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Discrete Abelian Gauge Symmetries

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Abstract

We investigate the possibility to gauge discrete Abelian symmetries. An algebraic approach to understanding general Abelian discrete groups, which govern the coupling structure of a physical theory is presented. In particular, the embedding of a general Abelian discrete group $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ into a general Abelian gauge group $U(1)^k$ via spontaneous symmetry breaking of the continuous group is elaborated in detail. A promising candidate for the embedding of any discrete gauge symmetry is string theory. The algebraic approach to general discrete Abelian groups establishes new possibilities of controlling the coupling structure in string derived model building. We discuss phenomenological consequences of discrete Abelian symmetries arising in string derived MSSM models. A simple discrete R-symmetry, $\mathbb{Z}_4^R$, which contains matter parity as non-anomalous subgroup, is capable of resolving multiple issues such as dimension four and five proton decay as well as the μ-problem.

Zusammenfassung

Wir untersuchen diskrete Abelsche Eichsymmetrien. Eine algebraische Sichtweise allgemeiner diskreter Abelscher Gruppen, welche die Kopplungsstruktur einer physikalischen Theorie bestimmen, wird dargelegt. Insbesondere wird die Einbettung einer allgemeinen diskreten Abelschen Gruppe $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ in eine allgemeine Abelsche Eichgruppe $U(1)^k$, mittels spontaner Symmetriebrechung der kontinuierlichen Gruppe, erarbeitet. Ein viel versprechender Kandidat für die Einbettung einer jeglichen diskreten Eichsymmetrie ist String Theorie. Der algebraische Zugang zu allgemeinen diskreten abelschen Gruppen eröffnet neue Möglichkeiten die Kopplungsstruktur in Modellen, die einen stringtheoretischen Ursprung haben, zu kontrollieren. Wir diskutieren die Auswirkungen von diskreten Abelschen Symmetrien auf die Phänomenologie von MSSM Modellen mit stringtheoretischem Ursprung. Eine simple diskrete R-Symmetrie, $\mathbb{Z}_4^R$, welche 'matter parity' als nicht anomale Untergruppe enthält, ist in der Lage mehrere offene Fragen, wie Dimension vier und fünf Protonzerfall sowie das μ-Problem, zu lösen.

Contents

Chapter 1

Introduction

Symmetries are an obvious concept of nature as we can observe many examples in our everyday life. Since physics is devoted to describing nature as accurately as possible, symmetries consequently have to play a certain role in physics. Yet, far from being limited to occasional appearances in physics, symmetries rather occur to be a guiding principle for theoretical physics. Although every physicist most likely has a fair notion of what the term "symmetry" means, it is useful to be more specific and define it exactly. Loosely speaking one would equalize a symmetry with some kind of invariance. From a rigorous mathematical point of view, the notion of a symmetry is inseparably connected with the concept of a group [1]. For G a group and M a nonempty set, the map

$$\Phi \ : \ G \times M \to M \ ,$$

which shall have the homomorphic properties

$$\begin{aligned} \Phi(e, m) &= m \\ \Phi\left(g, \Phi(h, m)\right) &= \Phi(gh, m) \end{aligned}$$

for all $m \in M$ and $g, h \in G$ defines a *group action* of G on M. Here, e denotes the neutral element of G. If the group action of G leaves some given mathematical structure on M invariant, G is called a *symmetry group* and Φ a *symmetry transformation.*

Symmetries are either continuous or discrete, Abelian or non-Abelian, global or local. In a physical theory they can appear as spacetime symmetries or internal symmetries, they can be equipped with a grading, yet they can even be broken. Most of these attributes provide information about the symmetry group G, or its group action Φ.

Continuous symmetry transformations require the symmetry group to be a Lie-group. Conveniently, these can be described by their Lie-algebra, i.e. by a finite set of generators. Continuous symmetries enjoy a reputation in physics, since invariance of the action induces a conservation law for each symmetry due to Noether's theorem. In contrast, the group action of discrete symmetries cannot be described by continuous transformations. While for a global symmetry the transformation law is the same everywhere in spacetime, a local symmetry allows the transformation law to vary smoothly for different points of the spacetime manifold. This additional freedom can be thought of as a choice of reference frame at every spacetime point and reflects a redundancy in the description of the physical system, which is the basis of a gauge theory. As long as the transformation only depends on spacetime we have an internal symmetry, while a spacetime symmetry shows a non-trivial transformation of the spacetime coordinates themselves.

All of these different types of symmetries eventually played a role in the historical success of theoretical physics, which is a story of unification. Newton's mechanics unified Kepler's description of planetary motion with Galilei's law of falling bodies. This early theory already is dominated by symmetries. It is equipped with the Galilei group as spacetime symmetry group, which induces conservation of important physical quantities, such as momenta, angular momenta and energy.

Later Maxwell's theory of electromagnetism unified electric with magnetic forces. This theory was a remarkable achievement in many ways. First, it is invariant under Poincaré symmetry, an extended spacetime symmetry that paved the way for Einstein's special relativity, renewing the understanding of spacetime. Second, Maxwell's theory exhibits a realization of an Abelian gauge symmetry.

The next step in the unification process combined quantum mechanics, special relativity and the gauge principle to form the first quantum field theory, quantum electrodynamics, which consistently describes the electromagnetic force. Thereby gauge symmetry provides the theory with (bosonic) gauge fields acting as force carriers, here represented by the photon. The idea of generalizing the gauge theory ansatz to non-Abelian symmetry groups, brought up by Yang and Mills, finally provided the necessary ingredient to describe also the strong and the weak force by quantum field theoretical methods. The final step to obtain the remarkably successful Standard Model of elementary particle physics, based on the internal gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, was to break the $SU(2)_L \times U(1)_Y$ part of the gauge group spontaneously down to $U(1)_{em}$ by the Higgs mechanism, giving the observed mass to the weak interaction gauge bosons. However, the success of the Standard Model is not only due to gauge symmetry. Discrete spacetime symmetries have played an important role in the construction of the Standard Model. Experimental evidence of parity violation in weak processes motivated a chiral theory. The Standard Model with its three generations was able to yield a natural explanation for the observed CP violation due to the complex phase of the CKM matrix. Additionally, approximate global flavor symmetries of the Standard Model have helped to understand the phenomenology of baryons and mesons.

Despite its great success there are open questions indicating that the Standard Model is to be considered an effective theory, i.e. a low energy limit of some, yet unknown, more fundamental theory. The large parameter space of the Standard Model, neutrino oscillations and the origin of dark matter and dark energy ask for physics beyond the Standard Model. So does the hierarchy problem, i.e. the stabilization of the weak scale against the Planck scale, and, last but not least, the unsolved problem of quantizing gravity.

A prominent candidate for 'beyond the Standard Model physics' is the realization of supersymmetry, which is a graded symmetry. It is capable to solve the hierarchy problem and consequently has been implemented in a minimal way, resulting in the Minimal Supersymmetric Standard Model (MSSM). The MSSM arguably states the most acknowledged extension of the Standard Model, apparently because of its appealing feature of gauge coupling unification at a scale of 10^{16} GeV. Yet, the MSSM introduces further problems of its own, for instance, renormalizable gauge invariant operators lead to rapid proton decay.

At this point, internal discrete symmetries come into play, which were somewhat neglected in the progress of theoretical physics discussed so far. A simple discrete Abelian $\mathbb{Z}_2$ symmetry, known as matter parity or R-parity, is introduced in order to forbid the rapid proton decay operators. As a pleasant side effect, it yields a dark matter candidate, since now the lightest supersymmetric particle is stable. Due to its appealing effects, this global discrete symmetry was broadly accepted, although it was introduced ad hoc and lacked a theoretical origin.

Furthermore, arguments arose, which disfavor global discrete symmetries in high energy physics. They are expected to be violated by quantum gravity effects, however, it was shown that this cannot happen to discrete gauge symmetries – a new concept of discrete symmetry, which was thought of being a remnant of a spontaneously broken gauge group. In context of the MSSM such discrete gauge symmetries soon were studied from a bottom-up perspective, i.e. disregarding their concrete origin, yet taking seriously anomaly freedom, which is mandatory for gauge symmetries.

The only present candidate for the pending aim of consistently unifying all fundamental forces, including gravity, in a single theory is string theory. Desiring the MSSM as its low energy limit, heterotic string theory appears to be a suitable starting point, since it is automatically equipped with a large enough gauge group to

embed the Standard Model. In particular, orbifold compactifications of heterotic string theory are known for their ability to yield a low energy limit resembling the MSSM.
Yet, such a top-down approach to string derived MSSM models has to reduce the rank of the string gauge group, which typically entails the breaking of an Abelian gauge group $U(1)^k$. Clearly, this scenario potentially gives rise to discrete gauge symmetries, which then provide the MSSM limit with matter parity or yet further discrete symmetries bearing phenomenological consequences.
The main purpose of this thesis is to systematically analyze discrete Abelian gauge symmetries in full generality. Consequently, we will study the discrete symmetry patterns, which result from the spontaneous breaking of a general Abelian group $U(1)^k$. We will then use the acquired algebraic understanding of general discrete Abelian groups to discuss their phenomenological impact in string derived MSSM models.

1.1 Outline of the thesis

The thesis is organized as follows. In the next chapter, chapter 2, we specify the arguments motivating discrete gauge symmetries, given by topological quantum fluctuations and the domain wall problem. Then, we discuss the realization of discrete symmetries as global or gauged symmetries in the beyond the Standard Model physics literature. The implementation of global discrete symmetries will be represented by flavor physics, while we illustrate the discrete gauge approach by MSSM model building. Finally, we show that string theory provides low energy model building with a reasonable origin for discrete gauge symmetries. We will focus particularly on the possibility of discrete gauge symmetries arising from a spontaneously broken Abelian group.
Chapter 3 elaborates in detail the breaking of a general Abelian gauge group $U(1)^k$ down to a remnant Abelian discrete group. We begin with a simple example, the breaking of a single $U(1)$ gauge symmetry, which shows us how a discrete gauge symmetry can continue to remain. Next, we put this basic $U(1) \to \mathbb{Z}_q$ example on a theoretical footing. Then, we discuss the problem of generalizing to an arbitrary Abelian gauge group $U(1)^k$. In order to resolve this issue, we introduce the geometrical concept of the *charge lattice*, which will guide us towards a deep algebraic comprehension of discrete Abelian groups. In particular, the obvious notion of a lattice basis change in the geometrical charge lattice picture suggests an algebraic freedom of unimodular transformations. Via the *Smith normal form*, this leads to a description of the remnant discrete symmetries in terms of the invariant factor

decomposition of finitely generated Abelian groups. After visualizing our results by means of an illustrative example, we close the chapter with a comprehensive discussion of the associated algebraic aspects of discrete Abelian groups.

Chapter 4 deepens our understanding of discrete Abelian groups by focusing on redundant and equivalent configurations. A redundancy occurs if the discrete symmetry allowed by the vacuum of the spontaneously broken theory is not fully realized by the remaining field content. We explain how to calculate the true discrete symmetry group of the theory. By agreeing on the invariant factor decomposition, we have eliminated equivalent descriptions of the discrete Abelian group because of isomorphisms; yet, further freedom concerning the discrete charge assignment due to automorphisms remains. The automorphism group of discrete Abelian groups is known; we review its construction and give a simple, but non-trivial example by means of the automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_4$. We then show that the coupling structure of the remaining fields, which is governed by the discrete symmetries, can be expressed by means of linear congruence equations and we prove their invariance under automorphisms. We proceed by addressing another freedom in assigning discrete charges for theories with unbroken $U(1)$ factors, in the MSSM literature well known as hypercharge shifts. Finally, we illustrate the methods of this chapter by means of a concrete example.

With chapter 5 we turn the discussion towards phenomenological implementations and consequences of discrete Abelian symmetries. First, we review discrete anomaly constraints, which have to be fulfilled for any discrete gauge symmetry. Next, we discuss discrete R-symmetries, a possibility of any supersymmetric theory. We then explain the identification of matter- or R-parity, which is of great importance for MSSM model building. The algebraic understanding of discrete Abelian groups resolves this issue rigorously. Finally, we comment on F- and D-term constraints, which restrict the vacuum expectation value (VEV) assignment in supersymmetric theories if supersymmetry is to maintain unbroken.

In chapter 6 we discuss a string derived MSSM model, which is equipped with a discrete Abelian R-symmetry $\mathbb{Z}_4^R$. We begin with a short synopsis of orbifold compactifications of heterotic string theory and their ability to produce the exact MSSM spectrum. Enhanced discrete symmetries are an appealing possibility to ameliorate the phenomenology of such models. We address the impact of a simple $\mathbb{Z}_4^R$ symmetry, which is “anomalous”, i.e. its anomaly is canceled by the Green-Schwarz mechanism. This entails the breaking of the discrete symmetry at the non-perturbative level. Yet, a $\mathbb{Z}_2$ subgroup serving as matter parity remains unbroken, which resolves dimension four proton decay. The $\mathbb{Z}_4^R$ symmetry forbids dimension five proton decay and the μ-term at the perturbative level. Both quantities become reintroduced

by non-perturbative terms, however, highly suppressed. Thus the leading effect for proton decay accounts for dimension six operators and the μ-term is expected to be near the electroweak scale.
The last chapter contains our conclusions. The appendix provides fundamental definitions and theorems of basic algebra and group theory.
A short note on our notation: in order to improve readability, we parenthesize single upper indices to emphasize those from ordinary powers. Over repeated indices is to be summed if not stated otherwise. Boldface symbols $\boldsymbol{x}$ denote column vectors, while $\boldsymbol{x}^T$ indicates a row vector.

1.2 List of publications

Parts of this work have been published in refereed scientific journals, as listed below.

- Björn Petersen, Michael Ratz and Roland Schieren, "Patterns of remnant discrete symmetries", JHEP08(2009)111.
- Rolf Kappl, Björn Petersen, Stuart Raby, Michael Ratz, Roland Schieren and Patrick K. S. Vaudrevange, "String-derived MSSM vacua with residual R symmetries", arXiv:1012.4574 [hep-th], to appear in Nuclear Physics B.

The main idea of chapter 3 and section 4.1 have been presented in [2]. Section 4.2 and section 5.2 as well as the model of chapter 6 were discussed in [3].

Chapter 2

Global versus local discrete symmetries

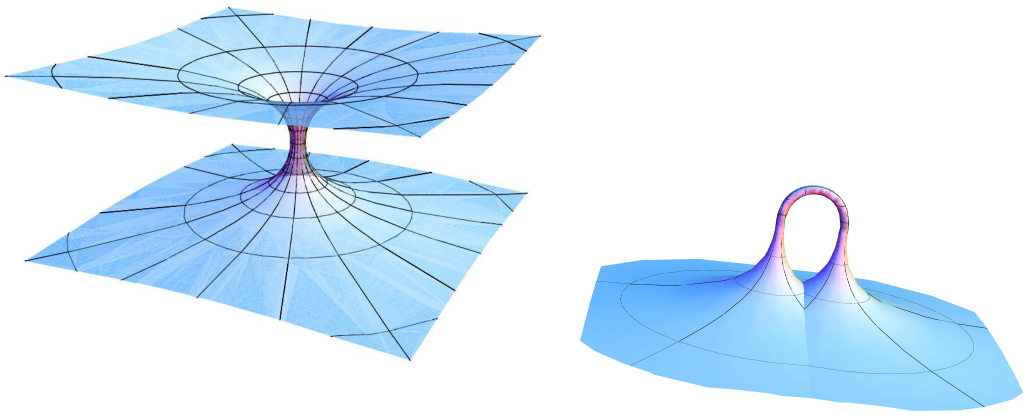
We start by summarizing why local discrete symmetries are favored against global discrete symmetries and discuss their perspectives in model building, giving ongoing research examples of fields where discrete symmetries of global and local type are implemented. As mentioned above, there are certain arguments and motivations to consider gauged discrete symmetries.

- Topological fluctuations of spacetime suggest violations of any global symmetry.
- Stable domain walls originating from global discrete symmetries in the early universe are phenomenologically problematic.
- String theory provides us with a large rank gauge group, giving plenty of possibilities for remnant gauged discrete symmetries.

Those arguments cannot exclude the existence of global symmetries in real physics definitely, but they strongly suggest a certain antipathy against global symmetries. The last point will motivate us to study discrete Abelian gauge symmetries in full generality, closing a gap in the literature. So far, the discussion of discrete Abelian groups was typically constricted to cyclic groups, a sub-category of general Abelian discrete groups.

2.1 Topological fluctuations

For any theory of quantum gravity to allow the description of black holes, the topology of spacetime needs to deviate from flat space. It is assumed that thus all possible spacetime topologies should be allowed [4], including non simply connected ones. For instance, black hole evaporation would be accompanied by a closed universe branching off from asymptotically flat spacetime. Such a closed universe, i.e. a compact manifold without boundary, could then connect to another universe (see figure 2.1(a)), or to the same one, therefore forming a handle, which results in a fundamental change of topology: from simply to non-simply connected (see figure 2.1(b)). These topology changing closed universes, also called wormholes, are



(a) Wormhole connecting two asymptotic flat regions of spacetime

(b) Spacetime handle (sketch)

Figure 2.1: Closed universes induce topological variations in spacetime.

known to be instable macroscopically [5]. However, they could contribute on the virtual level of quantum fluctuations – therefore, one speaks of topological fluctuations.

Now, if a wormhole branches off carrying away some amount of any conserved global charge, the charge conservation is violated for an observer in the 'parent' universe. Thus, a low energy effective theory will contain non-renormalizable operators, which break any global symmetry at some order in perturbation theory. These might be sufficiently suppressed, although for theoretical extensions, which lead in the direction of a fundamental theory, global symmetries should be considered as rather unreliable [6, 7].

However, this argument does not apply to a gauged symmetry; a closed universe needs to carry trivial charge under any gauge symmetry. This can be understood as follows. As an example, consider electric charge Q; via Gauss's law one can express it as a surface integral over the boundary S

$$Q = \int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} \ , \tag{2.1}$$

with electric field $\boldsymbol{E}$ and infinitesimal area element $\mathrm{d}\boldsymbol{A}$. But the wormhole has no boundary. Thus, its electric charge has to vanish and therefore we have charge conservation in each asymptotic region. The argument holds for any Yang-Mills gauge group [8], and even for discrete gauge symmetries [9, 10]. The notion that any discrete symmetry cannot be global, but has to be gauged in order to be compatible with quantum gravity, has been corroborated very recently in [11].

2.2 Domain walls

There is another, cosmological argument that calls global discrete symmetries into question: a spontaneously broken global discrete symmetry in the early universe can lead to a domain structure of the universe separated by stable domain walls. These are afflicted with surface mass density and thus would result in non observed anisotropies of the universe [12]. In contrast to ferromagnetic domain structures, domain walls do not emerge because of a favorable energetic state, but due to the degeneracy of the vacuum state. There is no reason for causally separated regions of the early universe to settle in any particular degenerate vacuum state. Hence, causally disconnected regions are expected to take different vacua, therefore building up the domain structure.

In contrast, discrete gauge symmetries – initially – are free of such problems. This is because in the gauged case, the degeneracy of the vacuum reflects the redundancy in the description of the system. In other words, the degenerate vacuum states become identified; that is, connected by a gauge transformation. Thus, once the spontaneous breaking of the discrete gauge symmetry occurs each domain is in the same physical state, since it may not depend on the gauge degree of freedom. So much for the argument, a concrete implementation has been established as well. A discrete gauge symmetry should have a continuous embedding. The continuous symmetry is broken at a high scale, resulting in the appearance of cosmic strings, while the breaking of the remnant discrete gauge symmetry at another scale gives rise to domain walls a priori. However, in the case of gauged discrete symmetries, the domain walls will be bounded by the strings, which leads to their vaporization [13, 14]. Yet, also for discrete gauge symmetries the domain wall problem becomes reintroduced if inflation occurs between the two breaking scales, which appears to be a likely scenario [13].

Nevertheless, we should remark that global discrete symmetries suffer from the domain wall problem generically, while for discrete gauge symmetries it is reintroduced only under certain assumptions.
A possibility to avoid the domain wall problem is to consider anomalous discrete symmetries, since in that case the degeneracy of the vacuum states becomes suspended by an energetic gap, which entails the annihilation of the domain structure [13]. This also holds for "anomalous" discrete gauge symmetries, i.e. symmetries only appearing to be anomalous at the perturbative level, yet their anomaly is canceled by the Green-Schwarz mechanism. Such a non-perturbative cancellation, facilitated by string theory, gives rise to an approximate discrete symmetry, broken by exponentially suppressed terms [15]. Those also generate an energy splitting between the domains, thus allowing for their annihilation under certain conditions, as discussed in [16].

2.3 Discrete symmetries in bottom-up physics

From a model building perspective, discrete symmetries constitute a popular approach to resolve phenomenological drawbacks. Typically, discrete symmetries tend to be imposed ad hoc in low energy effective theories, without a particular theoretical motivation of those symmetries. Limiting ourselves to 'beyond the Standard Model' physics, still a variety of issues in bottom-up models are attempted to be solved by discrete symmetries. Examples are multi-Higgs models [17, 18, 19], the strong CP problem [20, 21, 22], flavor physics and the MSSM. Below, we will present the latter two as representatives where discrete symmetries are introduced as global symmetries or where their gauged origin was studied extensively in the literature.
The question whether discrete spacetime symmetries like CP might have a gauged origin has been addressed in [23]. There, it has been argued that quantum gravitational violations of CP as a global symmetry would conflict with the smallness of the electric dipole moment of the neutron. It is shown that a "CP equivalent" can be gauged in the context of dimensional compactification, e.g. superstring theory. It then becomes spontaneously broken at a scale lower than 10^9 GeV, which preserves a tiny dipole moment and accounts for CP violation as well as resolves the strong CP problem. In the following, we will focus on internal symmetries.

2.3.1 Flavor physics

One major branch of particle physics where discrete symmetries have been utilized extensively is flavor physics. In the literature one can find a vast number of attempts

for (partially) solving the flavor puzzle, i.e. the quest for the origin of fermion masses and mixings, their hierarchies, and the difference of the flavor parameters in the neutrino sector. Unfortunately, most of those attempts do not consider such high energy arguments, which have been introduced above; that is, discrete flavor symmetries usually do not exhibit a gauged embedding but are imposed ad hoc. Some of those scenarios might arise as effective theories, where operators violating the global discrete symmetries are sufficiently suppressed, while others might not. Of course there are counterexamples, where these questions are addressed [24], in fact.
Why are discrete symmetries so popular within the context of flavor? In principle, the Froggatt-Nielsen mechanism [25] yields a promising way to explain fermion mass matrices and mixings by means of a continuous Abelian symmetry. By breaking a (gauged) flavor $U(1)$ at a high scale Λ_B close to, say, the Planck scale, a small parameter $\frac{\Lambda_B}{M_P}$ is naturally generated. The differences of magnitude of fermion mass matrix entries then are due to different powers of this small parameter, which are connected to the charge assignment under the broken Abelian group. However, Abelian symmetries that generate a realistic mass pattern often appear to be anomalous [26], at least naively. Yet, this is not necessarily in contradiction with an embedding of the Froggatt-Nielsen mechanism into a string derived model due to Green-Schwarz anomaly cancellation [27, 28, 29]. Yet, an Abelian symmetry cannot account for bi-tri-maximal mixing schemes in the lepton sector [30], which seems to be favored by recent neutrino oscillation data.
Ignoring the high energy arguments against global symmetries, discrete symmetries are rather appealing from a bottom-up perspective, since one does not need to worry about stringent anomaly constraints or Goldstone bosons while breaking the discrete symmetry. That horizontal symmetries have to be broken at some point has been proven in [31, 32] for the quark sector, else degenerate quarks or zero mixing angles would emerge. These authors also pointed out the value of Abelian discrete horizontal symmetries, giving non-trivial examples based on general discrete Abelian groups, i.e. non-cyclic ones.
In the lepton sector the implementation of discrete non-Abelian groups was studied extensively, see e.g. [33, 34, 35, 36, 37] or [38, 39] for reviews on the subject, however, attempts to provide a high energy origin of such non-Abelian discrete flavor symmetries [40] are rather few and far between.

