Universal Pricing Mechanism for Utility Maximization for Interference Coupled Systems

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***Abstract*—This paper investigates pricing mechanisms for utility maximization in interference coupled systems. An axiomatic framework of interference functions similar to the one proposed by Yates is utilized to capture interference coupling in wireless systems. Pricing mechanisms are used as a design tool to shift the solution outcome of a utility maximization problem to a desired point in the region. The paper explores the restrictions required on the class of utility functions and the restrictions on the class of interference functions such that a pricing mechanism can always guarantee the designer the ability of being able to shift the solution outcome to any desired point in the region, i.e. it is a universal pricing mechanism.**

I. Introduction

Utility maximization problems are frequently encountered in wireless systems. Game theoretic tools have often been used to analyze such problems. An interesting aspect of such problems is the ability to shift the solution outcome of the utility maximization problem to a desired point in the region. Pricing can be one possible tool for a system designer to shift the operating point of a wireless communication system to a desired region/point. Furthermore pricing could also be utilized for implementing a certain cost function for utilizing a resource, e.g. enforcing power constraints, energy efficiency, interference coordination and management etc. A variety of pricing schemes have been proposed in literature aiming to improve the performance of the Nash equilibrium in communication and wireless systems with respect to a particular setting and a particular criterion [1]–[5]. The topic of efficiency of such a Nash equilibrium has recently aroused interest [6], [7]. There exists a lot of literature in the economics community, which are occupied with finding the fundamental bounds for certain games where the outcome satisfies certain given criteria, e.g. [8] and references therein. Our reference list is by no means comprehensive and we have provided a brief overview of the vast literature available on the topic. In our work we differentiate ourselves from the rest of the literature by hunting for a universal pricing mechanism for the case of utility maximization for interference coupled systems. By a universal pricing mechanism we mean, that a mechanism which under certain requirements can achieve any operating point in a desired region.

The main contributions of our paper are as follows.

1) Theorem 1 shows that linear pricing in power p_k of user k is not sufficient for achieving all points in a desired region if we have interference coupled systems.
2) Theorem 2 and Theorem 3 establish the largest class of utility functions (the class of exponentially concave utility functions, described later in Definition 4), given linear interference functions, such that a pricing mechanism which is linear in β_k and logarithmic in p_k, the power of the k^{th} user is a universal pricing mechanism, i.e. the pricing mechanism can achieve all points in a desired region.
3) Theorem 4 and Theorem 5 establish the largest class of interference functions (the class of *log–convex* interference functions), described later in Definition 5, given utility functions from the class of exponentially concave utility functions, such that a pricing mechanism which is linear in β_k and logarithmic in p_k, the power of the k^{th} user is a universal pricing mechanism.

II. Interference Coupled Systems

A. Preliminaries and Notation

Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the l^{th} component. The notation $\boldsymbol{y} \geq 0$ implies that $y_l \geq 0$ for all components l. $\boldsymbol{x} \gneqq \boldsymbol{y}$ implies component-wise inequality with strict inequality for at least one component. Similar definitions hold for the reverse directions. $\boldsymbol{x} \neq \boldsymbol{y}$ implies that the vector differs in at least one component. Finally, let $\mathcal{U}$ imply a family of functions. The set of non–negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$.

B. Interference Functions

In a wireless system, the users' utilities can strongly depend on the underlying physical layer. An important measure for the link performance is the signal-to-interference(-plus-noise) (SINR) ratio. Consider K users with transmit powers $\boldsymbol{p} = [p_1, \ldots, p_K]^T$ and $\mathcal{K} := \{1, \ldots, K\}$. The noise power at each receiver is σ^2. Hence the SINR at each receiver depends on the *extended power vector* $\underline{\boldsymbol{p}} = [\boldsymbol{p}, \sigma^2]^T = [p_1, \ldots, p_K, \sigma^2]^T$.

