Method of moment estimation in the COGARCH(1,1) model

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Abstract

We suggest moment estimators for the parameters of a continuous time GARCH(1,1) process based on equally spaced observations. Using the fact that the increments of the COGARCH(1,1) process are strongly mixing with exponential rate, we show that the resulting estimators are consistent and asymptotically normal. We investigate the empirical quality of our estimators in a simulation study based on the variance gamma driven COGARCH(1,1) model. The estimated volatility with corresponding residual analysis is also presented. Finally, we fit the model to high-frequency data.

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1 Introduction

The GARCH(1,1) process is a model widely used by practitioners in the financial industry. It is defined as

$$Y_i = \sigma_i \epsilon_i \quad \text{with} \quad \sigma_i^2 = \beta + \lambda Y_{i-1}^2 + \delta \sigma_{i-1}^2 \,, \quad i \in \mathbb{N}, \tag{1.1}$$

where $\beta > 0, \lambda, \delta \geq 0$ and $(\epsilon_i)_{i\in\mathbb{N}}$ is an i.i.d. innovation sequence. This model captures some of the most prominent features in financial data, in particular in the volatility process. Empirical studies show that volatility changes randomly in time, has heavy or semi-heavy tails and clusters on high levels. These stylized features are modelled by the GARCH family as has been shown for the GARCH(1,1) process in detail in [14].

The modern treatment of stochastic volatility models is mostly in continuous time aiming at the analysis of high-frequency data. Approaches to create a continuous time GARCH model go back to [16] and we refer to [3] for an overview. Such processes are diffusion limits to discrete time GARCH models, where, unfortunately, many of the above features of the GARCH process are wiped out in the limit; see [5]. Since empirical work indicates upwards jumps in the volatility, a model driven by a Lévy process seems a natural approach. In [9, 10] such a model was suggested by iterating the volatility equation in (1.1) and replacing the noise variables ϵ_i by the jumps $\Delta L_t = L_t - L_{t-}$ of a Lévy process $L = (L_t)_{t\geq 0}$. A reparameterization, setting $\eta = -\log\delta$ and $\varphi = \lambda/\delta$, yields the following continuous time GARCH(1,1) model, where the parameter space is given by $\beta, \eta > 0$ and $\varphi > 0$ (the degenerate case $\varphi = 0$ will not be considered in this paper).

The COGARCH(1,1) process $G = (G_t)_{t\geq 0}$ is defined as the solution to the SDEs

$$\begin{aligned} dG_t &= \sigma_t \, dL_t \,, & (1.2)\\ d\sigma_{t+}^2 &= (\beta - \eta\sigma_t^2)\, dt + \varphi\, \sigma_t^2 d[L,L]_t^{(d)} \,, & (1.3) \end{aligned}$$

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where $[L,L]_t^{(d)} = \sum_{0<s\le t}(\Delta L_s)^2$, $t \ge 0$, is the discrete part of the quadratic variation process $([L,L]_t)_{t\ge 0}$ of the Lévy process L; we define $G_0 := 0$, and σ_0^2 is taken to be independent of L. Throughout we assume that L is càdlàg, and we denote by ν_L the Lévy measure of L, which is assumed to be non-zero, and by $\tau_L^2 \ge 0$ the variance of the Brownian motion component of L (see [18] for the basic definitions and notations concerning Lévy processes). Whereas the process G is taken as being càdlàg, for the volatility process we assume càglàd sample paths.

The quantity σ_t^2 is called the *instantaneous volatility* or *spot volatility*, which is assumed to be stationary and latent. In contrast to classical stochastic volatility models, it is not independent of the process, which drives the price process. On the contrary, L drives both, the volatility and the price process. Note that G jumps at the same times as L does with jump size $\Delta G_t = \sigma_t \Delta L_t$, and that ΔL_t is independent of $\sigma_t = \sigma_{t-}$.

If our data consist of returns over time intervals of fixed length $r > 0$, we denote

$$G_t^{(r)} := G_t - G_{t-r} = \int_{(t-r,t]} \sigma_s \, dL_s\,, \quad t \ge r\,, \tag{1.4}$$

and $(G_{ri}^{(r)})_{i\in\mathbb{N}}$ describes an equidistant sequence of such non-overlapping returns. Calculating the corresponding quantity for the volatility yields

$$\begin{aligned}\sigma_{ri}^{2(r)} &:= \sigma_{ri}^2 - \sigma_{r(i-1)}^2 = \int_{(r(i-1),ri]} \left((\beta - \eta\sigma_s^2)\, ds + \varphi\, \sigma_s^2 \, d[L,L]_s^{(d)} \right) \\ &= \beta r - \eta \int_{(r(i-1),ri]} \sigma_s^2 \, ds + \varphi \int_{(r(i-1),ri]} \sigma_s^2 \, d[L,L]_s^{(d)}\,. \end{aligned} \tag{1.5}$$

It is also worth noting that the stochastic process

$$R_t = \sum_{0<s\le t} \sigma_s^2 (\Delta L_s)^2 = \int_{(0,t]} \sigma_s^2 \, d[L,L]_s^{(d)}\,, \quad t \ge 0\,,$$

is the discrete part of the quadratic variation $[G,G]_t = \int_0^t \sigma_s^2 \, d[L,L]_s$, $t \ge 0$, of G, so that $\int_{(r(i-1),ri]} \sigma_s^2 \, d[L,L]_s^{(d)}$ in (1.5) corresponds to the jump part of the quadratic variation of G accumulated during $(r(i-1), ri]$.

The goal of this paper is to estimate the model parameters β, η, φ. Moreover, we shall present a simple estimate of the volatility. This estimation approach was first presented in a preliminary form in the diploma thesis of [20]. We would like to mention that [15] developed an MCMC estimation procedure for the COGARCH(1,1) model, which works also for irregularly spaced observations. The approach is, however, restricted to finite variational driving processes L.

The paper is organised as follows. In the next section we present some preliminary results regarding the moment structure of the COGARCH$(1,1)$ process. Then, in Section 3, we introduce a moment estimator for the parameter vector and derive its asymptotic properties. The estimator is applied in Section 4 to a simulated COGARCH$(1,1)$ process driven by an (infinite-activity) variance gamma process, and a method for estimating the volatility process is outlined. Finally, Section 5 is concerned with an empirical data analysis, and concludes with some extension of the model to include leverage effects. Longer proofs are presented in the appendices.

2 Preliminary results

An important role is played by the *auxiliary process*

$$X_t = \eta t - \sum_{0<s\le t} \log(1 + \varphi\,(\Delta L_s)^2)\,, \quad t \ge 0\,. \tag{2.1}$$

The stationary volatility process has, for instance, the representation

$$\sigma_t^2 = \left(\beta \int_0^t e^{X_s} ds + \sigma_0^2\right) e^{-X_{t-}}, \quad t \geq 0, \tag{2.2}$$

with $\beta > 0$ and $\sigma_0^2 \stackrel{d}{=} \beta \int_0^\infty e^{-X_t} dt$, independent of L. The auxiliary process $(X_t)_{t\geq 0}$ itself is a spectrally negative Lévy process of bounded variation with drift η, no Gaussian component (i.e. $\tau_X^2 = 0$), and Lévy measure ν_X given by

$$\nu_X [0, \infty) = 0, \quad \nu_X (-\infty, -x] = \nu_L \left(\{ y \in \mathbb{R} : |y| \geq \sqrt{(e^x - 1)/\varphi} \} \right), \quad x > 0.$$

We shall also need the Laplace transform $\mathbb{E}e^{-sX_t} = e^{t\Psi(s)}$ with Laplace exponent

$$\Psi(s) = -\eta s + \int_{\mathbb{R}} ((1 + \varphi x^2)^s - 1)\, \nu_L(dx), \quad s \geq 0. \tag{2.3}$$

The Laplace exponent was calculated in Lemma 4.1 in [9]. For fixed $s \geq 0$ the Laplace transform $\mathbb{E}e^{-sX_t}$ is finite for one and hence all $t > 0$, if and only if the integral appearing in (2.3) is finite. This is equivalent to $\mathbb{E}|L_1|^{2s} < \infty$. In particular, there exists a stationary version of the volatility process, if $\Psi(s) \leq 0$ for some $s > 0$ (cf. [10], Section 3).

One of the advantages of the COGARCH(1,1) is that its second order structure is well-known. In the following result we present the moments of $G_t^{(r)}$, which are independent of t by stationarity: expressions (2.4) and (2.6) have been already proved in Proposition 5.1 of [9], however, under additional assumptions such as bounded variation of L for (2.6). In Appendix A we shall give a different proof under less restrictive assumptions and also calculate the fourth moment of G.