2.3.2 The MSSM

The supersymmetrization of the Standard Model arranges each Standard Model field and its superpartner in G_{SM} supermultiplets and adds an additional Higgs doublet. In table 2.1, the matter and Higgs superfields with generation index $i = 1, 2, 3$ are presented.

Q_i	$\bar{U}_i$	$\bar{D}_i$	L_i	$\bar{E}_i$	H_d	H_u
$(\mathbf{3},\mathbf{2})_{\frac{1}{3}}$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{4}{3}}$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$(\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_{2}$	$(\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{2})_{1}$

Table 2.1: Matter and Higgs superfields and their $SU(3)_C \times SU(2)_L$ quantum numbers. The subscripts denote hypercharge Y.

Here we have expressed the entire matter field content as left chiral superfields; that is, right handed fields as left handed antifields. The renormalizable superpotential allowed by gauge invariance consists of the following terms [41]

$$W = h^E_{ij}\, L_i H_d \bar{E}_j + h^D_{ij}\, Q_i H_d \bar{D}_j + h^U_{ij}\, Q_i H_u \bar{U}_j + \mu\, H_d H_u \tag{2.2a}$$

$$+ \lambda_{ijk}\, L_i L_j \bar{E}_k + \lambda'_{ijk}\, L_i Q_j \bar{D}_k + \lambda''_{ijk}\, \bar{U}_i \bar{D}_j \bar{D}_k + \kappa_i\, L_i H_u\ . \tag{2.2b}$$

Yet, the terms in the second line (2.2b) violate lepton number, except for the λ'' term, which violates baryon number. This leads to rapid proton decay, which is phenomenologically unacceptable. Therefore, the MSSM is equipped with a discrete $\mathbb{Z}_2$ symmetry, either R-parity R_p acting on the component fields or equivalently matter parity $\mathcal{M}_p$ acting on the superfields, which both forbid all terms in (2.2b). Under R_p all Standard Model fields are even and all superpartners odd, while for matter parity all matter superfields are odd and Higgs as well as vector superfields are even. A pleasant side effect of the discrete symmetry is that the lightest supersymmetric particle (LSP) is stable and yields a natural cold dark matter candidate. Furthermore, superpartners can only be produced in even numbers.

However, one should be aware that for rapid proton decay to be forbidden, either lepton or baryon number violating terms need to vanish in (2.2b). Yet, within such R-parity violating scenarios, the stability of the LSP is lost and there are stringent bounds on the size of those R_p violating terms due to baryogenesis [41].

Even if the whole line (2.2b) is forbidden, non-renormalizable operators of dimension five and six are dangerous for the experimental bounds on proton decay [42].

The idea of gauged discrete symmetries has been taken seriously by Ibáñez and Ross [43, 44], who attempted to classify the anomaly free cyclic discrete gauge symmetries of the MSSM. They found an alternative to R-parity, baryon triality B_3, which forbids baryon number violating dimension four and five operators and in turn allows for R_p violating ones. Finally, Dreiner, Luhn and Thormeier [45] suggested proton-hexality P_6, which is just the direct product of $\mathcal{M}_p$ and B_3, isomorphic to a cyclic group of order six $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Thus P_6 cherry-picks the appealing properties of

both, matter parity and baryon triality.
It was then tried to obtain such discrete gauge symmetries for the MSSM from a top-down perspective [46], also in context of Grand Unification [47, 48], where proton-hexality turned out to be GUT incompatible [49, 50]. Discrete R-symmetries tend to be more agreeable with the appealing idea of Grand Unification [51]. Under certain assumptions, it was shown that there is a unique GUT compatible discrete symmetry of the MSSM: $\mathbb{Z}_4^R$ [52]. One of these assumptions is a perturbatively forbidden μ-term, which becomes reintroduced at the non-perturbative level, therefore being well suppressed. In order for this to happen, the discrete symmetry forbidding the μ-term has to be "anomalous", i.e. the anomaly is canceled by a Green-Schwarz mechanism. We will demonstrate a similar scenario below, presenting a string derived $\mathbb{Z}_4^R$ symmetry with promising phenomenological features.

2.4 Discrete symmetries from string theory

From a string inspired top-down approach to phenomenology, the need for discrete symmetries in order to suppress proton decay was stressed early by Witten [53]. There, it was also noted that such discrete symmetries have to emerge out of string theory rather than being introduced ad hoc. However, the utilization of discrete symmetries naturally arising by the breakdown of the large rank gauge group, present in certain string theories, was somewhat limited to its non-Abelian parts.
In this work, we will pursue another idea. We will study the remnant discrete gauge symmetries, which potentially arise by breaking the Abelian parts of the gauge group. The breaking of the rank 16 gauge group $E_8 \times E_8$ or $SO(32)$, provided by heterotic string theory, down to the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$ has to be rank reducing, which naturally entails the spontaneous breaking of some amount of $U(1)$ factors [54]. Our discussion focuses on orbifold compactifications of heterotic string theory due to their capabilities to draw the bridge to low energy phenomenology [55, 56, 57, 58]. Compactifying on an orbifold breaks the gauge group, yet preserves its rank. Hence, such a general Abelian gauge group, $U(1)^k$, typically remains to be broken via the Higgs mechanism at a later stage [59].
In what follows, we will systematically elaborate the remnant Abelian discrete symmetry group arising by spontaneously breaking a general Abelian gauge group $U(1)^k$. Of course, the breaking of non-Abelian gauge groups can yield further non-Abelian discrete symmetries and even an additional Abelian discrete symmetry group. However, the systematical approach concerning non-Abelian discrete gauge symmetries

seems involved[1], since the breaking mechanism depends on the representation of the non-Abelian group. Our approach is free of this issue, because all irreducible representations of finite Abelian groups are 1-dimensional [54]. If remnant discrete Abelian groups of non-Abelian gauge groups are found for any concrete model, howsoever, they can easily be incorporated into our approach.

[1] Except the trivial case of a remnant discrete center (see appendix A.3) of a non-Abelian group, which will be left invariant by any VEV in the adjoint representation, since the elements of the adjoint are equal to elements of the group itself and the center is defined to commute with all group elements. For a VEV out of an arbitrary irreducible representation transforming under $SU(N)$ as a (m, n) tensor, it is trivial that the center $\mathbb{Z}_N$ is left invariant if $m - n = 0 \mod N$, as can also be found in [60].

Chapter 3

Patterns of Abelian discrete gauge symmetries

This chapter addresses the breaking of a general Abelian gauge group $U(1)^k$ down to a discrete subgroup. Since the only Abelian discrete subgroup of a $U(1)$ is a cyclic (see appendix A.4) $\mathbb{Z}_n$ group, it is not surprising that the Abelian discrete subgroup of $U(1)^k$ will be representable by a direct product of cyclic groups. However, not every product of cyclic groups is itself isomorphic to a cyclic group, so that new structures arise.

First, an illustrative example will show how to break an Abelian single $U(1)$ gauge symmetry so that a discrete group remains. This example is then put on a theoretical footing that allows to calculate systematically the discrete symmetry group respected by the vacuum of the broken phase. Generalizing to the case $U(1)^k$ will lead to the concept of charge lattices, a suggestive picture manifesting the metamorphoses, which an arbitrary Abelian discrete symmetry group is able to undergo. Finally, we will fix the resulting ambiguities in terms of the Smith normal form, and shed light on the obtained symmetry patterns from an algebraic point of view.

3.1 Introductory example

In order to get a notion of how to break a continuous symmetry down to a discrete symmetry in a physical theory, the following intuitive simplest example [9, 61] is helpful. Consider two complex scalar fields ψ and ϕ of different charge under a common $U(1)$ gauge symmetry

$$\psi \mapsto e^{\mathrm{i}\alpha(x)}\psi \ , \tag{3.1}$$

$$\phi \mapsto e^{\mathrm{i}q\alpha(x)}\phi \ . \tag{3.2}$$

Besides the standard renormalizable Lagrangian for both of the scalars (and a kinetic term for the gauge fields), we can write down the following gauge invariant interaction terms with coupling constants β, γ, δ

$$\beta\, \phi\phi^*\, \psi\psi^* \,, \qquad \gamma\, \psi^q \phi^* \,, \qquad \delta\, \psi^{*q}\phi \tag{3.3}$$

and powers of these. Now, assume the $U(1)$ symmetry is spontaneously broken due to a vacuum expectation value $\langle\phi\rangle = v$, developed by the q charged field ϕ,

$$\phi(x) \xrightarrow{\text{SSB}} v + \phi^{'}(x) \ . \tag{3.4}$$

As a consequence, the Lagrangian of the broken theory contains the terms $v\psi^q$ and $v\psi^{*q}$, which are obviously no more $U(1)$ invariant, though, they still are manifestly invariant under the transformation

$$\psi \mapsto e^{\frac{2\pi \mathrm{i}}{q}\, m}\psi \ , \quad m \in \mathbb{Z} \ , \tag{3.5}$$

which corresponds to the residual discrete Abelian symmetry $\mathbb{Z}_q$.
For a low energy observer having no knowledge of the field ϕ, (3.5) appears to be an ordinary global discrete symmetry. It is its local origin why it is called discrete gauge symmetry.

3.2 Systematical approach

We will now put this mechanism, which we understood by looking at the explicit coupling structure so far, on a neat theoretical footing. Let us reconsider the simple case of a single $U(1)$ local theory from above. In order to break it, let the q charged[1] field attain a VEV, just as in (3.4). To which subgroups does the VEV break the theory? That is a common problem in the quantum field theory literature, it has to be checked whether there are subgroups, which leave the VEV invariant. Usually one is looking for invariant generators $\alpha^{(i)}$ at the infinitesimal level

$$\alpha^{(i)}\langle\phi\rangle = 0 \ , \tag{3.6}$$

i.e. for continuous invariant subgroups. In the present case of a single $U(1)$ this reads $\alpha v = 0$, which of course can only be satisfied trivially as $U(1)$ has no continuous subgroups. Yet, one could imagine that the VEV respects some discrete subgroup. Therefore, we have to switch to the finite level of group elements, which promotes

[1] For (notational) convenience we will consider integer charges throughout this work.

(3.6) to

$$e^{\mathrm{i}q\alpha}v = v \ . \tag{3.7}$$

This equation obviously has the solutions

$$\alpha = \tfrac{2\pi m}{q} \ , \quad m \in \mathbb{Z} \tag{3.8}$$

so that some field ψ with charge p under the very same $U(1)$ still has the invariant transformation

$$\psi \mapsto e^{2\pi \mathrm{i} \frac{p}{q} m}\psi \tag{3.9}$$

after the breaking. Since the discrete charge p is mapped onto itself for some integer values of m, it becomes defined modulo n (see appendix A.2), i.e. we have a residual $\mathbb{Z}_n$ symmetry, where $n = q$ at first sight. However, if p and q have a greatest common divisor, $\mathrm{GCD}(p, q) \neq 1$, i.e. if p and q are not coprime, the symmetry of the theory will certainly be reduced to $n' = \frac{q}{\mathrm{GCD}(p,q)}$ with discrete charge $p' = \frac{p}{\mathrm{GCD}(p,q)}$. Such cases of reduced or redundant discrete symmetries will be discussed extensively in the next chapter. Note that so far the maximal value of n is limited by the charge of the VEV, i.e. if one wants to have a remnant $\mathbb{Z}_q$ symmetry one should give a charge q field a VEV; this is a well known statement about cyclic discrete gauge symmetries.

In principle, we can make the same ansatz for a general Abelian gauge symmetry, however, we will see that the discrete symmetry group then is not that easy to read off.

3.3 General Abelian discrete gauge symmetries

We will now extend this idea and seek for remnant discrete subgroups of general Abelian $U(1)^k$ theories. First, let us specify a generic field content. Fields attaining non-zero VEVs will be called $\phi^{(i)}, i = 1, \ldots, a$, whereas remaining 'matter' fields will be named $\psi^{(l)}, l = 1, \ldots, b$. Now that we have k $U(1)$'s the fields transform by a linear combination of all the $U(1)$ factors

$$\phi^{(i)} \mapsto e^{\mathrm{i}q_j(\phi^{(i)})\alpha^{(j)}}\phi^{(i)} \ , \quad j = 1, \ldots, k \tag{3.10}$$

$$\psi^{(l)} \mapsto e^{\mathrm{i}q_j(\psi^{(l)})\alpha^{(j)}}\psi^{(l)} \ , \quad j = 1, \ldots, k \tag{3.11}$$

with continuous generators $\alpha^{(j)}$ and associated charges $q_j(\phi^{(i)}) = q_{ij}$. Thus, invariance of the VEVs in this setup reads

$$e^{\mathrm{i}q_j(\phi^{(i)})\alpha^{(j)}}v^{(i)} = v^{(i)} \ , \tag{3.12}$$

which tells us

$$q_{ij}\alpha^{(j)} = 2\pi m_i\,, \quad m_i \in \mathbb{Z}\,. \tag{3.13}$$

We can write this set of linear equations in matrix form

$$\begin{pmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & & \vdots \\ q_{a1} & \cdots & q_{ak} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = 2\pi \begin{pmatrix} m_1 \\ \vdots \\ m_a \end{pmatrix} \,. \tag{3.14}$$

Note that the matrix $Q_\phi = (q_{ij})$, composed of the $U(1)$ charges belonging to the $\phi^{(i)}$ fields, is quadratic, i.e. $k = a$, if there are as many linearly independent VEVs as $U(1)$ factors. Let us consider this special case for now, at a later stage (section 3.7) it will be easy to include the deviant situations where $k \neq a$. Rewriting equation (3.14) in terms of Q_ϕ, to which we will simply refer as the 'charge matrix' in the following, we obtain the compact form

$$Q_\phi\,\boldsymbol{\alpha} = 2\pi\,\boldsymbol{m}\,, \tag{3.15}$$

where we have grouped the $U(1)$ generators $\alpha^{(j)}$ as well as the integers m_i in corresponding vectors $\boldsymbol{\alpha}$ and $\boldsymbol{m}$. Benefiting from the restriction of a quadratic charge matrix, due to $k = a$, we can solve for $\boldsymbol{\alpha}$ by means of the inverse[2] charge matrix Q_ϕ^{-1}. In this context, remember the classical adjoint from linear algebra, which fulfills

$$Q\,\mathrm{adj}(Q) = \det(Q)\,\mathbb{1}\,. \tag{3.16}$$

It is of importance once matrices are defined over a ring (e.g. $\mathbb{Z}$) rather than a field. In that case, it serves as an almost inverse for matrices with $\det(Q) \neq 1$. Yet, a non integer inverse of Q_ϕ is unproblematic here. In fact, rational values for the $\boldsymbol{\alpha}$ entries are desirable, in order to obtain non-trivial discrete symmetries. Thus, we can use (3.16) to write

$$\boldsymbol{\alpha} = 2\pi \underbrace{\frac{\mathrm{adj}(Q_\phi)}{\det(Q_\phi)}}_{Q_\phi^{-1}}\,\boldsymbol{m}\,. \tag{3.17}$$

Note that at this point, the $\alpha^{(j)}$ take discrete values, just as in (3.8), which makes them "discrete generators", i.e. generators of remnant discrete symmetries. In other words, the former $U(1)$ generators become fixed by the invariance claim of the VEVs. But the actual discrete symmetry of the theory will be determined by the transfor-

[2] Since the $U(1)$'s are supposed to be independent and the rows of the charge matrix are linearly independent by assumption, Q_ϕ has full rank and thus the inverse of Q_ϕ exists.

mation of the remaining field content

$$\psi^{(l)} \mapsto e^{2\pi\mathrm{i}\,\boldsymbol{q}^T(\psi^{(l)})\,\frac{\mathrm{adj}(Q_\phi)}{\det(Q_\phi)}\,\boldsymbol{m}}\psi^{(l)} \ . \tag{3.18}$$

Let us examine this condition more closely for l fixed. One observes $k = a$ cyclic discrete symmetry groups $\mathbb{Z}_{n_i}$, which are distinguished by the m_i, where the n_i are given by the denominators of each summand in the exponential of (3.18). Those cyclic factors span the overall Abelian discrete symmetry group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. If some n_i happen to be one, the corresponding trivial[3] symmetry groups $\mathbb{Z}_1$ is to be discarded. Since the entries of the matrix $\mathrm{adj}(Q_\phi)$ are in $\mathbb{Z}$ again, $\mathbb{Z}_{\det(Q_\phi)}$ constitutes a (naive, as will become clear later) upper limit for each of the cyclic groups. However, a smaller symmetry pattern can (and will) be encountered in the following cases:

- $q_i(\psi^{(l)})$ cancels against $\det(Q_\phi)$
- the columns of $\mathrm{adj}(Q_\phi)$ contain common factors, which cancel down $\det(Q_\phi)$.

Although the first point appears to be rather intelligible (we will discuss it thoroughly in section 4), the latter one is cumbersome to explore in general. Furthermore, we do not know how to identify remnant discrete symmetries for $k \neq a$ within this framework.

3.4 Geometrical perspective

Due to the issues encountered above, let us change our point of view slightly and focus on a geometrical approach. Apart from gaining an intuitive visualization of the problem via a lattice description, the geometrical interpretation will draw the bridge to an algebraic formulation, provided by the theorem of finitely generated Abelian groups.

The idea of approaching arithmetic, algebraic or number theoretical problems by geometrical means goes back to Minkowski [62], founding the "Geometry of Numbers". The following states a very basic but illustrative example, how to solve an arithmetical problem via a geometrical interpretation. Which integers n can be written as the sum of two integer squares? That is, for $n \in \mathbb{Z}$ fixed and $p, q \in \mathbb{Z}$, solve the equation

$$n = p^2 + q^2 \ . \tag{3.19}$$

[3] Since it only consists of the identity.

From a geometrical point of view, the equation $n = x^2 + y^2$ with $x, y \in \mathbb{R}$ corresponds to the circle of radius $\sqrt{n}$ and center $(0, 0)$ in the $\mathbb{R}^2$ plane. By charting all integer coordinate points in $\mathbb{R}^2$, which gives a lattice pattern, the solutions of equation (3.19) can be read off immediately. The (p, q) are just the coordinates of the lattice points lying on the circle (see figure 3.1).

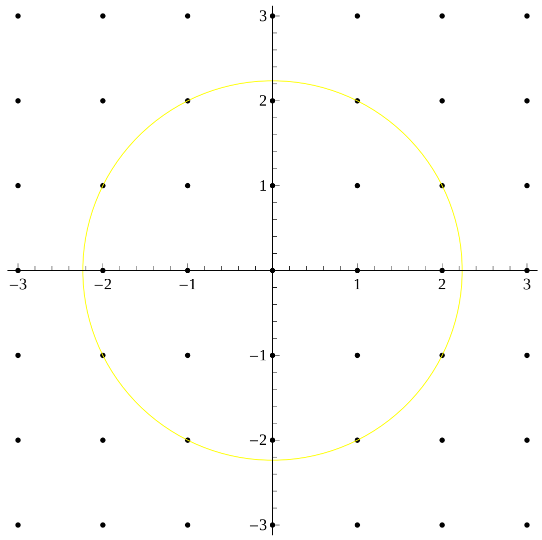


Figure 3.1: Solving (3.19) geometrically: for $n = 5$, eight different integer solutions can be read off.

Here, the fundamental concept [63] of the "Geometry of Numbers" has been used intuitively: the concept of a lattice. Instead of dealing with the somewhat cumbersome ring of integers by itself, we rather considered the integers as a subset of an appropriate embedding (here $\mathbb{R}^2$), which is equipped with vector space properties.

Definition 1 *Let $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$ be linearly independent vectors in k-dimensional real Euclidean space, then the set of points*

$$\{m_1\boldsymbol{\lambda}^{(1)} + \cdots + m_k\boldsymbol{\lambda}^{(k)} \ : m_i \in \mathbb{Z}\} \tag{3.20}$$

is called a lattice with basis $\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(k)}$.

3.4.1 The charge lattice

Following this concept, we may consider $\mathbb{R}^k$ with the set of charge vectors $\{\boldsymbol{q}(\phi^{(i)})\}$ defining a k-dimensional[4] lattice. So far, we studied the case $k = a$ of a square

[4] Or lower dimensional in case of linear dependencies among the charge vectors.

charge matrix Q_ϕ with full rank. In this case the charge vectors $\{\boldsymbol{q}(\phi^{(i)})\}$ span a basis $\{\boldsymbol{\lambda}^{(i)}\}$ of the lattice, i.e. the charge lattice is given by

$$\{m_i\boldsymbol{\lambda}^{(i)} \ : \ m_i \in \mathbb{Z}\} = \{m_i\boldsymbol{q}(\phi^{(i)}) \ : \ m_i \in \mathbb{Z}\} \ . \tag{3.21}$$

Hence, the basis vectors $\boldsymbol{\lambda}^{(i)T}$ of the charge lattice equal the rows of the charge matrix Q_ϕ. Therefore, the columns of the inverse Q_ϕ^{-1} define the dual (also called reciprocal or polar) lattice basis $\{\boldsymbol{\lambda}_i^*\}$, such that

$$\boldsymbol{\lambda}^{(i)T}\boldsymbol{\lambda}_j^* = \delta_j^{(i)} \ . \tag{3.22}$$

With the dual lattice at hand, the transformation of the remaining fields $\psi^{(l)}$ can be recast in terms of the charge lattice. The exponent of (3.18) then reads

$$\mathrm{i}\boldsymbol{q}^T(\psi^{(l)})\,\boldsymbol{\alpha} = 2\pi\mathrm{i}\,\left(\boldsymbol{q}^T(\psi^{(l)})\boldsymbol{\lambda}_1^* m_1 + \cdots + \boldsymbol{q}^T(\psi^{(l)})\boldsymbol{\lambda}_a^* m_a\right) \ . \tag{3.23}$$

Thus, in order to have a discrete symmetry factor $\mathbb{Z}_{n_i}$, one is seeking for situations where some charge lattice coordinate of a remaining field $\boldsymbol{q}(\psi^{(l)})\boldsymbol{\lambda}_i^*$ is rational, i.e. the charge vector $\boldsymbol{q}^T(\psi^{(l)})$ does not lie on the charge lattice. However, any coupling $(\psi^{(1)})^{x_1}\cdots(\psi^{(b)})^{x_b}$ of the broken phase, with field powers $x_l \in \{0,1,2,\cdots\}$, has to lie on a lattice node – this is the translation of gauge invariance in the unbroken theory. Comparing the fraction structure of (3.23) for all l yields a direct product of cyclic groups $\prod_{i=1}^k \mathbb{Z}_{n_i'}$ as discrete symmetry group. In this process the $\mathrm{GCD}(q_i(\psi^{(1)}),\ldots,q_i(\psi^b),n_i)$ is to be canceled for each i, resulting in a common denominator n_i' for all values of l. This corresponds to evaluating (3.18) "by hand" and in principle results in a valid discrete symmetry group of the broken theory. It is free of uncontrollable cancellations worrying us at the end of section 3.3, since the structure of Q_ϕ^{-1}, and thus $\mathrm{adj}(Q_\phi)$, now is encoded in the dual basis. However, this manual method still is incompatible with the general case $k \neq a$ of non square charge matrices. This is because for linearly dependent VEV setups, the charge vectors do not form a basis of the charge lattice. A proper basis can be found by lattice reduction methods [64], which perform a change of the lattice basis.

Do we actually have the freedom to change the lattice basis? Recapitulating the procedure above, we expect to obtain a different product of cyclic groups for a different (dual) basis of the charge lattice, since the denominator structure of (3.23) may change. Yet, this is not an immediate contradiction, since discrete Abelian groups have various isomorphic descriptions, which will become clear shortly.