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The resulting SINR of user k is

$$\mathrm{SINR}_k(\underline{\boldsymbol{p}}) = \frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})} = \gamma_k, \quad (1)$$

where $\mathcal{I}_k$ is the interference (plus noise) as a function of $\underline{\boldsymbol{p}}$. In order to model interference coupling, we shall follow the axiomatic approach proposed in [9], [10]. The general interference functions possess the properties of conditional positivity, scale invariance and monotonicity with respect to the power component and strict monotonicity with respect to the noise component. For further details, kindly refer to the Appendix VII. Certain examples, where the interference function framework has been utilized are as follows: beamforming [11]–[13], CDMA [14], base station assignment, robust design [15], [16], transmitter optimization [17], [18] and characterization of the Pareto boundary [19]. The framework can be used to combine power control [20] and adaptive receiver strategies. Certain examples, where this has been successfully achieved are as follows. In [21] it was proposed to incorporate admission control to avoid unfavorable interference scenarios. In [22] it was proposed to adapt the quality of service (QoS) requirements to certain network conditions. In [23] a power control algorithm using fixed-point iterations was proposed for a modified cost function, which permits control of convergence behavior by adjusting fixed weighting parameters.

C. Impact of Interference Coupling

Users in wireless systems coupled by interference are intrinsically competitive. Each of them is principally interested in maximizing their own utility and have little or no regard for the utilities of the other users and for the entire system utility. Such neglect of course does not come for free. Such a characterization is accompanied by a pre–condition that there must be at least one user $k \in \mathcal{K}$ who sees interference from another user $j \in \mathcal{K}$ and $j \neq k$, i.e. it must not be possible to completely orthogonalize all the users in the system. If the users are completely orthogonal, then they are coupled only by the constraints on the resource allocation strategy and there is no "competition" in the sense we describe in this section.

Example 1. Consider the function $g_k(\underline{\boldsymbol{p}}) = \log(\underline{p}_k/\mathcal{I}_k(\underline{\boldsymbol{p}}))$. The function

$$u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) = \sum_{k \in \mathcal{K}} \omega_k g_k(\underline{\boldsymbol{p}}), \quad (2)$$

for all $\boldsymbol{\omega} > \mathbf{0}$ is not jointly concave with respect to $\underline{\boldsymbol{p}}$. Furthermore the problem of maximizing the function $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega})$ specified in (2) is not a convex optimization problem even for linear interference functions, e.g. $\mathcal{I}_k(\underline{\boldsymbol{p}}) = \sum_{l \in \mathcal{K}} v_{kl}\underline{p}_l + \sigma_k^2$, where v_{kl} is the link–gain between transmitter l and receiver k.

[24] implies that if g_k is the rate of user k, then the following sum of weighted rate maximization problem cannot be jointly concave in its current form.

III. PRICING MECHANISMS

In this section we shall formally introduce what we mean by a pricing mechanism and elaborate on the topic before presenting universal pricing mechanisms and investigating classes of utility functions and classes of interference functions where such universal pricing mechanisms are permissible. Consider the function $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega})$, where

$$u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) = \sum_{k \in \mathcal{K}} \omega_k g_k\left(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}\right). \quad (3)$$

The function presented in (3) is a general utility maximization problem as a function of SINR. Such a problem is frequently encountered in wireless systems. In our paper, "utility" can represent certain arbitrary performance measures, which depend on the SINR by a strictly monotonic increasing continuous function g_k defined on $\mathbb{R}_+$. The utility of user k is

$$g_k(\gamma_k) = g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}), \quad k \in \mathcal{K}. \quad (4)$$

An example of the above case is capacity: $g_k(x) = \log(1+x)$ and effective bandwidth $g_k(x) = x/(1+x)$ [25]. With respect to the utility maximization problem presented in (3) we present two pricing mechanisms below.

1) Linear pricing in β_k and linear pricing p_k: $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} \beta_k p_k$
2) Linear pricing in β_k and logarithmic pricing in p_k: $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} \beta_k \log(p_k)$

Let $\mathcal{U}$ represent a family of functions $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega})$. Let $\mathcal{F}$ represent a family of functions

$$u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} f_k(\beta_k, \underline{p}_k). \quad (5)$$

We now present a formal definition of what we mean by a pricing mechanism.

Definition 1. *Pricing Mechanism*: A *pricing mechanism* is a mapping from $\mathcal{U}$ to $\mathcal{F}$.

