ENERGY CONSERVING MODEL FOR THE CHANNEL POWER-GAIN IN WIRELESS MULTI-USER DOWNLINK SCENARIOS

Michel T. Ivrlač and Josef A. Nossek

Institute for Circuit Theory and Signal Processing
Technische Universität München
{ivrlac, nossek}@tum.de

ABSTRACT

Established models for the prediction of channel power gain in radio communication systems have got a slight drawback which grows into a real problem in a theoretical large system analysis of downlink scenarios, where the number of user terminals is allowed to grow unboundedly. The problem is that the model predictions of channel power gain do not take into account the existence of additional receivers. As the number of terminals grows unboundedly, the sum of the predicted receive powers increases unboundedly, and eventually becomes larger than the total transmit power. In a passive propagation medium – like air – such model errors conflict with the principle of conservation of energy, when applied to asymptotic large system analysis. We propose a simple generic modification which – while applicable to any of the existent channel power gain models – corrects this model error and is intended as a first step towards the provision of a channel power gain model that allows for a meaningful large system analysis. We present a fully worked out example which, besides providing interesting insight, is also intended to demonstrate that mathematical tractability is not seriously affected by the proposed modification.

1. INTRODUCTION

In radio communications, the channel power gain (CPG) is a function of the distance between the transmitting and the receiving antennas. In a free-space scenario, for example, the CPG decreases with the square of this distance [1]. In applications, like network planning, it is obvious that models for the accurate prediction of CPG are required [2]. But even in a purely theoretical analysis those models are of considerable interest, especially in the context of the multi-user downlink (broadcast channel). The reason is that the user specific channel power gains impact performance of scheduling and signal processing algorithms [3, 4].

There are many CPG models published and in use today. See, for instance, [5–10], and the references therein. What all these models have in common is that they actually assume a single-user system, since the predicted average channel gains do not depend on whether there exist other terminals. In practical multi-user downlink scenarios, such kind of model error is of little importance, though. The number of users is usually low enough such that it can safely be assumed that the existence of one user's receiving antenna has negligible effect on the electro-magnetic field, and therefore, does not change the receive power of another user's receive antenna in any noticeable way. However, in an asymptotic system analysis, where the number of terminals is allowed to grow unboundedly, this behavior of the established models turns into a real problem. That this is so, can be most clearly seen from the fact that for a large enough number of terminals, the sum of the predicted receive powers exceeds the transmit power. Of course, this is impossible in a passive medium, like air. As a consequence, the conventional CPG models should not be used »out of the box« in an asymptotic system analysis because of the inherent conflict with the principle of conservation of energy.

In this paper, we propose a simple, generic modification which can be applied to any CPG model making it compliant with the principle of conservation of energy. When the number of terminals is small enough, such that conflict with this principle is avoided well already by the conventional models, the proposed modification will have only a negligible effect. The original model's behavior for small systems is preserved while making it applicable for a large system analysis. After providing its definition and basic properties, we demonstrate its application to asymptotic system analysis of a multi-user downlink with an opportunistic proportional fair scheduling. Besides providing interesting insight, this example serves to demonstrate that the mathematical tractability is not affected seriously by the proposed modification.

2. ENERGY CONSERVING CHANNEL POWER GAIN MODEL

In the following, we consider a multi-user downlink scenario, where a basestation communicates with a number of (at least two) users.

Definition 1 (Channel power-gain in a multi-user downlink)**.** *The channel power-gain of a user j of the multi-user down-*

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link is defined as the ratio

$$\rho_j(d_j) = \frac{\overline{P}_{\mathrm{R},j}}{P_{\mathrm{T}}}, \tag{1}$$

of the average received power $\overline{P}_{\mathrm{R},j}$ and the transmit power P_{T}. The channel power-gain essentially depends on the distance d_j between the terminal and the basestation.

Conventional models for the prediction of the channel power gain essentially assume a single-user context, with one transmitter and one receiver. When they are nevertheless applied to a multi-user context, we use the term *single-user channel power gain* to tell it apart from the true channel power gain.

Definition 2 (Single-user channel power gain)**.** *We define the single-user channel power gain of user j as:*

$$\tilde{\rho}_j(d_j) = \left.\frac{\overline{P}_{\mathrm{R},j}}{P_{\mathrm{T}}}\right|_{\text{Only user } j \text{ exists.}} \tag{2}$$

Because the single user channel power gain may conflict with the principle of conservation of energy, we propose the simple, generic modification:

Definition 3 (Energy conserving channel power gain)**.** *Let us denote with $\tilde{\rho}_j(d_j)$, where $0 < \tilde{\rho}_j(d_j) < 1$, the single-user channel power gain of user j in a multi-user downlink, where d_j, with $d_j > 0$, is the distance between the transmitter (the basestation) and the receiver of user j. The $\tilde{\rho}_j(d_j)$ are computed according to any standard channel power gain model. Furthermore, let all d_j be pairwise different, i.e., $d_j \neq d_{i \neq j}$. The energy conserving channel power gain is defined by:*

$$\rho_j(d_j) \;=\; \tilde{\rho}_j(d_j) \prod_{d_i < d_j} \Big(1 - \tilde{\rho}_i(d_i)\Big). \tag{3}$$

The idea behind this definition can easily be understood from the following Theorem.

