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Parametric Order Reduction of Proportionally Damped Second Order Systems

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Abstract

In this paper, structure preserving order reduction of proportionally damped and undamped second order systems is presented. The discussion is based on *Second Order Krylov Subspace* method and it is shown that for systems with a proportional damping, the damping matrix does not contribute to the projection matrices and the reduction can be carried out using the classical Krylov subspaces. As a result of direct projection, the reduced order model is parameterized in terms of the damping coefficients.

Keywords: Order reduction, Second order systems, Proportional damping, Moment Matching, Second order Krylov subspaces.

1 Introduction

Model order reduction based on Krylov subspaces [2, 7] has been originally developed for the reduction of first order systems. However, quite often in engineering, it is necessary to deal with second order systems as they are results of common modern modeling techniques [12, 13, 17]. A possibility to apply the Krylov subspace methods to second order system requires a transformation to first order (the so called linearization) which is undesirable since the structure of the original system is destroyed during model reduction.

In [4, 18], Krylov subspaces were used to reduce second order systems while preserving their structure. This approach has been revisited by different authors in recent years proposing alternative approaches and some improvements [8, 19]. Also, special model reduction methods based on the so called *Second Order Krylov Subspaces* have been developed to treat second order systems directly [3, 11, 15, 16].

In the present paper, by considering a special class of second order systems, namely proportionally damped, a simplified alternative to the structure preserving order reduction of second order systems is suggested.

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The large-scale models considered here are assumed to be given in the form,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\mathbf{z}(t), \end{cases} \tag{1}$$

with n second order differential equations, m inputs and p outputs. The total order of the system is $N = 2n$ and the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ are called mass, damping and stiffness matrices, respectively. In addition, it is assumed that the damping is proportional, i.e. $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$, which is widely used in engineering [6]. In practice, the coefficients α and β are chosen based on experimental results and previous experience and can vary for the same structural model depending on the external conditions. These facts pose an additional requirement for simulation, that is, the free variation of α and β without repeating the reduction procedure.

One of the reasons of its widespread usage is that proportional damping does not change the eigenspace of the original undamped problem [6]. For instance, when one uses the mode superposition method for model reduction for a model with proportional damping [1], then:

i. modal analysis is performed for the original undamped system,

ii. a few most important modes are selected,

iii. α and β are used as parameters for the reduced system.

Based on this idea, in [9, 14] a model reduction method, which preserves α and β as parameters, has been suggested. It consists of first performing a Krylov-based model reduction for the original undamped model, and then applying the generated projection matrices to the proportionally damped system to obtain the final reduced model.

The validity of the approach has been empirically shown in [9, 14], however without any mathematical proof on moment matching. One of the goals of this paper is to bridge this gap by providing this missing proof: It is shown that the projection matrix used for the reduction of proportionally damped system by moment matching is independent of the damping matrix and thus the parameters α and β.

The paper is organized as follows. In the following section, the second order Krylov subspace method is reviewed and in section 3, the order reduction of proportionally damped systems is discussed. Reduction of undamped systems is the focus of section 4 and in section 5.2, the proposed approach is applied to reduce a technical system.

2 Order Reduction of Second Order Systems

The reduction approach considered in this paper is based on matching some of the characteristic parameters of the original and reduced systems that are called moments. The moments are defined as the negative coefficients of the Taylor series expansion of the transfer matrix [7]. Assuming that $\mathbf{K}$ is nonsingular, the moments (about zero) of the second order system (1) are,

$$\mathbf{m}_i = \mathbf{C}\left(\mathbf{A}^{-1}\mathbf{E}\right)^i \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{L}\mathbf{K}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{M}\mathbf{K}^{-1} \\ \mathbf{I} & -\mathbf{D}\mathbf{K}^{-1} \end{bmatrix}^i \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}.$$

The original system (1) is reduced by applying a projection, $\mathbf{z} = \mathbf{V}\mathbf{z}_r$, with $\mathbf{V} \in \mathbb{R}^{n\times q}$, $q < n$, and then multiplying the state equation by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{n\times q}$, resulting in a reduced model of order $Q = 2q$,

$$\begin{cases} \mathbf{W}^T\mathbf{M}\mathbf{V}\ddot{\mathbf{z}}_r + \mathbf{W}^T\mathbf{D}\mathbf{V}\dot{\mathbf{z}}_r + \mathbf{W}^T\mathbf{K}\mathbf{V}\mathbf{z}_r = \mathbf{W}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{V}\mathbf{z}_r. \end{cases} \tag{2}$$

