2\eta n + 2$ we can apply Proposition 8.22 to infer that there is a red C_5 touching R which contradicts Fact 4.

Fact 7. *If* $M_{DD'}$ *and* $M_{D'D''}$ *are substantial, then* $K[M_{DD'}, M_{D'D''}]$ *and* $K[R_{D''}, R_D]$ *are green and* $K[M_{DD'} \cup M_{D'D''}, R_{D''} \cup R_D]$ *is red. Moreover, if* $v \in R_{D'}$ *then* $K[v, M_{DD'} \cup M_{D'D''}]$ *and* $K[v, R]$ *are monochromatic, with distinct colours.*

By Fact 6 every vertex in $R_{D''} \cup R_D$ sends some green edges to R and hence the maximality of M implies that $K[M_{DD'} \cup M_{D''D'}, R_{D''} \cup R_D]$ is red. Since there is no red triangle touching R, the graphs $K[M_{DD'} \cap D, M_{D''D'} \cap D']$, $K[M_{DD'} \cap D', M_{D''D'} \cap D'']$, and $K[R_{D''}, R_D]$ are green. Using Proposition 8.22 we get similarly as before that also edges in $K[M_{DD'} \cap D, M_{D'D''} \cap D'']$ are green, since otherwise there was a red C_5 touching R. It remains to show the second part of Fact 7. By Fact 6 the graph $K[v, R]$ is monochromatic. Moreover, applying Fact 6 once to $M_{DD'}$ and once to $M_{D''D'}$, we obtain that $K[v, R]$ and $K[v, M_{DD'} \cup M_{D''D'}]$ are monochromatic graphs of distinct colours.

Now we have gathered enough structural information to show that K is extremal.

Fact 8. *K is in spider configuration with parameter $\tilde{\eta}$.*

We first argue that we can assume without loss of generality that

$$C \text{ always plays the rôle of } D' \text{ in Fact 6.} \qquad (*)$$

Indeed, by Fact 7 this is the case if, besides M_{BC}, the block M_{AC} is substantial. If M_{AC} (and hence also M_{AB}) is not substantial on the other hand then it might be the case that B plays the rôle of D' in Fact 6. Then however we may delete at most $\tilde{\eta}n$ vertices from R_B in order to guarantee $|R_B| \le |R_C|$ and then the following argument still works with B and C interchanged.

To obtain the spider configuration set $A_1 := R_A$, $B_1 := R_B$, let C_1 be the set of those vertices $v \in R_C$ such that $K[v, M_{BC}]$ is red, let $C_C := R_C \setminus C_1$, and define $D_{D'} := M_{DD'} \cap D$ for all $D, D' \in \{A, B, C\}$ with $D \neq D'$. If any of the sets we just defined has less than $\tilde{\eta}n$ vertices delete all vertices in this set. Finally, define A_2, B_2, C_2 as in the definition of the spider configuration (Definition 8.5). Observe that this together with Fact 6 implies that $K[C_C, M_{BC}]$ is green and $K[C_C, R]$ is red.

Now let $\{X, Y, Z\} = \{A, B, C\}$ arbitrarily. Clearly we have $|X_1 \cup X_2| \ge (1-3\tilde{\eta})n \ge (1-\eta')n$. Moreover $K[X_1, Y_2]$ is η-complete. We next verify that this graph is also red. We distinguish two cases. First assume that $Y \neq C$. In this case $X_1 \subseteq R_X$ and $Y_2 = Y_X \cup Y_Z \subseteq (M_{XY} \cap Y) \cup (M_{YZ} \cap Y)$. We have $Y_Z \neq \emptyset$ only if M_{YZ} is substantial and then Fact 5 implies that $K[R_X, M_{YZ}]$ is red. Similarly $Y_X \neq \emptyset$ only if M_{XY} is substantial. By $(*)$ Fact 6 implies that then $K[R_X, M_{XY}]$ is red if $X \neq C$. By the definition of C_1 we also get that $K[X_1, M_{XY}]$ is red if $X = C$. Thus all edges between X_1 and Y_2 are red as desired. If $Y = C$ on the other hand then $X_1 \subseteq R_X$ and $Y_2 = Y_X \cup Y_Z \cup C_C \subseteq (M_{XY} \cap Y) \cup (M_{XZ} \cap Y) \cup C_C$. Analogous to the argument in the first case the graphs $K[R_X, M_{YZ}]$ and $K[R_X, M_{XY}]$ are red (since $X \neq C$). As noted above in addition all edges between R and C_C are red and so $K[X_1, Y_2]$ is also red in this case.

We finish the proof of Fact 8 (and hence Lemma 8.23) by checking that we have a spider configuration with colour $c =$ red. Observe that the graph $K[A_1 \cup B_1 \cup C_1, A_2 \cup B_2 \cup C_2]$ is connected and bipartite. We now verify Conditions 1–4 of the spider configuration. For Condition 2 assume that $C_C \neq \emptyset$. Fact 7 and the definition of C_2 imply then that M_{AB} is not substantial and hence $|A_B| = 0$. Moreover, since $|R_A| \ge |R_B| \ge |R_C|$ we get the first part of Condition 1, and $|D'_D| = |D_{D'}|$ is clearly true by definition. By Fact 1 we have $n - |M_{D''D} \cup M_{D''D'}| = |R_{D''}| > |M_{DD'}|$ which implies $n - |D''_2| > |D_{D'}|$ unless $D'' = C$ and $C_C \neq \emptyset$ (if $D'' \neq C$ or $C_C = \emptyset$ then $|M_{DD'}| = |D_{D'}|$). And if $C_C \neq \emptyset$ Condition 2 implies $|D_{D'}| = |A_B| = 0$ and thus we also get $n - |D''_2| > |D_{D'}|$ in this case. This establishes Condition 1. To see Condition 3, note that if A_2 is non-empty then either M_{AB} or M_{AC} are substantial. Since in addition M_{BC} is substantial by assumption we conclude from Fact 7 that there is a green triangle connected to M_{BC} and hence to the green matching M. As K is not $\frac{3}{4}(1-\eta')n$-odd this implies $\frac{1}{2}|A_B \cup A_C \cup B_A \cup B_C \cup C_A \cup C_B| \le |M| < \frac{3}{4}(1-\eta')n$. It remains to verify Condition 4. Assume, for a contradiction, that $C_1 \neq \emptyset$ and $|A_1 \cup B_1 \cup C_1| \ge (1-\eta)\frac{3}{2}n$ and $|B_1 \cup C_1| > (1-\eta)\frac{3}{4}n$. As $|R_A| \ge |R_B| \ge |R_C| \ge |C_1|$ and $C_1 \neq \emptyset$ all these sets have size at least $\tilde{\eta}n$ and so $A_1 = R_A$, $B_1 = R_B$ and $C_1 \subseteq R_C$. By Fact 6 and the definition of C_1 the graph $K[A_1, B_1, C_1]$ is (η, green)-complete and thus contains a green triangle by Proposition 8.21(c) and is connected by (b) of the same proposition. Observe that this implies that any matching in $K[A_1, B_1, C_1]$ is connected and odd. We will show that $K[A_1, B_1, C_1]$ contains a green matching of size at least $\frac{3}{4}(1-\eta')n$ contradicting the fact that K is not $\frac{3}{4}(1-\eta')n$-odd. We distinguish two cases. If $|A_1| \geq |B_1 \cup C_1|$ an easy greedy algorithm guarantees a green matching of size $|B_1 \cup C_1| - \eta n > (1-3\eta)\frac{3}{4}n \geq \frac{3}{4}(1-\eta')n$ in $K[A_1, B_1 \cup C_1]$. If $|A_1| \leq |B_1 \cup C_1|$ on the other hand there is a green matching covering at least $|A_1 \cup B_1 \cup C_1| - 4\eta n - 1 > (1-4\eta)\frac{3}{2}n - 1 \geq \frac{3}{2}(1-\eta')n$ vertices in $K[A_1, B_1, C_1]$ by Proposition 8.20. □

We will now use Lemma 8.23 to prove Lemma 8.8.

Proof of Lemma 8.8. Let η' be given and set $\tilde{\eta} := \eta'/15$. Let $\eta_{8.23}$ and n_0 be provided by Lemma 8.23 for input $\eta'_{8.23} = \tilde{\eta}$ and set $\eta := \min\{\eta_{8.23}, \tilde{\eta}/5\}$. Let $K = (A \dot{\cup} B \dot{\cup} C, E)$ be a non-extremal coloured member of $\mathcal{K}_n^\eta$ with partition classes and assume for a contradiction that K is not $(1-\eta')3n/4$-odd.

Our first step is to show that K has big green and red connected matchings.

Fact 1. *K has even connected matchings M_r and M_g in red and green, respectively, with $|M_r|, |M_b| \geq (1-\tilde{\eta})n$.*

Assume for a contradiction that a maximum matching M in red has size less than $(1-\tilde{\eta})n$. By Lemma 8.23 applied with $\tilde{\eta}$ we conclude that there is an odd connected matching M' with $|M'| > |M|$. On the other hand K is not $\big((1-\eta')3n/4\big)$-good, hence $|M'| < (1-\eta')3n/4$. Another application of Lemma 8.23 with $\tilde{\eta} \leq \eta'$ thus provides us with a red connected matching of size bigger than $|M'|$ which contradicts the maximality of M. We conclude that there is a red connected matching M_r, and by symmetry also a green connected matching M_g, of size at least $(1-\tilde{\eta})n$. Clearly, M_r and M_g are even since K is not $\big((1-\eta')3n/4\big)$-good.

Let R be the component of M_r and G be the component of M_g in K. Fact 1 states, that R and G are bipartite. We observe in the following fact that both R and G substantially intersect all three partition classes. For this purpose define $D_r := D \cap V(R)$ and $D_g := D \cap V(G)$, and further $\bar{D}_r := D \setminus D_r$ and $\bar{D}_g := D \setminus D_g$ for all $D \in \{A, B, C\}$.

Fact 2. *For all $D \in \{A, B, C\}$ and $c \in \{r, g\}$ we have $|D_c| \geq 2\tilde{\eta}n$.*

Indeed, assume without loss of generality, that $|A_r| < 2\tilde{\eta}n$ which implies $|\bar{A}_r| > (1-2\tilde{\eta})n$. As $|M_r| \geq (1-\tilde{\eta})n$ it follows that $|B_r| > (1-3\tilde{\eta})n$ and $|C_r| > (1-3\tilde{\eta})n$. By definition all edges between $\bar{A}_r$ and $B_r \cup C_r$ are green and thus K is in pyramid configuration with tunnel, pyramids $(B_r, \bar{A}_r)$ and $(C_r, \emptyset)$, and parameter $3\tilde{\eta} < \eta'$, which is a contradiction.

Next we strengthen the last fact by showing that at most one of the sets $\bar{D}_c$ with $D \in \{A, B, C\}$ and $c \in \{r, g\}$ is significant.

Fact 3. *There is at most one set $D \in \{A, B, C\}$ and colour $c \in \{r, g\}$ such that $|\bar{D}_c| \geq \tilde{\eta}n$.*

If such a D and c exist we assume, without loss of generality, $D = A$ and $c = r$. Hence, for the proof of Fact 3, assume that $|\bar{A}_r| \geq \tilde{\eta}n$. First we show that

$$|\bar{B}_r|, |\bar{C}_r| < \tfrac{\tilde{\eta}}{2}n. \tag{8.17}$$

Assume for a contradiction and without loss of generality that $|\bar{B}_r| \geq \frac{\tilde{\eta}}{2}n$. By definition, all edges in $E(\bar{A}_r, C_r \dot{\cup} B_r)$ and $E(\bar{B}_r, C_r \dot{\cup} A_r)$ are green. Since $|\bar{A}_r|, |\bar{B}_r| \geq \tilde{\eta}n > 2\eta n$ by assumption and $|A_r|, |B_r| \geq 2\tilde{\eta}n/2 > 2\eta n$ (by Fact 2) we can apply Proposition 8.21(b) to infer that the graph with edges $E(\bar{A}_r, C_r \dot{\cup} B_r)$ and $E(\bar{B}_r, C_r \dot{\cup} A_r)$ is connected. As M_r is even we conclude that all edges in $E(A_r, C_r)$, $E(B_r, C_r)$, and $E(A_r, B_r)$ are red. Since $|A_r|, |B_r|, |C_r| \geq \tilde{\eta}n > 2\eta n$ by Fact 2 we infer from Proposition 8.21(c) that the graph $K[A_r, B_r, C_r] \subseteq R$ contains a red triangle which contradicts the fact that M_r is even.

Thus it remains to show that $|\bar{D}_g| < \tilde{\eta}n$ for all $D \in \{A, B, C\}$. By (8.17) and Fact 2 we have $|B_r \cap B_g|, |C_r \cap C_g| > \frac{\tilde{\eta}}{2}n > \eta n$ which implies that there is an edge in $E(B_r \cap B_g, C_r \cap C_g)$. By assumption we also have $|\bar{A}_r| \geq \tilde{\eta}n > 2\eta n$ and thus each pair of vertices in $B_r \dot{\cup} C_r$ has a common neighbour in $\bar{A}_r$ by (a) of Proposition 8.21. By definition of $\bar{A}_r$ all edges in $E(\bar{A}_r, B_r \dot{\cup} C_r)$ are green, and therefore we conclude that all edges in $E(B_r \cap B_g, C_r \cap C_g)$ are red since otherwise there would be a green triangle connected to M_g. Accordingly $|\bar{A}_g| \leq 2\eta n < \tilde{\eta}n/2$ since otherwise we could equally argue that all edges in $E(B_r \cap B_g, C_r \cap C_g)$ are green, a contradiction. Therefore $|A_g| \geq (1 - \tilde{\eta}/2)n$. As $|\bar{A}_r| \geq \tilde{\eta}n$ this implies $|A_g \cap \bar{A}_r| \geq \tilde{\eta}/2n > \eta n$ and from (8.17) we also get $|B_r \cap \bar{B}_g| \geq \tilde{\eta}/2n > \eta n$. Thus there is an edge in $E(A_g \cap \bar{A}_r, B_r \cap \bar{B}_g)$. However, this edge can neither be red since it connects $\bar{A}_r$ and B_r, nor green since it connects $\bar{B}_g$ and A_g, a contradiction. Therefore $|\bar{B}_g| < \tilde{\eta}n$ and by symmetry also $|\bar{C}_g| < \tilde{\eta}n$ which finishes the proof of Fact 3.

We label the vertices in each of the bipartite graphs R and G according to their bipartition class by 1 and 2. In the remaining part of the proof we examine the distribution of these bipartition classes over the partition classes of K. Let F_{ij} denote the set of vertices in $V(R) \cap V(G)$ with label i in R and label j in G for $i, j \in [2]$. Let further F_{0j} be the set of vertices in $\bar{A}_r \cap V(G)$ that have label j in G for $j \in [2]$. Next we observe that each of the sets F_{ij} with $i, j \in [2]$ is essentially contained in one partition class of K.

Fact 4. *For all $i, j \in [2]$ there is at most one partition class $D \in \{A, B, C\}$ of K with $|F_{ij} \cap D| \geq \tilde{\eta}n$. Moreover $E(F_{0j}, F_{ij}) = \emptyset$.*

To prove the first part of Fact 4 assume for a contradiction that $|F_{ij} \cap A|, |F_{ij} \cap B| \geq \tilde{\eta}n$. Then there would be an edge in $K[A \cap F_{ij}, B \cap F_{ij}]$ since $\tilde{\eta} > \eta$. This contradicts the fact that F_{ij} is independent by definition. For the second part observe that an edge in $E(F_{0j}, F_{ij})$ can neither be red as such an edge would connect vertices from $\bar{A}_r$ to R nor green since $F_{0j} \cup F_{ij}$ lies in one bipartition class j of G.

Fact 5. *There are $X, Y \in \{A, B, C\}$ with $X \neq Y$ and indices $b, b', c, c' \in [2]$ with $bb' \neq cc'$ such that $|F_{bb'} \cap X|, |F_{cc'} \cap Y| \geq (1 - 5\tilde{\eta})n$ and $|F_{0b'}|, |F_{0c'}| \leq \tilde{\eta}n$.*

We divide the proof of this fact into three cases: The first case deals with $\bar{A}_r \neq \emptyset$, the second one with $\bar{A}_r = \emptyset$ and the additional assumption that there are $D \in \{A, B, C\}$ and $ij \neq i'j' \in [2]$ such that $|D \cap F_{ij}|, |D \cap F_{i'j'}| \geq \tilde{\eta}n$. The third and remaining case treats the situation when $\bar{A}_r = \emptyset$ and for each $D \in \{A, B, C\}$ there is at most one index pair (i, j) with $|D \cap F_{ij}| \geq \tilde{\eta}n$.

For the first case, let $j \in [2]$ be such that $F_{0j} \neq \emptyset$. Observe that then the second part of Fact 4 implies that $|F_{1j} \cap (B \cup C)|, |F_{2j} \cap (B \cup C)| < \eta n$. Let $c' = b' \in [2]$ with $c' \neq j$. Then, because Fact 3 implies that $|\bar{B}_r|, |\bar{B}_g|, |\bar{C}_r|, |\bar{C}_g| < \tilde{\eta}n$, we have that $|B \cap (F_{1b'} \cup F_{2b'})| \geq (1 - 4\tilde{\eta})n$ and $|C \cap (F_{1c'} \cup F_{2c'})| \geq (1 - 4\tilde{\eta})n$. Thus there is a $b \in [2]$ such that $|B \cap F_{bb'}| \geq \tilde{\eta}n$. Let $c' \in [2]$ with $c' \neq b'$. The first part of Fact 4 implies that $|C \cap F_{bc'}| < \tilde{\eta}n$, thus $|C \cap F_{cc'}| \geq (1 - 5\tilde{\eta})n \geq \tilde{\eta}n$. By symmetry we also get $|B \cap F_{bb'}| \geq (1 - 5\tilde{\eta})n$. This proves the first part of the statement for the first case. To see the second part, observe that if $F_{0b'} \neq \emptyset$, then $|F_{1b'} \cap (B \cup C)|, |F_{2b'} \cap (B \cup C)| < \eta n$ by Fact 4, a contradiction.

The second part of the second and third cases is straightforward as $F_{0,1}, F_{0,2} \subseteq \bar{A}_r = \emptyset$. To see the first part of the second case let D be as specified above and $\{X, Y\} = \{A, B, C\} \setminus \{D\}$. The first part of Fact 4 implies that $|F_{ij} \cap X|, |F_{i'j'} \cap X|, |F_{ij} \cap Y|, |F_{i'j'} \cap Y| < \tilde{\eta}n$. Thus $|(F_{ij'} \cup F_{i'j}) \cap X| \geq (1 - 2\tilde{\eta})n - 2\tilde{\eta}n$, as $|\bar{X}_r|, |\bar{X}_g| < \tilde{\eta}n$. Without loss of generality, let ij' be such that $|X \cap F_{ij'}| \geq \tilde{\eta}n$. We set $b := i$, $b' := j'$, $c = i'$ and $c' := j$. The rest of the proof is similar to the first case, proving that then $|Y \cap F_{cc'}| \geq (1 - 5\tilde{\eta})n$ and by symmetry that $|X \cap F_{bb'}| \geq (1 - 5\tilde{\eta})n$.

It remains to prove the first part of the third case. For this observe that for all $D \in \{A, B, C\}$

we have that $|D \cap \bigcup_{(i'j') \neq (i,j)} F_{i',j'}| < 3\tilde{\eta}n$, where i, j are as specified in the definition of the third case. Observe also that $|\bar{D}_r|, |\bar{D}_g| < \tilde{\eta}n$. This implies $|D \cap F_{i,j}| \geq (1-5\tilde{\eta})n$, as desired. Hence, for $X = B$ and $Y = C$ we obtain indices b, b', c, c' such that $|X \cap F_{bb'}|, |Y \cap F_{i'j'}| \geq (1-5\tilde{\eta})n$, with $bb' \neq cc'$ by Fact 5.

This brings us to the last step which shows that K is extremal, a contradiction.

Fact 6. *K is in pyramid configuration with parameter η'.*

Let $X, Y \in \{A, B, C\}$ and $b, b', c, c' \in [2]$ be as in Fact 5. Let $Z \in \{A, B, C\} \setminus \{X, Y\}$. Assume without loss of generality that $b = b' = 1$. Thus Fact 5 states that $|F_{11} \cap X| \geq (1-5\tilde{\eta})n$ and $|F_{01}| \leq \tilde{\eta}n$. We distinguish two cases. First, assume that $c' = 2$ and set $\bar{c} := 3 - c$. By Fact 5 this implies $|F_{c2} \cap Y| \geq (1-5\tilde{\eta})n$ and $|F_{02}| \leq \tilde{\eta}n$ and thus $|(F_{\bar{c}2} \cup F_{21}) \cap Z| \geq (1-5\tilde{\eta})n$ by Fact 4. Moreover $E(F_{11} \cap X, F_{21} \cap Z)$ forms an η-complete red bipartite graph since $F_{11} \cup F_{21}$ is an independent set in G. Similarly $E(F_{c2} \cap Y, F_{\bar{c}2} \cap Z)$ forms an η-complete red bipartite graph. Further, if $c = 2$ then $E(F_{c2} \cap Y, F_{21} \cap Z)$ and $E(F_{11} \cap X, F_{\bar{c}2} \cap Z)$ form η-complete green bipartite graphs (leading to crossings) and if $c = 1$ then $E(F_{11} \cap X, F_{c2} \cap Y)$ forms an η-complete green bipartite graph (leading to a tunnel). Therefore, in both sub-cases, K is in pyramid configuration with parameter $5\tilde{\eta} \leq \eta'$ and pyramids $(F_{11} \cap X, F_{21} \cap Z)$ and $(F_{c2} \cap Y, F_{\bar{c}2} \cap Z)$, unless one of the sets $F_{21} \cap Z$ and $F_{\bar{c}2} \cap Z$ has size at most $10\tilde{\eta}n$. In this case, however, we can simply replace this set by the empty set and still obtain a pyramid configuration with parameter at most $15\tilde{\eta} \leq \eta'$.

In the case $c' = 1$ we have $c = 2$. Fact 5 guarantees that $|F_{21} \cap Y| \geq (1-5\tilde{\eta})n$. Since $|F_{01}| \leq \tilde{\eta}n$ we conclude from Fact 4 that $|(F_{12} \cup F_{22} \cup F_{02}) \cap Z| \geq (1-5\tilde{\eta})n$. Similarly as before $E(F_{11} \cap X, (F_{12} \cup F_{02}) \cap Z)$ and $E(F_{21} \cap Y, F_{22} \cap Z)$ form η-complete green bipartite graphs and $E(F_{11} \cap X, F_{21} \cap Y)$ forms an η-complete red bipartite graph. Accordingly we also get a pyramid configuration with parameter $5\tilde{\eta} \leq \eta'$ in this case, where the pyramids are $(F_{11} \cap X, (F_{12} \cup F_{02}) \cap Z)$ and $(F_{21} \cap Y, F_{22} \cap Z)$ unless, again, $(F_{12} \cup F_{02}) \cap Z$ or $F_{22} \cap Z$ are too small in which case we proceed as above. □

8.6.2 Extremal configurations

Our aim in this section is to provide a proof of Lemma 8.9. This proof naturally splits into two cases concerning pyramid and spider configurations, respectively. The former is covered by Proposition 8.24, the latter by Proposition 8.25.

Proposition 8.24. *Lemma 8.9 is true for pyramid configurations.*

Proof. Given η' set $\eta = \eta'/3$. Let K be a coloured graph from $\mathcal{K}_n^\eta$ that is in pyramid configuration with parameter η and pyramids (D_1, D_1') and (D_2, D_2') such that the requirements of (E1) in Definition 8.5 are met for colours c and c'.

Fact 1. *If the pyramid configuration has crossings then K is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good.*

Indeed, by Proposition 8.19 there is a matching M of colour either c or c' and size at least $(1-2\eta)\frac{1}{2}n$ in $K[D_1, D_2]$. Note further, that the pyramid configuration with crossings is symmetric with respect to the colours c and c' and hence we may suppose, without loss of generality, that M is of colour c and that $|D_1'| \geq (1-\eta)\frac{1}{2}n$. As $K[D_1, D_1']$ and $K[D_2, D_2']$ are (η, c)-complete, there are c-coloured matchings M_1 and M_2 in $K[D_1, D_1']$ and $K[D_2 \setminus M, D_2']$, respectively, of size at least $\min\{|D_1'|, |D_1|\} - \eta n$ and $\min\{|D_2'|, |D_2 \setminus M|\} - \eta n$, respectively. This implies

$$|M| + |M_1| + |M_2| \geq (1-3\eta)\frac{3}{2}n = (1-\eta')\frac{3}{2}n.$$

Observe that, depending on the size of M, either $M \cup M_2$ or $M_1 \cup M_2$ is a matching of size at least $(1-3\eta)n = (1-\eta')n$. Now, the union of M, M_1, and M_2 forms a 2-fork system F and since $K[D_1, D_1']$ and $K[D_2, D_2']$ are (η, c)-complete the bipartite graph formed by these two graphs and M is connected and has partition classes $D_1 \cup D_2'$ and $D_2 \cup D_1'$. It follows that F has size $|M| + |M_1| + |M_2| \geq (1-\eta')\frac{3}{2}n$.

Fact 2. *If the pyramid configuration has a c'-tunnel and if there is a matching M of colour c' and size at least $(1-\eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ or in $K[D_2, D_1' \cup D_2']$ then K is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good in colour c'.*

As K has a c'-tunnel, there is a connected matching M' of colour c' and size at least $|D_1| - \eta n \geq (1-\eta')n$ in $K[D_1, D_2]$. We will extend the matching M' (which is a 1-fork-system) to a 2-fork system. Without loss of generality assume that the matching M promised by Fact 2 is in $K[D_1, D_1' \cup D_2']$. As $M \cap D_1$ and D_2 are non-negligible the bipartite graph $K[M \cap D_1, D_2]$ is connected by Proposition 8.21(b) and thus M is connected. Hence $M \cup M'$ forms a connected 2-fork system centered in D_1 and of size $|M'| + |M| \geq (1-\eta')\frac{3}{2}n$.

Fact 3. *If the pyramid configuration has a c'-tunnel but no crossings and there is no matching of colour c' and size at least $(1-\eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ or in $K[D_2, D_1' \cup D_2']$ then K is $((1-\eta')n, (1-\eta')\frac{3}{2}n, 3)$-good in colour c.*

To obtain the 3-fork system note that Proposition 8.19 implies that there are matchings M_1 and M_2 of colour c and sizes at least $(1-\eta')\frac{1}{2}n$ in $K[D_1, D_1' \cup D_2']$ and $K[D_2, D_1' \cup D_2']$, respectively. The union of M_1 and M_2 forms a 2-fork system F centered in $D_1' \cup D_2'$ covering at least $(1-\eta')\frac{1}{2}n$ vertices in D_1 and at least $(1-\eta')\frac{1}{2}n$ vertices in D_2. We can assume without loss of generality that $|D_1'| \geq (1-\eta)\frac{1}{2}n \geq (1-\eta')\frac{1}{2}n$. As $K[D_1, D_1']$ is (η, c)-complete and $|D_1 \setminus F| \leq (1-\eta')n - (1-\eta')\frac{1}{2}n = (1-\eta')\frac{1}{2}n$ we can greedily find a matching between D_1' and $D_1 \setminus F$ covering all but at most ηn vertices of $D_1 \setminus F$. Its union with F forms a 3-fork system F' centered in $D_1' \cup D_2'$ covering at least $(1-\eta')n$ vertices in D_1 and at least $(1-\eta')\frac{1}{2}n$ vertices in D_2, implying that F' has size at least $(1-\eta')\frac{3}{2}n$. The graph $K[D_1, D_1'] \cup K[D_2, D_2']$ clearly contains a matching M of size at least $|D_1' \cup D_2'| - \eta n \geq (1-\eta')n$ in colour c.

Since the pyramid configuration has no crossings there are edges of colour c in $K[D_1, D_2'] \cup K[D_2, D_1']$. Together with the fact that D_1, D_2, D_1', and D_2' are non-negligible, we obtain that the bipartite graphs $K[D_1, D_1' \cup D_2']$ and $K[D_2, D_1' \cup D_2']$ are connected by (b) of Proposition 8.21. Thus the matching M and the fork system F' are both connected. □

Proposition 8.25. *Lemma 8.9 is true for spider configurations.*

Proof. Given η' set $\eta = \eta'/5$ and let K be a coloured graph from $\mathcal{K}_n^\eta$ that is in spider configuration with parameter η, i. e., it satisfies (E2) of Definition 8.5. In this proof we construct only matchings and fork systems of colour c. Observe that these are connected by definition. We distinguish two cases.

Case 1: First assume that $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$. We will show that in this case our configuration contains both a connected matching of size at least $(1-\eta')n$ and a connected 3-fork system of size at least $(1-\eta')\frac{3}{2}n$. We need the following auxiliary observation.

Fact 1. *If $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$ then $A_B = B_A = \emptyset$. Moreover $|A_1| + \eta n \geq |B_C| = |C_B|$ and $|B_1| + \eta n \geq |A_C| = |C_A|$.*

Indeed, by Condition 3 of (E2) either $A_2 = \emptyset$ and hence $A_B \subseteq A_2$ is empty or $|A_2 \cup B_2 \cup (C_2 \setminus C_C)| \leq (1-\eta)\frac{3}{2}n$. In the second case we conclude from $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$ that

$$|A_1 \cup A_2| + |B_1 \cup B_2| + |C_1 \cup (C_2 \setminus C_C)| < (1-\eta)3n.$$

As $|A_1 \cup A_2|, |B_1 \cup B_2|, |C_1 \cup C_2| \geq (1-\eta)n$ it follows that $|C_1 \cup (C_2 \setminus C_C)| < (1-\eta)n$ and thus $C_C \neq \emptyset$. By Condition 2 of (E2) we get $A_B = \emptyset$. For the second part of the fact observe that Condition 1 of (E2) states that $n - |A_2| \geq |B_C| = |C_B|$ and thus we conclude $|A_1| \geq (1-\eta)n - |A_2| \geq |B_C| - \eta n = |C_B| - \eta n$. The inequality $|B_1| \geq |A_C| - \eta n$ is established in the same way.

Fact 2. *If $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$ then K is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n), 3\big)$-good.*

From Condition 1 of (E2) we infer that $|A_C| < n - |B_2| \leq |B_1| + \eta n$ and Fact 1 implies that $|A_1| + |A_C| = |A_1| + |A_2| \geq (1-\eta)n$ and $|C_1| + |C_2| \geq (1-\eta)n$. We thus conclude from $|A_1 \cup B_1 \cup C_1| < (1-\eta)\frac{3}{2}n$ that

$$|A_C| - \eta n < |B_1| < (1-\eta)\tfrac{3}{2}n - |A_1 \cup C_1| < |A_C| + |C_2| - \eta n.$$

This (together with the fact that $K[B_1, A_C]$ and $K[B_1, C_2]$ are (η, c)-complete) justifies that there is a c-coloured matching M_1 in $K[B_1, A_C \cup C_2]$ covering all vertices of B_1 and all but at most ηn vertices of A_C. Further, by Fact 1 we know that $|B_C| \leq |A_1| + \eta n$ and hence we can find a matching M_2 of colour c in (the (η, c)-complete graph) $K[B_C, A_1]$ covering all but at most ηn vertices of B_C. The matching $M := M_1 \cup M_2$ satisfies

$$|M| \geq |B_1| + |B_C| - \eta n = |B_1| + |B_2| - \eta n \geq (1-\eta)n - \eta n \geq (1-\eta')n,$$

where the equality follows from Fact 1. Next, we extend the matching M to a connected 3-fork system of colour c and size at least $(1-\eta')\frac{3}{2}n$ in the following way. Consider maximal matchings M_3, M_4, and M_5 in $K[B_1, C_A \setminus M_1]$, $K[A_1, C_B \setminus M_1]$ and $K[A_1, C_C \setminus M_1]$, respectively. By Fact 1 we infer that M_3 and M_4 each cover all but at most ηn vertices of $C_A \setminus M_1$ and $C_B \setminus M_1$, respectively. As $|C_C| \leq |C_C \cup C_1| \leq |A_1|$ by Condition 1 of (E2) the matching M_5 covers all but at most ηn vertices of C_C.

Then the union $M \cup M_3 \cup M_4 \cup M_5$ is a 3-fork-system F centered in $A_1 \cup B_1$ and covering all but at most $5\eta n$ vertices of $A_C \cup B_C \cup C_A \cup C_B \cup C_C = A_2 \cup B_2 \cup C_2$. Thus F has size at least $(1-\eta)3n - |A_1 \cup B_1 \cup C_1| - 5\eta n \geq (1-\eta')\frac{3}{2}n$.

Case 2: Now we turn to the case $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$. We further divide this case into two sub-cases, treating $C_1 = \emptyset$ and $C_1 \neq \emptyset$, respectively.

Fact 3. *If $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$ and $C_1 = \emptyset$ then K is $\big((1-\eta')n, (1-\eta')\frac{3}{2}n, 2\big)$-good.*

By definition $|C_2| \geq (1-\eta)n - |C_1| = (1-\eta)n$ in this case. Therefore, using the fact that $K[A_1, C_2]$ and $K[B_1, C_2]$ are (η, c)-complete, we can greedily construct a maximal matching M_A in $K[A_1, C_2]$ and a maximal matching M'_B in $K[B_1, C_2 \setminus M_A]$ such that the matching $M := M_A \cup M'_B$ covers C_2 (as $|A_1 \cup B_1| = |A_1 \cup B_1 \cup C_1| > |C_2| + \eta n$) and thus has size at least $(1-\eta)n$. Then we extend M'_B to a maximal matching M_B in $K[B_1, C_2]$. Observe that M_A and M_B cover all but at most ηn vertices of A_1 and B_1, respectively. Thus the 2-fork system $F := M_A \cup M_B$ has size at least $|A_1 \cup B_1| - 2\eta n = |A_1 \cup B_1 \cup C_1| - 2\eta n \geq (1-\eta')\frac{3}{2}n$.