Proposition 2.1. *Suppose that the Lévy process $(L_t)_{t\geq 0}$ has finite variance and zero mean, and that $\Psi(1) < 0$. Let $(\sigma_t^2)_{t\geq 0}$ be the stationary volatility process, so that $(G_t)_{t\geq 0}$ has stationary increments. Then $\mathbb{E}(G_t^2) < \infty$ for all $t \geq 0$, and for every $t, h \geq r > 0$ it holds*

$$\mathbb{E}(G_t^{(r)}) = 0, \quad \mathbb{E}(G_t^{(r)})^2 = \frac{\beta r}{|\Psi(1)|}\mathbb{E}(L_1^2), \quad \mathrm{Cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0. \tag{2.4}$$

If further $\mathbb{E}(L_1^4) < \infty$ and $\Psi(2) < 0$, then $\mathbb{E}(G_t^4) < \infty$ for all $t \geq 0$ and, if additionally the Lévy measure ν_L of L is such that $\int_{\mathbb{R}} x^3 \nu_L(dx) = 0$, then it holds for every $t, h \geq r > 0$

$$\begin{aligned}\mathbb{E}(G_t^{(r)})^4 &= 6\mathbb{E}(L_1^2)\frac{\beta^2}{\Psi(1)^2}\left(\frac{2\eta}{\varphi} + 2\tau_L^2 - \mathbb{E}(L_1^2)\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\left(r - \frac{1 - e^{-r|\Psi(1)|}}{|\Psi(1)|}\right) \\ &\quad + \frac{2\beta^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right) r + 3\frac{\beta^2}{\Psi(1)^2}(\mathbb{E}(L_1^2))^2 r^2 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathrm{Cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) &= \mathbb{E}(L_1^2)\frac{\beta^2}{|\Psi(1)|^3}\left(\frac{2\eta}{\varphi} + 2\tau_L^2 - \mathbb{E}(L_1^2)\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right) \\ &\quad \times \left(1 - e^{-r|\Psi(1)|}\right)\left(e^{r|\Psi(1)|} - 1\right) e^{-h|\Psi(1)|} > 0. \end{aligned} \tag{2.6}$$

Lemma 2.2. *Under the conditions of Proposition 2.1 the process $((G_{ri}^{(r)})^2)_{i\in\mathbb{N}}$ has for each fixed $r > 0$ the autocorrelation structure of an* ARMA(1,1) *process.*

Proof. Denote by $\gamma(h) = \mathbb{C}\mathrm{ov}((G_{ri}^{(r)})^2, (G_{r(i+h)}^{(r)})^2)$, $h \in \mathbb{N}_0$, the autocovariance function and by $\rho(h) = \mathbb{C}\mathrm{orr}((G_{ri}^{(r)})^2, (G_{r(i+h)}^{(r)})^2)$, $h \in \mathbb{N}_0$, the autocorrelation function of the discrete time process $((G_{ri}^{(r)})^2)_{i\in\mathbb{N}}$ as defined in (1.4). Then

$$\frac{\rho(h)}{\rho(1)} = \frac{\gamma(h)}{\gamma(1)} = e^{-(h-1)r|\Psi(1)|}, \quad h \geq 1 .$$

Moreover, for $h = 1$ we get

$$\rho(1) = \frac{\gamma(1)}{\mathbb{V}\mathrm{ar}(G_r^2)} .$$

Recalling the autocorrelation function of an ARMA(1,1) process (see e.g. [1], Exercise 3.16), we identify $e^{-r|\Psi(1)|}$ as the autoregressive root ϕ. The moving average root θ can be determined by matching $\rho(1) = (1+\phi\theta)(\phi+\theta)/(1+\theta^2+2\phi\theta)$.

□

Remark 2.3. From Corollary 4.1 of [9] we know for $k \in \mathbb{N}$ the moment $\mathbb{E}(\sigma^{2k})$ of the stationary volatility process, which exists if and only if $\mathbb{E}(L_1^{2k}) < \infty$ and $\Psi(k) < 0$. In particular, if $\mathbb{E}(L_1^4) < \infty$ and $\Psi(2) < 0$, then for $t, h \geq 0$

$$\mathbb{E}(\sigma_t^2) = \frac{\beta}{|\Psi(1)|} \quad \text{and} \quad \mathbb{E}(\sigma_t^4) = \frac{2\beta^2}{|\Psi(1)\Psi(2)|}, \tag{2.7}$$

$$\mathbb{C}\mathrm{ov}(\sigma_t^2, \sigma_{t+h}^2) = \beta^2 \left(\frac{2}{|\Psi(1)\Psi(2)|} - \frac{1}{\Psi(1)^2} \right) e^{-h|\Psi(1)|} = \mathbb{V}\mathrm{ar}(\sigma_t^2)\, e^{-h|\Psi(1)|} . \tag{2.8}$$

Econometric literature suggests that volatility is quite persistent, which would imply that $e^{-|\Psi(1)|}$ is close to 1; i.e. $\Psi(1) < 0$ near 0. This should be kept in mind, when estimating the model parameters.

3 Method of moment estimation

3.1 Identifiability of the model parameters

We aim at estimation of the model parameters (β, η, φ) from a sample of equally spaced returns by matching empirical autocorrelation function and moments to their theoretical counterparts given in Proposition 2.1. In our next result we show that the parameters are identifiable by this estimation procedure for driving Lévy processes L as in Proposition 2.1. We assume throughout that $\mathbb{E}(L_1) = 0$ and $\mathbb{V}\mathrm{ar}(L_1) = 1$; furthermore, we assume that the variance τ_L^2 of the Brownian motion component in L is known. This last assumption is crucial for our analysis and we will comment on it in Section 4, when setting up our simulation study. For the sake of simplicity we set $r = 1$.

Theorem 3.1. *Suppose $(L_t)_{t\geq 0}$ is a Lévy process such that $\mathbb{E}(L_1) = 0$, $\mathbb{V}\mathrm{ar}(L_1) = 1$, the variance τ_L^2 of the Brownian motion component of L is known with $0 \leq \tau_L^2 < \mathbb{V}\mathrm{ar}(L_1) = 1$, $\mathbb{E}(L_1^4) < \infty$ and $\int_{\mathbb{R}} x^3 \nu_L(dx) = 0$. Assume also that $\Psi(2) < 0$, and denote by $(G_i^{(1)})_{i\in\mathbb{N}}$ the stationary increment process of the* COGARCH(1,1) *process with parameters $\beta, \eta, \varphi > 0$. Let $\mu, \gamma(0), k, p > 0$ be constants such that*

$$\begin{aligned}
\mathbb{E}((G_i^{(1)})^2) &= \mu, \\
\mathbb{V}\mathrm{ar}((G_i^{(1)})^2) &= \gamma(0), \\
\rho(h) \;=\; \mathbb{C}\mathrm{orr}((G_i^{(1)})^2, (G_{i+h}^{(1)})^2) &= k e^{-hp}, \quad h \in \mathbb{N}.
\end{aligned}$$

Define

$$\begin{aligned}
M_1 &:= \gamma(0) - 2\mu^2 - 6\frac{1-p-e^{-p}}{(1-e^p)(1-e^{-p})} k\,\gamma(0), && (3.1)\\
M_2 &:= \frac{2k\gamma(0)p}{M_1(e^p-1)(1-e^{-p})}. && (3.2)
\end{aligned}$$

Then $M_1, M_2 > 0$, and the parameters β, η, φ are uniquely determined by $\mu, \gamma(0), k$ and p and are given by the formulas

$$\begin{aligned}
\beta &= p\,\mu, && (3.3)\\
\varphi &= p\sqrt{1+M_2} - p, && (3.4)\\
\eta &= p\sqrt{1+M_2}\,(1-\tau_L^2) + p\,\tau_L^2 = p + \varphi(1-\tau_L^2). && (3.5)
\end{aligned}$$

Proof. Since $r = \mathbb{E}(L_1^2) = 1$, we obtain from Proposition 2.1

$$\begin{aligned}
\mu &= \frac{\beta}{|\Psi(1)|}, && (3.6)\\
\gamma(0) &= 6\frac{\beta^2}{|\Psi(1)|^3}\left(\frac{2\eta}{\varphi} + 2\tau_L^2 - 1\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\left(|\Psi(1)| - 1 + e^{-|\Psi(1)|}\right) \\
&\quad + \frac{2\beta^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right) + \frac{2\beta^2}{\Psi(1)^2} && (3.7)\\
&=: \beta^2\widetilde{\gamma}(0), \\
p &= |\Psi(1)|, && (3.8)\\
k &= \frac{\widetilde{\gamma}^{-1}(0)}{|\Psi(1)|^3}\left(\frac{2\eta}{\varphi} + 2\tau_L^2 - 1\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\left(1 - e^{-|\Psi(1)|}\right)\left(e^{|\Psi(1)|} - 1\right) && (3.9)
\end{aligned}$$

Then (3.6) and (3.8) immediately give (3.3). Inserting (3.9) in (3.7) and using (3.6) and (3.8) we obtain

$$\gamma(0) = 6\frac{p-1+e^{-p}}{(1-e^{-p})(e^p-1)} k\gamma(0) + \frac{2\mu^2 p^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{p}\right) + 2\mu^2.$$

By definition of M_1 and (A.5) we see that

$$M_1 = \frac{2\mu^2 p^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{p}\right) = \frac{2\mu^2 p^2}{\varphi^2}\,\frac{\varphi^2}{|\Psi(2)|p}\int_{\mathbb{R}^4} x^4\,\nu_L(dx) > 0,$$

so that

$$\frac{2}{|\Psi(2)|} - \frac{1}{p} = \frac{M_1\varphi^2}{2\mu^2 p^2}.$$

Inserting this in (3.9) and using (3.3) gives

$$k\gamma(0) = \frac{2\eta\varphi^{-1} + 2\tau_L^2 - 1}{p^3} \frac{M_1\varphi^2}{2} (1 - e^{-p})(e^p - 1),$$

so that

$$0 < pM_2 = \frac{2k\gamma(0)p^2}{M_1(e^p - 1)(1 - e^{-p})} = \frac{2\eta\varphi^{-1} + 2\tau_L^2 - 1}{p}\varphi^2 = \left(2 + \frac{\varphi}{p}\right)\varphi,$$

where we used

$$p = |\Psi(1)| = \eta - \varphi(\mathbb{E}(L_1^2) - \tau_L^2) \tag{3.10}$$

from (2.3). Solving this quadratic equation for φ gives (3.4), which together with (3.10) implies (3.5).