For the choice of $\mathbf{V}$ and $\mathbf{W}$ to match the moments, the *Second Order Krylov Subspaces* [15] that are defined as,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1) = colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{q-1}\}, \tag{3}$$

$$\text{where} \begin{cases} \mathbf{P}_0 = \mathbf{G}_1 \ , \ \mathbf{P}_1 = \mathbf{A}_1\mathbf{P}_0 \\ \mathbf{P}_i = \mathbf{A}_1\mathbf{P}_{i-1} + \mathbf{A}_2\mathbf{P}_{i-2}, \ i = 2, 3, \cdots \end{cases} \tag{4}$$

where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n\times n}, \mathbf{G}_1 \in \mathbb{R}^{n\times m}$ are constant matrices, can be used. The columns of $\mathbf{G}_1$ are called the starting vectors and the matrices $\mathbf{P}_i$ are called the basic blocks.

Theorem 1. *If the columns of the matrix* $\mathbf{V}$ *used in (2) form a basis for the input Second Order Krylov Subspace* $\mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ *and* $\mathbf{W}$ *is chosen such that* $\mathbf{K}_r$ *is nonsingular, then the first* q *moments (*$\mathbf{m}_0$ *to* $\mathbf{m}_{q-1}$*) of the original and reduced models match.*

In one-sided methods as mentioned in Theorem 1, $\mathbf{W} = \mathbf{V}$ is a typical choice that have also some advantages in preserving stability [11, 16]. The number of matching parameters can be increased by using two second order Krylov subspaces for the choice of $\mathbf{V}$ and $\mathbf{W}$ but the details are omitted here.

By matching the moments about zero, the low frequency behavior of the original system is well approximated. However, to approximate the higher frequency behavior, the moments about $s_0 \neq 0$ are to be matched. It can be shown that this is achieved by substituting the matrices $\mathbf{K}$ by $\mathbf{K}_{s_0} = \mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M}$ and $\mathbf{D}$ by $\mathbf{D}_{s_0} = \mathbf{D}+2s_0\mathbf{M}$, in the corresponding Second Order Krylov Subspace [16]. In this case the condition of non-singularity of $\mathbf{K}$ is substituted by non-singularity of $\mathbf{K}_{s_0}$. In other words, s_0 should not be a quadratic eigenvalue of the triple $(\mathbf{M}, \mathbf{D}, \mathbf{K})$.

3 Proportionally Damped Systems

Let the second order system (1) be proportionally damped. First, it is shown how for this family of second order systems, the second order krylov subspaces used for moment matching about zero can be reduced to the classical Krylov subspaces without affecting the moment matching property.

Theorem 2. *If* $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$ *with* $\alpha \neq 0$ *then,*

$$\mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}) = \mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$$

Proof: Let $\mathbf{P}_i$ and $\hat{\mathbf{P}}_i$ be the basic blocks of the second order and standard Krylov subspaces, respectively. It is shown that the basic blocks of the two subspaces span the same space by proving that the i-th basic block of one subspace can be written as a linear combination of the first i blocks of the other.

The starting vectors are clearly the same, $\mathbf{P}_0 = \hat{\mathbf{P}}_0$. For the next basic block, we have,

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1}\mathbf{G} = \mathbf{K}^{-1}(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{K}^{-1}\mathbf{G} \\ &= \alpha\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{G} + \beta\mathbf{K}^{-1}\mathbf{G} = \alpha\hat{\mathbf{P}}_1 + \beta\hat{\mathbf{P}}_0 \end{aligned}$$

Now consider that $\mathbf{P}_i = \sum_{j=0}^{i} c_j\hat{\mathbf{P}}_j$ for $i = 0, \cdots, k-1$. For $i = k$, we have,

$$\begin{aligned} \mathbf{P}_k &= -\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_{k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{k-2} = -\mathbf{K}^{-1}(\alpha\mathbf{M} + \beta\mathbf{K})\mathbf{P}_{k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{k-2} \\ &= (-\alpha\mathbf{K}^{-1}\mathbf{M} - \beta)\sum_{j=0}^{k-1} c_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-2} c_j\hat{\mathbf{P}}_j = \alpha\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j - \beta\sum_{j=0}^{k-1} c_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k-1} c_j\hat{\mathbf{P}}_j. \end{aligned}$$

The proof is completed by induction. ■

Theorem 2 shows that the projection matrix $\mathbf{V}$ can be calculated using the conventional Krylov subspace and applied directly to reduce the original second order system. Here, it should be noted that during this work an alternative proof to Theorem 2, which does not use the second order Krylov methods, has been published in [5].

