Weighted Sum Rate Maximization in the MIMO MAC with Linear Transceivers: Asymptotic Results

Raphael Hunger, Michael Joham, Christian Guthy, and Wolfgang Utschick
Associate Institute for Signal Processing, Technische Universität München, 80290 Munich, Germany
Telephone: +49 89 289-28508, Fax: +49 89 289-28504, Email: {hunger,joham,guthy,utschick}@tum.de

***Abstract*— We present asymptotic results for the weighted sum rate maximization in a MIMO multiple access channel with individual power constraints when the noise power at the receiver becomes small and linear filtering is applied at the transceivers. The key parameter that determines the optimum signaling strategy is the number of antennas at the base station. If there are more antennas than the user terminals have in sum, the asymptotically optimum transmit covariance matrices are scaled identities, whereas scaled rank-deficient projectors have to be chosen when the base station does not have enough degrees of freedom. We derive the optimum transmit covariance matrices for both antenna configurations and shed some light on the impact of the solutions on the underlying rate region structure. In addition, the impact of the optimum transmission strategy on the convex hull of the underlying rate region is demonstrated.**

I. Introduction

Weighted sum rate maximization is a technique to optimize the throughput of a system taking the priorities of the users into account for the allocation of the resources. Thereby, different criteria like importance of the subscribers, buffer queue states, or fairness can be incorporated into the total throughput optimization. If nonlinear successive interference cancellation is applied at the receiver in the multiple access channel with individual power constraints, a strategy to achieve the sum capacity can be found in [1], whereas an arbitrarily weighted sum rate is maximized in [2]. The optimum signaling in case of a simpler receiver type that only applies linear filtering is unknown so far due to the inherent nonconcavity of the weighted sum rate utility with linear filtering. Nonetheless, linear receivers are of interest since latency is not an issue contrary to successive interference cancelation, where all streams first need to be decoded before their impact on the received signal can be subtracted. Several heuristic or only locally optimum approaches are available for the MIMO broadcast channel which differ from the considered setup due to the sum power constraint instead of the individual ones. An efficient implementation of an algorithm that is targeted at maximizing the weighted sum rate in a successive fashion with individual power constraints in the MAC is given in our companion paper [3]. Despite the nonconvexity of the optimization problem, we will present theoretical results on the asymptotic behavior and the asymptotically optimum transmit strategy in this contribution for the case when the noise power goes down to zero. Such an asymptotic analysis for the broadcast channel counterpart with a sum power constraint has been covered in [4], [5] and was later extended in [6], [7].

Notation: Matrices and vectors are upper and lower case bold, respectively. $\mathbb{S}_M$ denotes the set of $M \times M$ positive semidefinite matrices and $\mathbf{I}_M$ is the $M \times M$ identity matrix. The operators $\|\cdot\|_{\mathrm{F}}$, $|\cdot|$, $(\cdot)^*$, and $(\cdot)^{\mathrm{H}}$ stand for Frobenius norm, determinant, conjugate, and Hermitian transposition, respectively.

II. System Model

We focus on the uplink of a K user MIMO multiple access channel, where the B_k dimensional symbol vector $\boldsymbol{s}_k \in \mathbb{C}^{B_k}$ of user k is mapped to its M_k antennas by the matrix $\boldsymbol{T}_k \in \mathbb{C}^{M_k \times B_k}$. We assume mutually independent symbol vectors having an identity covariance matrix and the individual power constraint for the k-th user reads as $\|\boldsymbol{T}_k\|_{\mathrm{F}}^2 \leq P_k$, $k \in \{1,\ldots,K\}$. The precoded symbol vector $\boldsymbol{T}_k\boldsymbol{s}_k$ propagates over the frequency flat channel $\boldsymbol{H}_k \in \mathbb{C}^{N \times M_k}$ to the N-antenna base station, where zero mean additive Gaussian noise $\boldsymbol{\eta} \in \mathbb{C}^N$ with covariance matrix $\boldsymbol{C}_{\boldsymbol{\eta}} = \sigma^2 \boldsymbol{C}'_{\boldsymbol{\eta}}$ is added. Here, the trace of $\boldsymbol{C}'_{\boldsymbol{\eta}}$ is normalized to $\mathrm{tr}(\boldsymbol{C}'_{\boldsymbol{\eta}}) = N$ and the scalar noise variance factor σ^2 will go down to zero for our asymptotic analysis. We will make use of the following definition:

Definition II.1. *Two functions f and g are said to be strongly asymptotically equivalent, if*

$$\lim_{\sigma^2 \to 0} \left[f(\sigma^2) - g(\sigma^2) \right] = 0,$$

and we shall use the notation $f \cong g$.