Theorem 1. *The energy conserving channel power gain from Definition 3 can equivalently be written as:*

$$\rho_j(d_j) \;=\; \left(1 - \sum_{d_i < d_j} \rho_i(d_i)\right) \tilde{\rho}_j(d_j). \tag{4}$$

Proof. See Appendix A. □

We can observe from (4) that the part of the transmit power which is »eaten up« by receivers which are positioned closer to the basestation cannot be consumed anymore by receivers which happen to be located farther away.

Theorem 2 (Energy conservation)**.** *For any $d_j < \infty$, it holds true that:*

$$\sum_{d_i < d_j} \rho_i(d_i) \;\leq\; 1, \tag{5}$$

where equality holds only if the number of users is not finite.

Proof. See Appendix B. □

This shows that in no circumstances the sum of the received powers of all finitely or infinitely many terminals can exceed the transmit power, which makes the model (3) obey the principle of conservation of energy, even when applied within an asymptotic large system analysis.

3. APPLICATION EXAMPLE: PROPORTIONAL FAIR OPPORTUNISTIC SCHEDULING

Let us have a look at how the proposed channel power gain model can be applied in an asymptotic large system analysis of a multi-user downlink system. Suppose there are a number of K equal terminals located in a finite space with

$$d_1 < d_2 < d_3 < \cdots < d_K\,, \tag{6}$$

located far enough from the transmitter such that $d_K/d_1 \approx 1$. The single-user channel power gain $\tilde{\rho}$ is therefore almost the same for each terminal. Applying the proposed model from (3) leads to the channel power gains:

$$\rho_j = \tilde{\rho} \cdot \big(1 - \tilde{\rho}\big)^{j-1}\,, \tag{7}$$

for $j \in \{1, 2, \ldots, K\}$. Note that

$$\sum_{j=1}^{\infty} \rho_j = 1, \quad \text{and} \quad \forall K < \infty : \; \sum_{j=1}^{K} \rho_j < 1, \tag{8}$$

independent of $\tilde{\rho}$. Therefore, as the number of terminals approaches infinity, the sum of all received powers approaches the transmit power from below.

On top of this average channel power gain let there be an independent and identically distributed random fading power gain γ_j for each terminal. Take note that the all too common choice of an exponential probability density function (pdf) for the γ_j, i.e., Rayleigh Fading, is *not* an option in this context. Because it allows for arbitrarily large fading power gains, it asks for trouble with the principle of conservation of energy. Since trouble with this principle is exactly what we wanted to avoid in the first place, the pdf of the γ_j *has to have a finite support*. In this paper, we choose the simplest pdf with finite support, the uniform distribution:

$$\mathrm{pdf}_{\gamma_j}(x) = \mathrm{pdf}_{\gamma}(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1\,, \\ 0 & \text{else.} \end{cases} \tag{9}$$

The transmitter decides to communicate with only one terminal at a time. Proportional fair scheduling [11] selects the terminal j_* among the K terminals which momentarily has the largest fading power gain:

$$j_* = \arg \max_{j \in \{1,2,\ldots K\}} \gamma_j\,. \tag{10}$$

Because of the »proportional fair« nature of the scheduling, each of the K terminals has the same chance to be scheduled at a given time. Hence, the average channel power gain of *the scheduled terminal* is given by

$$\overline{\rho_{\text{sched}}} = \mathrm{E}\left[\rho_j\right] \cdot \mathrm{E}\left[\gamma_{j*}\right], \tag{11}$$

where E[.] is the expectation operation, and

$$\begin{aligned} \mathrm{E}\left[\rho_j\right] &= \frac{1}{K}\sum_{j=1}^{K}\rho_j = \frac{\widetilde{\rho}}{K}\sum_{j=1}^{K}\left(1-\widetilde{\rho}\right)^{j-1} \\ &= \frac{1-\left(1-\widetilde{\rho}\right)^K}{K}. \end{aligned} \tag{12}$$

Note that $\mathrm{E}\left[\rho_j\right] \to 0$, as $K \to \infty$, since more and more terminals compete for a finite transmit power resource. Because $\mathrm{E}\left[\gamma_{j*}\right] \leq 1$, it also follows that $\overline{\rho_{\text{sched}}} \to 0$, as $K \to \infty$. In this way, it is not good for the average scheduled power gain to have too many terminals around. »Too much competition kills the sport«, so to speak.