3.4.2 Lattice bases and their unimodular transformations

Such a lattice basis change, as pictured in figure 3.2, is well known [65] to be performed by unimodular transformations

$$\boldsymbol{\lambda}'^{(i)} = M_{ij}\boldsymbol{\lambda}^{(j)} \ . \tag{3.24}$$

Such a transformation is defined as follows:

Definition 2 (Unimodular matrix) *A $k\times k$ matrix M with integer entries, which is invertible over the ring of integers $\mathbb{Z}$, is called a unimodular matrix. We write $(M_{ij}) \in GL_k(\mathbb{Z})$. Unimodular matrices have $\det(M) = \pm 1$.*

Due to the unit determinant those unimodular transformations preserve the volume of the fundamental region of the charge lattice, which is given by the determinant of the charge matrix.

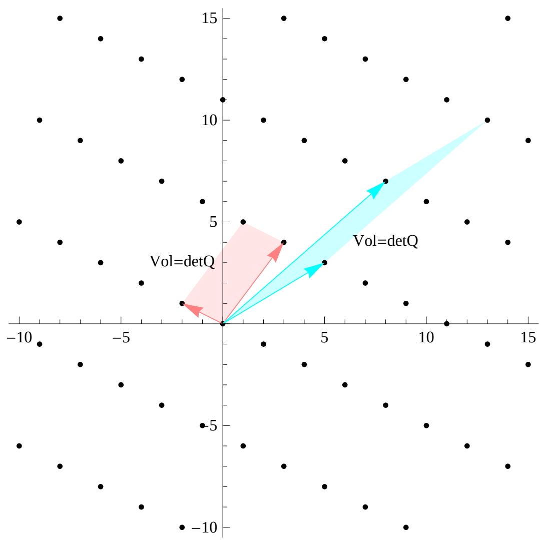


Figure 3.2: Basis change of the charge lattice. Each fundamental region is pigmented.

Indeed, it is true that a change of lattice basis for the charge lattice does not affect the breaking structure as will be shown now. Forget about the unimodular transformations for short and consider an arbitrary integer matrix M acting on the charge lattice basis. This corresponds to a transformation of the charge vectors

$$q'_j(\phi^{(i)}) = q'_{ij} = M_{im}q_{mj} \ , \quad \text{i.e.} \quad Q'_\phi = MQ_\phi \ , \tag{3.25}$$

which entails a redefinition of the VEVs

$$v^{(i)} \to v'^{(i)} = \prod_{j=1}^{k=a} (v^{(j)})^{M_{ij}} \ . \tag{3.26}$$

Therefore, the breaking condition (3.12), which determines the remnant discrete symmetries, is shifted as well

$$e^{\mathrm{i}q_j(\phi^{(i)})\alpha^{(j)}} v^{(i)} = v^{(i)} \quad \to \quad e^{\mathrm{i}q'_j(\phi^{(i)})\alpha^{(j)}} v'^{(i)} = v'^{(i)} \ . \tag{3.27}$$

Of course a product of VEVs, as in (3.26), can again be a valid VEV, but how do we need to restrict M so that the new VEV and charge setup does not alter the breaking pattern?

First of all, M has to be integer in order for Q'_ϕ to be a proper charge matrix. Moreover, on the one hand, the original breaking condition forces the generators to be

$$\boldsymbol{\alpha} = 2\pi Q^{-1} \boldsymbol{m} \ , m_i \in \mathbb{Z} \ , \tag{3.28}$$

as was elaborated in (3.17). On the other hand, the shifted breaking condition yields

$$\boldsymbol{\alpha} = 2\pi Q'^{-1}_\phi \boldsymbol{m} = 2\pi Q^{-1}_\phi \underbrace{M^{-1} \boldsymbol{m}}_{\boldsymbol{m}'} \ . \tag{3.29}$$

Thus, in order to recover the original breaking equation (3.28), the m'_i necessarily need to be integer, which means M^{-1} has to be integer. But it is well known that the set of integer and integer invertible matrices are exactly the unimodular ones (i.e. we have a complimentary unit determinant).

Let us study (3.29) more closely. It certainly is necessary that the $m'_i = M^{-1}_{ij} m_j$ are integer, which is the case if M^{-1} is an integer matrix. But is that sufficient for equivalence with (3.28)? Actually not, if M^{-1} only was integer, it could happen that $m'_i = c \cdot m''_i$, where m''_i still is integer. Then m'_i does not take values in whole $\mathbb{Z}$, but only in the ideal $c\mathbb{Z}$. Hence, c would be subject to cancel the denominator structure of Q^{-1}_ϕ, which determines the discrete symmetry. However, therefore the rows of M^{-1} need to have the greatest common divisor c. But this is impossible for unimodular M, since $\det(M) = \pm 1$ ensures that the row entries are relative prime, i.e. the greatest common divisor of each row is one (the same holds for columns).

Since this perception is of great importance for this work, and will reappear frequently, let us put it on a solid mathematical footing. The objects $\boldsymbol{m}$, which we want to map, are elements in $\mathbb{Z}^k$, an additive group, and we are provided with a compatible scalar multiplication $\mathbb{Z} \times \mathbb{Z}^k \to \mathbb{Z}^k$. This defines almost a vector space,

however, since scalar multiplication is defined over the ring $\mathbb{Z}$ instead of a field we only have a module, in fact – more precisely a $\mathbb{Z}$-module.
The map

$$M \; : \; \mathbb{Z}^k \to \mathbb{Z}^k \tag{3.30}$$

$$m_i \mapsto M_{ij} m_j \; , \quad \text{where } (M_{ij}) \in \mathrm{GL}_k(\mathbb{Z}) \; , \tag{3.31}$$

is a $\mathbb{Z}$-module isomorphism, since it is bijective (invertible over $\mathbb{Z}$) and respects the $\mathbb{Z}$-module structure, i.e.

$$M(\boldsymbol{m}+\boldsymbol{n}) = M(\boldsymbol{m}) + M(\boldsymbol{n}) \, , \quad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^k \tag{3.32}$$

$$M(c\boldsymbol{m}) = cM(\boldsymbol{m}) \, , \quad c \in \mathbb{Z} \; . \tag{3.33}$$

It has to be avoided that M maps $\mathbb{Z}^k \to c_1\mathbb{Z} \times \cdots \times c_k\mathbb{Z}$ with at least one $c_i \neq 1$. Abstract algebra tells us that for $X \xrightarrow{f} Y$, X covers Y, if f is a surjective homomorphism of the underlying structure. But the map M is even an isomorphism of the $\mathbb{Z}$-module, thus the target and the domain are identical, as desired.

Remark 1 *For $M \in GL_k(\mathbb{Z})$ and $m_i \in \mathbb{Z}$*

$$m'_i = M_{ij} m_j \quad \textit{covers } \mathbb{Z} \; . \tag{3.34}$$

Having established this, let us resume the key message of the above elaboration. It has been shown that a basis change of the charge lattice does not modify the breaking pattern, i.e. one has the freedom to shift the charge matrix $Q_\phi \to MQ_\phi$. It is of crucial importance that M is a unimodular transformation. Finally, this freedom allows to generalize the breaking condition (3.28) to

$$MQ_\phi \, \boldsymbol{\alpha} = 2\pi \, \boldsymbol{m} \; . \tag{3.35}$$

However, the charges of the remaining fields $\psi^{(l)}$ are not affected in the process of a charge lattice basis change. Since we can also consider a redefined charge matrix $Q'_\phi = MQ_\phi$ instead of absorbing M into $\boldsymbol{m}'$, it is clear that we will potentially experience another discrete symmetry setup for each unimodular transformation M, because the denominator structure of Q'^{-1}_ϕ is different (yet, not canceled). As already mentioned above, from an algebraic point of view, different equivalent products of cyclic groups describing the very same discrete Abelian group are not surprising because of isomorphisms among these.
Yet, the urgency of an unambiguous description of the remnant discrete symmetry group becomes manifest.

3.5 Smith normal form

A very important tool, which will guide us in direction of a distinct description of discrete Abelian symmetries, is given by the Smith normal form technique. It is algebraic knowledge that an integer matrix can be diagonalized by means of unimodular matrices. The proof [66] of the following theorem, specifying this kind of diagonalization, is connected[5] to the fundamental theorem of finitely generated Abelian groups, with which we will become acquainted shortly.

Theorem 1 (Smith normal form) *Let A be any integer $m \times n$ matrix. There exist unimodular matrices $M \in GL_m(\mathbb{Z})$ and $N \in GL_n(\mathbb{Z})$ such that*

$$MAN = D = \mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0) \ , \tag{3.36}$$

is a unique diagonal integer matrix with $d_i \neq 0$ for $i = 1, \ldots, r$ and $d_i | d_j$ for $i \leq j$. A diagonal $m \times n$ matrix is defined to have (i, i) entries d_i and zeros elsewhere. D is called ***Smith normal form*** *of A.*

We have seen that the generalized breaking condition (3.35) is already endowed with a left hand side unimodular matrix M, thus we can bring Q_ϕ into Smith normal form by insertion of the identity in terms of $\mathbb{1} = NN^{-1}$, where N is unimodular

$$MQ_\phi N \, N^{-1} \boldsymbol{\alpha} = 2\pi \, \boldsymbol{m} \ . \tag{3.37}$$

Since N^{-1} is unimodular as well, and hence integer, it is clear that $N^{-1}\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ is just another linear combination of generators. Thus we have the distinct expression

$$D_\phi \boldsymbol{\alpha}' = 2\pi \, \boldsymbol{m} \ . \tag{3.38}$$

That is, we have brought the charge matrix into Smith normal form, i.e. diagonal shape, by exploiting the freedom to perform unimodular transformations onto the breaking condition, which was stated in (3.35).

3.6 The discrete symmetry of the vacuum

Let us for now still consider the simple case $k = a$, such that the rank of D_ϕ is $r = k$. Thus the inverse of the charge matrix in Smith normal form D_ϕ is given by $D_\phi^{-1} = \mathrm{diag}(1/d_1, \ldots, 1/d_r)$, i.e. we can read off the discrete symmetries from the

[5] Another nice proof for theorem 1 can be found in [67], however, disregarding the connection to finitely generated Abelian groups.

solution

$$\boldsymbol{\alpha}' = 2\pi\,\mathrm{diag}(1/d_1, \ldots, 1/d_k)\boldsymbol{m} \tag{3.39}$$

directly. This is because now, in the redefined generator basis $\boldsymbol{\alpha}'$, the discrete symmetries disentangle, i.e. each α'_i is assigned to a proper denominator d_i. Thus,

$$\mathcal{G} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k} \tag{3.40}$$

is an elegant way to describe the discrete symmetry group, which leaves the VEV setup invariant. The symmetry group decomposes into cyclic groups whose orders are given by the d_i, which divide each other.

A remarkable point about this description, apart from its uniqueness via the Smith normal form, is its simplicity due to the diagonal form of (3.39). The discrete generators α'_i are obviously orthogonal, at the expense of the former $U(1)$ generators orthogonality. This can be visualized nicely in terms of the geometrically affected charge lattice picture. We will present an explicit example in section 3.8.

In this distinct notation, we see that the permissible set of cyclic groups $\mathbb{Z}_{n_i}$ allowed by the VEV structure fulfills $\prod_{i=1}^{a} n_i = \prod_{i=1}^{k} d_i = \det(Q_\phi)$. But this is the order $|\mathcal{G}|$ (see section 3.10) of the entire discrete symmetry group (3.40). Thus, at this point, it becomes clear that the highest achievable symmetry is one single $\mathbb{Z}_{\det(Q_\phi)}$, any cyclic subgroup of $\mathcal{G}$ will be of smaller order. This statement is a first merit of the geometrical and algebraic perspective; at the end of section 3.3 we found a much less specific bound.

To resume, we elaborated a convenient description of the remnant discrete Abelian symmetry group $\mathcal{G}$, which is respected by the VEV setup of a spontaneously broken general Abelian gauge symmetry $U(1)^k$.

Again, the actual discrete symmetry of the broken theory might be reduced by redundancies. In the process of identifying the remnant discrete Abelian symmetry group $\mathcal{G}$, we had to redefine the discrete generators as $\boldsymbol{\alpha}'$. Therefore, the discrete charges of the remaining fields $\psi^{(l)}$ have to be expressed in the same basis, which can be achieved easily

$$\boldsymbol{q}(\psi^{(l)}) N\, N^{-1}\, \boldsymbol{\alpha} = \boldsymbol{q}'(\psi^{(l)})\, \boldsymbol{\alpha}' \ . \tag{3.41}$$

The discrete charges $q'_j(\psi^{(l)})$ form the elements transforming under the discrete group (3.40). The transformation law

$$\psi^{(l)} \mapsto \exp\left(\mathrm{i}\boldsymbol{q}'(\psi^{(l)})\, \boldsymbol{\alpha}'\right)\, \psi^{(l)} = \exp\left(2\pi\mathrm{i}\, \boldsymbol{q}'(\psi^{(l)})\, \mathrm{diag}(1/d_1, \ldots, 1/d_k)\boldsymbol{m}\right)\, \psi^{(l)} \tag{3.42}$$

manifests the possibility of redundancies, e.g. if $q'_i(\psi^{(l)})|d_i \;\forall\; l$ the permissible symmetry $\mathcal{G}$ will be reduced. We will discuss redundancies and the resulting discrete symmetry group of the theory thoroughly in chapter 4.

3.7 Rectangular charge matrices

The key point about the Smith normal form approach is its universality – it is capable to deal with charge matrices of any shape. In case of linear dependencies among the VEVs, Q_ϕ is rank deficient and D_ϕ will only have $r < a$ diagonal entries, thus D_ϕ will contain negligible zero rows yielding no restrictions. If r becomes even smaller than k, or if we started with a setup where $a < k$, there will remain $k - r$ unbroken $U(1)$'s, since D_ϕ will show $k-r$ zero columns, which clearly do not restrict the $\alpha'_{r+1}, \ldots, \alpha'_k$ corresponding to the unbroken $k - r$ $U(1)$ generators.
Summarily, the presented method resolves the problem of rectangular shaped charge matrices, where $k \neq a$, automatically – one can drop zero rows respective columns, since these are redundant concerning the quest for the invariant discrete subgroups. Only the quadratic $r \times r$ submatrix is of importance, which ensures invertibility.
This immediately manifests an interesting result: a single $U(1)$ theory can only generate a charge matrix of maximal rank $r = 1$. It is thus clear that the remnant discrete symmetry group in this case is cyclic[6], since it can be expressed as a single $\mathbb{Z}_d$.

3.8 Intermediate example

Let us picture the geometrical interpretation of the presented construction by means of some concrete charge setup. It is purpose-built to improve the understanding of the results achieved so far. Further issues, yet undiscussed, are disguised intentionally.
The most interesting situation clearly is given by a linearly dependent VEV setup, as is the case for $k < a$. The easiest nontrivial example is to consider a $U(1)^2$ theory. Take three fields, which gain non-zero VEVs, inducing the charge matrix

$$Q_\phi = \begin{pmatrix} 5 & 14 \\ 9 & 1 \\ 8 & 7 \end{pmatrix} \quad . \tag{3.43}$$

[6] Of course the cyclic nature of the discrete Abelian group can be disguised by isomorphisms, e.g. $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. However, this does not change the fact that it is generated by only one element – more complex discrete Abelian groups cannot arise out of a single $U(1)$. This is somewhat mistakable expressed in [68] and does not agree with [69].

These fields span a nontrivial charge lattice, but they do not form a basis (see figure 3.3). This is because of the additional VEV, which of course is linearly dependent, though not a linear combination of the other two VEVs [7]. The volume of the

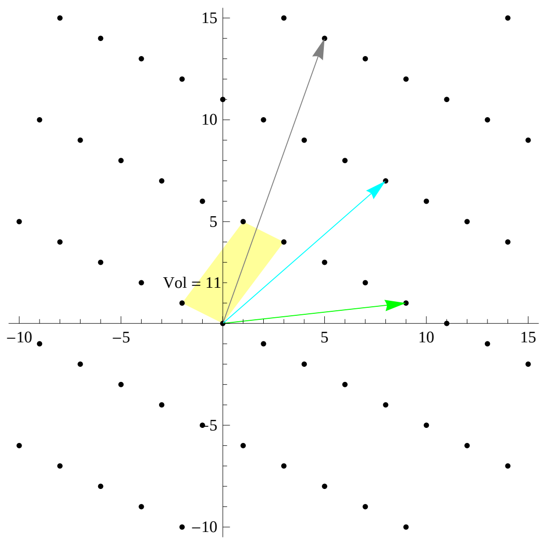


Figure 3.3: Charge setup of the three VEV acquiring fields and the corresponding charge lattice. The fundamental region is pigmented.

fundamental region, which is given by $\det(Q_\phi)$, takes the value Vol $= 11$, so the maximal discrete symmetry respected by the VEVs is $\mathbb{Z}_{11}$. Since this is prime we immediately know that the Smith normal form of the charge matrix D_ϕ will look like

$$\begin{pmatrix} 1 & 0 \\ 0 & 11 \\ 0 & 0 \end{pmatrix} , \tag{3.44}$$

because the diagonal entries need to divide each other. Nevertheless, let us perform the procedure step by step in order to shed light on the mechanism regarding the lattice point of view. Hence, we are to bring the charge matrix into diagonal form by means of two unimodular matrices M and N. The action of M on Q_ϕ eliminates

[7] Since the set of lattice points itself has $\mathbb{Z}$-module structure (as we already noted above remark 1), this becomes possible. Linear dependence in context of modules is defined just as in vector spaces, i.e. a family of $\mathbb{Z}$-module elements $\{\boldsymbol{q}_i\}$ is linearly dependent, if $c_i\boldsymbol{q}_i = 0$ with $c_i \neq 0\ \forall i$ and $c_i \in \mathbb{Z}$. But in contrast to vector spaces one element of a linearly dependent family is not necessary linearly dependent on the others [70].

the additional row, i.e. the linearly dependent VEV,

$$MQ_\phi = \begin{pmatrix} 1 & -6 \\ 0 & 11 \\ 0 & 0 \end{pmatrix} , \qquad \text{with } M = \begin{pmatrix} 0 & 1 & -1 \\ -1 & -3 & 4 \\ 5 & 7 & -11 \end{pmatrix} , \tag{3.45}$$

therefore choosing a particular basis of the charge lattice (figure 3.4). The right

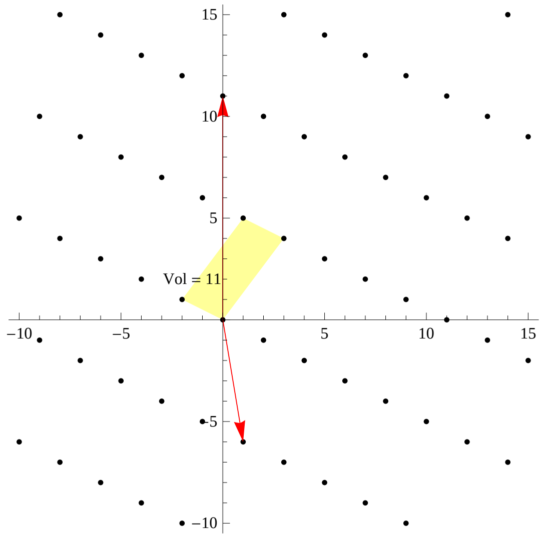


Figure 3.4: The action of the left unimodular matrix M on the charge matrix picks a lattice basis.

action of N on MQ_ϕ, which entails a change of generators $\boldsymbol{\alpha}' = N^{-1}\boldsymbol{\alpha}$, finally diagonalizes Q_ϕ

$$D_\phi \boldsymbol{\alpha}' = MQ_\phi N \boldsymbol{\alpha}' = \begin{pmatrix} 1 & 0 \\ 0 & 11 \\ 0 & 0 \end{pmatrix} \boldsymbol{\alpha}' , \qquad \text{where } N = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} . \tag{3.46}$$

The change of generator basis from $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}'$ reshapes the charge lattice to an orthogonal form (see figure 3.5). Now we can drop the redundant zero row of D_ϕ and solve for $\boldsymbol{\alpha}'$ by the inverse D_ϕ^{-1}

$$\boldsymbol{\alpha}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{pmatrix} 2\pi \, \boldsymbol{m} . \tag{3.47}$$

Finally, the denominator structure of (3.47) tells us that the permissible discrete symmetry is $\mathcal{G} = \mathbb{Z}_{11}$, which can not be reduced any further by the charge alignment

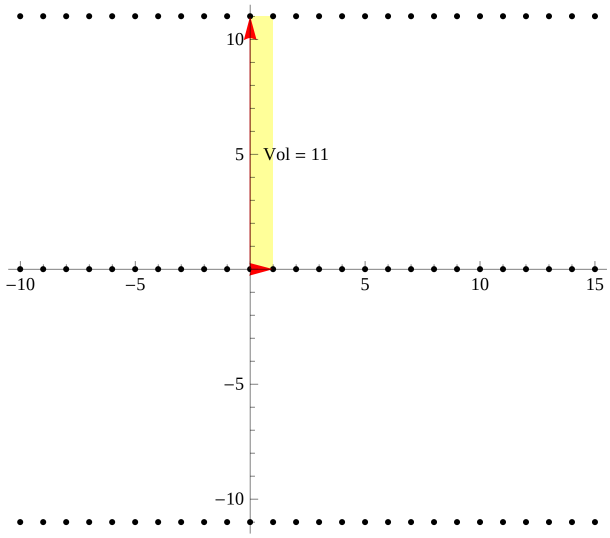


Figure 3.5: In terms of the transformed generators $\boldsymbol{\alpha}' = N^{-1}\boldsymbol{\alpha}$ the charge lattice has an orthogonal basis.

of any remaining fields, since it is prime.

3.9 Couplings

In fact, the modifications we performed to achieve the description of the permissible discrete Abelian symmetry group $\mathcal{G}$ via the Smith normal form was twice a basis change, one for the lattice basis and one for the generator basis, both represented by unimodular matrices M and N. We will show now that these transformations have no effect on the coupling conditions; that is, couplings again have to lie on the orthogonalized lattice. Remember that gauge invariance ensures that a coupling, which was part of the Lagrangian prior to giving the $\phi^{(i)}$ fields VEVs, now lies on the charge lattice. Thus, a general coupling of the broken phase $(\psi^{(1)})^{x_1} \cdots (\psi^{(b)})^{x_b}$ implies

$$x_1 \boldsymbol{q}(\psi^{(1)}) + x_2 \boldsymbol{q}(\psi^{(2)}) + \cdots + x_b \boldsymbol{q}(\psi^{(b)}) = m_i \boldsymbol{\lambda}^{(i)} , \tag{3.48}$$

where the lattice basis is generally elected by a particular unimodular matrix M acting on the charge vectors $\boldsymbol{q}(\phi^{(i)})$, as was illustrated in section 3.8,

$$\boldsymbol{\lambda}^{(i)} = M \boldsymbol{q}(\phi^{(i)}) . \tag{3.49}$$

Note that even in case of a square charge matrix of full rank, i.e. even if the charge vectors already describe a charge lattice basis, M is nontrivial in general, since the

Smith normal form is achieved only by the combination of left and right hand side unimodular transformation. Thus we can write (3.48) more compactly as

$$\boldsymbol{x}^T Q_\psi = \boldsymbol{m}^T M Q_\phi \ , \tag{3.50}$$

where $(Q_\psi)_{ij} = q_j(\psi^{(i)})$ is the charge matrix of the remaining 'matter' fields. Multiplication from the right by N yields

$$\boldsymbol{x}^T Q'_\psi = \boldsymbol{m}^T D_\phi \ , \tag{3.51}$$

with proper discrete charges $(Q'_\psi)_{ij} = q'_j(\psi^{(i)})$ of the 'matter' fields, as defined in (3.41), with respect to the generator basis $\boldsymbol{\alpha}'$. Equation (3.51) manifests that couplings again do lie on the orthogonal lattice.