III. Asymptotic Weighted Sum Rate Maximization

In [8], it is stated that in the high power regime of a MIMO broadcast channel with linear filtering, as many data streams have to be multiplexed as degrees of freedom are available in order not to sacrifice the multiplexing gain. The same reasoning can be adopted to the MIMO multiple access channel such that for any positive weight vector $\boldsymbol{w} > \mathbf{0}$, there is a threshold on the noise variance below which as many data streams as degrees of freedom are available in the system have to be multiplexed for maximizing the weighted sum rate $\sum_{k=1}^{K} w_k R_k$. In other words, $M := \sum_{k=1}^{K} M_k$ streams have to be active in the asymptotic limit if $N \geq M$, whereas only N streams may be activated when $N < M$. In the following, we

This work was supported by Deutsche Forschungsgemeinschaft (DFG) under fund Ut 36/9-1.

differentiate these two antenna configurations. For both cases, the rate of user k under linear filtering can be expressed as

$$R_k = \log_2 \left| \mathbf{I}_{M_k} + \boldsymbol{H}_k^{\mathrm{H}} \Big(\boldsymbol{C}_{\boldsymbol{\eta}} + \sum_{\ell \neq k} \boldsymbol{H}_\ell \boldsymbol{Q}_\ell \boldsymbol{H}_\ell^{\mathrm{H}} \Big)^{-1} \boldsymbol{H}_k \boldsymbol{Q}_k \right|, \quad (1)$$

where $\boldsymbol{Q}_k = \boldsymbol{T}_k \boldsymbol{T}_k^{\mathrm{H}} \in \mathbb{S}_{M_k}$ denotes the transmit covariance matrix of user k. Applying the matrix inversion lemma two times, above rate expression can be reformulated to

$$R_k = -\log_2 \left| \boldsymbol{E}_k^{\mathrm{T}} \Big(\mathbf{I}_B + \frac{1}{\sigma^2} \boldsymbol{T}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \boldsymbol{T} \Big)^{-1} \boldsymbol{E}_k \right|, \quad (2)$$

with the composite channel and precoder matrices

$$\boldsymbol{H} = [\boldsymbol{H}_1, \ldots, \boldsymbol{H}_K] \in \mathbb{C}^{N \times M},$$
$$\boldsymbol{T} = \mathbf{blockdiag}\{\boldsymbol{T}_k\}_{k=1}^K \in \mathbb{C}^{M \times B},$$

where $B := \sum_{k=1}^K B_k$ denotes the total number of active streams and $\boldsymbol{E}_k$ the k-th block unit matrix defined via

$$\boldsymbol{E}_k^{\mathrm{T}} = [\mathbf{0}, \ldots, \mathbf{0}, \mathbf{I}_{B_k}, \mathbf{0}, \ldots, \mathbf{0}] \in \{0,1\}^{B_k \times B}$$

which extracts the kth main diagonal block of a matrix when left hand side multiplied by $\boldsymbol{E}_k^{\mathrm{T}}$ and right hand side multiplied by $\boldsymbol{E}_k$.

A. Enough Degrees of Freedom at the Base Station

If the base station is equipped with enough antennas, all M streams can be activated when the noise variance is low by letting every user transmit as many data streams as he has antennas, i.e., $B_k = M_k \ \forall k$ and $B = M$. In combination with a noise-variance independent power allocation for the K users, the multiplexing gain of every user corresponds to his number of antennas, which is a prerequisite for maximizing any weighted sum rate with positive weights if $N \geq M$. For any $N \times M$ channel realization $\boldsymbol{H}$ with $\mathrm{rank}(\boldsymbol{H}) = M$, there is a threshold for the noise variance below which the optimum transmission strategy is to have M streams active. Letting the noise variance σ^2 go down to zero, the eigenvalues of $\frac{1}{\sigma^2} \boldsymbol{T}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \boldsymbol{T}$ in (2) become much larger than one and we may neglect the identity matrix inside the inverse. Thus, we end up with the strong asymptotic equivalence

$$R_k \cong \log_2 \left| \frac{\boldsymbol{Q}_k}{\sigma^2} \right| - \log_2 \left| \boldsymbol{E}_k^{\mathrm{T}} \big(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \big)^{-1} \boldsymbol{E}_k \right|, \quad (3)$$