□

We conclude from (3.3)–(3.5) that our model parameter vector (β, η, φ) is a continuous function of the first two moments $\mu, \gamma(0)$ and the parameters of the autocorrelation function p and k. Hence, by continuity, consistency of the moments will immediately imply consistency of the corresponding plug-in estimates for (β, η, φ).

3.2 The estimation algorithm

The parameters are estimated under the following assumptions:

(H1) We have equally spaced observations G_i, $i = 0, \ldots, n$, giving return data $G_i^{(1)} = G_i - G_{i-1}$, $i = 1, \ldots, n$.

(H2) $\mathbb{E}(L_1) = 0$ and $\mathbb{V}\mathrm{ar}(L_1) = 1$, i.e. σ^2 can be interpreted as the volatility.

(H3) The variance τ_L^2 of the Brownian motion component of L is known and in $[0, 1)$.

(H4) $\int_{\mathbb{R}} x^3\, \nu_L(dx) = 0$, $\mathbb{E}(L_1^4) < \infty$ and $\Psi(2) < 0$.

Define the parameter vectors $\boldsymbol{\theta} := (k, p)$ and $\boldsymbol{\vartheta} := (\beta, \varphi, \eta)$, where k and p are as in Theorem 3.1.

Remark 3.2. In Theorem 3.1, under the chosen conditions, $\rho(h) > 0$ for all $h \in \mathbb{N}$. Furthermore, it was shown that M_1 and M_2 are strictly positive. However, this does not imply that the corresponding empirical estimates are strictly positive. As we shall prove in Theorem 3.8 the above estimators are strongly consistent, this implies for almost all sufficiently large sample paths that the empirical estimates will be strictly positive and all parameter estimates are well-defined.

Algorithm 1

(1) Calculate the moment estimator

$$\widehat{\mu}_n := \frac{1}{n} \sum_{i=1}^{n} (G_i^{(1)})^2$$

of μ and for fixed $d \geq 2$ the empirical autocovariances $\widehat{\gamma}_n := (\widehat{\gamma}_n(0), \widehat{\gamma}_n(1), \ldots, \widehat{\gamma}_n(d))^T$ as

$$\widehat{\gamma}_n(h) := \frac{1}{n} \sum_{i=1}^{n-h} \left((G_{i+h}^{(1)})^2 - \widehat{\mu}_n \right) \left((G_i^{(1)})^2 - \widehat{\mu}_n \right), \quad h = 0, \ldots, d\,.$$

(2) Compute the empirical autocorrelations $\widehat{\boldsymbol{\rho}}_n := (\widehat{\gamma}_n(1)/\widehat{\gamma}_n(0), \ldots, \widehat{\gamma}_n(d)/\widehat{\gamma}_n(0))^T$.

(3) For fixed $d \geq 2$ define the mapping $H : \mathbb{R}_+^{d+2} \to \mathbb{R}$ by

$$H(\widehat{\boldsymbol{\rho}}_n, \boldsymbol{\theta}) := \sum_{h=1}^{d} (\log(\widehat{\rho}_n(h)) - \log k + ph)^2 .$$

Compute the least squares estimator

$$\widehat{\boldsymbol{\theta}}_n := \mathrm{argmin}_{\boldsymbol{\theta} \in \mathbb{R}_+^2} H(\widehat{\boldsymbol{\rho}}_n, \boldsymbol{\theta}) . \tag{3.11}$$

(4) Define the mapping $J : \mathbb{R}_+^4 \to [0, \infty)^3$ by

$$J(\mu, \gamma(0), \boldsymbol{\theta}) \quad := \quad \begin{cases} (p\mu, p\sqrt{1 + M_2} - p, p\sqrt{1 + M_2}\,(1 - \tau_L^2) + p\,\tau_L^2) & \text{if} \quad p, M_2 > 0\,, \\ (0, 0, 0) & \text{otherwise,} \end{cases} \tag{3.12}$$

where M_2 is defined as in (3.2). Compute the estimator

$$\widehat{\boldsymbol{\vartheta}}_n = J(\widehat{\mu}_n, \widehat{\gamma}_n(0), \widehat{\boldsymbol{\theta}}_n) .$$

In part (3), alternatively, we could also have based the least squares estimation on the autocovariance function. It turned out, however, that the estimators chosen as above are considerably more accurate. The reason for this is that k is independent of β (see (3.9)) in contrast to $k_\gamma := \mathrm{Cov}((G_i^{(1)})^2, (G_{i+1}^{(1)})^2)\, e^p$.

In addition to Remark 3.2 we emphasize that for a stationary model the parameter p has to be strictly positive. But if we compute the unrestricted minimum of $H(\widehat{\boldsymbol{\rho}}_n, \boldsymbol{\theta})$ we get

$$\widehat{p}_n^* \quad := \quad -\frac{\sum_{h=1}^{d} \left(\log(\widehat{\rho}_n(h)) - \overline{\log(\widehat{\rho}_n)} \right) \left(h - \frac{d+1}{2} \right)}{\sum_{h=1}^{d} \left(h - \frac{d+1}{2} \right)^2} \tag{3.13}$$

$$\widehat{k}_n \quad := \quad \exp\left\{ \overline{\log(\widehat{\rho}_n)} + \frac{d+1}{2} \widehat{p}_n^* \right\} , \tag{3.14}$$

with $\overline{\log(\widehat{\rho}_n)} := \frac{1}{d} \sum_{h=1}^{d} \log(\widehat{\rho}_n(h))$, and $\widehat{p}_n^*$ may be negative. As a remedy we define the estimator of p as

$$\widehat{p}_n := \max\{\widehat{p}_n^*, 0\} \tag{3.15}$$

and take $\widehat{p}_n = 0$ as an indication that the data is non-stationary.

Defining the mapping $S : \mathbb{R}_+^{d+1} \to \mathbb{R}_+^2$ by the equations (3.13)–(3.15) and noting that $\widehat{\rho}_n(h) = \widehat{\gamma}_n(h)/\widehat{\gamma}_n(0)$ presents the least squares estimator $\widehat{\boldsymbol{\theta}}_n := (\widehat{k}, \widehat{p})$ as a function of $\widehat{\gamma}_n$:

$$\widehat{\boldsymbol{\theta}}_n = S(\widehat{\gamma}_n) . \tag{3.16}$$

3.3 Asymptotic properties of the moment estimators

Strong mixing properties guarantee strong consistency and asymptotic normality of the empirical moments. In this section we summarize the necessary results, which we prove in Appendix B. The definition of the strong mixing coefficient can also be found in the Appendix. The following remark is the starting point of our analysis.

Remark 3.3. Let $\sigma^2 := (\sigma_t^2)_{t\geq 0}$ be the strictly stationary volatility process given by (2.2). Then σ_+^2 falls into the class of generalised Ornstein-Uhlenbeck processes; see [11], Section 5. Consequently, the result of [4] applies giving that σ^2 is exponentially β-mixing. This implies in particular that σ^2 is strongly mixing (also called α-mixing) with exponentially decreasing rate.

In the following theorem we show that also the COGARCH(1,1) process satisfies a strong mixing condition.

Theorem 3.4. *Suppose that $(L_t)_{t\geq 0}$ is such that $\mathbb{E}(L_1^4) < \infty$ and the parameters of the* COGARCH(1,1) *process satisfy $\Psi(2) < 0$. Let $(\sigma_t^2)_{t\geq 0}$ be the strictly stationary volatility process given as solution to* (1.3)*. Then for every $r > 0$ the process $(G_{ir}^{(r)})_{i\in\mathbb{N}}$ is α-mixing with exponentially decreasing rate.*

Since we assumed in the above theorem that σ^2 is strictly stationary, the return process is also strictly stationary and together with the strong mixing property this implies that $(G_{ir}^{(r)})_{i\in\mathbb{N}}$ is ergodic. This enables us to apply Birkhoff's ergodic theorem to give strong consistency of the empirical moments and autocovariance function of $((G_i^{(1)})^2)_{i\in\mathbb{N}}$:

Corollary 3.5. *Under the same conditions as in Theorem 3.4 we obtain for $n \to \infty$*

$$\widehat{\mu}_n \overset{\text{a.s.}}{\to} \mathbb{E}((G_t^{(1)})^2)\,, \quad \widehat{\gamma}_n \overset{\text{a.s.}}{\to} \gamma\,. \tag{3.17}$$

Corollary 3.5 will imply strong consistency of the estimators $(\widehat{\vartheta}_n)_{n\in\mathbb{N}}$, as stated in (3.21) below. To obtain asymptotic normality of the empirical estimates we want to apply a central limit theorem for strongly mixing processes.