3.10 Algebraic viewpoint

What we have achieved in the preceding sections is in fact the description of the underlying discrete Abelian group in terms of the invariant factor decomposition, a well known concept in algebra [71]. A discrete Abelian group does not have a unique description due to various isomorphisms. Yet, two representations are outstanding, namely the invariant factor decomposition and the elementary divisor decomposition. Both are of course isomorphic and thus equivalent. They arise as two equivalent possibilities to state the key concept of basic algebra: the *fundamental theorem of finitely generated Abelian groups.* This, in turn, is a special case of the more abstract algebraic *structure theorem for finitely generated modules over a principal ideal domain.*

Since both decompositions will be important in the remainder of this work, e.g. for addressing phenomenological questions like the identification of matter parity, we will present both of them and discuss their relation among each other.

3.10.1 Invariant factors

Let us first state the theorem in terms of the invariant factor decomposition.

Theorem 2 (Fundamental theorem of finitely generated Abelian groups) *Let G be a finitely generated Abelian group. Then*

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \ , \tag{3.52}$$

with integers $r \geq 0$ and $n_i \geq 2$ such that $n_i | n_{i+1}$ for $1 \leq i \leq s-1$.

Note that the *free rank* or *Betti number* r is zero throughout this work, since we only consider finite groups. Finitely generated but infinite groups have non-zero Betti number. The integers $n_1, \dots, n_s$ are known as the *invariant factors* of G and the invariant factor decomposition is unique. Since the order of a single cyclic group $\mathbb{Z}_{n_i}$ is n_i, the order of G is given by the product of the invariant factors.

3.10.2 Elementary divisors

Next, we state the same theorem in terms of the elementary divisor decomposition, also known as primary decomposition.

Theorem 3 *Let G be a finite Abelian group of order $n > 1$ with (unique) factorization into powers of distinct primes*[8] $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. *Then*

(1) $G \cong H_1 \times \dots \times H_k$ *with* $|H_i| = p_i^{\alpha_i}$

(2) $H_i \cong \mathbb{Z}_{p_i^{\beta_{i1}}} \times \dots \times \mathbb{Z}_{p_i^{\beta_{it}}}$ *with* $1 \leq \beta_{i1} \leq \dots \leq \beta_{it}$ *and* $\beta_{i1} + \dots + \beta_{it} = \alpha_i$ *(where t depends on i).*

The integers $p_i^{\beta_{ij}}$ are called the *elementary divisors* of G, and the H_i are the *Sylow p_i subgroups* of G (see appendix A.4). Since $p_i^{\beta_{ij}} | p_i^{\beta_{ij+1}}$ if and only if $\beta_{ij+1} \geq \beta_{ij}$, (2) tells us that the elementary divisors of G are the invariant factors of the Sylow p_i subgroups as we run over all $i \in 1, \dots, k$.
Yet, theorem 3 (2) does not state an isomorphism to all possible $\alpha_i = \beta_{i1} + \dots + \beta_{it}$ decompositions. Quite the contrary, the β_{ij} are fixed for some concrete H_i, since different lists of elementary divisors belong to non isomorphic groups, due to their uniqueness. Hence, one can find all different discrete Abelian groups of given order by listing all possible partitions of the α_i.
For instance, take the groups of order p^3. We find three different, non isomorphic discrete Abelian groups due to the integer decompositions of the power 3.

Prime power partition	Distinct Abelian groups
3	$\mathbb{Z}_{p^3}$
$2, 1$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$
$1, 1, 1$	$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

Generalizing from groups of prime power order to arbitrary ones is easy, let again $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ with c_i the number of partitions of α_i. Then the count of distinct discrete Abelian groups of order n adds to $\prod_{i=1}^{k} c_i$.

[8] Due to the fundamental theorem of arithmetic (see appendix A.1).

3.10.3 Obtaining elementary divisors from invariant factors

Let us outline the connection between the two equivalent theorems from above. Therefore, we need the following

Proposition 1 *Let m, n be positive integers.*

(1) $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ *if and only if* $GCD(m, n) = 1$.

(2) Let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$. *Then* $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}}$.

Assume we have given the invariant factor decomposition of G by

$$G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s} \tag{3.53}$$

where G is of order $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n_1 \ldots n_s$. We can factor each $n_i = p_1^{\beta_{i1}} \ldots p_k^{\beta_{ik}}$, thus

$$\mathbb{Z}_{n_i} \cong \mathbb{Z}_{p_1^{\beta_{i1}}} \times \cdots \times \mathbb{Z}_{p_k^{\beta_{ik}}} \ , \tag{3.54}$$

where all $\beta_{ij} = 0$ can be neglected because they only contribute a trivial factor $\mathbb{Z}_1 = 1$ to the product expansion of G. Hence, the elementary divisors of G are

$$p_j^{\beta_{ij}} \ , \quad \text{with } 1 \leq j \leq k,\ 1 \leq i \leq s \text{ and } \beta_{ij} \neq 0 \ . \tag{3.55}$$

From the charge lattice picture point of view, the invariant factor decomposition is most natural, since the discrete symmetries are orthogonal in that basis. However, elementary divisors will be important at a later stage, too. The automorphism group of finitely generated Abelian groups will most conveniently be studied in terms of Sylow H_i groups. Furthermore, phenomenological problems may require the elementary divisor decomposition; for instance, the identification of matter parity can be restricted on H_2, as we will see in section 5.3.

Chapter 4

Redundancies & equivalences

So far, we successfully elaborated the breaking of a general Abelian gauge group down to remnant discrete symmetries. We were able to describe the remaining Abelian discrete group, which respects the VEV configuration responsible for the breakdown of the continuous symmetries, by means of its invariant factors. However, we already encountered various hints that this symmetry might be larger than the actual symmetry of the Lagrangian, which depends on the remaining field content of the theory. In this chapter, we will analyze such redundant field configurations and show how to eliminate redundancies.

By agreeing upon the invariant factor or elementary divisor account, we already suppressed some amount of equivalent descriptions given by isomorphisms. Yet, there are further equivalent alignments among these decompositions themselves, because of automorphisms. From the physical point of view, an automorphism of a given discrete Abelian group corresponds to an equivalent discrete charge assignment. We will review the description of the automorphism group of finite Abelian groups and discuss their connection to the elimination of redundancies. Finally, we study hypercharge shifts, another kind of equivalent charge assignment, which arises if an unbroken continuous Abelian group coexists with a discrete Abelian group, and illustrate the presented methods by means of a concrete example.

4.1 Redundant field configurations

Let us first face the problem of redundancies; that is, the reduction of the permissible discrete symmetry group $\mathcal{G}$ (3.40) by the remaining 'matter' field charge configuration. As an intelligible access, suppose we have only one 'matter' field ψ transforming under a cyclic discrete group $\mathcal{G} = \mathbb{Z}_d$. Then the discrete symmetry reduces, if the discrete charge $q'(\psi)$ divides d, since one can cancel out the $\mathrm{GCD}(q'(\psi), d)$ in

the transformation law (3.42). Hence, the cyclic group $\mathbb{Z}_d$ is reduced to $\mathbb{Z}_{d'}$ with $d' = \frac{d}{\text{GCD}(q'(\psi),d)}$, because the field content is able to generate the subgroup $\mathbb{Z}_{d'}$ of $\mathbb{Z}_d$, only [72]. Thus, the order of $\mathcal{G}$ becomes reduced.
Let us picture this for the concrete case $d = 4$, i.e. consider a $\mathbb{Z}_4$ with one field of charge two. The order of $\mathbb{Z}_4$ is four, which means there are four elements in this group. Yet, the charge two field can only reach two of those elements, namely $\{2, 0\}$, because of the mod 4 constraint.

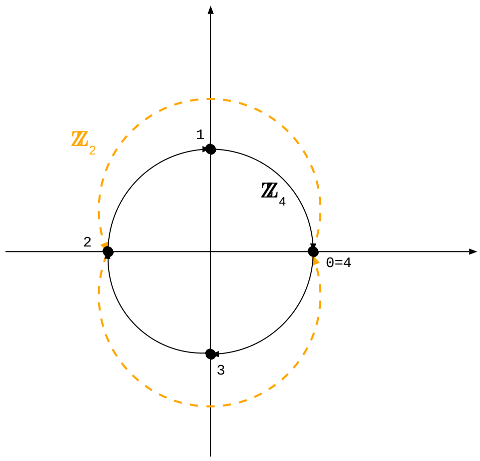


Figure 4.1: A charge two field embedded into a $\mathbb{Z}_4$. The field can only reach the subgroup $\mathbb{Z}_2$, thus the description is redundant.

The order of the field content thus is two and, hence, smaller than the order of the discrete group, manifesting the redundancy (see figure 4.1). Once the common factor two of the field's charge and the group order is canceled, we arrive at a neat description, namely a charge one field transforming under a $\mathbb{Z}_2$.

4.1.1 Eliminating redundancies

For non cyclic Abelian discrete symmetry groups, seeking for redundancies is somewhat more involved. We will present a procedure, which is capable to eliminate redundancies for any given charge setup under some arbitrary Abelian discrete symmetry group $\mathcal{G}$. It may be given in invariant factor decomposition, but does not have to. For our purpose, we will demonstrate the mechanism by continuing with the resulting situation of equations (3.42) and (3.40); that is, 'matter' fields

$$\psi^{(l)} \mapsto e^{\mathrm{i}\boldsymbol{q}'(\psi^{(l)})\,\boldsymbol{\alpha}'}\psi^{(l)} = e^{2\pi\mathrm{i}\, q_i'(\psi^{(l)})\,\frac{m_i}{d_i}}\,\psi^{(l)} \tag{4.1}$$

transforming under the permissible discrete symmetry $\mathcal{G} = d_1 \times \cdots \times d_r$ with $d_i | d_{i+1}$. For the case of an arbitrary Abelian discrete symmetry group, one can think of the d_i as uncorrelated.
Now, according to the transformation law (4.1), the quantity

$$Q'_\psi \begin{pmatrix} \frac{m_1}{d_1} \\ \vdots \\ \frac{m_r}{d_r} \end{pmatrix}, \quad \text{where } (Q'_\psi)_{li} = q'_i(\psi^{(l)}) \tag{4.2}$$

flocks the permissible $\mathbb{Z}_{d_i}$ symmetries and the corresponding discrete charges of the $\psi^{(l)}$ fields together. If Q'_ψ was diagonal, one could cancel common factors immediately. Thus, the idea is to diagonalize Q'_ψ by means of the Smith normal form. The emerging unimodular matrices have no common factor in their rows or columns, as we noted in section 3.4.2, hence the redundancy remains in the diagonal part, so to speak. One could try to put Q'_ψ into Smith normal form directly, but then the right unimodular matrix would mix the d_i, which is rather cumbersome in order to read off the discrete symmetries. A much more convenient method is to introduce a unit in terms of

$$Q'_\psi \underbrace{\begin{pmatrix} \frac{d}{d_1} & & \\ & \ddots & \\ & & \frac{d}{d_r} \end{pmatrix}}_{U^{-1}} \underbrace{\begin{pmatrix} \frac{d_1}{d} & & \\ & \ddots & \\ & & \frac{d_r}{d} \end{pmatrix}}_{U} \begin{pmatrix} \frac{m_1}{d_1} \\ \vdots \\ \frac{m_r}{d_r} \end{pmatrix}, \tag{4.3}$$

where d is the least common multiple (LCM) of the d_i. Since here $d_i | d_{i+1}$, we have $\text{LCM}(d_1, \ldots, d_r) = d_r$, of course, but that is not the case for an Abelian discrete group $\mathcal{G}$, which is not given in invariant factor decomposition. Now, since $d_i | d$, the matrix U^{-1} is integer and thus $Q'_\psi U^{-1}$ can be brought into Smith normal form

$$M_\psi^{-1} D_\psi N_\psi^{-1} \begin{pmatrix} \frac{m_1}{d} \\ \vdots \\ \frac{m_r}{d} \end{pmatrix}. \tag{4.4}$$

But due to $\det(N_\psi^{-1}) = \pm 1$ the row entries of N_ψ^{-1} are relative prime and therefore $m'_i = (N_\psi^{-1})_{ij} m_j$ covers $\mathbb{Z}$ again, as discussed in section 3.4.2 remark 1. Thus we obtain

$$M_\psi^{-1} \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_r \end{pmatrix} \begin{pmatrix} \frac{m'_1}{d} \\ \vdots \\ \frac{m'_r}{d} \end{pmatrix}, \tag{4.5}$$

with $s_i|s_{i+1}$. Now common factors can be canceled deliberately: let us write $s_i = a_i \cdot \mathrm{GCD}(s_i, d)$ and $d = d'_i \cdot \mathrm{GCD}(s_i, d)$, such that we have

$$\underbrace{M_\psi^{-1} \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \end{pmatrix}}_{Q_\psi^F} \begin{pmatrix} \frac{m'_1}{d'_1} \\ \vdots \\ \frac{m'_r}{d'_r} \end{pmatrix} . \tag{4.6}$$

Comparing with (4.2) one can read off the final discrete symmetry group[1]

$$\mathcal{G}^F = \mathbb{Z}_{d'_1} \times \cdots \times \mathbb{Z}_{d'_r} \qquad \text{now with } d'_{i+1}|d'_i \, , \tag{4.7}$$

and the final matter charge matrix Q_ψ^F. Note that we could equally well drop the matrix $\mathrm{diag}(a_1, \ldots, a_r)$, since a_i and d'_i are relative prime. Best, exemplify this in one dimension: consider one charge 1 field transforming under $\mathbb{Z}_3$. Multiplying the elements of $\mathbb{Z}_3$ with the coprime $a = 2$, and keeping in mind the mod 3 constraint, the element 'charge 1' of order three is mapped onto the element 'charge 2' and vice versa, while the order one element 'charge 0' is left invariant, see table 4.1.

$\mathbb{Z}_3$	$\times 2$ mod 3	$\mathbb{Z}_3$	Order
0	$\rightarrow$	0	1
1	$\rightarrow$	2	3
2	$\rightarrow$	1	3

Table 4.1: Two equivalent charge assignments of $\mathbb{Z}_3$

In fact, this is an automorphism of $\mathbb{Z}_3$, so both charge setups are equivalent. Same holds for the choice of charges Q_ψ^F or M_ψ^{-1} in (4.6). We will postpone the proof that such transformations leave the coupling structure invariant to section 4.3, until we have discussed automorphisms thoroughly. But first, let us note on some details of the presented redundancy elimination method.

4.1.2 Remarks on the elimination procedure

For the sake of completeness, note that the above approach always renders a – possibly rectangular – diagonal matrix D_ψ with at most r diagonal entries. If there happen to be less than r diagonal entries, i.e. some $s_i = a_i = 0$, all remaining 'matter' fields are uncharged under the corresponding cyclic factors $\mathbb{Z}_{d'_i}$, which consequently

[1] If some $d'_i = 1$, the corresponding trivial factors $\mathbb{Z}_1$ again drop out of $\mathcal{G}^F$.

drop out of $\mathcal{G}^F$. Note that we have skipped potential zero rows and columns, beyond the block diagonal structure $\text{diag}(s_1, \ldots, s_r)$ of D_ψ in the above derivation, since those do not yield any information with respect to $\mathcal{G}^F$.
We are now able to state some interesting consequences of redundancy eliminations:

- There will be at most as many cyclic $\mathbb{Z}_{d'_i}$ factors in $\mathcal{G}^F$ as 'matter' fields $\psi^{(l)}$.

- As necessary condition for a reduction of $\mathcal{G}$ to $\mathcal{G}^F < \mathcal{G}$, the volume of the charge lattice $\det Q_\phi$ and the volume of the remaining 'matter' fields charge lattice[2] $\det Q'_\psi = \det Q_\psi$ must not be coprime.

- This condition is not sufficient.

The first point is pretty intuitive, as we are familiar with it from the continuous case. One can not define more, say, $U(1)$'s than transforming, i.e. charged, fields.
The second point can be seen as follows. If $\mathbb{Z}_{d_i}$ is redundant, the i-th column of Q'_ψ has to possess a common factor c such that $c|d_i$. Therefore, on the one hand, the volume of the 'matter' fields charge lattice can be written as $\det Q'_\psi = c \cdot \det \tilde{Q}_\psi$. On the other hand, the volume of the ordinary charge lattice decomposes as $\det Q_\phi = \prod_{i=1}^r d_i = c \cdot \det \tilde{Q}_\phi$, i.e. c can be factored out of both lattice volumes. Thus their greatest common divisor can not be one.
Yet, discrete symmetry reduction does not follow automatically once the lattice volumes show a GCD. Compare the determinants of the matrices preceding $\boldsymbol{m}$ and $\boldsymbol{m}'$ in (4.2) and (4.6). This is valid, since $\boldsymbol{m}$ and $\boldsymbol{m}'$ vary only by a determinant one transformation. One obtains the ratio

$$\frac{\det Q'_\psi}{\det Q_\phi} = \frac{\prod_i a_i}{\prod_i d'_i} \ . \tag{4.8}$$

Although a_i and d'_i are coprime for each i separately, the two products on the right hand side can have a common factor. Hence, we find that after the reduction mechanism both lattice volumes can still have a GCD; thus this cannot yield a sufficient criterion for the appearance of a redundancy.
Finally, note that Q_ψ will not be a square matrix in general. Thus, in order to compute the lattice volume, a lattice reduction is necessary, i.e. a change of lattice basis, as described in section 3.4. One can as well compute the Smith normal form of Q_ψ and take the product of its invariant factors.

[2] Defined analogously to the VEV fields charge lattice from section 3.4.1.

4.2 Automorphisms

What we concealed in the discussion so far is that the unimodular matrices, which render the Smith normal form, are not unique. That means, once the (unique) Smith normal form of some charge matrix is found

$$D_\phi = M Q_\phi N \ , \tag{4.9}$$

the same result could be obtained by

$$D_\phi = M' Q_\phi N' \ , \tag{4.10}$$

with $M \neq M'$ and $N \neq N'$. Since the discrete charges Q'_ψ of the remaining 'matter' fields, transforming under $\mathcal{G}$, are obtained by multiplying N onto the original 'matter' field charges Q_ψ, as stated in (3.41), one will encounter differing but equivalent sets of discrete charges for different choices of N. A mapping among such unequal charge setups has to be an automorphism, in order to ensure equivalence. Therefore, the question is how many different, automorphic charge assignments are there, and how to identify them?
In order to investigate this question, it is notable that the multiplicative group of unimodular matrices $GL_{n\times n}(\mathbb{Z})$ is generated by two elements [73]

$$U_a = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad U_b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ (-1)^n & 0 & 0 & \dots & 0 \end{pmatrix} \ . \tag{4.11}$$

As a consequence, one can always find some particular unimodular matrix U, which connects two different unimodular matrices like

$$M = UM' \ . \tag{4.12}$$

Thus, seeking for all unimodular matrices U, V, which leave the Smith normal form D invariant

$$UDV = D \ , \tag{4.13}$$

renders all possible M', N', since

$$M^{-1} D N^{-1} = M'^{-1} U D V N'^{-1} \ , \tag{4.14}$$

with $U^{-1}M' = M$ and $N'V^{-1} = N$. Exploiting (4.13) is possible analytically for $U, V \in GL_{2\times 2}(\mathbb{Z})$, however, rather cumbersome in the case of higher $n \times n$ matrices.

Moreover, in table 4.1 we already encountered one automorphic mapping among discrete charges for the cyclic group $\mathbb{Z}_3$, which was certainly not achieved by a unimodular transformation, since it had determinant two. Thus the set of automorphic charges will be even larger than those ascribed to the different possibilities of obtaining the Smith normal form. Luckily, the complete description of the automorphism group of finite Abelian groups is already known.

4.2.1 Description of the automorphism group

For any group G, an automorphism of G is an isomorphism from G onto itself. The set of all automorphisms $\mathrm{Aut}(G)$ forms a group under composition of automorphisms. For G a finite Abelian group, an explicit construction of $\mathrm{Aut}(G)$ is available due to [74]. We will review the definitions and theorems needed to adopt this construction, for proofs refer to the original work.

The first step is to realize that it is sufficient to have a description of $\mathrm{Aut}(H_i)$, the automorphisms of the Sylow p_i subgroups of G.

Lemma 1 *For finite Abelian groups H and K of relative prime order*

$$Aut(H \times K) \cong Aut(H) \times Aut(K)\ . \tag{4.15}$$

Since H_i is of order $p_i^{\alpha_i}$ and G decomposes into a direct product over all distinct Sylow p_i subgroups, with $i \in 1, \ldots, k$, due to theorem 3, the above lemma tells us that the automorphism group of G decomposes in the very same way

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(H_1) \times \cdots \times \mathrm{Aut}(H_k)\ . \tag{4.16}$$

Thus, we can constrict the investigation of automorphisms of finite Abelian groups to $\mathrm{Aut}(H_i)$.

A proper element of H_i is a row vector $(\bar{q}_1, \ldots, \bar{q}_t)$ with $\bar{q}_i \in \mathbb{Z}_{p^{\beta_i}} \cong \mathbb{Z}/p^{\beta_i}\mathbb{Z}$ and $1 \leq \beta_1 \leq \cdots \leq \beta_t$. Here, the bar denotes the standard quotient map, which assigns each vector entry to the corresponding residue class.

Definition 3 *The standard quotient mapping $\pi_i : \mathbb{Z} \to \mathbb{Z}_{p^{\beta_i}}$ is given by $\pi_i(q) = \bar{q}$ and let $\pi : \mathbb{Z}^n \to H_i$ be $\pi(q_1, \ldots, q_t) = (\pi_1(q_1), \ldots, \pi_t(q_t))$.*

We would call such an element of H_i a discrete 'matter' field charge under the Sylow p_i subgroup $\mathbb{Z}_{p^{\beta_1}} \times \cdots \times \mathbb{Z}_{p^{\beta_t}}$ of the discrete Abelian symmetry group $\mathcal{G}$, but let us ignore the physical interpretation for now, in order to simplify the notation. Next, we define the set of matrices

Definition 4

$$R_p = \{A \in \mathbb{Z}^{t\times t} \,:\, p^{\beta_i - \beta_j} | a_{ij} \,\forall\, 1 \le j \le i \le t\}\ , \tag{4.17}$$

which exhibits ring structure. Now, the endomorphisms of H_i are constructed as a quotient mapping of an element of H_i multiplied by a matrix $A \in R_p$.

Theorem 4 *The endomorphisms of H_i are given by the mapping*

$$(\bar{q}_1, \ldots, \bar{q}_t) \mapsto \pi\left((q_1, \ldots, q_t)A^T\right)\ , \tag{4.18}$$

where $A \in R_p$.

Finally, the elements of $\mathrm{Aut}(H_i)$ are identified as the following subgroup of $\mathrm{End}(H_i)$.

Theorem 5 *An endomorphisms of H_i is an automorphism if and only if A mod p (entry-wise) $\in GL_t(\mathbb{Z}_p)$.*

Note that the invertible $t \times t$ matrices over $\mathbb{Z}_p$ are not restricted to have determinant one, but rather may take any value, which is not a multiple of p, since p is identified with zero.
Now, in order to come to terms with so much formal development, an easy example will be helpful.

4.2.2 Automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_4$

Therefore, let us consider the discrete symmetry group $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_4$. This is the simplest nontrivial example. It allows us to write down explicitly all automorphic transformations, yet, their number increases[3] dramatically for multiple and/or larger prime powers. But this choice of example has another convenient advantage. It is given in invariant factor decomposition as we would obtain it due to the lattice approach, however, it is in primary decomposition as well, since both are the same here. Thus, for the moment we do not need to address the problem of obtaining the charges $Q_\psi^{(p)}$, which transform under H_i, out of the charge setup Q_ψ with respect to $\mathcal{G}$. Furthermore, we have only one distinct prime $p = 2$, so that we only need to study H_2 and do not have to repeat the same steps for other H_i according to (4.16).

[3] An explicit formula which counts the number of automorphisms of H_i is presented in [74].