which, without loss of optimality, is maximized by the choice

$$\boldsymbol{Q}_k = \frac{P_k}{M_k} \mathbf{I}_{M_k}, \quad (4)$$

so every user has to consume his complete power budget. In the low noise regime, the data rate R_k of user k only depends on his own transmit covariance matrix $\boldsymbol{Q}_k$, and not on other covariance matrices, see (3). The resulting asymptotic rate reads as

$$R_k \cong M_k \log_2 \frac{P_k}{\sigma^2 M_k} - \log_2 \left| \boldsymbol{E}_k^{\mathrm{T}} \big(\boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \big)^{-1} \boldsymbol{E}_k \right|, \quad (5)$$

and interestingly, this rate is asymptotically optimum for *any* choice of the weight vector $\boldsymbol{w} > \mathbf{0}$ of the weighted sum rate maximization! This clearly differs from the asymptotic behavior of the broadcast channel with a sum power constraint [6], [7]. There, both the transmit covariance matrices and the obtained rate tuples do depend on the weight vector. The geometric interpretation of the underlying rate region is that there has to be a sharp edge of a hyper-cuboid at that position, and any supporting hyperplane defined by the weight vector $\boldsymbol{w} > \mathbf{0}$ touches the rate region at that single point in the asymptotic low noise limit, see the first three figures in Section IV.

B. Too Few Degrees of Freedom at the Base Station

This type of antenna configuration has so far never been investigated in any asymptotic analysis, even in the broadcast channel. If at most N streams are active in a system with $M > N$ rendering zero forcing possible, the $B \times B$ matrix $\boldsymbol{T}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \boldsymbol{T}$ has rank B and is thus invertible. As a consequence, the rate R_k of user k is strongly asymptotically equivalent to

$$R_k \cong -B_k \log_2 \sigma^2 - \log_2 \left| \boldsymbol{E}_k^{\mathrm{T}} \big(\boldsymbol{T}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \boldsymbol{T} \big)^{-1} \boldsymbol{E}_k \right|, \quad (6)$$

and thus, every of his B_k streams asymptotically contributes to the weighted sum rate via $-w_k \log_2 \sigma^2$ plus some constant that does not depend on σ^2. Therefore, streams have to be allocated to the users with the largest weights to maximize the weighted sum rate when the noise variance σ^2 goes to zero. Let $s[1], \ldots, s[K]$ denote a permutation of the user indices such that the weights are sorted in a nonincreasing fashion, i.e., $w_{s[1]} \geq \ldots \geq w_{s[K]}$. If all weights are *different*, the asymptotically optimum user selection starts by allocating $\min\{N, M_{s[1]}\}$ streams to user $s[1]$. If $N \leq M_{s[1]}$, all streams are allocated to user $s[1]$, otherwise, we start to fill the remaining $N - M_{s[1]}$ streams to user $s[2]$ with the second largest weight, but no more than $M_{s[2]}$, and so on. When some users share the same weight, all possible combinations of stream allocation have to be probed.

In the following, we investigate a two user system where $N < M_1 + M_2$, i.e., the base station has less antennas than both users have in sum. Furthermore, we assume either that user one applies full multiplexing with $B_1 = M_1 < N$ and user two multiplexes $B_2 = N - M_1 < M_2$ streams (when $w_1 \geq w_2$), or that user two applies full multiplexing with $B_2 = M_2 < N$ and user one multiplexes only $B_1 = N - M_2 < M_1$ streams (when $w_1 \leq w_2$). Thus, the composite precoder matrix $\boldsymbol{T}$ is no longer square and cannot be inverted. However, we may still neglect the identity matrix in the inverse of (2) and apply the block inversion lemma for partitioned matrices to $(\boldsymbol{T}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{C}_{\boldsymbol{\eta}}'^{-1} \boldsymbol{H} \boldsymbol{T})^{-1}$ to extract the two main diagonal blocks of the inverse. The upper left one reads as

$$\Big[\boldsymbol{T}_1^{\mathrm{H}} \boldsymbol{H}_1^{\mathrm{H}} \big[\mathbf{I} - \boldsymbol{H}_2 \boldsymbol{T}_2 (\boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{T}_2)^{-1} \boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \big] \boldsymbol{H}_1 \boldsymbol{T}_1 \Big]^{-1} \quad (7)$$

and the lower right one as

$$\Big[\boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \big[\mathbf{I} - \boldsymbol{H}_1 (\boldsymbol{H}_1^{\mathrm{H}} \boldsymbol{H}_1)^{-1} \boldsymbol{H}_1^{\mathrm{H}} \big] \boldsymbol{H}_2 \boldsymbol{T}_2 \Big]^{-1}. \quad (8)$$