Now consider the sub-case when $C_1 \neq \emptyset$.

Fact 4. *If $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$ and $C_1 \neq \emptyset$ then $|B_1 \cup C_1| \leq (1-\eta)\frac{3}{4}n$ and we have $|C_2| \geq (1-\eta)\frac{1}{4}n$ and $|C_1| \leq |B_2| - \eta n$.*

The first inequality follows from Condition 4 of (E2). Accordingly $|C_2| \geq (1-\eta)n - |C_1| \geq (1-\eta)\frac{1}{4}n$ which establishes the second inequality. For the third inequality we use that $|B_1 \cup B_2| \geq (1-\eta)n$ by definition and so

$$|C_1| \leq (1-\eta)\tfrac{3}{4}n - |B_1| \leq (1-\eta)\tfrac{3}{4}n - (1-\eta)n + |B_2| \leq |B_2| - \eta n.$$

Fact 5. *If* $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$ *and* $C_1 \neq \emptyset$ *then there is a matching* M *of size at least* $(1-\eta)n$ *and colour* c *covering* C_1*.*

Let M_1 be a maximal matching in $K[C_1, B_2]$. We conclude from Fact 4 that M_1 covers C_1. Let M_2 be a maximal matching in $K[C_2, A_1 \cup B_1]$. As $|C_2| \leq n - |C_1| \leq |A_1 \cup B_1| - \eta n$ the matching M_2 covers C_2. Setting $M := M_1 \cup M_2$, we obtain a matching of size $|M| = |C_1| + |C_2| \geq (1-\eta)n$ as required.

Fact 6. *If* $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$ *and* $C_1 \neq \emptyset$*, then there is a* 3*-fork system of colour* c *and of size at least* $(1-\eta')\frac{3}{2}n$*.*

Let M be the matching from Fact 5. Clearly, we can greedily construct a 2-fork system F' in the (η, c)-complete graph $K[C_2, (A_1 \cup B_1) \setminus M]$ which either is of size $2|C_2|$ or covers all but at most ηn vertices of $|(A_1 \cup B_1) \setminus M|$. Then $F := M \cup F'$ forms a 3-fork system. If the former case occurs we infer from Fact 4 that F is of size at least $(1-\eta)n + 2|C_2| \geq (1-\eta)\frac{3}{2}n$. In the latter case F covers all but at most ηn vertices of $A_1 \cup B_1 \cup C_1$ and thus has size at least $(1-\eta)\frac{3}{2}n - \eta n \geq (1-2\eta)\frac{3}{2}n$. We conclude that K is $((1-\eta')n, (1-\eta')\frac{3}{2}n, 3)$-good also in the sub-case $|A_1 \cup B_1 \cup C_1| \geq (1-\eta)\frac{3}{2}n$ and $C_1 \neq \emptyset$. □

8.7 Sparser tripartite graphs

As noted earlier our proof of Theorem 8.1 applies to suitably chosen (sparser) subgraphs of $K_{n,n,n}$ as well. More precisely, for any fixed $p \in (0,1)$ the same method can be used to show that asymptotically almost surely $\mathcal{G}_p(n,n,n) \to \mathcal{T}_t^{\Delta}$, where $\mathcal{G}_p(n,n,n)$ is a random tripartite graph with edge probability p and partition classes of size n, and where $t \leq (1-\mu)n/2$ and $\Delta \leq n^{\alpha}$ for a small positive $\alpha = \alpha(\mu, p)$. Indeed, standard methods can be used to show that the following holds asymptotically almost surely for $G = \mathcal{G}_p(n,n,n)$ with partition classes $V_1 \dot\cup V_2 \dot\cup V_3$ and for any $\zeta > 0$:

- G has at most $4pn^2$ edges.
- $e(U,W) \geq p|U||W|/2$ for all $U \subseteq V_i$ and $W \subseteq V_j$, $i \neq j$, with $\min\{|U|, |W|\} > \zeta n$.

The first property guarantees that we obtain a graph with few edges. We claim further that these two properties imply that $G \to \mathcal{T}_k^{\Delta}$. To see this we proceed as in the proof of Theorem 8.1 and apply the regularity lemma on the coloured graph G. We then colour an edge in the reduced graph $\mathbb{G}$ by green or red, respectively, if the corresponding cluster pair is regular and has density at least $p/4$ in green or red. Using the two properties from above it is not difficult to verify that $\mathbb{G}$ is a coloured tripartite graph that is η-complete. Hence, from this point on, we can use the strategy described in the proof of Theorem 8.1, apply our structural lemma, Lemma 8.7, the assignment lemma, Lemma 8.13, and the embedding lemma, Lemma 8.12.

One may ask whether this approach can be pushed even further and consider random tripartite graphs $\mathcal{G}_p(n,n,n)$ with edge probabilities $p(n)$ that tend to zero as n goes to infinity. It is likely that similar methods can be used in this case in conjunction with the regularity method for sparse graphs (see, e.g., [41, 57]).

Chapter 9

Embedding into sparse graphs

All embedding problems we considered so far in this thesis were concerned with dense host graphs and applied the dense regularity method. In this section we turn to sparse graphs and sparse regularity.

The sparse regularity method, which is based on the sparse regularity lemma (Lemma 3.15), recently received a considerable amount of attention. A series of results about the nature and the properties of sparse regular pairs were established with important applications in the theory of random graphs (see [41, 59]). However, proving sparse counterparts of the embedding tools that worked well in the dense regularity method seems to be difficult already for constant-size subgraphs. Embedding problems in systems of sparse regular pairs for graphs of constant size are considered in [58, 39, 38, 40, 77]. Here we use techniques developed in [8, 61] for the embedding of graphs that are linear in the size of the host graph. We combine them with the methods from Chapter 5 in order to obtain a result that can be considered a sparse analogue of Theorem 5.1 for bipartite graphs.

We consider subgraphs $G = (V, E)$ of the random graph $\Gamma = \mathcal{G}_{n,p}$ and show that G contains all almost spanning bipartite bounded-degree graphs H with sublinear bandwidth if $p = p(n) = o(1)$ is sufficiently large and G has the following property: Each vertex $v \in V$ has more than half of its neighbours in Γ as neighbours in G.

Theorem 9.1. *For each $\eta, \gamma > 0$ and $\Delta > 1$ there exist positive constants β and c such that the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$. Every spanning subgraph $G = (V, E)$ of Γ with $\deg_G(v) \geq (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$ contains a copy of every graph H on $m = (1 - \eta)n$ vertices with maximum degree $\Delta(H) \leq \Delta$ and bandwidth $\mathrm{bw}(H) \leq \beta n$.*

In the proof of this theorem we will, again, use two structural lemmas that provide partitions of G and H, respectively. In addition we will apply two embedding lemmas, the constraint blow-up lemma and the connection lemma. The constraint blow-up lemma is a sparse bipartite analogue of the blow-up lemma (Lemma 3.9) in the sense that it embeds (almost) spanning bipartite graphs into (sparse) p-dense pairs. The connection lemma embeds graphs that are much smaller (similar as the partial embedding lemma, Lemma 3.10) but can cope with the following situation: Each vertex x that will be embedded by this lemma is equipped with a (possibly very small) candidate set from which its image in the embedding will be chosen. In the proof of Theorem 9.1 we will first apply the constrained blow-up lemma to embed most vertices of H into G and then the connecting lemma for finishing the embedding.

Observe that this strategy is different from the one we employed for dense problems using the general embedding lemma (Lemma 3.12): In the proof of Lemma 3.12 (see Section 3.12) we *first* applied the partial embedding lemma and *then* the blow-up lemma. The reason for this change is as follows. In the dense case the application of the partial embedding lemma created image restrictions for the application of the blow-up lemma. These image restrictions are

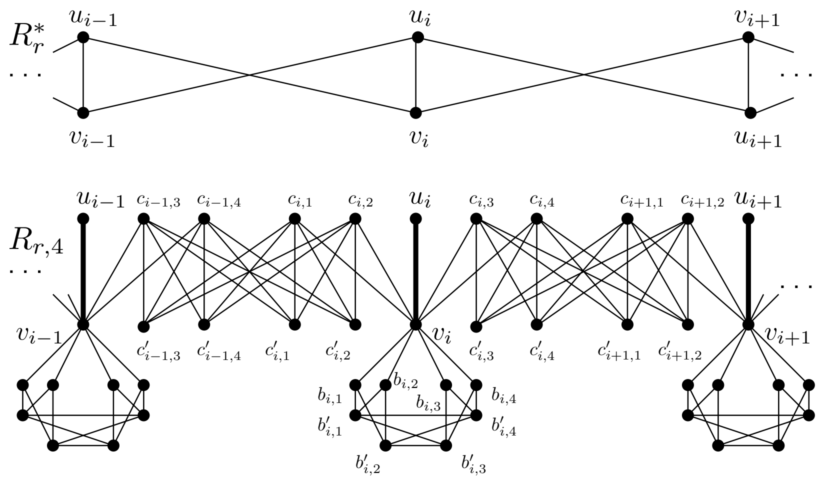


Figure 9.1: The ladder R_r^* and the spin graph $R_{r,t}$ for the special case $t = 2$.

neighbourhoods of vertices (hosting previously embedded vertices of H) in the host graph G. Because G is a dense graph we can guarantee that these neighbourhoods are of linear size, that is, big enough to be used as image restriction in the blow-up lemma. In the sparse case such neighbourhoods may be tiny and cannot be handled by Lemma 9.4. To solve this problem we apply this lemma first, without any image restrictions. This creates restrictions for the connecting lemma, which we will take care of with the help of the candidate sets mentioned above.

The four main lemmas and the idea for the proof of Theorem 9.1 are given in the following section, and the details of this proof in Section 9.4. To prepare this proof we will first examine some relevant properties of random graphs in Section 9.2 and consider neighbourhood and inheritance properties in p-dense pairs in Section 9.3.

9.1 Main Lemmas

In this section we will formulate the main lemmas for the proof of Theorem 9.1. In order to state them we first need to define two (families of) special graphs.

For $r, t \in \mathbb{N}$, t even, let $U = \{u_1, \dots, u_r\}$, $V = \{v_1, \dots, v_r\}$, $C = \{c_{i,j}, c'_{i,j} : i \in [r], j \in [2t]\}$, and $B = \{b_{i,j}, b'_{i,j} : i \in [r], j \in [2t]\}$. Let the *ladder* $R_r^* = (U \dot\cup V, E(R_r^*))$ have edges $E(R_r^*) := \{u_i v_j : i, j \in [r], |i - j| \leq 1\}$ and let the *spin graph* $R_{r,t} = (U \dot\cup V \dot\cup C \dot\cup B, E(R_{r,t}))$ be the graph with the following edge set (see Figure 9.1):

$$E(R_{r,t}) := \bigcup_{\substack{i,i' \in [r], i' \neq 1 \\ j,j' \in [2t] \\ k,k' \in [t] \\ \ell,\ell' \in [t+1,2t]}} \Big(\Big\{ u_i v_i,\ b_{i,k} b'_{i,k'},\ b_{i,\ell} b'_{i,\ell'},\ c_{i,k} c'_{i,k'},\ c_{i,\ell} c'_{i,\ell'} \Big\} \cup \Big\{ b_{i,j} v_i,\ c_{i,j} v_i \Big\} \cup \Big\{ b'_{i,k} b'_{i,\ell},\ c_{i'-1,\ell} c'_{i',k},\ c_{i'-1,\ell} c'_{i',k} \Big\} \Big).$$

Now we can state our four main lemmas, two partition lemmas and two embedding lemmas. We start with the lemma for G, which constructs a partition of the host graph G. This lemma is a consequence of the sparse regularity lemma (Lemma 3.18) and asserts a p-dense partition of G such that its reduced graph contains a spin graph. We will indicate below why this is useful for the embedding of H. The lemma for G produces clusters of very different sizes: A set of larger clusters U_i and V_i which we call *big clusters* and which will accommodate most of the vertices of H later, and a set of smaller clusters $B_{i,j}$,$B'_{i,j}$, $C_{i,j}$, and $C'_{i,j}$. The $B_{i,j}$ and $B'_{i,j}$ are called *balancing clusters* and the $C_{i,j}$ and $C'_{i,j}$ *connecting clusters*. They will be used to host a small number of vertices of H. These vertices balance and connect the pieces of H that are embedded into the big clusters (see also the explanations in Sections 1.3.2 and 1.3.4). The proof of Lemma 9.2 is given in Section 9.5.

In the formulation of this lemma (and also in the lemma for H below) we abuse the notation in the following sense. For two sets A and B and a number x we write $|A| := |B| \geq x$ by which we simultaneously mean that A is defined to be the set B and that the size $|A| = |B|$ of this set is at least x.

Lemma 9.2 (Lemma for G)**.** *For all integers $t, r_0 > 0$ and reals $\eta, \gamma > 0$ there are positive reals η' and d such that for all $\varepsilon > 0$ there is r_1 such that the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with* $\log^4 n/(pn) = o(1)$*. Let $G = (V, E)$ be a spanning subgraph of Γ with $\deg_G(v) \geq (\frac{1}{2}+\gamma)\deg_\Gamma(v)$ for all $v \in V$. Then there is $r_0 \leq r \leq r_1$, a subset V_0 of V with $|V_0| \leq \varepsilon n$, and a mapping g from $V \setminus V_0$ to the spin graph $R_{r,t}$ such that for every $i \in [r], j \in [2t]$ we have*

(G1) $|U_i| := |g^{-1}(u_i)| \geq (1-\eta)\frac{n}{2r}$ *and* $|V_i| := |g^{-1}(v_i)| \geq (1-\eta)\frac{n}{2r}$,

(G2) $|C_{i,j}| := |g^{-1}(c_{i,j})| \geq \eta'\frac{n}{2r}$ *and* $|C'_{i,j}| := |g^{-1}(c'_{i,j})| \geq \eta'\frac{n}{2r}$,
$|B_{i,j}| := |g^{-1}(b_{i,j})| \geq \eta'\frac{n}{2r}$ *and* $|B'_{i,j}| := |g^{-1}(b'_{i,j})| \geq \eta'\frac{n}{2r}$,

(G3) *the pair $(g^{-1}(x), g^{-1}(y))$ is (ε, d, p)-dense for all $xy \in E(R_{r,t})$.*

Our second lemma provides a partition of H that fits to the structure of the partition of G generated by Lemma 9.2. We will first state this lemma and then explain the different properties it guarantees.

Lemma 9.3 (Lemma for H)**.** *For all integers Δ there is an integer $t > 0$ such that for any $\eta > 0$ and any integer $r \geq 1$ there is $\beta > 0$ such that the following holds for all integers m and all bipartite graphs H on m vertices with $\Delta(H) \leq \Delta$ and* $\mathrm{bw}(H) \leq \beta m$*. There is a homomorphism h from H to the spin graph $R_{r,t}$ such that for every $i \in [r], j \in [2t]$*

(H1) $|\tilde{U}_i| := |h^{-1}(u_i)| \leq (1+\eta)\frac{m}{2r}$ *and* $|\tilde{V}_i| := |h^{-1}(v_i)| \leq (1+\eta)\frac{m}{2r}$,

(H2) $|\tilde{C}_{i,j}| := |h^{-1}(c_{i,j})| \leq \eta\frac{m}{2r}$ *and* $|\tilde{C}'_{i,j}| := |h^{-1}(c'_{i,j})| \leq \eta\frac{m}{2r}$,
$|\tilde{B}_{i,j}| := |h^{-1}(b_{i,k})| \leq \eta\frac{m}{2r}$ *and* $|\tilde{B}'_{i,j}| := |h^{-1}(b'_{i,k})| \leq \eta\frac{m}{2r}$,

(H3) $\tilde{C}_{i,j}$, $\tilde{C}'_{i,j}$, $\tilde{B}_{i,j}$, *and* $\tilde{B}'_{i,j}$ *are 3-independent in H,*

(H4) $\deg_{\tilde{V}_i}(y) = \deg_{\tilde{V}_i}(y') \leq \Delta - 1$ *for all* $yy' \in \binom{\tilde{C}_{i,j}}{2} \cup \binom{\tilde{B}_{i,j}}{2}$,
$\deg_{\tilde{C}_i}(y) = \deg_{\tilde{C}_i}(y')$ *for all* $y, y' \in \tilde{C}'_{i,j}$,
$\deg_{L(i,j)}(y) = \deg_{L(i,j)}(y')$ *for all* $y, y' \in \tilde{B}'_{i,j}$,

where $\tilde{C}_i := \bigcup_{k\in[2t]} \tilde{C}_{i,k}$ and $L(i,j) := \bigcup_{k\in[2t]} \tilde{B}_{i,k} \cup \bigcup_{k<j} \tilde{B}'_{i,k}$. Further, let $\tilde{X}_i$ with $i \in [r]$ be the set of vertices in $\tilde{V}_i$ with neighbours outside $\tilde{U}_i$. Then

(H5) $|\tilde{X}_i| \leq \eta|\tilde{V}_i|$.

This lemma asserts a homomorphism h from H to a spin graph $R_{r,t}$. Recall that $R_{r,t}$ is contained in the reduced graph of the p-dense partition provided by Lemma 9.2. As we will see we can fix the parameters in this lemma such that, when we apply it together with Lemma 9.2, the homomorphism h further has the following property. The number $\tilde{L}$ of vertices that it maps to a vertex a of the spin graph is less than the number L contained in the corresponding cluster A provided by Lemma 9.2 (compare (G1) and (G2) with (H1) and (H2)). If A is a big cluster, then the numbers L and $\tilde{L}$ differ only slightly (these vertices will be embedded using the constrained blow-up lemma), but for balancing and connecting clusters A the number $\tilde{L}$ is much smaller than L (this is necessary for the embedding of these vertices using the connection lemma). With property (H5) Lemma 9.3 further guarantees that only few edges of H are not assigned either to two connecting or balancing clusters, or to two big clusters. This is helpful because it implies that we do not have to take care of "too many dependencies" between the applications of the blow-up lemma and the connection lemma. The remaining properties (H3)–(H4) of Lemma 9.3 are technical but required for the application of the connection lemma (see conditions (B) and (C) of Lemma 9.5).

The vertices in $\tilde{C}_{i,j}$ and $\tilde{C}'_{i,j}$ are also called *connecting vertices* of H, the vertices in $\tilde{B}_{i,j}$ and $\tilde{B}'_{i,j}$ *balancing vertices.*

We next describe the two embedding lemmas, the constrained blow-up lemma (Lemma 9.4) and the connection lemma (Lemma 9.5), which we would like to use on the partitions of G and H provided by Lemmas 9.2 and 9.3. The connecting lemma will be used to embed the connecting and balancing vertices to the connecting and balancing clusters *after* all the other vertices were embedded to the big clusters with the help of the constrained blow-up lemma.

The constrained blow-up lemma states that bipartite graphs H with bounded maximum degree can be embedded into a p-dense pair $G = (U, V)$ whose cluster sizes are just slightly bigger than the partition classes of H. This lemma further guarantees the following. If we specify in one of the partition classes of H a small family of small special sets and in the corresponding cluster of G a small family of small forbidden sets, then no special set is mapped to a forbidden set. This property will be crucial in the application of this lemma together with the connection lemma in the proof of Theorem 9.1 in order to handle the "dependencies" between these applications. The proof of this Lemma is given in Section 9.7 and relies on techniques developed in [8].

Lemma 9.4 (Constrained blow-up lemma)**.** *For every integer $\Delta > 1$ and for all positive reals d and η there exist positive constants ε and μ such that for all positive integers r_1 there is c such that for all integers $1 \le r \le r_1$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \ge c(\log n/n)^{1/\Delta}$. Let $G = (U, V) \subseteq \Gamma$ be an (ε, d, p)-dense pair with $|U|, |V| \ge n/r$ and let H be a bipartite graph on vertex classes $\tilde{U} \dot\cup \tilde{V}$ of sizes $|\tilde{U}|, |\tilde{V}| \le (1-\eta)n/r$ and with $\Delta(H) \le \Delta$. Moreover, suppose that there is a family $\mathcal{H} \subseteq \binom{\tilde{V}}{\Delta}$ of* special Δ-sets *in $\tilde{V}$ such that each $\tilde{v} \in \tilde{V}$ is contained in at most Δ special sets and a family $\mathcal{B} \subseteq \binom{V}{\Delta}$ of* forbidden Δ-sets *in V with $|\mathcal{B}| \le \mu |V|^{\Delta}$. Then there is an embedding of H into G such that no special set is mapped to a forbidden set.*

Our last main lemma, the connection lemma (Lemma 9.5), embeds (partitioned) graphs H into graphs G forming a system of p-dense pairs. In contrast to the blow-up lemma, however, the graph H has to be much smaller than the graph G now (see condition (A)). In addition, each vertex $\tilde{y}$ of H is equipped with a candidate set $C(\tilde{y})$ in G from which the connection lemma will choose the image of $\tilde{y}$ in the embedding. Lemma 9.5 requires that these candidate sets are big (condition (D)) and that pairs of candidate sets that correspond to an edge of H form p-dense pairs (condition (E)). The remaining conditions ((B) and (C)) are conditions on the neighbourhoods and degrees of the vertices in H (with respect to the given partition of H). For their statement we need the following additional definition.

For a graph H on vertex set $\tilde{V} = \tilde{V}_1\dot\cup\dots\dot\cup\tilde{V}_t$ and $y \in \tilde{V}_i$ with $i \in [t]$ define the *left degree* of v with respect to the partition $\tilde{V}_1\dot\cup\dots\dot\cup\tilde{V}_t$ to be $\mathrm{ldeg}(y; \tilde{V}_1, \dots, \tilde{V}_t) := \sum_{j=1}^{i-1} \deg_{\tilde{V}_j}(y)$. When clear from the context we may also omit the partition and simply write $\mathrm{ldeg}(y)$.

Lemma 9.5 (Connection lemma). *For all integers $\Delta > 1$, $t > 0$ and reals $d > 0$ there are ε, $\xi > 0$ such that for all positive integers r_1 there is $c > 1$ such that for all integers $1 \le r \le r_1$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \ge c(\log n/n)^{1/\Delta}$. Let $G \subseteq \Gamma$ be any graph on vertex set $W = W_1\dot\cup\dots\dot\cup W_t$ and let H be any graph on vertex set $\tilde{W} = \tilde{W}_1\dot\cup\dots\dot\cup\tilde{W}_t$. Suppose further that for each $i \in [t]$ each vertex $\tilde{w} \in \tilde{W}_i$ is equipped with an arbitrary set $X_{\tilde{w}} \subseteq V(\Gamma) \setminus W$ with the property that the indexed set system $\big(X_{\tilde{w}} \colon \tilde{w} \in \tilde{W}_i\big)$ consists of pairwise disjoint sets and such that the following holds. We define the* external degree *of $\tilde{w}$ to be* $\mathrm{edeg}(\tilde{w}) := |X_{\tilde{w}}|$, *its* candidate set $C(\tilde{w}) \subseteq W_i$ *to be $C(\tilde{w}) := N^{\cap}_{W_i}(X_{\tilde{w}})$, and require that*

(A) *$|W_i| \ge n/r$ and $|\tilde{W}_i| \le \xi n/r$,*
(B) *$\tilde{W}_i$ is a 3-independent set in H,*
(C) $\mathrm{edeg}(\tilde{w}) + \mathrm{ldeg}(\tilde{w}) = \mathrm{edeg}(\tilde{v}) + \mathrm{ldeg}(\tilde{v})$ *and* $\deg_H(\tilde{w}) + \mathrm{edeg}(\tilde{w}) \le \Delta$ *for all $\tilde{w}, \tilde{v} \in \tilde{W}_i$,*
(D) *$|C(\tilde{w})| \ge ((d - \varepsilon)p)^{\mathrm{edeg}(\tilde{w})}|W_i|$ for all $\tilde{w} \in \tilde{W}_i$, and*
(E) *$(C(\tilde{w}), C(\tilde{v}))$ forms an (ε, d, p)-dense pair for all $\tilde{w}\tilde{v} \in E(H)$.*

Then there is an embedding of H into G such that every vertex $\tilde{w} \in \tilde{W}$ is mapped to a vertex in its candidate set $C(\tilde{w})$.

The proof of this lemma is inherent in [61]. We adapt it to our setting in Section 9.8.

9.2 Stars in random graphs

In this section we formulate two lemmas concerning properties of random graphs that will be useful when analysing neighbourhood properties of p-dense pairs in the following section. More precisely, we consider the following question here. Given a vertex set X in a random graph $\Gamma = \mathcal{G}_{n,p}$ together with a family $\mathcal{F}$ of pairwise disjoint ℓ-sets in $V(\Gamma)$. Then we would like to determine how many pairs (x, F) with $x \in X$ and $F \in \mathcal{F}$ have the property that x lies in the common neighbourhood of the vertices in F.

Definition 9.6 (stars). *Let $G = (V, E)$ be a graph, X be a subset of V and $\mathcal{F}$ be a family of pairwise disjoint ℓ-sets in $V \setminus X$ for some ℓ. Then the number of* stars *in G between X and $\mathcal{F}$ is*

$$\#\,\mathrm{stars}^G(X, \mathcal{F}) := \Big|\,\big\{\, (x, F) \colon\ x \in X,\ F \subseteq N_G(x) \,\big\}\,\Big|. \tag{9.1}$$

Observe that in a random graph $\Gamma = \mathcal{G}_{n,p}$ and for fixed sets X and $\mathcal{F}$ the random variable $\#\,\mathrm{stars}^{\Gamma}(X, \mathcal{F})$ has binomial distribution $\mathrm{Bi}(|X||\mathcal{F}|, p^{\ell})$. This will be used in the proofs of the following lemmas. The first of these lemmas states that in $\mathcal{G}_{n,p}$ the number of stars between X and $\mathcal{F}$ does not exceed its expectation by more than seven times as long as X and $\mathcal{F}$ are not too small. This is a straight-forward consequence of Chernoff's inequality.

Lemma 9.7 (star lemma for big sets). *For all positive integers Δ, and positive reals ν there is c such that if $p \ge c(\log n/n)^{1/\Delta}$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V.*

Let X be any subset of V and $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $\nu n \leq |X| \leq |\mathcal{F}| \leq n$, then

$$\#\,\text{stars}^{\Gamma}(X,\mathcal{F}) \leq 7p^{\Delta}|X||\mathcal{F}|$$

Proof. Given Δ and ν let c be such that $7c^{\Delta}\nu^2 \geq 3\Delta$. From Chernoff's inequality (2.1) we know that $\mathbb{P}[Y \geq 7\,\mathbb{E}\,Y] \leq \exp(-7\,\mathbb{E}\,Y)$ for a binomially distributed random variable Y. We conclude that for fixed X and $\mathcal{F}$

$$\begin{aligned}\mathbb{P}\left[\#\,\text{stars}^{\Gamma}(X,\mathcal{F}) > 7p^{\Delta}|X||\mathcal{F}|\right] &\leq \exp(-7p^{\Delta}|X||\mathcal{F}|)\\ &\leq \exp(-7c^{\Delta}(\log n/n)\nu^2 n^2) \leq \exp(-3\Delta n \log n)\end{aligned}$$

by the choice of c. Thus the probability that there are sets X and $\mathcal{F}$ violating the assertion of the lemma is at most

$$2^n n^{\Delta n} \exp(-3\Delta n \log n) \leq \exp(2\Delta n \log n - 3\Delta n \log n)$$

which tends to 0 as n tends to infinity. □

We will also need a variant of this lemma for smaller sets X and families $\mathcal{F}$ which is provided in the next lemma. As a trade-off the bound on the number of stars provided by this lemma will be somewhat worse. This lemma almost appears in this form in [61]. The only (slight) modification that we need here is that X is allowed to be bigger than $\mathcal{F}$. However, the same proof as presented in [61] still works for this modified version. We delay it to Section 9.9.

Lemma 9.8 (star lemma for small sets)**.** *For all positive integers Δ and positive reals ξ there are positive constants ν and c such that if $p \geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V. Let X be any subset of V and $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $|X| \leq \nu n p^{\Delta}|\mathcal{F}|$ and $|X|, |\mathcal{F}| \leq \xi n$, then*

$$\#\,\text{stars}^{\Gamma}(X,\mathcal{F}) \leq p^{\Delta}|X||\mathcal{F}| + 6\xi n p^{\Delta}|\mathcal{F}|. \tag{9.2}$$

9.3 Joint neighbourhoods in p-dense pairs

As discussed in Section 3.4 of Chapter 3 it follows directly from the definition of p-denseness that sub-pairs of dense pairs form again dense pairs. For applying Lemma 9.4 and Lemma 9.5 together, we will need corresponding results on joint neighbourhoods in systems of dense pairs (see Lemmas 9.10 and 9.13). In order to state them it is necessary to first introduce some notation.

Let $G = (V,E)$ be a graph, $\ell, T > 0$ be integers, p, ε, d be positive reals, and X, Y, $Z \subseteq V$ be disjoint vertex sets. Recall that for a set B of vertices from V and a vertex set $Y \subseteq V$ we call the set $N_Y^{\cap}(B) = \bigcap_{b\in B} N_Y(b)$ the *joint neighbourhood* of (the vertices in) B in Y.

Definition 9.9 (Bad and good vertex sets)**.** *Let G, ℓ, T, p, ε, d, X, Y, and Z be as above. We define the following family of ℓ-sets in Y with small joint neighbourhood in Z:*

$$\text{bad}^{G,\ell}_{\varepsilon,d,p}(Y,Z) := \left\{B \in \binom{Y}{\ell} \colon\ |N_Z^{\cap}(B)| < (d-\varepsilon)^{\ell} p^{\ell} |Z|\right\}. \tag{9.3}$$

If (Y,Z) has p-density $d_{G,p}(Y,Z) \geq d-\varepsilon$, then all ℓ-sets $T \in \binom{Y}{\ell}$ that are not in $\text{bad}^{G,\ell}_{\varepsilon,d,p}(Y,Z)$ *are called* p-good *in (Y,Z). Let further*

$$\text{Bad}^{G,\ell}_{\varepsilon,d,p}(X,Y,Z)$$

be the family of ℓ-sets $B \in \binom{X}{\ell}$ that contain an ℓ'-set $B' \subseteq B$ with $\ell' > 0$ such that either $|N_Y^{\cap}(B')| < (d-\varepsilon)^{\ell'} p^{\ell'} |Y|$ or $(N_Y^{\cap}(B'), Z)$ is not (ε, d, p)-dense in G.

The following Lemma states that p-dense pairs in random graphs have the property that most ℓ-sets have big common neighbourhoods. Results of this type (with a slightly smaller exponent in the edge probability p) were established in [58]. The proof of Lemma 9.10 is given in Section 9.9.

Lemma 9.10 (joint neighbourhood lemma)**.** *For all integers $\Delta, \ell \geq 1$ and positive reals d, ε' and μ, there is $\varepsilon > 0$ such that for all $\xi > 0$ there is $c > 1$ such that if $p \geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1 \geq \xi p^{\Delta-1} n$, $n_2 \geq \xi p^{\Delta-\ell} n$ let $G = (X \dot\cup Y, E)$ be any bipartite subgraph of Γ with $|X| = n_1$ and $|Y| = n_2$. If (X,Y) is an (ε, d, p)-dense pair, then $|\operatorname{bad}_{\varepsilon',d,p}^{G,\ell}(X,Y)| \leq \mu n_1^{\ell}$.*

Thus we know that typical vertex sets in dense pairs inside random graphs are p-good. In the next lemma we observe that families of such p-good vertex sets exhibit strong expansion properties.

Given ℓ and p we say that a bipartite graph $G = (X \dot\cup Y, E)$ is *(A, f)-expanding* if, for any family $\mathcal{F} \subseteq \binom{X}{\ell}$ of pairwise disjoint p-good ℓ-sets in (X,Y) with $|\mathcal{F}| \leq A$, we have $|N_Y^{\cap}(\mathcal{F})| \geq f|\mathcal{F}|$.

Lemma 9.11 (expansion lemma)**.** *For all positive integers Δ and positive reals d and ε, there are positive ν and c such that if $p \geq c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. Let $G = (X \dot\cup Y, E)$ be a bipartite subgraph of Γ. If (X,Y) is an (ε, d, p)-dense pair, then (X,Y) is $(1/p^{\Delta}, \nu n p^{\Delta})$-expanding.*

Proof. Given ℓ, d, ε, set $\delta := d - \varepsilon$, $\xi := \delta^{\Delta}/7$ and let ν' and c be the constants from Lemma 9.8 for this Δ and ξ. Further, choose ν such that $\nu \leq \xi$ and $\nu \leq \nu'$. Let $\mathcal{F} \subseteq \binom{X}{\Delta}$ be a family of pairwise disjoint p-good Δ-sets with $|\mathcal{F}| \leq 1/p^{\Delta}$. Let $U = N_Y^{\cap}(\mathcal{F})$ be the joint neighbourhood of $\mathcal{F}$ in Y. We wish to show that $|U| \geq (\nu n p^{\Delta})|\mathcal{F}|$. Suppose the contrary. Then $|U| < \nu' n p^{\Delta} |\mathcal{F}|$, $|U| < \nu n p^{\Delta} |\mathcal{F}| \leq \nu n \leq \xi n$ and $|\mathcal{F}| \leq 1/p^{\Delta} \leq c^{\Delta} n/\log n \leq \xi n$ for n sufficiently large and so we can apply Lemma 9.8 with parameters Δ and ξ to U and $\mathcal{F}$. Since every member of $\mathcal{F}$ is p-good in (X,Y), we thus have

$$\begin{aligned} \delta^{\Delta} p^{\Delta} n |\mathcal{F}| &\leq \#\operatorname{stars}^{G}(U, \mathcal{F}) \leq \#\operatorname{stars}^{\Gamma}(U, \mathcal{F}) \overset{(9.2)}{\leq} p^{\Delta}|U||\mathcal{F}| + 6\xi n p^{\Delta}|\mathcal{F}| \\ &< p^{\Delta}(\nu n p^{\Delta})|\mathcal{F}||\mathcal{F}| + 6\xi n p^{\Delta}|\mathcal{F}| \leq \nu n p^{\Delta}|\mathcal{F}| + 6\xi n p^{\Delta}|\mathcal{F}| \leq 7\xi n p^{\Delta}|\mathcal{F}|, \end{aligned}$$

which tells that $\delta^{\Delta} < 7\xi$, a contradiction. □

In the remainder of this section we are interested in the inheritance of p-denseness to sub-pairs (X', Y') of p-dense pairs (X,Y) in a graph $G = (V,E)$. It comes as a surprise that even for sets X' and Y' that are much smaller than the sets considered in the definition of p-denseness, such sub-pairs are typically dense. Phenomena of this type were observed in [58, 37].