In each of the pricing mechanisms presented above: f is a function of a certain scalar parameter β_k and the power p_k for user k. In each of these cases the pricing mechanism is a tool, which the designer could utilize to shift the operating point of the system to a desired point. An example of such a framework for the purpose of energy–efficiency is [26].

We now explain what me mean by the *pricing problem*. Given $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) \in \mathcal{U}$ and given a power vector $\underline{\boldsymbol{p}}$, the *pricing problem* is to choose a vector $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}(\underline{\boldsymbol{p}})$ and a K–tuple $\boldsymbol{f} = [f_1, \ldots, f_K]$, such that

$$\sup_{\tilde{\boldsymbol{p}} \in \mathbb{R}_+^K} \left(u(\underline{\tilde{\boldsymbol{p}}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} f_k(\hat{\beta}_k, \tilde{p}_k)\right) = u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) - \sum_{k \in \mathcal{K}} f_k(\hat{\beta}_k, p_k). \quad (6)$$

For the purpose of solving the *pricing problem*, such that every possible point can be an operating point we introduce the definition of a universal pricing mechanism below.

Definition 2. *Universal pricing mechanism*: A pricing mechanism is said to be a *universal pricing mechanism*, if it chooses the same point in the range $\mathcal{F}$, independent of the choice of $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) \in \mathcal{U}$.

In our paper we consider an interference coupled wireless system, where the users always report their true utilities to the game designer (system designer). The system works as a two step process:

1) The users report their utilities $g_1, \ldots, g_K$ to the system designer.
2) The system designer wants to achieve a certain objective, e.g. $\max_{\boldsymbol{p} \in \mathcal{P}} u(\underline{\boldsymbol{p}}, \boldsymbol{\omega})$, where $u(\underline{\boldsymbol{p}}, \boldsymbol{\omega})$ is as defined in (3) and $\mathcal{P}$ is a set which is result of certain power constraints on the system. The system designer now allocates the resources. In this case the powers $\boldsymbol{p}$ to the users.

We now introduce two families of utility functions, which we shall utilize while analyzing pricing mechanisms.

Definition 3. $\mathcal{C}onc$ is the family of all strictly monotonic increasing, continuous functions g, such that $g(x)$ is concave.

If g is strictly monotonic increasing, then $g(x)$ is also concave, i.e. the concavity of the function $g(e^y)$ is a stronger requirement. To utilize this requirement, we introduce the the family of functions $\mathcal{EC}onc$ below.

Definition 4. $\mathcal{EC}onc$ is the family of all strictly monotonic increasing, continuous functions g, such that $g(e^x)$ is concave.

Based on the system model described above and the introduced classes of utility functions, we are interested in tackling the following problems.

Problem 1. For a given family of utility functions and for a certain structure of interference coupling in the system, is it possible to design a pricing mechanism, such that every possible point can be an operating point?

Problem 2. For a system with linear interference functions, what is the largest class of utility functions, such that we can have a universal pricing mechanism?

Problem 3. For a system with utility functions in the family $\mathcal{EC}onc$, are there restrictions on the interference coupling of the systems systems

1) orthogonal or non–orthogonal systems and
2) dependency (classes of interference functions),

such that we can have a universal pricing mechanism?

Equipped with the necessary definitions and concepts we now go about investigating the Problems 1, 2 and 3 presented above.

IV. Universal Pricing Mechanisms – Linear Interference Functions

The work in [27] states, that a pricing mechanism, which is linear in p_k and linear in β_k, is sufficient to achieve every possible operating point in multiuser orthogonal systems. We investigate the possibility of extending the result from [27] to the case of non–orthogonal systems when we have utility functions from the class $\mathcal{EC}onc$.

A. Universal Pricing Mechanism – Non–orthogonal Systems

In this section we begin with utility functions in the class $\mathcal{EC}onc$ and a non–orthogonal system with linear interference functions. We check if for such a system a pricing mechanism, which is linear in p_k and linear in β_k is a universal pricing mechanism.