Now, let us pick an arbitrary non redundant charge setup. The simplest choice is the 2×2 unit, since it clearly generates the group. Thus we have

$$Q_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \text{which means} \qquad \begin{array}{c|cc} & \mathbb{Z}_2 & \mathbb{Z}_{2^2} \\ \hline \psi_1 & 1 & 0 \\ \psi_2 & 0 & 1 \end{array} . \tag{4.19}$$

Next, for $\beta_1 = 1$ and $\beta_2 = 2$ the matrix ring (4.17) consists of the set R_2 of 2×2 matrices

$$R_2 = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \; : \; a, b, c, d \, \in \, \mathbb{Z} \right\} , \tag{4.20}$$

which allow us to write the endomorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_4$ as

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_2^T \right) \bigg|_\pi = R_2^T\big|_\pi = \begin{pmatrix} a & 2c \\ b & d \end{pmatrix} \bigg|_\pi , \tag{4.21}$$

due to theorem 4. Here, π projects onto the (least) residue of the corresponding congruence class. That is, the first column can take values of (least) residues mod 2, and the second column mod 4. Thus the endomorphisms are

$$\left\{ \begin{pmatrix} a & 2c \\ b & d \end{pmatrix} \; : \; a, b, c \, \in \, \{0, 1\} \, \text{and} \, d \, \in \, \{0, 1, 2, 3\} \right\} . \tag{4.22}$$

Finally, theorem 5 tells us that those matrices above, which are entry-wise mod 2 still invertible, form the group $\text{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_4)$, given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \right\} . \tag{4.23}$$

Hence, any (least residue) charge setup Q_ψ of $\mathbb{Z}_2 \times \mathbb{Z}_4$ multiplied by one of these matrices from the right results in another equivalent charge assignment.

4.3 Coupling equations

One should keep in mind that the discrete symmetries studied here are determined by the coupling structure of the original theory, which was subject to spontaneous symmetry breaking. In the broken phase, the 'matter' field couplings are just what remains when all other fields acquired VEVs. Therefore, those couplings lie on the charge lattice. Viewed the other way round, the discrete symmetry group, parametrized by the charge lattice, determines, which matter couplings are realizable. In order to read off whether some desirable or unwanted 'matter' coupling exists in the given discrete symmetry setup, it is useful to examine "coupling equations"; that is, solve the coupling conditions from (3.51)

$$\boldsymbol{x}^T\, Q'_\psi = \boldsymbol{m}^T\, D_\phi \;, \tag{4.24}$$

with $\boldsymbol{x}^T$ taken to be unknown. From a model building perspective, one might have some parametric freedom allowing to assign charges deliberately. In those cases, full control over the coupling equations enables to "forbid" dangerous couplings, e.g. for proton decay, by choosing charges such that dangerous operators have overall charge under the remnant discrete symmetry. On the other hand, dangerous couplings forbidden only at the level of the original continuous gauge symmetry can appear effectively in the broken phase, if not protected by discrete symmetries.
Before thinking about solutions to coupling equations, we would like to express those conditions in terms of the redundancy free language. By following the same steps as in section 4.1.1 we see that (3.51) takes the form

$$\boldsymbol{x}^T\, Q^F_\psi = \boldsymbol{m}'^T\, D'_\phi \;, \quad \text{with } D'_\phi = \begin{pmatrix} d'_1 & & \\ & \ddots & \\ & & d'_r \end{pmatrix} \tag{4.25}$$

after reduction of redundancies, while all performed transformations preserve the set of solutions.

4.3.1 Systems of linear congruences

Now, let us study such systems of linear equations in general. The most generic type is of the form

$$x_i B_{ij} - g_j = m_j \cdot d_j \text{ (no sum over } j) \tag{4.26}$$

or equivalently

$$x_i B_{ij} = g_j \,(\text{mod } d_j) \;, \tag{4.27}$$

known as systems of linear congruences. Let $i \in \{1, \ldots, b\}$ and $j \in \{1, \ldots, r\}$. Note that (4.25) actually takes a simpler form, namely $g_j = 0$. However, let us analyze general systems of linear congruences, since discrete R-symmetries, which will be studied at a later stage, induce a non-trivial g_j.
Defining $\text{LCM}(d_1, \ldots, d_r) = d$ and $d = d_j \cdot r_j \; \forall\, j$ (no sum), the system of congruences can be recast as

$$x_i A_{ij} - k_j = m_j \cdot d \;, \tag{4.28}$$

with $A_{ij} = B_{ij} \cdot r_j$ (no sum) and $k_j = g_j \cdot r_j$ (no sum), now having a common modulus d. Note that from the coupling equation point of view, the least common multiple description we used in (4.3) for the elimination of redundancies was nothing else than switching to the common modulus.

Now, up to the modulus d the system of congruences is a system of Diophantine equations

$$x_i A_{ij} = k_j \ . \tag{4.29}$$

In fact, the solution procedure is identical and known for a long time [75, 76], see [77] for a modern treatment and [78] for a computational approach. The main idea is to diagonalize A by means of the Smith normal form, then solutions are rather obvious. Let us become acquainted with that idea by examining systems of Diophantine equations first. We can bring A in (4.29) into Smith normal form

$$\left(\boldsymbol{x}^T M^{-1}\right)\ (M\ A\ N) = \boldsymbol{k}^T N\ , \tag{4.30}$$

with unimodular matrices M and N. For the primed objects $\boldsymbol{x}'^T = \boldsymbol{x}^T M^{-1}$ and $\boldsymbol{k}'^T = \boldsymbol{k}^T N$ the system of equations takes the simple form

$$\boldsymbol{x}'^T C = \boldsymbol{k}'^T\ , \tag{4.31}$$

where C has diagonal entries $c_1, \dots, c_s$ (its invariant factors) and zeros elsewhere. Thus, one can conclude immediately that this system is solvable only if c_i divides k'_i and $k'_i = 0$ for $i > s$. Of course, the solution then can be retranslated back to the unprimed objects.

The additional modulus of (4.28) can be handled with the very same approach. Again, A is to be brought into Smith normal form by means of unimodular matrices M and N

$$\left(\boldsymbol{x}^T M^{-1}\right)\ (M\ A\ N) - \boldsymbol{k}^T N = \underbrace{\boldsymbol{m}^T N}_{\boldsymbol{m}'} \cdot d\ , \tag{4.32}$$

where each $m'_j = m_i N_{ij}$ again covers $\mathbb{Z}$, as discussed in section 3.4.2 remark 1. Therefore, one obtains the congruence equations

$$x'_i c_i = k'_i \bmod d \quad \forall\ i = \{1, \dots, s\}\ \text{(no sum)} \tag{4.33}$$

$$k'_i = 0 \bmod d \quad \forall\ i > s\ . \tag{4.34}$$

According to the congruence theorem (see theorem 8 in appendix A.2), equation (4.33) has a solution if and only if $\mathrm{GCD}(c_i, d) | k'_i$. Hence, a system of congruences of the type

$$x_i B_{ij} = 0\ (\mathrm{mod}\ d) \tag{4.35}$$

always has solutions, since everything divides zero. Thus, the coupling equations (4.25) are always solvable.

4.3.2 Invariance under automorphisms

We have seen in section 4.2 that the redundancy reduction method picks one arbitrary out of all automorphic charge setups under the redundancy free symmetry group $\mathcal{G}_F$. As a consistency check, let us show that this is indeed a valid operation by proving the invariance of the coupling structure under automorphic transformations. As a first step, let us show that the multiplication of a congruence equation by numbers, which are relative prime to the modulus results in an equivalent congruence equation. In fact, this is what we postponed to prove in section 4.1.1 where we encountered a first automorphism in table 4.1.
The assertion for congruence equations is

$$ca = cb \bmod d \quad \Leftrightarrow \quad a = b \bmod \frac{d}{\mathrm{GCD}(c,d)} \ . \tag{4.36}$$

Let us assume the left hand side. We can rewrite it as

$$ca - cb = e \cdot d \quad \text{where } e \in \mathbb{Z} \ . \tag{4.37}$$

Dividing by $g = \mathrm{GCD}(c, d)$ gives

$$\frac{c}{g}(a - b) = e \cdot \frac{d}{g} \ , \tag{4.38}$$

where all fractions are still integer numbers, since $g = \mathrm{GCD}(c, d)$. This just tells us that $\frac{d}{g} | \frac{c}{g}(a - b)$. But, since $\mathrm{GCD}(\frac{d}{g}, \frac{c}{g}) = 1$, i.e. $\frac{d}{g} \nmid \frac{c}{g}$, we know that $\frac{d}{g} |(a - b)$, which can be written as

$$a = b \bmod \frac{d}{g} \ . \tag{4.39}$$

In order to prove the other direction just follow the same steps backwards.
Hence, if c and d are coprime, then

$$ca = cb \bmod d \quad \Leftrightarrow \quad a = b \bmod d \ , \tag{4.40}$$

i.e. it is valid to ‘cancel’ coprimes of the modulus, or to multiply the congruence equation with them. This will not change the modulus.
Next, assume that we have the coupling equations given in terms of the primary decomposition. We do not know how to transform the charge matrix Q'_ψ to its equiv-

alents $Q^{(p)}_\psi$ transforming under the H_i, yet. This will be elaborated in section 5.3. But we do know that the coupling equations will have the form

$$\boldsymbol{x}^T Q^{(p)}_\psi = \begin{pmatrix} m_1 p^{\beta_1} & \dots & m_t p^{\beta_t} \end{pmatrix} \tag{4.41}$$

for each H_i. Given a solution $\boldsymbol{x}^T$, can we substitute $Q^{(p)}_\psi$ by another automorphic charge matrix? Indeed, we can. In order to see it, let us first switch to the common modulus, just as we did for the redundancy elimination method. Therefore, we have to multiply (4.41) by $\mathrm{diag}(p^{\beta_t-\beta_1}, p^{\beta_k-\beta_2}, \dots, 1)$ from the right.
An automorphism is composed by the matrix ring R_p. The matrices $A \in R_p$ have the property $p^{\beta_i-\beta_j}|a_{ij}$ for $i \geq j$, which can be recast as $a_{ij} = p^{\beta_i} a'_{ij} p^{-\beta_j}$ (no sum) with $a'_{ij} \in \mathbb{Z}$. Hence, it is always possible to find an $A' \in \mathbb{Z}^{t\times t}$ such that

$$A = PA'P^{-1} \ , \tag{4.42}$$

where $P = \mathrm{diag}(p^{\beta_1}, \dots, p^{\beta_t})$ and, of course, $\det(A) = \det(A')$. Thus, we can equip (4.41) with a common modulus in terms of

$$\boldsymbol{x}^T Q^{(p)}_\psi p^{\beta_t} P^{-1} = \begin{pmatrix} m_1 & \dots & m_t \end{pmatrix} p^{\beta_t} \ . \tag{4.43}$$

Next, we multiply from the right by A'^T. This is an admissible operation: since A shall be related to an automorphism, theorem 5 tells us that $p \nmid \det(A)$, because else A mod p would not be invertible over $\mathbb{Z}_p$. Thus, $\mathrm{GCD}(\det(A'), p^{\beta_t}) = 1$ and therefore there exists a number s, such that $s \cdot \det(A') = 1 \mod p^{\beta_k}$, i.e. an inverse $A'^{-1}\big|_\pi = s \cdot \mathrm{adj}(A')$ under the canonical projection π.
On our intuitive way facing the problem of discrete Abelian gauge symmetries, we already defined discrete charges to be (least) representatives of residue classes in section 3.2. Hence, the coupling equations always involve an application of π implicitly.
Thus, we can write

$$\boldsymbol{x}^T Q^{(p)}_\psi p^{\beta_t} P^{-1} A'^T P = \begin{pmatrix} m_1 & \dots & m_t \end{pmatrix} A'^T p^{\beta_t} P \ , \tag{4.44}$$

where we multiplied by P as well. Now, there are two crucial steps in order to see that an automorphism of the charge matrix leaves the coupling equation (4.41) unchanged. First, remember what we have shown above: a congruence equation can be multiplied by any coprime of the modulus without changing the modulus. That is, equation (4.36) tells that a congruence equation can equivalently be written as

$$ca - cb = ed\,,\ e \in \mathbb{Z} \quad \text{or} \quad ca - cb = e'd\,, \quad \text{with } e' = c \cdot e \in c\mathbb{Z} \ , \tag{4.45}$$

as long as $\text{GCD}(c,d) = 1$. For a congruence equation with prime power modulus any ideal $n\mathbb{Z}$ is acceptable as long as it does not equal $p\mathbb{Z}$. In that sense, we can group $\begin{pmatrix} m_1 & \dots & m_t \end{pmatrix} A'^T = \begin{pmatrix} m'_1 & \dots & m'_t \end{pmatrix}$, because $p \nmid \det(A')$ and thus $m'_i \notin p\mathbb{Z}$. Second, we may combine $P^{-1}A'^T P$ in (4.44) to an element $A \in R_p$, which is related to an automorphism due to (4.42). Note that so far, the object $p^{\beta_t}P^{-1}$ was an entity because P^{-1} itself is not of integer entries and thus not a well defined operation within the congruence equations. However, now that we grouped $A^T = P^{-1}A'^T P$, which is integer itself, p^{β_t} on the left hand side of (4.44) will cancel against the same factor on the right hand side, again due to (4.36). Consequently, we are left with

$$\boldsymbol{x}^T Q_\psi^{(p)} A^T = \begin{pmatrix} m_1 p^{\beta_1} & \dots & m_t p^{\beta_t} \end{pmatrix} \ , \tag{4.46}$$

which is what we wanted to show, since $(Q_\psi^{(p)} A^T)\big|_\pi \in \text{Aut}(H_i)$.

4.4 Hypercharge shifts

Theories with remaining unbroken $U(1)$ factors exhibit yet another freedom concerning the discrete charge assignment. We have seen in section 3.7 that such unbroken $U(1)$ symmetries detach themselves from the formation of discrete symmetries. However, once we have figured out the actual discrete symmetry group of such a theory, which still possesses $U(1)$ invariance, the corresponding discrete charges of the transforming fields can be shifted by multiples of their $U(1)$ charges. This provides additional freedom, unavailable in theories invariant under discrete symmetries exclusively. The effect is easy to understand. As explained in sections 3.3 – 3.7, the case arises by breaking a $U(1)^k$ theory to the discrete group $\mathbb{Z}_{d_1} \times \dots \times \mathbb{Z}_{d_r}$ by a rank $r < k$ charge matrix of VEV fields, so that we are left with $k - r$ remaining $U(1)$ generators $\alpha_{r+1}, \dots, \alpha_k$.
For simplicity, let us assume a redundancy free setup and only one remaining $U(1)$ gauge group. The continuous and discrete phases of the transforming fields are given by

$$\psi^{(l)} \mapsto \exp\left(2\pi\mathrm{i}\left(q'_1(\psi^{(l)})\frac{m_1}{d_1} + \dots + q'_r(\psi^{(l)})\frac{m_r}{d_r}\right)\right) \exp\left(\mathrm{i}q(\psi^{(l)})\alpha(x)\right) \ \psi^{(l)} \tag{4.47}$$

with discrete $\mathbb{Z}_{d_i}$ charges q'_i and continuous $U(1)$ charge q. Since all couplings are still invariant under the remaining $U(1)$, a redefinition of the corresponding generator

$$\alpha(x) \mapsto \tilde{\alpha}(x) + 2\pi c \tag{4.48}$$

for arbitrary c will, of course, not change the theory due to $U(1)$ invariance. We can split $c = \frac{n_1}{d_1} + \cdots + \frac{n_r}{d_r}$ and plug (4.48) into the transformation (4.47). But we are free to group the constant part $\exp(2\pi\mathrm{i}\, c)$ with the discrete charges, combining to

$$\psi^{(l)} \mapsto \exp\left(2\pi\mathrm{i} \sum_{i=1}^{r} \frac{q_i'(\psi^{(l)})\, m_i + n_i q(\psi^{(l)})}{d_i} + \mathrm{i}\, \tilde{\alpha}(x)\right) \psi^{(l)} \ . \tag{4.49}$$

Remember that the $m_i \in \mathbb{Z}$ just reflect the equivalence relation modulo d_i of the discrete charges, hence, as long as $n_i q(\psi^{(l)})$ is integer for all i, (4.49) allows us to shift the discrete charges of a field by multiples of the fields (integer normalized) $U(1)$ charges.
For more than one remaining $U(1)$ the weighting of the shift can be any linear combination of the available $U(1)$ charges, therefore allowing for various shifting possibilities.
The case of one remaining $U(1)$ is of particular phenomenological interest, since every low energy model of particle physics has to exhibit $U(1)_Y$, i.e. hypercharge, invariance. Hypercharge shifts are commonly practiced in the discrete gauge symmetry literature [43, 45, 69, 31, 32, 51]. As discussed earlier, the most prominent target for discrete gauge symmetries is given by the MSSM, which requires discrete symmetries manifestly in order to suppress phenomenologically excluded couplings mediating rapid proton decay. For later convenience, let us give the MSSM hypercharge assignment according to table 2.1 in integer normalization at this point.

Superfield	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
Hypercharge $3Y$	1	-4	2	-3	6	-3	3

Table 4.2: Integer normalized hypercharges of the MSSM superfields.

4.5 Final example

Let us close this chapter by an example, which illustrates the key points we worked out herein. Since we want to draw the bridge to phenomenological model building in the next chapters, let us construct an example with the field content of the MSSM. Because we are missing important tools of model building, yet – like anomaly considerations – we will not be able to judge the viability of the exemplary model and, hence, should consider it as a toy model.

Let us take the matter (and Higgs) field content from table 4.2, charged under the Standard Model gauge group and assume a further $U(1)^2$ gauge symmetry, which will be broken by the Standard Model singlet fields ϕ_1, ϕ_2 acquiring VEVs at some high scale. Under the extra $U(1)$'s, call them $U(1)_A$ and $U(1)_X$, these shall have the following charges

	$U(1)_A$	$U(1)_X$
ϕ_1	4	6
ϕ_2	2	6

, which gives $$Q_\phi = \begin{pmatrix} 4 & 6 \\ 2 & 6 \end{pmatrix} . \tag{4.50}$$

This leads to the Smith normal form

$$D_\phi = M\, Q_\phi\, N = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} , \tag{4.51}$$

where the unimodular matrices M, N are not unique according to section 4.2; we can pick them, for instance, to be

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} . \tag{4.52}$$

From (3.39) we know that the permissible discrete symmetry, respected by the VEVs, is $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_6$. Now, the question is whether this is indeed the discrete symmetry of this toy model, or whether it might be redundant. In order to check this, we need to specify the $U(1)_A \times U(1)_X$ charges of the remaining 'matter' fields, which give rise to their discrete charges. Let us take the following setup

	$U(1)_A$	$U(1)_X$
Q	0	6
$\bar{U}$	3	2
$\bar{D}$	−1	4
L	0	2
$\bar{E}$	1	2
H_d	−1	2
H_u	1	4

(4.53)

stating the matrix structure of Q_ψ, analogously to (4.50). First, we have to multiply this matrix by N from the right in order to obtain the discrete charges Q'_ψ transforming under $\mathbb{Z}_2 \times \mathbb{Z}_6$. Then, we multiply the first column of Q'_ψ by a factor 3, which gives us a common modulus in return, such that the discrete symmetry group now is given by $\mathbb{Z}_6 \times \mathbb{Z}_6$. Next, we compute the Smith normal form resulting in

$$M_\psi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 6 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{m_1}{6} \\ \frac{m_2}{6} \end{pmatrix} \quad \text{with} \quad m_1, m_2 \in \mathbb{Z} \,, \tag{4.54}$$

according to (4.5). Thus, that factor of 6 can be canceled, which turns the second $\mathbb{Z}_6$ into a trivial $\mathbb{Z}_1$ manifesting a redundancy. The unimodular transformation matrix M_ψ^{-1}, now the final discrete charge matrix, is a rather huge 7×7 matrix, however, we are only interested in the first row, which states the discrete charges under the remaining $\mathbb{Z}_6$. Modulo 6 the charges read

$$Q_\psi^F = \begin{pmatrix} 0 & 5 & 1 & 2 & 5 & 5 & 1 \end{pmatrix}^T \,. \tag{4.55}$$

Let us contemplate a bit about this result. We have seen that the permissible discrete symmetry $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_6$, which we obtained by breaking $U(1)_A \times U(1)_X$, is not fully realized by the transforming field content. We rather found the reduced discrete symmetry $\mathcal{G}^F = \mathbb{Z}_6$. On our way, we made use of the Smith normal form and redefined the discrete 'matter' charges by means of the unimodular matrix M_ψ^{-1}, which we know not to be unique. Another choice, however, would lead to an automorphic charge. We have learned to calculate the automorphism group in section 4.2, so let us try it for this example.

From section 4.3.2 we gained the notion that we should be allowed to multiply all charges of (4.55) by a coprime of 6; that is, 5. Taking then the least residue we should obtain another equivalent charge setup. However, the proper description of the automorphism group from section 4.2.1 is based on the primary decomposition of discrete Abelian groups. Thus, in order to explore the automorphisms of (4.55) systematically, we need to find a suitable isomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_2 \times \mathbb{Z}_3$ first. Therefore, we have to map a generator (see apendix A.4) of $\mathbb{Z}_6$, e.g. 1, onto a generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$, e.g. $(1, 1)$. The mapping of all other elements is then fixed one by one.

Now we are able to examine the automorphisms of H_2 and H_3 separately, according to (4.16). Since both consist of one primary factor only, $H_2 = \mathbb{Z}_2$ and $H_3 = \mathbb{Z}_3$, the structure of the R_p matrices from (4.17) is fairly simple. Those are just integer numbers. Finally, theorem 5 tells us that the automorphisms are those numbers, which are mod p invertible over $\mathbb{Z}_p$. Hence, for $\mathbb{Z}_2$ we only find 1 – the trivial automorphism – and for $\mathbb{Z}_3$ we have 1 and 2. Thus, there is one non-trivial automorphism,

$\mathbb{Z}_6$	$\cong$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	Order
1	$\rightarrow$	$(1,1)$	6
2	$\rightarrow$	$(0,2)$	3
3	$\rightarrow$	$(1,0)$	2
4	$\rightarrow$	$(0,1)$	3
5	$\rightarrow$	$(1,2)$	6
0	$\rightarrow$	$(0,0)$	1

Table 4.3: Isomorphism mapping the invariant factor decomposition onto the primary decomposition

which consists of multiplying the $\mathbb{Z}_3$ charges by two, resulting in the exchange of charges

$$(1,1) \leftrightarrow (1,2) \quad \text{and} \quad (0,2) \leftrightarrow (0,1) \tag{4.56}$$

in table 4.3. On the one hand, this corresponds to the second possible isomorphism, mapping the $\mathbb{Z}_6$ generator 1 onto $(1,2)$, which states another suitable generator of $\mathbb{Z}_2 \times \mathbb{Z}_3$, since it also has order six. On the other hand, keeping the isomorphism fixed, it obviously corresponds to multiplying the $\mathbb{Z}_6$ charges by 5, which agrees with our notion what the automorphisms of $\mathbb{Z}_6$ should be. In fact, this is a well know result in the algebraic literature [71], the automorphism structure of cyclic groups is much simpler than for general Abelian finite groups. In particular, $\mathrm{Aut}(\mathbb{Z}_6) \cong (\mathbb{Z}/6\mathbb{Z})^\times = \{\bar{1}, \bar{5}\}$, as explained in appendix A.4.
To conclude, we have found that the $\mathbb{Z}_6$ charge assignment of the MSSM superfields (4.55) of this toy model is automorphic to

Superfield	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
$\mathbb{Z}_6$	0	1	5	4	1	1	5

, (4.57)

which is exactly the charge setup under the discrete symmetry "proton-hexality", introduced by [45]. Yet, remember that this is a toy model constructed to exemplify redundancies and automorphisms. We will see in section 5.1 that it is phenomenologically not viable because of anomaly constraints.