In the following, a special class of systems with a damping only proportional to the stiffness is discussed.

Theorem 3. *If* $\mathbf{D} = \beta\mathbf{K}$ $(\alpha = 0)$ *then,*

$$\mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}) = \mathcal{K}_{\frac{q}{2}}(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$$

Proof: Following the proof of Theorem 2, in this case $\mathbf{P}_i = -\beta\mathbf{P}_{i-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{i-2}$. The starting vectors are clearly the same. For the next blocks,

$$\begin{aligned}\mathbf{P}_1 &= -\beta\mathbf{P}_0 = -\beta\hat{\mathbf{P}}_0,\\ \mathbf{P}_2 &= -\beta\mathbf{P}_1 - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_0 = \beta\hat{\mathbf{P}}_0 + \beta\hat{\mathbf{P}}_1.\end{aligned}$$

Now consider that $\mathbf{P}_{2i} = \sum_{j=0}^{i} c_j\hat{\mathbf{P}}_j$ and $\mathbf{P}_{2i+1} = \sum_{j=0}^{i} d_j\hat{\mathbf{P}}_j$ for $i = 0, \cdots, k-1$. For $i = k$, we have,

$$\begin{aligned}\mathbf{P}_{2k} &= -\beta\mathbf{P}_{2k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{2k-2} = -\beta\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-1} c_j\hat{\mathbf{P}}_j\\ &= -\beta\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j,\\ \mathbf{P}_{2k+1} &= -\beta\mathbf{P}_{2k} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{2k-1} = \beta^2\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \beta\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j\\ &= \beta^2\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \beta\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k} d_j\hat{\mathbf{P}}_j.\end{aligned}$$

The proof is completed by induction showing that,

$$\begin{aligned}&colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{2k+1}\} =\\ &colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{2k}\} \subset colspan\left\{\hat{\mathbf{P}}_0, \hat{\mathbf{P}}_1, \cdots, \hat{\mathbf{P}}_k\right\}.\end{aligned}$$

■

Theorem 3 shows also that if $\alpha = 0$, deleting all the odd indexed basic blocks from the second order Krylov subspace does not affect the subspace and reduces the order of the reduced system to q (instead of $2q$) while matching the same number of moments. From the preceding two theorems, it is also remarked, that the projection matrices are independent of the damping matrix, and thus of α and β.

These results can be generalized for the case of matching the moments about $s_0 \neq 0$. The system matrices involved in the calculation of the second order Krylov subspaces for this case are:

$$\mathbf{M}_{s_0} = \mathbf{M}, \quad (5)$$

$$\mathbf{D}_{s_0} = \beta\mathbf{K} + (2s_0 + \alpha)\mathbf{M}, \quad (6)$$

$$\mathbf{K}_{s_0} = (1 + s_0\beta)\mathbf{K} + (s_0^2 + s_0\alpha)\mathbf{M}. \quad (7)$$

By proper manipulation of the equations (5)-(7), the matrix $\mathbf{D}_{s_0}$ can be rewritten as $\gamma\mathbf{K}_{s_0}+\lambda\mathbf{M}_{s_0}$ with

$$\gamma = \frac{\beta}{1+s_0\beta},\ \ \lambda = 2s_0 + \alpha - \frac{\beta(s_0^2+s_0\alpha)}{1+s_0\beta}$$

Thus, the modified system is still proportionally damped, and by using Theorem 2,

$$\mathcal{K}_q(-\mathbf{K}_{s_0}^{-1}\mathbf{D}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{M}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{G}) = \mathcal{K}_q(-\mathbf{K}_{s_0}^{-1}\mathbf{M}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{G}). \tag{8}$$

A convenient choice of the expansion point for this class of systems would be $s_0 = -\frac{\alpha}{2}$, as it results in a damping only proportional to the stiffness and makes Theorem 3 applicable here and consequently reduces by half the order of the reduced model.