Here, we will consider sub-pairs induced by neighbourhoods of vertices $v \in V$ (which may or may not be in $X \dot\cup Y$), i.e., sub-pairs (X', Y') where X' (or Y' or both) is the neighbourhood of v in Y (or in X). Further, we only consider the case when G is a subgraph of a random graph $\mathcal{G}_{n,p}$.

In [61] an inheritance result of this form was obtained for triples of dense pairs. More precisely, the following holds for subgraphs G of $\mathcal{G}_{n,p}$. For sufficiently large vertex set X, Y, and Z in G such that (X,Y) and (Y,Z) form p-dense pairs we have that most vertices $x \in X$ are such that $(N_Y(x), Y)$ forms again a p-dense pair (with slightly changed parameters). If, moreover, also (Y,Z) forms a p-dense pair, then also $(N_Y(x), N_Z(x))$ is typically a p-dense pair.

Lemma 9.12 (inheritance lemma for vertices [61]). *For all integers $\Delta > 0$ and positive reals d_0, ε' and μ there is ε such that for all $\xi > 0$ there is $c > 1$ such that if $p > c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For n_1, $n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-2} n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. If (X,Y) and (Y,Z) are (ε, d, p)-dense pairs in G with $d \geq d_0$, then there are at most μn_1 vertices $x \in X$ such that $(N(x) \cap Y, Z)$ is not an (ε', d, p)-dense pair in G.*

If, additionally, (X,Z) is (ε, d, p)-dense and n_1, n_2, $n_3 \geq \xi p^{\Delta-2} n$, then there are at most μn_1 vertices $x \in X$ such that $(N(x) \cap Y, N(x) \cap Z)$ is not an (ε', d, p)-dense pair in G. □

In order to combine the restricted blow-up lemma (Lemma 9.4) and the connection lemma (Lemma 9.5) in the proof of Theorem 9.1 we will need a version of this result for ℓ-sets. Such a lemma, stating that joint neighbourhoods of certain ℓ-sets form again p-dense pairs, can be obtained by an inductive argument from the first part of Lemma 9.12. We defer its proof to Section 9.9.

Lemma 9.13 (inheritance lemma for ℓ-sets). *For all integers $\Delta, \ell > 0$ and positive reals d_0, ε', and μ there is ε such that for all $\xi > 0$ there is $c > 1$ such that if $p > c(\frac{\log n}{n})^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1, n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-\ell-1} n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. Assume further that (X,Y) and (Y,Z) are (ε, d, p)-dense pairs with $d \geq d_0$. Then*

$$\left|\mathrm{Bad}^{G,\ell}_{\varepsilon',d,p}(X,Y,Z)\right| \leq \mu n_1^{\ell}.$$

9.4 Proof of Theorem 9.1

In this section we present a proof of Theorem 9.1 that combines our four main lemmas, the lemma for G (Lemma 9.2), the lemma for H (Lemma 9.3), the restricted blow-up lemma (Lemma 9.4), and the connection lemma (Lemma 9.5). This proof follows the outline given in Section 9.1. In addition we will apply the inheritance lemma for ℓ-sets (Lemma 9.13), which supplies an appropriate interface between the restricted blow-up lemma and the connection lemma.

Proof of Theorem 9.1. We first set up the constants. Given η, γ, and Δ let t be the constant promised by the lemma for H (Lemma 9.3) for input Δ. Set

$$\eta_G := \eta/10, \qquad \text{and} \qquad r_0 = 1\,, \tag{9.4}$$

and apply the lemma for G (Lemma 9.2) with input t, r_0, η_G, and γ in order to obtain η'_G and d. Next, the connection lemma (Lemma 9.5) with input Δ, $2t$, and d provides us with $\varepsilon_{\mathrm{CL}}$, and ξ_{CL}. We apply the constrained blow-up lemma (Lemma 9.4) with Δ, d, and $\eta/2$ in order to obtain $\varepsilon_{\mathrm{BL}}$ and μ_{BL}. With this we set

$$\eta_H := \min\{\eta/10,\ \xi_{\mathrm{CL}}\eta'_G,\ 1/(\Delta+1)\}. \tag{9.5}$$

Choose $\mu > 0$ such that

$$100t^2\mu \leq \eta_{\mathrm{BL}}, \tag{9.6}$$

and apply Lemma 9.13 with Δ and $\ell = \Delta - 1$, $d_0 = d$, $\varepsilon' = \varepsilon_{\mathrm{CL}}$, and μ to obtain $\varepsilon_{9.13}$. Let

$$\xi_{9.13} := \eta'_G/2r \tag{9.7}$$

and continue the application of Lemma 9.13 with $\xi_{9.13}$ to obtain $c_{9.13}$. Now we can fix

$$\varepsilon := \min\{\varepsilon_{\mathrm{CL}}, \varepsilon_{\mathrm{BL}}, \varepsilon_{9.13}\} \tag{9.8}$$

and continue the application of Lemma 9.2 with input ε to get r_1. Let $\hat{r}_{\mathrm{BL}}$ and $\hat{r}_{\mathrm{CL}}$ be such that

$$\frac{2r_1}{1-\eta_G} \leq \hat{r}_{\mathrm{BL}} \qquad \text{and} \qquad \frac{2r_1}{\eta_G} \leq \hat{r}_{\mathrm{CL}} \tag{9.9}$$

and let c_{CL} and c_{BL} be the constants obtained from the continued application of Lemma 9.5 with r_1 replaced by $\hat{r}_{\mathrm{CL}}$ and Lemma 9.4 with r_1 replaced by $\hat{r}_{\mathrm{BL}}$, respectively.

We continue the application of Lemma 9.3 with input η_H. For each $r \in [r_1]$ Lemma 9.3 provides a value β_r , among all of which we choose the smallest one and set β to this value. Finally, we set $c := \max\{c_{\mathrm{BL}}, c_{\mathrm{CL}}, c_{9.13}\}$.

Consider a graph $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$. Then Γ a.a.s. satisfies the properties stated in Lemma 9.2, Lemma 9.4, Lemma 9.5, and Lemma 9.13, with the parameters previously specified. We assume in the following that this is the case and show that then also the following holds. For all subgraphs $G \subseteq \Gamma$ and all graphs H such that G and H have the properties required by Theorem 9.1 we have $H \subseteq G$. To summarise the definition of the constants above, we assume, more precisely, that Γ satisfies the conclusion of the following lemmas:

- (L9.2) Lemma 9.2 for parameters t, $r_0 = 1$, η_G, γ, η'_G, d, ε, and r_1, i.e., if G is any spanning subgraph of Γ satisfying the requirements of Lemma 9.2, then we obtain a partition of G as specified in the lemma and with these parameters,
- (L9.4) Lemma 9.4 for parameters Δ, d, $\eta/2$, $\varepsilon_{\mathrm{BL}}$, μ_{BL}, and $\hat{r}_{\mathrm{BL}}$,
- (L9.5) Lemma 9.5 for parameters Δ, $2t$, d, $\varepsilon_{\mathrm{CL}}$, ξ_{CL}, and $\hat{r}_{\mathrm{CL}}$,
- (L9.13) Lemma 9.13 for parameters Δ, $\ell = \Delta - 1$, $d_0 = d$, $\varepsilon' = \varepsilon_{\mathrm{CL}}$, μ, $\varepsilon_{9.13}$, and $\xi_{9.13}$.

Now suppose we are given a graph $G = (V, E) \subseteq \Gamma$ with $\deg_G(v) \geq (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$ and $|V| = n$, and a graph $H = (\tilde{V}, \tilde{E})$ with $|\tilde{V}| = (1-\eta)n$. Before we show that H can be embedded into G we will use the lemma for G (Lemma 9.2) and the lemma for H (Lemma 9.3) to prepare G and H for this embedding.

First we use the fact that Γ has property (L9.2). Hence, for the graph G we obtain an r with $1 \leq r \leq r_1$ from Lemma 9.2, together with a set $V_0 \subseteq V$ with $|V_0| \leq \varepsilon n$, and a mapping $g\colon V \setminus V_0 \to R_{r,t}$ such that (G1)–(G3) of Lemma 9.2 are fulfilled. For all $i \in [r], j \in [2t]$ let U_i, V_i, $C_{i,j}$, $C'_{i,j}$, $B_{i,j}$, and $B'_{i,j}$ be the sets defined in Lemma 9.2. Recall that these sets were called big clusters, connecting clusters, and balancing clusters. With this the graph G is prepared for the embedding. We now turn to the graph H.

We assume for simplicity that $2r/(1-\eta_G)$ and $r/(t\eta'_G)$ are integers and define

$$r_{\mathrm{BL}} := 2r/(1-\eta_G) \qquad \text{and} \qquad r_{\mathrm{CL}} := 2r/\eta'_G\,. \tag{9.10}$$

We apply Lemma 9.3 which we already provided with Δ and η_H. For input H this lemma provides a homomorphism h from H to $R_{r,t}$ such that (H1)–(H5) of Lemma 9.3 are fulfilled.

For all $i \in [r], j \in [2t]$ let $\tilde{U}_i$, $\tilde{V}_i$, $\tilde{C}_{i,j}$, $\tilde{C}'_{i,j}$, $\tilde{B}_{i,j}$, $\tilde{B}'_{i,j}$, and $\tilde{X}_i$ be the sets asserted by Lemma 9.3. Further, set $C_i := C_{i,1}\dot{\cup}\dots\dot{\cup}C_{i,2t}$, $\tilde{C}_i := \tilde{C}_{i,1}\dot{\cup}\dots\dot{\cup}\tilde{C}_{i,2t}$, that is, C_i consists of connecting clusters and $\tilde{C}_i$ of connecting vertices. Define C'_i, $\tilde{C}'_i$, B_i, $\tilde{B}_i$, B'_i, and $\tilde{B}'_i$ analogously (B_i consists of balancing clusters and $\tilde{B}_i$ of balancing vertices).

Our next goal will be to appeal to property (L9.4) which asserts that we can apply the constrained blow-up lemma (Lemma 9.4) for each p-dense pair (U_i, V_i) with $i \in [r]$ individually and embed $H[\tilde{U}_i\dot{\cup}\tilde{V}_i]$ into this pair. For this we fix $i \in [r]$. We will first set up special Δ-sets $\mathcal{H}_i$ and forbidden Δ-sets $\mathcal{B}_i$ for the application of Lemma 9.4. The idea is as follows. With the help of Lemma 9.4 we will embed all vertices in $\tilde{U}_i\dot{\cup}\tilde{V}_i$. But all connecting and balancing vertices of H remain unembedded. They will be handled by the connection lemma, Lemma 9.5, later on. However, these two lemmas cannot operate independently. If, for example, a connecting vertex $\tilde{y}$ has three neighbours in $\tilde{V}_i$, then these neighbours will be embedded already to vertices v_1, v_2, v_3 in V_i (by the blow-up lemma) when we want to embed $\tilde{y}$. Accordingly the image of $\tilde{y}$ in the embedding is confined to the joint neighbourhood of the vertices v_1, v_2, v_3 in G. In other words, this joint neighbourhood will be the candidate set $C(\tilde{y})$ in the application of Lemma 9.5. This lemma requires, however, that candidate sets are not too small (condition (D) of Lemma 9.5) and, in addition, that candidate sets of vertices joined by an edge induce p-dense pairs (condition (E)). Hence we need to prepare for these requirements. This will be done via the special and forbidden sets. The family of special sets $\mathcal{H}_i$ will contain neighbourhoods in $\tilde{V}_i$ of connecting or balancing vertices $\tilde{y}$ of H (observe that such vertices do not have neighbours in $\tilde{U}_i$, see Figure 9.1). The family of forbidden sets $\mathcal{B}_i$ will consist of sets in V_i which are “bad” for the embedding of these neighbourhoods in view of (D) and (E) of Lemma 9.5 (recall that Lemma 9.4 does not map special sets to forbidden sets). Accordingly, $\mathcal{B}_i$ contains Δ-sets that have small common neighbourhoods or do not induce p-dense pairs in one of the relevant balancing or connecting clusters. We will next give the details of this construction of $\mathcal{H}_i$ and $\mathcal{B}_i$.

We start with the special Δ-sets $\mathcal{H}_i$. As explained, we would like to include in the family $\mathcal{H}_i$ all neighbourhoods of vertices $\tilde{w}$ of vertices outside $\tilde{U}_i\dot{\cup}\tilde{V}_i$. Such neighbourhoods clearly lie entirely in the set $\tilde{X}_i$ provided by Lemma 9.3. However, they need not necessarily be Δ-sets (in fact, by (H4) of Lemma 9.3, they are of size at most $\Delta - 1$). Therefore we have to “pad” these neighbourhoods in order to obtain Δ-sets. This is done as follows. We start by picking an arbitrary set of $\Delta|\tilde{X}_i|$ vertices (which will be used for the “padding”) in $\tilde{V}_i \setminus \tilde{X}_i$. We add these vertices to $\tilde{X}_i$ and call the resulting set $\tilde{X}'_i$. This is possible because (H5) of Lemma 9.3 and (9.5) imply that $|\tilde{X}'_i| \le (\Delta + 1)|\tilde{X}_i| \le (\Delta + 1)\eta_H|\tilde{V}_i| \le |\tilde{V}_i|$.

Now let $\tilde{Y}_i$ be the set of vertices in $\tilde{B}_i\dot{\cup}\tilde{C}_i$ with neighbours in $\tilde{V}_i$. These are the vertices for whose neighbourhoods we will include Δ-sets in $\mathcal{H}_i$. It follows from the definition of $\tilde{X}_i$ that $|\tilde{Y}_i| \le \Delta|\tilde{X}_i|$. Let $\tilde{y} \in \tilde{Y}_i \subseteq \tilde{B}_i \cup \tilde{C}_i$. By the definition of $\tilde{X}_i$ we have $N_H(\tilde{y}) \subseteq \tilde{X}_i$. Next, we let

$$\tilde{X}_{\tilde{y}} \text{ be the set of neighbours of } \tilde{y} \text{ in } \tilde{V}_i \tag{9.11}$$

As explained, $\tilde{y}$ has strictly less than Δ neighbours in $\tilde{V}_i$ and hence we choose additional vertices from $\tilde{X}'_i \setminus \tilde{X}_i$. In this way we obtain for each $\tilde{y} \in \tilde{Y}_i$ a Δ-set $N_{\tilde{y}} \in \tilde{X}'_i$ with

$$N_{\tilde{X}_i}(\tilde{y}) = N_{\tilde{V}_i}(\tilde{y}) = \tilde{X}_{\tilde{y}} \subseteq N_{\tilde{y}}\,. \tag{9.12}$$

We make sure, in this process, that for any two different $\tilde{y}$ and $\tilde{y}'$ we never include the same additional vertex from $\tilde{X}'_i \setminus \tilde{X}_i$. This is possible because $|\tilde{X}'_i \setminus \tilde{X}_i| \ge \Delta|\tilde{X}_i| \ge |\tilde{Y}_i|$. We can thus guarantee that

$$\text{each vertex in } \tilde{X}_i' \text{ is contained in at most } \Delta \text{ sets } N_{\tilde{y}}. \tag{9.13}$$

The family of special Δ-sets for the application of Lemma 9.4 on (U_i, V_i) is then

$$\mathcal{H}_i := \{N_{\tilde{y}} \colon \tilde{y} \in \tilde{Y}_i\}. \tag{9.14}$$

Note that this is indeed a family of Δ-sets encoding all neighbourhoods in $\tilde{U}_i \dot\cup \tilde{V}_i$ of vertices outside this set.

Now we turn to the family $\mathcal{B}_i$ of forbidden Δ-sets. Recall that this family should contain sets that are forbidden for the embedding of the special Δ-sets because their joint neighbourhood in a (relevant) balancing or connecting cluster is small or does not induce a p-dense pair. More precisely, we are interested into Δ-sets S that have one of the following properties. Either S has a small common neighbourhood in some cluster from B_i or from C_i (observe that only balancing vertices from $\tilde{B}_i$ and connecting vertices from $\tilde{C}_i$ have neighbours in $\tilde{V}_i$). Or the neighbourhood $N_D^{\cap}(S)$ of S in a cluster D from B_i or C_i, respectively, is such that $(N_D^{\cap}(S), D')$ is not p-dense for some cluster D' from $B_i' \cup B_{i+1}'$ or $C_i' \cup C_{i+1}'$ (observe that edges between balancing vertices run only between $\tilde{B}_i$ and $\tilde{B}_i' \cup \tilde{B}_{i+1}'$ and edges between connecting vertices only between $\tilde{C}_i$ and $\tilde{C}_i' \cup \tilde{C}_{i+1}'$).

For technical reasons, however, we need to digress from this strategy slightly: We want to bound the number of Δ-sets in $\mathcal{B}_i$ with the help of the inheritance lemma for ℓ-sets, Lemma 9.13, later. Notice that, thanks to the lower bound on n_2 in Lemma 9.13, this lemma cannot be applied (in a meaningful way) for Δ-sets. But it can be applied for $(\Delta-1)$-sets. Therefore, we will not consider Δ-sets directly but first construct an auxiliary family of $(\Delta-1)$-sets and then, again, "pad" these sets to obtain a family of Δ-sets. Observe that the strategy outlined while setting up the special sets $\mathcal{H}_i$ still works with these $(\Delta-1)$-sets: neighbourhoods of connecting or balancing vertices in $\tilde{V}_i$ are of size at most $\Delta-1$ by (H4) of Lemma 9.3.

But now let us finally give the details. We first define the auxiliary family of $(\Delta-1)$-sets as follows:

$$\begin{aligned} \mathcal{B}_i' := &\bigcup_{\substack{i' \in \{i, i+1\}, j, j' \in [2t] \\ (c_{i,j}, c'_{i',j'}) \in R_{r,t}}} \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i, C_{i,j}, C'_{i',j'}) \quad \cup \\ &\bigcup_{\substack{j, j' \in [2t] \\ (b_{i,j}, b'_{i,j'}) \in R_{r,t}}} \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i, B_{i,j}, B_{i,j'}). \end{aligned} \tag{9.15}$$

We will next bound the size of this family by appealing to property (L9.13), and hence Lemma 9.13, with the tripartite graphs $G[V_i, C_{i,j}, C'_{i',j'}]$ and $G[V_i, B_{i,j}, B'_{i,j'}]$ with indices as in the definition of $\mathcal{B}_i'$. For this we need to check the conditions appearing in this lemma. By the definition of $R_{r,t}$ and (G3) of Lemma 9.2 all pairs $(C_{i,j}, C'_{i',j'})$ and $(B_{i,j}, B'_{i,j'})$ appearing in the definition of $\mathcal{B}_i'$ as well as the pairs $(V_i, C_{i,j})$ and $(V_i, B_{i,j})$ with $j \in [2t]$ are (ε, d, p)-dense. For the vertex sets of these dense pairs we know $|V_i|, |C'_{i',j'}|, |B'_{i,j'}| \geq \eta_G' n/2r \geq \xi_{9.13} p^{\Delta-1} n$ and $|C_{i,j}|, |B_{i,j}| \geq \eta_G' n/2r = \xi_{9.13} n$ by (G1) and (G2) of Lemma 9.2 and (9.7). Thus, since $\varepsilon \leq \varepsilon_{9.13}$, property (L9.13) implies that the family

$$\mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i, C_{i,j}, C'_{i',j'}), \text{ and } \mathrm{Bad}^{G,\Delta-1}_{\varepsilon_{\mathrm{CL}},d,p}(V_i, B_{i,j}, B'_{i,j'})$$

is of size $\mu|V_i|^{\Delta-1}$ at most. It follows from (9.15) that $|\mathcal{B}'_i| \le 8t^2\mu|V_i|^{\Delta-1}$ which is at most $\mu_{\text{BL}}|V_i|^{\Delta-1}$ by (9.6). The family of forbidden Δ-sets is then defined by

$$\mathcal{B}_i := \mathcal{B}'_i \times V_i \quad \text{and we have} \quad |\mathcal{B}_i| \le \mu_{\text{BL}}|V_i|^{\Delta} . \tag{9.16}$$

Having defined the special and forbidden Δ-sets we are now ready to appeal to (L9.4) and use the constrained blow-up Lemma (Lemma 9.4) with parameters Δ, d, $\eta/2$, ε_{BL}, μ_{BL}, $\hat{r}_{\text{BL}}$, and r_{BL} separately for each pair $G_i := (U_i, V_i)$ and for $H_i := H[\tilde{U}_i \dot\cup \tilde{V}_i]$. Let us quickly check that the constant r_{BL} and the graphs G_i and H_i satisfy the required conditions. Observe first, that $1 \le r_{\text{BL}} = 2r/(1-\eta_G) \le 2r_1/(1-\eta_G) \le \hat{r}_{\text{BL}}$ by (9.10) and (9.9). Moreover (U_i, V_i) is an $(\varepsilon_{\text{BL}}, d, p)$-dense pair by (G3) of Lemma 9.2 and (9.8). (G1) implies

$$|U_i| \ge (1-\eta_G)\frac{n}{2r} \overset{(9.10)}{=} \frac{n}{r_{\text{BL}}}$$

and similarly $|V_i| \ge n/r_{\text{BL}}$. By (H1) of Lemma 9.3 we have

$$\begin{aligned} |\tilde{U}_i| &\le (1+\eta_H)\frac{m}{2r} \le (1+\eta_H)(1-\eta)\frac{n}{2r} \le (1+\eta_H-\eta)\frac{n}{2r} \overset{(9.4),(9.5)}{\le} (1-\tfrac{1}{2}\eta-\eta_G)\frac{n}{2r} \\ &\le (1-\tfrac{1}{2}\eta)(1-\eta_G)\frac{n}{2r} \overset{(9.10)}{=} (1-\tfrac{1}{2}\eta)\frac{n}{r_{\text{BL}}} \end{aligned}$$

and similarly $|\tilde{V}_i| \le (1-\frac{\eta}{2})n/r_{\text{BL}}$. For the application of Lemma 9.4, let the families of special and forbidden Δ-sets be defined in (9.14) and (9.16), respectively. Observe that (9.13) and (9.16) guarantee that the required conditions (of Lemma 9.4) are satisfied. Consequently there is an embedding of H_i into G_i for each $i \in [r]$ such that no special Δ-set is mapped to a forbidden Δ-set. Denote the united embedding resulting from these r applications of the constrained blow-up lemma by $f_{\text{BL}} : \bigcup_{i\in[r]} \tilde{U}_i \cup \tilde{V}_i \to \bigcup_{i\in[r]} U_i \cup V_i$.

It remains to verify that f_{BL} can be extended to an embedding of all vertices of H into G. We still need to take care of the balancing and connecting vertices. For this purpose we will, again, fix $i \in [r]$ and use property (L9.5) which states that the conclusion of the connection lemma (Lemma 9.5) holds for parameters Δ, $2t$, d, ε_{CL}, ξ_{CL}, and $\hat{r}_{\text{CL}}$. We will apply this lemma with input r_{CL} to the graphs $G'_i := G[W_i]$ and $H'_i := H[\tilde{W}_i]$ where W_i and $\tilde{W}_i$ and their partitions for the application of the connection lemma are as follows (see Figure 9.2). Let $W_i := W_{i,1} \dot\cup \ldots \dot\cup W_{i,8t}$ where for all $j \in [t], k \in [2t]$ we set

$$\begin{aligned} W_{i,j} &:= C_{i,t+j} , & W_{i,t+j} &:= C_{i+1,j} , & W_{i,2t+j} &:= C'_{i,t+j} , \\ W_{i,3t+j} &:= C'_{i+1,j} , & W_{i,4t+k} &:= B_{i,k} , & W_{i,6t+k} &:= B'_{i,k} . \end{aligned}$$

(*This means that we propose the clusters in the following order to the connection lemma. The connecting clusters without primes come first, then the connecting clusters with primes, then the balancing clusters without primes, and finally the balancing clusters with primes.*)

The partition $\tilde{W}_i := \tilde{W}_{i,1} \dot\cup \ldots \dot\cup \tilde{W}_{i,8t}$ of the vertex set $\tilde{W}_i$ of H'_i is defined accordingly, i.e., for all $j \in [t], k \in [2t]$ we set

$$\begin{aligned} \tilde{W}_{i,j} &:= \tilde{C}_{i,t+j} , & \tilde{W}_{i,t+j} &:= \tilde{C}_{i+1,j} , & \tilde{W}_{i,2t+j} &:= \tilde{C}'_{i,t+j} , \\ \tilde{W}_{i,3t+j} &:= \tilde{C}'_{i+1,j} , & \tilde{W}_{i,4t+k} &:= \tilde{B}_{i,k} , & \tilde{W}_{i,6t+k} &:= \tilde{B}'_{i,k} . \end{aligned}$$

To check whether we can apply the connecting lemma observe first that

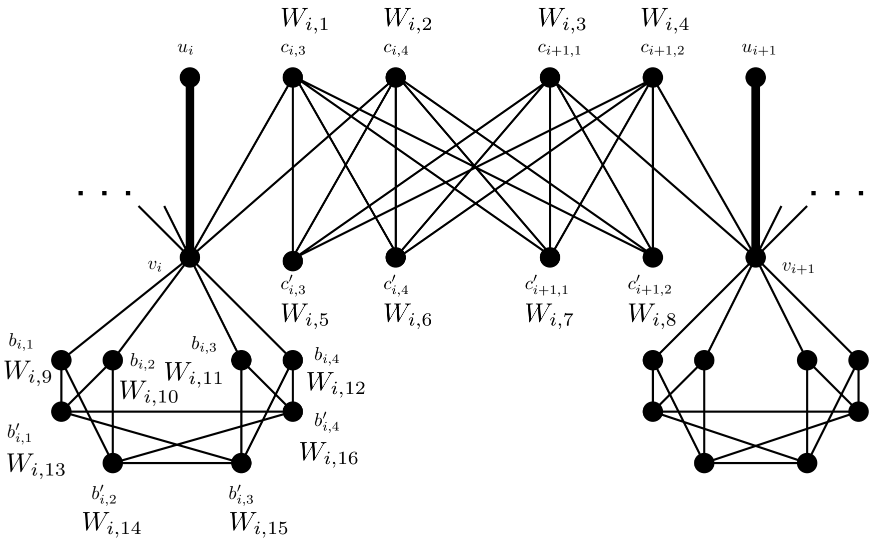


Figure 9.2: The partition $W_i = W_{i,1}\dot\cup \ldots \dot\cup W_{i,8t}$ of $G'_i = G[W_i]$ for the special case $t = 2$.

$$1 \leq 2r/\eta_G \leq 2r_1/\eta_G \leq \hat{r}_{\mathrm{CL}}$$

by (9.9). For $\tilde{y} \in \tilde{W}_{i,j}$ with $j \in [8t]$ recall from (9.11) (using that each vertex in H has neighbours in at most one set $\tilde{V}_{i'}$, see Figure 9.1) that

$$\tilde{X}_{\tilde{y}} \text{ is the set of neighbours of } \tilde{y} \text{ in } \tilde{V}_i \cup \tilde{V}_{i+1} \text{ and set } X_{\tilde{y}} := f_{\mathrm{BL}}(\tilde{X}_{\tilde{y}}). \tag{9.17}$$

Then the indexed set system $\left(\tilde{X}_{\tilde{y}} \colon \tilde{y} \in \tilde{W}_{i,j}\right)$ consists of pairwise disjoint sets because $\tilde{W}_{i,j}$ is 3-independent in H by (H3) of Lemma 9.3. Thus also $\left(X_{\tilde{y}} : \tilde{y} \in \tilde{W}_{i,j}\right)$ consists of pairwise disjoint sets, as required by Lemma 9.5. Now let the external degree and the candidate set of $\tilde{y} \in \tilde{W}_{i,j}$ be defined as in Lemma 9.5, i.e.,

$$\mathrm{edeg}(\tilde{y}) := |X_{\tilde{y}}| \qquad \text{and} \qquad C(\tilde{y}) := N^{\cap}_{W_{i,j}}(X_{\tilde{y}})\,. \tag{9.18}$$

Observe that this implies $C(\tilde{y}) = W_{i,j}$ if $\tilde{X}_{\tilde{y}} = \emptyset$ and hence $X_{\tilde{y}} = \emptyset$. Now we will check that conditions (A)–(E) of Lemma 9.5 are satisfied. From (G2) of Lemma 9.2 and (H2) of Lemma 9.3 it follows that

$$|W_{i,j}| \overset{\text{(G2)}}{\geq} \eta'_G \frac{n}{2r} \overset{(9.10)}{=} \frac{n}{r_{\mathrm{CL}}} \qquad \text{and}$$
$$|\tilde{W}_{i,j}| \overset{\text{(H2)}}{\leq} \eta_H \frac{m}{2r} \leq \eta_H \frac{n}{2r} \overset{(9.10)}{=} \frac{\eta_H}{\eta'_G} \frac{n}{r_{\mathrm{CL}}} \overset{(9.5)}{\leq} \xi_{\mathrm{CL}} \frac{n}{r_{\mathrm{CL}}}$$

and thus we have condition (A). By (H3) of Lemma 9.3 we also get condition (B) of Lemma 9.5. Further, it follows from (H4) of Lemma 9.3 that $\mathrm{edeg}(\tilde{y}) = \mathrm{edeg}(\tilde{y}')$ and $\mathrm{ldeg}(\tilde{y}) = \mathrm{ldeg}(\tilde{y}')$ for all $\tilde{y}, \tilde{y}' \in \tilde{W}_{i,j}$ with $j \in [8t]$. In addition $\Delta(H) \leq \Delta$ and hence

$$\begin{aligned}\deg_{H'_i}(\tilde{y}) + \mathrm{edeg}(\tilde{y}) &\overset{(9.18)}{=} |N_{\tilde{W}_i}(\tilde{y})| + |X_{\tilde{y}}| \\ &\overset{(9.17)}{=} |N_{\tilde{W}_i}(\tilde{y})| + |N_{\tilde{V}_i \cup \tilde{V}_{i+1}}(\tilde{y})| \leq \deg_H(\tilde{y}) \leq \Delta\end{aligned}$$

and thus condition (C) of Lemma 9.5 is satisfied. To check conditions (D) and (E) of Lemma 9.5 observe that for all $\tilde{y} \in \tilde{C}'_{i',j}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$ we have $C(\tilde{y}) = C'_{i',j}$ as $\tilde{y}$ has no neighbours in $\tilde{V}_i$ or $\tilde{V}_{i+1}$ and hence the external $\text{edeg}(\tilde{y}) = 0$ (see (9.17) and (9.18)). Thus (D) is satisfied for $\tilde{y} \in \tilde{C}'_{i',j}$, and similarly for $\tilde{y} \in \tilde{B}'_{i',j}$. For all $\tilde{y} \in \tilde{C}_{i,j}$ with $t < j \leq 2t$ on the other hand we have $\tilde{X}_{\tilde{y}} \subseteq N_{\tilde{y}} \in \binom{\tilde{V}_i}{\Delta}$ by (9.11). Recall that $N_{\tilde{y}}$ was a special Δ-set in the application of the restricted blow-up lemma on $G_i = (U_i, V_i)$ and $H_i = H[\tilde{U}_i \dot\cup \tilde{V}_i]$ owing to (9.14). Therefore $N_{\tilde{y}}$ is not mapped to a forbidden Δ-set in $\mathcal{B}_i \subseteq \binom{V_i}{\Delta}$ by f_{BL} and thus, by (9.15), to no Δ-set in $\text{Bad}^{G,\Delta-1}_{\varepsilon_{\text{CL}},d,p}(V_i, C_{i,j}, C'_{i',j'}) \times V_i$ with $i' \in \{i, i+1\}, j, j' \in [2t]$ and $(c_{i,j}, c'_{i',j'}) \in R_{r,t}$. We infer that the set $f_{\text{BL}}(\tilde{X}_{\tilde{y}}) = X_{\tilde{y}} \in \binom{V_{i(\tilde{y})}}{\text{edeg}(\tilde{y})}$ satisfies $|N^{\cap}_{C_{i,j}}(X_{\tilde{y}})| \geq (d - \varepsilon_{\text{CL}})^{\text{edeg}(\tilde{y})} p^{\text{edeg}(\tilde{y})} |C_{i,j}|$ and is such that

$$(N^{\cap}_{C_{i,j}}(X_{\tilde{y}}), C'_{i',j'}) \text{ is } (\varepsilon_{\text{CL}}, d, p)\text{-dense}$$
$$\text{for all } i' \in \{i, i+1\}, j, j' \in [2t] \text{ with } (c_{i,j}, c'_{i',j'}) \in R_{r,t}. \qquad (9.19)$$

Since we chose $C(\tilde{y}) = N^{\cap}(X_{\tilde{y}}) \cap C_{i,j}$ in (9.18) we get condition (D) of Lemma 9.5 also for $\tilde{y} \in \tilde{C}_{i,j}$ with $t < j \leq 2t$. For $\tilde{y} \in \tilde{C}_{i+1,j}$ with $j \in [t]$ the same argument applies with $\tilde{X}_{\tilde{y}} \subseteq N_{\tilde{y}} \in \binom{\tilde{V}_{i+1}}{\Delta}$, and for $\tilde{y} \in \tilde{B}_{i,j}$ with $j \in [2t]$ the same argument applies with $\tilde{X}_{\tilde{y}} \subseteq N_{\tilde{y}} \in \binom{\tilde{V}_i}{\Delta}$.