Note that we assumed $\boldsymbol{C}'_{\boldsymbol{\eta}} = \mathbf{I}$ in above and all following expressions for a simpler notation which can always be achieved by pre-whitening all channels via $\boldsymbol{H}_k \rightarrow \boldsymbol{C}'^{-\frac{1}{2}}_{\boldsymbol{\eta}} \boldsymbol{H}_k$. Plugging (8) into the expression in (2), the rate of user two asymptotically reads as

$$R_2 \cong \log_2 \left| \frac{1}{\sigma^2} \boldsymbol{T}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{T}_2 \right|, \tag{9}$$

where $\bar{\boldsymbol{H}}_2 = [\mathbf{I} - \boldsymbol{H}_1(\boldsymbol{H}_1^{\mathrm{H}} \boldsymbol{H}_1)^{-1} \boldsymbol{H}_1^{\mathrm{H}}] \boldsymbol{H}_2$ is the channel of user two projected into the null space of the channel $\boldsymbol{H}_1$ of user one. Since $\boldsymbol{T}_1$ has full rank and is thus invertible, the asymptotic expression for R_2 does not depend on the actual choice for $\boldsymbol{T}_1$. In contrast, the asymptotic behavior of R_1 still depends on $\boldsymbol{T}_2$ since the null space projector in (7) explicitly varies with $\boldsymbol{T}_2$ for $B_2 < M_2$. Using (7) for the rate term in (2) and neglecting the identity in the inverse, few manipulations of the resulting expression together with arguments similar to (3) and (4) lead to

$$R_1 \cong M_1 \log_2 \frac{P_1}{\sigma^2 M_1} + \log_2 \left| \boldsymbol{H}_1^{\mathrm{H}} \boldsymbol{H}_1 \right| + \log_2 \frac{\left| \boldsymbol{T}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{T}_2 \right|}{\left| \boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{T}_2 \right|}.$$

Again, all users have to transmit with full power to maximize their rates. Combining the last two equations, the asymptotically optimum weighted sum rate $w_1 R_1 + w_2 R_2$ can now be decomposed into a summand depending on $\boldsymbol{T}_2$ and a constant, which does not depend on $\boldsymbol{T}_2$:

$$w_1 R_1 + w_2 R_2 \cong w_1 \log_2 \frac{\left| \boldsymbol{T}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{T}_2 \right|^{1+\frac{w_2}{w_1}}}{\left| \boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{T}_2 \right|} + \text{const.}$$

The precoder dependent part of the asymptotic weighted sum rate entails the optimization

$$\underset{\boldsymbol{T}_2 \in \mathbb{C}^{M_2 \times B_2}}{\text{maximize}} \frac{\left| \boldsymbol{T}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{T}_2 \right|^{\alpha}}{\left| \boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{T}_2 \right|} \quad \text{s.t.: } \|\boldsymbol{T}_2\|_{\mathrm{F}}^2 = P_2 \tag{10}$$

with $\alpha \in (1, 2]$ since letting user one multiplex all of his M_1 streams and user two the remaining $N - M_1$ can only be asymptotically optimum for weights $w_2 \leq w_1$. For $\alpha > 1$, the constraint in (10) is active, and the associated Lagrangian function reads as

$$\mathcal{L}(\boldsymbol{T}_2, \mu) = \frac{\left| \boldsymbol{T}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{T}_2 \right|^{\alpha}}{\left| \boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{T}_2 \right|} - \mu \left[\operatorname{tr}(\boldsymbol{T}_2^{\mathrm{H}} \boldsymbol{T}_2) - P_2 \right]. \tag{11}$$

The optimum Lagrangian multiplier $\check{\mu}$ can be computed via

$$\begin{aligned} \operatorname{tr}\left[\boldsymbol{T}_2^{\mathrm{H}} \frac{\partial \mathcal{L}(\boldsymbol{T}_2, \mu)}{\partial \boldsymbol{T}_2^*} \right] \Big|_{\boldsymbol{T}_2 = \check{\boldsymbol{T}}_2, \mu = \check{\mu}} = 0 \ \Leftrightarrow \\ \check{\mu} = \frac{B_2}{P_2} (\alpha - 1) \frac{\left| \check{\boldsymbol{T}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \check{\boldsymbol{T}}_2 \right|^{\alpha}}{\left| \check{\boldsymbol{T}}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \check{\boldsymbol{T}}_2 \right|}, \end{aligned} \tag{12}$$