4 Undamped Systems

In this section, the second order system (1) is considered to be undamped and thus $\mathbf{D} = 0$. The second order Krylov subspace for moment matching about zero for undamped systems is $\mathcal{K}_q(\mathbf{0}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ resulting in the following projection matrix,

$$\begin{aligned} colspan(\mathbf{V}) &= colspan\left\{-\mathbf{K}^{-1}\mathbf{G}, \mathbf{0}, \mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{G}, \mathbf{0}, \dots\right\} \\ &= \mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}) \end{aligned}$$

This fact is also clear from Theorem 3 when setting $\beta = 0$.

It is well known, based on (9) and (3), that the moments can be expressed as a function of the basic blocks $\mathbf{P}_i$ of the second order Krylov subspace. As a consequence, the odd-indexed moments are zero as their corresponding $\mathbf{P}_i$ are zero. Another way to examine this fact is to calculate the moments of the undamped second order system about zero:

$$\mathbf{m}_i = \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix} \tag{9}$$

By simple matrix manipulation, it is clearly seen that for all odd i, the corresponding moment is zero.

An advantage of applying a direct projection for the reduction of second order systems is that if the original model is undamped, the reduced system is undamped too. This is helpful to match the zero moments automatically as stated by the following remark:

Remark 1. If an undamped second order system is reduced by a one-sided method and moment matching to order $2q$ ($\mathbf{W} = \mathbf{V} \in \mathbb{R}^{n\times q}$), $2q$ moments match with the odd-indexed one among them equal to zero.

When matching the moments about $s_0 \neq 0$ for an undamped system, the involved matrices in the calculations of the second order Krylov subspaces are the following:

$$\begin{aligned} &\mathbf{M}_{s_0} = \mathbf{M},\ \mathbf{D}_{s_0} = 2s_0\mathbf{M}, \\ &\mathbf{K}_{s_0} = \mathbf{K} + s_0^2\mathbf{M}. \end{aligned}$$

It can be clearly seen that this is just a special case of proportional damping with the matrix $\mathbf{D}_{s_0}$ only proportional to the matrix $\mathbf{M}_{s_0}$. Hence, the Krylov subspace of equation (8) should be used for the reduction procedure.

5 Experimental Results and Discussions

The clamped beam model described in [10] has been used to demonstrate the effectiveness of the suggested method. The mechanical model shown in Fig. 1 is a typical structure whose generic layout corresponds to, e.g., atomic force microscopy tips as well as radio frequency switches and filters. The real-life system consists of the beam and a counter electrode placed below it. A voltage source generates a potential difference between the two electrodes that creates an attraction force between them and therefore results in a deformation of the flexible beam. In the considered model, this attraction force, shown in Fig. 1 as −F(t), is applied directly to the right end of the beam. It has to be pointed out, that with these assumptions, only the mechanical problem has been considered.

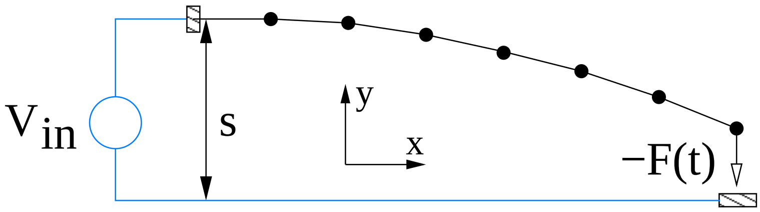

Figure 1: A conducting clamped beam with counter electrode below right end.

5.1 Modeling of the elastic beam

The model of the beam is extracted under some approximations like numerical discretization, constraints on the degrees of freedom and material properties [10]. The partial differential equation is then approximated by an ordinary differential equation through finite element discretization. For symmetry reasons, the beam motion can be constrained to a plane, yielding a two-dimensional (2D) motion. In this case, three possible beam deflections can be observed [20]:

- **Torsional displacements:** A rotation about the beams longitudinal axis.
- **Axial displacements:** Compression or expansion of the beam along its longitudinal axis.
- **Flexural displacements:** Deflecting the beam out of its plane undeformed axis.

It is assumed that the beam deflection is small, so that geometric nonlinearities can be neglected. This allows to impose another constraint on the beam motion: x and y deflection are decoupled. It also assumed that the possible deflections are smaller than the distance between the beams so that no contact occurs. The material used is assumed to be isotropic and ideally elastic with no plastic deformation or brittle fracture. As common in micro-mechanics, gravity may be neglected.

A Lagrangian formulation is used to determine the equations of motion following the treatment in [20] for calculation of the energies. The damping matrix $\mathbf{D}$ is usually calculated by a linear combination of the stiffness and mass matrices using the mode-preserving Rayleigh damping formulation which results in a proportionally damped system. More details of the implementation are available in [10].