Proposition 3.6. *Let the same conditions hold as in Theorem 3.4. Assume further*

(H5) *There exists a positive constant $\delta > 0$ such that $\mathbb{E}\left(G_1^{8+\delta}\right) < \infty$.*

Then as $n \to \infty$,

$$\sqrt{n}\left(\begin{bmatrix} \widehat{\mu}_n \\ \widehat{\gamma}_n \end{bmatrix} - \begin{bmatrix} \mu \\ \gamma \end{bmatrix}\right) \quad \overset{d}{\longrightarrow} \quad N_{d+2}\left(\mathbf{0}, \Sigma\right), \tag{3.18}$$

where the covariance Σ has components

$$\begin{aligned}\Sigma_{k+2,l+2} \quad &= \quad \mathrm{Cov}((G_1^{(1)})^2(G_{1+k}^{(1)})^2, (G_1^{(1)})^2(G_{1+l}^{(1)})^2) \\ &\quad\; +2\sum_{j=1}^{\infty}\mathrm{Cov}((G_1^{(1)})^2(G_{1+k}^{(1)})^2, (G_{1+j}^{(1)})^2(G_{1+l+j}^{(1)})^2)\end{aligned}$$

for $k, l = 0, \ldots, d$,

$$\Sigma_{1,k+2} = \mathrm{Cov}((G_1^{(1)})^2, (G_1^{(1)})^2(G_{1+k}^{(1)})^2) + 2\sum_{j=1}^{\infty}\mathrm{Cov}((G_1^{(1)})^2, (G_{1+j}^{(1)})^2(G_{1+k+j}^{(1)})^2)$$

for $k = 0, \ldots, d$ and $\Sigma_{1,1} = \gamma(0) + 2\sum_{h=1}^{\infty} k_\gamma e^{-ph}$.

Proof. We will first concentrate on the asymptotic behaviour of $(\widehat{\mu}_n, \gamma_n^*)$, where $\gamma_n^* = (\gamma_n^*(0), \ldots, \gamma_n^*(d))$ and $\gamma_n^*(h) = \frac{1}{n}\sum_{i=1}^{n}[(G_{i+h}^{(1)})^2 - \mu][(G_i^{(1)})^2 - \mu]$, $h = 0, \ldots, d$. Denote

$$\mathbf{Y}_i := ((G_i^{(1)})^2, [(G_i^{(1)})^2 - \mu]^2, [(G_i^{(1)})^2 - \mu][(G_{i+1}^{(1)})^2 - \mu], \ldots, [(G_i^{(1)})^2 - \mu][(G_{i+d}^{(1)})^2 - \mu])^T.$$

For (3.18) to hold for $(\widehat{\mu}_n, \gamma_n^*)$ in place of $(\widehat{\mu}_n, \widehat{\gamma}_n)$, by the Cramér-Wold device, we have to show that as $n \to \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\lambda}^T \mathbf{Y}_i - \boldsymbol{\lambda}^T \begin{bmatrix} \mu \\ \gamma \end{bmatrix}\right) \xrightarrow{d} N(0, \boldsymbol{\lambda}^T \Sigma \boldsymbol{\lambda}), \tag{3.19}$$

for all vectors $\boldsymbol{\lambda} \in \mathbb{R}^{d+2}$ such that $\boldsymbol{\lambda}^T \Sigma \boldsymbol{\lambda} > 0$. But as strong mixing is preserved under linear transformations as well as the rate, the sequence $(\boldsymbol{\lambda}^T \mathbf{Y}_i)_{i \in \mathbb{N}}$ is strongly mixing with exponentially decaying rate. Hence we get $\sum_{k=1}^{\infty} \{\alpha_{\boldsymbol{\lambda}^T \mathbf{Y}}(k)\}^c < \infty$ for every $c > 0$, and since $\mathbb{E}|Y_i|^{2+\delta} < \infty$ for some $\delta > 0$ by (H5), the central limit theorem for strongly mixing processes is applicable (see Theorem 18.5.3 in [8]).

Therefore, as $n \to \infty$,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\lambda}^T \mathbf{Y}_i - \boldsymbol{\lambda}^T \begin{bmatrix} \mu \\ \gamma \end{bmatrix}\right) \xrightarrow{d} N(0, \sigma^2),$$

with

$$\sigma^2 := \mathbb{V}\mathrm{ar}(\boldsymbol{\lambda}^T \mathbf{Y}_1) + 2\sum_{i=1}^{\infty} \mathbb{C}\mathrm{ov}(\boldsymbol{\lambda}^T \mathbf{Y}_1, \boldsymbol{\lambda}^T \mathbf{Y}_{1+i}). \tag{3.20}$$

Evaluation of (3.20) and rearranging with respect to $\boldsymbol{\lambda}$ shows $\sigma^2 = \boldsymbol{\lambda}^T \Sigma \boldsymbol{\lambda}$. Observing that $\sqrt{n}\left(n^{-1}\sum_{i=1}^{n} \boldsymbol{\lambda}^T \mathbf{Y}_i - \boldsymbol{\lambda}^T \binom{\widehat{\mu}_n}{\widehat{\gamma}_n}\right)$ converges in probability to zero as $n \to \infty$ for every $\boldsymbol{\lambda} \in \mathbb{R}^{d+2}$ such that $\boldsymbol{\lambda}^T \Sigma \boldsymbol{\lambda} > 0$ (see e.g. the proof of Proposition 7.3.4 in [1]), it follows that $(\widehat{\mu}_n, \widehat{\gamma}_n)$ has the same asymptotic behaviour as $(\widehat{\mu}_n, \gamma_n^*)$, giving (3.18).

□

Applying the delta method (see Theorem 3.1 in [19]), we obtain:

Corollary 3.7. *Let the same conditions hold as in Proposition 3.6. Then as $n \to \infty$,*

$$\sqrt{n}(\widehat{\boldsymbol{\rho}}_n - \boldsymbol{\rho}) \xrightarrow{d} N_d(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{æ}}).$$

The following theorem gives asymptotic normality of our parameter estimates. The true parameter vector and the corresponding moments are form now on indicated by $\boldsymbol{\vartheta}_0$, μ_0 and γ_0 respectively. We shall also denote by $P_{\boldsymbol{\vartheta}_0}$ the probability with respect to the parameter vector $\boldsymbol{\vartheta}_0$.

Theorem 3.8. *Let the same conditions hold as in Theorem 3.4. Assume that (H1)–(H4) are satisfied. For $S(\gamma)$ as in (3.16), define the mapping $Q : \mathbb{R}^{d+2} \to \mathbb{R}^3$ by $(\mu, \boldsymbol{\gamma}^T) \mapsto Q((\mu, \boldsymbol{\gamma}^T)) := J(\mu, \gamma(0), S(\boldsymbol{\gamma}))$. Then as $n \to \infty$,*

$$\widehat{\boldsymbol{\vartheta}}_n \xrightarrow{\text{a.s.}} \boldsymbol{\vartheta}_0. \tag{3.21}$$

Assume additionally (H5). Then, under $P_{\boldsymbol{\vartheta}_0}$, as $n \to \infty$,

$$\sqrt{n}(\widehat{\boldsymbol{\vartheta}}_n - \boldsymbol{\vartheta}_0) \xrightarrow{d} \partial_{(\mu,\boldsymbol{\gamma})} Q((\mu_0, \boldsymbol{\gamma}_0))\, N_{d+2}(\mathbf{0}, \Sigma), \tag{3.22}$$

where Σ is as in Proposition 3.6.

Proof. Strong consistency of $\widehat{\boldsymbol{\vartheta}}_n$ follows from (3.17) and the fact that the mapping Q is continuous in $(\mu, \boldsymbol{\gamma})$. Since $(\widehat{\mu}_n, \widehat{\boldsymbol{\gamma}}_n)$ is asymptotically normal and Q is differentiable at $(\mu_0, \boldsymbol{\gamma}_0)$, we can apply the delta method and the asymptotic normality of $\widehat{\boldsymbol{\vartheta}}_n$ follows from (3.18).

□

4 Simulation study

In this section we investigate the small sample behaviour of the moment estimators of Algorithm 1. As the driving Lévy process L we choose a variance gamma (VG) process which has infinite activity and justifies the choice of $\tau_L = 0$, i.e. we assume that L has no Gaussian component. This seems reasonable since the VG process itself is used to model stock log-prices, cf. [12]. Consequently, (H3) is satisfied. Assumption (H2) requires that the mean of L is zero and the variance is equal to one, so that the characteristic function at time $t \geq 0$ is given by

$$\mathbb{E}(e^{iuL_t}) = \left(1 + \frac{u^2}{2C}\right)^{-tC}$$

for $C > 0$. The Lévy measure of L has the Lebesgue density

$$\nu_L(dx) = \frac{C}{|x|} \exp\left(-(2C)^{1/2}|x|\right) dx\,, \quad x \neq 0\,. \tag{4.1}$$

Inserting (4.1) into (**??**) we obtain

$$\Psi(1) = -\eta + \varphi \quad \text{and} \quad \Psi(2) = -2\eta + 2\varphi + 3\varphi^2 C^{-1}\,. \tag{4.2}$$

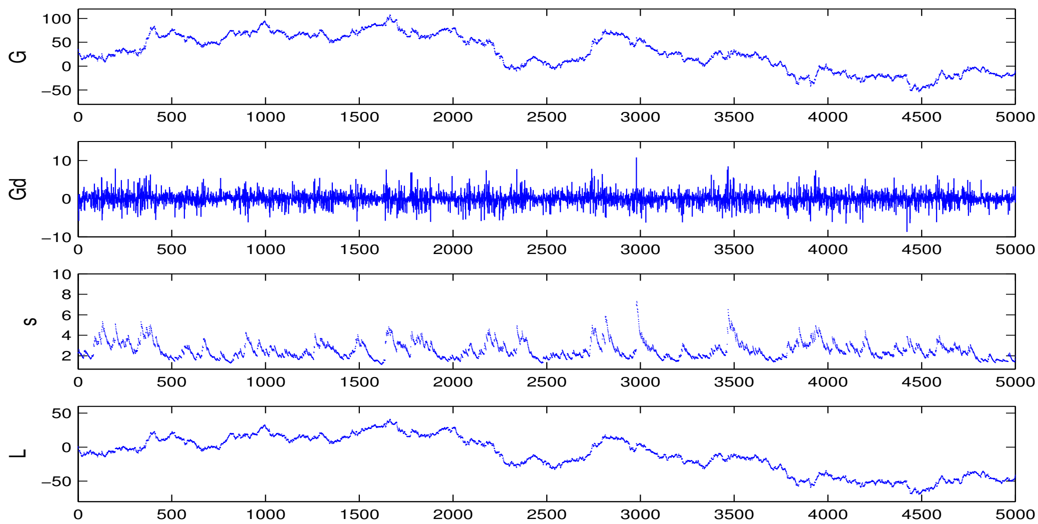

Figure 1: Simulated VG driven COGARCH(1,1) process $(G_t)_{0\leq t\leq 5000}$ with parameters $\beta = 0.04, \eta = 0.053$ and $\varphi = 0.038$ (*first*), the differenced COGARCH(1,1) process $(G_t^{(1)})$ of order 1 (*second*), the volatility process (σ_t^2) (*third*), the driving VG process (L_t) with parameter $C = 1$ (*last*).