so $\check{\mu}$ is $(\alpha - 1) B_2 / P_2$ times the optimum utility and vanishes only for $\alpha = 1$. Reinserting $\check{\mu}$ into the partial derivative of $\mathcal{L}(\boldsymbol{T}_2, \mu)$ with respect to $\boldsymbol{T}_2^*$ and left-hand side multiplying by the Hermitian optimum precoder $\check{\boldsymbol{T}}_2^{\mathrm{H}}$ yields

$$\check{\boldsymbol{T}}_2^{\mathrm{H}} \frac{\partial \mathcal{L}(\boldsymbol{T}_2, \mu)}{\partial \boldsymbol{T}_2^*} \Big|_{\boldsymbol{T}_2 = \check{\boldsymbol{T}}_2, \mu = \check{\mu}} = \mathbf{0} \ \Leftrightarrow \ \check{\boldsymbol{T}}_2^{\mathrm{H}} \check{\boldsymbol{T}}_2 = \frac{P_2}{B_2} \cdot \mathbf{I}_{B_2}, \tag{13}$$

so the optimum $\check{\boldsymbol{T}}_2$ asymptotically maximizing the weighted sum rate is a weighted partial isometry and $\check{\boldsymbol{T}}_2 \check{\boldsymbol{T}}_2^{\mathrm{H}}$ is a scaled projector. For single stream transmission, the optimum precoding vector $\check{\boldsymbol{t}}_2$ can be found by computing a sequence of dominant eigenvectors and the utility in (10) then corresponds to the principal eigenvalue, which will be shown in the following.

Setting the derivative of $\mathcal{L}(\boldsymbol{t}_2, \mu)$ in (11) with respect to $\boldsymbol{t}_2^*$ to zero and evaluating the resulting expression at the optimum precoder $\check{\boldsymbol{t}}_2$ and the optimum Lagrangian multiplier $\check{\mu}$ from (12), we obtain

$$\left(\alpha \beta(\check{\boldsymbol{t}}_2)^{\alpha - 1} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 - \frac{\alpha - 1}{P_2} \beta(\check{\boldsymbol{t}}_2)^{\alpha} \mathbf{I} \right) \check{\boldsymbol{t}}_2 = u(\check{\boldsymbol{t}}_2) \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \check{\boldsymbol{t}}_2, \tag{14}$$

with the precoder-dependend utility

$$u(\boldsymbol{t}_2) = \frac{(\boldsymbol{t}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{t}_2)^{\alpha}}{\boldsymbol{t}_2^{\mathrm{H}} \boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \boldsymbol{t}_2}$$

and the Hermitian form substitution $\beta(\boldsymbol{t}_2) = \boldsymbol{t}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 \boldsymbol{t}_2$. The dependency of $\beta(\check{\boldsymbol{t}}_2)$ on the optimum precoder $\check{\boldsymbol{t}}_2$ prevents from a closed form solution of (14) for $\check{\boldsymbol{t}}_2$ via a single eigenvalue decomposition. Nonetheless, the utility $u(\boldsymbol{t}_2)$ can be maximized by an iterative principal eigenvector computation based on the KKT condition in (14). To this end, the optimum precoder $\check{\boldsymbol{t}}_2$ in the Hermitian form $\beta(\check{\boldsymbol{t}}_2)$ in (14) is replaced by the precoder $\boldsymbol{t}_2^{(n)}$ of iteration n, whereas the expression $\check{\boldsymbol{t}}_2$ in the two matrix-vector products in (14) is replaced by $\boldsymbol{t}_2^{(n+1)}$, i.e., by the precoder for the next iteration $n + 1$. This yields the eigenvalue problem

$$\boldsymbol{K}^{(n)} \boldsymbol{t}_2^{(n+1)} = \lambda^{(n+1)} \boldsymbol{t}_2^{(n+1)}, \tag{15}$$

where the matrix $\boldsymbol{K}^{(n)}$ is defined in analogy to (14) as

$$\boldsymbol{K}^{(n)} := \left[\boldsymbol{H}_2^{\mathrm{H}} \boldsymbol{H}_2 \right]^{-1} \left(\alpha \beta(\boldsymbol{t}_2^{(n)})^{\alpha - 1} \bar{\boldsymbol{H}}_2^{\mathrm{H}} \bar{\boldsymbol{H}}_2 - \frac{\alpha - 1}{P_2} \beta(\boldsymbol{t}_2^{(n)})^{\alpha} \mathbf{I} \right)$$