Theorem 1. *Let utility functions $g_1, \ldots, g_K \in \mathcal{C}onc$ be arbitrary (not constant). Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary linear interference functions, such that the system is not orthogonal. There exists a vector $\boldsymbol{\omega} > \mathbf{0}$, such that not every power vector $\underline{\boldsymbol{p}} > \mathbf{0}$ is supportable by a pricing mechanism, which is linear in β_k and linear in p_k for all users $k \in \mathcal{K}$.*

Proof: Assume that Theorem 1 is not true, i.e. there exists utility functions $g_1, \ldots, g_K \in \mathcal{C}onc$ and linear interference functions $\mathcal{I}_1, \ldots, \mathcal{I}_K$ (non–orthogonal system), such that for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ the following statement holds. For every power vector $\boldsymbol{p} > \mathbf{0}$ there exists a $\hat{\boldsymbol{p}} = \beta(\boldsymbol{p})$ such that $\sum_{k \in \mathcal{K}} \Big(\omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}) - \beta_k p_k \Big) = \sup_{\tilde{\boldsymbol{p}} > \mathbf{0}} \big(\sum_{k \in \mathcal{K}} \omega_k g_k(\frac{\tilde{p}_k}{\mathcal{I}_k(\tilde{\boldsymbol{p}})}) - \sum_{k \in \mathcal{K}} \beta_k \tilde{p}_k \big) = \mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta})$. Then, we have that

$$\sum_{k \in \mathcal{K}} \omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}) = \inf_{\boldsymbol{\beta} \in \mathbb{R}^K} \big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) + \sum_{k \in \mathcal{K}} \beta_k p_k \big).$$

Let us choose power vectors $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} > \mathbf{0}$ arbitrarily and choose $\boldsymbol{p}(\lambda) = (1-\lambda)\boldsymbol{p}^{(1)} + \lambda \boldsymbol{p}^{(2)}$. We have

$$\begin{aligned}
\sum_{k \in \mathcal{K}} \omega_k g_k(\frac{p_k(\lambda)}{\mathcal{I}_k(\underline{\boldsymbol{p}}(\lambda))}) \quad &= \quad \inf_{\boldsymbol{\beta} \in \mathbb{R}^K} \Big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) \\
&\qquad + (1-\lambda) \sum_{k \in \mathcal{K}} \beta_k p_k^{(1)} \\
&\qquad + \lambda \sum_{k \in \mathcal{K}} \beta_k p_k^{(2)} \Big) \\
&= \quad \inf_{\boldsymbol{\beta} \in \mathbb{R}^K} \Big((1-\lambda)\big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) \\
&\qquad + \sum_{k \in \mathcal{K}} \beta_k p_k^{(1)} \big) \\
&\qquad + \lambda \big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) + \sum_{k \in \mathcal{K}} \beta_k p_k^{(2)} \big) \Big) \\
&\geq \quad (1-\lambda) \inf_{\boldsymbol{\beta} \in \mathbb{R}^K} \big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) \\
&\qquad + \sum_{k \in \mathcal{K}} \beta_k p_k^{(1)} \big) \\
&\qquad + \lambda \inf_{\boldsymbol{\beta} \in \mathbb{R}^K} \big(\mathcal{G}_{\boldsymbol{\omega}}(\boldsymbol{\beta}) + \sum_{k \in \mathcal{K}} \beta_k p_k^{(2)} \big) \\
&= \quad (1-\lambda) \sum_{k \in \mathcal{K}} \omega_k g_k(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}}^{(1)})}) \\
&\qquad + \lambda \sum_{k \in \mathcal{K}} \omega_k g_k(\frac{p_k^{(2)}}{\mathcal{I}_k(\underline{\boldsymbol{p}}^{(2)})}),
\end{aligned}$$

i.e. the function $\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})})$ is jointly concave for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$, i.e. $g_k(\frac{p_k}{\mathcal{I}_k(\boldsymbol{p})})$ is jointly concave. This is in direct contradiction to reference [24, Theorem 4]. ■

The above result shows that linear pricing in β_k and linear pricing in p_k is a universal pricing mechanism only for orthogonal systems. The above result partly addresses Problem 1 and partly addresses Problem 3. We shall now investigate in the next section the possibility of having a universal pricing mechanism for the largest class of utility functions, given systems with linear interference functions.

B. Universal Pricing Mechanisms – Largest Class of Utility Functions

We now present Lemmas 1 and 2 (these results are part of [28] and have been presented here for completeness of the analysis), which we shall require to answer Problem 2.