Chapter 5

Discrete symmetries and phenomenological model building

In the last chapters, Abelian discrete gauge symmetries were studied on a rather abstract footing, partitioning the field content of the underlying theory into VEV attaining and non VEV attaining fields. Challenging the described systematics through some concrete physical model, it is clear that the coupling structure will not be governed solely by the discrete Abelian symmetries, but by further constraints like e.g. unbroken internal gauge symmetries. Since these are model dependent, they have to be implemented as cases arise.
However, some phenomenological constraints with respect to the discrete symmetries are shared among all, or at least most models, such that a separate discussion deems appropriate. For instance, the question of (discrete) anomaly freedom needs to be addressed for every consistent quantum theory. Another problem, which we will discuss below, is to what extent R-symmetries fit into the described systematics. At least for the large class of MSSM related models, the identification of matter parity is of great importance. Finally, in supersymmetric theories VEVs have to be carefully aligned in order not to break supersymmetry.

5.1 (Discrete) anomalies

An anomaly – in a nutshell – is a violation of a symmetry of the classical Lagrangian by quantum effects. Although first encountered by badly divergent triangle diagrams [79], pictured in figure 5.1, anomalies are manifest only in the path integral approach, as a deviant transformation of the path integral measure. This was first noted by Fujikawa [80, 81] for Yang-Mills gauge theories including chiral fermions. Under a chiral transformation treating left and right handed fields differently

$$\psi \to e^{\mathrm{i}\alpha(x)\gamma_5}\psi \ , \tag{5.1}$$

where $\alpha(x) = \alpha^a(x)t_a$ comprises the generators t_a of the gauge group, the path integral measure transforms non-trivially

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to \mathcal{J}^{-2}\mathcal{D}\psi\mathcal{D}\bar{\psi} \ . \tag{5.2}$$

The Jacobian can be recast in terms of $\mathcal{A}$, the anomaly function,

$$\mathcal{J} = e^{-\mathrm{i}\int d^4x\,\mathcal{A}(x;\alpha)} \ . \tag{5.3}$$

For $\mathcal{J}$ to be trivial, $\mathcal{A}(x;\alpha)$ has to vanish or $\int d^4x\,\mathcal{A}(x;\alpha)$ needs to be an integer multiple of 2π. The anomaly function evaluates to

$$\mathcal{A}(x;\alpha) = \frac{g^2}{32\pi^2}\mathrm{tr}\left(\alpha(x)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \ , \tag{5.4}$$

where $F_{\mu\nu}$ denotes the field strength corresponding to the Abelian or non-Abelian gauge group to which the axial current couples and the trace runs over all internal indices. The anomaly function in four dimensions is (non-trivially) related to the triangle loop diagram, coupling the axial current to two gauge group vertices (see figure 5.1).

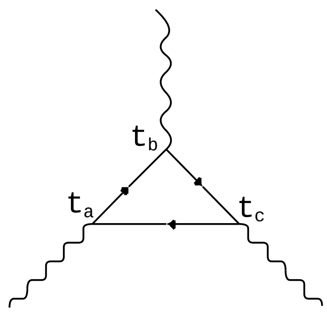


Figure 5.1: Triangle diagram coupling an axial current to gauge currents.

Apart from gauge anomalies, also gravitational anomalies are known [82]. For a theory containing gauged chiral currents to be consistent, it is of great importance that the gauge as well as the gravitational anomalies vanish, else gauge invariance or general covariance, respectively, will be broken [83]. If the anomaly belongs to a global chiral current, the consequence is only the violation of this symmetry. Such global anomalies are known to be relevant for physical effects, a prominent example being the global chiral anomaly of QCD. It corresponds to the triangle diagram, which couples the global chiral current to two electromagnetic currents, resulting in a measurable contribution to the decay rate $\Gamma(\pi^0 \to \gamma\gamma)$ [84, 79].

5.1.1 Continuous anomaly constraints

With the help of the index theorem [85]

$$\int d^4x \, \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F^{(a)}_{\mu\nu} F^{(b)}_{\rho\sigma} \mathrm{tr}(T_a T_b) \in \mathbb{Z} \,, \tag{5.5}$$

certain anomaly constraints can be derived. Here, the generators T_a belong to the fundamental representation, for which we use the common normalization $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$.
Consider, for instance, a chiral $U(1)$ coupling to two gauge currents, i.e. the $U(1) - G - G$ anomaly. The Jacobian

$$\mathcal{J} = \exp\left\{ -\mathrm{i} \int d^4x \, \alpha(x) \, \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F^{(a)}_{\mu\nu} F^{(b)}_{\rho\sigma} \mathrm{tr}(t_a t_b) \right\} \tag{5.6}$$

needs to be trivial for anomaly freedom. One has to consider all possible fermions running in the loop of figure 5.1, which do not necessarily live in the fundamental representation. In order to use the index theorem for all fermion representations $\boldsymbol{r}_f$, we write

$$\mathrm{tr}\left(t_a(\boldsymbol{r}_f) t_b(\boldsymbol{r}_f)\right) = 2l(\boldsymbol{r}_f) \cdot \tfrac{1}{2}\delta_{ab} \,. \tag{5.7}$$

Then, the index theorem together with the mean value theorem tells that in general

$$\sum_{\boldsymbol{r}_f} q_f \, l(\boldsymbol{r}_f) = 0 \,, \tag{5.8}$$

for (5.6) to yield one. Of course, all fermions combined in some representation $\boldsymbol{r}_f$ under G contribute only one times their $U(1)$ charge, which is pointed up by the sum over $\boldsymbol{r}_f$.
Such an anomaly constraint can be derived for the gravitational anomaly $U(1) - \mathrm{grav} - \mathrm{grav}$ as well,

$$\sum_f q_f = 0 \,, \tag{5.9}$$

where the charge of every fermion is counted, independent of their gauge group representation structure.
Yet, there is another way to cancel non vanishing anomalies for string derived mod-

els, known as the Green-Schwarz mechanism [86]. At the level of the ten dimensional superstring, non-vanishing anomalies cancel against an anomalous variation of the antisymmetric second rank tensor field B. After compactification, the theory remains anomaly free. However, anomalous looking $U(1)$ gauge groups may appear in the compactified four dimensional theory [87], e.g. orbifold compactifications of heterotic string theory typically provide exactly one such "anomalous" $U(1)$ factor [88] (more precisely, a basis can always be found, in which at most one $U(1)$ appears anomalous). The "anomaly" at four dimensional level cancels against the remnants of the B field, once again. This entails the breakdown of the "anomalous" symmetry, though, since the corresponding gauge boson acquires mass [87, 88, 89]. In particular, the "anomalous" symmetry will be violated by non-perturbative effects [90].

5.1.2 Discrete anomaly constraints

By the same token, discrete symmetries can be anomalous or non-anomalous [91]. For a discrete symmetry taken to be global, an anomaly means that the global discrete symmetry will be violated by certain operators and thus is broken.
The situation is much more serious for a discrete gauge symmetry. Since such discrete symmetries typically possess a gauged origin, and anomaly freedom is hereditary [92, 93], one is required to keep the low energy theory free of discrete anomalies, for the sake of the high energy theory's consistency. This aspect of discrete gauge symmetries has been discussed early in the literature [43, 44], however, a path integral approach leading to rigorous statements like (5.8) has been elaborated only more recently [91]. For instance, one can evaluate the $\mathbb{Z}_n - G - G$ anomaly constraint similar to the $U(1)$ case. Once again, the Jacobian is given by (5.6), yet, for a $\mathbb{Z}_n$ it is always possible to express $\alpha(x)$ as $\frac{2\pi m}{n} q'$, with $m \in \mathbb{Z}$ and q' denoting discrete charge. This is because representations of finite Abelian groups are completely reducible and the irreducible representations are one dimensional. Then, for the Jacobian to be trivial, it is sufficient to have

$$\sum_{\boldsymbol{r}_f} q'_f \, 2l(\boldsymbol{r}_f) = k \cdot n \,, \tag{5.10}$$

with $k \in \mathbb{Z}$ and q'_f the discrete $\mathbb{Z}_n$ charges of the fermions in each representation of G. This can be rewritten as a congruence equation

$$\sum_{\boldsymbol{r}_f} q'_f \, l(\boldsymbol{r}_f) = 0 \mod \frac{n}{2} \,. \tag{5.11}$$

Analogously, one can deduce the constraint

$$\sum_f q'_f = 0 \mod \frac{n}{2} \tag{5.12}$$

for the $\mathbb{Z}_n$ − grav − grav anomaly. Note that one can omit the denominator of the modulus in (5.11, 5.12) if n is odd, since 2 is coprime to any odd number, as has been shown in section 4.3.2. On the other hand, if n is even, the modulus is integer in any case. Thus the congruence equations (5.11, 5.12) are well defined.
But, pretty much like anomalies of continuous symmetries, discrete anomalies can be canceled by the Green-Schwarz mechanism [15, 94]. For instance, one can consider a remnant $\mathbb{Z}_n$ symmetry of a broken "anomalous" $U(1)$. This remnant discrete symmetry can be non-anomalous, i.e. fulfill the discrete anomaly constraints (5.11, 5.12), or it can appear anomalous [94]. Then, by the same token as in the continuous case, this remnant "anomalous" discrete symmetry is expected to be broken by non-perturbative terms. However, even though this "anomalous" $\mathbb{Z}_n$ constitutes a discrete gauge symmetry, its violation of the discrete anomaly constraints (5.11, 5.12) does not signal an inconsistency of the high energy theory, because of the Green-Schwarz mechanism, which cancels the anomaly of its gauged origin.
Yet, "anomalous" discrete symmetries from string theory do not necessarily possess an "anomalous" gauge embedding, e.g. in orbifold compactifications they can arise as remnants of the higher dimensional Lorentz group [52].

5.1.3 Example: proton-hexality

Let us revisit the example from section 4.5. We are now able to check the anomalies of this setup. Concerning the potential gauge embedding $U(1)_A \times U(1)_X$, we immediately detect from the charge assignment (4.53) that this cannot be anomaly free. All the $U(1)_X$ charges are positive and thus incapable to solve the constraints (5.8, 5.9). But, $U(1)_A$ is anomalous as well; let us check the gauge anomalies exemplarily. All fields in (4.53), which are charged under $SU(2)$ or $SU(3)$ live in fundamental representations, thus we can skip the overall Dynkin index $\frac{1}{2}$. We will write the anomaly contribution of each field as a product $a \cdot b \cdot c$, where a states the multiplicity due to generation independence, b denotes the multiplicity from not considered gauge group representations[1] and c the fields $U(1)_A$ charge. Given this notation, we

[1] For instance, there are 2 fermions in the Q doublet contributing to the $U(1) - SU(3) - SU(3)$ anomaly, or a color factor 3 of Q for $U(1) - SU(2) - SU(2)$.

can table the anomaly contributions as

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u	Σ
$U(1)_A - [SU(3)]^2$	$3{\cdot}2{\cdot}0$	$3{\cdot}1{\cdot}3$	$3{\cdot}1{\cdot}(-1)$					6
$U(1)_A - [SU(2)]^2$	$3{\cdot}3{\cdot}0$			$3{\cdot}1{\cdot}0$		$1{\cdot}1{\cdot}(-1)$	$1{\cdot}1{\cdot}1$	0

, (5.13)

which shows a $U(1)_A - SU(3) - SU(3)$ anomaly. Since the gauge anomalies are not universal, i.e. not the same, there is no hope for a Green-Schwarz cancellation. Hence, this toy model is phenomenologically not viable.

However, the resulting discrete symmetry, proton-hexality P_6, is a candidate for phenomenological model building, since it is free of discrete anomalies. Indeed, taking the charge assignment (4.57) we find the following contributions to the discrete gauge and gravitational anomalies.

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u	Σ
$\mathbb{Z}_6 - [SU(3)]^2$	$3{\cdot}2{\cdot}0$	$3{\cdot}1{\cdot}1$	$3{\cdot}1{\cdot}5$					18
$\mathbb{Z}_6 - [SU(2)]^2$	$3{\cdot}3{\cdot}0$			$3{\cdot}1{\cdot}4$		$1{\cdot}1{\cdot}1$	$1{\cdot}1{\cdot}5$	18
$\mathbb{Z}_6 - [\mathrm{grav}]^2$	$3{\cdot}6{\cdot}0$	$3{\cdot}3{\cdot}1$	$3{\cdot}3{\cdot}5$	$3{\cdot}2{\cdot}4$	$3{\cdot}1{\cdot}1$	$1{\cdot}2{\cdot}1$	$1{\cdot}2{\cdot}5$	93

(5.14)

Again, since all fermions live in the corresponding fundamental representation, the factor $2l(\boldsymbol{r}_f)$ in (5.10) drops out and thus the discrete gauge anomalies need to vanish modulo 6, while the discrete gravitational anomaly (5.12) has to be 0 mod $\frac{6}{2}$, which (5.14) obviously meets.

Note that the discrete anomaly freedom of proton-hexality does not contradict the unsuccessful gauge embedding of the toy model from section 4.5. In fact, it is impossible to embed proton-hexality into one $U(1)$ gauge group without introducing new massless fermions (which then become massive by breaking the gauge embedding). This can be seen as follows. Assuming the breaking $U(1)_{P_6} \to \mathbb{Z}_6$, the continuous charges of the matter fields are given by the discrete charges (4.57) of proton-hexality modulo six. This is because independently of how many VEVs break the single $U(1)_{P_6}$, the right unimodular "matrix" N, which defines the $\mathbb{Z}_6$ charges via (3.42) is constricted to be the integer ± 1. Repeating the anomaly calculation for the $U(1)_{P_6}$ charges results in

	Σ
$U(1)_{P_6} - [SU(3)]^2$	$18 + 6l_1$
$U(1)_{P_6} - [SU(2)]^2$	$18 + 6l_2$
$U(1)_{P_6} - [\mathrm{grav}]^2$	$93 + 6l_3$

, with $l_i \in \mathbb{Z}$. (5.15)

According to the continuous anomaly constraint (5.8, 5.9) each Σ value has to vanish. This is not problematic for the gauge anomalies, but not possible for the gravitational anomaly. Including hypercharge shifts does not alter this result.
The discrete anomaly freedom only tells us that there is a gauge embedding – it does not have to be an Abelian one. In other words, for embedding a discrete gauge symmetry into an Abelian gauge symmetry, the conditions (5.11, 5.12) are necessary but not sufficient criteria.

5.2 Remnant discrete R-symmetries

Supersymmetric theories allow for so called R-symmetries; that is, symmetries of the supersymmetry algebra whose generators do not commute with the supersymmetry generators. To be more specific, let us consider the case of minimal amount of supersymmetry, i.e. $N = 1$. In this case, the maximal continuous R-symmetry consists of a single $U(1)_R$, such that the $U(1)$ generator R yields the commutation relations

$$[Q_A, R] = Q_A \tag{5.16}$$

$$[\bar{Q}^{\dot{A}}, R] = -\bar{Q}^{\dot{A}} , \tag{5.17}$$

with the supersymmetry generator $Q_A, \bar{Q}^{\dot{A}}$ written in Weyl notation, i.e. $A = 1, 2$ (we use the conventions of [95]). The commutation relations fix the R-charges of the supersymmetry generators

$$Q_A \rightarrow e^{\mathrm{i}\alpha R} Q_A e^{-\mathrm{i}\alpha R} = e^{-\mathrm{i}\alpha} Q_A \tag{5.18}$$

$$\bar{Q}^{\dot{A}} \rightarrow e^{\mathrm{i}\alpha R} \bar{Q}^{\dot{A}} e^{-\mathrm{i}\alpha R} = e^{\mathrm{i}\alpha} \bar{Q}^{\dot{A}} \tag{5.19}$$

to be -1 for the pairing $Q, \bar{\theta}$ and $+1$ for $\bar{Q}, \theta$, which means the Grassmann variables, extending spacetime to superspace, transform non-trivially under $U(1)_R$

$$\theta_A \rightarrow e^{\mathrm{i}\alpha} \theta_A \tag{5.20}$$

$$\bar{\theta}^{(\dot{A})} \rightarrow e^{-\mathrm{i}\alpha} \bar{\theta}^{(\dot{A})} . \tag{5.21}$$

Thus, fermionic component fields transform differently than bosonic components of the very same superfield. Similarly, a discrete symmetry is of R type if the Grassmann variables transform non-trivially under the discrete rotations.
In principle, we can break a continuous R-symmetry down to discrete subgroups by assigning VEVs to scalar fields with non-vanishing R charge, in full analogy to section 3.3. Depending on the transformation behavior of the θ's, we obtain discrete

symmetries of R or of ordinary non-R type. However, having more Abelian non-R symmetries to come along, such that we want to know the discrete subgroup of $U(1)^k \times U(1)_R$, we need to rely on Smith normal form techniques, which essentially mix the R and non-R generators by means of unimodular mixing matrices. Thus, the question arises, which of the cyclic factors of the remnant discrete symmetry group $\mathcal{G} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ are of R-type? This is easily answered by promoting the Grassmann variables to dynamical fields and grouping them with the other remaining ψ fields. Then, (3.41) reveals which $\mathbb{Z}_{d_i}$ obtained an R fraction, since the θ's are charged under those discrete symmetries. Of course, this also holds for the redundancy techniques of chapter 4, such that we are able to control discrete R-symmetries during their algebraic transitions.
Analogously, the charge of the Grassmann variables under R-symmetries demands a modification of the charge lattice picture when it comes to couplings contained in the superpotential W. This is because we need to compensate the R charge of the measure $d^2\theta = -\frac{1}{4}\, d\theta_A\, d\theta^{(A)}$, such that

$$\mathcal{L} \supset \int d^2\theta\, W + \text{h.c.} \tag{5.22}$$

is uncharged, which implies that W needs to carry opposite R-charge than the Grassmann measure. The effect of integration and differentiation is equal for Grassmann variables, as indicated by the Berenzin integral

$$\int d\theta\, \theta = 1 \ . \tag{5.23}$$

But the identity, of course, does not transform under any symmetry. Thus, we can conclude that the R-transformation of the measure has to be opposite in sign than the θ transformation [96]

$$\int d\theta \to e^{-\mathrm{i}\alpha} \int d\theta \qquad \text{and thus} \qquad \int d^2\theta \to e^{-2\mathrm{i}\alpha} \int d^2\theta \ . \tag{5.24}$$

Hence, with the definitions of (5.20, 5.21), $U(1)_R$ invariance would require net R charge 2 for each superpotential coupling due to (5.24). Consequently, an allowed coupling of a remnant $\mathbb{Z}_q^R$ symmetry, after $U(1)_R$ breaking, would not have to lie on a conventional charge lattice dot as required by gauge invariance in the ordinary case (see section 3.4.1), but rather on a sort of "translated lattice" (see figure 5.2), which is not even a proper lattice, since it does not comprise the zero.
Note that this modification of the lattice picture only arises because we are considering superpotential couplings instead of ordinary couplings at the Lagrangian level.

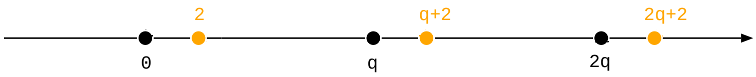


Figure 5.2: Charge lattice (black) versus “translated lattice” (orange) for a cyclic $\mathbb{Z}_q^R$.

Since the Grassmann measure in (5.22) now carries discrete R charge, the lattice points shift by an constant amount. For the general case of an arbitrary discrete symmetry $\mathcal{G} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ containing non-R as well as R-type cyclic factors, the charge lattice becomes translated only in the R directions. This can just as well be compensated by adding a “dummy” field χ to each coupling having twice the charge of the Grassmann variable θ, which was promoted to a transforming field. Hence, it automatically adds the required amount of discrete R charge of the measure in each lattice direction.

Let us illustrate these points by means of a brief example. Consider a $U(1)^2$ theory, which shall be broken by the charge matrix

$$\begin{array}{c|c} & U(1)_R \times U(1)_X \\ \hline Q_\phi & \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \\ \hline \theta & \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array} \quad , \tag{5.25}$$

where the charge assignment of θ (no matter which $\theta^{(A)}$, both transform identically) indicates that the first $U(1)_R$ is of R-type. The Smith normal form of this charge matrix is given by

$$M\,Q_\phi\,N = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \quad , \qquad \text{with e.g.} \qquad M = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} , \; N = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \quad , \tag{5.26}$$

such that the remnant discrete symmetry results in $\mathbb{Z}_3 \times \mathbb{Z}_9$. Now, are those discrete symmetries of R or non-R type? In order to answer this question, we have to figure out the discrete charges of the ‘matter’ field charge matrix Q_ψ, which we extended by the θ charges, as explained above. Remember from (3.41) that we have to multiply by N from the right to obtain the discrete charges

$$\begin{array}{ccc} Q_\psi & & Q'_\psi \\ \begin{pmatrix} 1 & 0 \end{pmatrix} & \xrightarrow{N} & \begin{pmatrix} 1 & 7 \end{pmatrix} \end{array} \quad . \tag{5.27}$$

The discrete charge of θ, aligned under the else unspecified Q'_ψ, seems to be non-trivial for both cyclic factors of the discrete symmetry group, suggesting that the whole discrete group is of R-type. Yet, from section 3.7 we know that a single $U(1)$ can only be liable for a cyclic group. However, $\mathbb{Z}_3 \times \mathbb{Z}_9$ has two invariant factors and thus is not cyclic. Hence, our notion tells us to be suspicious about both cyclic factors to be of R-type, since we started out with one $U(1)_R$ only. Indeed, in (5.27) we are misled by the usual algebraic hide-and-seek: it is possible to find an automorphism, which renders only one R-type cyclic factor. Relying on the automorphism techniques studied in section 4.2.1, we know that the automorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_9$ can be represented by the matrices $A^T = \begin{pmatrix} a & 3c \\ b & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{Z}$ and A (entry wise mod 3) invertible over $\mathbb{Z}_3$. Thus,

$$\boldsymbol{q}'^T(\theta) \mapsto \boldsymbol{q}'^T(\theta') = \boldsymbol{q}'^T(\theta) A^T = \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{5.28}$$

rotates θ completely in the $\mathbb{Z}_9$ direction. Note that a mere $\mathbb{Z}_3^R$ cannot be achieved, since then $3c + 7d \overset{!}{=} 0 \bmod 9$, with three inequivalent possible values for $c = 0, 1, 2$. We know from Bézout (see appendix A.2) that each case has one incongruent solution for d, which turns out to be a multiple of 3 each time and thus cannot account for an automorphism, because A (entry wise mod 3) is not even unimodular in these cases.

Having applied the above automorphism to Q'_ψ and $\boldsymbol{q}'(\theta)$, we now want to figure out the couplings of the superpotential allowed by the discrete symmetry. Therefore, we can either translate the charge lattice by $2 \cdot 1$ in $\mathbb{Z}_9^R$ direction and identify those $\boldsymbol{x}$ for which $\psi_1^{x_1} \dots \psi_b^{x_b}$ lies on the 'translated lattice'. Or, more conveniently, we can look for modified couplings $\chi \psi_1^{x_1} \dots \psi_b^{x_b}$, which lie on the ordinary charge lattice, where the "dummy" field χ imitates the charge of the measure $d^2\theta$. Algebraically, allowed values for the coupling exponents x_l can be calculated via systems of linear congruence equations, now with $g_j = 2$, as discussed in section 4.3.1. Yet, not every lattice point accounts for an allowed coupling, since the exponents $x_1, \dots, x_b$ need to be positive due to the holomorphy of W.

Note that one does not have to speculate about the coupling structure of the superpotential in case of a "$\mathbb{Z}_2^R$", since the measure transforms trivially under this symmetry. Hence, it is indistinguishable from an ordinary $\mathbb{Z}_2$. Therefore, $\mathbb{Z}_2$ symmetries (like R-parity) are always considered as ordinary non-R discrete symmetries [97].

5.3 Seeking for matter parity

The question whether matter parity (or equivalently R-parity) is a symmetry of a given model is of exceeding importance for MSSM related model building. Tracking a suitable $\mathbb{Z}_2$ symmetry with the correct transformation behavior for all fields has been an involved issue in the literature so far, see e.g. the discussion in [57]. One of the main achievements of our elaborations is that this task can be solved systematically now. Although we do need the whole machinery we developed to implement the search for matter parity, we will benefit from already having worked out most of the important steps.