Lemma 1. *The average channel power-gain of a scheduled user is given by:*

$$\overline{\rho_{\text{sched}}}(K) = \frac{1-\left(1-\widetilde{\rho}\right)^K}{K+1}. \tag{13}$$

Proof. See Appendix C. □

The optimum number K_{opt} of users can be defined as:

$$K_{\text{opt}} = \arg\max_{K>1} \overline{\rho_{\text{sched}}}(K), \tag{14}$$

since this number of users maximizes throughput.

Theorem 3. *The optimum number K_{opt} of users has the following asymptotic behavior:*

$$K_{\text{opt}} \to \sqrt{\frac{2}{\widetilde{\rho}}} \quad \text{as} \quad \widetilde{\rho} \to 0. \tag{15}$$

Proof. See Appendix D. □

Setting $\widetilde{\rho} = 10^{-6}$, the maximum throughput is achieved with approximately 1.4×10^3 terminals. This particular number of terminals ensures the best compromise between a high multi-user diversity (large K) and not too much competition for the finite transmit power resource (small K). For $\widetilde{\rho} = 10^{-12}$, the optimum number of terminals is more than a million already.

4. CONCLUSION

In the asymptotic analysis of the downlink of multi-user radio communication systems, where the number of user terminals is allowed to grow unboundedly, conventional models for the calculation of channel power gain cannot be applied directly, since they are in conflict with the principle of conservation of energy. We propose a generic modification applicable to any conventional channel power gain model which assures compliance with the principle of conservation of energy. With an exemplary, fully worked out asymptotic system analysis, it is demonstrated that there is an optimum compromise between multi-user diversity on the one hand, and multi-user competition for finite transmit power resources, on the other hand. It should be pointed out here that – besides the discussed channel power gain models – conventional models for shadowing and small-scale fading (like log-normal shadowing or the popular Rayleigh fading) have to be adapted properly, as they, too, can conflict with the principle of conservation of energy.

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A. PROOF OF THEOREM 1

Without loss of generality, let $d_1 < d_2 < \cdots < d_j$. We have to prove that

$$1-\sum_{i=1}^{j-1} \widetilde{\rho}_i(d_i) \prod_{k=1}^{i-1} \left(1-\widetilde{\rho}_k(d_k)\right) = \prod_{i=1}^{j-1} \left(1-\widetilde{\rho}_i(d_i)\right), \quad (16)$$

for $j \in \mathbb{N} \setminus \{1\}$. Obviously, for $j = 2$, the equality holds. We just have to show that equality still holds for $j \leftarrow j+1$, provided that it holds for j.

$$1-\sum_{i=1}^{j} \widetilde{\rho}_i(d_i) \prod_{k=1}^{i-1} \left(1-\widetilde{\rho}_k(d_k)\right) =$$

$$\underbrace{1-\sum_{i=1}^{j-1} \widetilde{\rho}_i(d_i) \prod_{k=1}^{i-1} \left(1-\widetilde{\rho}_k(d_k)\right)}_{\prod_{i=1}^{j-1} \left(1-\widetilde{\rho}_i(d_i)\right)} - \widetilde{\rho}_j(d_j) \prod_{i=1}^{j-1} \left(1-\widetilde{\rho}_i(d_i)\right)$$

$$= \prod_{i=1}^{j} \left(1-\widetilde{\rho}_i(d_i)\right).$$

B. PROOF OF THEOREM 2

From (4) it follows that

$$\sum_{d_i < d_j} \rho_i(d_i) = 1 - \underbrace{\frac{\rho_j(d_j)}{\widetilde{\rho}_j(d_j)}}_{\geq 0} \leq 1. \quad (17)$$

Take note that $\rho_j(d_j)/\widetilde{\rho}_j(d_j) \geq 0$ is a direct consequence of $0 < \widetilde{\rho}_j(d_j) < 1$ (by definition) and the relationship from (3) which makes $\rho_j(d_j) \geq 0$, where equality only holds for infinitely many users.