The first condition of (H4) is satisfied by symmetry. The only delicate point for choosing the parameters β, η and φ is the last condition of (H4). As indicated in Remark 1.5 the autocovariance function of $(G^{(1)})^2$ should not decrease too fast as is observed in empirical observations. From Proposition 2.1 we know that this is implied by $\Psi(1) < 0$ close to zero. Setting $\beta = 0.04$, $\eta = 0.053$ and $\varphi = 0.038$ gives $\Psi(1) = -0.015$ and $\Psi(2) = -0.0257$ which are satisfactory values. Condition (H5) requires for G a finite moment of higher order than the eighth, which is the case if $\mathbb{E}(L_1^{8+2\delta}) < \infty$ and the $(4+\delta)$-moment of the volatility is finite i.e. $\psi(4+\delta) < 0$. The VG process has finite moments of all orders for every $C > 0$, but for given η and φ the finiteness of $\mathbb{E}(\sigma_1^{8+2\delta})$ depends on C, since

$$\Psi(4) = -4\eta + 4\varphi + 18\varphi^2 C^{-1} + 120\varphi^3 C^{-2} + 630\varphi^4 C^{-3}$$

has to be strictly negative (this then implies $\Psi(4+\delta) < 0$ for some $\delta > 0$ by continuity of Ψ, cf. [18], Lemma 26.4). Therefore we choose $C = 1$, resulting in $\Psi(4) = -0.0261$.

We will perform the estimation procedure for two different sample sizes, namely 5 000 and 20 000. The estimates $\widehat{p}_n$ (3.13) and $\widehat{k}_n$ (3.14) are sensitive to the choice d of lags used and to outliers in the empirical autocorrelation function. Based on experience for linear models (recall Lemma 2.2), it seems reasonable to choose $d \approx \sqrt{n}$. Numerical experiments have indeed shown that d equal to 50 is sufficient for both our sample sizes. Moreover, we performed a robust linear regression (see e.g. Chapter 7 in [7]) to estimate the parameters, i.e. they are estimated by an iteratively reweighted least squares algorithm instead of ordinary least squares. The resulting estimates are not only less sensitive to outliers in the data, but also to the number of lags d taken into account.

4.1 Estimation results

We first simulate 1 000 samples of $n = 5\,000$ equidistant observations of $G^{(1)}$. Table 1 summarizes the estimation results of our simulation study concerning the parameters β, η and φ.

The empirical mean of all the estimated parameter values $\widehat{\beta}_n$, $\widehat{\eta}_n$ and $\widehat{\varphi}_n$ is shown in the first line, with the empirical standard deviations in brackets. We also estimated mean square error (MSE) and mean absolute error (MAE), again with the standard deviation in brackets. The corresponding results for a sample size of $n = 20\,000$ observations are reported in the last three lines of Table 1.

n=5 000	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\varphi}$
Mean	0.04172 (0.00073)	0.04897 (0.00068)	0.03329 (0.00046)
MSE	0.00053 (0.00003)	0.00048 (0.00002)	0.00023 (0.00001)
MAE	0.01772 (0.00046)	0.01724 (0.00043)	0.01208 (0.00029)
n=20 000	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\varphi}$
Mean	0.04309 (0.00043)	0.05311 (0.00038)	0.03689 (0.00026)
MSE	0.00019 ($9 \cdot 10^{-6}$)	0.00015 ($8 \cdot 10^{-6}$)	0.00007 ($4 \cdot 10^{-6}$)
MAE	0.01089 (0.00028)	0.00954 (0.00024)	0.00651 (0.00017)

Table 1: Estimated mean, MSE and MAE for $\widehat{\beta}, \widehat{\eta}$ and $\widehat{\varphi}$ and corresponding estimated standard deviations in brackets. The true values are $\beta = 0.04$, $\eta = 0.053$ and $\varphi = 0.038$.

The three estimators $\widehat{\eta}_n, \widehat{\varphi}_n$ and $\widehat{\beta}_n$ show a similar power. This is not surprising since they are all mappings of $(\widehat{\mu}_n, \widehat{\gamma}_n)$. Actually $\widehat{\beta}_n$ seems to have the largest small sample variance. Whereas for $n = 5\,000$ the estimated bias of $\widehat{\beta}_n$ is the smallest, for $n = 20\,000$ estimated bias and variance of $\widehat{\beta}_n$ are the largest among the three estimators. When one compares the estimates for the different sample sizes it can be seen that the MSE reduces for all three estimators, when the sample size is increased, and the reduction is roughly of a factor of four which would correspond to the asymptotic properties of the estimators.

From (4.2) we know that $\Psi(1)$ is equal to $\Psi(1) = -\eta + \varphi$. Thus these two parameters give important characteristics of the model concerning stationarity and the rate p of decrease of the autocovariance and autocorrelation function. In case of $\widehat{p}_n > 0$, which indicates that the data is stationary, it is also clear from (3.12), that the estimated parameters will always correspond to a stationary model, since $p > 0$ implies $\Psi(1) = -\eta + \varphi < 0$ and the same identity holds for the estimated parameters.

4.2 Estimation of the volatility σ_t^2

Recall from (1.5) for $r = 1$,

$$\sigma_i^2 = \sigma_{i-1}^2 + \beta - \eta \int_{(i-1,i]} \sigma_s^2 ds + \varphi \sum_{i-1<s\le i} \sigma_s^2 (\Delta L_s)^2 , \quad i \in \mathbb{N} . \tag{4.3}$$

Since σ_s is latent and ΔL_s is usually not observable, we have to approximate the integral and the sum on the right hand side. For the integral we use a simple Euler approximation

$$\int_{(i-1,i]} \sigma_s^2 ds \approx \sigma_{i-1}^2 , \quad i \in \mathbb{N} .$$

As we observe G only at integer times we approximate

$$\sum_{i-1<s\le i} \sigma_s^2 (\Delta L_s)^2 \approx (G_i - G_{i-1})^2 = (G_i^{(1)})^2 , \quad i \in \mathbb{N} .$$

An estimate of the volatility process $(\sigma_t^2)_{t\ge 0}$ can therefore be calculated recursively by

$$\widehat{\sigma}_i^2 = \widehat{\beta} + (1-\widehat{\eta})\widehat{\sigma}_{i-1}^2 + \widehat{\varphi}\,(G_i^{(1)})^2 , \quad i \in \mathbb{N} . \tag{4.4}$$

Note that $\widehat{\sigma}_i$ defines the conditional variance of a discrete time GARCH(1,1) model, which implies that we have to require $0 < \eta < 1$. The estimator (4.4) is plotted in Figure 2 together with the theoretical $(\sigma_t^2)_{t\ge 0}$ for one simulation.

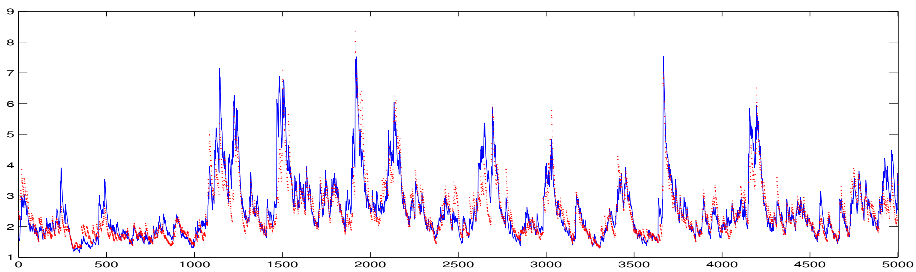

Figure 2: Sample paths of σ_t^2 (solid line) and $\widehat{\sigma}_t^2$ (dotted line) of one simulation.

Next we investigate the goodness of fit of our estimation method by a residual analysis. The estimated residuals are given by $G_i^{(1)}/\widehat{\sigma}_{i-1}$ for $i = 1, \ldots, n$. Since we assumed a symmetric jump distribution with zero mean, the residuals should be symmetric around zero and their mean should be close to zero. Furthermore, if the volatility has been estimated correctly, we expect the standard deviation to be close to one.

Consequently, we estimated mean, MSE, MAE and the corresponding standard deviations for the mean, the standard deviation and the skewness of the residuals $G_i^{(1)}/\widehat{\sigma}_{i-1}$ based on 1 000 simulations. The results for both sample sizes are reported in Table 2 and indicate a reasonable fit.