and depends only on the precoder of iteration n. The precoder $\boldsymbol{t}_1^{(n+1)}$ of the next iteration is chosen to be the principal eigenvector of $\boldsymbol{K}^{(n)}$ corresponding to the *largest* eigenvalue $\lambda^{(n+1)}$, because in the limit $n \rightarrow \infty$, the eigenvalue $\lambda^{(n+1)}$ corresponds to the utility $u(\boldsymbol{t}_2^{(n+1)})$ which of course shall be maximized. The monotonic convergence of the utility $u(\boldsymbol{t}_2^{(n)})$ and the precoder $\boldsymbol{t}_2^{(n)}$ is shown in [9].

So far we have shown how to compute the optimum precoder $\check{\boldsymbol{t}}_2$ in case user one has a higher priority than user two, i.e., $w_1 > w_2$. If $w_2 > w_1$, the same derivation remains valid if the two user indices are interchanged, so $\check{\boldsymbol{t}}_1$ can be computed analogously. In case of equal weights $w_1 = w_2$, both stream configurations have to be probed to find the maximum sum rate. It is not known in advance but rather depends on the channel realization which user needs to apply single stream beamforming to achieve the maximum sum rate.

IV. Graphical Visualization

The scenario with enough degrees of freedom at the base station is shown in Fig. 1. The two users with $M_1 = M_2 = 2$

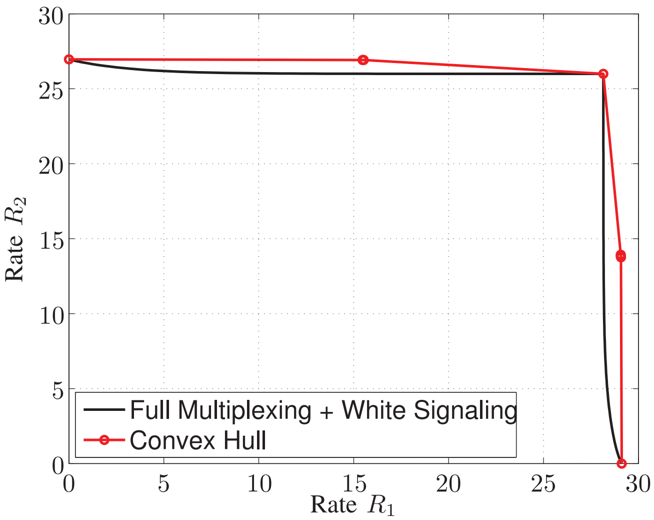

Fig. 1. Convex hull of two user rate region for an $N = 5$ antenna base station serving two users with $M_1 = M_2 = 2$ antennas each. The noise variance is $10 \log_{10} \sigma^2 = -37\text{dB}$.

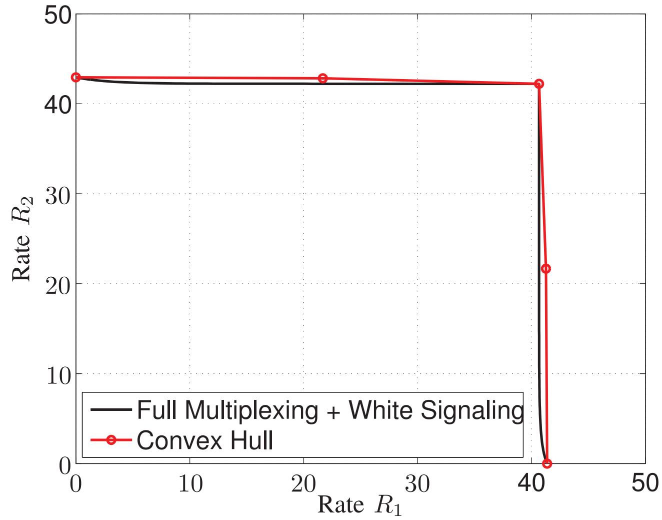

Fig. 2. Convex hull of two user rate region for an $N = 5$ antenna base station serving two users with $M_1 = M_2 = 2$ antennas each. The noise variance is $10 \log_{10} \sigma^2 = -57\text{dB}$.