Now it will be easy to see that we get (E) of Lemma 9.5. Indeed, recall again that $C(\tilde{y}) = C'_{i',j'}$ for all $\tilde{y} \in \tilde{C}'_{i',j'}$ and $C(\tilde{y}) = B'_{i',j'}$ for all $\tilde{y} \in \tilde{B}'_{i',j'}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$. In addition, the mapping h constructed by Lemma 9.3 is a homomorphism from H to $R_{r,t}$. Hence (9.19) and property (G3) of Lemma 9.2 assert that condition (E) of Lemma 9.5 is satisfied for all edges $\tilde{y}\tilde{y}'$ of $H'_i = H[\tilde{W}_i]$ with at least one end, say $\tilde{y}$, in a cluster $\tilde{C}'_{i',j'}$ or $\tilde{B}'_{i',j'}$. This is true because then $C(\tilde{y}) = W_{i,k}$ where $\tilde{W}_{i,k}$ is the cluster containing $\tilde{y}$, and $C(\tilde{y}') = N^{\cap}(X_{\tilde{y}'}) \cap W_{i,k'}$ where $\tilde{W}_{i,k'}$ is the cluster containing $\tilde{y}'$. Moreover, since h is a homomorphism all edges $\tilde{y}\tilde{y}'$ in $H'_i = H[\tilde{W}_i]$ have at least one end in a cluster $\tilde{C}'_{i',j'}$ or $\tilde{B}'_{i',j'}$.

So conditions (A)–(E) are satisfied and we can apply Lemma 9.5 to get embeddings of $H'_i = H[\tilde{W}_i]$ into $G'_i = G[W_i]$ for all $i \in [r]$ that map vertices $\tilde{y} \in \tilde{W}_i$ (i.e. connecting and balancing vertices) to vertices $y \in W_i$ in their candidate sets $C(\tilde{y})$. Let f_{CL} be the united embedding resulting from these r applications of the connection lemma and denote the embedding that unites f_{BL} and f_{CL} by f.

To finish the proof we verify that f is an embedding of H into G. Let $\tilde{x}\tilde{y}$ be an edge of H. By definition of the spin graph $R_{r,t}$ and since the mapping h constructed by Lemma 9.3 is a homomorphism from H to $R_{r,t}$ we only need to distinguish the following cases for $i \in [r]$ and $j, j' \in [2t]$ (see also Figure 9.1):

case 1: If $\tilde{x} \in \tilde{V}_i$ and $\tilde{y} \in \tilde{U}_i$, then $f(\tilde{x}) = f_{\text{BL}}(\tilde{x})$ and $f(\tilde{y}) = f_{\text{BL}}(\tilde{y})$ and thus the constrained blow-up lemma guarantees that $f(\tilde{x})f(\tilde{y})$ is an edge of G_i.

case 2: If $\tilde{x} \in \tilde{W}_i$ and $\tilde{y} \in \tilde{W}_i$, then $f(\tilde{x}) = f_{\text{CL}}(\tilde{x})$ and $f(\tilde{y}) = f_{\text{CL}}(\tilde{y})$ and thus the connection lemma guarantees that $f(\tilde{x})f(\tilde{y})$ is an edge of G'_i.

case 3: If $\tilde{x} \in \tilde{V}_i$ and $\tilde{y} \in \tilde{W}_i$, then either $\tilde{y} \in \tilde{C}_{i,j}$ or $\tilde{y} \in \tilde{B}_{i,j}$ for some j. Moreover, $f(\tilde{x}) = f_{\text{BL}}(\tilde{x})$ and therefore by (9.18) the candidate set $C(\tilde{y})$ of $\tilde{y}$ satisfies $C(\tilde{y}) \subseteq N_{C_{i,j}}(f(\tilde{x}))$ or $C(\tilde{y}) \subseteq N_{B_{i,j}}(f(\tilde{x}))$, respectively. As $f(\tilde{y}) = f_{\text{CL}}(\tilde{y}) \in C(\tilde{y})$ we also get that $f(\tilde{x})f(\tilde{y})$ is an edge of G in this case.

It follows that f maps all edges of H to edges of G, which finishes the proof of the theorem. □

9.5 A p-dense partition of G

For the proof of the Lemma for G we shall apply the minimum degree version of the sparse regularity lemma (Lemma 3.18). Observe that this lemma guarantees that the reduced graph of the regular partition we obtain is dense. Thus we can apply Theorem 5.1 to this reduced graph. In the proof of Lemma 9.2 we use this theorem to find a copy of the ladder R_r^* in the reduced graph (the graphs R_r^* and $R_{r,t}$ are defined in Section 9.1 on page 130, see also Figure 9.1). Then we further partition the clusters in this ladder to obtain a regular partition whose reduced graph contains a spin graph $R_{r,t}$. Recall that this partition will consist of a series of so-called *big cluster* which we denote by U_i and V_i, and a series of smaller clusters called *balancing clusters* $B_{i,j}$, $B'_{i,j}$ and *connecting clusters* $C_{i,j}$, $C'_{i,j}$ with $i \in [r]$, $j \in [2t]$. We will now give the details.

Proof of Lemma 9.2. Given t, r_0, η, and γ choose η' such that

$$\frac{\eta}{5} + \left(\frac{4}{\gamma} + 2\right) t \cdot \eta' \leq \frac{\eta}{2} \tag{9.20}$$

and set $d := \gamma/4$. Let β and k_{BK} be the constants provided by Theorem 5.1 for input $r_{\mathrm{BK}} := 2$, $\Delta = 3$ and $\gamma/2$. For input ε set

$$r'_0 := \max\{2r_0 + 1, k_{\mathrm{BK}}, 3/\beta, 6/\gamma, 2/\varepsilon, 10/\eta\} \tag{9.21}$$

and choose ε' such that

$$\varepsilon'/\eta' \leq \varepsilon/2, \qquad \text{and} \qquad \varepsilon' \leq \min\{\gamma/4, \eta/10\}. \tag{9.22}$$

Lemma 3.18 applied with $\alpha := \frac{1}{2} + \gamma$, ε', r'_0 then gives us the missing constant r_1.

Assume that Γ is a typical graph from $\mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$, in the sense that it satisfies the conclusion of Lemma 3.18, and let $G = (V, E) \subseteq \Gamma$ satisfy $\deg_G(v) \geq (\frac{1}{2}+\gamma)\deg_\Gamma(v)$ for all $v \in V$. Lemma 3.18 applied with $\alpha = \frac{1}{2} + \gamma$, ε', r'_0, and d to G gives us an (ε', d, p)-dense partition $V = V'_0 \dot\cup V'_1 \dot\cup \ldots \dot\cup V'_{r'}$ of G with reduced graph R' with $|V(R')| = r'$ such that $2r_0 + 1 \leq r'_0 \leq r' \leq r_1$ and with minimum degree at least $(\frac{1}{2}+\gamma-d-\varepsilon')r' \geq (\frac{1}{2}+\frac{\gamma}{2})r'$ by (9.22). If r' is odd, then set $V_0 := V'_0 \dot\cup V'_{r'}$ and $r := (r'-1)/2$, otherwise set $V_0 := V'_0$ and $r := r'/2$. Clearly $r_0 \leq r \leq r_1$, the graph $R := R'[2r]$ still has minimum degree at least $(\frac{1}{2} + \frac{\gamma}{3})2r$ and $|V_0| \leq \varepsilon' n + (n/r'_0) \leq (\eta/5)n$ by the choice of r'_0 and ε'. It follows from Theorem 5.1 applied with $\Delta = 3$ and $\gamma/2$ that R contains a copy of the ladder R_r^* on $2r$ vertices (R_r^* has bandwidth $2 \leq \beta \cdot 2r$ by the choice of r'_0in (9.21)). This naturally defines an equipartite mapping f from $V \setminus V_0$ to the ladder R_r^*: f maps each vertex of G in cluster V_i to vertex i of R_r^*. We will show that subdividing the clusters $f^{-1}(i)$ for all $i \in V(R_r^*)$ will give the desired mapping g.

To this end let us first rename the vertices of the graph $R'[2r]$ to $\{u_1, v_1, \ldots, u_r, v_r\}$ according to the spanning copy R_r^*. We will now construct the balancing clusters $B_{j,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [2t]$ and afterwards turn to the connecting clusters $C_{j,j}$ and $C'_{i,j}$ and big clusters U_j and V_i with $i \in [r]$, $j \in [2t]$.

Notice that $\delta(R) \geq (\frac{1}{2} + \frac{\gamma}{3})2r$ implies that every edge $u_i v_i$ of $R_r^* \subseteq R$ is contained in more than γr triangles in R. Therefore, we can choose vertices w_i of R for all $i \in [r]$ such that $u_i v_i w_i$ forms a triangle in R and no vertex of R serves as w_i more than $2/\gamma$ times. We continue by choosing in cluster $f^{-1}(u_i)$ arbitrary disjoint vertex sets $B_{i,j}$, …, $B_{i,t}$, $B'_{i,t+1}$, …, $B'_{i,2t}$, of size $\eta' n/(2r)$ each, for all $i \in [r]$. We will show below that $f^{-1}(u_i)$ is large enough so that

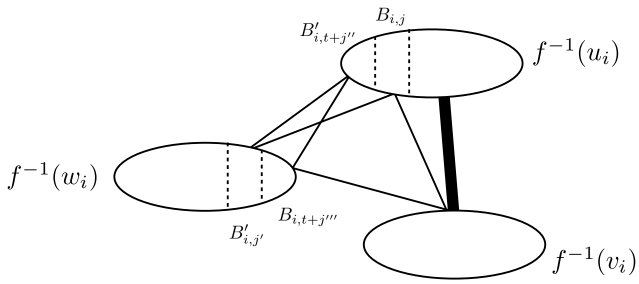


Figure 9.3: Cutting off a set of balancing clusters from $f^{-1}(u_i)$ and $f^{-1}(w_i)$. These clusters build p-dense pairs (thanks to the triangle $u_iv_iw_i$ in R) in the form of a C_5.

these sets can be chosen. We then remove all vertices in these sets from $f^{-1}(u_i)$. Similarly, we choose in cluster $f^{-1}(w_i)$ arbitrary disjoint vertex sets $B_{i,t+1}, \ldots, B_{i,2t}$, $B'_{i,1}, \ldots, B'_{i,t}$, of size $\eta' n/(2r)$ each, for all $i \in [r]$. We also remove these sets from $f^{-1}(w_i)$. Observe that this construction asserts the following property. For all $i \in [r]$ and $j, j', j'', j''' \in [t]$ each of the pairs $(f^{-1}(v_i), B_{i,j})$, $(B_{i,j}, B'_{i,j'})$, $(B'_{i,j'}, B'_{i,t+j''})$, $(B'_{i,t+j''}, B_{i,t+j'''})$, and $(B_{i,t+j'''}, f^{-1}(v_i))$ is a sub-pair of a p-dense pair corresponding to an edge of $R[\{u_i, v_i, w_i\}]$ (see Figure 9.3). Accordingly this is a sequence of p-dense pairs in the form of a C_5, as needed for the balancing clusters in view of condition (G3) (see also Figure 9.1). Hence we call the sets $B_{i,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [t]$ balancing clusters from now on and claim that they have the required properties. This claim will be verified below.

We now turn to the construction of the connecting clusters and big clusters. Recall that we already removed balancing clusters from all clusters $f^{-1}(u_i)$ and possibly from some clusters $f^{-1}(v_i)$ (because v_i might have served as $w_{i'}$) with $i \in [r]$. For each $i \in [r]$ we arbitrarily partition the remaining vertices of cluster $f^{-1}(u_i)$ into sets $C_{i,1}\dot\cup \ldots \dot\cup C_{i,2t}\dot\cup U_i$ and the remaining vertices of cluster $f^{-1}(v_i)$ into sets $C'_{i,1}\dot\cup \ldots \dot\cup C'_{i,2t}\dot\cup V_i$ such that $|C_{i,j}|, |C'_{i,j}| = \eta' n/(2r)$ for all $i \in [r], j \in [2t]$. This gives us the connecting and the big clusters and we claim that also these clusters have the required properties. Observe, again, that for all $i \in [r]$, $i' \in \{i-1, i, i+1\} \setminus \{0\}$, $j, j' \in [2t]$ each of the pairs (U_i, V_i), $(C_{i',j}, V_i)$, and $(C_{i,j}, C'_{i,j'})$ is a sub-pair of a p-dense pair corresponding to an edge of R^*_r (see Figure 9.4).

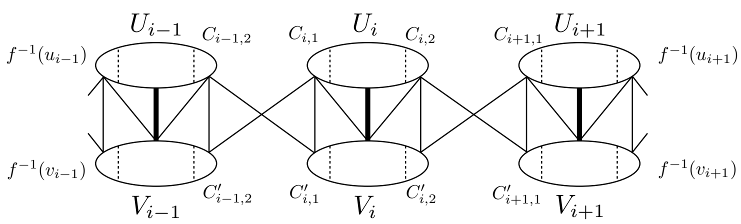


Figure 9.4: Partitioning the remaining vertices of cluster $f^{-1}(u_i)$ and $f^{-1}(v_i)$ into sets $C_{i,1}\dot\cup C_{i,2}\dot\cup U_i$ and $C'_{i,1}\dot\cup C'_{i,2}\dot\cup V_i$ (for the special case $t = 1$). These clusters form p-dense pairs (thanks to the ladder R^*_r in R) as indicated by the edges.

We will now show that the balancing clusters, connecting clusters and big clusters satisfy conditions (G1)–(G3). Note that condition (G2) concerning the sizes of the connecting and balancing clusters is satisfied by construction. To determine the sizes of the big clusters observe that from each cluster V'_j with $j \in [2r]$ vertices for at most $2t \cdot 2/\gamma$ balancing clusters were removed. In addition, at most $2t$ connecting clusters were split off from V'_j. Therefore we get

$$|V_i|, |U_i| \geq \left(1 - \frac{\eta}{5}\right)\frac{n}{2r} - \left(\frac{4}{\gamma} + 2\right) t \cdot \eta' \frac{n}{2r} \geq (1-\eta)\frac{n}{2r}$$

by (9.20) and (9.22). This is condition (G1). It remains to verify condition (G3). It can easily be checked that for all $xy \in E(R_{r,t})$ the corresponding pair $(g^{-1}(x), g^{-1}(y))$ is a subpair of some cluster pair $(f^{-1}(x'), f^{-1}(y'))$ with $x'y' \in E(R)$ by construction. In addition all big, connecting, and balancing clusters are of size at least $\eta' n/(2r)$. Hence we have $|g^{-1}(x)| \geq \eta' |f^{-1}(x')|$ and $|g^{-1}(y)| \geq \eta' |f^{-1}(y')|$. We conclude from Proposition 3.16 that $(g^{-1}(x), g^{-1}(y))$ is (ε, d, p)-dense since $\varepsilon'/\eta' \leq \varepsilon$ by (9.22). This finishes the verification of (G3). □

9.6 A partition of H

The theorem of Hajnal and Szemerédi (Theorem 1.4) states that every graph G with $\delta(G) \geq \frac{r-1}{r} n$ contains a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r. In fact Hajnal and Szemerédi obtained a more general result and determined the minimum degree that forces a certain number of vertex disjoint K_r copies in G. In addition their result guarantees that the remaining vertices can be covered by copies of K_{r-1}.

Another way to express this, which actually resembles the original formulation, is obtained by considering the complement $\bar{G}$ of G and its maximum degree. Then, so the theorem asserts, the graph $\bar{G}$ contains a certain number of vertex disjoint independent sets of almost equal sizes. In other words, $\bar{G}$ admits a vertex colouring such that the sizes of the colour classes differ by at most 1. Such a colouring is also called *equitable colouring.*

Theorem 9.14 (Hajnal & Szemerédi [48])**.** *Let $\bar{G}$ be a graph on n vertices with maximum degree $\Delta(\bar{G}) \leq \Delta$. Then there is an equitable vertex colouring of G with $\Delta + 1$ colours.* □

In the proof of Lemma 9.3 that we shall present in this section we will use this theorem in order to guarantee property (H3). This will be the very last step in the proof, however. First, we need to take care of the remaining properties.

Before we start, let us agree on some terminology that will turn out to be useful in the proof of Lemma 9.3. When defining a homomorphism h from a graph H to a graph R, we write $h(S) := z$ for a set S of vertices in H and a vertex z in R to say that all vertices from S are mapped to z. Recall that we have a bandwidth hypothesis on H. Consider an ordering of the vertices of H achieving its bandwidth. Then we can deal with the vertices of H in this order. In particular, we can refer to vertices as the *first* or *last* vertices in some set, meaning that they are the vertices with the smallest or largest label from this set.

We start with the following proposition.

Proposition 9.15. *Let $\bar{R}$ be the following graph with six vertices and six edges:*

$$\bar{R} := \left(\{z^0, z^1, \ldots, z^5\}, \{z^0z^1, z^1z^2, z^2z^3, z^3z^4, z^4z^5, z^5z^1\}\right).$$

For every real $\bar{\eta} > 0$ there exists a real $\bar{\beta} > 0$ such that the following holds: Consider an arbitrary bipartite graph $\bar{H}$ with $\bar{m}$ vertices, colour classes Z^0 and Z^1, and $\mathrm{bw}(\bar{H}) \leq \bar{\beta}\bar{m}$ *and*

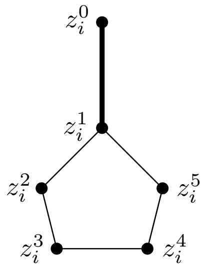


Figure 9.5: The graph $\bar{R}$ in Proposition 9.15.

denote by T the union of the first $\bar{\beta}\bar{m}$ vertices and the last $\bar{\beta}\bar{m}$ vertices of H. Then there exists a homomorphism $\bar{h}\colon V(\bar{H}) \to V(\bar{R})$ from $\bar{H}$ to $\bar{R}$ such that for all $j \in \{0,1\}$ and all $k \in [2,5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \le |\bar{h}^{-1}(z^j)| \le \frac{\bar{m}}{2} + \bar{\eta}\bar{m}\,, \tag{9.23}$$

$$|\bar{h}^{-1}(z^k)| \le \bar{\eta}\bar{m}\,, \tag{9.24}$$

$$h(T \cap Z^j) = z^j. \tag{9.25}$$

Roughly speaking, Proposition 9.15 shows that we can find a homomorphism from a bipartite graph $\bar{H}$ to a graph $\bar{R}$ which consists an edge z^0z^1 which has an attached 5-cycle (see Figure 9.5 for a picture of $\bar{R}$) in such a way that most of the vertices of $\bar{H}$ are mapped about evenly to the vertices z^0 and z^1. If we knew that the colour classes of $\bar{H}$ were of almost equal size, then this would be a trivial task, but since this is not guaranteed, we will have to make use of the additional vertices $z^2, \dots, z^5$.

Proof of Proposition 9.15. Given $\bar{\eta}$, choose an integer $\ell \ge 6$ and a real $\bar{\beta} > 0$ such that

$$\frac{5}{\ell} < \bar{\eta} \quad \text{and} \quad \bar{\beta} := \frac{1}{\ell^2}. \tag{9.26}$$

For the sake of a simpler exposition we assume that $\bar{m}/\ell$ and $\bar{\beta}\bar{m}$ are integers. Now consider a graph $\bar{H}$ as given in the statement of the proposition. Partition $V(\bar{H})$ along the ordering induced by the bandwidth labelling into sets $\bar{W}_1, \dots, \bar{W}_\ell$ of sizes $|\bar{W}_i| = \bar{m}/\ell$ for $i \in [\ell]$. For each $\bar{W}_i$, consider its last $5\bar{\beta}\bar{m}$ vertices and partition them into sets $X_{i,1}, \dots, X_{i,5}$ of size $|X_{i,k}| = \bar{\beta}\bar{m}$. For $i \in [\ell]$ let

$$W_i := \bar{W}_i \setminus (X_{i,1} \cup \dots \cup X_{i,5}), \quad W := \bigcup_{i=1}^{\ell} W_i,$$

and note that

$$L := |W_i| = \frac{\bar{m}}{\ell} - 5\bar{\beta}\bar{m} \overset{(9.26)}{=} \left(\frac{1}{\ell} - \frac{5}{\ell^2}\right)\bar{m} \ge \frac{1}{\ell^2}\bar{m} \overset{(9.26)}{=} \bar{\beta}\bar{m}.$$

For $i \in [\ell]$, $j \in \{0,1\}$, and $1 \le k \le 5$ let

$$W_i^j := W_i \cap Z^j, \quad X_{i,k}^j := X_{i,k} \cap Z^j.$$

Thanks to the fact that $\mathrm{bw}(\bar{H}) \leq \bar{\beta}\bar{m}$, we know that there are no edges between W_i and $W_{i'}$ for $i \neq i' \in [\ell]$. In a first round, for each $i \in [\ell]$ we will either map all vertices from W_i^j to z^j for both $j \in \{0,1\}$ (call such a mapping a *normal* embedding of W_i) or we map all vertices from W_i^j to z^{1-j} for both $j \in \{0,1\}$ (call this an *inverted* embedding). We will do this in such a way that the difference between the number of vertices that get sent to z^0 and the number of those that get sent to z^1 is as small as possible. Since $|W_i| \leq L$ the difference is therefore at most L. If, in addition, we guarantee that both W_1 and W_ℓ receive a normal embedding, it is at most $2L$. So, to summarize and to describe the mapping more precisely: there exist integers $\varphi_i \in \{0,1\}$ for all $i \in [\ell]$ such that $\varphi_1 = 0 = \varphi_\ell$ and the function $h : W \to \{z^0, z^1\}$ defined by

$$h(W_i^j) := \begin{cases} z^j & \text{if } \varphi_i = 0, \\ z^{1-j} & \text{if } \varphi_i = 1, \end{cases}$$

is a homomorphism from $\bar{H}[W]$ to $\bar{R}[\{z^0, z^1\}]$, satisfying that for both $j \in \{0,1\}$

$$\begin{aligned} |h^{-1}(z^j)| &\leq \frac{\ell L}{2} + 2L = \left(\frac{\ell}{2} + 2\right)\frac{\bar{m}}{\ell} - \left(\frac{\ell}{2} + 2\right)5\bar{\beta}\bar{m} \\ &\overset{(9.26)}{=} \frac{\bar{m}}{2} + \bar{m}\left(\frac{2}{\ell} - \frac{5}{2}\frac{1}{\ell} - \frac{10}{\ell^2}\right) \leq \frac{\bar{m}}{2}\,. \end{aligned} \tag{9.27}$$

In the second round we extend this homomorphism to the vertices in the classes $X_{i,k}$. Recall that these vertices are by definition situated after those in W_i and before those in W_{i+1}. The idea for the extension is simple. If W_i and W_{i+1} have been embedded in the same way by h (either both normal or both inverted), then we map all the vertices from all $X_{i,k}$ to z^0 and z^1 accordingly. If they have been embedded in different ways (one normal and one inverted), then we walk around the 5-cycle $z^1, \ldots, z^5, z^1$ to switch colour classes.

Here is the precise definition. Consider an arbitrary $i \in [\ell]$. Since $h(W_i^0)$ and $h(W_i^1)$ are already defined, choose (and fix) $j \in \{0,1\}$ in such a way that $h(W_i^j) = z^1$. Note that this implies that $h(W_i^{1-j}) = z^0$. Now define $h_i : \bigcup_{k=0}^{5} X_{i,k} \to \bigcup_{k=1}^{5}\{z^k\}$ as follows:

Suppose first that $\varphi_i = \varphi_{i+1}$. Observe that in this case we must also have $h(W_{i+1}^j) = z^1$ and $h(W_{i+1}^{1-j}) = z^0$. So we can happily define for all $k \in [5]$

$$h_i(X_{i,k}^j) = z^1 \quad \text{and} \quad h_i(X_{i,k}^{1-j}) = z^0.$$

Now suppose that $\varphi_i \neq \varphi_{i+1}$. Since we are still assuming that j is such that $h(W_i^j) = z^1$ and thus $h(W_i^{1-j}) = z^0$, the fact that $\varphi_i \neq \varphi_{i+1}$ implies that $h(W_{i+1}^j) = z^0$ and $h(W_{i+1}^{1-j}) = z^1$. In this case we define h_i as follows:

$h(W_i^{1-j})$ $= z^0$	$h_i(X_{i,1}^{1-j})$ $:= z^2$	$h_i(X_{i,2}^{1-j})$ $:= z^2$	$h_i(X_{i,3}^{1-j})$ $:= z^4$	$h_i(X_{i,4}^{1-j})$ $:= z^4$	$h_i(X_{i,5}^{1-j})$ $:= z^1$	$h(W_{i+1}^{1-j})$ $= z^1$
$h(W_i^j)$ $= z^1$	$h_i(X_{i,1}^j)$ $:= z^1$	$h_i(X_{i,2}^j)$ $:= z^3$	$h_i(X_{i,3}^j)$ $:= z^3$	$h_i(X_{i,4}^j)$ $:= z^5$	$h_i(X_{i,5}^j)$ $:= z^5$	$h(W_{i+1}^j)$ $= z^0$

Finally, we set $\bar{h} : V(\bar{H}) \to V(\bar{R})$ by letting $\bar{h}(x) := h(x)$ if $x \in W_i$ for some $i \in [\ell]$ and $\bar{h}(x) := h_i(x)$ if $x \in X_{i,k}$ for some $i \in [\ell]$ and $k \in [5]$.

In order to verify that this is a homomorphism from $\bar{H}$ to the sets $\bar{R}$, we first let

$$X_{i,0}^0 := W_i^0, X_{i,0}^1 := W_i^1, X_{i,6}^0 := W_{i+1}^0, X_{i,6}^1 := W_{i+1}^1.$$

Using this notation, it is clear that any edge xx' in $\bar{H}[W_i \cup \bigcup_{k=1}^{5} X_{i,k} \cup W_{i+1}]$ with $x \in Z^j$ and $x' \in Z^{1-j}$ is of the form

$$xx' \in (X^j_{i,k} \times X^{1-j}_{i,k}) \cup (X^j_{i,k} \times X^{1-j}_{i,k+1}) \cup (X^j_{i,k+1} \times X^{1-j}_{i,k})$$

for some $k \in [0, 6]$. It is therefore easy to check in the above table that $\bar{h}$ maps xx' to an edge of R.

We conclude the proof by showing that the cardinalities of the preimages of the vertices in R match the required sizes. In the second round we mapped a total of

$$\ell \cdot 5\bar{\beta}\bar{m} \overset{(9.26)}{=} \frac{5}{\ell}\bar{m} \overset{(9.26)}{\leq} \bar{\eta}\bar{m}$$

additional vertices from $\bar{H}$ to the vertices of $\bar{R}$, which guarantees that

$$|\bar{h}^{-1}(z^j)| \overset{(9.27)}{\leq} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \text{ for all } j \in \{0,1\}, \quad |\bar{h}^{-1}(z^k)| \leq \bar{\eta}\bar{m} \text{ for all } k \in [2,5].$$

Finally, the lower bound in (9.23) immediately follows from the upper bounds:

$$|\bar{h}^{-1}(z^j)| \geq \bar{m} - |\bar{h}^{-1}(z^{1-j})| - \sum_{k=2}^{5} |\bar{h}^{-1}(z^k)| \geq \frac{\bar{m}}{2} - 5\bar{\eta}\bar{m}.$$

□

We remark that Proposition 9.15 (and thus Lemma 9.3) would remain true if we replaced the 5-cycle in $\bar{R}$ by a 3-cycle. However, we need the properties of the 5-cycle in the proof of the main theorem. Now we will prove Lemma 9.3.

Proof of Lemma 9.3. Given the integer Δ, set $t := (\Delta+1)^3(\Delta^3+1)$. Given a real $0 < \eta < 1$ and integers m and r, set $\bar{\eta} := \eta/20 < 1/20$ and apply Proposition 9.15 to obtain a real $\bar{\beta} > 0$. Choose $\beta > 0$ sufficiently small so that all the inequalities

$$\frac{1}{r} - 4\beta \geq \beta/\bar{\beta}, \quad 4\beta r \leq \frac{\eta}{20r}, \quad 16\Delta\beta r \leq \eta\left(\frac{1}{r} - 4\beta\right)\left(\frac{1}{2} - 5\bar{\eta}\right) \tag{9.28}$$

hold. Again, we assume that m/r and βm are integers.

Next we consider the spin graph $R_{r,t}$ with $t = 1$, i.e., let $R := R_{r,1}$. For the sake of simpler reference, we will change the names of its vertices as follows: For all $i \in [r]$ we set (see Figure 9.6)

$$z_i^0 := u_i, z_i^1 := v_i, z_i^2 := b_{i,1}, z_i^3 := b'_{i,1}, z_i^4 := b'_{i,2}, z_i^5 := b_{i,2},$$

$$q_i^2 := c_{i,1}, q_i^3 := c'_{i,1}, q_i^4 := c_{i,2}, q_i^5 := c'_{i,2}.$$

Note that for every $i \in [r]$ the graph $R[\{z_i^0, \ldots, z_i^5\}]$ is isomorphic to the graph $\bar{R}$ defined in Proposition 9.15.

Partition $V(H)$ along the ordering (induced by the bandwidth labelling) into sets $\bar{S}_1, \ldots, \bar{S}_r$ of sizes $|\bar{S}_i| = m/r$ for $i \in [r]$.

Define sets $T_{i,k}$ for $i \in [r]$ and $k \in [0,5]$ with $|T_{i,k}| = \beta m$ such that $T_{i,0} \cup \cdots \cup T_{i,4}$ contain the last $5\beta m$ vertices of $\bar{S}_i$ and $T_{i,5}$ the first βm vertices of $\bar{S}_{i+1}$ (according to the ordering).

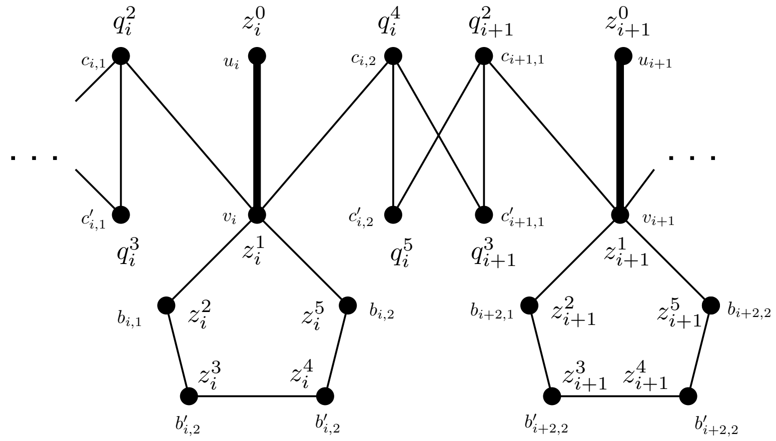


Figure 9.6: The subgraph $R[\{z_i^0,\dots,z_i^5,q_i^4,q_i^5,q_{i+1}^2,q_{i+1}^3,z_{i+1}^0,\dots,z_{i+1}^5\}]$ of $R_{r,1}$ in the proof of Lemma 9.3.

Set $S_i := \bar{S}_i \setminus (T_{i,1}\cup\dots\cup T_{i,4})$ and observe that this implies that $T_{i,0}$ is the set of the last βm vertices of S_i and $T_{i,5}$ is the set of the first βm vertices in $\bar{S}_{i+1}$. Set

$$\bar{m} := |S_i| = (m/r) - 4\beta m = \left(\frac{1}{r} - 4\beta\right) m \overset{(9.28)}{\geq} \beta m/\bar{\beta}, \quad \text{thus} \quad \bar{\beta}\bar{m} \geq \beta m. \tag{9.29}$$

Denote by Z^0 and Z^1 the two colour classes of the bipartite graph H. For $i\in[\ell]$, $k\in[0,5]$ and $j\in[0,1]$ let

$$S_i^j := S_i \cap Z^j, \quad T_{i,k}^j := T_{i,k}\cap Z^j.$$

Now for each $i\in[r]$ apply Proposition 9.15 to $\bar{H}_i := H[S_i]$ and $\bar{R}_i := R[\{z_i^0,\dots,z_i^5\}]$. Observe that

$$\mathrm{bw}(\bar{H}_i) \leq \mathrm{bw}(H) \leq \beta m \overset{(9.29)}{\leq} \bar{\beta}\bar{m},$$

so we obtain a homomorphism $\bar{h}_i : S_i \to \{z_i^0,\dots,z_i^5\}$ of $\bar{H}_i$ to $\bar{R}_i$. Combining these yields a homomorphism

$$\bar{h} : \bigcup_{i=1}^{r} S_i \to \bigcup_{i=1}^{r}\{z_i^0,\dots,z_i^5\},$$

$$\text{from} \quad H[\bigcup_{i=1}^{r} S_i] \quad \text{to} \quad R[\bigcup_{i=1}^{r}\{z_i^0,\dots,z_i^5\}]$$

with the property that for every $i\in[r]$, $j\in[0,1]$ and $k\in[2,5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \overset{(9.23)}{\leq} |\bar{h}^{-1}(z_i^j)| \overset{(9.23)}{\leq} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \leq \left(1+\frac{\eta}{10}\right)\frac{m}{2r} \quad \text{and}$$
$$|\bar{h}^{-1}(z_i^k)| \overset{(9.24)}{\leq} \bar{\eta}\bar{m} \leq \frac{\eta}{10}\frac{m}{2r}.$$

Thanks to (9.29), we know that $\bar{\beta}\bar{m} \geq \beta m$, and therefore applying the information from (9.25) in Proposition 9.15 yields that for all $i\in[r]$ and $j\in[0,1]$

$$\bar{h}(T_{i,0}^j) = z_i^j \quad \text{and} \quad \bar{h}(T_{i,5}^j) = z_{i+1}^j.$$

In the second round, our task is to extend this homomorphism to the vertices in $\bar{S}_i \setminus S_i$ by defining a function

$$h_i : T_{i,1} \cup \cdots \cup T_{i,4} \to \{z_i^1, q_i^4, q_i^5, q_{i+1}^2, q_{i+1}^3, z_{i+1}^1\}$$

for each $i \in [r]$ as follows:

$\bar{h}(T_{i,0}^0) = z_i^0$	$h_i(T_{i,1}^0) := q_i^4$	$h_i(T_{i,2}^0) := q_i^4$	$h_i(T_{i,3}^0) := q_{i+1}^2$	$h_i(T_{i,4}^0) := q_{i+1}^2$	$\bar{h}(T_{i,5}^0) = z_{i+1}^0$
$\bar{h}(T_{i,0}^1) = z_i^1$	$h_i(T_{i,1}^1) := z_i^1$	$h_i(T_{i,2}^1) := q_i^5$	$h_i(T_{i,3}^1) := q_{i+1}^3$	$h_i(T_{i,4}^1) := z_{i+1}^1$	$\bar{h}(T_{i,5}^1) = z_{i+1}^1$

Now set $h(x) := \bar{h}(x)$ if $x \in S_i$ for some $i \in [r]$ and $h(x) := h_i(x)$ if $x \in T_{i,k}$ for some $i \in [r]$ and $k \in [4]$.