5.2 Reduced order modeling of the beam

We consider a model of order $N = 15992$ with $n = 7996$ second order differential equations with a proportional damping[1]. The output of the system is the state number 5996 which is the displacement of the point on the last two-third of the beam. The reduction is carried out for two cases: the undamped case and a damped model with $\alpha = 100,\ \beta = 10^{-7}$.

The original model is reduced to different orders by the proposed approach of this paper and by applying the Second Order Krylov method. In Table 1, the elapsed time of the two approaches are compared when reducing to order 3.

Table 1: Comparison between the reduced order models

Damping	Order $Q = 2q$	Method	elapsed time
Undamped	6	Proposed approach	0.037 s
Undamped	6	Second Order Krylov	0.26 s
Damped	6	Proposed approach	0.037 s
Damped	6	Second Order Krylov	0.18 s

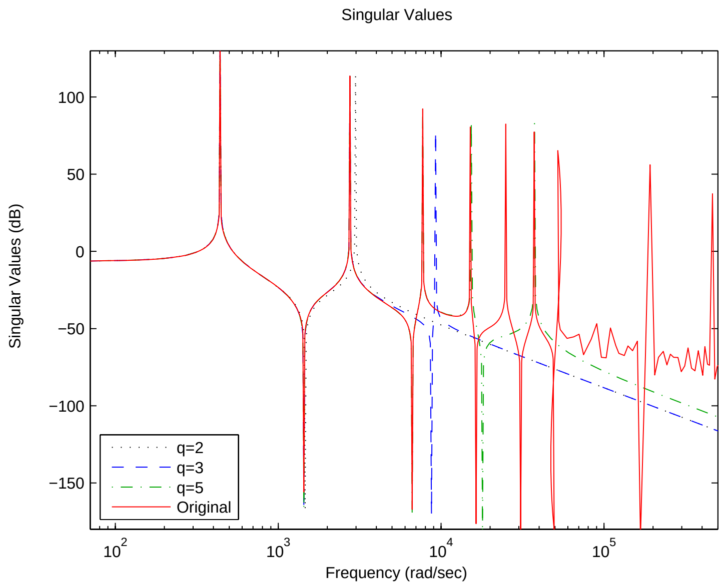

Figure 2: Frequency response of the reduced systems of the undamped model.

Since the proposed approach is independent of damping, less calculation is required in each step and the calculation is faster. Even if the damping is zero, for matching the moments about zero, the second order Krylov method requires double number of iterations since half of the columns of the projection matrix should be deleted. Furthermore, the new approach finds the projection matrix only once and independent of the damping while for second order Krylov method, it should be calculated separately for every choice of the damping coefficients.

[1] The model can be downloaded from Oberwolfach Model Reduction Benchmark Collection available online at http://www.imtek.uni-freiburg.de/simulation/benchmark/

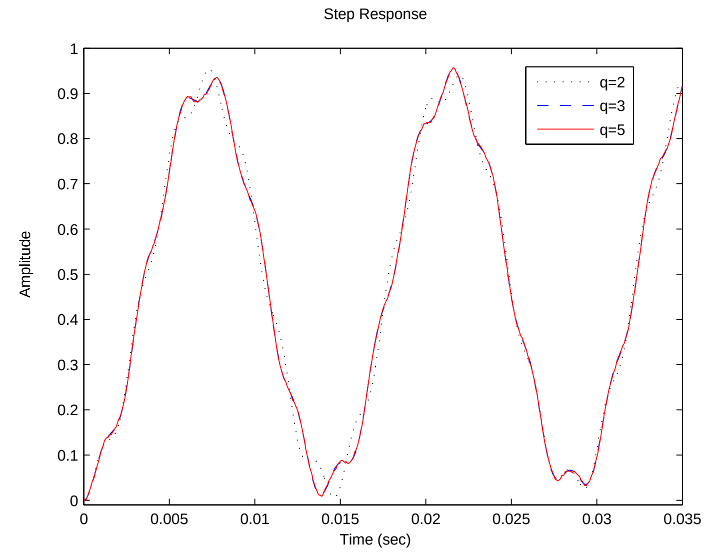

Figure 3: Step Response of the reduced systems of the undamped model.

The step and frequency responses of the reduced systems for the undamped case are shown in Figures 2 and 3.

The simulation results of the proportionally damped model can be seen in Figures 4 and 5.