antennas each are served by an $N = 5$ antenna base station and the individual transmit powers p_1 and p_2 are upper bounded by $p_1 \leq P_1 = 1$ and $p_2 \leq P_2 = 1$, respectively, and the noise variance is given by $10 \log_{10} \sigma^2 = -37\text{dB}$. The black curve without marker shows the achievable rate region when both users apply full multiplexing with transmit covariance matrices $\boldsymbol{Q}_1 = p_1/2 \cdot \mathbf{I}_2$ and $\boldsymbol{Q}_2 = p_2/2 \cdot \mathbf{I}_2$. The almost horizontal part of this curve corresponds to the case when user two transmits with full power $(p_2 = P_2)$ and user one varies his power from $p_1 = 0$ to $p_1 = P_1$, whereas for the almost vertical part user two varies his power from $p_2 = 0$ to $p_2 = P_2$ with $p_1 = P_1$ fixed. The sharp 90-degree edge at the rate pair $R_1 \approx 28$ bits per channel use and $R_2 \approx 26$ bits per channel use shows that the weighted sum rate maximizer is independent of the chosen weight vector $\boldsymbol{w} > \mathbf{0}$ in case of full multiplexing as derived in Section III-A. In addition to the black curve depicting the full multiplexing transmission strategy, the convex hull of the rate region is shown by the red curve with the circle marker. As we can see, the angle of intersection at the sharp edge is slightly larger than 90 degrees. For extreme weight ratios $w_1/w_2 \gg 1$ or $w_1/w_2 \ll 1$ and the given noise variance, the weighted sum rate might be maximized when only one user applies full multiplexing and the other one applies single stream beamforming according to Section III-B, see the circle markers for the respective rate pairs. However, with diminishing noise variance, the angle of intersection at the full multiplexing edge becomes smaller and smaller, and in the limit $\sigma^2 \to 0$, it converges to 90 degrees, such that the full multiplexing configuration leads to the largest weighted sum rate for any positive weight vector $\boldsymbol{w}$. When the noise variance is reduced to $10 \log_{10} \sigma^2 = -57\text{dB}$, the angle of intersection at the full multiplexing edge has almost reached 90 degrees, see Fig. 2. In contrast, a larger noise variance of $10 \log_{10} \sigma^2 = -17\text{dB}$ leads to the fact that the beamforming configuration for one user yields the largest weighted sum rate even for moderately different user priorities w_1 and w_2, see Fig. 3. Despite the fact that the full multiplexing configuration still features the sharp edge at the rate pair $R_1 \approx 14$ bits per channel use and $R_2 \approx 12$ bits per channel use, the edge of the convex hull shows an obtuse angle of about 150 degrees. As a consequence, the full multiplexing configuration is the optimum stream configuration only for rate ratios not too far away from one.

For all reasonably small noise variances, the black curve also represents the boundary of the rate region when time sharing is not considered as long as the individual rates are larger than the ones obtained by the single-stream configuration. In Fig. 1 for example, the black curve represents the boundary of the rate region for rates $R_1 > 16$ bits per channel use and $R_2 > 15$ bits per channel use.

Reducing the number of antennas deployed at the base station to $N = 3$ leads to the rate region shown in Fig. 4. There, the full multiplexing configuration does not contribute to the convex hull of the rate region, only the single-stream beamforming configuration for one user from Section III-B is relevant for the given setup. Besides the two rate pairs where only one component is different from zero, i.e., where only one user is active, only few positive rate pairs contribute to the convex hull of the rate region. Even for different weight vectors, almost the same rate point maximizes the weighted sum rate for one of the two possible stream configurations where only one user applies full multiplexing whereas the other one uses single-stream beamforming. As a consequence, the convex hull almost looks like a pentagon. The line segment whose slope is approximately -45 degree resembles the lack of the full multiplexing configuration. Rate pairs on that line segment feature a multiplexing gain of only three compared to a multiplexing gain of four at the sharp edge in Fig. 1–Fig. 3.

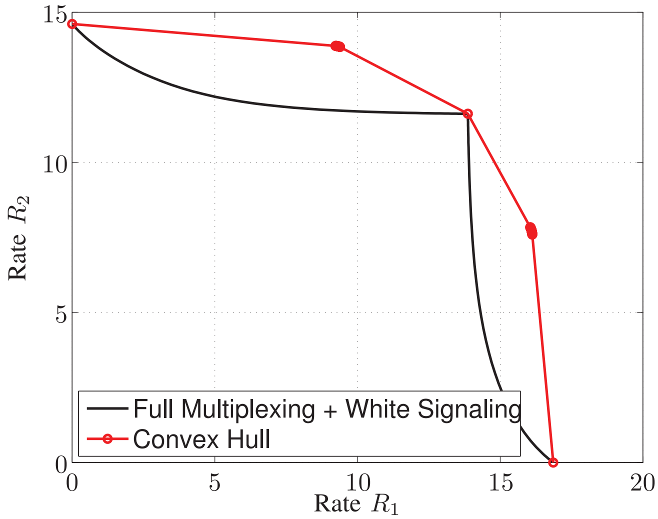

Fig. 3. Convex hull of two user rate region for an $N = 5$ antenna base station serving two users with $M_1 = M_2 = 2$ antennas each. The noise variance is $10 \log_{10} \sigma^2 = -17$dB.