Let us verify that h is a homomorphism from H to R. For edges xx' with both endpoints inside a set S_i we do not need to check anything because here $h(x) = \bar{h}(x)$ and $h(x') = \bar{h}(x')$ and we know from Proposition 9.15 that $\bar{h}$ is a homomorphism. Due to the bandwidth condition $\mathrm{bw}(H) \leq \beta m$, any other edge xx' with $x \in Z^0$ and $x' \in Z^1$ is of the form

$$xx' \in (T_{i,k}^0 \times T_{i,k}^1) \cup (T_{i,k}^0 \times T_{i,k+1}^1) \cup (T_{i,k+1}^0 \times T_{i,k}^1)$$

for some $i \in [\ell]$ and $0 \leq k, k+1 \leq 5$. It is therefore easy to check in the above table that h maps xx' to an edge of R.

What can we say about the cardinalities of the preimages? In the second round we have mapped $4\beta mr$ additional vertices from H to vertices in R, hence for any vertex z in R with $z \notin \{z_i^0, z_i^1\}$, $i \in [\ell]$ we have

$$|h_i^{-1}(z)| \leq 4\beta mr \overset{(9.28)}{\leq} \frac{\eta}{10}\frac{m}{2r}, \tag{9.30}$$

and therefore the required upper bounds immediately follow from (9.6).

At this point we have found a homomorphism h from H to $R = R_{r,1}$ of which we know that it satisfies properties (H1) and (H2).

So far we have been working with the graph $R = R_{r,1}$, and therefore we know which vertices have been mapped to $u_i = z_i^0$ and $v_i = z_i^1$:

$$\tilde{U}_i := h^{-1}(u_i) = h^{-1}(z_i^0) \quad \text{and} \quad \tilde{V}_i := h^{-1}(v_i) = h^{-1}(z_i^1).$$

Moreover for $i \in [r]$ and $k \in [2,5]$ set

$$Z_i^k := h^{-1}(z_i^k) \quad \text{and} \quad Q_i^k := h^{-1}(q_i^k)$$

Let us deal with property (H5) next. By definition, a vertex in $\tilde{X}_i \subseteq \tilde{V}_i$ must have at least one neighbour in Q_i^2 or Q_i^4 or Z_i^2 or Z_i^5. We know from (9.30) that the two latter sets contain at most $4\beta mr$ vertices each, and each of their vertices has at most Δ neighbours. Thus

$$\begin{aligned} |\tilde{X}_i| &\leq \Delta \cdot 16\beta mr \overset{(9.28)}{\leq} \eta\left(\frac{1}{r} - 4\beta\right)\left(\frac{1}{2} - 5\bar{\eta}\right) m \overset{(9.29)}{=} \eta\bar{m}\left(\frac{1}{2} - 5\bar{\eta}\right) \\ &\overset{(9.6)}{\leq} \eta|\bar{h}^{-1}(z_i^1)| \leq \eta|h^{-1}(z_i^1)| = \eta|\tilde{V}_i|, \end{aligned}$$

which shows that (H5) is also satisfied.

Next we would like to split up the sets Z_i^k and Q_i^k for $i \in [r]$ and $k \in [2,5]$ into smaller sets in order to meet the additional requirements (H3) and (H4). This means that we need to partition them further into sets of vertices which have no path of length 1, 2, or 3 between them and which have the same degree into certain sets.

To achieve this, first denote by H^3 the 3rd power of H. Then an upper bound on the maximum degree of H^3 is obviously given by

$$\Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)(\Delta - 1) \leq \Delta^3.$$

Hence H^3 has a vertex colouring $c : V(H) \to \mathbb{N}$ with at most $\Delta^3 + 1$ colours. Notice that a set of vertices that receives the same colour by c forms a 3-independent set in H. To formalize this argument, we define a 'fingerprint' function

$$f : \bigcup_{i=1}^{r} \bigcup_{k=2}^{5} (Z_i^k \cup Q_i^k) \to [0, \Delta] \times [0, \Delta] \times [0, \Delta] \times [\Delta^3 + 1]$$

as follows:

$$f(y) := \begin{cases} \Big(\deg_{\tilde{V}_i}(y), \deg_{Q_i^2 \cup Q_i^4}(y), \deg_{Z_i^2}(y), c(y)\Big) & \text{if } y \in \Big(\bigcup_{k=2}^{5}(Q_i^k \cup Z_i^k)\Big) \setminus Z_i^4, \\ \Big(\deg_{\tilde{V}_i}(y), \deg_{Q_i^2 \cup Q_i^4}(y), \deg_{Z_i^3 \cup Z_i^5}(y), c(y)\Big) & \text{if } y \in Z_i^4, \end{cases}$$

for some $i \in [r]$.

Recall that we defined $t := (\Delta + 1)^3(\Delta^3 + 1)$, so let us identify the codomain of f with the set $[t]$. Now for $i \in [r]$ and $j \in [t]$ we set

$$\begin{aligned} \tilde{B}_{i,j} &:= Z_i^2 \cap f^{-1}(j), & \tilde{B}_{i,t+j} &:= Z_i^5 \cap f^{-1}(j) \\ \tilde{B}'_{i,j} &:= Z_i^3 \cap f^{-1}(j), & \tilde{B}'_{i,t+j} &:= Z_i^4 \cap f^{-1}(j) \\ \tilde{C}_{i,j} &:= Q_i^2 \cap f^{-1}(j), & \tilde{C}_{i,t+j} &:= Q_i^4 \cap f^{-1}(j) \\ \tilde{C}'_{i,j} &:= Q_i^3 \cap f^{-1}(j), & \tilde{C}'_{i,t+j} &:= Q_i^5 \cap f^{-1}(j). \end{aligned}$$

Observe, for example, that for $y \in \tilde{B}_{i,j}$ the third component of $f(y)$ is exactly equal to $\deg_{L(i,j)}(y)$. Now, for any

$$yy' \in \binom{\tilde{C}_{i,j}}{2} \cup \binom{\tilde{B}_{i,j}}{2} \cup \binom{\tilde{C}'_{i,j}}{2} \cup \binom{\tilde{B}'_{i,j}}{2},$$

we have $f(y) = j = f(y')$ and hence any of the parameters required in (H3) and (H4) have the same value for y and y'.

The only thing missing before the proof of Lemma 9.3 is complete is that we need to guarantee that every $y \in Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ has at most $\Delta - 1$ neighbours in $\tilde{V}_i$, as required in the first line of (H4). If a vertex y does not satisfy this, it must have *all* its Δ neighbours in $\tilde{V}_i$. Since by definition of $\tilde{V}_i$ these neighbours have been mapped to z_i^1, we can map y to z_i^0 (instead of mapping it to z_i^2, z_i^5, q_i^2 or q_i^4).

Even if, in this way, all of the vertices in $Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ would have to be mapped to z_i^0, (9.30) assures us that these are at most $4\frac{\eta}{10}\frac{m}{2r}$ vertices. Since by (9.6) at most $(1 + \frac{\eta}{10})\frac{m}{2r}$ have already been mapped to z_i^0 in the first round and by (9.30) at most $\frac{\eta}{10}\frac{m}{2r}$ in the second round, this does not violate the upper bound in (H1). □

9.7 The constrained blow-up lemma

As explained earlier the proof of the constrained blow-up lemma uses techniques developed in [8, 86] adapted to our setting. In fact, the proof we present here follows the embedding strategy used in the proof of [8, Theorem 1.5]. This strategy is roughly as follows. Assume we want to embed the bipartite graph H on vertex set $\tilde{U}\dot{\cup}\tilde{V}$ into the host graph G on vertex set $U\dot{\cup}V$. Then we consider injective mappings $f\colon \tilde{V} \to V$, and try to find one that can be extended to $\tilde{U}$ such that the resulting mapping is an embedding of H into G. For determining whether a particular mapping f can be extended in this way we shall construct an auxiliary bipartite graph B_f, a so-called candidate graph (see Definition 9.16), which contains a matching covering one of its partition classes if and only if f can be extended. Accordingly, our goal will be to check whether B_f contains such a matching M which we will do by appealing to Hall's condition. On page 154 we will explain the details of this part of the proof, determine necessary conditions for the application of Hall's theorem, and collect them in form of a matching lemma (Lemma 9.23). It will then remain to show that there is a mapping f such that B_f satisfies the conditions of this matching lemma. This will require most of the work. The idea here is as follows.

We will show that mappings f usually have the necessary properties as long as they do not map neighbourhoods $N_H(\tilde{u}) \subseteq \tilde{V}$ of vertices in $\tilde{u} \in \tilde{U}$ to certain "bad" spots in V. The existence of (many) mappings that avoid these "bad" spots is verified with the help of a hypergraph packing lemma (Lemma 9.21). This lemma states that half of all possible mappings f avoid *almost all* "bad" spots and can easily be turned into mappings f' avoiding *all* "bad" spots with the help of so-called switchings.

Candidate graphs

If we have injective mappings $f\colon \tilde{V} \to V$ as described in the previous paragraph we would like to decide whether f can be extended to an embedding of H into G. Observe that in such an embedding each vertex $\tilde{u} \in \tilde{U}$ has to be embedded to a vertex $u \in U$ such that the following holds. The neighbourhood $N_H(\tilde{u})$ has its image $f(N_H(\tilde{u}))$ in the set $N_G(u)$. Determining which vertices u are "candidates" for the embedding of $\tilde{u}$ in this sense gives rise to the following bipartite graph.

Definition 9.16 (candidate graph)**.** *Let H and G be bipartite graphs on vertex sets $\tilde{U}\dot{\cup}\tilde{V}$ and $U\dot{\cup}V$, respectively. For an injective function $f\colon \tilde{V} \to V$ we say that a vertex $u \in U$ is an* f-candidate *for $\tilde{u} \in \tilde{U}$ if and only if $f(N_H(\tilde{u})) \subseteq N_G(u)$.*

The candidate graph $B_f(H,G) := (\tilde{U}\dot{\cup}U, E_f)$ *for f is the bipartite graph with edge set*

$$E_f := \left\{\tilde{u}u \in \tilde{U} \times U\colon\ u \text{ is an } f\text{-candidate for } \tilde{u}\right\} .$$

Now it is easy to see that the mapping f described above can be extended to an embedding of H into G if and only if the corresponding candidate graph has a matching covering $\tilde{U}$. Clearly, if the candidate graph $B_f(H,G)$ of f has vertices $\tilde{u} \in \tilde{U}$ of degree 0, then $B_f(H,G)$ has no such matching and hence f cannot be extended. More generally we would like to avoid that $\deg_{B_f(H,G)}(\tilde{u})$ is too small. Notice that this means precisely that $N_H(\tilde{u})$ should not be mapped to a set $B \subseteq V$ by f, that has a small common neighbourhood in G. These sets B are the "bad" spots (see the beginning of this section) that should be avoided by f.

We explained above that, in order to avoid "bad" spots, we will have to change certain mappings f slightly. The exact definition this operation is as follows.

Definition 9.17 (switching). *Let $f, f'\colon X \to Y$ be injective functions. We say that f' is obtained from f by a* switching *if there are $u, v \in X$ with $f'(u) = f(v)$ and $f'(v) = f(u)$ and $f(w) = f'(w)$ for all $w \notin \{u, v\}$. The* switching distance $\mathrm{d_{sw}}(f, f')$ *of f and f' is at most s if the mapping f' can be obtained from f by a sequence of at most s switchings.*

These switchings will alter the candidate graph corresponding to the injective function slightly (but not much, see Lemma 9.19). In order to quantify this we further define the neighbourhood distance between two bipartite graphs B and B' which determines the number of vertices (in one partition class) whose neighbourhoods differ in B and B'.

Definition 9.18 (neighbourhood distance). *Let $B = (U \dot\cup \tilde{U}, E)$, $B' = (U \dot\cup \tilde{U}, E')$ be bipartite graphs. We define the* neighbourhood distance *of B and B' with respect to $\tilde{U}$ as*

$$\mathrm{d}_{N(\tilde{U})}(B, B') := \big|\{\tilde{u} \in \tilde{U}\colon\ N_B(\tilde{u}) \neq N_{B'}(\tilde{u})\}\big|.$$

The next simple lemma now examines the effect of switchings on the neighbourhood distance of candidate graphs and shows that functions with small switching distance correspond to candidate graphs with small neighbourhood distance.

Lemma 9.19 (switching lemma). *Let H and G be bipartite graphs on vertex sets $\tilde{U} \dot\cup \tilde{V}$ and $U \dot\cup V$, respectively, such that $\deg_H(\tilde{v}) \leq \Delta$ for all $\tilde{v} \in \tilde{V}$ and let $f, f'\colon \tilde{V} \to V$ be injective functions with switching distance $\mathrm{d_{sw}}(f, f') \leq s$. Then the neighbourhood distance of the candidate graphs $B_f(H, G)$ and $B_{f'}(H, G)$ satisfies*

$$\mathrm{d}_{N(\tilde{U})}\Big(B_f(H, G), B_{f'}(H, G)\Big) \leq 2s\Delta\,.$$

Proof. We proceed by induction on s. For $s = 0$ the lemma is trivially true. Thus, consider $s > 0$ and let g be a function with $\mathrm{d_{sw}}(f, g) \leq s - 1$ and $\mathrm{d_{sw}}(g, f') = 1$. Define

$$N(f, f') := \Big\{\tilde{u} \in \tilde{U}\colon N_{B_f(H,G)}(\tilde{u}) \neq N_{B_{f'}(H,G)}(\tilde{u})\Big\}\,.$$

Clearly, $|N(f, f')| = \mathrm{d}_{N(\tilde{U})}(B_f(H, G), B_{f'}(H, G))$ and $N(f, f') \subseteq N(f, g) \cup N(g, f')$. By induction hypothesis we have $|N(f, g)| \leq 2(s - 1)\Delta$. The remaining switching from g to f' interchanges only the images of two vertices from $\tilde{V}$, say $\tilde{v}_1$ and $\tilde{v}_2$. It follows that

$$N(g, f') = \Big\{\tilde{u} \in N_H(\tilde{v}_1) \cup N_H(\tilde{v}_2)\colon N_{B_g(H,G)}(\tilde{u}) \neq N_{B_{f'}(H,G)}(\tilde{u})\Big\}\,,$$

which implies $|N(g, f')| \leq 2\Delta$ and therefore we get $|N(f, f')| \leq 2s\Delta$. □

A hypergraph packing lemma

The main ingredient to the proof of the constrained blow-up lemma is the following hypergraph packing result (Lemma 9.21). To understand what this lemma says and how we will apply it, recall that we would like to embed the vertex set $\tilde{U}$ of H into the vertex set U of G such that subsets of $\tilde{U}$ that form neighbourhoods in the graph H avoiding certain "bad" spots in U. If H is a Δ-regular graph, then these neighbourhoods form Δ-sets. In this case, as we will see, also the "bad" spots form Δ-sets. Accordingly, we have to solve the problem of packing the neighbourhood Δ-sets $\mathcal{N}$ and the "bad" Δ-sets $\mathcal{B}$, which is a hypergraph packing problem. Lemma 9.21 below states that this is possible under certain conditions. One of these conditions is that the "bad" sets should not "cluster" too much (although there might be many of them). The following definition makes this precise.

Definition 9.20 (corrupted sets)**.** *For $\Delta \in \mathbb{N}$ and a set V let $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a collection of Δ-sets in V and let x be a positive real. We say that all $B \in \mathcal{B}$ are x-*corrupted *by $\mathcal{B}$. Recursively, for $i \in [\Delta - 1]$ an i-set $B \in \binom{V}{i}$ in V is called x-corrupted by $\mathcal{B}$ if it is contained in more than x of the $(i+1)$-sets that are x-corrupted by $\mathcal{B}$.*

Observe that, if a vertex $v \in V$ is not x-corrupted by $\mathcal{B}$, then it is also not x'-corrupted by $\mathcal{B}$ for any $x' > x$.

The hypergraph packing lemma now implies that $\mathcal{N}$ and $\mathcal{B}$ can be packed if $\mathcal{B}$ contains no corrupted sets. In fact this lemma states that *half of all* possible ways to map the vertices of $\mathcal{N}$ to $\mathcal{B}$ can be turned into such a packing by performing a sequence of few switchings.

Lemma 9.21 (hypergraph packing lemma [86])**.** *For all integers $\Delta \geq 2$ and $\ell \geq 1$ there are positive constants $\eta_{9.21}$, and $n_{9.21}$ such that the following holds. Let $\mathcal{B}$ be a Δ-uniform hypergraph on $n' \geq n_{9.21}$ vertices such that no vertex of $\mathcal{B}$ is $\eta_{9.21}n'$-corrupted by $\mathcal{B}$. Let $\mathcal{N}$ be a Δ-uniform hypergraph on $n \leq n'$ vertices such that no vertex of $\mathcal{N}$ is contained in more than ℓ edges of $\mathcal{N}$.*

Then for at least half of all injective functions $f \colon V(\mathcal{N}) \to V(\mathcal{B})$ there are packings f' of $\mathcal{N}$ and $\mathcal{B}$ with switching distance $\mathrm{d_{sw}}(f, f') \leq \sigma n$. □

When applying this lemma we further make use of following lemma which helps us to bound corruption.

Lemma 9.22 (corruption lemma)**.** *Let $n,\Delta > 0$ be integers and μ and η be positive reals. Let V be a set of size n and $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a family of Δ-sets of size at most μn^{Δ}. Then at most $(\Delta!/\eta^{\Delta-1})\mu n$ vertices are ηn-corrupted by $\mathcal{B}$.*

Proof. For $i \in [\Delta]$ let $\mathcal{B}_i$ be the family of all those i-sets $B' \in \binom{V}{i}$ that are ηn-corrupted by $\mathcal{B}$. We will prove by induction on i (starting at $i = \Delta$) that

$$|\mathcal{B}_i| \leq \frac{\Delta!/i!}{\eta^{\Delta-i}}\mu n^i. \tag{9.31}$$

For $i = 1$ this establishes the lemma. For $i = \Delta$ the assertion is true by assumption. Now assume that (9.31) is true for $i > 1$. By definition every $B' \in \mathcal{B}_{i-1}$ is contained in more than ηn sets $B \in \mathcal{B}_i$. On the other hand, clearly every $B \in \mathcal{B}_i$ contains at most i sets from $\mathcal{B}_{i-1}$. Double counting thus gives

$$\eta n\,|\mathcal{B}_{i-1}| \leq \left|\left\{(B', B) : B' \in \mathcal{B}_{i-1}, B \in \mathcal{B}_i, B' \subseteq B\right\}\right| \leq i\,|\mathcal{B}_i| \overset{(9.31)}{\leq} i\frac{\Delta!/i!}{\eta^{\Delta-i}}\mu n^i,$$

which implies (9.31) for i replaced by $i - 1$. □

A matching lemma

We indicated earlier that we are interested into determining whether a candidate graph has a matching covering one of its partition classes. In order to do so we will make use of the following matching lemma which is an easy consequence of Hall's theorem. This lemma takes two graphs B and B' as input that have small neighbourhood distance. In our application these two graphs will be candidate graphs that correspond to two injective mappings f and f' with small switching distance (such as promised by the hypergraph packing lemma, Lemma 9.21).

Recall that Lemma 9.19 guarantees that mappings with small switching distance correspond to candidate graphs with small neighbourhood distance.

The matching lemma asserts that B' has the desired matching if certain vertex degree and neighbourhood conditions are satisfied. These conditions are somewhat technical. They are tailored exactly to match the conditions that we establish for candidate graphs in the proof of the constrained blow-up lemma (see Claims 9.26–9.28).

Lemma 9.23 (matching lemma)**.** *Let $B = (\tilde{U}\dot{\cup}U, E)$ and $B' = (\tilde{U}\dot{\cup}U, E')$ be bipartite graphs with $|U| \geq |\tilde{U}|$ and $\mathrm{d}_{N(\tilde{U})}(B, B') \leq s$. If there are positive integers x and n_1, n_2, n_3 such that*

(i) $\deg_{B'}(\tilde{u}) \geq n_1$ *for all* $\tilde{u} \in \tilde{U}$,
(ii) $|N_{B'}(\tilde{S})| \geq x|\tilde{S}|$ *for all* $\tilde{S} \subseteq \tilde{U}$ *with* $|\tilde{S}| \leq n_2$
(iii) $e_{B'}(\tilde{S}, S) \leq \frac{n_1}{n_3}|\tilde{S}||S|$ *for all* $\tilde{S} \subseteq \tilde{U}$, $S \subseteq U$ *with* $xn_2 \leq |S| < |\tilde{S}| < n_3$,
(iv) $|N_B(S) \cap \tilde{S}| > s$ *for all* $\tilde{S} \subseteq \tilde{U}$, $S \subseteq U$ *with* $|\tilde{S}| \geq n_3$ *and* $|S| > |U| - |\tilde{S}|$,

then B' has a matching covering $\tilde{U}$.

Proof. We will check Hall's condition in B' for all sets $\tilde{S} \subseteq \tilde{U}$. We clearly have $|N_{B'}(\tilde{S})| \geq |\tilde{S}|$ for $|\tilde{S}| \leq xn_2$ by (ii) (if $|\tilde{S}| > n_2$, then consider a subset of $\tilde{S}$ of size n_2).

Next, consider the case $xn_2 < |\tilde{S}| < n_3$. Set $S := N_{B'}(\tilde{S})$ and assume, for a contradiction, that $|S| < |\tilde{S}|$. Since $|S| < |\tilde{S}| < n_3$ we have $|S|/n_3 < 1$. Therefore, applying (i), we can conclude that

$$e_{B'}(\tilde{S}, S) = \sum_{\tilde{u} \in \tilde{S}} |N_{B'}(\tilde{u})| \geq n_1|\tilde{S}| > \frac{n_1}{n_3}|\tilde{S}||S|,$$

which is a contradiction to (ii). Thus $|N_{B'}(\tilde{S})| \geq |\tilde{S}|$.

Finally, for sets $\tilde{S}$ of size at least n_3 set $S := U \setminus N_{B'}(\tilde{S})$ and assume, again for a contradiction, that $|N_{B'}(\tilde{S})| < |\tilde{S}|$. This implies $|S| > |U| - |\tilde{S}|$. Accordingly we can apply (iv) to $\tilde{S}$ and S and infer that $|N_B(S) \cap \tilde{S}| > s$. Since $\mathrm{d}_{N(\tilde{U})}(B, B') \leq s$, at most s vertices from $\tilde{U}$ have different neighbourhoods in B and B' and so

$$\begin{aligned} \left|N_{B'}(S) \cap \tilde{S}\right| &= \left|\left\{\tilde{u} \in \tilde{S} : N_{B'}(\tilde{u}) \cap S \neq \emptyset\right\}\right| \\ &\geq \left|\left\{\tilde{u} \in \tilde{S} : N_B(\tilde{u}) \cap S \neq \emptyset\right\}\right| - s = \left|N_B(S) \cap \tilde{S}\right| - s > 0, \end{aligned}$$

which is a contradiction as $S = U \setminus N_{B'}(\tilde{S})$. □

Proof of Lemma 9.4

Now we are almost ready to present the proof of the constrained blow-up lemma (Lemma 9.4). We just need one further technical lemma as preparation. This lemma considers a family of pairwise disjoint Δ-sets $\mathcal{S}$ in a set S and states that a random injective function from S to a set T usually has the following property. The images $f(\mathcal{S})$ of sets in $\mathcal{S}$ "almost" avoid a small family of "bad" sets $\mathcal{T}$ in T.

Lemma 9.24. *For all positive integers Δ and positive reals β and μ_S there is $\mu_T > 0$ such that the following holds. Let S and T be disjoint sets, $\mathcal{S} \subseteq \binom{S}{\Delta}$ be a family of pairwise disjoint Δ-sets in S with $|\mathcal{S}| \leq \frac{1}{\Delta}(1 - \mu_S)|T|$, and $\mathcal{T} \subseteq \binom{T}{\Delta}$ be a family of Δ-sets in T with $|\mathcal{T}| \leq \mu_T |T|^{\Delta}$.*

Then a random injective function $f\colon S \to T$ satisfies $|f(\mathcal{S}) \setminus \mathcal{T}| > (1-\beta)|\mathcal{S}|$ with probability at least $1 - \beta^{|\mathcal{S}|}$.

Proof. Given Δ, β, and μ_S choose

$$\mu_T := \sqrt[\beta]{\beta}\left(\frac{e}{\beta}\left(\frac{\Delta}{\mu_S}\right)^{\Delta}\right)^{-1}. \tag{9.32}$$

Let S, T, $\mathcal{S}$, and $\mathcal{T}$ be as required and let f be a random injective function from S to T. We consider f as a consecutive random selection (without replacement) of images for the elements of S where the images of the elements of the (disjoint) sets in $\mathcal{S}$ are chosen first. Let S_i be the i-th such set in $\mathcal{S}$. Then the probability that f maps S_i to a set in $\mathcal{T}$, which we denote by p_i, is at most

$$p_i \le \frac{|\mathcal{T}|}{\binom{|T|-(i-1)\Delta}{\Delta}} \le \frac{\mu_T|T|^{\Delta}}{\binom{\mu_S|T|}{\Delta}} \le \mu_T\frac{|T|^{\Delta}}{\left(\frac{\mu_S|T|}{\Delta}\right)^{\Delta}} = \mu_T\left(\frac{\Delta}{\mu_S}\right)^{\Delta} =: p\,,$$

where the second inequality follows from $(i-1)\Delta \le |\bigcup\mathcal{S}| \le (1-\mu_S)|T|$. Let Z be a random variable with distribution $\mathrm{Bi}(|\mathcal{S}|, p)$. It follows that $\mathbb{P}[|f(\mathcal{S}) \cap \mathcal{T}| \ge z] \le \mathbb{P}[Z \ge z]$. Since

$$\mathbb{P}[Z \ge z] \le \binom{|\mathcal{S}|}{z}p^z < \left(\frac{e|\mathcal{S}|p}{z}\right)^z,$$

we infer that

$$\mathbb{P}\Big[|f(\mathcal{S}) \cap \mathcal{T}| \ge \beta|\mathcal{S}|\Big] < \left(\frac{ep}{\beta}\right)^{\beta|\mathcal{S}|} = \left(\frac{e\mu_T}{\beta}\left(\frac{\Delta}{\mu_S}\right)^{\Delta}\right)^{\beta|\mathcal{S}|} \overset{(9.32)}{=} \beta^{|\mathcal{S}|},$$

which proves the lemma since $|f(\mathcal{S}) \cap \mathcal{T}| \ge \beta|\mathcal{S}|$ holds iff $|f(\mathcal{S}) \setminus \mathcal{T}| \le (1-\beta)|\mathcal{S}|$. □

Now we can finally give the proof of Lemma 9.4.

Proof of Lemma 9.4. We first define a sequence of constants. Given Δ, d, and η fix $\Delta' := \Delta^2+1$. Choose β and σ such that

$$\beta^{\frac{1}{7}(\frac{d}{2})^{\Delta}} \le \frac{1}{5} \qquad \text{and} \qquad \frac{(1-\beta)d^{\Delta}}{100^{\Delta}} \ge 2\sigma \tag{9.33}$$

Apply the hypergraph packing lemma, Lemma 9.21, with input Δ, $\ell = 2\Delta+1$, and σ to obtain constants $\eta_{9.21}$, and $n_{9.21}$. Next, choose $\eta'_{9.21}$, μ_{BL}, and μ_S such that

$$\frac{\eta'_{9.21}}{1-\eta} \le \eta_{9.21}\,, \qquad \frac{\Delta! \cdot 2\mu_{\mathrm{BL}}}{(\eta'_{9.21})^{\Delta-1}} \le \eta\,, \qquad \frac{1}{\Delta'} \le \frac{1}{\Delta}(1-\mu_S)\,. \tag{9.34}$$

Lemma 9.24 with input Δ, β, μ_S provides us with a constant μ_T. We apply Lemma 9.10 two times, once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_{\mathrm{BL}}/\Delta'$ and once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_T$ and get constants $\varepsilon_{9.10}$ and $\tilde{\varepsilon}_{9.10}$, respectively. Now we can fix the promised constant ε such that

$$\varepsilon \le \min\left\{\frac{\varepsilon_{9.10}}{\Delta'}, \frac{d}{2\Delta}\right\}, \qquad \text{and} \qquad \frac{\varepsilon\Delta'}{\eta(1-\eta)} < \min\{d, \tilde{\varepsilon}_{9.10}\}. \tag{9.35}$$

As last input let r_1 be given and set

$$\xi_{9.10} := \eta(1-\eta)/(r_1\Delta'). \tag{9.36}$$

Next let $c_{9.10}$ be the maximum of the two constants obtained from the two applications of Lemma 9.10, that we started above, with the additional parameter $\xi_{9.10}$. Further, let ν and $c_{9.11}$ be the constants from Lemma 9.11 for input Δ, d, and ε, and let $c_{9.7}$ be the constant from Lemma 9.7 for input Δ and ν. Finally, we choose $c = \max\{c_{9.10}, c_{9.11}, c_{9.7}\}$. With this we defined all necessary constants.

Now assume we are given any $1 \leq r \leq r_1$, and a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p \geq c(\log n/n)^{1/\Delta}$, where, without loss of generality, n is such that

$$(1-\eta')\frac{n}{r} \geq n_{9.21}. \tag{9.37}$$

Then, with high probability, the graph Γ satisfies the assertion of the different lemmas concerning random graphs, that we started to apply in the definition of the constants. More precisely, by the choice of the constants above,

(P1) Γ satisfies the assertion of Lemma 9.7 for parameters Δ and ν, i.e., for any set X and any family $\mathcal{F}$ with the conditions required in this lemma, the conclusion of the lemma holds.

(P2) Similarly Γ satisfies the assertion of Lemma 9.10 for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_{\mathrm{BL}}/\Delta'$, $\varepsilon_{9.10}$, and $\xi_{9.10}$. The same holds for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_T$, $\tilde{\varepsilon}_{9.10}$, and $\xi_{9.10}$.

(P3) Γ satisfies the assertion of Lemma 9.11 for parameters Δ, d, ε, and ν.

In the following we will assume that Γ has these properties and show that it then also satisfies the conclusion of the constrained blow-up lemma, Lemma 9.4.

Let $G \subseteq \Gamma$ and H be two bipartite graphs on vertex sets $U\dot\cup V$ and $\tilde{U}\dot\cup\tilde{V}$, respectively, that fulfil the requirements of Lemma 9.4. Moreover, let $\mathcal{H} \subseteq \binom{\tilde{V}}{\Delta}$ be the family of special Δ-sets, and $\mathcal{B} \subseteq \binom{V}{\Delta}$ be the family of forbidden Δ-sets. It is not difficult to see that, by possibly adding some edges to H, we can assume that the following holds.

($\tilde{\mathrm{U}}$) All vertices in $\tilde{U}$ have degree *exactly* Δ.

($\tilde{\mathrm{V}}$) All vertices in $\tilde{V}$ have degree maximal $\Delta + 1$.

Our next step will be to split the partition class U of G and the corresponding partition class $\tilde{U}$ of H into Δ' parts of equal size. From the partition of H we require that no two vertices in one part have a common neighbour. This will guarantee that the neighbourhoods of two different vertices from one part form disjoint vertex sets (which we need because we would like to apply Lemma 9.11 later, in the proof of Claim 9.26, and Lemma 9.11 asserts certain properties for families of disjoint vertex sets).

Let us now explain precisely how we split U and $\tilde{U}$. We assume for simplicity that $|\tilde{U}|$ and $|U|$ are divisible by Δ' and partition the sets U arbitrarily into Δ' parts $U = U_1\dot\cup\ldots U_{\Delta'}$ of equal size, i.e., sets of size at least $n/(r\Delta')$. Similarly let $\tilde{U} = \tilde{U}_1\dot\cup\ldots\dot\cup\tilde{U}_{\Delta'}$ be a partition of $\tilde{U}$ into sets of equal size such that each $\tilde{U}_j$ is 2-independent in H. Such a partition exists by the Theorem of Hajnal and Szemerédi (Theorem 9.14) applied to $H^2[\tilde{U}]$ because the maximum degree of H^2 is less than $\Delta' = \Delta^2 + 1$.

In Claim 9.25 below we will assert that there is an embedding f' of $\tilde{V}$ into V that can be extended to *each* of the $\tilde{U}_j$ separately such that we obtain an embedding of H into G. To this end we will consider the candidate graphs $B_{f'}(H_j, G_j)$ defined by f' (see Definition 9.16) and show, that there is an f' such that each $B_{f'}(H_j, G_j)$ has a matching covering $\tilde{U}_j$. This, as discussed earlier, will ensure the existence of the desired embedding. For preparing this argument, we first need to exclude some vertices of V which are not suitable for such an embedding. For identifying these vertices, we define the following family of Δ-sets which contains $\mathcal{B}$ and all sets in V that have a small common neighbourhood in some $\tilde{U}_j$.