C. PROOF OF LEMMA 1

The probability density function of γ_{j*} can be found from its cumulative distribution function (cdf):

$$\mathrm{cdf}_{\gamma_{j*}}(x) = \Pr[\gamma_{j*} < x] =$$

$$K \int_{\gamma_1=0}^{x} \int_{\gamma_2=0}^{\gamma_1} \int_{\gamma_3=0}^{\gamma_1} \cdots \int_{\gamma_K=0}^{\gamma_1} \prod_{j=1}^{K} \mathrm{pdf}_\gamma(\gamma_j)\, \mathrm{d}\gamma_j\,, \quad (18)$$

$$= K \int_{\gamma_1=0}^{x} \mathrm{pdf}_\gamma(\gamma_1) \left(\mathrm{cdf}_\gamma(\gamma_1)\right)^{K-1} \mathrm{d}\gamma_1\,,$$

from which follows immediately:

$$\mathrm{pdf}_{\gamma_{j*}}(x) = K \cdot \mathrm{pdf}_\gamma(x) \cdot \left(\mathrm{cdf}_\gamma(x)\right)^{K-1}. \quad (19)$$

From (9) and (19) we then obtain:

$$\mathrm{pdf}_{\gamma_{j*}}(x) = \begin{cases} K \cdot x^{K-1} & \text{for } 0 \leq x \leq 1\,, \\ 0 & \text{else.} \end{cases} \quad (20)$$

The average fading power-gain of the scheduled user j_* then computes to:

$$\mathrm{E}\left[\gamma_{j*}\right] = K \int_{x=0}^{1} x^K \,\mathrm{d}x\,, \quad (21)$$

$$= \frac{K}{K+1}. \quad (22)$$

With (11) and (12) the claim of the lemma follows.

D. PROOF OF THEOREM 3

The optimization problem (14) can be solved by treating the number K of terminals as a real-valued variable, and by computing the first derivative of $\overline{\rho_{\mathrm{sched}}}(K)$ with respect to K:

$$F(\widetilde{\rho}, K) \overset{\mathrm{def}}{=} \frac{\partial \overline{\rho_{\mathrm{sched}}}}{\partial K}$$

$$= \frac{-1 + \left(1-\widetilde{\rho}\right)^K \left(1 + \left(1+K\right) \log_{\mathrm{e}}\left(1-\widetilde{\rho}\right)\right)}{\left(K+1\right)^2}. \quad (23)$$

While solving for its roots in closed form appears to be difficult, we can nevertheless find the solution for small values of $\widetilde{\rho}$ in the following way. Let us first obtain a Taylor series expansion of $F(\tilde{\rho}, K)$ around the point $(0, \tilde{K})$:

$$F(\tilde{\rho}, K) \approx F(0, \tilde{K}) +$$

$$+ \begin{pmatrix} \tilde{\rho} \\ K - \tilde{K} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{\partial F}{\partial \tilde{\rho}} \\ \frac{\partial F}{\partial K} \end{pmatrix} \Bigg|_{\substack{\tilde{\rho} = 0 \\ K = \tilde{K}}} +$$

$$\frac{1}{2} \begin{pmatrix} \tilde{\rho} \\ K - \tilde{K} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \frac{\partial^2 F}{(\partial \tilde{\rho})^2} & \frac{\partial^2 F}{\partial \tilde{\rho} \partial K} \\ \frac{\partial^2 F}{\partial K \partial \tilde{\rho}} & \frac{\partial^2 F}{(\partial K)^2} \end{pmatrix} \Bigg|_{\substack{\tilde{\rho} = 0 \\ K = \tilde{K}}} \begin{pmatrix} \tilde{\rho} \\ K - \tilde{K} \end{pmatrix}, \quad (24)$$

which computes to

$$-\tilde{\rho} \frac{-2 + 4K - \left(6 - \tilde{\rho}\right) \tilde{K} + \tilde{\rho} \left(\tilde{K}^3 + 3\tilde{K}^2 - 1\right)}{2 \left(1 + \tilde{K}\right)^3}. \quad (25)$$

The root of (25) with respect to K has to equal $\tilde{K}$, in order to make the Taylor expansion (24) centered around its root:

$$\tilde{K} = \frac{2 + \tilde{\rho}}{4} + \frac{6 - \tilde{\rho}}{4} \tilde{K} - \frac{3\tilde{\rho}}{4} \tilde{K}^2 - \frac{\tilde{\rho}}{4} \tilde{K}^3. \quad (26)$$

Solving (26) for $\tilde{K}$ we find

$$\tilde{K} \in \left\{ -1, \ -1 \pm \sqrt{\frac{2}{\tilde{\rho}} + 2} \right\}. \quad (27)$$

Since we are looking for a positive solution, it follows:

$$\tilde{K} = -1 + \sqrt{\frac{2}{\tilde{\rho}} + 2}. \quad (28)$$

Since $\overline{\rho_{\text{sched}}}(K)$ is continuous in K, and the Taylor series expansion is centered around its root with respect to K, it follows that

$$K_{\text{opt}} \to \tilde{K} \quad \text{as} \quad \tilde{\rho} \to 0. \quad (29)$$

From this and (28) then follows the claim of the Theorem:

$$K_{\text{opt}} \to \sqrt{\frac{2}{\tilde{\rho}}} \quad \text{as} \quad \tilde{\rho} \to 0. \quad (30)$$