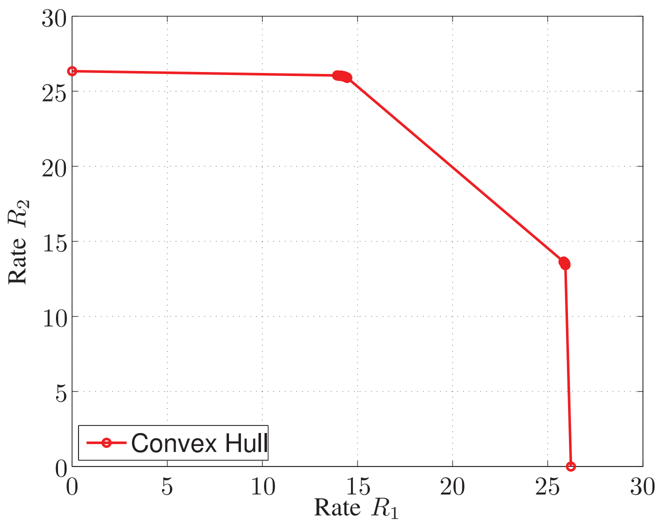

Fig. 4. Convex hull of two user rate region for an $N = 3$ antenna base station serving two users with $M_1 = M_2 = 2$ antennas each. The noise variance is $10 \log_{10} \sigma^2 = -37$dB.

V. Conclusion

We analyzed the asymptotic weighted sum rate maximization in the MIMO multiple access channel with individual power constraints for setups with both enough and too few degrees of freedom for overall full multiplexing. It turned out that the transmitters either have to apply white signaling in case of enough degrees of freedom or a scaled projector as transmit covariance matrix when not enough degrees of freedom are available for the user to apply full multiplexing.

References

[1] W. Yu, W. Rhee, S. Boyd, and J. M. Cioffi, "Iterative Water-filling for Gaussian Vector Multiple-Access Channels," *IEEE Transactions on Information Theory*, vol. 50, pp. 145–152, January 2004.

[2] M. Kobayashi and G. Caire, "Iterative Waterfilling for the Weighted Rate Sum Maximization in MIMO MAC," in *Signal Processing Advances in Wireless Communications (SPAWC 2006)*, July 2006.

[3] C. Guthy, W. Utschick, R. Hunger, and M. Joham, "Weighted Sum Rate Maximization in the MIMO MAC with Linear Transceivers: Algorithmic Solutions," in *43rd Asilomar Conference on Signals, Systems, and Computers (Asilomar 2009)*, November 2009.

[4] J. Lee and N. Jindal, "Dirty Paper Coding vs. Linear Precoding for MIMO Broadcast Channels," in *40th Asilomar Conference on Signals, Systems, and Computers (Asilomar 2006)*, October 2006, pp. 779–783.

[5] ——, "High SNR Analysis for MIMO Broadcast Channels: Dirty Paper Coding Versus Linear Precoding," *IEEE Transactions on Information Theory*, vol. 53, no. 12, pp. 4787–4792, December 2007.

[6] R. Hunger and M. Joham, "On the Asymptotic Optimality of Block-Diagonalization for the MIMO BC under Linear Filtering," in *Workshop on Smart Antennas (WSA 09)*, Berlin, February 2009.

[7] ——, "An Asymptotic Analysis of the MIMO Broadcast Channel under Linear Filtering," in *Conference on Information Sciences and Systems (CISS 2009)*, March 2009.

[8] R. Hunger, D. A. Schmidt, and M. Joham, "A Combinatorial Approach to Maximizing the Sum Rate in the MIMO BC with Linear Precoding," in *42nd Asilomar Conference on Signals, Systems, and Computers (Asilomar 2008)*, October 2008.

[9] R. Hunger, P. de Kerret, and M. Joham, "An Algorithm for Maximizing a Quotient of Two Hermitian Form Determinants with Different Exponents," *Submitted to ICASSP 2010.*