Define $\mathcal{B}' := \mathcal{B} \cup \bigcup_{j\in\Delta'} \mathcal{B}_j$ where

$$\mathcal{B}_j := \left\{ B \in \binom{V}{\Delta} :\ \left|N_G^{\cap}(B) \cap U_j\right| < (\tfrac{1}{2}d)^{\Delta} p^{\Delta} |U_j| \right\} \stackrel{(9.3)}{=} \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V, U_j). \tag{9.38}$$

We claim that we obtain a set $\mathcal{B}'$ that is not much larger than $\mathcal{B}$. Indeed, by Proposition 3.16 the pair

$$(V, U_j) \text{ is } (\varepsilon\Delta', d, p)\text{-dense for all } j \in [\Delta'], \tag{9.39}$$

and $\varepsilon\Delta' \le \varepsilon_{9.10}$ by (9.35). Moreover we have $|U_j| \ge n/(r\Delta') \ge n/(r_1\Delta') \ge \xi_{9.10} n$ by (9.36). We can thus use the fact that our random graph Γ satisfies property (P2) (with $\mu = \mu_{\mathrm{BL}}/\Delta'$) on the bipartite subgraph $G[V \dot\cup U_j]$ and conclude that $|\mathcal{B}_j| \le \mu_{\mathrm{BL}} |V|^{\Delta}/\Delta'$. Since $|\mathcal{B}| \le \mu_{\mathrm{BL}}|V|^{\Delta}$ by assumption we infer

$$|\mathcal{B}'| \le \mu_{\mathrm{BL}}|V|^{\Delta} + \Delta' \cdot \mu_{\mathrm{BL}}|V|^{\Delta}/\Delta' = 2\mu_{\mathrm{BL}}|V|^{\Delta}.$$

Set

$$V' := V \setminus V'' \quad \text{with} \quad V'' := \left\{ v \in V :\ v \text{ is } \eta'_{9.21}|V|\text{-corrupted by } \mathcal{B}' \right\} \tag{9.40}$$

and delete all sets from $\mathcal{B}'$ that contain vertices from V''. This determines the set V'' of vertices that we exclude from V for the embedding. We will next show that we did not exclude too many vertices in this process. For this we use the corruption lemma, Lemma 9.22. Indeed, Lemma 9.22 applied with n replaced by $|V|$, with Δ, $\mu = 2\mu_{\mathrm{BL}}$, and $\eta'_{9.21}$ to V and $\mathcal{B}'$ implies that

$$|V''| \le \frac{\Delta!}{(\eta'_{9.21})^{\Delta-1}} 2\mu_{\mathrm{BL}}|V| \stackrel{(9.34)}{\le} \eta|V| \quad \text{and thus} \quad n' := |V'| \ge (1-\eta)|V|. \tag{9.41}$$

Let

$$H_j := H\big[\tilde{U}_j \dot\cup \tilde{V}\big] \qquad \text{and} \qquad G_j := G\big[U_j \dot\cup V'\big].$$

Now we are ready to state the claim announced above, which asserts that there is an embedding f' of the vertices in $\tilde{V}$ to the vertices in V' such that the corresponding candidate graphs $B_{f'}(H_j, G_j)$ have matchings covering $\tilde{U}_j$. As we will shall show, this claim implies the assertion of the constrained blow-up lemma. Its proof, which we will provide thereafter, requires the matching lemma (Lemma 9.23), and the hypergraph packing lemma (Lemma 9.21).

Claim 9.25. *There is an injection $f' : \tilde{V} \to V'$ with $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$ such that for all $j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ has a matching covering $\tilde{U}_j$.*

Let us show that proving this claim suffices to establish the constrained blow-up lemma. Indeed, let $f' : \tilde{V} \to V'$ be such an injection and denote by $M_j : \tilde{U}_j \to U_j$ the corresponding matching in $B_{f'}(H_j, G_j)$ for $j \in [\Delta]$. We claim that the function $g \colon \tilde{U} \dot\cup \tilde{V} \to U \dot\cup V$, defined by

$$g(\tilde{w}) = \begin{cases} M_j(\tilde{w}) & \tilde{w} \in \tilde{U}_j, \\ f'(\tilde{w}) & \tilde{w} \in \tilde{V}, \end{cases}$$

is an embedding of H into G. To see this, notice first that g is injective since f' is an injection and all M_j are matchings. Furthermore, consider an edge $\tilde{u}\tilde{v}$ of H with $\tilde{u} \in \tilde{U}_j$ for some $j \in [\Delta']$ and $\tilde{v} \in \tilde{V}$ and let

$$u := g(\tilde{u}) = M_j(\tilde{u}) \qquad \text{and} \qquad v := g(\tilde{v}) = f'(\tilde{v}).$$

It follows from the definition of M_j that $\tilde{u}u$ is an edge of the candidate graph $B_{f'}(H_j, G_j)$. Hence, by the definition of $B_{f'}(H_j, G_j)$, u is an f'-candidate for $\tilde{u}$, i.e.,

$$f'\left(N_{H_j}(\tilde{u})\right) \subseteq N_{G_j}(u).$$

Since $v = f'(\tilde{v}) \in f'\left(N_{H_j}(\tilde{u})\right)$ this implies that uv is an edge of G. Because f' also satisfies $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$ the embedding g also meets the remaining requirement of the constrained blow-up lemma that no special Δ-set is mapped to a forbidden Δ-set. □

For completing the proof of Lemma 9.4, we still need to prove Claim 9.25 which we shall be occupied with for the remainder of this section. We will assume throughout that we have the same setup as in the preceding proof. In particular all constants, sets, and graphs are defined as there.

For proving Claim 9.25 we will use the matching lemma (Lemma 9.23) on candidate graphs $B = B_f(H_j, G_j)$ and $B' = B_{f'}(H_j, G_j)$ for injections $f, f' \colon \tilde{V} \to V'$. As we will see, the following three claims imply that there are suitable f and f' such that the conditions of this lemma are satisfied. More precisely, Claim 9.26 will take care of conditions (i) and (ii) in this lemma, Claim 9.27 of condition (iii), and Claim 9.28 of condition (iv). Before proving these claims we will show that they imply Claim 9.25.

The first claim states that *many* injective mappings $f \colon \tilde{V} \to V'$ can be turned into injective mappings f' (with the help of a few switchings) such that the candidate graphs $B_{f'}(H_j, G_j)$ for f' satisfy certain degree and expansion properties.

Claim 9.26. *For* at least half *of all injections* $f : \tilde{V} \to V'$ *there is an injection* $f' : \tilde{V} \to V'$ *with* $d_{\mathrm{sw}}(f, f') \le \sigma n/r$ *such that the following is satisfied for all* $j \in [\Delta']$*. For all* $\tilde{u} \in \tilde{U}_j$ *and all* $\tilde{S} \subseteq \tilde{U}_j$ *with* $|\tilde{S}| \le p^{-\Delta}$ *we have*

$$\deg_{B_{f'}(H_j,G_j)}(\tilde{u}) \ge (\tfrac{d}{2})^{\Delta} p^{\Delta} |U_j| \qquad \text{and} \qquad |N_{B_{f'}(H_j,G_j)}(\tilde{S})| \ge \nu n p^{\Delta} |\tilde{S}|. \tag{9.42}$$

Further, no special Δ*-set from* $\mathcal{H}$ *is mapped to a forbidden* Δ*-set from* $\mathcal{B}$ *by* f'*.*

The second claim asserts that *all* injective mappings f' are such that the candidate graphs $B_{f'}(H_j, G_j)$ do not contain sets of certain sizes with too many edges between them.

Claim 9.27. All *injections* $f' : \tilde{V} \to V'$ *satisfy the following for all* $j \in [\Delta']$ *and all* $S \subseteq U_j$*,* $\tilde{S} \subseteq \tilde{U}_j$*. If* $\nu n \le |S| < |\tilde{S}| < \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|$*, then*

$$e_{B_{f'}(H_j,G_j)}(\tilde{S}, S) \le 7p^{\Delta}|\tilde{S}||S|.$$

The last of the three claims states that for *random* injective mappings f the graphs $B_{f'}(H_j, G_j)$ have edges between any pair of large enough sets $S \subseteq U_j$ and $\tilde{S} \subseteq \tilde{U}_j$.

Claim 9.28. *A* random *injection* $f : \tilde{V} \to V'$ *a.a.s. satisfies the following. For all* $j \in [\Delta']$ *and all* $S \subseteq U_j$*,* $\tilde{S} \subseteq \tilde{U}_j$ *with* $|\tilde{S}| \ge \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|$ *and* $|S| > |U_j| - |\tilde{S}|$ *we have*

$$\left| N_{B_f(H_j,G_j)}(S) \cap \tilde{S} \right| > 2\sigma n/r.$$

Proof of Claim 9.25. Our aim is to apply the matching lemma (Lemma 9.23) to the candidate graphs $B_f(H_j, G_j)$ and $B_{f'}(H_j, G_j)$ for all $j \in [\Delta']$ with carefully chosen injections f and f'.

Let $f : \tilde{V} \to V'$ be an injection satisfying the assertions of Claim 9.26 and Claim 9.28 and let f' be the injection promised by Claim 9.26 for this f. Such an f exists as at least half of all injections from $\tilde{V}$ to V' satisfy the assertion of Claim 9.26 and almost all of those satisfy the assertion of Claim 9.28. We will now show that for all $j \in [\Delta']$ the conditions of Lemma 9.23 are satisfied for input

$$B = B_f(H_j, G_j), \qquad B' = B_{f'}(H_j, G_j), \qquad s = 2\sigma n/r\,,$$
$$x = \nu n p^{\Delta}, \qquad n_1 = (\tfrac{d}{2})^{\Delta} p^{\Delta} |U_j|, \qquad n_2 = p^{-\Delta}, \qquad n_3 = \tfrac{1}{7}(\tfrac{d}{2})^{\Delta}|U_j|,$$

Claim 9.26 asserts that $\mathrm{d_{sw}}(f, f') \le \sigma n/r$. Since $\tilde{U}_j$ is 2-independent in H we have $\deg_{H_j}(\tilde{v}) \le 1$ for all $\tilde{v} \in \tilde{V}$. Thus the switching lemma, Lemma 9.19, applied to H_j and G_j and with s replaced by $\sigma n/r$ implies

$$\mathrm{d}_{N(\tilde{U}_j)}(B, B') = \mathrm{d}_{N(\tilde{U}_j)}\Big(B_f(H_j, G_j), B_{f'}(H_j, G_j)\Big) \le 2\sigma n/r = s$$

Moreover, by Claim 9.26, for all $\tilde{u} \in \tilde{U}_j$ we have

$$\deg_{B'}(\tilde{u}) = \deg_{B_{f'}(H_j,G_j)}(\tilde{u}) \ge (\tfrac{d}{2})^{\Delta} p^{\Delta} |U_j| = n_1$$

and thus condition (i) of Lemma 9.23 holds true. Further, we conclude from Claim 9.26 that $|N_{B'}(\tilde{S})| \ge x|\tilde{S}|$ for all $\tilde{S} \subseteq \tilde{U}_j$ with $|\tilde{S}| < p^{-\Delta} = n_2$. This gives condition (ii) of Lemma 9.23. In addition, Claim 9.27 states that for all $S \subseteq U_j$, $\tilde{S} \subseteq \tilde{U}_j$ with $xn_2 = \nu n \le |S| < |\tilde{S}| < \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j| = n_3$ we have

$$e_{B'}(\tilde{S}, S) = e_{B_{f'}(H_j,G_j)}(\tilde{S}, S) \le 7p^{\Delta}|\tilde{S}||S| = \frac{n_1}{n_3}|\tilde{S}||S|$$

and accordingly also condition (iii) of Lemma 9.23 is satisfied. To see (iv), observe that the choice of f and Claim 9.28 assert

$$\Big|N_B(S) \cap \tilde{S}\Big| = \Big|N_{B_f(H_j,G_j)}(S) \cap \tilde{S}\Big| > 2\sigma n/r = s$$

for all $S \subseteq U_j$, $\tilde{S} \subseteq \tilde{U}_j$ with $|\tilde{S}| \ge \frac{1}{7}(\frac{d}{2})^{\Delta}|U_j| = n_3$ and $|S| > |U| - |\tilde{S}|$. Therefore, all conditions of Lemma 9.23 are satisfied and we infer that for *all* $j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ with f' as chosen above has a matching covering $\tilde{U}$. Moreover, by Claim 9.26, f' maps no special Δ-set to a forbidden Δ-set. This establishes Claim 9.25. □

It remains to show Claims 9.26–9.28. We start with Claim 9.26. For the proof of this claim we apply the hypergraph packing lemma (Lemma 9.21).

Proof of Claim 9.26. Notice that ($\tilde{\mathrm{U}}$) on page 157 implies that $N_H(\tilde{u})$ contains exactly Δ elements for each $\tilde{u} \in \tilde{U}$. Hence we may define the following family of Δ-sets. Let

$$\mathcal{N} := \Big\{N_H(\tilde{u})\colon\ \tilde{u} \in \tilde{U}\Big\} \cup \mathcal{H} \subseteq \binom{\tilde{V}}{\Delta}.$$

We want to apply the hypergraph packing lemma (Lemma 9.21) with Δ, with ℓ replaced by $2\Delta + 1$, and with σ to the hypergraphs with vertex sets $\tilde{V}$ and V' and edge sets $\mathcal{N}$ and $\mathcal{B}'$, respectively (see (9.38) on page 158). We will first check that the necessary conditions are satisfied.

Observe that

$$|V'| \overset{(9.41)}{\geq} (1-\eta')|V| \geq (1-\eta')n/r \overset{(9.37)}{\geq} n_{9.21}, \qquad \text{and} \qquad |\tilde{V}| \leq |V'|\,.$$

Furthermore, a vertex $\tilde{v} \in \tilde{V}$ is neither contained in more than Δ sets from $\mathcal{H}$ nor is $\tilde{v}$ in $N_H(\tilde{u})$ for more than $\Delta+1$ vertices $\tilde{u} \in \tilde{U}$ (by ($\tilde{\text{V}}$) on page 157). Therefore the condition Lemma 9.21 imposes on $\mathcal{N}$ is satisfied with ℓ replaced by $2\Delta+1$. Moreover, according to (9.40) no vertex in V' is $\eta'_{9.21}|V|$-corrupted by $\mathcal{B}'$. Since

$$\eta'_{9.21}|V| \overset{(9.41)}{\leq} \eta'_{9.21}(1-\eta)^{-1}n' \overset{(9.34)}{\leq} \eta_{9.21}n',$$

this (together with the observation in Definition 9.20) implies that no vertex in V' is $\eta_{9.21}n'$-corrupted by $\mathcal{B}'$ and therefore all prerequisites of Lemma 9.21 are satisfied.

It follows that the conclusion of Lemma 9.21 holds for at least half of all injective functions $f\colon \tilde{V} \to V'$, namely that there are packings f' of (the hypergraphs with edges) $\mathcal{N}$ and $\mathcal{B}$ with switching distance $\mathrm{d_{sw}}(f,f') \leq \sigma|\tilde{V}| \leq \sigma n/r$. Clearly, such a packing f' does not send any special Δ-set from $\mathcal{H}$ to any forbidden Δ-set from $\mathcal{B}$. Our next goal is to show that f' satisfies the first part of (9.42) for all $j \in [\Delta']$ and $\tilde{u} \in \tilde{U}_j$. For this purpose, fix j and $\tilde{u}$. The definition of the candidate graph $B_{f'}(H_j,G_j)$, Definition 9.16, implies

$$\begin{aligned}\deg_{B_{f'}(H_j,G_j)}(\tilde{u}) &= \Big|\Big\{u \in U_j\colon\ f'\left(N_{H_j}(\tilde{u})\right) \subseteq N_{G_j}(u)\Big\}\Big| \\ &= \Big|\Big\{u \in U_j\colon\ u \in N^{\cap}_{G_j}\Big(f'\left(N_{H_j}(\tilde{u})\right)\Big)\Big\}\Big| \\ &= \Big|N^{\cap}_{G_j}\Big(f'\left(N_{H_j}(\tilde{u})\right)\Big)\Big| \geq (\tfrac{1}{2}d)^{\Delta}p^{\Delta}|U_j|\,.\end{aligned}$$

where the first inequality follows from the fact that $N_{H_j}(\tilde{u}) \in \mathcal{N}$ and thus, as f' is a packing of $\mathcal{N}$ and $\mathcal{B}'$, we have $f'(N_{H_j}(\tilde{u})) \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V,U_j) \subseteq \mathcal{B}'$ (see the definition of $\mathcal{B}'$ in (9.38)). This in turn means that all Δ-sets $f'(N_{H_j}(\tilde{u}))$ with $\tilde{u} \in \tilde{U}_j$ are p-good (see Definition 9.9) in (V,U_j), because (V,U_j) has p-density at least $d-\varepsilon\Delta' \geq \frac{d}{2}$ by (9.39) and (9.35). With this information at hand we can proceed to prove the second part of (9.42). Let $\tilde{S} \subseteq \tilde{U}_j$ with $\tilde{S} < 1/p^{\Delta}$ and consider the family $\mathcal{F} \subseteq \binom{V}{\Delta}$ with

$$\mathcal{F} := \{f'(N_H(\tilde{u}))\colon\ \tilde{u} \in \tilde{S}\}.$$

Because U_j is 2-independent in H the sets $N_H(\tilde{u})$ with $\tilde{u} \in \tilde{S}$ form a family of disjoint Δ-sets in $\tilde{V}$. It follows that also the sets $f'(N_H(\tilde{u}))$ with $\tilde{u} \in \tilde{S}$ form a family of disjoint Δ-sets in V. By (P3) on page 157 the conclusion of Lemma 9.11 holds for Γ. We conclude that the pair (V,U_j) is $(1/p^{\Delta},\nu np^{\Delta})$-expanding. Since $|\mathcal{F}| = |\tilde{S}| < 1/p^{\Delta}$ by assumption and all members of $\mathcal{F}$ are p-good in (V,U_j) this implies that $|N^{\cap}_{U_j}(\mathcal{F})| \geq \nu np^{\Delta}|\mathcal{F}|$. On the other hand $N^{\cap}_{U_j}(\mathcal{F}) = N_{B_{f'}(H_j,G_j)}(\tilde{S})$ by the definition of $B_{f'}(H_j,G_j)$ and $\mathcal{F}$ and thus we get the second part of (9.42). □

Recall that property (P1) states that Γ satisfies the conclusion of Lemma 9.7 for certain parameters. We will use this fact to prove Claim 9.27.

Proof of Claim 9.27. Fix $f' : \tilde{V} \to V'$, $j \in [\Delta']$, $S \subseteq U_j$, and $\tilde{S} \subseteq \tilde{U}_j$ with $\nu n \le |S| < |\tilde{S}| < \frac{1}{7}(\frac{d}{2})^\Delta |U_j|$. For the candidate graphs $B_{f'}(H_j, G_j)$ of f' we have

$$\begin{aligned} e_{B_{f'}(H_j,G_j)}(\tilde{S}, S) &= \left|\left\{\tilde{u}u \in \tilde{S} \times S\colon\ f'\big(N_H(\tilde{u})\big) \subseteq N_G(u)\right\}\right| \\ &\overset{(9.1)}{=} \#\,\mathrm{stars}^G\left(S, \left\{f'\big(N_H(\tilde{u})\big)\colon\ \tilde{u} \in \tilde{S}\right\}\right) \\ &\le \#\,\mathrm{stars}^\Gamma\left(S, \left\{f'\big(N_H(\tilde{u})\big)\colon\ \tilde{u} \in \tilde{S}\right\}\right) = \#\,\mathrm{stars}^\Gamma(S, \mathcal{F}'), \end{aligned}$$

where $\mathcal{F}' := \{f'\big(N_H(\tilde{u})\big)\colon\ \tilde{u} \in \tilde{S}\}$. As before the sets $f'(N_H(\tilde{u}))$ with $\tilde{u} \in \tilde{S}$ form a family of $|\tilde{S}|$ disjoint Δ-sets in V'. Since $\nu n \le |S| < |\tilde{S}| = |\mathcal{F}'| \le n$ we can appeal to property (P1) (and hence Lemma 9.7) with the set $X := S$ and the family $\mathcal{F}'$ and infer that

$$e_{B_{f'}(H_j,G_j)}(\tilde{S}, S) \le \#\,\mathrm{stars}^\Gamma(S, \mathcal{F}') \le 7p^\Delta |\mathcal{F}'||S| = 7p^\Delta |\tilde{S}||S|$$

as required. □

Finally, we prove Claim 9.28. For this proof we will use the fact that Δ-sets in p-dense graphs have big common neighbourhoods (the conclusion of Lemma 9.10 holds by property (P2)) together with Lemma 9.24.

Proof of Claim 9.28. Let f be an injective function from $\tilde{V}$ to V'. First, consider a fixed $j \in [\Delta']$ and fixed sets $S \subseteq U_j$, $\tilde{S} \subseteq \tilde{U}_j$ with $|\tilde{S}| \ge \frac{1}{7}(\frac{d}{2})^\Delta |U_j|$ and $|S| > |U_j| - |\tilde{S}|$. Define

$$\mathcal{S} := \{N_{H_j}(\tilde{u})\colon\ \tilde{u} \in \tilde{S}\} \qquad \text{and} \qquad \mathcal{T} := \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S).$$

and observe that

$$\begin{aligned} \left|N_{B_f(H_j,G_j)}(S) \cap \tilde{S}\right| &= \left|\left\{\tilde{u} \in \tilde{S}\colon\ \exists u \in S \text{ with } f\left(N_{H_j}(\tilde{u})\right) \subseteq N_{G_j}(u)\right\}\right| \\ &= \left|\left\{\tilde{u} \in \tilde{S}\colon\ N^\cap_{G_j}\Big(f\left(N_{H_j}(\tilde{u})\right)\Big) \cap S \neq \emptyset\right\}\right| \\ &\ge \left|\left\{\tilde{u} \in \tilde{S}\colon\ f\left(N_{H_j}(\tilde{u})\right) \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S)\right\}\right| = |f(\mathcal{S}) \setminus \mathcal{T}| \end{aligned}$$

since all Δ-sets $B \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V', S)$ satisfy $|N^\cap_{G_j}(B) \cap S| \ge (\frac{d}{2})^\Delta p^\Delta |S| > 0$. Thus, for proving the claim, it suffices to show that a random injection $f : \tilde{V} \to V'$ violates $|f(\mathcal{S}) \setminus \mathcal{T}| > 2\sigma n/r$ with probability at most $5^{-|U_j|}$ because this implies that f violates the conclusion of Claim 9.28 for *some* $j \in [\Delta']$, and *some* $S \subseteq U_j$, $\tilde{S} \subseteq \tilde{U}_j$ with probability at most $\mathcal{O}(2^{|U_j|} 2^{|\tilde{U}_j|} . 5^{-|U_j|}) = o(1)$. For this purpose, we will use the fact that the pair (V', S) is p-dense. Indeed, observe that

$$|S| > |U_j| - |\tilde{S}| > |U_j| - |\tilde{U}_j| = \frac{|U| - |\tilde{U}|}{\Delta'} \ge \frac{\eta|U|}{\Delta'}$$

by the assumptions of the constrained blow-up lemma, Lemma 9.4. As $|V'| \ge (1-\eta)|V|$ by (9.41) we can apply Proposition 3.16 twice to infer from the (ε, d, p)-density of (V, U) that (V', S) is $(\tilde{\varepsilon}, d, p)$-dense with $\tilde{\varepsilon} := \varepsilon\Delta'/(\eta(1-\eta))$. Furthermore $\tilde{\varepsilon} \le \tilde{\varepsilon}_{9.10}$ by (9.35) and

$$|V'| \overset{(9.41)}{\ge} (1-\eta)\frac{n}{r} \overset{(9.36)}{\ge} \xi_{9.10} n, \qquad \text{and} \qquad |S| > \frac{\eta|U|}{\Delta'} \ge \frac{\eta n}{r\Delta'} \overset{(9.36)}{\ge} \xi_{9.10} n.$$

Hence we conclude from (P2) on page 157 (with $\mu = \mu_T$) that $|\mathcal{T}| = |\operatorname{bad}^{G,\Delta}_{d/2,d,p}(V', S)| \le \mu_T |V'|^{\Delta}$. In addition

$$\tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta}|U_j| \le |\tilde{S}| = |\mathcal{S}| \le |\tilde{U}_j| \le (1-\eta)\frac{n}{\Delta'} \overset{(9.41)}{\le} \frac{|V'|}{\Delta'} \overset{(9.34)}{\le} \frac{1}{\Delta}(1-\mu_S)|V'|. \tag{9.43}$$

Thus, we can apply Lemma 9.24 with Δ, β, and μ_S to $S = \tilde{V}$, $T = V'$, and to $\mathcal{S}$ and $\mathcal{T}$ and conclude that f violates

$$|f(\mathcal{S}) \setminus \mathcal{T}| > (1-\beta)|\mathcal{S}| \overset{(9.43)}{\ge} (1-\beta)\tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta}|U_j| \ge \frac{(1-\beta)d^{\Delta}n}{7\cdot 2^{\Delta} r \Delta'} \ge \frac{(1-\beta)d^{\Delta}n}{100^{\Delta} r} \overset{(9.33)}{\ge} 2\sigma\frac{n}{r}$$

with probability at most

$$\beta^{|\mathcal{S}|} \le \beta^{\frac{1}{7}(\frac{d}{2})^{\Delta}|U_j|} \le 5^{-|U_j|}$$

where the first inequality follows from (9.43) and the second from (9.33). □

9.8 The connection lemma

The proof of Lemma 9.5 which we present in this section is inherent in the proof of [61, Lemma 18]. The only difference is that we have a somewhat more special set-up here (given by the pre-defined partitions and candidate sets). This set-up however is chosen exactly in such a way that this proof continues to work if we adapt the involved parameters accordingly.

Proof of Lemma 9.5. For the proof of Lemma 9.5 we use an inductive argument and embed a partition class of H into the corresponding partition class of G one at a time. Before describing this strategy we will define two graph properties $\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$ and $\mathrm{STAR}_p(k, \xi, \nu)$, which a random graph $\Gamma = \mathcal{G}_{n,p}$ enjoys a.a.s. for suitable sets of parameters. Then we will set up these parameters accordingly and define all other constants involved in the proof.

For a fixed n-vertex graph Γ, fixed positive reals d_0, ε', μ, ε, ξ, and ν, a fixed integer k, and a function $p = p(n)$ we define the following properties of Γ.

$\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$ *We say that* Γ *has property* $\mathrm{D}_p(d_0, \varepsilon', \mu, \varepsilon, \xi)$ *if it satisfies the property stated in Lemma 9.12 with these parameters and with* Δ*, i.e., whenever* $G = (X \dot\cup Y \dot\cup Z, E)$ *is a tripartite subgraph of* Γ *with the required properties, then it satisfies the conclusion of this lemma.*

$\mathrm{STAR}_p(k, \xi, \nu)$ *Similarly* Γ *has property* $\mathrm{STAR}_p(k, \xi, \nu)$ *if* Γ *has the property stated in Lemma 9.8 with* Δ *replaced by* k*, with parameters* ξ*,* ν*, and for* $p = p(n)$*.*

Now we set up the constants. Let Δ, t and d be given and assume without loss of generality that $d \le \frac{1}{4}$. First we set

$$\mu = \frac{1}{4\Delta^2} \tag{9.44}$$

and we fix ε_i for $i = t, t-1, \dots, 0$ by setting

$$\begin{aligned} \varepsilon_t &= \frac{d}{12\Delta t}, \qquad d_0 := d, \quad \text{and} \\ \varepsilon_{i-1} &= \min\left\{\varepsilon(\Delta, d_0, \varepsilon' = \varepsilon_i, \mu),\, \varepsilon_i\right\} \text{ for } i = t, \dots, 1, \end{aligned} \tag{9.45}$$

where $\varepsilon(\Delta-1, d_0, \varepsilon' = \varepsilon_i, \mu)$ is given by Lemma 9.12. We choose $\varepsilon := \varepsilon_0$ and $\xi := (d/100)^{\Delta}$ and receive r_1 as input. For each $k \in [\Delta]$ and each $r' \in [r_1]$ Lemma 9.8 with Δ replaced by

k and with ξ replaced by ξ/r' provides positive constants $\nu(k,r')$ and $c(k,r')$. Let ν be the minimum among the $\nu(k,r')$ and let $c_{9.8}$ be the maximum among the $c(k,r')$ as we let both k and r' vary. Similarly Lemma 9.12 with input $\Delta-1$, d_0, $\varepsilon'=\varepsilon_i$, μ, and ξ replaced by ξ/r' provides constants $c'(i,r')$ for $i\in[0,t]$ and $r'\in[r_1]$. We let $c_{9.12}$ be the maximum among these $c'(i,r')$. Then we fix $c:=\max\{c_{9.8},c_{9.12}\}$, and receive $r\in[r_1]$ as input. Finally, we set

$$\xi_{9.8}:=\xi_{9.12}:=\xi/r=(d/100)^{\Delta}(1/r)\,. \tag{9.46}$$

This finishes the definition of the constants.

Let $p=p(n)\geq c(\log n/n)^{1/\Delta}$ and let Γ be a graph from $\mathcal{G}_{n,p}$. By Lemma 9.8, Lemma 9.12, and the choice of constants the graph Γ a.a.s. satisfies properties $\mathrm{D}_p(d,\varepsilon_i,\mu,\varepsilon_{i-1},\xi_{9.12})$ for all $i\in[t]$, and properties $\mathrm{STAR}_p(k,\xi_{9.8},\nu)$ for all $k\in[\Delta]$. In the remainder of this proof we assume that Γ has these properties and show that then Γ also satisfies the conclusion of Lemma 9.5.

Let $G\subseteq\Gamma$ and H be arbitrary graphs satisfying the requirements stated in the lemma on vertex sets $W=W_1\dot\cup\ldots\dot\cup W_t$ and $\tilde W=\tilde W_1\dot\cup\ldots\dot\cup\tilde W_t$, respectively. Let $h\colon\tilde W\to[t]$ be the "partition function" for the vertex partition of H, i.e.,

$$h(\tilde w)=j\quad\text{if and only if}\quad\tilde w\in\tilde W_j.$$

For an integer $i\leq h(\tilde w)$ we denote by

$$\mathrm{ldeg}^i(\tilde w):=\big|N_H(\tilde w)\cap\{\tilde x\in\tilde W\colon\ h(\tilde x)\leq i\}\big|$$

the left degree of $\tilde w$ with respect to $\tilde W_1\dot\cup\ldots\dot\cup\tilde W_i$. Clearly $\mathrm{ldeg}^{h(\tilde w)}(\tilde w)=\mathrm{ldeg}(\tilde w)$. Before we continue, recall that each vertex $\tilde w\in\tilde W_i$ is equipped with a set $X_{\tilde w}\subseteq V(\Gamma)\setminus W$ and that we defined an external degree $\mathrm{edeg}(\tilde w)=|X_{\tilde w}|$ of $\tilde w$ as well as a candidate set $C(\tilde w)=N^{\cap}_{W_i}(X_{\tilde w})\subseteq W_i$. In the course of our embedding procedure, that we will describe below, we shall shrink this candidate set but keep certain invariants as we explain next.

We proceed inductively and embed the vertex class $\tilde W_i$ into W_i one at a time, for $i=1,\ldots,t$. To this end, we verify the following statement $(\mathcal{S}_i)$ for $i=0,\ldots,t$.

($\mathcal{S}_i$) There exists a partial embedding φ_i of $H[\bigcup_{j=1}^i\tilde W_j]$ into $G[\bigcup_{j=1}^i W_j]$ such that for every $\tilde z\in\bigcup_{j=i+1}^t\tilde W_j$ there exists a candidate set $C_i(\tilde z)\subseteq C(\tilde z)$ given by
   (a) $C_i(\tilde z)=\bigcap\{N_G(\varphi_i(\tilde x))\colon\ \tilde x\in N_H(\tilde z)\text{ and }h(\tilde x)\leq i\}\cap C(\tilde z)$,

   and satisfying
   (b) $|C_i(\tilde z)|\geq(dp/2)^{\mathrm{ldeg}^i(\tilde z)}|C(\tilde z)|$, and
   (c) for every edge $\{\tilde z,\tilde z'\}\in E(H)$ with $h(\tilde z),h(\tilde z')>i$ the pair $\big(C_i(\tilde z),C_i(\tilde z')\big)$ is (ε_i,d,p)-dense in G.

Statement $(\mathcal{S}_i)$ ensures the existence of a partial embedding of the first i classes $\tilde W_1,\ldots,\tilde W_i$ of H into G such that for every unembedded vertex $\tilde z$ there exists a candidate set $C_i(\tilde z)\subseteq C(\tilde z)$ that is not too small (see part (b)). Moreover, if we embed $\tilde z$ into its candidate set, then its image will be adjacent to all vertices $\varphi_i(\tilde x)$ with $\tilde x\in(\tilde W_1\cup\cdots\cup\tilde W_i)\cap N_H(\tilde z)$ (see part (a)). The last property, part (c), says that for edges of H such that none of the endvertices are embedded already the respective candidate sets induce (ε,d',p)-dense pairs for some positive d'. This property will be crucial for the inductive proof.

Remark. In what follows we shall use the following convention. Since the embedding of H into G will be divided into t rounds, we shall find it convenient to distinguish among the vertices of H. We shall use $\tilde{x}$ for vertices that have already been embedded, $\tilde{y}$ for vertices that will be embedded in the current round, while $\tilde{z}$ will denote vertices that we shall embed at a later step.

Before we verify $(\mathcal{S}_i)$ for $i = 0, \ldots, t$ by induction on i we note that $(\mathcal{S}_t)$ implies that H can be embedded into G in such a way that every vertex $\tilde{w} \in \tilde{W}$ is mapped to a vertex in its candidate set $C(\tilde{w})$. Consequently, verifying $(\mathcal{S}_t)$ concludes the proof of Lemma 9.5.

Basis of the induction: $i = 0$. We first verify $(\mathcal{S}_0)$. In this case φ_0 is the empty mapping and for every $\tilde{z} \in \tilde{W}$ we have, according to (a), $C_0(\tilde{z}) = C(\tilde{z})$, as there is no vertex $\tilde{x} \in N_H(\tilde{z})$ with $h(\tilde{x}) \leq 0$. Property (b) holds because $C_0(\tilde{z}) = C(\tilde{z})$ and $\mathrm{ldeg}^0(\tilde{z}) = 0$ for every $\tilde{z} \in \tilde{W}$. Finally, property (c) follows from the property that $(C(\tilde{z}), C(\tilde{z}'))$ is (ε_0, d, p)-dense by (E) of Lemma 9.5.