The correlation of the squared residuals was checked by performing a Ljung-Box test for each sample. For $n = 5\,000$ we computed the test statistic based on $70 \approx \sqrt{5\,000}$ lags and had to reject the null hypothesis of no correlation 140 times out of 1 000 simulations at the 0.05 level. Whereas for $n = 20\,000$ the test statistics were computed using $140 \approx \sqrt{20\,000}$ lags and the null hypothesis was rejected 137 times out of 1 000 simulations again at the 0.05 level.

n=5 000	mean($G_n^{(1)}/\widehat{\sigma}_{n-1}$)	std($G_n^{(1)}/\widehat{\sigma}_{n-1}$)	skewness($G_n^{(1)}/\widehat{\sigma}_{n-1}$)
Mean	0.00011 (0.00044)	1.00931 (0.00021)	-0.00152 (0.00428)
MSE	0.00019 ($9 \cdot 10^{-6}$)	0.00012 ($8 \cdot 10^{-6}$)	0.01838 (0.00098)
MAE	0.01110 (0.00027)	0.00945 (0.00020)	0.10671 (0.00264)
n=20 000	mean($G_n^{(1)}/\widehat{\sigma}_{n-1}$)	std($G_n^{(1)}/\widehat{\sigma}_{n-1}$)	skewness($G_n^{(1)}/\widehat{\sigma}_{n-1}$)
Mean	0.00018 (0.00022)	0.01078 (0.00014)	-0.00285 (0.00205)
MSE	0.00005 ($2 \cdot 10^{-6}$)	0.00013 ($4 \cdot 10^{-6}$)	0.00422 (0.00019)
MAE	0.00577 (0.00013)	0.01079 (0.00014)	0.05119 (0.00126)

Table 2: Estimated mean, MSE and MAE for the mean, standard deviation and skewness of the residuals with corresponding estimated standard deviations in brackets.

5 Real data analysis

The COGARCH(1,1) model will be fitted to five minutes log-returns of three different stocks, which are General Motors (GM), Cisco and Intel. We have tick-by-tick data of the *Trades and Quotes* database of the New York Stock Exchange (NYSE) and Nasdaq. The GM stock is from NYSE, whereas Cisco and Intel belong to Nasdaq. The data spans over 4 months starting in February 2002. We considered only the prices between 9.35am and 4pm to compute the five minutes log-returns based on previous tick interpolation. There were 83 trading days between the beginning of February and the end of May 2002. Hence each of the series has a total length of 6 391 data points. This is part of a data set, which was analysed in [2] with respect to the extreme dependence structure of the three stocks.

The effect of seasonality is common in high frequency data and also appears in the raw data. Therefore, the data was deseasonalised by a median filter, which is explained in Section 4.2 in [2]. The resulting time series are shown in Figure 3.

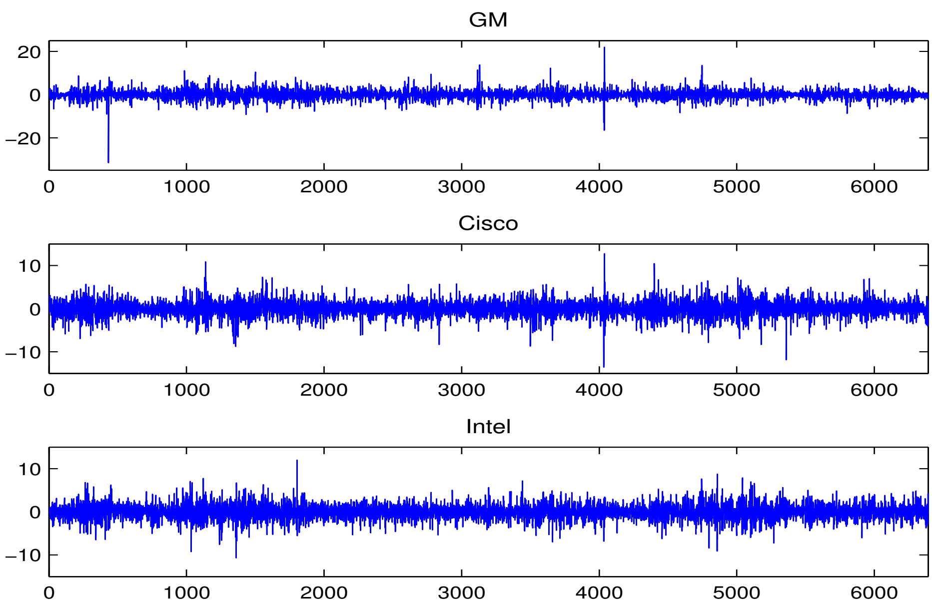

Figure 3: Deseasonalised 5 minutes log-returns of GM (*top*), Cisco (*middle*) and Intel (*bottom*).

An application of Algorithm 1 produces moment estimates of β, η and φ under the assumption that the driving Lévy process es of each stock have no Brownian component. The results are shown in Table 3.

	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\varphi}$
GM	0.1091	0.1625	0.1357
Cisco	0.0621	0.0328	0.0126
Intel	0.0180	0.0396	0.0336

Table 3: $\widehat{\beta}, \widehat{\eta}$ and $\widehat{\varphi}$ for the GM, Cisco and Intel data.

To investigate the model fit, we performed a Ljung-Box test for squared residuals of all three data sets. The test statistics used 80 lags of the corresponding empirical autocorrelation function. The null hypothesis was not rejected for GM and Intel at the 0.05 level. For the GM squared residuals the p-value was 0.35, whereas for Intel it was only 0.27. The test statistic for the Cisco squared residuals was equal to 202.62, which led to a rejection of the null hypothesis, since the test had a critical value of 101.87 at the 0.05 level. This result is also obvious from Figure 4 were the empirical autocorrelation function of the squared residuals are plotted on the right, showing significant correlations of the Cisco residuals.

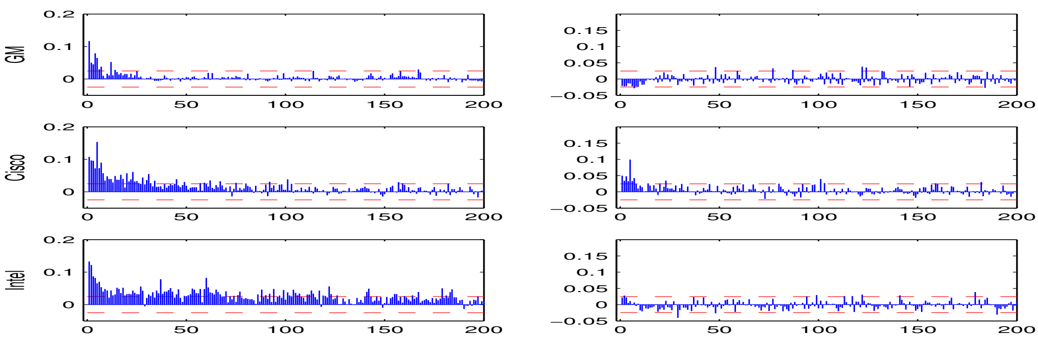

Figure 4: Empirical acf of the squared 5 minutes log-returns (*left*) and the squared residuals (*right*) of GM (*top*), Cisco (*middle*) and Intel (*bottom*).

The estimated mean, standard deviation and skewness of the residuals are summarized in Table 4. The numbers show that the mean and variation of the residuals are according to our model, but that the residuals are significantly skewed. This skewness can also be seen in Figure 5 showing estimates of the log density for all three datasets.

	mean$(G_n^{(1)}/\widehat{\sigma}_{n-1})$	std$(G_n^{(1)}/\widehat{\sigma}_{n-1})$	skewness$(G_n^{(1)}/\widehat{\sigma}_{n-1})$
GM	-0.0143	1.0785	-0.3714
Cisco	-0.0015	0.9832	-0.2082
Intel	-0.0002	1.0100	-0.0626

Table 4: Mean, standard deviation and skewness of the GM, Cisco and Intel residuals.

It does not come as a surprise as it is a well-known fact that financial data are skewed. Although we will not deal with the problem in this first paper on COGARCH(1,1) estimation, we want to discuss the assumptions, which prevent the modelling of skewness and also indicate some remedy to be worked out in detail in future work.

The observed skewness indicates that the first condition of (H4) requiring that $\int_{\mathbb{R}} x^3 \nu_L(dx) = 0$ is violated. This introduces a bias into our estimates as in the calculation of $\mathbb{E}(G_t^4)$ the last term in (A.2) does not disappear.

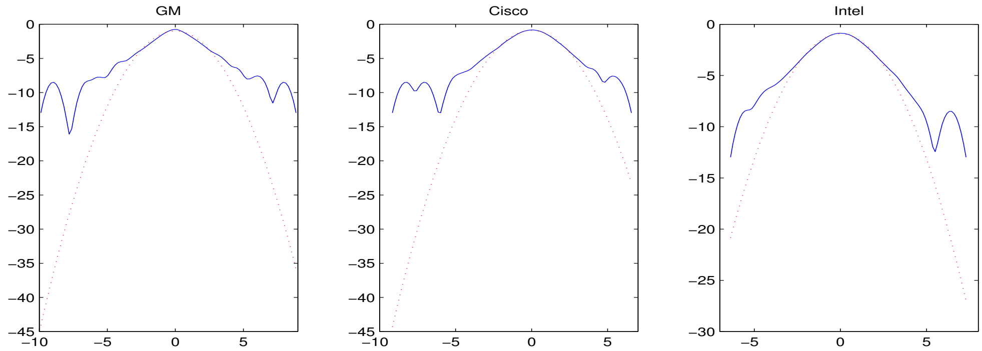

Figure 5: Kernel estimates of the log density (*solid line*) of the squared GM (*left*), Cisco (*middle*) and Intel (*right*) residuals together with a normal log density (*dashed line*) with mean and variance of the corresponding residual series.