The outline of seeking for $\mathcal{M}_p$ is very simple. Given a valid discrete Abelian symmetry group, we have to check whether there is a $\mathbb{Z}_2$ subgroup with charge 1 for all matterlike chiral superfields and charge 0 for all vector or Higgs superfields, so that the matter fields are odd and the rest even under $\mathcal{M}_p$.

Of course, tracking such a particular charge assignment predominantly accounts for automorphisms, and we know from section 4.2.1 lemma 1 that we can conveniently restrict ourselves to the subgroup H_2 of the discrete Abelian group. Note that it is not sufficient to look at the $\mathbb{Z}_2$ subgroups only! Take for instance $\mathbb{Z}_2 \times \mathbb{Z}_4$, we know all its automorphisms from section 4.2.2, which act non-trivially on the $\mathbb{Z}_2$ subgroup. In contrast, considering the $\mathbb{Z}_2$ subgroup alone, there are no non-trivial automorphisms. One might have the (wrong) idea to split the $\mathbb{Z}_4$ into $\mathbb{Z}_2$ factors, but $\mathbb{Z}_2 \times \mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the latter states a completely different group, which we learned in section 3.10.

Hence, we need to know the structure of H_2, which means we need the primary decomposition of the discrete Abelian group. However, an arbitrary discrete Abelian group, which might have arisen by breaking $U(1)^k$, but can as well have further contributions, e.g. from orbifolding or remnants of spontaneously broken non-Abelian groups, is generally neither in elementary divisor decomposition nor in invariant factors decomposition. In section 3.10.2 we found that one cannot read off the elementary divisor decomposition as long as the invariant factors are unknown. Yet, the redundancy elimination procedure from section 4.1.1 renders the invariant factors, even for an arbitrary discrete Abelian group, and keeping clear of redundant symmetries is obligatory anyway.

Having at hand the invariant factors, section 3.10.3 tells how to calculate the corresponding elementary divisors. Each invariant factor has to be factorized into powers of distinct primes. The direct product of all, in such a manner split, invariant factors can be rearranged by the distinct primes p_i, now appearing with various exponents β_{ij} such that we obtain the Sylow subgroups $H_i = \mathbb{Z}_{p_i^{\beta_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{\beta_{it}}}$, with charac-

teristic elementary divisors $p_i^{\beta_{ij}}$.
Now that we have figured out the structure of H_2, we can build the matrices A corresponding to automorphisms of H_2, as explained in section 4.2.1. Letting those act on the 'matter' field charges under H_2 reveals whether there is the desired matter parity charge structure contained in one of the $\mathbb{Z}_2$ subgroups of H_2, or not.
However, we do not know what the charges of the 'matter' fields are under H_2, yet. Let us fill this gap now. We have already studied how charges map among isomorphisms of cyclic groups in section 4.5 table 4.3. One just has to map a generator of the cyclic group in invariant factor decomposition onto a generator of its primary decomposition. Taking succeeding powers of each generator builds up the whole group in each decomposition and yields a one-to-one mapping of all the group elements between both decompositions.
In case of multiple invariant factors, we have more than one generator. Yet, this does not complicate the issue much, since one can construct a canonical basis of generators, disentangling the cyclic factors. The general Abelian discrete group $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ is generated by the r elements $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Each of these generators with the 1 at position i gives rise to the cyclic subgroup $\mathbb{Z}_{d_i}$, and any possible combination of multiple generators creates the remainder of the general Abelian discrete group $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$.
Now, since these generators are orthogonal by definition, we can use the isomorphism mapping of a cyclic group (e.g. table 4.3) for each invariant factor separately, which yields the generators of the primary decomposition. Again, constructing the entire group by means of the generators in each decomposition side by side gives us a one-to-one mapping of group elements.
Let us illustrate this procedure by means of a simple example. The easiest non-trivial case is $\mathbb{Z}_2 \times \mathbb{Z}_6$, which is already of order 12. We want to obtain the complete map of group elements from the invariant factor decomposition $\mathbb{Z}_2 \times \mathbb{Z}_6$ to the primary decomposition, which is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, according to section 3.10.3. As explained above, we take the canonical orthogonal generators and decompose each cyclic factor separately. We know the mapping of $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ from table 4.3, so we can map the orthogonal canonical generators from invariant factor to primary decomposition

$$\begin{array}{c|c|c} \mathbb{Z}_2 \times \mathbb{Z}_6 & \cong & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \\ \hline (1,0) & \rightarrow & (1,0,0) \\ (0,1) & \rightarrow & (0,1,1) \end{array} . \qquad (5.29)$$

That was already the main effort, the rest of the group elements are mapped one-to-one by taking powers of these generators

$$
\begin{array}{c|c|c}
\mathbb{Z}_2 \times \mathbb{Z}_6 & \cong & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \\
\hline
(0,0) & \to & (0,0,0) \\
(0,2) & \to & (0,0,2) \\
(0,3) & \to & (0,1,0) \\
(0,4) & \to & (0,0,1) \\
(0,5) & \to & (0,1,2) \\
\hline
(1,1) & \to & (1,1,1) \\
(1,2) & \to & (1,0,2) \\
(1,3) & \to & (1,1,0) \\
(1,4) & \to & (1,0,1) \\
(1,5) & \to & (1,1,2)
\end{array}\ , \tag{5.30}
$$

where the first block accounts to powers of each generator alone, and the second block is due to their combinations. Taken together, (5.29) and (5.30) states a complete map of all 12 elements.

Not all group elements need to be part of a given concrete 'matter' field charge matrix, but we now know how to translate those, which are present, into primary decomposition. Let us continue this example by assuming three fields ψ_M, ψ_H, ψ_V, which shall represent a matter, Higgs and vector superfield. These shall have the following discrete charges under $\mathbb{Z}_2 \times \mathbb{Z}_6$

$$
\begin{array}{c|c}
 & \mathbb{Z}_2 \times \mathbb{Z}_6 \\
\hline
\begin{array}{c} \psi_M \\ \psi_H \\ \psi_V \end{array} & \begin{pmatrix} 0 & 3 \\ 1 & 1 \\ 1 & 5 \end{pmatrix}
\end{array}\ , \tag{5.31}
$$

which states the charge matrix Q'_ψ. As we have seen, this translates to the primary decomposition as

$$
\begin{array}{c|c}
 & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \\
\hline
Q'_\psi & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}
\end{array}\ . \tag{5.32}
$$

Yet, the charge assignment of both $\mathbb{Z}_2$ factors does not correspond to matter parity; we would like to have charge one for the matter field and charge zero else. In order to check whether the desired charge setup is automorphic to the one above, we can constrict ourselves to $H_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e we can forget about the third column in

Q'_ψ. This gives us the charge matrix under H_2, which we introduced as $Q_\psi^{(2)}$ in section 4.3.2. According to section 4.2.1, the automorphisms of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are given by 2×2 matrices, which are (entry wise modulo two) invertible over $\mathbb{Z}_2$, acting from the right onto $Q_\psi^{(2)}$. Certainly, the required matrix A, such that

$$Q_\psi^{(2)} A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.33}$$

is of this type and manifests the suitability of the first $\mathbb{Z}_2$ for $\mathcal{M}_p$. Hence, this exemplified matter parity search was successful.

However, in case one does not find a $\mathbb{Z}_2$ subgroup with the desired charge setup by considering automorphisms, it does not mean that there is no matter parity, yet. Remember that we still have the freedom of $U(1)_Y$ shifts within the context of MSSM models, which we have discussed in section 4.4.

For instance, consider the setup of proton-hexality, introduced in section 4.5. From (4.57) and table 4.3 we can read off the H_2 charges of the MSSM matter and Higgs fields immediately

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
$H_2 \cong \mathbb{Z}_2$	0	1	1	0	1	1	1

. (5.34)

This is not the required charge setup for matter parity and the automorphisms are trivial in this case. Yet, shifting by one times the integer normalized hypercharge assignment from table 4.2 gives

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
$\mathcal{M}_p$	1	1	1	1	1	0	0

, (5.35)

which shows that indeed, $P_6 \cong \mathcal{M}_p \times \mathbb{Z}_3$.

For large discrete Abelian groups, the described steps are tedious to perform, but they can easily be automatized. An implementation as a `Mathematica`-package can be found in [98].

Yet, if the search for matter parity is unsuccessful on this footing, its appearance is not fully excluded. A loophole is given by discrete Abelian groups, which are further broken to a matter parity subgroup. We will present an example for such a scenario in chapter 6, where an "anomalous" discrete symmetry is broken by non-perturbative terms leaving a non-anomalous subgroup serving as matter parity. However, if the full discrete symmetry group is known and H_2 is trivial, then matter parity is impossible to achieve.

5.4 Supersymmetric vacua

The construction of manifestly supersymmetric actions is constricted to D-terms of general superfields and F-terms of chiral superfields, since these map into a total spacetime derivative under infinitesimal supersymmetry transformations. Thus, a general renormalizable supersymmetric Lagrangian, written in terms of chiral superfields, is given by the D-term of $\Phi_i \Phi_i^\dagger$ and the F-term of an analytic function containing left chiral superfields Φ_i only, known as the superpotential $W(\Phi_i)$, and the Hermitian conjugate thereof. In component field notation one can read off the scalar potential $V(\phi_i, \phi_i^\dagger)$, which takes the (off-shell) form

$$V = \frac{1}{2} D^{(a)} D^{(a)} + F_i F_i^\dagger \tag{5.36}$$

for a supersymmetric gauge theory, where the index a belongs to the gauge group generators $t^{(a)}$. The "equations of motion" for the auxiliary fields read

$$\begin{aligned} D^{(a)} &= -g\, \phi_i^\dagger t_{ij}^{(a)} \phi_j \ , & (5.37) \\ F_i &= -\frac{\partial W^\dagger}{\partial \Phi_i^\dagger}\bigg| \ . & (5.38) \end{aligned}$$

In presence of an "anomalous" $U(1)_A$ originating from string theory, as discussed above, the corresponding D-term becomes extended [99, 100] by the so called Fayet-Iliopoulos term ξ

$$D^{(A)} = q_i^{(A)} |\phi_i|^2 + \xi \ , \tag{5.39}$$

where $\xi = \frac{g^2 M_P^2}{192\pi^2} \operatorname{Tr} q^{(A)}$.

Now, for a supersymmetric vacuum configuration, the D- and F-terms must vanish, else supersymmetry will be broken, i.e.

$$\begin{aligned} \langle D^{(a)} \rangle &= 0 \ , & (5.40) \\ \langle F_i \rangle &= 0 \ . & (5.41) \end{aligned}$$

In particular, once we want to give VEVs to Standard Model singlet fields (e.g. in order to obtain discrete symmetries) at high scales, where supersymmetry should not be broken, we have to assign the VEVs in such a way that (5.40, 5.41) holds. Note that the constraints (5.40) and (5.41) are not independet in general. For a non-Abelian supersymmetric gauge theory a solution to the F-terms always induces

a solution for the D-terms [101].
It is well known, that gauge invariant holomorphic monomials correspond to D-flat directions [102, 103]. Let us exemplify this by means of a (non-anomalous) $U(1)^k$ gauge theory. The D-constraints (5.40) read

$$q_i^{(a)}|\langle\phi_i\rangle|^2 = 0 \qquad \forall\ a = 1, \ldots, k\ , \tag{5.42}$$

whereas the gauge invariance of a holomorphic monomial

$$\prod_i \Phi_i^{n_i}\ , \qquad \text{with } n_i \geq 0\ , \tag{5.43}$$

actually requires

$$n_i q_i^{(a)} = 0 \qquad \forall\ a \text{ and } n_i \geq 0\ . \tag{5.44}$$

Consequently, $\langle\phi_i\rangle = \sqrt{n_i}\, v$, with v a constant, implies $\langle D^{(a)}\rangle = 0$. Thus, every gauge invariant holomorphic monomial corresponds to a D-flat direction, and it can be shown that the converse also holds [103]. Hence, seeking all D-flat directions reduces to solving (5.44), which can be recast in matrix form

$$\begin{pmatrix} q_1^{(1)} & \cdots & q_m^{(1)} \\ \vdots & & \vdots \\ q_1^{(k)} & \cdots & q_m^{(k)} \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_m \end{pmatrix} = 0\ , \qquad n_i \in \mathbb{N}\ , \tag{5.45}$$

where we have assumed that there are m different fields Φ_i. That is, we are back at solving linear Diophantine equations, however, over strictly positive integers this time. This is a non-trivial task, which can be accomplished by means of Hilbert basis methods, as presented in [3].
Finally, in order to cancel the FI D-term, if present, it is obviously necessary to have at least one holomorphic gauge invariant monomial with anomalous charge of opposite sign than ξ.

Chapter 6

Discrete symmetries in string derived MSSM models

The most promising candidate for the unification of quantum mechanics and general relativity is string theory. Apart from a consistent theory of quantum gravity, it predicts spacetime supersymmetry in order to incorporate fermions, forming a superstring theory. A Yang-Mills gauge group, compatible with the superstring, comprises the Standard Model gauge group. Furthermore, string theory automatically yields extra dimensions, which have to be compactified to obtain four dimensional low energy physics. Although string theory is not fully formulated yet, an important step for its verification is to show the existence of a low energy limit resulting in the Standard Model. Unfortunately, string theory allows for a huge number of compactification possibilities, giving rise to the picture of the string landscape. Quasi-realistic vacua are very rare among all possible vacua, and, not at least due to the huge energetic gap between the string and the weak scale, a precise string theoretical description of all Standard Model parameters appears unlikely. However, the question whether generic properties of low scale physics can be reproduced at all is of great interest. In this regard, orbifold compactifications of heterotic string theory are a convenient starting point.
In the following, we will identify the discrete Abelian symmetry structure of an explicitly string derived low energy model with the exact spectrum of the MSSM. Studying the phenomenological consequences of the discrete Abelian symmetries, now traceable due to the methods and techniques we elaborated in the preceding chapters, we will find appealing results like highly suppressed proton decay and a solution to the μ-problem.

6.1 String derived MSSM models

Although a detailed understanding of orbifold models leading to a low energy theory with the exact spectrum of the MSSM is not mandatory for the discussion of phenomenological consequences of their remnant discrete symmetries, we want to outline the origin of such string derived MSSM models.

6.1.1 Heterotic string theory

Heterotic string theory is a somewhat peculiar construction of closed strings only, which treats the left-moving and right-moving sector essentially different [104]. It is the only known superstring theory, which is able to provide the gauge group $E_8 \times E_8$, besides $SO(32)$ also known to be permissible by anomaly cancellation in 10 dimensions. For a successful embedding of the low energy gauge group of the Standard Model, the unification chain

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8 \tag{6.1}$$

is favored against $SO(32)$.

The left- and right-moving modes of bosonic string theory, due to conformal anomaly freedom living in 26 critical dimensions, and superstring theory living in 10 critical dimensions, are completely decoupled. Therefore, it is possible to construct a hybrid string theory with a bosonic string structure for the, say, left-movers, and a superstring structure for the right-movers. The right-moving superstring induces $N = 1$ spacetime supersymmetry. This setup is called heterotic string theory. The connection between the dimensions of the left- and right-moving sector can be performed by the equivalent fermionic or bosonic construction. The first manifests that heterotic string theory is 10 dimensional, while the latter demonstrates the appearance of the gauge group. The explicit constructions can be found in e.g. [105]. From the point of view of the bosonic construction, 16 dimensions of the left-movers have to be compactified on a torus in order to match dimensions. The 16 dimensional torus lattice has to be a Euclidean even self-dual lattice for one loop modular invariance. There are only two such lattices, giving rise to the gauge groups $SO(32)$ or $E_8 \times E_8$. Hence, the gauge group in 10 spacetime dimensions is a consequence of the excessive 16 compactified bosonic degrees of freedom.

6.1.2 Orbifold compactifications

In order to make contact to four dimensional low energy physics we obviously have to compactify six further dimensions. Yet, simple torus compactifications lead to unrealistic low energy theories. The geometry of the internal six dimensional space significantly affects the four dimensional effective theory, such that the requirement of four dimensional $N = 1$ spacetime supersymmetry basically accounts for Calabi-Yau or orbifold compactifications. Since we are interested in the detailed phenomenology of the effective low energy theory, orbifold compactifications are much more suitable. This is because orbifolds are flat except for singular points and thus the explicit form of the metric is known, in contrast to most Calabi-Yau manifolds.

An orbifold is the quotient space M/G of a smooth manifold M and its discrete group G of non freely acting isometries. This means that points of M, which are connected by a transformation corresponding to an element of G, become identified on M/G. Since there are fixed points in M with respect to G, i.e. there are points in M, which are left invariant by the action of some non-trivial element of G, the quotient space has singularities, which makes it an orbifold instead of a manifold.

Let us focus on toroidal orbifolds in the following, which means we take a six-torus T^6 and mod out the finite symmetry group of its torus lattice, called the point group P. The torus itself can be viewed as the quotient of $\mathbb{R}^6$ and a lattice Γ, the torus lattice, such that $T^6 \cong \mathbb{R}^6/\Gamma$. One defines the space group S to be the semi-direct product of the point group P and the lattice translations of Γ. Hence, the orbifold can be written as the quotient space $\mathbb{O} = \mathbb{R}^6/S$.

Now, considering an orbifold compactification of heterotic string theory [106, 107], i.e. taking the internal manifold to be an orbifold, allows for new string modes, which fulfill the boundary condition of closed strings. Besides the usual boundary condition for closed strings in ten dimensional flat space, which are now called the untwisted modes, the strings can also close up to a space group transformation. String modes that close only modulo a non-trivial space group transformation are called twisted modes. These turn out to be localized at the fixed points of the orbifold.

Modular invariance further requires the action of the orbifold to take effect on the gauge degrees of freedom, living in the left-moving sector. Therefore, orbifold compactifications are able to break the huge gauge symmetry of the 10 dimensional heterotic theory, yet, without touching the rank of the gauge group. Hence, the four dimensional gauge group will generically consist of some amount of $U(1)$ factors. Several of these $U(1)$'s can appear to be "anomalous" [99], however, a basis in which at most one "anomalous" $U(1)_A$ remains can always be found [88]. The

would-be anomaly is canceled by the Green-Schwarz mechanism, of course. Yet, $U(1)_A$ induces an FI D-term, as explained in section 5.4, which is of great importance for low energy phenomenology. In order to preserve supersymmetry just below the string scale, this FI D-term has to be canceled, which requires to assign large VEVs to certain fields. In fact, any Standard Model singlet field may obtain a VEV as long as F- and D-flatness is ensured. These VEVs can give large masses to unwanted states of the spectrum as well as spontaneously break the gauge group, reducing its rank. In particular, the Abelian part of the gauge group after compactification, $U(1)^k$, can be broken.
This is where our mechanism described in the preceding chapters takes effect. The knowledge of the remaining discrete Abelian group ameliorates the understanding of the low energy limit significantly.

6.1.3 Towards realistic MSSM limits

Although obtaining the Standard Model gauge group from heterotic orbifold compactifications is straightforward, the appearance of the exact MSSM spectrum is rather challenging. Such models, equipped with the chiral matter content of the MSSM, yet, without chiral exotics and all vectorlike exotics decoupling from the low energy theory, have been presented in [55, 56]. At the orbifold point, one has differing gauge groups at the fixed points and the bulk, i.e. untwisted sector. The intersection of these gives the four dimensional gauge group, which contains the Standard Model gauge group. There are three generations of Standard Model matter including the right handed neutrino, which arise from the twisted and untwisted sectors in a way, providing a large top Yukawa coupling. Assigning VEVs to Standard Model singlet fields along D- and F-flat directions further breaks the gauge group to G_{SM} times a true hidden sector, needed for low energy supersymmetry breaking, e.g. via gaugino condensation. In particular, cancellation of the FI term induces VEVs close to the GUT scale. Such large VEVs account for the decoupling of the $U(1)$ gauge bosons and the vectorlike exotics, because they induce large mass terms. Numerous orbifold models with such a potentially realistic structure have been found [58].
Yet, for a viable model, more detailed issues like the flavor structure, existence of matter parity, absence of rapid proton decay, and the μ-problem have to be addressed. Searches for models providing these features have been performed, e.g. in [57]. In this context, the role of discrete symmetries turned out to be quite helpful. The remnant discrete symmetries arising by the breaking of the Abelian part of the gauge group $U(1)^k$ are now fully understood due to the methods described in this work. Therefore, the quest for matter parity has been simplified. Moreover,

additional discrete symmetries of R or non-R type can take effect on the coupling structure, enhancing the phenomenology of the string derived model.
In the following, we will present a particularly interesting discrete symmetry, $\mathbb{Z}_4^R$, which is able to provide matter parity, suppress dimension five proton decay operators as well as resolve the μ-problem simultaneously.

6.2 Discrete phenomenology in $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

We will now discuss a phenomenologically appealing vacuum configuration of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification with the exact MSSM spectrum. The orbifold construction is very similar to the model presented in [108]. Yet, our vacuum differs significantly. Albeit being equipped with matter parity, found by the methods described in the preceding chapters, the configuration in [108] describes only a semi realistic vacuum, since all Higgs candidates attain large masses. In contrast, our vacuum forbids mass terms for the MSSM Higgs fields at all order in perturbation theory due to the discrete symmetry $\mathbb{Z}_4^R$. Since this symmetry is "anomalous", i.e. its anomaly is canceled by a discrete version of the Green-Schwarz mechanism, a light μ-term is reintroduced at the non-perturbative level. The $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4^R$ is non-anomalous, though, and serves as matter parity. Thus, dimension four proton decay operators are forbidden by the matter parity subgroup and dimension five proton decay operators are highly suppressed by $\mathbb{Z}_4^R$. The model has further realistic properties such as full rank Yukawa couplings.

6.2.1 Vacuum configuration exhibiting $\mathbb{Z}_4^R$

After orbifolding there are discrete R and non-R symmetries as well as continuous non-R symmetries. The discrete symmetries do not necessarily have an embedding into an internal gauge group, as they can arise as remnants of the ten dimensional Lorentz group [109]. However, in order to be discrete gauge symmetries they certainly have to be anomaly free, or a potential "anomaly" has to be canceled by the Green-Schwarz mechanism [91]. In particular, the gauge group of the model considered here is $G_{\text{SM}} \times [SU(3) \times SU(2) \times SU(2)]_{\text{hid}} \times U(1)^8$ at the orbifold point. Choosing a vacuum, i.e. assigning VEVs to Standard Model singlet fields, further breaks the gauge group. In order to preserve supersymmetry the VEVs are aligned in such a way that the F- and D-terms vanish, including the cancellation of the FI term.
The vacuum considered here is chosen such that the remnant symmetry group is $G_{\text{SM}} \times [SU(2)]_{\text{hid}} \times \mathbb{Z}_4^R$, where the remnant discrete symmetry equals the one pre-

sented in [52]. Thus, $\mathbb{Z}_4^R$ is "anomalous", i.e. its anomaly is canceled by a discrete version of the Green-Schwarz mechanism, yet, it has a non-anomalous subgroup resulting in matter parity. A remnant true hidden sector can be useful for low energy supersymmetry breaking, e.g. via gaugino condensation [110, 111]. The spectrum of this configuration is given by table 6.1.

Field	Q_i	$\bar{U}_i$	d_i	$\bar{d}_i$	ℓ_i	$\bar{\ell}_i$	$\bar{E}_i$	x_i	$\bar{x}_i$	y_i	z_i	s_i
max(i)	3	3	3	6	9	6	3	5	5	6	6	37
$SU(3)_c$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$SU(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$U(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0
$SU(3)_{\text{hid}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$SU(2)_{\text{hid}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
$SU(2)_{\text{hid}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$

Table 6.1: Field labels and their Standard Model and hidden sector quantum numbers. The $d_i/\bar{d}_i$ decompose into quarks and exotics, while the $\ell_i/\bar{\ell}_i$ split into leptons and Higgs candidates.