Induction step: $i \to i+1$. For the inductive step, we suppose that $i < t$ and assume that statement $(\mathcal{S}_i)$ holds; we have to construct φ_{i+1} with the required properties. Our strategy is as follows. In the first step, we find for every $\tilde{y} \in \tilde{W}_{i+1}$ an appropriate subset $C'(\tilde{y}) \subseteq C_i(\tilde{y})$ of its candidate set such that if $\varphi_{i+1}(\tilde{y})$ is chosen from $C'(\tilde{y})$, then the new candidate set $C_{i+1}(\tilde{z}) := C_i(\tilde{z}) \cap N_G(\varphi_{i+1}(\tilde{y}))$ of every "right-neighbour" $\tilde{z}$ of $\tilde{y}$ will not shrink too much and property (c) will continue to hold.

Note, however, that in general $|C'(\tilde{y})| \leq |C_i(\tilde{y})| = o(n) \ll |\tilde{W}_{i+1}|$ (if $\mathrm{ldeg}^i(\tilde{y}) \geq 1$) and, hence, we cannot "blindly" select $\varphi_{i+1}(\tilde{y})$ from $C'(\tilde{y})$. Instead, in the second step, we shall verify Hall's condition to find a system of distinct representatives for the indexed set system $\big(C'(\tilde{y})\colon\ \tilde{y} \in \tilde{W}_{i+1}\big)$ and we let $\varphi_{i+1}(\tilde{y})$ be the representative of $C'(\tilde{y})$. (A similar idea was used in [11, 85].) We now give the details of those two steps.

First step: For the first step, fix $\tilde{y} \in \tilde{W}_{i+1}$ and set

$$N_H^{i+1}(\tilde{y}) := \{\tilde{z} \in N_H(\tilde{y})\colon\ h(\tilde{z}) > i+1\}\,.$$

A vertex $v \in C_i(\tilde{y})$ will be "bad" (i.e., we shall not select v for $C'(\tilde{y})$) if there exists a vertex $\tilde{z} \in N_H^{i+1}(\tilde{y})$ for which $N_G(v) \cap C_i(\tilde{z})$, in view of (b) and (c) of $(\mathcal{S}_{i+1})$, cannot play the rôle of $C_{i+1}(\tilde{z})$.

We first prepare for (b) of $(\mathcal{S}_{i+1})$. Fix a vertex $\tilde{z} \in N_H^{i+1}(\tilde{y})$. Since $(C_i(\tilde{y}), C_i(\tilde{z}))$ is an (ε_i, d, p)-dense pair by (c) of $(\mathcal{S}_i)$, Proposition 3.17 implies that there exist at most $\varepsilon_i |C_i(\tilde{y})| \leq \varepsilon_t |C_i(\tilde{y})|$ vertices v in $C_i(\tilde{y})$ such that

$$|N_G(v) \cap C_i(\tilde{z})| < \big(d - \varepsilon_t\big) p |C_i(\tilde{z})|\,.$$

Repeating the above for all $\tilde{z} \in N_H^{i+1}(\tilde{y})$, we infer from (a) and (b) of $(\mathcal{S}_i)$, that there are at most $\Delta \varepsilon_t |C_i(\tilde{y})|$ vertices $v \in C_i(\tilde{y})$ such that the following fails to be true for some $\tilde{z} \in N_H^{i+1}(\tilde{y})$:

$$\begin{aligned} |N_G(v) \cap C_i(\tilde{z})| &\geq \big(d - \varepsilon_t\big) p |C_i(\tilde{z})| \\ &\overset{(a),(b)}{\geq} (d - \varepsilon_t)\, p \left(\frac{dp}{2}\right)^{\mathrm{ldeg}^i(\tilde{z})} |C(\tilde{z})| \overset{(9.45)}{\geq} \left(\frac{dp}{2}\right)^{\mathrm{ldeg}^{i+1}(\tilde{z})} |C(\tilde{z})|\,. \end{aligned} \tag{9.47}$$

For property (c) of $(\mathcal{S}_{i+1})$, we fix an edge $e = \{\tilde{z}, \tilde{z}'\}$ with $h(\tilde{z})$, $h(\tilde{z}') > i+1$ and with at least one end vertex in $N_H^{i+1}(\tilde{y})$. There are at most $\Delta(\Delta-1) < \Delta^2$ such edges. Note that if both vertices $\tilde{z}$ and $\tilde{z}'$ are neighbours of $\tilde{y}$, i.e., $\tilde{z}$, $\tilde{z}' \in N_H^{i+1}(\tilde{y})$, then

$$\max\left\{ \operatorname{ldeg}^i(\tilde{y}) + \operatorname{edeg}(\tilde{y}), \operatorname{ldeg}^i(\tilde{z}) + \operatorname{edeg}(\tilde{z}), \operatorname{ldeg}^i(\tilde{z}') + \operatorname{edeg}(\tilde{z}')\right\} \le \Delta - 2\,,$$

by (C) of Lemma 9.5 and because all three vertices $\tilde{y}$, $\tilde{z}$, and $\tilde{z}'$ have at least two neighbours in $\tilde{W}_{i+1} \cup \cdots \cup \tilde{W}_t$. From property (b) of $(\mathcal{S}_i)$, and (A) and (D) of Lemma 9.5 we infer for all $\tilde{w} \in \{\tilde{y}, \tilde{z}, \tilde{z}'\}$ that

$$|C_i(\tilde{w})| \overset{\text{(b)}}{\ge} \left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\tilde{w})} |C(\tilde{w})| \overset{\text{(A),(D)}}{\ge} \left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\tilde{w})+\operatorname{edeg}(\tilde{w})} \frac{n}{r} \overset{(9.46)}{\ge} \xi_{9.12} p^{\Delta-2} n.$$

Furthermore, Γ has property $\mathrm{D}_p(d, \varepsilon_{i+1}, \mu, \varepsilon_i, \xi_{9.12})$ by assumption. This implies that there are at most $\mu|C_i(\tilde{y})|$ vertices $v \in C_i(\tilde{y})$ such that the pair $(N_G(v) \cap C_i(\tilde{z}), N_G(v) \cap C_i(\tilde{z}'))$ fails to be $(\varepsilon_{i+1}, d, p)$-dense.

If, on the other hand, say, only $\tilde{z} \in N_H^{i+1}(\tilde{y})$ and $\tilde{z}' \notin N_H^{i+1}(y)$, then

$$\begin{aligned} \max\left\{ \operatorname{ldeg}^i(\tilde{y}) + \operatorname{edeg}(\tilde{y}), \operatorname{ldeg}^i(\tilde{z}') + \operatorname{edeg}(\tilde{z}')\right\} &\le \Delta - 1 \\ \text{and} \quad \operatorname{ldeg}^i(\tilde{z}) + \operatorname{edeg}(\tilde{z}) &\le \Delta - 2. \end{aligned}$$

Consequently, similarly as above,

$$\min\left\{|C_i(\tilde{y})|\,, |C_i(\tilde{z}')|\right\} \ge \xi_{9.12} p^{\Delta-1} n \quad \text{and} \quad |C_i(\tilde{z})| \ge \xi_{9.12} p^{\Delta-2} n$$

and we can appeal to the fact that Γ has property $\mathrm{D}_p(d, \varepsilon_{i+1}, \mu, \varepsilon_i, \xi_{9.12})$ to infer that there are at most $\mu|C_i(\tilde{y})|$ vertices $v \in C_i(\tilde{y})$ such that $(N_G(v) \cap C_i(\tilde{z}), C_i(\tilde{z}'))$ fails to be $(\varepsilon_{i+1}, d, p)$-dense. For a given $v \in C_i(\tilde{y})$, let $\hat{C}_i(\tilde{z}) = C_i(\tilde{z}) \cap N_G(v)$ if $\tilde{z} \in N_H^{i+1}(\tilde{y})$ and $\hat{C}_i(\tilde{z}) = C_i(\tilde{z})$ if $\tilde{z} \notin N_H^{i+1}(\tilde{y})$, and define $\hat{C}_i(\tilde{z}')$ analogously.

Summarizing the above we infer that there are at least

$$(1 - \Delta\varepsilon_t - \Delta^2\mu)|C_i(\tilde{y})| \tag{9.48}$$

vertices $v \in C_i(\tilde{y})$ such that

(b') $|N_G(v) \cap C_i(\tilde{z})| \ge (dp/2)^{\operatorname{ldeg}^{i+1}(\tilde{z})}|C(\tilde{z})|$ for every $\tilde{z} \in N_H^{i+1}(\tilde{y})$ (see (9.47)) and

(c') $(\hat{C}_i(\tilde{z}), \hat{C}_i(\tilde{z}'))$ is $(\varepsilon_{i+1}, d, p)$-dense for all edges $\{\tilde{z}, \tilde{z}'\}$ of H with $h(\tilde{z})$, $h(\tilde{z}') > i+1$ and $\{\tilde{z}, \tilde{z}'\} \cap N_H^{i+1}(\tilde{y}) \ne \emptyset$.

Let $C'(\tilde{y})$ be the set of those vertices v from $C_i(\tilde{y})$ satisfying properties (b') and (c') above. Recall that $\operatorname{ldeg}^i(\tilde{y}) + \operatorname{edeg}(\tilde{y}) = \operatorname{ldeg}^i(\tilde{y}') + \operatorname{edeg}(\tilde{y}')$ for all $\tilde{y}$, $\tilde{y}' \in \tilde{W}_{i+1}$ and set

$$k := \operatorname{ldeg}^i(\tilde{y}) + \operatorname{edeg}(\tilde{y}) \text{ for some } \tilde{y} \in \tilde{W}_{i+1}. \tag{9.49}$$

Since $\tilde{y} \in \tilde{W}_{i+1}$ was arbitrary, we infer from property (b) of $(\mathcal{S}_i)$, properties (A) and (D) of Lemma 9.5, and the choices of μ and ε_t that

$$\begin{aligned} |C'(\tilde{y})| &\overset{(9.48)}{\ge} (1 - \Delta\varepsilon_t - \Delta^2\mu)|C_i(\tilde{y})| \overset{\text{(b)}}{\ge} (1 - \Delta\varepsilon_t - \Delta^2\mu)\left(\frac{dp}{2}\right)^{\operatorname{ldeg}^i(\tilde{y})} |C(\tilde{z})| \\ &\overset{\text{(A),(D)}}{\ge} (1 - \Delta\varepsilon_t - \Delta^2\mu)\left(\frac{dp}{2}\right)^k \frac{n}{r} \overset{(9.44),(9.45)}{\ge} \left(\frac{dp}{10}\right)^k \frac{n}{r}\,. \end{aligned} \tag{9.50}$$

Second step: We now turn to the aforementioned second part of the inductive step. Here we ensure the existence of a system of distinct representatives for the indexed set system

$$\mathcal{C}_{i+1} := \left(C'(\tilde{y}) \colon \; \tilde{y} \in \tilde{W}_{i+1}\right).$$

We shall appeal to *Hall's condition* and show that for every subfamily $\mathcal{C}' \subseteq \mathcal{C}_{i+1}$ we have

$$|\mathcal{C}'| \leq \left| \bigcup_{C' \in \mathcal{C}'} C' \right|. \tag{9.51}$$

Because of (9.50), assertion (9.51) holds for all families $\mathcal{C}'$ with $1 \leq |\mathcal{C}'| \leq (dp/10)^k n/r$.

Thus, consider a family $\mathcal{C}' \subseteq \mathcal{C}_i$ with $|\mathcal{C}'| > (dp/10)^k n/r$. For every $\tilde{y} \in \tilde{W}_{i+1}$ we have a set $\tilde{K}(\tilde{y})$ of $\mathrm{ldeg}^i(\tilde{y})$ already embedded vertices of H such that $\tilde{K}(\tilde{y}) = N_H(\tilde{y}) \setminus N_H^{i+1}(\tilde{y})$. Let $K'(\tilde{y}) := \varphi_i(\tilde{K}(\tilde{y}))$ be the image of $\tilde{K}(\tilde{y})$ in G under φ_i. Recall that $\tilde{y}$ is equipped with a set $X_{\tilde{y}} \subseteq V(\Gamma) \setminus W$ of size $\mathrm{edeg}(\tilde{y})$ in Lemma 9.5. We have $\mathrm{ldeg}^i(\tilde{y}) + \mathrm{edeg}(\tilde{y}) = k$ by (9.49). Hence, when we add the vertices of $X_{\tilde{y}}$ to $K'(\tilde{y})$ we obtain a set $K(\tilde{y}) = \{u_1(\tilde{y}), \dots, u_k(\tilde{y})\}$ of k vertices in Γ. Note that for two distinct vertices $\tilde{y}$, $\tilde{y}' \in \tilde{W}_{i+1}$ the sets $\tilde{K}(\tilde{y})$ and $\tilde{K}(\tilde{y}')$ are disjoint. This follows from the fact that the distance in H between $\tilde{y}$ and $\tilde{y}'$ is at least four by the 3-independence of $\tilde{W}_{i+1}$ (cf. (B) of Lemma 9.5) and if $\tilde{K}(\tilde{y}) \cap \tilde{K}(\tilde{y}') \neq \emptyset$, then this distance would be at most two. In addition $\left(X_{\tilde{y}} : \tilde{y} \in \tilde{W}_{i+1}\right)$ consists of pairwise disjoint sets by hypothesis. Consequently, the sets $K(\tilde{y})$ and $K(\tilde{y}')$ are disjoint as well and, therefore,

$$\mathcal{F} := \{K(\tilde{y}) \colon \; C'(\tilde{y}) \in \mathcal{C}'\} \subseteq \{K(\tilde{y}) \colon \; \tilde{y} \in \tilde{W}_{i+1}\} \subseteq \binom{V(\Gamma)}{k}$$

is a family of $|\mathcal{C}'|$ pairwise disjoint k-sets in $V(\Gamma)$. Moreover, $C(\tilde{y}) = N^{\cap}_{W_i}(X_{\tilde{y}})$ by definition and so (a) of $(\mathcal{S}_i)$ implies

$$C'(\tilde{y}) \subseteq C(\tilde{y}) \cap \bigcap_{v \in K'(\tilde{y})} N_\Gamma(v) = \bigcap_{v \in K(\tilde{y})} N_\Gamma(v).$$

Let

$$U = \bigcup_{C'(\tilde{y}) \in \mathcal{C}'} C'(\tilde{y}) \subseteq W_{i+1},$$

and suppose for a contradiction that

$$|U| < |\mathcal{C}'| = |\mathcal{F}|. \tag{9.52}$$

We now use the fact that Γ has property $\mathrm{STAR}_p(k, \xi_{9.8}, \nu)$ and apply it to U and $\mathcal{F}$ (see Lemma 9.8). By assumption $|U| < |\mathcal{F}| \leq \nu n p^k |\mathcal{F}|$. We deduce that

$$\#\,\mathrm{stars}^\Gamma(U, \mathcal{F}) \leq p^k |U||\mathcal{F}| + 6\xi_{9.8} n p^k |\mathcal{F}|.$$

On the other hand, because of (9.50), we have

$$\#\,\mathrm{stars}^\Gamma(U, \mathcal{F}) \geq \left(\frac{dp}{10}\right)^k \frac{n}{r} |\mathcal{F}|.$$

Combining the last two inequalities we infer from property (A) of Lemma 9.5 that

$$|U| \geq \left(\left(\frac{d}{10}\right)^k \frac{1}{r} - 6\xi_{9.8}\right) n \overset{(9.46)}{\geq} \xi_{9.8} n \overset{(9.46)}{=} \xi \frac{n}{r} \overset{(A)}{\geq} |\tilde{W}_{i+1}| \geq |\mathcal{C}'|,$$

which contradicts (9.52). This contradiction shows that (9.52) does not hold, that is, Hall's condition (9.51) does hold. Hence, there exists a system of representatives for $\mathcal{C}_{i+1}$, i.e., an injective mapping $\psi\colon \tilde{W}_{i+1} \to \bigcup_{\tilde{y} \in \tilde{W}_{i+1}} C'(\tilde{y})$ such that $\psi(\tilde{y}) \in C'(\tilde{y})$ for every $\tilde{y} \in \tilde{W}_{i+1}$.

Finally, we extend φ_i. For that we set

$$\varphi_{i+1}(\tilde{w}) = \begin{cases} \varphi_i(\tilde{w})\,, & \text{if } \tilde{w} \in \bigcup_{j=1}^{i} \tilde{W}_j, \\ \psi(\tilde{w})\,, & \text{if } \tilde{w} \in \tilde{W}_{i+1}\,. \end{cases}$$

Note that every $\tilde{z} \in \bigcup_{j=i+2}^{t} \tilde{W}_j$ has at most one neighbour in $\tilde{W}_{i+1}$, as otherwise there would be two vertices $\tilde{y}$ and $\tilde{y}' \in \tilde{W}_{i+1}$ with distance at most 2 in H, which contradicts property (B) of Lemma 9.5. Consequently, for every $\tilde{z} \in \bigcup_{j=i+2}^{t} \tilde{W}_j$ we have

$$C_{i+1}(\tilde{z}) = \begin{cases} C_i(\tilde{z})\,, & \text{if } N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \emptyset, \\ C_i(\tilde{z}) \cap N_G(\varphi_{i+1}(\tilde{y}))\,, & \text{if } N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \{\tilde{y}\}. \end{cases}$$

by (a) of $(\mathcal{S}_{i+1})$. In what follows we show that φ_{i+1} and $C_{i+1}(\tilde{z})$ for every $\tilde{z} \in \bigcup_{j=i+2}^{t} \tilde{W}_j$ have the desired properties and validate $(\mathcal{S}_{i+1})$.

First of all, from (a) of $(\mathcal{S}_i)$, combined with $\varphi_{i+1}(\tilde{y}) \in C'(\tilde{y}) \subseteq C_i(\tilde{y})$ for every $\tilde{y} \in \tilde{W}_{i+1}$ and the property that $\big(\varphi_{i+1}(\tilde{y})\colon\ \tilde{y} \in \tilde{W}_{i+1}\big)$ is a system of distinct representatives, we infer that φ_{i+1} is indeed a partial embedding of $H[\bigcup_{j=1}^{i+1} W_j]$.

Next we shall verify property (b) of $(\mathcal{S}_{i+1})$. So let $\tilde{z} \in \bigcup_{j=i+2}^{t} \tilde{W}_j$ be fixed. If $N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \emptyset$, then $C_{i+1}(\tilde{z}) = C_i(\tilde{z})$, $\mathrm{ldeg}^{i+1}(\tilde{z}) = \mathrm{ldeg}^{i}(\tilde{z})$, which yields (b) of $(\mathcal{S}_{i+1})$ for that case. If, on the other hand, $N_H(\tilde{z}) \cap \tilde{W}_{i+1} \neq \emptyset$, then there exists a unique neighbour $\tilde{y} \in \tilde{W}_{i+1}$ of H (owing to the 3-independence of W_{i+1} by property (B) of Lemma 9.5). As discussed above we have $C_{i+1}(\tilde{z}) = C_i(\tilde{z}) \cap N_G(\varphi_{i+1}(\tilde{y}))$ in this case. Since $\varphi_{i+1}(\tilde{y}) \in C'(\tilde{y})$, we infer directly from (b') that (b) of $(\mathcal{S}_{i+1})$ is satisfied.

Finally, we verify property (c) of $(\mathcal{S}_{i+1})$. Let $\{\tilde{z}, \tilde{z}'\}$ be an edge of H with $\tilde{z}, \tilde{z}' \in \bigcup_{j=i+2}^{t} \tilde{W}_j$. We consider three cases, depending on the size of $N_H(\tilde{z}) \cap \tilde{W}_{i+1}$ and of $N_H(\tilde{z}') \cap \tilde{W}_{i+1}$. If $N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \emptyset$ and $N_H(\tilde{z}') \cap \tilde{W}_{i+1} = \emptyset$, then part (c) of $(\mathcal{S}_{i+1})$ follows directly from part (c) of $(\mathcal{S}_i)$ and $\varepsilon_{i+1} \geq \varepsilon_i$, combined with $C_{i+1}(\tilde{z}) = C_i(\tilde{z})$, $C_{i+1}(\tilde{z}') = C_i(\tilde{z}')$. If $N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \{\tilde{y}\}$ and $N_H(\tilde{z}') \cap \tilde{W}_{i+1} = \emptyset$, then (c) of $(\mathcal{S}_{i+1})$ follows from (c') and the definition of $C_{i+1}(\tilde{z})$ and $C_{i+1}(\tilde{z}')$. If $N_H(\tilde{z}) \cap \tilde{W}_{i+1} = \{\tilde{y}\}$ and $N_H(\tilde{z}') \cap \tilde{W}_{i+1} = \{\tilde{y}'\}$, then $\tilde{y} = \tilde{y}'$, as otherwise there would be a $\tilde{y}$-$\tilde{y}'$-path in H with three edges, contradicting the 3-independence of $\tilde{W}_{i+1}$. Consequently, (c) of $(\mathcal{S}_{i+1})$ follows from (c') and the definition of $C_{i+1}(\tilde{z})$ and $C_{i+1}(\tilde{z}')$.

We have therefore verified (a)–(c) of $(\mathcal{S}_i)$, thus concluding the induction step. The proof of Lemma 9.5 follows by induction. □

9.9 Proofs of auxiliary lemmas

In this section we provide all proofs that were postponed earlier, namely those of Lemma 9.8, Lemma 9.10, and Lemma 9.13.

Proof of Lemma 9.8

This proof makes use of a Chernoff bound for the binomially distributed random variable $\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F})$ appearing in this lemma (cf. Definition 9.6 and the discussion below this definition).

Proof of Lemma 9.8. Given Δ and ξ let ν and c be constants satisfying

$$\begin{aligned} -6\xi\log(2\xi) &\le -(6\xi-2\sqrt{\nu})\log\xi, \qquad & 2\nu &\le (\sqrt{\nu}-2\nu), \\ \Delta+1-6\xi c^{\Delta} &\le -1, \qquad \text{and} & \Delta &\le \nu c^{\Delta}. \end{aligned} \tag{9.53}$$

First we estimate the probability that there are X and $\mathcal{F}$ with $|\mathcal{F}|\ge n/\log n$ fulfilling the requirements of the lemma but violating (9.2). Chernoff's inequality $\mathbb{P}[Y\ge\mathbb{E}\,Y+t]\le\exp(-t)$ for a binomially distributed random variable Y and $t\ge 6\,\mathbb{E}\,Y$ (see (2.3)) implies

$$\mathbb{P}\Big[\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F})\ge p^{\Delta}|X||\mathcal{F}|+6\xi np^{\Delta}|\mathcal{F}|\Big]\le\exp(-6\xi np^{\Delta}|\mathcal{F}|)\le\exp(-6\xi c^{\Delta}|\mathcal{F}|\log n)$$

for fixed X and $\mathcal{F}$ since $6\xi np^{\Delta}|\mathcal{F}|\ge 6p^{\Delta}|X||\mathcal{F}|$. As the number of choices for $\mathcal{F}$ and X can be bounded by $\sum_{f=n/\log n}^{\xi n} n^{\Delta f}$ and $2^n\le\exp(n)$, respectively, the probability we want to estimate is at most

$$\sum_{f=\frac{n}{\log n}}^{\xi n}\exp\Big(\Delta f\log n+n-6\xi c^{\Delta}f\log n\Big)\le\sum_{f=\frac{n}{\log n}}^{\xi n}\exp\big(f\log n(\Delta+1-6\xi c^{\Delta})\big),$$

which does not exceed $\xi n\exp(-n)$ by (9.53) and thus tends to 0 as n tends to infinity.

It remains to establish a similar bound on the probability that there are X and $\mathcal{F}$ with $|\mathcal{F}|<n/\log n$ fulfilling the requirements of the lemma but violating (9.2). For this purpose we use that

$$\mathbb{P}[Y\ge t]\le q^t\binom{m}{t}\le\exp\Big(-t\log\frac{t}{3qm}\Big)$$

for a random variable Y with distribution $\mathrm{Bi}(m,q)$ and infer for fixed X and $\mathcal{F}$

$$\begin{aligned}\mathbb{P}\Big[\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F})\ge p^{\Delta}|X||\mathcal{F}|+6\xi np^{\Delta}|\mathcal{F}|\Big]&\le\mathbb{P}\Big[\#\,\mathrm{stars}^{\Gamma}(X,\mathcal{F})\ge 6\xi np^{\Delta}|\mathcal{F}|\Big]\\ &\le\exp\left(-6\xi np^{\Delta}|\mathcal{F}|\log\frac{2\xi n}{|X|}\right)\le\exp\left(-2\sqrt{\nu}np^{\Delta}|\mathcal{F}|\log\frac{n}{|X|}\right).\end{aligned}$$

because $-6\xi\log(2\xi)\le-(6\xi-2\sqrt{\nu})\log\xi\le(6\xi-2\sqrt{\nu})\log(n/|X|)$ by (9.53). The number of choices for $\mathcal{F}$ and X in total can be bounded by

$$\begin{aligned}\sum_{f=1}^{\frac{n}{\log n}}\sum_{x=1}^{\nu np^{\Delta}f}n^{\Delta f}\binom{n}{x}&\le\sum_{f=1}^{\frac{n}{\log n}}\sum_{x=1}^{\nu np^{\Delta}f}\exp\Big(\Delta f\log n+\nu np^{\Delta}f\log\frac{en}{x}\Big)\\ &\le\sum_{f=1}^{\frac{n}{\log n}}\sum_{x=1}^{\nu np^{\Delta}f}\exp\Big(2\nu np^{\Delta}f\log\frac{en}{x}\Big)\le\sum_{f=1}^{\frac{n}{\log n}}\sum_{x=1}^{\nu np^{\Delta}f}\exp\Big(\sqrt{\nu}np^{\Delta}f\log\frac{n}{x}\Big)\end{aligned}$$

where the second inequality follows from $\Delta \log n \leq \nu c^{\Delta} \log n \leq \nu n p^{\Delta}$ and the last from $2\nu \log e \leq (\sqrt{\nu} - 2\nu) \log(n/x)$ by (9.53). Therefore the probability under consideration is at most

$$\sum_{f=1}^{\frac{n}{\log n}} \sum_{x=1}^{\nu p^{\Delta} n f} \exp\left(\sqrt{\nu} n p^{\Delta} f \log \frac{n}{x} - 2\sqrt{\nu} n p^{\Delta} f \log \frac{n}{x}\right) \leq n^2 \exp\left(-\sqrt{\nu} \log n \frac{n}{\log n}\right).$$

□

Proof of Lemma 9.10

We will use the following simple proposition about cuts in hypergraphs. This proposition generalises the well known fact that any graph G admits a vertex partition into sets of roughly equal size such that the resulting cut contains at least half the edges of G.

Proposition 9.29. *Let $\mathcal{G} = (V, \mathcal{E})$ be an ℓ-uniform hypergraph with m edges and n vertices such that $n \geq 3\ell$. Then there is a partition $V = V_1 \dot\cup V_2$ with $|V_1| = \lfloor 2n/3 \rfloor$ and $|V_2| = \lceil n/3 \rceil$ such that at least $m \cdot \ell / 2^{\ell+2}$ edges in $\mathcal{E}$ are 1-crossing, i.e., they have exactly one vertex in V_2.*

Proof. Let X be the number of $\frac{1}{3}$-cuts of V, i.e., cuts $V = V_1 \dot\cup V_2$ with $|V_1| = \lfloor 2n/3 \rfloor$ and $|V_2| = \lceil n/3 \rceil$. For a fixed edge B there are precisely 2^{ℓ} ways to distribute its vertices over $V_1 \dot\cup V_2$ out of which exactly ℓ are such that B is 1-crossing. Further, for r fixed vertices of B exactly $\binom{n-\ell}{\lceil n/3 \rceil - r}$ of all $\frac{1}{3}$-cuts have exactly these vertices in V_2. It is easy to check that

$$\binom{n-\ell}{\lceil n/3 \rceil - r} \leq 4 \binom{n-\ell}{\lceil n/3 \rceil - 1} \qquad \text{for all } 0 \leq r \leq \ell\,.$$

It follows that B is 1-crossing for at least an $\frac{1}{4}\ell/(2^{\ell})$ fraction of all $\frac{1}{3}$-cuts. Now assume that all $\frac{1}{3}$-cuts have less than $m \cdot \ell/2^{\ell+2}$ edges that are 1-crossing. Then double counting gives

$$m \cdot \frac{\ell}{2^{\ell+2}} \cdot X > \sum_{B \in \mathcal{E}} \#\{\, \tfrac{1}{3}\text{-cuts s.t. } B \text{ is 1-crossing} \,\} \geq m \cdot \frac{1}{4} \frac{\ell}{2^{\ell}} \cdot X$$

which is a contradiction. □

In the proof of Lemma 9.10 we need to estimate the number of “bad” ℓ-sets in a vertex set X. For this purpose we will use Proposition 9.29 to obtain a partition of X into sets $X = X_1 \dot\cup X_2$ such that a substantial proportion of all these bad ℓ-sets will be 1-crossing and X_1 is not too small. In this way we obtain many $(\ell - 1)$-sets in X_1 most of which will, as we show, be similarly bad as the ℓ-sets we started with. This will allow us to prove Lemma 9.10 by induction.

Proof of Lemma 9.10. Let Δ and d be given. Let Γ be an n-vertex graph, let ℓ be an integer, let ε', μ, ε, ξ be positive real numbers, and let $p = p(n)$ be a function. We say that Γ has property $P_{\ell}(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if Γ has the property stated in Lemma 9.10 with parameters ε', μ, ε, ξ, $p(n)$ and with parameters and Δ and d. Similarly, Γ has property $D(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if it satisfies the conclusion of Lemma 9.12 with these parameters and with Δ and $d_0 := d$. For any fixed $\ell > 0$, we denote by $(\mathcal{P}_{\ell})$ the following statement.

($\mathcal{P}_\ell$) *For all $\varepsilon', \mu > 0$ there is ε such that for all $\xi > 0$ there is $c > 1$ such that a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p > c(\frac{\log n}{n})^{1/\Delta}$ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ with probability $1 - o(1)$.*

We prove that ($\mathcal{P}_\ell$) holds for every fixed $\ell > 0$ by induction on ℓ. The case $\ell = 1$ is an easy consequence of Proposition 3.17 which states that in *all* (ε, d, p)-dense pairs most vertices have a large neighbourhood.

For the inductive step assume that ($\mathcal{P}_{\ell-1}$) holds. We will show that this implies ($\mathcal{P}_\ell$). We start by specifying the constants appearing in statement ($\mathcal{P}_\ell$). Let ε' and μ be arbitrary positive constants. Set $\varepsilon'_{\ell-1} := \varepsilon'$ and $\mu_{\ell-1} := \frac{1}{100}\mu\frac{\ell}{2^{\ell+2}}$. Let $\varepsilon_{\ell-1}$ be given by ($\mathcal{P}_{\ell-1}$) for input parameters $\varepsilon'_{\ell-1}$ and $\mu_{\ell-1}$. Set $\varepsilon'_{9.12} := \varepsilon_{\ell-1}$ and let $\varepsilon_{9.12}$ be as promised by Lemma 9.12 with parameters $\varepsilon'_{9.12}$ and $\mu_{9.12} := \frac{1}{2}$. Define $\varepsilon := \mu_{\ell-1}\varepsilon_{9.12}\varepsilon'_{\ell-1}$. Next, let ξ be an arbitrary parameter provided by the adversary in Lemma 9.10 and choose $\xi_{\ell-1} := \xi(d - \varepsilon)$ and $\xi_{9.12} := \mu_{\ell-1}\xi$. Finally, let $c_{\ell-1}$ and $c_{9.12}$ be given by ($\mathcal{P}_{\ell-1}$) and by Lemma 9.12, respectively, for the previously specified parameters together with $\xi_{\ell-1}$ and $\xi_{9.12}$. Set $c := \max\{c_{\ell-1}, c_{9.12}\}$. We will prove that with this choice of ε and c the statement in ($\mathcal{P}_\ell$) holds for the input parameters ε', μ, and ξ.

Let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. By ($\mathcal{P}_{\ell-1}$) and Lemma 9.12, and by the choice of the parameters the graph Γ has properties

$$P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n)) \qquad \text{and} \qquad D(\varepsilon'_{9.12}, \mu_{9.12}, \varepsilon_{9.12}, \xi_{9.12}, p(n))$$

with probability $1-o(1)$ if n is large enough. We will show that a graph Γ with these properties also satisfies $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Let $G = (X\dot\cup Y, E)$ be an arbitrary subgraph of such a Γ where $|X| = n_1$ and $|Y| = n_2$ with $n_1 \geq \xi p^{\Delta-1}n$, $n_2 \geq \xi p^{\Delta-\ell}n$, and (X, Y) is an (ε, d, p)-dense pair.