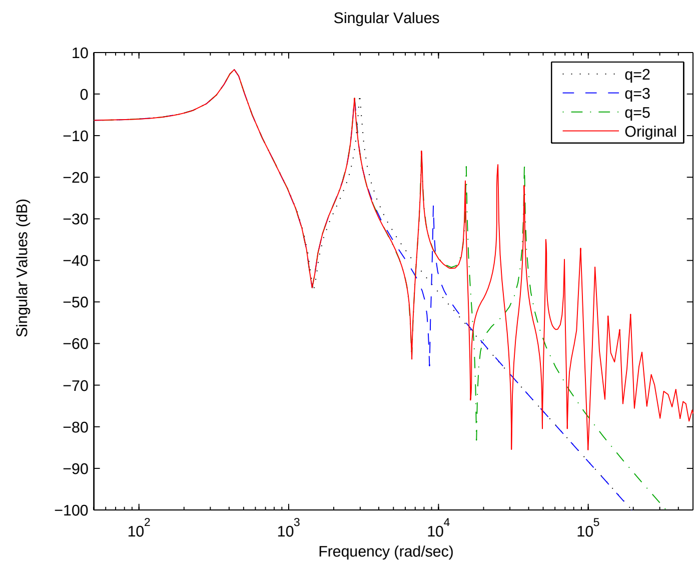

Figure 4: Frequency response of the reduced systems of the damped model.

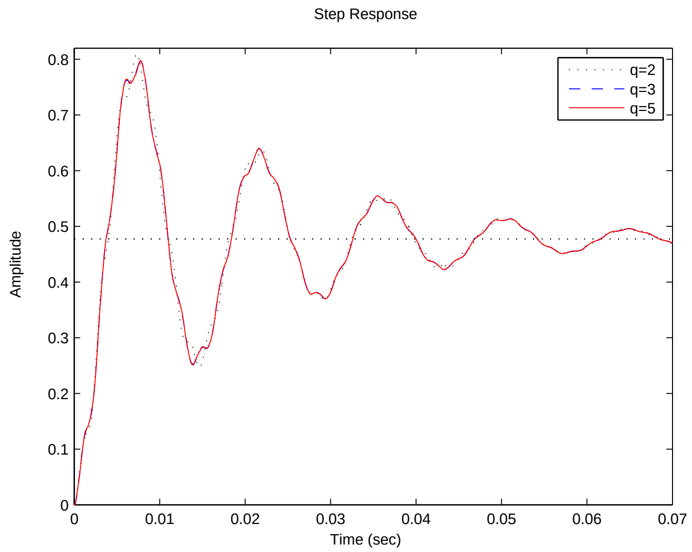

Figure 5: Step Response of the reduced systems of the damped model.

For both cases, the step response is approximated very well by a reduced model of order $Q = 6$ ($q = 3$). In fact, by increasing the order, the step response changes very slightly.

In the frequency response, it is remarked that by going to higher orders better accuracy at higher frequencies is achieved. Because of preserving the second order structure, the slope of the bode plots at high frequencies is $-40dB/decade$.

6 Conclusions

In this paper, it was proved that if the damping of the original second order system is proportional to the mass and stiffness matrices, the projection matrix can be calculated using the standard Krylov subspace.

The proposed approach not only preserves the second order structure but also carries out all calculations in the dimension of the second order system without going to state space. It is stressed that in the case of proportional damping, the projection matrices, that guarantee moment matching, are calculated independently from α and β, unlike the second order Krylov methods that require a new calculation of the projection matrices each time the damping coefficients are changed. In other words, despite the theoretical and computational simplicity of the new method compared to the methods of [3, 11, 15], it still allows us to achieve exactly the same results and to preserve α and β as parameters in the symbolic form during the reduction procedure. Furthermore, as it is based on the classical Krylov subspaces, the new proposed method takes advantage of the numerical reliable and well-known algorithms of standard Krylov subspace method like Arnoldi and Lanczos to calculate the required projection matrices.

Finally, in Table 2, the number of matching parameters for the different cases discussed in this paper is summarized.

Table 2: Number of matched moments when reducing using a one sided-method ($\mathbf{W} = \mathbf{V}$) to order $2q$

System	α	s_0	Matched Moments
Undamped ($\mathbf{D} = 0$)	-	0	2q
	-	$\neq 0$	q
Prop. Damped ($\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$)	0	0	2q
	$\neq 0$	0	q
	$\neq 0$	$\neq 0$	q

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