Instead of including the term $\int_{\mathbb{R}} x^3 \nu_L(dx)$ in a statistical analysis we suggest to extend the model by an extra term in a similar fashion as [6] for the discrete time GARCH process, to model the leverage in the market explicitly; thus taking care of the effect directly. Consequently, we extend the volatility model (1.3) for $\rho > 0$ to

$$d\sigma_{t+}^2 = (\beta - \eta\sigma_t^2)\, dt + \varphi\, \sigma_t^2 d[L, L]_t^{(d)} + \rho\, \sigma_t^2 dU_t\,, \tag{5.1}$$

where

$$U_t := \sum_{\substack{0<s\le t \\ \Delta L_s < 0}} (\Delta L_s)^2$$

and ρ is a positive constant. Then

$$\Delta\sigma_t^2 = \begin{cases} \varphi\sigma_t^2(\Delta L_t)^2, & \text{if } \Delta L_t > 0, \\ (\varphi + \rho)\sigma_t^2(\Delta L_t)^2, & \text{if } \Delta L_t < 0, \end{cases}$$

so that a negative jump of L gives rise to a higher increase of the volatility than a positive jump of the same modulus does. Note that $(U_t)_{t\ge 0}$ is a subordinator, and so is

$$M_t := \varphi[L, L]_t^{(d)} + \rho U_t = \sum_{\substack{0<s\le t \\ \Delta L_s > 0}} \varphi(\Delta L_s)^2 + \sum_{\substack{0<s\le t \\ \Delta L_s < 0}} (\varphi + \rho)(\Delta L_s)^2,$$

so that (5.1) can be rewritten as

$$d\sigma_{t+}^2 = (\beta - \eta\sigma_t^2)\, dt + \sigma_t^2\, dM_t,$$

an expression similar to (1.3).

Acknowledgement

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Appendix

A Calculating the moments

Proof of Proposition 2.1. Since L has finite variance and zero mean, it is a square integrable martingale. Further, $\Psi(1) < 0$ implies $\mathbb{E}(\sigma_t^2) = \frac{\beta}{|\Psi(1)|} < \infty$ by (2.7), and it follows easily from the properties of the stochastic integral that

$$\mu := \mathbb{E}(G_t^2) = \mathbb{E}[G,G]_t = \mathbb{E}\int_0^t \sigma_s^2\, d[L,L]_s = \mathbb{E}[L,L]_1 \int_0^t \mathbb{E}(\sigma_s^2)\, ds\,,$$

giving that $\mathbb{E}(G_t^2)$ is finite and has the form specified in (2.4). The remaining equations in (2.4) are shown as in Proposition 5.1 of [9].

Suppose that $\mathbb{E}(L_1^4) < \infty$ and $\Psi(2) < 0$. Then $\mathbb{E}(G_t^4)$ is finite by the Burkholder-Davis-Gundy inequality, cf. [17], p. 222, since

$$\mathbb{E}\left([G,G]_t^2\right) = \mathbb{E}\left(\int_0^t \sigma_s^2\, d[L,L]_s\right)^2$$

is finite as a consequence of $\mathbb{E}(\sigma_t^4) < \infty$ and $\mathbb{E}(L_1^4) < \infty$.

Now suppose additionally that $\int_{\mathbb{R}} x^3\, \nu_L(dx) = 0$. To calculate the value of $\mathbb{E}(G_t^4)$, observe that by integration by parts,

$$G_t^2 = 2\int_0^t G_{s-}\, dG_s + [G,G]_t = 2\int_0^t G_{s-}\sigma_s\, dL_s + \int_0^t \sigma_s^2\, d[L,L]_s, \tag{A.1}$$

$$\begin{aligned} G_t^4 &= 2\int_0^t G_{s-}^2\, dG_s^2 + [G^2,G^2]_t \\ &= 4\int_0^t G_{s-}^3\sigma_s\, dL_s + 2\int_0^t G_{s-}^2\sigma_s^2\, d[L,L]_s \\ &\quad +4\int_0^t G_{s-}^2\sigma_s^2\, d[L,L]_s + \int_0^t \sigma_s^4\, d\big[[L,L],[L,L]\big]_s \\ &\quad +4\int_0^t G_{s-}\sigma_s^3\, d\big[[L,L],L\big]_s. \end{aligned} \tag{A.2}$$

Taking expectations in (A.2), the first and the last summand vanish due to the assumptions $\mathbb{E}L_1 = 0$ and $\int_{\mathbb{R}} x^3\, \nu_L(dx) = 0$, respectively, so that

$$\mathbb{E}(G_t^4) = 6\mathbb{E}(L_1^2)\int_0^t \mathbb{E}(G_{s-}^2\sigma_s^2)\, ds + \int_{\mathbb{R}} x^4\, \nu_L(dx) \int_0^t \mathbb{E}(\sigma_s^4)\, ds. \tag{A.3}$$

The expression $\mathbb{E}(G_{s-}^2\sigma_s^2)$ was already calculated in the proof of Proposition 5.1 in [9], however, under additional assumptions which required in particular bounded variation of L. The following calculations do not require these restrictions.

Let $Y_t := \int_0^t G_{s-}\sigma_s\, dL_s$, $t \geq 0$. Then $\mathbb{E}(Y_t) = 0$ for all $t \geq 0$, and integration by parts and substituting from (1.3) give

$$\begin{aligned} Y_t\sigma_{t+}^2 &= \int_0^t Y_{s-}\, d\sigma_{s+}^2 + \int_0^t \sigma_s^2\, dY_s + [\sigma_+^2, Y]_t \\ &= \int_0^t Y_{s-}(\beta - \eta\sigma_s^2)\, ds + \int_0^t Y_{s-}\varphi\sigma_s^2\, d[L,L]_t^{(d)} \\ &\quad + \int_0^t \sigma_s^3 G_{s-}\, dL_s + \left[\int_0^\cdot (\beta - \eta\sigma_s^2)\, ds + \int_0^\cdot \varphi\sigma_s^2\, d[L,L]_s^{(d)}, \int_0^\cdot G_{s-}\sigma_s\, dL_s\right]_t. \end{aligned}$$

Taking expectations gives

$$\begin{aligned}\mathbb{E}(Y_t\sigma_{t+}^2) &= \left(\varphi(\mathbb{E}(L_1^2)-\tau_L^2)-\eta\right)\int_0^t \mathbb{E}(Y_{s-}\sigma_s^2)\,ds + \mathbb{E}\int_0^t \varphi\sigma_s^3 G_{s-}\,d\sum_{0<u\le s}(\Delta L_u)^3\\ &= \left(\varphi(\mathbb{E}(L_1^2)-\tau_L^2)-\eta\right)\int_0^t \mathbb{E}(Y_s\sigma_{s+}^2)\,ds,\end{aligned}$$

where we used that $\int_{\mathbb{R}} x^3\,\nu_L(dx)=0$ and that $Y_{s-}\sigma_s^2 = Y_s\sigma_{s+}^2$ almost surely for fixed s. Solving this integral equation and using that $Y_0=0$ implies $\mathbb{E}(Y_0\sigma_{0+}^2)=0$, it follows that $\mathbb{E}(Y_t\sigma_{t+}^2)=0$ for all $t\ge 0$. Substituting

$$\int_0^t \sigma_s^2\,d[L,L]_s = \int_0^t \sigma_s^2\tau_L^2\,ds + \varphi^{-1}\left(\sigma_{t+}^2-\sigma_0^2-\int_0^t(\beta-\eta\sigma_s^2)\,ds\right)$$

from (1.3), equations (A.1) and (2.8) now give

$$\begin{aligned}\mathbb{E}(G_t^2\sigma_{t+}^2) &= \mathbb{E}\left(\sigma_{t+}^2\int_0^t\sigma_s^2\,d[L,L]_s\right)\\ &= (\tau_L^2+\varphi^{-1}\eta)\int_0^t \mathbb{E}(\sigma_t^2\sigma_s^2)\,ds + \varphi^{-1}\mathbb{E}(\sigma_t^4)-\varphi^{-1}\mathbb{E}(\sigma_t^2\sigma_0^2)-\varphi^{-1}\beta\mathbb{E}(\sigma_t^2)t\\ &= (\tau_L^2+\varphi^{-1}\eta)\mathbb{V}\mathrm{ar}(\sigma_0^2)\frac{1-e^{-t|\Psi(1)|}}{|\Psi(1)|} + \varphi^{-1}\mathbb{V}\mathrm{ar}(\sigma_0^2)(1-e^{-t|\Psi(1)|})\\ &\quad + \left((\tau_L^2+\eta\varphi^{-1})(\mathbb{E}(\sigma_0^2))^2-\beta\varphi^{-1}\mathbb{E}(\sigma_0^2)\right)t. \end{aligned}\tag{A.4}$$

Using (2.7), (2.8) and $\Psi(1) = -\eta+\varphi\left(\mathbb{E}(L_1^2)-\tau_L^2\right)$ then leads to

$$\mathbb{E}(G_t^2\sigma_{t+}^2) = \frac{\beta^2}{\psi(1)^2}\left(\frac{2}{|\Psi(2)|}-\frac{1}{|\Psi(1)|}\right)(\frac{2\eta}{\varphi}+2\tau_L^2-\mathbb{E}(L_1^2))(1-e^{-t|\Psi(1)|})+\frac{\beta^2}{\Psi(1)^2}\mathbb{E}(L_1^2)t.$$

This then implies (2.5), where we used (A.3), (2.8) and the fact that by (2.3)

$$\int_{\mathbb{R}} x^4\,\nu_L(dx) = \frac{\Psi(2)-2\Psi(1)}{\varphi^2}\,. \tag{A.5}$$

For the autocorrelation of the squared increments, observe that by equation (5.4) of [9] we have

$$\mathbb{C}\mathrm{ov}((G_t^{(r)})^2,(G_{t+h}^{(r)})^2) = \left(\frac{e^{r|\Psi(1)|}-1}{|\Psi(1)|}\right)\mathbb{E}(L_1^2)\mathbb{C}\mathrm{ov}(G_r^2,\sigma_r^2)\,e^{-h|\Psi(1)|} \tag{A.6}$$

(in [9] this was stated under the additional assumption that L is a quadratic pure jump process (i.e. $\tau_L^2=0$), but it can be seen that the proof given there holds true also for L having a Brownian motion component). This then implies (2.6) by (A.4), (2.4) and (2.7).