We would like to show that for $\mathcal{B}_\ell := \mathrm{bad}^{G,\ell}_{\varepsilon',d,p}(X, Y)$ we have $|\mathcal{B}_\ell| \leq \mu n_1^\ell$. Assume for a contradiction that this is not the case. By Proposition 9.29 there is a cut $X = X_1\dot\cup X_2$ with $|X_1| = \lfloor 2n_1/3\rfloor$ and $|X_2| = \lceil n_2/3\rceil$ such that at least $|\mathcal{B}_\ell| \cdot \ell/2^{\ell+2}$ of the ℓ-sets in $\mathcal{B}_\ell$ are 1-crossing, i.e., have exactly one vertex in X_2. By Proposition 3.17 there are less than $\varepsilon|X|$ vertices $x \in X_2$ such that $|N_Y(x)| < (d-\varepsilon)pn_2$. We delete all ℓ-sets from $\mathcal{B}_\ell$ that contain such a vertex or are not 1-crossing for $X = X_1\dot\cup X_2$ and call the resulting set $\mathcal{B}'_\ell$. It follows that

$$|\mathcal{B}'_\ell| \geq |\mathcal{B}_\ell|\tfrac{\ell}{2^{\ell+2}} - \varepsilon|X|n_1^{\ell-1} > \mu n_1^\ell \tfrac{\ell}{2^{\ell+2}} - \varepsilon n_1^\ell \geq 20\mu_{\ell-1}n_1^\ell. \tag{9.54}$$

Now, for each $v \in X_2$ we count the number of ℓ-sets $B \in \mathcal{B}'_\ell$ containing v. We delete all vertices v from X_2 for which this number is less than $|\mathcal{B}'_\ell|/(10n_1)$ and call the resulting set X'. Observe that the definition of $\mathcal{B}'_\ell$ implies that all vertices x in X' satisfy $|N_Y(x)| \geq (d-\varepsilon)pn_2$. Because $\mathcal{B}'_\ell$ contains only 1-crossing ℓ-sets we get

$$|\mathcal{B}'_\ell| \leq |X_2 \setminus X'|\frac{|\mathcal{B}'_\ell|}{10n_1} + |X'|n_1^{\ell-1} \leq \frac{|\mathcal{B}'_\ell|}{10} + |X'|n_1^{\ell-1}$$

and thus

$$|X'| \geq \frac{9}{10n_1^{\ell-1}}|\mathcal{B}'_\ell| \overset{(9.54)}{\geq} 10\mu_{\ell-1}n_1.$$

This together with Proposition 3.16 implies that the pairs (X', Y) and (Y, X_1) are $(\varepsilon_{9.12}, d, p)$-dense. In addition we have $|X'|, |X_1| \geq \mu_{\ell-1}n_1 \geq \mu_{\ell-1}\xi p^{\Delta-1}n = \xi_{9.12}p^{\Delta-1}n$ and $|Y| \geq \xi p^{\Delta-\ell}n \geq \xi_{9.12}p^{\Delta-2}n$. Because Γ has property $D(\varepsilon'_{9.12}, \mu_{9.12}, \varepsilon_{9.12}, \xi_{9.12}, p(n))$ we conclude for the tripartite graph $G[X'\dot\cup Y\dot\cup X_1]$ that there are at least $|X'| - \mu_{9.12}|X'| \geq 1$ vertices x in

X' such that $(N_Y(x), X_1)$ is $(\varepsilon'_{9.12}, d, p)$-dense. Let $x^* \in X'$ be one of these vertices and set $Y' := N_Y(x^*)$. Thus (Y', X_1) is $(\varepsilon'_{9.12}, d, p)$-dense and since X' only contains vertices with a large neighbourhood in Y we have $|Y'| \geq (d-\varepsilon)pn_2$. Furthermore, let $\mathcal{B}'_\ell(x^*)$ be the family of ℓ-sets in $\mathcal{B}'_\ell$ that contain x^*. Then $\mathcal{B}'_\ell(x^*)$ contains ℓ-sets with $\ell-1$ vertices in X_1 and with one vertex, the vertex x^*, in X_2 because $\mathcal{B}'_\ell$ contains only 1-crossing ℓ-sets. By definition of X' and because $x^* \in X'$ we have

$$|\mathcal{B}'_\ell(x^*)| \geq |\mathcal{B}'_\ell|/(10n_1) \overset{(9.54)}{\geq} 2\mu_{\ell-1}n_1^{\ell-1}. \tag{9.55}$$

For $B \in \mathcal{B}'_\ell(x^*)$ let $\Pi_{\ell-1}(B)$ be the projection of B to X_1. This implies that $\Pi_{\ell-1}(B)$ is an $(\ell-1)$-set in X_1. In addition $N_{Y'}(\Pi_{\ell-1}(B)) = N_Y(B)$ by definition of Y' and hence $\Pi_{\ell-1}(B)$ has less than $(d-\varepsilon')^\ell p^\ell n_2$ joint neighbours in Y' because $B \in \mathcal{B}'_\ell(x^*) \subseteq \mathcal{B}_\ell$. Accordingly the family $\mathcal{B}_{\ell-1}$ of all projections $\Pi_{\ell-1}(B)$ with $B \in \mathcal{B}'_\ell(x^*)$ is a family of size $|\mathcal{B}'_\ell(x^*)|$ and contains only $(\ell-1)$-sets B' with

$$|N_{Y'}(B')| \leq (d-\varepsilon')^\ell p^\ell n_2 \leq (d-\varepsilon')^{\ell-1}p^{\ell-1}|Y'| = (d-\varepsilon'_{\ell-1})^{\ell-1}p^{\ell-1}|Y'|.$$

This means $\mathcal{B}_{\ell-1} \subseteq \mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, Y')$. Recall that (X_1, Y') is $(\varepsilon'_{9.12}, d, p)$-dense by the choice of x^*. Because $|X| = n_1 \geq \xi p^{\Delta-1}n$ and

$$|Y'| \geq (d-\varepsilon)pn_2 \geq (d-\varepsilon)p \cdot \xi p^{\Delta-\ell}n = \xi_{\ell-1}p^{\Delta-(\ell-1)}n$$

we can appeal to $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ and conclude that

$$|\mathcal{B}'_\ell(x^*)| = |\mathcal{B}_{\ell-1}| \leq |\,\mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X, Y')| \leq \mu_{\ell-1}n_1^{\ell-1}\,,$$

contradicting (9.55).

Because G was arbitrary this shows that Γ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Thus $(\mathcal{P}_\ell)$ holds, which finishes the proof of the inductive step. □

Proof of Lemma 9.13

In this section we provide the proof of Lemma 9.13 which examines the inheritance of p-density to neighbourhoods of Δ-sets. For this purpose we will first establish a version of this lemma, Lemma 9.30 below, which only considers Δ-sets that are crossing in a given vertex partition.

We need some definitions. Let $G = (V, E)$ be a graph, X be a subset of its vertices, and $X = X_1\dot\cup \ldots \dot\cup X_T$ be a partition of X. Then, for integers $\ell, T > 0$, we say that an ℓ-set $B \subseteq X$ is *crossing* in $X_1\dot\cup \ldots \dot\cup X_T$ if there are indices $0 < i_1 < \cdots < i_\ell < T$ such that B contains exactly one element in X_{i_j} for each $j \in [\ell]$. In this case we also write $B \in X_{i_1} \times \cdots \times X_{i_\ell}$ (hence identifying crossing ℓ-sets with ℓ-tuples).

Now let p, ε, d be positive reals, and Y, $Z \subseteq V$ be vertex sets such that X, Y, and Z are mutually disjoint. Define

$$\mathrm{bad}^{G,\ell}_{\varepsilon,d,p}(X_1, \ldots, X_T; Y, Z)$$

to be the family of all those crossing ℓ-sets B in $X_1\dot\cup \ldots \dot\cup X_T$ that either satisfy $|N^\cap_Y(B)| < (d-\varepsilon)^\ell p^\ell |Y|$ or have the property that $(N^\cap_Y(B), Z)$ is not (ε, d, p)-dense in G. Further, let

$$\mathrm{Bad}^{G,\ell}_{\varepsilon,d,p}(X_1, \ldots, X_T; Y, Z)$$

be the family of crossing ℓ-sets B in $X_1\dot\cup \ldots \dot\cup X_T$ that contain an ℓ'-set $B' \subseteq B$ with $\ell' > 0$ such that $B' \in \mathrm{bad}^{G,\ell'}_{\varepsilon,d,p}(X_1, \ldots, X_T; Y, Z)$.

Lemma 9.30. *For all integers $\ell, \Delta > 0$ and positive reals d_0, ε', and μ there is ε such that for all $\xi > 0$ there is $c > 1$ such that if $p > c(\frac{\log n}{n})^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1, n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-\ell-1} n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. Assume further that $X = X_1 \dot\cup \ldots \dot\cup X_\ell$ with $|X_i| \geq \lfloor \frac{n_1}{\ell} \rfloor$ and that (X, Y) and (Y, Z) are (ε, d, p)-dense pairs with $d \geq d_0$. Then*

$$\left| \operatorname{bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \ldots, X_\ell; Y, Z) \right| \leq \mu n_1^\ell.$$

Proof. Let Δ and d_0 be given. For a fixed n-vertex graph Γ, a fixed integer ℓ and fixed positive reals ε', μ, ε, ξ, and a function $p = p(n)$ we say that we say that a graph Γ on n vertices has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ if Γ has the property stated in the lemma for these parameters and for Δ and d_0, that is, whenever $G = (X \dot\cup Y \dot\cup Z, E)$ is a tripartite subgraph of Γ with the required properties, then G satisfies the conclusion of the lemma. For any fixed $\ell > 0$, we denote by $(\mathcal{P}_\ell)$ the following statement.

($\mathcal{P}_\ell$) *For all $\varepsilon', \mu > 0$ there is ε such that for all $\xi > 0$ there is $c > 1$ such that a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p > c(\frac{\log n}{n})^{1/\Delta}$ has property $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$ with probability $1 - o(1)$.*

We prove that $(\mathcal{P}_\ell)$ holds for every fixed $\ell > 0$ by induction on ℓ.

The case $\ell = 1$ is an easy consequence of Lemma 9.12 and Proposition 3.17. Indeed, let ε' and μ be arbitrary, let $\varepsilon_{9.12}$ be as asserted by Lemma 9.12 for Δ, d_0, ε', and $\mu/2$ and fix $\varepsilon := \min\{\varepsilon_{9.12}, \varepsilon', \mu/2\}$. Let ξ be arbitrary and pass it on to Lemma 9.12 for obtaining c. Now, let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. Then, by the choice of parameters, Lemma 9.12 asserts that the graph Γ has the following property with probability $1 - o(1)$. Let $G = (X \dot\cup Y \dot\cup Z, E)$ be any subgraph with $X = X_1$ and $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$, where $n_1, n_3 \geq \xi p^{\Delta-1} n$ and $n_2 \geq \xi p^{\Delta-2} n$, and (X, Y) and (Y, Z) are (ε, d, p)-dense pairs. Then there are at most $\frac{\mu}{2} n_1$ vertices $x \in X$ such that $(N(x) \cap Y, Z)$ is not an (ε', d, p)-dense pair in G. Because $\varepsilon \leq \mu/2$, Proposition 3.17 asserts that in every such G there are at most $\frac{\mu}{2} n_1$ vertices $x \in X$ with $|N_Y(x)| < (d - \varepsilon')p|Y|$. This implies that

$$\left| \operatorname{bad}^{G,1}_{\varepsilon',d,p}(X_1; Y, Z) \right| \leq \mu n_1$$

holds with probability $1 - o(1)$ for all such subgraphs G of the random graph Γ. Accordingly we get $(\mathcal{P}_1)$.

For the inductive step assume that $(\mathcal{P}_{\ell-1})$ and $(\mathcal{P}_1)$ hold. We will show that this implies $(\mathcal{P}_\ell)$. Again, let ε' and μ be arbitrary positive constants. Let ε_1 be as promised in the statement $(\mathcal{P}_1)$ for parameters $\varepsilon'_1 := \varepsilon'$ and $\mu_1 := \mu/2$. Set $\varepsilon'_{\ell-1} := \min\{\varepsilon_1, \varepsilon', \frac{\mu}{4}\}$, and let $\varepsilon_{\ell-1}$ be given by $(\mathcal{P}_{\ell-1})$ for parameters $\varepsilon'_{\ell-1}$ and $\mu_{\ell-1} := \frac{\mu}{4}$. We define $\varepsilon := \varepsilon_{\ell-1}/(\ell+1)$. Next, let ξ be an arbitrary parameter and choose

$$\xi_1 := \min\{\xi/(\ell+1), (d_0 - \varepsilon'_{\ell-1})^{\ell-1}\xi\} \quad \text{and} \quad \xi_{\ell-1} := \xi/(\ell+1). \tag{9.56}$$

Finally, let c_1 and $c_{\ell-1}$ be given by $(\mathcal{P}_1)$ and $(\mathcal{P}_{\ell-1})$, respectively, for the previously specified parameters together with ξ_1 and $\xi_{\ell-1}$. Set $c := \max\{c_1, c_{\ell-1}\}$. We will prove that with this choice of ε and c the statement in $(\mathcal{P}_\ell)$ holds for the input parameters ε', μ, and ξ. For this purpose let $\Gamma = \mathcal{G}_{n,p}$ be a random graph. By $(\mathcal{P}_1)$ and $(\mathcal{P}_{\ell-1})$ and the choice of the parameters the graph Γ has properties $P_1(\varepsilon'_1, \mu_1, \varepsilon_1, \xi_1, p(n))$ and $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ with probability $1 - o(1)$. We will show that a graph Γ with these properties also satisfies $P_\ell(\varepsilon', \mu, \varepsilon, \xi, p(n))$. Let $G = (X \dot\cup Y \dot\cup Z, E)$ be an arbitrary subgraph of such a Γ where

$X = X_1 \dot\cup \dots \dot\cup X_\ell$, $|X| = n_1$, $|Y| = n_2$, $|Z| = n_3$, with $n_1, n_3 \geq \xi p^{\Delta-1} n$, $n_2 \geq \xi p^{\Delta-\ell-1} n$, and $|X_i| \geq \lfloor \frac{n_1}{\ell} \rfloor$, and assume that (X, Y) and (Y, Z) are (ε, d, p)-dense pairs for $d \geq d_0$.

We would like to bound $\mathcal{B}_\ell := \mathrm{bad}^{G,\ell}_{\varepsilon',d,p}(X_1, \dots, X_\ell; Y, Z)$. For this purpose let B' be a fixed $(\ell - 1)$-set and define

$$\mathcal{B}_{\ell-1} := \mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \dots, X_{\ell-1}; Y, Z) \cup \mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \dots, X_{\ell-1}; Y, X_\ell) \tag{9.57a}$$

$$\mathcal{B}_1(B') := \mathrm{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; N^\cap_Y(B'), Z). \tag{9.57b}$$

For an ℓ-set $B \in X_1 \times \dots \times X_\ell$ let further $\Pi_{\ell-1}(B)$ denote the $(\ell-1)$-set that is the projection of B to $X_1 \times \dots \times X_{\ell-1}$ and let $\Pi_\ell(B)$ be the vertex that is the projection of B to X_ℓ. Now, consider an ℓ-set B that is contained in $B \in \mathcal{B}_\ell$ but is such that $B' := \Pi_{\ell-1}(B) \notin \mathcal{B}_{\ell-1}$. Let $v = \Pi_\ell(B) \in X_\ell$ and $Y' := N^\cap_Y(B')$. We will show that then $v \in \mathcal{B}_1(B')$. Indeed, since $B' \notin \mathcal{B}_{\ell-1}$ it follows from (9.57a) that

$$B' \notin \mathrm{bad}^{G,\ell-1}_{\varepsilon'_{\ell-1},d,p}(X_1, \dots, X_{\ell-1}; Y, Z)$$

and thus $|Y'| \geq (d - \varepsilon'_{\ell-1})^{\ell-1} p^{\ell-1} n_2$. As $N^\cap_Y(B) = N^\cap_{Y'}(v)$ we conclude that

$$v \in \mathrm{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; Y', Z) = \mathcal{B}_1(B')$$

by (9.57b) because otherwise $(N^\cap_Y(B), Z)$ was (ε', d, p)-dense and we had

$$|N^\cap_Y(B)| \geq (d - \varepsilon')p|Y'| \geq (d - \varepsilon')p \cdot (d - \varepsilon'_{\ell-1})^{\ell-1} p^{\ell-1} n_2 \geq (d - \varepsilon')^\ell p^\ell n_2,$$

which contradicts $B \in \mathcal{B}_\ell$. Summarizing, we have

$$\begin{aligned} \mathcal{B}_\ell &= \{B \in \mathcal{B}_\ell : \Pi_{\ell-1}(B) \in \mathcal{B}_{\ell-1}\} \cup \{B \in \mathcal{B}_\ell : \Pi_{\ell-1}(B) \notin \mathcal{B}_{\ell-1}\} \\ &\subseteq (\mathcal{B}_{\ell-1} \times X_\ell) \cup \bigcup_{B' \notin \mathcal{B}_{\ell-1}} \{B'\} \times \mathcal{B}_1(B'). \end{aligned} \tag{9.58}$$

For bounding $\mathcal{B}_\ell$ we will thus estimate the sizes of $\mathcal{B}_{\ell-1}$ and $\mathcal{B}_1(B')$ for $B' \notin \mathcal{B}_{\ell-1}$. Let $X' := X_1 \dot\cup \dots \dot\cup X_{\ell-1}$. Since (X, Y) is (ε, d, p)-dense we conclude from Proposition 3.16 that (X', Y) and (X_ℓ, Y) are $(\varepsilon_{\ell-1}, d, p)$-dense pairs since $\varepsilon(\ell + 1) \leq \varepsilon_{\ell-1}$. Further, by the choice of $\xi_{\ell-1}$ we get $|X'|, |X_\ell| \geq n_1/(\ell + 1) \geq \xi_{\ell-1} p^{\Delta-1} n$ since $n_1 \geq \xi p^{\Delta-1} n$ by assumption. Thus we can use the fact that Γ has property $P_{\ell-1}(\varepsilon'_{\ell-1}, \mu_{\ell-1}, \varepsilon_{\ell-1}, \xi_{\ell-1}, p(n))$ once on the tripartite subgraph induced on $X' \dot\cup Y \dot\cup Z$ in G and once on the tripartite subgraph induced on $X' \dot\cup Y \dot\cup X_\ell$ in G and infer that

$$|\mathcal{B}_{\ell-1}| \leq 2 \cdot \mu_{\ell-1} n_1^{\ell-2} = \frac{\mu}{2} n_1^{\ell-2}. \tag{9.59}$$

For estimating $|\mathcal{B}_1(B')|$ for $B' \notin \mathcal{B}_{\ell-1}$ let $Y' := N^\cap_Y(B')$. Observe that this implies that (Y', Z) and (X_ℓ, Y') are (ε_1, d, p)-dense pairs because $\varepsilon'_{\ell-1} \leq \varepsilon_1$, and that

$$|Y'| \geq (d - \varepsilon'_{\ell-1})^{\ell-1} p^{\ell-1} n_2 \geq (d - \varepsilon'_{\ell-1})^{\ell-1} p^{\ell-1} \cdot \xi p^{\Delta-\ell-1} n \overset{(9.56)}{\geq} \xi_1 p^{\Delta-1} n.$$

By (9.56) $|X_\ell|, |Z| \geq \xi p^{\Delta-1} n/(\ell + 1) \geq \xi_1 p^{\Delta-1} n$. As Γ satisfies $P_1(\varepsilon'_1, \mu_1, \varepsilon_1, \xi_1, p(n))$ we conclude that

$$\left|\mathcal{B}_1(B')\right| \overset{(9.57b)}{=} |\mathrm{bad}^{G,1}_{\varepsilon',d,p}(X_\ell; Y', Z)| \leq \mu_1 n_1 \leq \frac{\mu}{2} n_1. \tag{9.60}$$

In view of (9.58), combining (9.59) and (9.60) gives

$$\left|\mathrm{bad}^{G,\ell}_{\varepsilon',d,p}(X_1,\dots,X_\ell;Y,Z)\right| = |\mathcal{B}_\ell| \le \frac{\mu}{2}n_1^{\ell-1}\cdot n_1 + n_1^{\ell-1}\cdot\frac{\mu}{2}n_1 = \mu n_1^\ell.$$

Because G was arbitrary this shows that Γ has property $P_\ell(\varepsilon',\mu,\varepsilon,\xi,p(n))$. Thus $(\mathcal{P}_\ell)$ holds which finishes the proof of the inductive step. □

In the proof of Lemma 9.13 we now first partition the vertex set X, in which we count bad ℓ-sets, arbitrarily into T vertex sets of equal size. Lemma 9.30 then implies that for all $\ell'\in[\ell]$ there are not many bad ℓ'-sets that are crossing in this partition. It follows that only few ℓ-sets in X contain a bad ℓ'-set for some $\ell'\in[\ell]$ (recall that in Definition 9.9 for $\mathrm{Bad}^{G,\ell}_{\varepsilon,d,p}(X,Y,Z)$ such ℓ'-sets are considered). Moreover, if T is sufficiently large then the number of non-crossing ℓ-sets is negligible. Hence we obtain that there are few bad sets in total.

Proof of Lemma 9.13. Given $\Delta,\ell,d_0,\varepsilon'$ and μ let T be such that $\mu T\ge 2$, fix $\mu_{9.30}:=\frac{1}{2}\mu/(\ell T^\ell)$. For $j\in[\ell]$ let ε_j be given by Lemma 9.30 with ℓ replaced by j and for Δ, d_0, ε', and $\mu_{9.30}$ and set $\varepsilon_{9.30}:=\min_{j\in[\ell]}\varepsilon_j$. Define $\varepsilon:=\varepsilon_{9.30}/(T+1)$. Now, in Lemma 9.13 let ξ be given by the adversary for this ε. Set $\xi_{9.30}:=\xi/(T+1)$, and let c be given by Lemma 9.30 for this $\xi_{9.30}$.

Let $\Gamma=\mathcal{G}_{n,p}$ with $p\ge c(\frac{\log n}{n})^{1/\Delta}$. Then a.a.s. the graph Γ satisfies the statement in Lemma 9.30 for parameters $j\in[\ell]$, Δ, d_0, ε', $\mu_{9.30}$, and $\xi_{9.30}$. Assume that Γ has this property for all $j\in[\ell]$. We will show that it then also satisfies the statement in Lemma 9.13.

Indeed, let G and X, Y, Z be arbitrary with the properties as required in Lemma 9.13. Let $X=X_1\dot\cup\dots\dot\cup X_T$ be an arbitrary partition of X with $|X_i|\ge\lfloor\frac{n_1}{T}\rfloor$. We will first show that there are not many bad crossing ℓ-sets with respect to this partition, i.e., we will bound the size of $\mathrm{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)$. By definition

$$\left|\,\mathrm{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right| \le \sum_{j\in[\ell]}\left|\,\mathrm{bad}^{G,j}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right|\cdot n_1^{\ell-j}.$$

Now, fix $j\in[\ell]$ and an index set $\{i_1,\dots,i_j\}\in\binom{[T]}{j}$ and consider the induced tripartite subgraph $G'=(X'\dot\cup Y\dot\cup Z,E')$ of G with $X'=X_{i_1}\dot\cup\dots\dot\cup X_{i_j}$. Observe that $|Y|\ge\xi_{9.30}p^{\Delta-j-1}n$, $|Z|\ge\xi_{9.30}p^{\Delta-1}n$, and $n_1':=|X'|\ge j\lfloor n_1/T\rfloor\ge\xi_{9.30}p^{\Delta-1}n$. By definition $\varepsilon(T+1)/j\le\varepsilon_{9.30}\le\varepsilon_j$ and so by Proposition 3.16 the pair (X',Y) is (ε_j,d,p)-dense. Thus, because Γ satisfies the statement in Lemma 9.30 for parameters j, Δ, d, ε', $\mu_{9.30}$, and $\xi_{9.30}$ we have that G' satisfies

$$\left|\,\mathrm{bad}^{G',j}_{\varepsilon',d,p}(X_{i_1},\dots,X_{i_j};Y,Z)\right|\le\mu_{9.30}(n_1')^j.$$

As there are $\binom{T}{j}$ choices for the index set $\{i_1,\dots,i_j\}$ this implies

$$\left|\,\mathrm{bad}^{G,j}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right|\le\binom{T}{j}\mu_{9.30}(n_1')^j\le T^j\mu_{9.30}n_1^j,$$

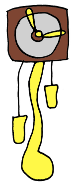
and thus

$$\left|\,\mathrm{Bad}^{G,\ell}_{\varepsilon',d,p}(X_1,\dots,X_T;Y,Z)\right|\le\sum_{j\in[\ell]}T^j\mu_{9.30}n_1^j\cdot n_1^{\ell-j}\le\frac{1}{2}\mu n_1^\ell.$$

The number of ℓ-sets in X that are not crossing with respect to the partition $X=X_1\dot\cup\dots\dot\cup X_T$ is at most $T\binom{n_1/T}{2}\binom{n_1}{\ell-2}\le\frac{1}{T}n_1^\ell\le\frac{1}{2}\mu n_1^\ell$ and so we get $|\,\mathrm{Bad}^{G,\ell}_{\varepsilon',d,p}(X,Y,Z)|\le\mu n_1^\ell$. □

Chapter 10

Concluding remarks



In this thesis we established a variety of results concerning spanning or almost spanning subgraphs (Chapter 8 is an exception) that are forced by different degree conditions.

In the introductory paragraphs on extremal graph theory in the introduction (Section 1.1.1), we also discussed classical results on the appearance of constant size subgraph in dense graphs. This raises the question, what happens in between. For example, one may ask which minimum degree ensures a copy of a graph H in G where H covers, say, 42% of G.

The Theorem of Corrádi and Hajnal (in a more general form than Theorem 1.3) answers this question for triangle factors. This result asserts that every graph G with minimum degree $\delta(G) \geq 2k$ contains k vertex disjoint cycles. In particular this implies that a triangle factor on $3(2\delta - n)$ vertices is forced if G has minimum degree δ.

Theorem 10.1 (Corrádi, Hajnal [26]). *Let G be a graph on n vertices with minimum degree $\delta(G) = \delta \in [\frac{1}{2}n, \frac{2}{3}n]$. Then G contains $2\delta - n$ vertex disjoint triangles.* □

At the other end of the spectrum we know from Theorem 1.6 that a spanning triangle factor appears at (roughly) the same minimum degree threshold as a spanning square of a path.

While the theorem of Corrádi and Hajnal determines the number t of disjoint triangles that are forced in a graph with minimum degree δn we (in joint work with Allen and Hladký [5]) recently established a corresponding result mediating between Turán's theorem and Theorem 1.6 (or Pósa's conjecture). More precisely, we determine the relationship between $\delta(G) = \delta n$ and the length p of a square-path P_p^2 (and a square-cycle C_p^2) that is forced in G. In contrast to Theorem 10.1 which states that t grows linearly with δ (when n is fixed) our main result implies that p as a function of δ behaves very differently: it is piece-wise linear but jumps at certain points (see Figure 10.1).

In order to quantify this precisely we need some (rather technical appearing) definitions. Given two positive integers n and δ with $\delta \in (\frac{1}{2}n, n-1]$, we define $\mathrm{rp}(n,\delta)$ to be the largest integer r such that $2\delta - n < \lfloor \delta/r \rfloor$. We let $\mathrm{rc}(n,\delta)$ be the smallest integer r such that $2\delta - n \geq \delta/(r+1)$. We then define the number $\mathrm{sp}(n,\delta)$ by

$$\mathrm{sp}(n,\delta) := \min\left\{ \left\lceil \frac{3}{2} \left\lceil \frac{\delta}{\mathrm{rp}(n,\delta)} \right\rceil + \frac{1}{2} \right\rceil, n \right\} ,$$

and the number $\mathrm{sc}(n,\delta) \leq \mathrm{sp}(n,\delta)$ by

$$\mathrm{sc}(n,\delta) := \min\left\{ \left\lfloor \frac{3}{2} \left\lceil \frac{\delta}{\mathrm{rc}(n,\delta)} \right\rceil \right\rfloor, n \right\} .$$

To understand where these numbers come from, consider the graph G on n vertices consisting of an independent set of size $n - \delta$ and r cliques that share the remaining vertices as equally

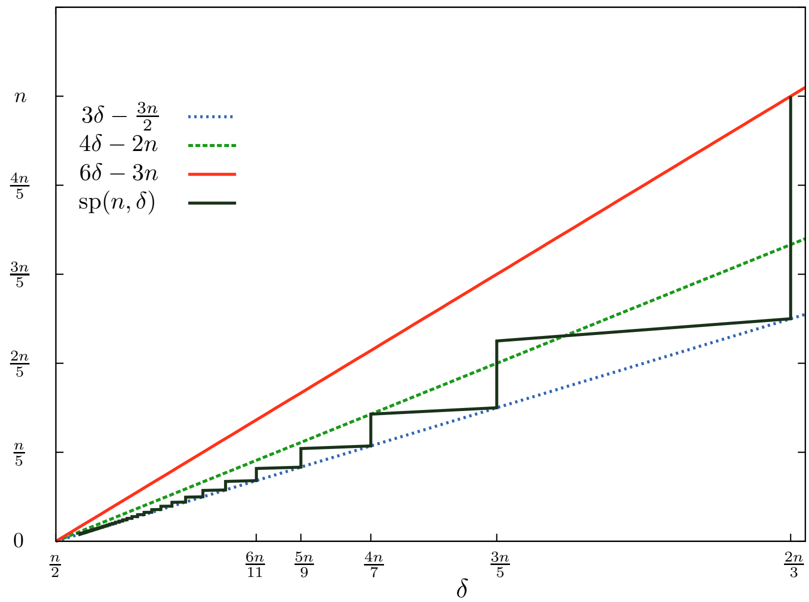


Figure 10.1: The behaviour of $\mathrm{sp}(n,\delta)$.

as possible, and that has all edges running between the independent set and any clique. Then it is not difficult to see that $\mathrm{rp}(n,\delta)$ is the biggest integer r such that the graph G constructed in this way has minimum degree δ and $\mathrm{sp}(n,\delta)$ is the length of a longest square-path in G for this r. In fact, our theorem states that this class of graphs provides (up to minor modifications) precisely the extremal graphs for the containment of square-paths at a particular minimum degree. Similar examples show that $\mathrm{sc}(n,\delta)$ is the maximum length of a square-cycle in a graph on n vertices that is forced at minimum degree δ.

Komlós, Sarközy, and Szemerédi [64] proved the Pósa conjecture (Conjecture 1.5) for large values of n (see Section 1.1.1, page 4). Their proof actually asserts the following stronger result.

Theorem 10.2 (Komlós, Sarközy, Szemerédi [64]). *There exists an integer n_0 such that for all integers $n > n_0$ the following holds. Suppose G is a graph of order n and minimum degree at least $2n/3$. Then G contains the square $C^2_{3\ell}$ of a cycle of length 3ℓ for any $3 \le 3\ell \le n$. Furthermore, if $\delta(G) \ge (2n+1)/3$ then G contains C^2_ℓ for any $\ell \in [n] \setminus \{1,2,5\}$.*

This theorem guarantees square-cycles of all lengths between 6 and n (curiously, the case $\ell = 5$ has to be excluded, because $C^5_\ell = K_5$, a graph with chromatic number 5). Similarly, our result guarantees square-cycles of length 3ℓ for all $\ell \le \mathrm{sc}(n,\delta)$. Furthermore, if any square-cycle with length not divisible by 3 is excluded from G, then we are even guaranteed much longer square-cycles in G.

Theorem 10.3 (Allen, Böttcher, Hladky [5]). *For any $\nu > 0$ there exists an integer n_0 such that for all integers n and δ with $n > n_0$, $\delta \in (n/2 + \nu n, n-1]$ the following holds. Suppose G is a graph of order n and with minimum degree $\delta(G) \ge \delta$. Then $P^2_{\mathrm{sp}(n,\delta)} \subseteq G$ and for any $\ell \in \mathbb{N}$ with $\ell \le \mathrm{sc}(n,\delta)/3$, $C^2_{3\ell} \subseteq G$. Furthermore, either $C^2_\ell \subseteq G$ for every $\ell \in \{3,4\} \cup [6, \mathrm{sc}(n,\delta)]$ or $C^2_{3\ell} \subseteq G$ for every integer $\ell \le 2\delta - n - \nu n$.*

Similar phenomena were observed for simple cycles in [4]. Obtaining higher-chromatic analogues appears to be more difficult. We believe that this theorem (and such higher chromatic analogues) may turn out useful when proving corresponding results for bounded-degree graphs with sublinear bandwidth, in the same spirit as the minimum degree threshold for spanning path powers provided by Theorem 5.6 was essential in the proof of Theorem 5.1.

Acknowledgements

These last 4 years have been an adventurous journey between Munich, Berlin, Prague, Jerusalem, and São Paulo, mainly. At each of these places a number of people have shaped this thesis, my work, and my life (probably not always in that order). I am indebted to them for support, inspiration, knowledge, and fun.

In particular, my appreciation goes to all members of M9 at TU München, of KAM at Charles University in Prague, and of the Einstein Institute at Hebrew University Jerusalem for providing a stimulating working environment, pencils, paper, and everything else that is necessary to do mathematics. In addition, I thank the groups of Mathias Schacht at Humboldt Universität Berlin, and of Yoshiharu Kohayakawa at USP in São Paulo for their hospitality during my visits, their interest in my research, and for valuable answers to many of my questions. I would like to thank Angelika Steger for refereeing this thesis.

For criticism, many helpful comments, and for encouraging words I owe thanks to Felix Fischer, Jan Foniok, Peter Heinig, Jan Hladký, Barbara Langfeld, Sybille Müller, and Andreas Würfl, who read earlier drafts of this thesis. I thank my sister, Heike Böttcher, for her creativity in interpreting mathematical subjects, when preparing the illustrations.

In Munich, I thank Jan Hladký, Diana Piguet, and Andreas Würfl for sharing ideas and insights during breakfast-seminars, in our office, or at my dinner table, Michael Ritter for his comfortingly friendly assistance with any technical problem I was able to come up with, Thomas Stolte for printing this thesis and for providing access to showers in the meanwhile, Felix Fischer for being around whenever I lost my key or my head, and Barbara Langfeld for tea, literature, and so much more that compensated for Bavarian culture.

Many thanks go to Yifaat Baron, Mark Berman, Anne Schober, and Fadi Youssef for making Jerusalem a home, and to Uri Shapira for his philosophical questions about mathematics and for making me addicted to Arabic coffee.

Moreover, I thank Jan Foniok for indispensable advice and humour, and for cooking for me when the deadline for submitting this thesis drew too close.

I am especially grateful to my family, for being supportive in uncountably many ways, for an admirable amount of patience, and for providing a “life out there” whenever I needed a break from ε’s.

Most thanks, however, go to my supervisor Anusch Taraz, who made it possible that I started, endured, and completed this thesis. His guidance, encouragement, and help were essential, in every respect.

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