Lemma 1. *$\psi(s) = c_1 exp(\mu s)$, with $c_1, \mu > 0$ is a transformation such that the function $g\big(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}))\big)$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$, for all linear interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ and for all utility functions $g \in \mathcal{EC}onc$.*

In [28] it was shown, that under certain intuitive restrictions, for all linear interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ and for all utility functions $g \in \mathcal{EC}onc$, the function $g\big(\psi(s_k)/\mathcal{I}_k(\psi(\boldsymbol{s}))\big)$ is jointly concave with respect to $\boldsymbol{s} \in \mathbb{R}^{K+1}$, if and only if $\psi(s) = c_1 exp(\mu s)$, with $c_1, \mu > 0$.

Lemma 2. *For all $\boldsymbol{\omega} > \mathbf{0}$, $\boldsymbol{\omega} = [\omega_1, \dots, \omega_K]^T$ with $\sum_{k\in\mathcal{K}} \omega_k = 1$, the function $\sum_{k\in\mathcal{K}} \omega_k g_k\big(p_k/\mathcal{I}_k(\underline{\boldsymbol{p}})\big)$ is jointly concave with respect to $\underline{\boldsymbol{p}}$, if and only if the utility functions $g_1(\underline{\boldsymbol{p}}), \dots, g_K((\underline{\boldsymbol{p}}))$ are all jointly concave.*

Theorem 2. *Linear pricing in β_k and logarithmic pricing in p_k, for all users $k \in \mathcal{K}$, is a universal pricing mechanism for all utility functions $g_1, \dots, g_K \in \mathcal{EC}onc$, for all linear interference functions and for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$.*

Proof: From Lemmas 1 and Lemma 2, we have that the function $\sum_{k\in\mathcal{K}} \omega_k g_k(p_k/\mathcal{I}_k(\underline{\boldsymbol{p}}))$ only after the transformation $p_k = e^{s_k}$ is jointly concave with respect to $\boldsymbol{s}$. By applying our pricing mechanism, which is linear in β_k and logarithmic in p_k (linear in s_k) we have the following expression:

$$\sum_{k\in\mathcal{K}} \omega_k g_k(\frac{e^{s_k}}{\mathcal{I}_k(e^{\boldsymbol{s}})}) - \sum_{k\in\mathcal{K}} \beta_k s_k, \quad g_k \in \mathcal{EC}onc. \tag{7}$$

It can be easily observed in (7), for each s_k by an appropriate choice of β_k every point $\boldsymbol{s}$ in the above can be achieved. Hence, we have our desired result. ■

Theorem 2 has completely answered Problem 2 and partly addressed Problem 1 for the class of utility functions $\mathcal{EC}onc$. In Theorem 2 we have proved that a pricing mechanism, which is linear in β_k and logarithmic in p_k, for all users $k \in \mathcal{K}$ is a universal pricing mechanism with the following two conditions being *sufficient* for the result:

1) The utility functions $g_1, \dots, g_K$ are in $\mathcal{EC}onc$.
2) $\mathcal{I}_1, \dots, \mathcal{I}_K$ are linear interference functions.

Our next result will show that these two conditions are not only *sufficient*, however also *necessary* for the pricing mechanism, which is linear in β_k and logarithmic in p_k to be a universal pricing mechanism.

Theorem 3. *Let utility functions $g_1, \dots, g_K \in \mathcal{C}onc$, such that at least one of the g_k, for $k \in \mathcal{K}$ is not in $\mathcal{EC}onc$. Then, there exist linear interference functions and a weight vector $\boldsymbol{\omega} > \mathbf{0}$ such that the following statement holds: not every power vector $\boldsymbol{p} > \mathbf{0}$ is supportable by linear pricing in β_k and logarithmic pricing in p_k, for all $k \in \mathcal{K}$.*