B Strong mixing property of $(G_{ir}^{(r)})_{i\in\mathbb{N}}$

We want to show that the sequence $(G_{ir}^{(r)})_{n\in\mathbb{N}}$ is α-mixing with exponentially fast decreasing mixing coefficients. Recall the following definitions.

Definition B.1. *For a stationary process $Y=(Y_s)_{s\in\mathbb{R}}$ define the σ-algebras $\mathcal{F}_1=\mathcal{F}_{(0,u]}^Y := \sigma((Y_s)_{0\le s\le u})$ and $\mathcal{F}_2=\mathcal{F}_{[u+t,\infty)}^Y := \sigma((Y_s)_{s\ge u+t})$ for any $u\ge 0$.*

(a) *Y is called α-*mixing *(or* strongly mixing*), if as $t \to \infty$*

$$\alpha(t) = \alpha(\mathcal{F}_1, \mathcal{F}_2) := \sup_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} |P(A \cap B) - P(A)P(B)| \to 0 .$$

*Y is called α-*mixing with exponential rate*, if $\alpha(t) \le Ke^{-at}$ for $K, a > 0$ for all $t \ge 0$.*

(b) *For $t \ge 0$ denote by $b\mathcal{F}^Y_{[t,\infty)}$ the set of bounded $\mathcal{F}^Y_{[t,\infty)}$-measurable random variables. Let $\| \cdot \|_{L^1(P)}$ and $\| \cdot \|_\infty$ be the L^1-norm under P and the supremum norm, respectively. Then Y is called $\widetilde{\alpha}$-*mixing*, if as $t \to \infty$*

$$\widetilde{\alpha}(t) = \widetilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) := \sup_{f \in b\mathcal{F}_2, \|f\|_\infty \le 1} \|\mathbb{E}(f|\mathcal{F}_1) - \mathbb{E}(f)\|_{L^1(P)} \to 0 .$$

*Y is called $\widetilde{\alpha}$-*mixing with exponential rate*, if $\widetilde{\alpha}(t) \le \widetilde{K}e^{-\widetilde{a}t}$ for $\widetilde{K}, \widetilde{a} > 0$ for all $t \ge 0$.*

The following result shows that any α-mixing property is equivalent to the corresponding $\widetilde{\alpha}$-mixing property.

Lemma B.2. *$\alpha(\mathcal{F}_1, \mathcal{F}_2) \le \widetilde{\alpha}(\mathcal{F}_1, \mathcal{F}_2) \le 6\alpha(\mathcal{F}_1, \mathcal{F}_2)$ holds for all $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$.*

For the left-hand inequality assume $A^* \in \mathcal{F}_1$ and $B^* \in \mathcal{F}_2$ are such that $\alpha(\mathcal{F}_1, \mathcal{F}_2) = |P(A^* \cap B^*) - P(A^*)P(B^*)|$. Then take $f = \chi_{B^*}$ as the indicator of the set B^*, which implies $|P(A^* \cap B^*) - P(A^*)P(B^*)| \le \|\mathbb{E}(\chi_{B^*}|A^*) - \mathbb{E}(\chi_{B^*})\|_{L^1(P)}$. See Lemma 3.5 in [13] for the right-hand inequality.

Proof of Theorem 3.4. We show that $(G^{(r)}_{ir})_{i \in \mathbb{N}}$ is $\widetilde{\alpha}$-mixing. Define the σ-algebra $\mathcal{F}^{dL}_I := \sigma(L_t - L_s : s, t \in I)$ for $I \subset \mathbb{R}$; i.e. generated from all increments of L over the interval I. Using similarly notation as in Definition B.1 for the σ-algebras generated by the return process $(G^{(r)}_{ir})_{i \in \mathbb{N}}$ and volatility process $(\sigma^2_t)_{t \ge 0}$, we can formulate the following inclusions

$$\mathcal{F}^{G^{(r)}}_{\{1,\dots,l\}} \subset \mathcal{F}^{dL}_{[0,lr]} \vee \mathcal{F}^{\sigma^2}_{[0,lr]} =: \mathcal{F}^{dL,\sigma^2}_{[0,lr]} ,$$

where $\vee$ denotes the σ-algebra generated by the union of the two σ-algebras, and

$$\mathcal{F}^{G^{(r)}}_{\{k+l,k+l+1,\dots\}} \subset \mathcal{F}^{dL}_{[(k+l-1)r,\infty)} \vee \mathcal{F}^{\sigma^2}_{[(k+l-1)r,\infty)} =: \mathcal{F}^{dL,\sigma^2}_{[(k+l-1)r,\infty)} .$$

Using these relations we get

$$\begin{aligned}
\widetilde{\alpha}_{G^{(r)}}(k) &:= \sup \left\{ \|\mathbb{E}(f|\mathcal{F}^{G^{(r)}}_{\{1,\dots,l\}}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}^{G^{(r)}}_{\{k+l,k+l+1,\dots\}}, \|f\|_\infty \le 1 \right\} \\
&\le \sup \left\{ \|\mathbb{E}(f|\mathcal{F}^{dL,\sigma^2}_{[0,lr]}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}^{dL,\sigma^2}_{[(k+l-1)r,\infty)}, \|f\|_\infty \le 1 \right\} \\
&= \sup \left\{ \|\mathbb{E}(f|\mathcal{F}^{dL,\sigma^2}_{[0,lr]}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}^{dL}_{[(k+l-1)r,\infty)} \vee b\mathcal{F}^{\sigma^2}_{\{(k+l-1)r\}}, \|f\|_\infty \le 1 \right\} \\
&= \sup \left\{ \|\mathbb{E}(f|\mathcal{F}^{\sigma^2}_{[0,lr]}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}^{dL}_{[(k+l-1)r,\infty)} \vee b\mathcal{F}^{\sigma^2}_{\{(k+l-1)r\}}, \|f\|_\infty \le 1 \right\} \\
&= \sup \left\{ \|\mathbb{E}(f|\mathcal{F}^{\sigma^2}_{[0,lr]}) - \mathbb{E}(f)\|_{L^1(P)} : f \in b\mathcal{F}^{d[L,L]^d}_{[(k+l-1)r,\infty)} \vee b\mathcal{F}^{\sigma^2}_{\{(k+l-1)r\}}, \|f\|_\infty \le 1 \right\} \\
&= \widetilde{\alpha}_{\sigma^2}((k-1)r).
\end{aligned}$$

The first equality (in the third line) holds since the driving process of σ^2 is the discrete part of the quadratic variation of L. $\mathcal{F}^{dL}_{[0,lr]}$ and $b\mathcal{F}^{dL}_{[(k+l-1)r,\infty)} \vee b\mathcal{F}^{\sigma^2}_{\{(k+l-1)r\}}$ are conditionally independent given $\mathcal{F}^{\sigma^2}_{[0,lr]}$, which is due to the Markov property of σ^2 (see Theorem 3.2 in

[9]) and the independence between $\mathcal{F}^{dL}_{[0,lr]}$ and $b\mathcal{F}^{dL}_{[(k+l-1)r,\infty)}$. This gives the second equality. The third one follows from the independence of $b\mathcal{F}^{dL}_{[(k+l-1)r,\infty)}$ and $b\mathcal{F}^{d[L,L]^d}_{[(k+l-1)r,\infty)}$ from $\mathcal{F}^{\sigma^2}_{[0,lr]}$ and again the Markov property of σ^2, whereas the last one is due to the fact that $b\mathcal{F}^{\sigma^2}_{[(k+l-1)r,\infty)} = b\mathcal{F}^{d[L,L]^d}_{[(k+l-1)r,\infty)} \vee b\mathcal{F}^{\sigma^2}_{\{(k+l-1)r\}}$.

Consequently, $(G^{(r)}_{ir})_{i\in\mathbb{N}}$ inherits the mixing properties from the volatility process σ^2. It has been shown in [4] that σ^2 is α-mixing with exponential rate, i.e. there exist constants $K, a > 0$ such that

$$\widetilde{\alpha}_{G^{(r)}}(k) \leq \widetilde{\alpha}_{\sigma^2}((k-1)r) \leq 6\alpha_{\sigma^2}((k-1)r) \leq 6Ke^{-a(k-1)r},$$

implying that $(G^{(r)}_{ir})_{i\in\mathbb{N}}$ is α-mixing with exponential rate.

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