The vacuum is determined by giving the following set of fields VEVs

$$\begin{aligned}\phi^{(i)} =\{ & s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, \\ & x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, \\ & y_3, y_4, y_5, y_6\} \ . \end{aligned} \tag{6.2}$$

In particular, all singlet fields s_i, which are uncharged under $\mathbb{Z}_4^R$, obtain a VEV. The non-Abelian G_{SM} singlet fields $x_i/\bar{x}_i$, y_i develop VEVs as $\mathbb{Z}_4^R$ invariant contractions. Lepton and Higgs fields are indistinguishable by their MSSM quantum numbers only. Due to the discrete symmetry $\mathbb{Z}_4^R$ we are able to discriminate between MSSM matter fields, Higgs fields, and exotics. Since the non-perturbative terms that break the $\mathbb{Z}_4^R$ have charge two [52], a $\mathbb{Z}_2$ symmetry is left unbroken. Former $\mathbb{Z}_4^R$ even/odd charges become even/odd elements of $\mathbb{Z}_2$. Thus, in order to obtain the matter parity charge assignment, i.e. all matter fields charge one and the Higgs fields charge zero under $\mathbb{Z}_2$, the $\mathbb{Z}_4^R$ charges of matter or Higgs fields need to be odd or even, respectively. This situation allows us to split the potential d-quarks $d_i/\bar{d}_i$ of table 6.1 into quarks and exotics as well as the $\ell_i/\bar{\ell}_i$ into leptons and Higgs candidates (see table 6.2).

Lepton doublets	d-quarks	Higgs candidates		exotic triplets	
L_i	$\bar{D}_i$	H_i	$\bar{H}_i$	δ_i	$\bar{\delta}_i$
ℓ_4, ℓ_6, ℓ_7	$\bar{d}_1, \bar{d}_3, \bar{d}_4$	$\ell_1, \ell_2, \ell_3, \ell_5, \ell_8, \ell_9$	$\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4, \bar{\ell}_5, \bar{\ell}_6$	d_1, d_2, d_3	$\bar{d}_2, \bar{d}_5, \bar{d}_6$

Table 6.2: Identification of MSSM matter fields versus Higgs candidates and exotics.

Thus we obtain three generations of MSSM matter fields. Next, the massless Higgs fields need to be identified.

6.2.2 Mass matrices and Yukawa couplings

The $\mathbb{Z}_4^R$ charges of the Higgs and exotic fields, given in table 6.3, fix the structure of the $\bar{H}_i - H_j$ and $\bar{\delta}_i - \delta_j$ mass matrices. Since an allowed coupling of the superpotential needs to have $\mathbb{Z}_4^R$ charge two, a mass matrix entry $\mathcal{M}_{ij}$ only can be non-zero, if the charges of the corresponding fields sum to two.

(a) $\mathbb{Z}_4^R$ charges of the Higgs candidates.

	H_1	H_2	H_3	H_4	H_5	H_6	$\bar{H}_1$	$\bar{H}_2$	$\bar{H}_3$	$\bar{H}_4$	$\bar{H}_5$	$\bar{H}_6$
$\mathbb{Z}_4^R$ charge	0	2	0	2	0	0	0	2	0	0	2	2

(b) $\mathbb{Z}_4^R$ charges of the exotics.

	δ_1	δ_2	δ_3	$\bar{\delta}_1$	$\bar{\delta}_2$	$\bar{\delta}_3$
$\mathbb{Z}_4^R$ charge	0	2	2	2	0	0

Table 6.3: The $\mathbb{Z}_4^R$ charges of the Higgs candidates and exotic triplets determine the structure of their mass matrices.

Thus, the Higgs mass matrix shows the following pattern

$$\mathcal{M}_H = \begin{pmatrix} 0 & \tilde{\phi}^1 & 0 & \tilde{\phi}^1 & 0 & 0 \\ \tilde{\phi}^1 & 0 & \tilde{\phi}^1 & 0 & \tilde{\phi}^1 & \tilde{\phi}^1 \\ 0 & \tilde{\phi}^1 & 0 & \tilde{\phi}^3 & 0 & 0 \\ 0 & \tilde{\phi}^3 & 0 & \tilde{\phi}^5 & 0 & 0 \\ \tilde{\phi}^3 & 0 & \tilde{\phi}^1 & 0 & \tilde{\phi}^1 & \tilde{\phi}^3 \\ \tilde{\phi}^3 & 0 & \tilde{\phi}^1 & 0 & \tilde{\phi}^3 & \tilde{\phi}^1 \end{pmatrix} . \tag{6.3}$$

The non-zero entries $\tilde{\phi}^n$ denote polynomials of order n in the $\phi^{(i)}$ fields, which have been explicitly computed by string selection rules. The rank of the $\bar{H}_i - H_j$ mass matrix $\mathcal{M}_H$ is five, such that one pair of Higgs fields H_u, H_d remains massless,

$$\begin{aligned} H_u &= c_1\,\bar{H}_1 + c_2\,\bar{H}_3 + c_3\,\bar{H}_4 \ , \\ H_d &= c_4\,H_1 + c_5\,H_3 + c_6\,H_5 + c_7\,H_6 \ , \end{aligned} \tag{6.4}$$

with coefficients c_i. In contrast, the $\bar{\delta}_i - \delta_j$ mass matrix of the exotics has full rank. It reads

$$\mathcal{M}_\delta = \begin{pmatrix} \tilde{\phi}^5 & 0 & 0 \\ 0 & \tilde{\phi}^1 & \tilde{\phi}^3 \\ 0 & \tilde{\phi}^3 & \tilde{\phi}^1 \end{pmatrix} \ . \tag{6.5}$$

Thus the exotic particles decouple from the low energy theory. Effective Yukawa couplings arise by singlet fields, which obtain VEVs, coupling to trilinear terms containing one of the massless Higgs

$$W_{\text{Yuk}} = \sum_{i=1,3,4} (Y_U^{(i)})^{mn}\, Q_m \bar{U}_n \bar{H}_i + \sum_{i=1,3,5,6} (Y_D^{(i)})^{mn}\, Q_m \bar{D}_n H_i + \sum_{i=1,3,5,6} (Y_E^{(i)})^{mn}\, L_m \bar{E}_n H_i \ . \tag{6.6}$$

At tree level, the Yukawa matrices have the following structure

$$\begin{aligned} Y_U^{(1)} &= \begin{pmatrix} \tilde{\phi}^2 & \tilde{\phi}^4 & \tilde{\phi}^6 \\ \tilde{\phi}^4 & \tilde{\phi}^2 & \tilde{\phi}^6 \\ \tilde{\phi}^6 & \tilde{\phi}^6 & 1 \end{pmatrix} , & Y_U^{(3)} &= \begin{pmatrix} 1 & \tilde{\phi}^6 & \tilde{\phi}^4 \\ \tilde{\phi}^6 & 1 & \tilde{\phi}^4 \\ \tilde{\phi}^4 & \tilde{\phi}^4 & \tilde{\phi}^2 \end{pmatrix} \\ Y_E^{(5)} = (Y_D^{(5)})^T &= \begin{pmatrix} \tilde{\phi}^6 & \tilde{\phi}^6 & \tilde{\phi}^6 \\ \tilde{\phi}^6 & \tilde{\phi}^6 & 1 \\ \tilde{\phi}^6 & 1 & \tilde{\phi}^4 \end{pmatrix} , & Y_E^{(6)} = (Y_D^{(6)})^T &= \begin{pmatrix} \tilde{\phi}^6 & \tilde{\phi}^6 & 1 \\ \tilde{\phi}^6 & \tilde{\phi}^6 & \tilde{\phi}^6 \\ 1 & \tilde{\phi}^6 & \tilde{\phi}^4 \end{pmatrix} \ . \end{aligned} \tag{6.7}$$

All Yukawas have full rank, yet the tree level $SU(5)$ GUT relations $Y_E^{(i)} = (Y_D^{(i)})^T$ are desirable only for the third generation, but not for the light generations. The contributions $Y_U^{(4)}$ and $Y_{E,D}^{(1,3)}$ are of high order in $\tilde{\phi}$, such that these can be neglected as we assume the $\tilde{\phi}$ VEVs to be small with respect to the string scale. Note that the block structure in the Yukawa matrices and the $\bar{\delta}_i - \delta_j$ mass matrix are relics of a non-Abelian discrete D_4 flavor symmetry of the underlying string model.

6.2.3 Non-perturbative violation of $\mathbb{Z}_4^R$

The discrete $\mathbb{Z}_4^R$ symmetry is "anomalous", in fact. That is, the discrete anomaly constraints, as introduced in section 5.1.2, are not satisfied. However, they are universal, such that an anomaly cancellation via the Green-Schwarz mechanism is possible. This can be checked easily, e.g. for the gauge anomalies. Since only massless fermions contribute to the anomaly, the calculation can be restricted to the MSSM matter fields (see table 6.4) and the massless Higgs fields (see (6.4) and table 6.3). Yet, because the symmetry in consideration is of R type, the gauginos

	Q_i	$\bar{U}_i$	$\bar{D}_i$	L_i	$\bar{E}_i$
$\mathbb{Z}_4^R$ charge	1	1	1	1	1

Table 6.4: $\mathbb{Z}_4^R$ charges of the MSSM matter fields.

are also charged and thus contribute to the anomaly [91]. With the convention of section 5.2, i.e. the θ have $\mathbb{Z}_4^R$ charge 1, the $\bar{\theta}$ charge -1 and thus the superpotential charge 2, we know that the gauginos have charge 1 and the fermions of chiral q charged superfields have charge $q-1$ (mod 4). The Dynkin index of the chiral fields is $\frac{1}{2}$, for the fundamental representation. The gauge fields live in the adjoint representation of the gauge group G, thus the Dynkin index is given by the quadratic Casimir $c_2(G) = N$ for $SU(N)$. Since all chiral matter superfields have charge one, the corresponding fermions are uncharged and do not contribute to the anomaly. Hence, the calculation reduces to

	gauginos	H_d	H_u	Σ
$\mathbb{Z}_4^R - [SU(3)]^2$	$c_2(SU(3)) \cdot 1$			3
$\mathbb{Z}_4^R - [SU(2)]^2$	$c_2(SU(2)) \cdot 1$	$\frac{1}{2} \cdot 3$	$\frac{1}{2} \cdot 3$	5 .

(6.8)

Remembering the discrete anomaly constraints (5.11) from section 5.1.2, the sum Σ has to vanish modulo $\frac{4}{2} = 2$ for discrete anomaly freedom. This is not the case, but the gauge anomalies are universal, i.e. identical modulo 2, which is the precondition for the Green-Schwarz mechanism to cancel the anomaly.
At the non-perturbative level, terms including the factor e^{-aS}, where a is a constant and S the dilaton superfield, arise in the superpotential. The dilaton (or more precisely its imaginary part, the axion) transforms non-trivially under the "anomalous" symmetry $\mathbb{Z}_4^R$ and plays a crucial role for its anomaly cancellation via the Green-Schwarz mechanism. Therefore, the non-perturbative terms of the superpotential are fully gauge invariant, including the "anomalous" symmetry. Once the dilaton develops a VEV, the non-perturbative terms violate this symmetry and hence break the $\mathbb{Z}_4^R$, yet, because of the exponential by highly suppressed terms. The matter parity subgroup of $\mathbb{Z}_4^R$, though, is non-anomalous and thus also holds at the non-perturbative level.

6.2.4 Proton decay

Because of the non-anomalous matter parity, dimension four proton decay operators are entirely forbidden, at the perturbative as well as at the non-perturbative level. This is phenomenologically necessary, since the stringent experimental limits on proton decay [112] require a theoretical explanation for the corresponding tininess of the dimension four operator coupling constants in (2.2b). Yet, forbidding all renormalizable proton decay operators is not sufficient to account for the experimental limits. Although effective non-renormalizable superpotential couplings [42], such as $\frac{1}{\Lambda}QQQL$ or $\frac{1}{\Lambda}\bar{U}\bar{U}\bar{D}\bar{E}$, are suppressed by the cut-off scale Λ, they still contribute at an unacceptable rate.
In the present model, the discrete $\mathbb{Z}_4^R$ symmetry forbids such dimension five operators at the perturbative level. They become reintroduced at the non-perturbative level, yet, exponentially suppressed. For instance, the lowest order occurrence of $QQQL$, which appears to be the most dangerous dimension five contribution [113], is given by

$$W_{\text{np}} \supset e^{-aS}\, QQQL\, \tilde{\phi}^{13} \ . \tag{6.9}$$

Due to the high suppression of these operators, the leading contribution to proton decay is given by dimension six operators.

6.2.5 Solution to the μ-problem

The bilinear Higgs coupling $\mu H_u H_d$ is a renormalizable gauge invariant term, which thus may enter the superpotential (see (2.2a)). The natural scale for the parameter μ would be the Planck scale. Yet, for viable spontaneous symmetry breaking $SU(2)_W \times U(1)_Y \to U(1)_{em}$ by the MSSM Higgs mechanism, the μ-term needs to be around the weak scale. The issue of obtaining such a small μ parameter is known as the μ-problem. There are many different solutions to this problem [114, 115]; we will proceed similar to [116]. That is, the μ-term is assumed to be absent in the superpotential, but appears at the non-renormalizable level

$$W = W_0 + \lambda W_0 H_u H_d \ , \tag{6.10}$$

where W_0 is the superpotential without the μ-term and λ is a parameter of order one in Planck units. This is a valid ansatz for the string derived MSSM model considered here, see the discussion in [117, 118]. The μ-term becomes effectively generated in this scenario, once the superpotential W_0 attains an expectation value

$$\mu = \lambda \langle W_0 \rangle \ . \tag{6.11}$$

Since the requirement of a vanishing cosmological constant implies $m_{3/2} \sim \langle W_0 \rangle$, the μ-term is of the right size, given by the gravitino mass $m_{3/2}$, which sets the scale of the soft terms.
In the present model, the μ-term is forbidden by $\mathbb{Z}_4^R$ in the perturbative superpotential, yet, it may appear at the non-perturbative level. Due to $\mathbb{Z}_4^R$, the perturbative part of the superpotential does not develop a VEV, such that only the non-perturbative part accounts for the μ-term.

Chapter 7

Conclusions

We have elaborated the breaking of a continuous Abelian gauge group $U(1)^k$ down to a possible remnant discrete group $\mathcal{G}$. Such a 'general discrete Abelian gauge symmetry' was found to be representable in a unique way as $\mathcal{G} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$ with $d_i | d_{i+1}$ and $r \leq k$, which is known as the invariant factor decomposition of finite Abelian groups in abstract algebra. The algebraic formulation was achieved via the geometrical concept of the 'charge lattice', which was used to parametrize the physical structure of symmetry breaking in a descriptive way. Transformations of the lattice basis by unimodular matrices motivated an orthogonalization of the charge lattice by means of the Smith normal form, an algebraic concept.

In a next step, redundant and equivalent structures were studied. A redundancy reflects the situation of transforming physical fields, i.e. discrete group elements, which are not capable of fully realizing the maximal discrete Abelian symmetry allowed by the vacuum. We presented a method to eliminate such redundancies, which results in the true discrete Abelian symmetry of the physical theory. Automorphisms of the discrete Abelian group $\mathcal{G}$ correspond to equivalent discrete charge assignments from a physical point of view. A description of the automorphism group of finite Abelian groups is known, which we used to resolve the identification of R-parity or matter parity for MSSM model building.

Finally, we discussed the role of discrete Abelian symmetries in heterotic orbifold compactifications, which lead to the exact matter spectrum of the MSSM in the low energy limit. Such string derived MSSM models are a natural candidate for remnant discrete Abelian gauge symmetries, since orbifolding preserves the rank of the string gauge group. This entails an unbroken continuous Abelian gauge group at the orbifold point, which is spontaneously broken by vacuum expectation values of Standard Model singlet fields, allowing for remnant discrete gauge symmetries. We demonstrated through a concrete model that particularly discrete Abelian R-

symmetries are phenomenologically appealing. The model exhibits a simple $\mathbb{Z}_4^R$ symmetry, which is “anomalous”, i.e. the anomaly is canceled by the Green-Schwarz mechanism, and thus violated at the non-perturbative level. It is able to forbid dimension four proton decay due to a non-anomalous matter parity subgroup and highly suppress dimension five proton decay as well as resolve the μ-problem.

Outlook

We leave the field of non-Abelian discrete gauge symmetries for future work. These have been studied from a bottom-up perspective, i.e. their anomaly constraints have been elaborated in [91]. Yet, a generic mechanism to obtain remnant discrete gauge symmetries from spontaneously broken non-Abelian gauge groups has not been elaborated so far. Generalizing the approach presented here to the non-Abelian case is not straight forward, since it involves solving equations containing matrix exponentials. Our mechanism benefited from discrete Abelian groups having one dimensional irreducible representations, which is not the case for non-Abelian groups. Discrete non-Abelian symmetries have the advantage of a richer structure, which might be necessary for involved model building issues. But in turn they loose the beauty of simplicity, which is inherent to discrete Abelian symmetries. Finally, we note that the knowledge of remnant discrete symmetries of non-Abelian gauge symmetries also concerns the discrete Abelian gauge symmetry group, since these remnants may contain discrete Abelian factors.
With regard to string derived MSSM model building, it is certainly interesting to further study the impact of discrete Abelian gauge symmetries by means of the presented methods. A next obvious step is to elaborate the role of the discrete Abelian symmetry group in different orbifold constructions, which are known to be phenomenologically promising, as for instance the $\mathbb{Z}_6$-II mini-landscape models in [58, 57].

Appendix A

Algebraic glossary

In this appendix, definitions, conventions and basic theorems are presented, which are silently assumed to be known throughout this thesis, but might not lie within the standard scope of physicists. It shall serve as a glossary, however, without the claim to be exhaustive, since common knowledge (as the definition of a group and alike) will be spared out. Consequently, we will not prove the stated theorems, refer to [71, 66, 67] for this purpose.

A.1 Basic arithmetic

Let us start with the fundamental theorem of arithmetic, which addresses the decomposition of integer numbers into products of prime powers.

Theorem 6 (Fundamental theorem of arithmetic) *Let n be a positive integer greater one. There exists a decomposition into prime numbers*

$$n = p_1^{\alpha_1} \dots p_s^{\alpha_s} \ , \tag{A.1}$$

which is unique up to ordering.

Next, we want to recall the notion of residue classes.

A.2 Congruences

For a positive integer n, define a relation over $\mathbb{Z}$ given by

$$a \sim b \text{ if and only if } n|(a-b) \ . \tag{A.2}$$

This is an equivalence relation. If $a \sim b$ one writes

$$a = b \bmod n \tag{A.3}$$

and says *a is congruent to b modulo n*. Therefore, $\mathbb{Z}$ can be partitioned into n equivalence classes

$$\bar{0}, \bar{1}, \ldots \overline{n-1} \ , \tag{A.4}$$

where the *congruence class* or *residue class* of a modulo n is given by

$$\bar{a} = \{a + cn \,|\, c \in \mathbb{Z}\} \ . \tag{A.5}$$

The integers modulo n are the set of these equivalence classes, usually symbolized by the quotient $\mathbb{Z}/n\mathbb{Z}$. The smallest non negative integer congruent to a mod n is called the *least residue* of a mod n.
Residue classes are closed under addition, subtraction and multiplication, i.e.

Theorem 7 *If $a_1 = b_1$ mod n and $a_2 = b_2$ mod n, then $a_1 + a_2 = b_1 + b_2$ mod n, $a_1 - a_2 = b_1 - b_2$ mod n and $a_1 a_2 = b_1 b_2$ mod n.*

Note, however, that division is not possible in general. Yet, division by coprimes of the modulus is permissible, as has been proven in section 4.3.2.
Solvability of linear congruence equations, i.e. the question for which integers x the equation

$$a\,x = b \bmod n \tag{A.6}$$

has solutions, is addressed by the following theorem.

Theorem 8 (Linear congruence theorem) *If $GCD(a, n)|b$, then $a\,x = b$ mod n has $GCD(a, n)$ incongruent solutions. Otherwise it has no solution.*

In fact, this theorem is nothing else than Bézout's identity from number theory, since (A.6) is equivalent to

$$a\,x - b = n\,y \ , \qquad \text{with } y \in \mathbb{Z} \ . \tag{A.7}$$

After Bézout, such a linear Diophantine equation has solutions if and only if $b = \mathrm{GCD}(a, n)$ or any integer multiple thereof.

A.3 Basic group theory

Since a group is a pair $(G, \star)$ of a set and a binary operation, one has to specify the operation $\star$ each time in principle. For convenience, we will adopt the usual notation

and write groups multiplicatively whenever talking about groups in general.
In order to avoid confusion, note that $\mathbb{Z}/n\mathbb{Z}$ is a group only under addition. However, it is isomorphic to $\mathbb{Z}_n$, the cyclic group (see definition 7) written multiplicatively, e.g. via the exponential mapping $a \in \mathbb{Z}/n\mathbb{Z} \mapsto e^{\frac{2\pi i a}{n}}$.
Before focusing on discrete groups, let us define the center of an arbitrary group.

Definition 5 (Center) *The center $Z(G)$ of a group G is its subgroup $\{g \in G \,|\, gh = hg \;\forall\, h \in G\}$.*

That is, the center is the set of elements of G commuting with all elements of G.

A.4 Fundamental terms of discrete groups

One has to distinguish carefully between the order of a group and the order of an element of a group:

Definition 6 (Order) *The order $|G|$ of a group G is the number of group elements. The order $|x|$ of an element x of a group is the smallest positive integer n, such that $x^n = 1$.*

Most likely, the simplest discrete groups are given by the notion of cyclic groups.

Definition 7 (Cyclic group) *A group G is cyclic, if it can be generated by one element*

$$G = \{x^n \,|\, n \in \mathbb{Z}\} \; . \tag{A.8}$$

With these definitions at hand, one can state the following important theorem.

Theorem 9 *Any two cyclic groups of the same order are isomorphic.*

It is immediately clear that the order of a generator must equal the order of the group that it generates. Although a cyclic group is generated by one element, there may be more than one generator. In fact, the number of distinct generators of a cyclic group is determined by the Euler φ function.

Definition 8 (Euler φ function) *For $n \in \mathbb{Z}^+$, the number of positive integers a, which are coprime to n such that $a \leq n$, is denoted by $\varphi(n)$.*

Since φ is linear for coprimes, i.e.

$$\varphi(nm) = \varphi(n)\varphi(m) \; , \qquad \text{if } \mathrm{GCD}(m,n) = 1 \; , \tag{A.9}$$

and obviously $\varphi(p) = p - 1$, one can deduce a formula for any integer due to the fundamental theorem of arithmetic, theorem 6,

$$\varphi(n) = \varphi(p_1^{\alpha_1}) \dots \varphi(p_s^{\alpha_s}) = p_1^{\alpha_1 - 1}(p_1 - 1) \dots p_s^{\alpha_s - 1}(p_s - 1) \ . \tag{A.10}$$

It is well known that the automorphisms of cyclic groups (i.e. a special case of the results discussed in section 4.2.1) are isomorphic to the group of residue classes having a multiplicative inverse, which equals the group of residue classes whose representatives are coprime to n

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \,|\, \mathrm{GCD}(a, n) = 1\} \ . \tag{A.11}$$

That is, $\mathrm{Aut}(\mathbb{Z}_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Since obviously $(\mathbb{Z}/n\mathbb{Z})^\times$ is of order $\varphi(n)$, there are $\varphi(n)$ automorphisms of $\mathbb{Z}_n$.
A rather important notion in finite group theory are p-groups (groups of order p^α with $\alpha \geq 1$) and their maximal subgroups, called Sylow p subgroups.

Definition 9 (Sylow p subgroup) *Let G be a group of order $p^\alpha m$ where the prime $p \nmid m$. Any subgroup of order p^α is called a Sylow p subgroup of G.*

All members of the set of Sylow p subgroups of a group G, $\mathrm{Syl}_p(G)$, are isomorphic (for fixed p), which follows from the Sylow theorems (see e.g. [71]).

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