Proof: Assume for the sake of obtaining a contradiction that Theorem 3 is not true, i.e. for all utility functions $g_1, \dots, g_K \in \mathcal{C}onc$, such that at least one g_k, with $k \in \mathcal{K}$ is not in $\mathcal{EC}onc$, for all linear interference functions for all $\boldsymbol{\omega} > \mathbf{0}$ all power vectors $\boldsymbol{p} > \mathbf{0}$ are supportable by linear pricing in β_k and logarithmic pricing in p_k, for $k \in \mathcal{K}$. Exactly as in the proof of Theorem 1 we conclude, that $\sum_{k\in\mathcal{K}} \frac{e^{s_k}}{\mathcal{I}_k(e^{\boldsymbol{s}})}$ is jointly concave for all linear interference functions for all $\boldsymbol{\omega} > \mathbf{0}$. Then, we conclude that for all linear interference functions, for all users $k \in \mathcal{K}$ the function $g_k(\frac{e^{s_k}}{\mathcal{I}_k(e^{\boldsymbol{s}})})$ is jointly concave. However, this implies that $g_k \in \mathcal{EC}onc$ for all $k \in \mathcal{K}$. This is in contradiction to the assumptions of Theorem 3. ■

We observe that the largest class of utility functions, such that the corresponding pricing problem is solvable – is the class $\mathcal{EC}onc$. Hence, we have concluded that Problem 1 is not solvable for utility functions outside the class $\mathcal{EC}onc$.

V. Universal Pricing Mechanism – Beyond Linear Interference Functions

In the previous section we have established the largest class of utility functions, namely $\mathcal{EC}onc$ which along with linear interference functions permit a universal pricing mechanism, which is linear in β_k and logarithmic in p_k, for all users $k \in \mathcal{K}$. In this section we shall fix our class of utility functions to $\mathcal{EC}onc$ and look for the largest possible class of interference functions, which permit a universal pricing mechanism, which is linear in β_k and logarithmic in p_k, for all users $k \in \mathcal{K}$. Before we begin searching for the largest class of interference function, we introduce the property of *log–convexity*, which we shall exploit in this section. *Log–convexity* is a useful property that allows one to apply convex optimization techniques to certain non–convex problems.

Definition 5. *Log–convex* interference function: An interference function $\mathcal{I} : \mathbb{R}_+^{K+1} \mapsto \mathbb{R}_+$ is said to be a *log–convex* interference function if $A1 - A4$ (appendix VII) are fulfilled and $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^{K+1}$.

Linear interference functions are also *log–convex* interference functions.

Let $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f : \mathbb{R}^{K+1} \mapsto \mathbb{R}_+$ is *log–convex* on $\mathbb{R}^K$ if and only if $\log f$ is convex or equivalently $f(\boldsymbol{s}(\lambda)) \le f(\boldsymbol{s}^{(1)})^{1-\lambda} f(\boldsymbol{s}^{(2)})^{1-\lambda}$, for all $\lambda \in (0,1)$, $\boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)} \in \mathbb{R}^K$, where $\boldsymbol{s}(\lambda) = (1-\lambda)\boldsymbol{s}^{(1)} + \lambda \boldsymbol{s}^{(2)}$, $\lambda \in (0,1)$. Note that the *log–convexity* in Definition 5 is based on a change of variable $\underline{\boldsymbol{p}} = \exp\{\boldsymbol{s}\}$ (component-wise exponential).

Such a technique has been previously used to exploit a "hidden convexity" of functions, which are otherwise non-convex.

We shall now present a Lemma 3 (this results is part of [28] and have been presented here for completeness of the analysis), which we shall require to answer the question posed in Problem 3.

Lemma 3. *The function* $\sum_{k\in\mathcal{K}}\omega_k g\big(e^{s_k}/\mathcal{I}_k(e^{\boldsymbol{s}})\big)$ *is jointly concave with respect to* $\boldsymbol{s}$*, for all weight vectors* $\boldsymbol{\omega} > \mathbf{0}$*, for all* $g \in \mathcal{EC}onc$*, if and only if* $\mathcal{I}_1, \dots, \mathcal{I}_K$ *are log–convex interference functions.*

Theorem 4. *Let utility functions* $g_1, \dots, g_K \in \mathcal{EC}onc$ *be arbitrarily chosen. Let* $\mathcal{I}_1, \dots, \mathcal{I}_K$ *be arbitrary log–convex interference functions. Then, for all* $\boldsymbol{\omega} > \mathbf{0}$ *the pricing mechanism, which is linear in* β_k *and logarithmic in* p_k*, for all* $k \in \mathcal{K}$ *solves the pricing problem for* $\sum_{k\in\mathcal{K}}\omega_k g_k\big(\frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})}\big)$.

Proof: From Lemmas 1 and 3 and following the same arguments as in the proof of Theorem 2. ■

In Theorem 4 we have proved that a pricing mechanism, which is linear in β_k and logarithmic in p_k, for all users $k \in \mathcal{K}$ is a universal pricing mechanism with the following two conditions being *sufficient* for the result:

1) The utility functions $g_1, \dots, g_K$ are in $\mathcal{EC}onc$.
2) $\mathcal{I}_1, \dots, \mathcal{I}_K$ are *log–convex* interference functions.

Our next result will show that these two conditions are not only *sufficient*, however also *necessary* for the pricing mechanism, which is linear in β_k and logarithmic in p_k to be a universal pricing mechanism.

Theorem 5. *Let utility functions* $g_1, \dots, g_K \in \mathcal{EC}onc$ *be arbitrarily chosen. Let* $\mathcal{I}_1, \dots, \mathcal{I}_K$ *be arbitrary interference functions, such that at least one interference function* $\mathcal{I}_k$*, for* $k \in \mathcal{K}$ *is not a log–convex interference function. Then, for all weight vectors* $\boldsymbol{\omega} > \mathbf{0}$ *the following statement holds: not every power vector* $\boldsymbol{p} > \mathbf{0}$ *is supportable by linear pricing in* β_k *and logarithmic pricing in* p_k*, for all* $k \in \mathcal{K}$.

Proof: The proof follows in a similar manner as in the proof of Theorem 3. ■

Theorems 4 and 5 have completely addressed Problem 3.

VI. Discussion

We have shown, that a pricing mechanism, which is linear in the power of the k^{th} user p_k and is linear in β_k is a universal pricing mechanism only for orthogonal systems, however is not restricted by the class of utility functions. We have shown, that linear pricing in β_k and logarithmic pricing in p_k is a universal pricing mechanism for utility functions from the class $\mathcal{EC}onc$ and for *log–convex* interference functions. Furthermore, we have shown that these are the largest classes of utility functions and interference functions, which allow this particular pricing mechanism as a universal pricing mechanism. As further work, we would like to show that under certain mild restrictions on the interference coupled system the only possible universal pricing mechanism is one which is linear in β_k and logarithmic in p_k for all users $k \in \mathcal{K}$, namely the converse direction of the results proved in our paper.

VII. Appendix: Interference Functions

Definition 6. *Interference functions*: We say that $\mathcal{I} : \mathbb{R}_+^{K+1} \mapsto \mathbb{R}_+$ is an *interference function* if the following axioms are fulfilled:

$A1$	conditional positivity	$\mathcal{I}(\underline{\boldsymbol{p}}) > 0$ if $\underline{\boldsymbol{p}} > \mathbf{0}$
$A2$	scale invariance	$\mathcal{I}(\alpha\underline{\boldsymbol{p}}) = \alpha\mathcal{I}(\underline{\boldsymbol{p}}), \forall\alpha \in \mathbb{R}_+$
$A3$	monotonicity	$\mathcal{I}(\underline{\boldsymbol{p}}) \geq \mathcal{I}(\hat{\underline{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \hat{\underline{\boldsymbol{p}}}$
$A4$	strict monotonicity	$\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\hat{\underline{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \hat{\underline{\boldsymbol{p}}}$, $\underline{p}_{K+1} > \hat{\underline{p}}_{K+1}$.

Note that we require that $\mathcal{I}(\underline{\boldsymbol{p}})$ is *strictly monotone* with respect to the last component $\underline{p}_{K+1}$. An example is $\mathcal{I}(\underline{\boldsymbol{p}}) = \boldsymbol{v}^T\boldsymbol{p} + \sigma^2$, where $\boldsymbol{v} \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients. The axiomatic framework A1-A4 is connected with the framework of *standard interference functions* [9]. The details about the relationship between the model A1-A4 and Yates' *standard interference functions* were discussed in [10]. For the purpose of this paper it is sufficient to be aware that there exists a connection between these two models and the results of this paper are applicable to *standard interference functions*.

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